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PARANORMED RIESZ DIFFERENCE SEQUENCE SPACES OF FRACTIONAL ORDER

TAJA YAYING[1]

Abstract. In this article we introduce paranormed Riesz difference sequence spaces of fractional order α, $r_0^t\left(p, \Delta^{(\alpha)}\right)$, $r_c^t\left(p, \Delta^{(\alpha)}\right)$ and $r_\infty^t\left(p, \Delta^{(\alpha)}\right)$ defined by the composition of fractional difference operator $\Delta^{(\alpha)}$, defined by $(\Delta^{(\alpha)}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k-i}$, and Riesz mean matrix R^t. We give some topological properties, obtain the Schauder basis and determine the α-, β- and γ- duals of the new spaces. Finally, we characterize certain matrix classes related to these new spaces.

1. Introduction

Throughout the paper $\Gamma(m)$ will denote the gamma function of all real numbers $m \notin \{0, -1, -2, \ldots\}$. $\Gamma(m)$ can be expressed as an improper integral given by

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx. \tag{1.1}$$

Using (1.1), we state some properties of gamma function which are used throughout the text:

1. for $m \in \mathbb{N}$, $\Gamma(m+1) = m!$;
2. for any real number $m \notin \{0, -1, -2, \ldots\}$, $\Gamma(m+1) = m\Gamma(m)$;
3. for particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!, \ldots$

Throughout the paper $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and let w be the space of all real valued sequences. By ℓ_∞, c_0 and c we mean the spaces all bounded, null and convergent

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Key words and phrases. Riesz difference sequence spaces, difference operator $\Delta^{(\alpha)}$, Schauder basis, α-, β-, γ- duals, matrix transformation.

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sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_1, cs and bs, we mean the spaces of absolutely summable, convergent series and bounded series, respectively. The space ℓ_1 is normed by $\sum_k |x_k|$ and the spaces cs and bs are normed by $\sup_n \left|\sum_{k=0}^n x_k\right|$. Here and henceforth, the summation without limit runs from zero to ∞. Also, let $e = \{1, 1, 1 \ldots\}$ and $e^{(k)}$ be the sequences whose only non-zero term is 1 in the k^{th} place for each $k \in \mathbb{N}$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $M = \max\{1, H\}$, where $H = \sup_k p_k$. Then, Maddox [43, 44] defined the sequence spaces $\ell_\infty(p)$, $c_0(p)$, $c(p)$ and $\ell(p)$ as follows:

$$\begin{aligned}
\ell_\infty(p) &= \left\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\
c_0(p) &= \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}, \\
c(p) &= \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\right\}
\end{aligned}$$

and

$$\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\},$$

which are complete spaces paranormed by

$$g(x) = \sup_{k\in\mathbb{N}} |x_k|^{\frac{p_k}{M}} \quad \text{and} \quad h(x) = \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}}.$$

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex entries. Then A defines a matrix mapping from X to Y if for every sequence $x = (x_k)$, the A-transform of x, i.e., $Ax = \{(Ax)_n\} \in Y$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad n \in \mathbb{N}. \tag{1.2}$$

The sequence space X_A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.3}$$

is called the domain of matrix A.

By (X, Y), we denote the class of all matrices A from X to Y. Thus $A \in (X, Y)$ if and only if the series on the R.H.S. of the (1.2) converges for each $n \in \mathbb{N}$ and $x \in X$ such that $Ax \in Y$ for all $x \in X$.

The notion of difference sequence space $X(\Delta)$ for $X = \{\ell_\infty, c, c_0\}$ was introduced by Kızmaz [40]. Since then several authors [15–19, 21–24] generalized the notion of difference operator Δ and studied various sequence spaces of integer order. However, for a positive proper fraction α, Baliarsingh and Dutta [10] (see also [11, 12, 20]) have defined a generalized fractional difference operator $\Delta^{(\alpha)}$ and its inverse as

$$(\Delta^{(\alpha)}x)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k-i}, \tag{1.4}$$

$$(\Delta^{(-\alpha)}x)_k = \sum_i (-1)^i \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} x_{k-i}. \tag{1.5}$$

Throughout the paper it is assumed that the series on the R.H.S. of (1.4) and (1.5) are convergent for $x = (x_k) \in w$. It is more convenient to express $\Delta^{(\alpha)}$ as a triangle

$$(\Delta^{(\alpha)})_{nk} = \begin{cases} (-1)^{n-k} \dfrac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Moreover, Dutta and Baliarsingh [20] also studied the paranormed difference sequence spaces of fractional order $X(\Gamma, \Delta^{\tilde{\alpha}}, u, p)$ for $X = \{c_0, c, \ell_\infty\}$, where

$$(\Delta^{\tilde{\alpha}}x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}.$$

Furthermore, Baliarsingh and Dutta [11] studied the sequence spaces $X(\Gamma, \Delta^{\tilde{\alpha}}, p)$ for $X = \{c_0, c, \ell_\infty\}$. For some nice papers on fractional difference operator and related sequence spaces, one may refer to [10–13, 20, 25–34] and the references mentioned therein.

Let (t_k) be a sequence of positive numbers and let

$$T_n = \sum_{k=0}^{n} t_k, \quad n \in \mathbb{N}.$$

The Riesz mean matrix $R^t = (r^t_{nk})$ was defined in [1, 3] as

$$r^t_{nk} = \begin{cases} \dfrac{t_k}{T_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

The Riesz sequence spaces r^t_∞, r^t_0 and r^t_c were introduced by Malkowsky [3] as follows:

$$r^t_\infty = (\ell_\infty)_{R^t}, \quad r^t_0 = (c_0)_{R^t} \quad \text{and} \quad r^t_c = (c)_{R^t}.$$

Altay and Başar [1] introduced the paranormed Riesz sequence spaces $r^t(p)$ as

$$r^t(p) = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k \right|^{p_n} < \infty \right\}.$$

The paranormed Riesz sequence spaces $r^t_\infty(p)$, $r^t_0(p)$ and $r^t_c(p)$ were studied by Altay and Başar [2] as follows:

$$r^t_\infty(p) = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k \right|^{p_n} < \infty \right\},$$

$$r_0^t(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k \right|^{p_n} = 0 \right\} \text{ and}$$
$$r_c^t(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k - l \right|^{p_n} = 0 \text{ for some } l \in \mathbb{R} \right\}.$$

Since then various authors studied Riesz sequence spaces. One may refer to [1–7] and the references cited therein for more studies on Riesz sequence spaces. Following Altay and Başar [1, 2] and Baliarsingh [12], we construct a more generalised Riesz paranormed difference sequence spaces of fractional order and study in detail the related problems.

2. Riesz Difference Operator of Fractional Order and Sequence Spaces

In this section, we define the product matrix $R^t(\Delta^{(\alpha)})$, obtain its inverse, introduce paranormed Riesz difference sequence spaces of fractional order $r_\infty^t\left(p, \Delta^{(\alpha)}\right)$, $r_c^t\left(p, \Delta^{(\alpha)}\right)$ and $r_0^t\left(p, \Delta^{(\alpha)}\right)$ and give some topological properties of the spaces.

Combining the Riesz mean matrix R^t and the difference operator $\Delta^{(\alpha)}$, we obtain a new product matrix $R^t(\Delta^{(\alpha)}) = (\tilde{r}_{nk}^t)$ given by

$$\tilde{r}_{nk}^t = \begin{cases} \sum\limits_{i=k}^{n} (-1)^{i-k} \dfrac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)} \dfrac{t_i}{T_n}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Equivalently,

$$R^t(\Delta^{(\alpha)}) = \begin{bmatrix} 1 & 0 & 0 & \dots \\ \dfrac{t_0}{T_1} - \alpha\dfrac{t_1}{T_1} & \dfrac{t_1}{T_1} & 0 & \dots \\ \dfrac{t_0}{T_2} - \alpha\dfrac{t_1}{T_2} + \dfrac{\alpha(\alpha-1)}{2!}\dfrac{t_2}{T_2} & \dfrac{t_1}{T_2} - \alpha\dfrac{t_2}{T_2} & \dfrac{t_2}{T_2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, by simple calculation, one may obtain the inverse of the matrix $R^t(\Delta^{(\alpha)})$ as given in the following lemma.

Lemma 2.1. *The inverse of the product matrix $R^t(\Delta^{(\alpha)})$ is given by*

$$(R^t(\Delta^{(\alpha)}))^{-1}_{nk} = \begin{cases} (-1)^{n-k} \sum\limits_{j=k}^{k+1} \dfrac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \dfrac{T_k}{t_j}, & \textit{if } 0 \le k < n, \\ \dfrac{T_n}{t_n}, & \textit{if } k = n, \\ 0, & \textit{if } k > n. \end{cases}$$

Let us define a sequence $y = (y_n)$ which will be frequently used as the $R^t(\Delta^{(\alpha)})$-transform of the sequence $x = (x_k)$ as follows:

$$y_n = \sum_{k=0}^{n-1} \left[\sum_{i=k}^{n} (-1)^{i-k} \frac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)} \frac{t_i}{T_n} \right] x_k + \frac{t_n}{T_n} x_n, \quad n \in \mathbb{N}. \tag{2.1}$$

Now, we define the paranormed Riesz difference sequence spaces of fractional order α, $r_\infty^t\left(p, \Delta^{(\alpha)}\right)$, $r_c^t\left(p, \Delta^{(\alpha)}\right)$ and $r_0^t\left(p, \Delta^{(\alpha)}\right)$ as follows:

$$\begin{aligned}
r_\infty^t\left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in \ell_\infty(p) \right\}, \\
r_c^t\left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in c(p) \right\}, \\
r_0^t\left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in c_0(p) \right\}.
\end{aligned}$$

Using the notation (1.3), the above sequence spaces may be rewritten as:

$$\begin{aligned}
r_\infty^t\left(p, \Delta^{(\alpha)}\right) &= (\ell_\infty(p))_{R^t(\Delta^{(\alpha)})}, \\
r_c^t\left(p, \Delta^{(\alpha)}\right) &= (c(p))_{R^t(\Delta^{(\alpha)})}, \\
r_0^t\left(p, \Delta^{(\alpha)}\right) &= (c_0(p))_{R^t(\Delta^{(\alpha)})}.
\end{aligned}$$

The above sequence spaces reduce to the following classes of sequence spaces in the special cases of α and $p = (p_k)$:

1. if $\alpha = 0$ then above classes reduce to $X(p)$ for $X = \{r_\infty^t, r_c^t, r_0^t\}$ as studied by Altay and Başar [2], which further reduce to X in the case of $p = (p_k) = e$ as studied by Malkowsky [3];
2. if $\alpha = 1$ then above classes reduce to $X(p, \Delta^{(1)})$ for $X = \{r_\infty^t, r_c^t, r_0^t\}$, where $(\Delta^{(1)}x)_k = x_k - x_{k-1}$;
3. if $\alpha = m$ then above classes reduce to $X(p, \Delta^{(m)})$ for $X = \{r_\infty^t, r_c^t, r_0^t\}$, where $(\Delta^{(m)}x)_k = \sum\limits_{j=0}^{m} (-1)^j \binom{m}{j} x_{m-j}$.

We begin with the following result.

Lemma 2.2. *The operator $R^t(\Delta^{(\alpha)}) : w \to w$ is linear.*

Proof. The proof is a routine verification and hence omitted. □

Theorem 2.1. *The sequence space $r_0^t\left(\Delta^{(\alpha)}\right)$ is a linear metric space paranormed by*

$$g_{\Delta^{(\alpha)}}(x) = \sup_{k\in\mathbb{N}} \left| \left(R^t(\Delta^{(\alpha)})x \right)_k \right|^{\frac{p_k}{M}}. \tag{2.2}$$

$g_{\Delta^{(\alpha)}}$ is paranorm for the spaces $r_\infty^t(p, \Delta^{(\alpha)})$ and $r_c^t(p, \Delta^{(\alpha)})$ only in the trivial case, with $\inf p_k > 0$ when $r_\infty^t(p, \Delta^{(\alpha)}) = r_\infty^t(\Delta^{(\alpha)})$ and $r_c^t(p, \Delta^{(\alpha)}) = r_0^t(\Delta^{(\alpha)})$.

Proof. We prove the theorem for the space $r_0^t(\Delta^{(\alpha)})$.

Clearly, $g_{\Delta^{(\alpha)}}(\theta) = 0$ and $g_{\Delta^{(\alpha)}}(-x) = g_{\Delta^{(\alpha)}}(x)$ for all $x \in r_0^t\left(\Delta^{(\alpha)}\right)$. To show the linearity of $g_{\Delta^{(\alpha)}}$ with respect to coordinate wise addition and scalar multiplication, we take any two sequences $u, v \in r_0^t(p, \Delta^{(\alpha)})$ and scalars α_1 and α_2 in $\mathbb{R}$. Since $R^t(\Delta^{(\alpha)})$ is linear and using Maddox [45], we get

$$\begin{aligned}
&g_{\Delta^{(\alpha)}}(\alpha_1 u + \alpha_2 v)\\
&= \sup_k \left| \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)} \frac{t_i}{T_k} \right] (\alpha_1 u_j + \alpha_2 v_j) + \frac{t_k}{T_k}(\alpha_1 u_k + \alpha_2 v_k) \right|^{\frac{p_k}{M}}\\
&\leq \max\{1, |\alpha_1|\} \sup_k \left| \left(R^t(\Delta^{(\alpha)}) u \right)_k \right|^{\frac{p_k}{M}} + \max\{1, |\alpha_2|\} \sup_k \left| \left(R^t(\Delta^{(\alpha)}) v \right)_k \right|^{\frac{p_k}{M}}\\
&= \max\{1, |\alpha_1|\} g_{\Delta^{(\alpha)}}(u) + \max\{1, |\alpha_2|\} g_{\Delta^{(\alpha)}}(v).
\end{aligned}$$

This follows the subadditivity of $g_{\Delta^{(\alpha)}}$, i.e.,

$$g_{\Delta^{(\alpha)}}(x+y) \leq g_{\Delta^{(\alpha)}}(x) + g_{\Delta^{(\alpha)}}(y), \quad \text{for all } x, y \in r_0^t\left(p, \Delta^{(\alpha)}\right).$$

Let $\{x^n\}$ be any sequence of points in $r_0^t(p, \Delta^{(\alpha)})$ such that $g_{\Delta^{(\alpha)}}(x^n - x) \to 0$ and also (β_n) be any sequence of scalars such that $\beta_n \to \beta$ as $n \to \infty$. Then by using the subadditivity of $g_{\Delta^{(\alpha)}}$, we get

$$g_{\Delta^{(\alpha)}}(x^n) \leq g_{\Delta^{(\alpha)}}(x) + g_{\Delta^{(\alpha)}}(x^n - x).$$

Now, since $\{g_{\Delta^{(\alpha)}}(x^n)\}$ is bounded, we have

$$\begin{aligned}
g_{\Delta^{(\alpha)}}(\beta_n x^n - \beta x) &= \sup_k \Bigg| \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)} \frac{t_i}{T_k} \right] (\beta_n x_j^n - \beta x_j)\\
&\qquad + \frac{t_k}{T_k}(\beta_n x_k^n - \beta x_k) \Bigg|^{\frac{p_k}{M}}\\
&\leq |\beta_n - \beta|^{\frac{p_k}{M}} g_{\Delta^{(\alpha)}}(x^n) + |\beta|^{\frac{p_k}{M}} g_{\Delta^{(\alpha)}}(x^n - x)\\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, scalar multiplication for $g_{\Delta^{(\alpha)}}$ is continuous. Consequently, $g_{\Delta^{(\alpha)}}$ is a paranorm on the sequence space $r_0^t(p, \Delta^{(\alpha)})$. This completes the proof of the theorem. ☐

Theorem 2.2. *The sequence space $r_0^t(p, \Delta^{(\alpha)})$ is a complete linear metric space paranormed by $g_{\Delta^{(\alpha)}}$ defined in* (2.2)*.*

Proof. Let $x^i = \{x_k^{(i)}\}$ be any Cauchy sequence in $r_0^t(p, \Delta^{(\alpha)})$. Then for $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$ such that

$$g_{\Delta^{(\alpha)}}(x^i - x^j) < \varepsilon,$$

for all $i, j \geq N_0(\varepsilon)$. This implies that $\{(R^t(\Delta^{(\alpha)})x^0)_k, (R^t(\Delta^{(\alpha)})x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for each fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, the sequence $((R^t(\Delta^{(\alpha)})x^i)_k)$ converges. We assume that $(R^t(\Delta^{(\alpha)})x^i)_k \to (R^t(\Delta^{(\alpha)})x)_k$ as $i \to \infty$. Now, for each $k \in \mathbb{N}$, $j \to \infty$ and $i \geq N_0(\varepsilon)$, it is clear that

$$\left| (R^t(\Delta^{(\alpha)})x^i)_k - (R^t(\Delta^{(\alpha)})x)_k \right| < \frac{\varepsilon}{2}. \tag{2.3}$$

Again, $x^i = \{x_k^{(i)}\} \in r_0^t(p, \Delta^{(\alpha)})$. This implies that

$$\left|(R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}, \tag{2.4}$$

for all $k \in \mathbb{N}$. Therefore, using (2.3) and (2.4), we obtain

$$\begin{aligned}\left|(R^t(\Delta^{(\alpha)})x)_k\right|^{\frac{p_k}{M}} &\leq \left|(R^t(\Delta^{(\alpha)})x)_k - (R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} + \left|(R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for all $i \geq N_0(\varepsilon)$. This shows that the sequence $((R^t(\Delta^{(\alpha)})x)_k)$ belongs to the space $c_0(p)$. Since (x^i) is any arbitrary Cauchy sequence, the space $r_0^t(p, \Delta^{(\alpha)})$ is complete. □

Theorem 2.3. *The paranormed Riesz difference sequence spaces $r_0^t(p, \Delta^{(\alpha)})$, $r_c^t(p, \Delta^{(\alpha)})$ and $r_\infty^t(p, \Delta^{(\alpha)})$ are linearly isomorphic to $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. We prove the result for the space $r_\infty^t(p, \Delta^{(\alpha)})$. Using the notation (2.1), we define a mapping $\varphi : r_\infty^t(p, \Delta^{(\alpha)}) \to \ell_\infty(p)$ by $x \mapsto y = \varphi x$. Clearly, φ is linear and $x = 0$ whenever $\varphi x = 0$. Thus, φ is injective.

Let $y = (y_k) \in \ell_\infty(p)$ and using (2.1) define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1}\left[\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{T_j}{t_i}y_j\right] + \frac{T_k}{t_k}y_k, \quad k \in \mathbb{N}. \tag{2.5}$$

Then

$$\begin{aligned}g_{\Delta^{(\alpha)}}(x) &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}(-1)^{i-j}\frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)}\frac{t_i}{T_k}\right]x_j + \frac{t_k}{T_k}x_k\right|^{\frac{p_k}{M}} \\ &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{\frac{p_k}{M}} \\ &= \sup_{k\in\mathbb{N}}|y_k|^{\frac{p_k}{M}} < \infty,\end{aligned}$$

where

$$\delta_{kj} = \begin{cases}1, & \text{if } k = j, \\ 0, & \text{if } k \neq j.\end{cases}$$

Thus, $x \in r_\infty^t(p, \Delta^{(\alpha)})$. Consequently, φ is surjective and paranorm preserving. Thus, $r_\infty^t(p, \Delta^{(\alpha)}) \cong \ell_\infty(p)$. □

3. Schauder Basis

In this section, we shall construct the Schauder basis for the sequence spaces $r_0^t(p,\Delta^{(\alpha)})$ and $r_c^t(p,\Delta^{(\alpha)})$.

We recall that a sequence (x_k) of a normed space $(X,\|\cdot\|)$ is called a Schauder basis for X if for every $u\in X$ there exist a unique sequence of scalars (a_k) such that

$$\lim_{n\to\infty}\left\|u-\sum_{k=0}^{n}a_kx_k\right\|=0.$$

Theorem 3.1. *Let $\lambda_k(t)=(R^t(\Delta^{(\alpha)})x)_k$ for all $k\in\mathbb{N}$ and $0<p_k\le H<\infty$. Define the sequence $b^{(k)}(t)=(b_n^{(k)}(t))$ of the elements of the space $r_0^t(p,\Delta^{(\alpha)})$ for every fixed $k\in\mathbb{N}$ by*

$$b_n^{(k)}(t)=\begin{cases}\displaystyle\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{T_j}{t_i}, & \text{if } k<n,\\ \dfrac{T_n}{t_n}, & \text{if } k=n,\\ 0, & k>n.\end{cases}$$

Then

(a) *the sequence $(b^{(k)}(t))$ is basis for the space $r_0^t(p,\Delta^{(\alpha)})$ and every $x\in r_0^t(p,\Delta^{(\alpha)})$ has a unique representation of the form*

$$x=\sum_k\lambda_k(t)b^{(k)}(t); \tag{3.1}$$

(b) *the set $\{(R^t(\Delta^{(\alpha)}))^{-1}e,b^{(k)}(t)\}$ is a basis for the space $r_c^t(p,\Delta^{(\alpha)})$ and every $x\in r_c^t(p,\Delta^{(\alpha)})$ has a unique representation of the form*

$$x=le+\sum_k|\lambda_k(t)-l|\,b^{(k)}(t),$$

where $l=\lim_{k\to\infty}(R^t(\Delta^{(\alpha)})x)_k$.

Proof. (a) By the definition of $R^t(\Delta^{(\alpha)})$ and $b^{(k)}(t)$, it is clear that

$$\Big(R^t(\Delta^{(\alpha)})b^{(k)}(t)\Big)=e^{(k)}\in c_0(p), \tag{3.2}$$

for $0<p_k\le H<\infty$. Let $x\in r_0^t(p,\Delta^{(\alpha)})$ and for every non-negative integer m, we put

$$x^{[m]}=\sum_{k=0}^{m}\lambda_k(t)b^{(k)}(t). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$R^t(\Delta^{(\alpha)})x^{[m]}=\sum_{k=0}^{m}\lambda_k(t)R^t(\Delta^{(\alpha)})b^{(k)}(t)=(R^t(\Delta^{(\alpha)})x)_ke^{(k)},$$

and

$$\left(R^t(\Delta^{(\alpha)})(x - x^{[m]})\right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (R^t(\Delta^{(\alpha)})x)_i, & \text{if } i > m. \end{cases}$$

Now, for $\varepsilon > 0$ there exists an integer m_0 such that

$$\sup_{i \geq m} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$\begin{aligned} g_{\Delta^{(\alpha)}}\left(x - x^{[m]}\right) &= \sup_{i \geq m} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} \\ &\leq \sup_{i \geq m_0} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

for all $m \geq m_0$.

To show the uniqueness of the representation, we suppose that

$$x = \sum_k \mu_k(t) b^{(k)}(t).$$

Then, we have

$$\begin{aligned} (R^t(\Delta^{(\alpha)})x)_n &= \sum_k \mu_k(t) \left(R^t(\Delta^{(\alpha)}) b^{(k)}(t)\right)_n \\ &= \sum_k \mu_k(t) e_n^{(k)} = \mu_n(t), \quad n \in \mathbb{N}. \end{aligned}$$

This contradicts the fact that $(R^t(\Delta^{(\alpha)})x)_k = \lambda_k(t)$, $k \in \mathbb{N}$. Thus, the representation (3.1) is unique.

(b) The proof is analogous to the previous theorem and hence omitted. ☐

4. α-, β- AND γ-DUALS

In this section we shall compute α-, β- and γ-duals of $r_0^t(\Delta^{(\alpha)})$, $r_c^t(\Delta^{(\alpha)})$ and $r_\infty^t(\Delta^{(\alpha)})$. Note that the notation α used for α-dual has different meaning to that of the operator $\Delta^{(\alpha)}$.

For the sequence spaces X and Y, define multiplier sequence space $M(X, Y)$ by

$$M(X, Y) = \{p = (p_k) \in w : px = (p_k x_k) \in Y, \text{ for all } x = (x_k) \in X\}.$$

Then the α-, β- and γ-duals of X are given by

$$X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, cs), \quad X^\gamma = M(X, bs),$$

respectively. Now, we give the following lemmas given in [41] which will be used to obtain the duals. Throughout $\mathcal{F}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 4.1. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statement hold:*

(a) $A \in (\ell_\infty(p), \ell(q))$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} B^{\frac{1}{p_k}} \right|^{q_n} < \infty, \quad \textit{for all integers } B>1 \textit{ and } q_n \geq 1 \textit{ for all } n;$$

(b) $A \in (\ell_\infty(p), \ell_\infty(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| B^{\frac{1}{p_k}} \right)^{q_n} < \infty, \quad \textit{for all integers } B>1;$$

(c) $A \in (\ell_\infty(p), c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{\frac{1}{p_k}} < \infty, \quad \textit{for all integers } B>1,$$

$$\textit{exists } (\alpha_k) \subset \mathbb{R} \textit{ such that } \lim_{n\to\infty} \left(\sum_k |a_{nk}-\alpha_k| B^{\frac{1}{p_k}} \right)^{q_n} = 0, \quad \textit{for all } B>1;$$

(d) $A \in (\ell_\infty(p), c_0(q))$ *if and only if*

$$\lim_{n\to\infty} \left(\sum_k |a_{nk}| B^{\frac{1}{p_k}} \right)^{q_n} = 0, \quad \textit{for all integers } B>1.$$

Lemma 4.2. *Let $A=(a_{nk})$ be an infinite matrix. Then, the following statement hold:*

(a) $A \in (c_0(p), \ell_\infty(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| B^{\frac{-1}{p_k}} \right)^{q_n} < \infty, \quad \textit{for all integers } B>1; \tag{4.1}$$

(b) $A \in (c_0(p), c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{\frac{-1}{p_k}} < \infty \ , \quad \textit{for all integers } B>1, \tag{4.2}$$

$$\textit{exists } (\alpha_k) \subset \mathbb{R} \textit{ such that } \sup_{n\in\mathbb{N}} \sum_k |a_{nk}-\alpha_k| M^{\frac{1}{q_n}} B^{\frac{-1}{p_k}} < \infty, \tag{4.3}$$

for all integers $M, B > 1$,

$$\textit{exists } (\alpha_k) \subset \mathbb{R} \textit{ such that } \lim_{n\to\infty} |a_{nk}-\alpha_k|^{q_n} = 0, \quad \textit{for all } k\in\mathbb{N}; \tag{4.4}$$

(c) $A \in (c_0(p), c_0(q))$ *if and only if*

$$\textit{exists } (\alpha_k) \subset \mathbb{R} \textit{ such that } \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{\frac{1}{q_n}} B^{\frac{-1}{p_k}} < \infty, \quad \textit{for all integers } M, B>1, \tag{4.5}$$

$$\textit{exists } (\alpha_k) \subset \mathbb{R} \textit{ such that } \lim_{n\to\infty} |a_{nk}|^{q_n} = 0, \quad \textit{for all } k\in\mathbb{N}. \tag{4.6}$$

Lemma 4.3. *Let $A=(a_{nk})$ be an infinite matrix. Then the following statement hold:*

(a) $A \in (c(p), \ell_\infty(q))$ *if and only if* (4.1) *holds and*

$$\sup_{n\in\mathbb{N}} \left|\sum_k a_{nk}\right|^{q_n} < \infty;$$

(b) $A \in (c(p), c(q))$ *if and only if* (4.2), (4.3) *and* (4.4) *hold and*

$$\text{exists } \alpha \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \left|\sum_k a_{nk} - \alpha\right|^{q_n} = 0;$$

(c) $A \in (c(p), c_0(q))$ *if and only if* (4.5) *and* (4.6) *hold and*

$$\lim_{n\to\infty} \left|\sum_k a_{nk}\right|^{q_n} = 0.$$

Theorem 4.1. *Define the sets* $\nu_1(p)$, $\nu_2(p)$, $\nu_3(p)$, $\nu_4(p)$, $\nu_5(p)$ *and* $\nu_6(p)$ *as follows:*

$$\nu_1(p) = \bigcap_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| B^{\frac{1}{p_k}} < \infty \Bigg\},$$

$$\nu_2(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty \text{ and } \left(\frac{a_k T_k}{t_k} B^{\frac{1}{p_k}}\right) \in c_0 \right\},$$

$$\nu_3(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty \text{ and } \left\{ \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right\} \in \ell_\infty \right\},$$

$$\nu_4(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| B^{\frac{-1}{p_k}} < \infty \Bigg\},$$

$$\nu_5(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sum_n \left| \sum_k \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| < \infty \Bigg\},$$

$$\nu_6(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{-1}{p_k}} < \infty \right\},$$

where

$$\Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) = \frac{a_k}{t_k} + \sum_{j=k+1}^{n} (-1)^{j-k} a_j \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}. \tag{4.7}$$

Then

$$\left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\alpha = \nu_1(p), \quad \left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\beta = \nu_2(p), \quad \left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\gamma = \nu_3(p),$$

$$\begin{aligned}\left[r_c^t(p,\Delta^{(\alpha)})\right]^\alpha =&\nu_4(p)\cap\nu_5(p), \quad \left[r_c^t(p,\Delta^{(\alpha)})\right]^\beta = \nu_6(p)\cap cs,\\ \left[r_c^t(p,\Delta^{(\alpha)})\right]^\gamma =&\nu_6(p)\cap bs, \quad \left[r_0^t(p,\Delta^{(\alpha)})\right]^\alpha = \nu_4(p),\\ \left[r_0^t(p,\Delta^{(\alpha)})\right]^\beta =& \left[r_0^t(p,\Delta^{(\alpha)})\right]^\gamma = \nu_6(p).\end{aligned}$$

Proof. We prove the theorem for the space $r_\infty^t(p,\Delta^{(\alpha)})$. Consider the sequence $a=(a_k)\in w$ and $x=(x_k)$ is as defined in (2.5), then we have

$$\begin{aligned} a_nx_n =& \sum_{j=0}^{n-1}\left[\sum_{i=j}^{j+1}(-1)^{n-j}\frac{\Gamma(-\alpha+1)}{(n-i)!\Gamma(-\alpha-n+i+1)}\frac{T_j}{t_i}a_ny_j\right]+\frac{T_n}{t_n}a_ny_n\\ =& (Gy)_n, \quad \text{for each } n\in\mathbb{N}, \end{aligned} \tag{4.8}$$

where $G=(g_{nk})$ is a matrix defined by

$$g_{nk}=\begin{cases}\displaystyle\sum_{j=k}^{k+1}(-1)^{n-k}\frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)}\frac{T_k}{t_j}a_n, & \text{if } 0\le k<n,\\ \dfrac{T_n}{t_n}a_n, & \text{if } k=n,\\ 0, & \text{if } k>n.\end{cases}$$

Thus, we deduce from (4.8) that $ax=(a_nx_n)\in\ell_1$ whenever $x=(x_k)\in r_\infty^t(p,\Delta^{(\alpha)})$ if and only if $Gy\in\ell_1$ whenever $y=(y_k)\in\ell_\infty(p)$. This yields that $a=(a_n)\in\left[r_\infty^t(p,\Delta^{(\alpha)})\right]^\alpha$ if and only if $G\in(\ell_\infty(p),\ell_1)$.

Thus, by using Lemma 4.1(a) with $q_n=1$ for all n, we conclude that

$$\left[r_\infty^t(p,\Delta^{(\alpha)})\right]^\alpha=\nu_1(p).$$

Now, consider the following equation

$$\begin{aligned}\sum_{k=0}^n a_kx_k =& \sum_{k=0}^n a_k\left[\sum_{j=0}^{k-1}\left(\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{T_j}{t_i}y_j\right)+\frac{T_k}{t_k}y_k\right]\\ =& \sum_{k=0}^{n-1}y_kT_k\left(\frac{a_k}{t_k}+\sum_{j=k+1}^n(-1)^{j-k}a_j\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}\right)+\frac{T_n}{t_n}a_ny_n\\ =& \sum_{k=0}^{n-1}y_kT_k\Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)+\frac{T_n}{t_n}a_ny_n \qquad (4.9)\\ =& (Hy)_n, \quad \text{for each } n\in\mathbb{N}, \qquad (4.10)\end{aligned}$$

where $H = (h_{nk})$ is a matrix defined by

$$h_{nk} = \begin{cases} \Delta^{(\alpha)}\left(\dfrac{a_k}{t_k}\right) T_k, & \text{if } 0 \le k < n, \\ \dfrac{T_n}{t_n} a_n, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

and $\Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)$ is as defined in (4.7). Thus, we deduce from (4.10) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in r^t_\infty(p, \Delta^{(\alpha)})$ if and only if $Hy \in c$ whenever $y = (y_k) \in \ell_\infty(p)$. Therefore, by using Lemma 4.1 (c) with $q = (q_n) = 1$, we get that

$$\sum_k \left| \Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{T_k}{t_k} a_k B^{\frac{1}{p_k}} = 0.$$

Thus, $\left[r^t_\infty(p, \Delta^{(\alpha)})\right]^\beta = \nu_2(p)$.

Similarly, by using Lemma 4.1 (b), with $q_n = 1$ for all n, we can deduce that $\left[r^t_\infty(p, \Delta^{(\alpha)})\right]^\gamma = \nu_3(p)$. This completes the proof of the theorem. The duals of the other spaces can be obtained by the similar proceedings and using Lemma 4.2 and 4.3. ☐

5. Matrix Transformations

In this section, we give certain results regarding matrix transformation of the Riesz sequence spaces of fractional order to $X(p)$ where $X = \{\ell_\infty, c, c_0\}$. Let $q = (q_n)$ be a non-decreasing bounded sequence of positive real numbers. For brevity, we write

$$\Delta^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right) = \frac{a_{nk}}{t_k} + \sum_{j=k+1}^{n} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}$$

and

$$\Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right) = \frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i},$$

for all $n, k \in \mathbb{N}$. Let $x, y \in w$ be connected by the relation $y = R^t(\Delta^{(\alpha)})x$. Then we have by (4.9)

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \Delta^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right) T_k y_k + a_{nm} \frac{t_m}{T_m} y_m, \qquad n, m \in \mathbb{N}. \tag{5.1}$$

Now, let us consider the following conditions before we proceed:

$$\lim_{k\to\infty} \frac{a_{nk}}{t_k} T_k B^{\frac{1}{p_k}} = 0, \qquad \text{for all } n, B \in \mathbb{N}, \tag{5.2}$$

$$\sup_{n\in\mathbb{N}} \left[\sum_k \left| \Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} \right]^{q_n} < \infty, \qquad \text{for all } B \in \mathbb{N}, \tag{5.3}$$

$$\sup_{n\in\mathbb{N}} \sum_k \left| \Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty, \qquad \text{for all } B \in \mathbb{N}, \tag{5.4}$$

$$\text{(5.5)} \quad \text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty}\left[\sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right| B^{\frac{1}{p_k}}\right]^{q_n} = 0,$$

for all $B \in \mathbb{N}$,

$$\text{(5.6)} \quad \sup_{n\in\mathbb{N}}\left[\sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k\right| B^{\frac{-1}{p_k}}\right]^{q_n} < \infty, \quad \text{for all } B \in \mathbb{N},$$

$$\text{(5.7)} \quad \sup_{n\in\mathbb{N}}\left|\sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k\right|^{q_n} < \infty, \quad \text{for all } n \in \mathbb{N},$$

$$\text{(5.8)} \quad \text{exists } \alpha \in \mathbb{R} \text{ such that } \lim_{n\to\infty}\left|\sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha\right|^{q_n} = 0,$$

$$\text{(5.9)} \quad \text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty}\left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right|^{q_n} = 0, \quad \text{for all } k \in \mathbb{N},$$

$$\text{(5.10)} \quad \text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \sup_{n\in\mathbb{N}} L^{\frac{1}{q_n}}\sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right| B^{\frac{-1}{p_k}} < \infty,$$

for all L exists $B \in \mathbb{N}$.

Theorem 5.1. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) $A \in (r_\infty^t(p, \Delta^{(\alpha)}), \ell_\infty(q))$ *if and only if* (5.2) *and* (5.3) *hold;*
(b) $A \in (r_\infty^t(p, \Delta^{(\alpha)}), c(q))$ *if and only if* (5.2), (5.4) *and* (5.5) *hold;*
(c) $A \in (r_\infty^t(p, \Delta^{(\alpha)}), c_0(q))$ *if and only if* (5.2) *holds and* (5.5) *holds, with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Proof. We give the proof of (a) as the rest can be obtained in the similar manner. Let $A = (a_{nk}) \in (r_\infty^t(p, \Delta^{(\alpha)}), \ell_\infty(q))$ and $x = (x_k) \in r_\infty^t(p, \Delta^{(\alpha)})$. Consider equation (5.1). Since Ax exists and belongs to the space $\ell_\infty(q)$, therefore the necessity of the condition (5.2) is obvious. Now, letting $m \to \infty$ in equation (5.1), we straightly get

$$\begin{aligned} Ax &= \sum_k \left(\frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty}(-1)^{j-k}a_{nj}\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}\right)T_k y_k \\ \text{(5.11)} \qquad &= \sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k y_k. \end{aligned}$$

This implies that $A(R^t(\Delta^{(\alpha)}))^{-1}y \in \ell_\infty(q)$. That is, $A(R^t(\Delta^{(\alpha)}))^{-1} \in (\ell_\infty(p), \ell_\infty(q))$. Therefore, $A(R^t(\Delta^{(\alpha)}))^{-1}$ satisfies the lemma 4.1(b) which is equivalent to the condition (5.3). This shows the necessity of the condition (5.3).

Conversely, let the conditions (5.2) and (5.3) hold and $x \in r_\infty^t(p, \Delta^{(\alpha)})$. Then it is clear that Ax exists. Now, using equation (5.11) and the condition (5.3) with $B > \max\{1, \sup_k |y_k|^{p_k}\}$, we get

$$\|Ax\|_{\ell_\infty(q)} = \sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right|^{q_n}$$

$$= \sup_{n\in\mathbb{N}} \left| \sum_k \left(\frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)} \right) T_k y_k \right|^{q_n}$$
$$\leq \sup_{n\in\mathbb{N}} \left(\sum_k \left| \Delta_{\infty}^{(\alpha)} \left(\frac{a_{nk}}{t_k} \right) T_k y_k \right| \right)^{q_n}$$
$$\leq \sup_{n\in\mathbb{N}} \left(\sum_k \left| \Delta_{\infty}^{(\alpha)} \left(\frac{a_{nk}}{t_k} \right) T_k \right| B^{\frac{1}{p_k}} \right)^{q_n} < \infty.$$

This concludes that $A \in (r_{\infty}^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$. □

By the similar proceedings, we can derive the following results.

Theorem 5.2. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) $A \in (r_c^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$ *if and only if* (5.2), (5.6) *and* (5.7) *hold;*
(b) $A \in (r_c^t(p, \Delta^{(\alpha)}), c(q))$ *if and only if* (5.2), (5.8), (5.9) *and* (5.10) *hold and* (5.6) *also holds, with $q_n = 1$ for all $n \in \mathbb{N}$;*
(c) $A \in (r_c^t(p, \Delta^{(\alpha)}), c_0(q))$ *if and only if* (5.2) *holds and* (5.8), (5.9) *and* (5.10) *also hold, with $\alpha = 0$, $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 5.3. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) $A \in (r_0^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$ *if and only if* (5.2) *and* (5.6) *hold;*
(b) $A \in (r_0^t(p, \Delta^{(\alpha)}), c(q))$ *if and only if* (5.2), (5.9) *and* (5.10) *hold and* (5.6) *also holds, with $q_n = 1$ for all $n \in \mathbb{N}$;*
(c) $A \in (r_0^t(p, \Delta^{(\alpha)}), c_0(q))$ *if and only if* (5.2) *holds and* (5.9) *and* (5.10) *also hold, with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Conclusion

In this article, we introduce paranormed difference sequence spaces $r_{\infty}^t(\Delta^{(\alpha)})$, $r_c^t(\Delta^{(\alpha)})$ and $r_0^t(\Delta^{(\alpha)})$ of fractional order α, investigate their topological properties, Schauder basis, α-, β- and γ- duals and characterize the matrix classes related to these spaces. We conclude that the results obtained from the matrix domain of the product matrix $R^t(\Delta^{(\alpha)})$ are more general and extensive than the existent results of the previous authors. We expect that our results might be a reference for further studies in this field. In our next paper, we will investigate the results obtained from the matrix domain $R^t(\Delta^{(\alpha)})$ in the spaces ℓ_p of absolutely p-summable sequences, $1 \leq p < \infty$.

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GENERAL CLASSES OF SHRINKAGE ESTIMATORS FOR THE MULTIVARIATE NORMAL MEAN WITH UNKNOWN VARIANCE: MINIMAXITY AND LIMIT OF RISKS RATIOS

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Abstract. In this paper, we consider two forms of shrinkage estimators of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$ where σ^2 is unknown and estimated by the statistic S^2 ($S^2 \sim \sigma^2\chi^2_n$). Estimators that shrink the components of the usual estimator X to zero and estimators of Lindley-type, that shrink the components of the usual estimator to the random variable $\overline{X}$. Our aim is to improve under appropriate condition the results related to risks ratios of shrinkage estimators, when n and p tend to infinity and to ameliorate the results of minimaxity obtained previously of estimators cited above, when the dimension p is finite. Some numerical results are also provided.

1. Introduction

Shrinkage estimates are alternative estimates that use information from all studies to provide potentially better estimates for each study. While these estimates is biased, they have a considerably smaller variance, and thus tend to be better in terms of total mean squared error. For example, Xie et al. [21] introduced a class of semiparametric/parametric shrinkage estimators and established their asymptotic optimality properties, Hansen [9] compared the mean-squared error of ordinary least squares (OLS), James-Stein, and least absolute shrinkage and selection operator (Lasso) shrinkage estimators and shows that neither James-Stein nor Lasso uniformly dominates the other, Selahattin et al. [15] provided several alternative methods for derivation of the restricted ridge regression estimator (RRRE).

Key words and phrases. James-Stein estimator, multivariate Gaussian random variable, non-central chi-square distribution, quadratic risk, shrinkage estimator.

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Mean vector parameter estimation is an important problem in the context of shrinkage estimation and has been widely applied in many scientific and engineering problems. This fact is certainly reflected by the abundant literature on the subject, let us cite for instance. Stein [16] showed the inadmissibility of the usual estimator X of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ when the dimension of the space of the observations $p \geqslant 3$. James and Stein [10], introduced the class of shrinkage estimators $\delta_a = (1 - aS^2/\left\|X\right\|^2)X$, that improving the usual estimator X under the quadratic loss function. Many developments in this field has realized by Lindley [12], Baranchik [1], Stein [17] and Selahattin and Issam [13]. Tsukuma and Kubokawa [20] addresses the problem of estimating the mean vector of a singular multivariate normal distribution with an unknown singular covariance matrix. Selahattin and Issam [14], introduced and derived the optimal extended balanced loss function (EBLF) estimators and pridictors and discuss their performances.

When the dimension p is infinite, Casella and Hwang [4], studied the case where σ^2 is known ($\sigma^2 = 1$) and showed that if the limit of the ratio $\left\|\theta\right\|^2/p$ is a constant $c > 0$, then the risks ratios of the James-Stein estimator δ^{JS} and the positive-part of the James-Stein estimator δ^{JS+}, to the maximum likelihood estimator X, tend to a constant value $c/(1+c)$. Benmansour and Hamdaoui [2], have taken the same model given by Casella and Hwang [4], where the parameter σ^2 is unknown and they established the same results. Hamdaoui and Benmansour [6], considered the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). They studied the following class of shrinkage estimators $\delta_\phi = \delta^{JS} + l(S^2\phi(S^2, \|X\|^2)/\|X\|^2)X$, where l is a real parameter. The authors showed that, when the sample size n and the dimension of space parameters p tend to infinity, the estimators δ_ϕ have a lower bound $B_m = c/(1+c)$ and if the shrinkage function ϕ satisfies some conditions, the risks ratio $R(\delta_\phi, \theta)/R(X, \theta)$ attains this lower bound B_m, in particulary the risks ratios $R(\delta^{JS}, \theta)/R(X, \theta)$ and $R(\delta^{JS+}, \theta)/R(X, \theta)$. In Hamdaoui et al. [8], the authors studied the limit of risks ratios of two forms of shrinkage estimators. The first one has been introduced by Benmansour and Mourid [3], $\delta_\psi = \delta^{JS} + l(S^2\psi(S^2, \|X\|^2)/\|X\|^2)X$, where l is a real parameter and $\psi(\cdot, u)$ is a function with support $[0, b]$ and satisfies some conditions different from the one given in Hamdaoui and Benmensour [6]. The second is the polynomial form of shrinkage estimator introduced by Li and Kio [11]. Hamdaoui and Mezouar [7], studied the general class of shrinkage estimators $\delta_\phi = \left(1 - S^2\phi\left(S^2, \|X\|^2\right)/\left\|X\right\|^2\right)X$. They showed the same results given in Hamdaoui and Benmansour [6], with different conditions on the shrinkage function ϕ.

In this work, we consider the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ and independently of the observations X, we observe $S^2 \sim \sigma^2\chi^2_n$ an estimator of σ^2. It's well known that the quadratic risk of the usual estimator X is $p\sigma^2$. Consequently, any estimator of θ which has a quadratic risk less than $p\sigma^2$ dominate X, then it is minimax. We consider two different forms of shrinkage estimators of θ: estimators of the form $\delta^\psi = (1 - \psi(S^2, \left\|X\right\|^2)S^2/\left\|X\right\|^2)X$, and estimators of Lindley-type given by $\delta^\varphi =$

$(1-\varphi(S^2,T^2)S^2/T^2)(X-\overline{X})+\overline{X}$, that shrink the components of the maximum likelihood estimator X to the random variable $\overline{X}$. Our aim in this work is based on two points. First, when n and p tend to infinity, we give results of the limit of risks ratios of estimators defined above to the maximum likelihood estimator X, different from the one obtained in our published papers. The second point is to generalize and to improve the results of minimaxity obtained by Strawderman [18], Sun [19] and Hamdaoui and Benmansour [6].

The paper is outlined as follows: In Section 2, we consider the form of shrinkage estimators defined in (2.2) and we study the minimaxity and the limit of risks ratio to these estimators to the usual estimator X. In Section 3, we consider the second form of shrinkage estimators defined in (3.1) of Lindley-type. In this case, we follow the same steps as we treated the first form (2.2). In Section 4, we graphically illustrate some results given in this paper. In the end, we give an Appendix which contains technical lemmas used in the proofs of our results.

2. Shrinkage to Zero

Let $X\sim N_p\left(\theta,\sigma^2 I_p\right)$ where σ^2 is unknown and estimated by S^2 ($S^2\sim\sigma^2\chi^2_n$). The aim is to estimate θ by an estimator δ relatively at the quadratic loss function

$$L\left(\delta,\theta\right)=\left\|\delta-\theta\right\|_p^2,$$

with $\left\|\cdot\right\|_p$ is the usual norm in $\mathbb{R}^p$. We associate its risk function

$$R\left(\delta,\theta\right)=E_\theta\left(L\left(\delta,\theta\right)\right).$$

We denote the general form of a shrinkage estimator as follows

$$\delta_j^\phi\left(X,S^2\right)=\left(1-\phi\left(S^2,\|X\|^2\right)\right)X_j,\quad j=1,\ldots,p. \tag{2.1}$$

We recall that $\frac{\|X\|^2}{\sigma^2}\sim\chi^2_p\left(\lambda\right)$, where $\chi^2_p\left(\lambda\right)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda=\frac{\|\theta\|^2}{2\sigma^2}$. We also recall the following Lemma given by Fourdrinier et al. [5], that we will use often in the next.

Lemma 2.1. *Let $X\sim N_p\left(\theta,\sigma^2 I_p\right)$ with $\theta\in\mathbb{R}^p$. Then*

(a) *for $p\geq 3$ we have $E\left(\frac{1}{\|X\|^2}\right)=\frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right)$;*

(b) *for $p\geq 5$ we have $E\left(\frac{1}{\|X\|^4}\right)=\frac{1}{\sigma^4}E\left(\frac{1}{(p-2+2K)(p-4+2K)}\right)$,*

where $K\sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the Poisson's distribution of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

For the next, we need the following results obtained by Hamdaoui and Benmansour [6].

Proposition 2.1 (Hamdaoui and Benmansour [6])**.** *The risk of the estimator given in* (2.1) *is*

$$R\left(\delta^\phi(X,S^2),\theta\right)=\sigma^2E\left\{\phi_K^2\ \chi^2_{p+2K}-2\phi_K\left(\chi^2_{p+2K}-2K\right)+p\right\},$$

where $\phi_K = \phi\left(\sigma^2\chi^2_n, \sigma^2\chi^2_{p+2K}\right)$ *and* $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ *being the Poisson's distribution of parameter* $\frac{\|\theta\|^2}{2\sigma^2}$ *and* χ^2_n *is the central chi-square distribution with* n *degrees of freedom. Furthermore,* $R\left(\delta^{\phi}\left(X, S^2\right), \theta\right) \geq B_p\left(\theta\right)$ *with*

$$B_p\left(\theta\right) = \sigma^2\left\{p - 2 - E\left\{\frac{\left(p-2\right)^2}{p - 2 + 2K}\right\}\right\}.$$

We set by $b_p\left(\theta\right) = \frac{B_p(\theta)}{R(X,\theta)}$, it is clear that if $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c\ (>0)$, then

$$\lim_{p\to\infty} b_p\left(\theta\right) = \frac{c}{1+c}.$$

In the particular case where $\phi(S^2, \|X\|^2) = d\frac{S^2}{\|X\|^2}$ we have

$$\delta^d\left(X, S^2\right) = \left(1 - d\frac{S^2}{\|X\|^2}\right)X,$$

hence

$$R\left(\delta^d(X, S^2), \theta\right) = \sigma^2\left\{p + n\left[d^2(n+2) - 2d(p-2)\right]E\left(\frac{1}{p-2+2K}\right)\right\}.$$

For $d = \frac{p-2}{n+2}$ we obtain the James-Stein estimator which minimizes the risk of $\delta^d\left(X, S^2\right)$ whose quadratic risk is

$$R\left(\delta^{JS}(X, S^2), \theta\right) = \sigma^2\left\{p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\}.$$

Proposition 2.2 (Hamdaoui and Benmansour [6]). *If* $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, *then*

$$\lim_{n,p\to\infty}\frac{R\left(\delta^{\phi}(X, S^2), \theta\right)}{R(X,\theta)} \geq \frac{c}{1+c}$$

and

$$\lim_{n,p\to\infty}\frac{R\left(\delta^{JS}(X, S^2), \theta\right)}{R(X,\theta)} = \frac{c}{1+c}.$$

We note that from the Proposition 2.2, the risks ratio of any shrinkage estimator $\delta^{\phi}\left(X, S^2\right)$ of the form (2.1) dominating the James-Stein estimator $\delta^{JS}\left(X, S^2\right)$, to the maximum likelihood estimator attains the limiting lower bound $B_m = \frac{c}{1+c}\ (<1)$, when n and p tend simultaneously to infinity.

Now we rewrite the estimator in (2.1) by letting $\phi\left(S^2, \|X\|^2\right) = \psi\left(S^2, \|X\|^2\right)\frac{S^2}{\|X\|^2}$, as given by

$$\delta^{\psi}_j\left(X, S^2\right) = \left(1 - \psi\left(S^2, \|X\|^2\right)\frac{S^2}{\|X\|^2}\right)X_j, \quad j = 1, \dots, p. \tag{2.2}$$

Using the Proposition 2.1, the risk function of estimator $\delta^{\psi}(X, S^2)$ given in (2.2), is

$$\begin{aligned} R\left(\delta^{\psi}(X,S^2),\theta\right) &= \sigma^2 E\left\{\frac{\psi_K^2}{\sigma^2}\frac{(\sigma^2\chi_n^2)^2}{\left(\sigma^2\chi_{p+2K}^2\right)} - 2\psi_K\frac{(\sigma^2\chi_n^2)}{\left(\sigma^2\chi_{p+2K}^2\right)}\left(\chi_{p+2K}^2 - 2K\right) + p\right\} \\ &= p\sigma^2 + \sigma^2 E\left\{\chi_n^2\ \psi_K\left[\frac{\psi_K\ \chi_n^2}{\chi_{p+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p+2K}^2}\right)\right]\right\}, \end{aligned}$$

where $\psi_K = \psi\left(\sigma^2\chi_n^2\ , \sigma^2\chi_{p+2K}^2\right)$.

We write $\Delta_\psi = R\left(\delta^{\psi}(X,S^2),\theta\right) - R(X,\theta)$. As $R(X,\theta) = p\sigma^2$, then

$$\Delta_\psi = \sigma^2 E\left\{\chi_n^2\ \psi_K\left[\frac{\chi_n^2\ \psi_K}{\chi_{p+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p+2K}^2}\right)\right]\right\}. \tag{2.3}$$

2.1. **Limit of risks ratios.** In this part, we are interested in studying of the limit of risks ratios of estimators defined in (2.2), to the usual estimator X. So, we give results different from the one given in our published papers.

Theorem 2.1. *Assume that $\delta^{\psi}(X,S^2)$ is given in (2.2), such that $p \geq 3$ and ψ satisfies:*

(H) $\left|\dfrac{p-2}{n+2} - \psi\left(S^2, \|X\|^2\right)\right| \leq g(S^2)$ *a.s., where* $E\left\{g^2\left(\sigma^2\chi_{n+4}^2\right)\right\} = O\left(\frac{1}{n^2}\right)$*, when n is in the neighborhood of $+\infty$.*

If $\lim\limits_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$*, then*

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\psi}(X,S^2),\theta\right)}{R(X,\theta)} = \frac{c}{1+c}.$$

Proof. We note $\alpha = \frac{p-2}{n+2}$ and $\psi\left(S^2, \|X\|^2\right) = \psi$. As

$$R\left(\delta^{\psi}(X,S^2),\theta\right) = E\left\{\sum_{i=1}^{p}\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]^2\right\}$$

and

$$R\left(\delta^{JS}(X,S^2),\theta\right) = E\left\{\sum_{i=1}^{p}\left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]^2\right\},$$

then

$$\begin{aligned} \Delta_{JS} =& R\left(\delta^{\psi}(X,S^2),\theta\right) - R\left(\delta^{JS}(X,S^2),\theta\right) \\ =& E\left\{\sum_{i=1}^{p}\left\{\left(\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right] - \left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]\right)\right.\right. \\ & \times\left.\left.\left(\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right] + \left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]\right)\right\}\right\} \end{aligned}$$

$$\begin{aligned}
&=2E\left\{\sum_{i=1}^{p}\left(\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\right]\left[\left(1-\frac{(\alpha+\psi)}{2}\frac{S^2}{\|X\|^2}\right)X_i-\theta_i\right]\right)\right\}\\
&=2E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\left(1-\frac{(\alpha+\psi)}{2}\frac{S^2}{\|X\|^2}\right)\frac{S^2}{\|X\|^2}X_i^2\right]\right.\\
&\quad\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\;\theta_i\right]\right\}\\
&=2E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\left(1+\frac{(-\alpha+\alpha-\psi-\alpha)}{2}\frac{S^2}{\|X\|^2}\right)\frac{S^2}{\|X\|^2}X_i^2\right]\right.\\
&\quad\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\;\theta_i\right]\right\}\\
&=2E\left\{\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}\sum_{i=1}^{p}X_i^2\right]+\frac{1}{2}\left[(\alpha-\psi)^2\frac{S^4}{\|X\|^4}\sum_{i=1}^{p}X_i^2\right]\right.\\
&\quad\left.-\alpha\left[(\alpha-\psi)\frac{S^4}{\|X\|^4}\sum_{i=1}^{p}X_i^2\right]-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\;\theta_i\right]\right\}\\
&=2E\left\{(\alpha-\psi)S^2+\frac{1}{2}(\alpha-\psi)^2\frac{S^4}{\|X\|^2}-\alpha(\alpha-\psi)\frac{S^4}{\|X\|^2}\right.\\
&\quad\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\;\theta_i\right]\right\}.
\end{aligned}$$

Using the conditional expectation and the formula (2.7) given in Benmansour and Mourid [3], we have

$$\begin{aligned}
E\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}\langle X,\theta\rangle\right]&=E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\;\theta_i\right]\right\}\\
&=\lambda E\left[\left(\alpha-\psi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p+2}^2(\lambda)\right)\right)\frac{\chi_n^2}{\chi_{p+2}^2(\lambda)}\right],
\end{aligned}$$

where $\lambda=\frac{\|\theta\|^2}{\sigma^2}$. Then

$$\begin{aligned}
\Delta_{JS}\leq&2E\left\{\left[(|\alpha-\psi|)S^2\right]+\frac{1}{2}(\alpha-\psi)^2\frac{S^4}{\|X\|^2}+\alpha(|\alpha-\psi|)\frac{S^4}{\|X\|^2}\right.\\
&\left.+\lambda E\left[\left(\left|\alpha-\psi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p+2}^2(\lambda)\right)\right|\right)\frac{\chi_n^2}{\chi_{p+2}^2(\lambda)}\right]\right\}.
\end{aligned}$$

From the hypothesis (H) and the independence of two variables S^2 and $\|X\|^2$, we have

$$\Delta_{JS}\leq2E\left[S^2g\left(S^2\right)\right]+E\left[S^4g^2\left(S^2\right)\right]E\left(\frac{1}{\|X\|^2}\right)$$

$$
\begin{aligned}
& + 2\alpha E\left[S^4 g\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right) + 2\lambda E\left[S^2 g\left(S^2\right)\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right) \\
=& 2E\left[S^4 \frac{g\left(S^2\right)}{S^2}\right] + E\left[S^4 g^2\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right) \\
& + 2\alpha E\left[S^4 g\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right) + 2\lambda E\left[S^4 \frac{g\left(S^2\right)}{S^2}\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right).
\end{aligned}
$$

Using the Lemma 5.1 of the Appendix and the fact that $E\left(\frac{1}{\chi^2_p(\lambda)}\right) \leq \frac{1}{p-2}$, we obtain

$$
\begin{aligned}
& \Delta_{JS} \\
\leq & 2n(n+2)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + n(n+2)\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] E\left(\frac{1}{\chi^2_p(\lambda)}\right) \\
& + 2n(n+2)\sigma^2\left[\alpha E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] E\left(\frac{1}{\chi^2_p(\lambda)}\right) + \lambda E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right)\right] \\
\leq & 2n(n+2)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + \frac{n(n+2)}{p-2}\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \\
& + 2n\sigma^2 E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] + 2\lambda\frac{n(n+2)}{p}\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\Delta_{JS}}{p\sigma^2} \leq & \frac{2n(n+2)}{p} E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + \frac{n(n+2)}{p(p-2)} E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \\
& + \frac{2n}{p} E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] + \frac{2\lambda}{p}\frac{n(n+2)}{p} E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].
\end{aligned}
$$

From condition $E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] = O\left(\frac{1}{n^2}\right)$ and using the Schwarz inequality, when n is in the neighborhood of $+\infty$, we obtain

$$
\begin{aligned}
E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] & \leq E^{1/2}\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \times E^{1/2}\left[\frac{1}{\left(\chi^2_{n+4}\right)^2}\right] \\
& \leq \sqrt{M}\frac{1}{n} \times \sqrt{\frac{1}{n(n+2)}} \leq \sqrt{M}\frac{1}{n^2}
\end{aligned}
$$

and

$$
E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] \leq E^{1/2}\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \leq \sqrt{M}\frac{1}{n},
$$

where M is a real strictly positive. Then, when n is in the neighborhood of $+\infty$, we have

$$\frac{\Delta_{JS}}{p\sigma^2} \leq \frac{2(n+2)}{np}\sqrt{M} + \frac{n+2}{np(p-2)}M + \frac{2}{p}\sqrt{M} + \frac{2\lambda}{p\sigma^2} \cdot \frac{(n+2)}{np}M.$$

As $\lim\limits_{p\to+\infty} \frac{\lambda}{p} = \lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, then

$$\lim_{n,p\to+\infty} \frac{\Delta_{JS}}{p\sigma^2} \leq 0.$$

Using the Proposition 2.2, we have

$$\lim_{n,p\to+\infty} \frac{R\left(\delta^{\psi}\left(X,S^2\right),\theta\right)}{R(X,\theta)} = \frac{c}{1+c}. \qquad \square$$

Example 2.1. Let $\psi_1 = \frac{p-2}{n+2} - \frac{S^2}{(1+S^2)^2}$, therefore

$$\delta^{\psi_1}\left(X,S^2\right) = \left(1 - \left(\frac{p-2}{n+2} - \frac{S^2}{(1+S^2)^2}\right)\frac{S^2}{\|X\|^2}\right)X.$$

It is sufficient to take $g\left(S^2\right) = \frac{S^2}{(1+S^2)^2}$, then from the Lemma 5.1 of the Appendix, we have

$$\begin{aligned} E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] &= E\left[\frac{\left(\sigma^2\chi^2_{n+4}\right)^2}{\left(1+\sigma^2\chi^2_{n+4}\right)^4}\right] \\ &= (n+4)(n+6)\,\sigma^4 E\left[\frac{1}{\left(1+\sigma^2\chi^2_{n+8}\right)^4}\right] \\ &\leq \frac{(n+4)(n+6)}{\sigma^4}E\left[\frac{1}{\left(\chi^2_{n+8}\right)^4}\right] \\ &= \frac{1}{\sigma^4}\cdot\frac{(n+4)(n+6)}{n(n+2)(n+4)(n+6)} \overset{+\infty}{\sim} \frac{1}{\sigma^4}\cdot\frac{1}{n^2}. \end{aligned}$$

Thus,

$$E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] = O\left(\frac{1}{n^2}\right).$$

2.2. **Minimaxity.** In this part we study the minimaxity of estimators defined in (2.2). We give another results that improve the one given in Strawderman [18], Sun [19] and Hamdaoui and Benmansour [6].

Theorem 2.2. *Assume that $\delta^{\psi}\left(X,S^2\right)$ is given in (2.2), such that $p \geq 3$ and ψ satisfies:*

(a) $\psi\left(S^2, \|X\|^2\right)$ *is monotone non-decreasing in* $\|X\|^2$*;*

(b) $0 \leq \psi\left(S^2, \|X\|^2\right) \leq \frac{2(p-2)}{n+2}$.

A sufficient condition so that the estimator $\delta^{\psi}(X, S^2)$ is minimax is, for any k, $k = 0, 1, 2, \dots,$

$$E\left\{\psi\left(\sigma^2\chi^2_{n+4}\ ,\sigma^2\chi^2_{p+2k}\right)\right\} \le E\left\{\psi\left(\sigma^2\chi^2_{n+2}\ ,\sigma^2\chi^2_{p+2k}\right)\right\}.$$

Proof. From the formula (2.3) and the condition (b), we have

$$\Delta_{\psi} \le \sigma^2 E\left\{\chi^2_n\ \psi_K\left[\frac{\frac{2(p-2)}{n+2}\chi^2_n}{\chi^2_{p+2K}} - 2\left(1-\frac{2K}{\chi^2_{p+2K}}\right)\right]\right\}.$$

We will prove that the expectation on the right hand side being non-positive for any $K = k$, $k = 0, 1, 2, \dots$.

By using the conditional expectation, we obtain

$$\begin{aligned}\Delta_{\psi} &\le \sigma^2 E\left[E\left\{\psi_k\ \chi^2_n\left[\frac{\frac{2(p-2)}{n+2}\chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right]\middle|\ \chi^2_n\right\}\right]\\ &\le \sigma^2 E\left\{\chi^2_n\ E\left(\psi_k \mid \chi^2_n\right)\ E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right)\middle|\ \chi^2_n\right]\right\},\end{aligned}$$

the last inequality according to the condition (a) and the fact that the covariance of two functions one increasing and the other decreasing is non-positive.

Using the Lemma 2.1, we obtain

$$\begin{aligned}&E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right)\middle|\ \chi^2_n\right]\\ =&E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{p-2+2k} - 2 + \frac{4k}{p-2+2k}\right)\middle|\ \chi^2_n\right] = \frac{2(p-2)\left(\frac{\chi^2_n}{n+2}-1\right)}{p-2+2k}.\end{aligned}$$

Then

$$\begin{aligned}\Delta_{\psi} \le&\sigma^2 E\left\{\chi^2_n\ \frac{2(p-2)\left(\frac{\chi^2_n}{n+2}-1\right)}{p-2+2k}\ E\left(\psi_k \mid \chi^2_n\right)\right\}\\ =&\frac{2(p-2)\sigma^2}{p-2+2k}E\left\{\chi^2_n\ \left(\frac{\chi^2_n}{n+2}-1\right)\psi_k\right\}.\end{aligned}\tag{2.4}$$

From the Lemma 5.1 of the Appendix, we have
$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\psi_k\right\}=nE\left\{\psi\left(\sigma^2\chi_{n+4}^2\,,\sigma^2\chi_{p+2k}^2\right)-\psi\left(\sigma^2\chi_{n+2}^2\,,\sigma^2\chi_{p+2k}^2\right)\right\}.$$
Using the sufficient condition
$$E\left[\psi\left(\sigma^2\chi_{n+4}^2\,,\sigma^2\chi_{p+2k}^2\right)\right]\le E\left[\psi\left(\sigma^2\chi_{n+2}^2\,,\sigma^2\chi_{p+2k}^2\right)\right],$$
we obtain
$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\psi_k\right\}\le 0.$$
Thus,
$$\Delta_\psi\le 0.$$
□

Example 2.2. Let $\psi_2=\frac{2(p-2)}{n+2}\ln\left(1+S^2\right)\exp\left(-S^2\right)$, therefore
$$\delta^{\psi_2}\left(X,S^2\right)=\left(1-\frac{2\,(p-2)}{n+2}\frac{S^2\ln\left(1+S^2\right)\exp\left(-S^2\right)}{\|X\|^2}\right)X.$$

Remark 2.1. (i) Using the Lemma 5.2 of the Appendix, it is clear that if $\psi\left(S^2,\|X\|^2\right)$ is monotone non-increasing in S^2, then the sufficient condition:
$$E\left\{\psi\left(\sigma^2\chi_{n+4}^2\,,\sigma^2\chi_{p+2k}^2\right)\right\}\le E\left\{\psi\left(\sigma^2\chi_{n+2}^2\,,\sigma^2\chi_{p+2k}^2\right)\right\}$$
is satisfied. Thus, the theorem 2.2 gives an improvement of the results of minimaxity given in the first Theorem of Strawderman [18], Theorem 4.1 of Sun [19] and Theorem 4.1 of Hamdaoui and Benmansour [6].

(ii) Note that the James-Stein estimator satisfies the conditions of Theorem 2.2, thus Theorem 2.2 gives another proof of the minimaxity of the James-Stein estimator.

3. Estimator of Lindley-Type

Let the model be $X/\theta,\sigma^2\sim N_p\left(\theta,\sigma^2 I_p\right)$, where the parameters θ and σ^2 are unknown and σ^2 is estimated by S^2 ($S^2\sim\sigma^2\chi_n^2$). The aim is to estimate the mean $\theta=\left(\theta_1,\theta_2,\ldots,\theta_p\right)^t$ by shrinkage estimators of the form
$$\delta_j^\phi\left(X,S^2,T^2\right)=\left(1-\phi\left(S^2,T^2\right)\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\ldots,p, \tag{3.1}$$
where
$$\overline{X}=\frac{1}{p}\sum_{i=1}^{p}X_i\quad\text{and}\quad T^2=\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2,$$
with the two random variables S^2 and T^2 are independent. In the next, we follow the same steps that we treated in Section 2, then we give a similar results to those given in Section 2 with some changes in the proofs.

Lemma 3.1. *For any functions f and g of the two variables S^2 and T^2, such that all expectations of* (a) *and* (b) *exist, we have*

(a) $E\left\{f\left(S^2,T^2\right)\right\}=E\left\{f\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\right\}$;

(b) $E\left\{g\left(S^2,T^2\right)\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)\left(X_i-\overline{X}\right)\right\}=2\sigma^2E\left\{Kg\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\right\},$

where $K\sim P\left(\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2\right)$ *being the Poisson's distribution of parameter* $\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2$ *and* $\overline{\theta}=\frac{1}{p}\sum\limits_{i=1}^{p}\theta_i$.

Proof. Analogous to the proof of the Lemma 2.1 given by Sun [19]. □

The following proposition, gives the explicit formula of the risk of the estimator $\delta^{\phi}\left(X,S^2,T^2\right)$ given in (3.1). For the proof see Appendix.

Proposition 3.1. *Let* $\delta^{\phi}\left(X,S^2,T^2\right)$ *is given in* (3.1), *then for any* $p\geq 4$ *we have*

(i) $R\left(\delta^{\phi}(X,S^2,T^2),\theta\right)=\sigma^2\left\{\phi_K^2\chi_{p-1+2K}^2-2\phi_K\left(\chi_{p-1+2K}^2-2K\right)+p\right\}$;

(ii) $R\left(\delta^{\phi}(X,S^2,T^2),\theta\right)\geq B_p\left(\theta\right)$, *where*

$$\phi_K=\phi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\quad \text{and}\quad B_p\left(\theta\right)=\sigma^2E\left\{p-\frac{\left(\chi_{p-1+2K}^2-2K\right)^2}{\chi_{p-1+2K}^2}\right\};$$

(iii) *if* $c=\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2$ *exists, then*

$$\lim_{p\to+\infty}\frac{B_p\left(\theta\right)}{R\left(X,\theta\right)}=\lim_{p\to+\infty}\frac{B_p\left(\theta\right)}{p\sigma^2}=\lim_{p\to+\infty}b_p\left(\theta\right)=\frac{c}{1+c}.$$

Now, we consider the special case when $\phi\left(S^2,T^2\right)=d\frac{S^2}{T^2}$, where d is a constant, then the estimator given in (3.1) is written as

$$\delta_j^d\left(X,S^2,T^2\right)=\left(1-d\frac{S^2}{T^2}\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\ldots,p. \tag{3.2}$$

From the Proposition 3.1, we have

$$R\left(\delta^d(X,S^2,T^2),\theta\right)=\sigma^2\left\{p-\left[2dn\left(p-3\right)-d^2n\left(n+2\right)\right]E\left(\frac{1}{p-3+2K}\right)\right\}.$$

We note that when $d=0$, the estimator $\delta^0\left(X,S^2,T^2\right)$ given in (3.2) becomes the maximum likelihood estimator X, its risk equal $p\sigma^2$. In this case, the James-Stein estimator is obtained by minimizing the risk $R\left(\delta^d(X,S^2,T^2),\theta\right)$, the James-Stein estimator is given by

$$\delta_j^{JS}\left(X,S^2,T^2\right)=\left(1-\frac{p-3}{n+2}\frac{S^2}{T^2}\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\ldots,p. \tag{3.3}$$

Its risk is

$$R\left(\delta^{JS}(X,S^2,T^2),\theta\right)=\sigma^2\left\{p-\frac{n}{n+2}\left(p-3\right)^2E\left(\frac{1}{p-3+2K}\right)\right\}, \tag{3.4}$$

where $K\sim P\left(\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2\right)$.

Proposition 3.2. (a) *If $p \geq 4$, the James-Stein estimator $\delta^{JS}\left(X, S^2, T^2\right)$ given in* (3.3) *is minimax.*

(b) *If* $\lim\limits_{p \to +\infty} \sum\limits_{i=1}^{p} \left(\theta_i - \overline{\theta}\right)^2 / p\sigma^2 = c(>0)$, *then*

$$\lim_{n,p \to +\infty} \frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R(X, \theta)} = \frac{c}{1+c}.$$

Proof. (a) It is obviously from the formula (3.4).

(b) For $p \geq 6$ and from the Lemma 3.1 given by Sun [19], we have

$$\frac{1}{p-3+\frac{\sum_{i=1}^{p}\left(\theta_i - \overline{\theta}\right)^2}{\sigma^2}} \leq E\left(\frac{1}{p-3+2K}\right) \leq \frac{1}{p-5+\frac{\sum_{i=1}^{p}\left(\theta_i - \overline{\theta}\right)^2}{\sigma^2}},$$

then

$$\frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R(X, \theta)} \geq 1 - \frac{n}{n+2} \cdot \frac{(p-3)^2}{p^2} \cdot \frac{1}{\frac{p-5}{p} + \frac{\sum_{i=1}^{p}\left(\theta_i - \overline{\theta}\right)^2}{p\sigma^2}}$$

and

$$\frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R(X, \theta)} \leq 1 - \frac{n}{n+2} \cdot \frac{(p-3)^2}{p^2} \cdot \frac{1}{\frac{p-3}{p} + \frac{\sum_{i=1}^{p}\left(\theta_i - \overline{\theta}\right)^2}{p\sigma^2}}.$$

Thus,

$$\frac{c}{1+c} = 1 - \frac{1}{1+c} \leq \lim_{n,p \to +\infty} \frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R(X, \theta)} \leq 1 - \frac{1}{1+c} = \frac{c}{1+c}. \qquad \square$$

Remark 3.1. From Propositions 3.1 and 3.2, we note that the risks ratio of any shrinkage estimator $\delta^{\phi}\left(X, S^2, T^2\right)$ of the form (3.1) dominating the James-Stein estimator $\delta^{JS}\left(X, S^2, T^2\right)$, to the maximum likelihood estimator attains the limiting lower bound $B_m = \frac{c}{1+c}$, when n and p tend simultaneously to infinity.

Next, we consider the general form of shrinkage estimators of Lindley-type, defined by

$$\delta_j^{\varphi}\left(X, S^2, T^2\right) = \left(1 - \varphi\left(S^2, T^2\right)\frac{S^2}{T^2}\right)\left(X_j - \overline{X}\right) + \overline{X}, \quad j = 1, 2, \ldots, p. \tag{3.5}$$

We write $\Delta_{\varphi} = R\left(\delta^{\varphi}\left(X, S^2, T^2\right), \theta\right) - R(X, \theta)$. Then

$$\Delta_{\varphi} = \sigma^2 E\left\{\chi_n^2\ \varphi_K \left[\frac{\chi_n^2 \varphi_K}{\chi_{p-1+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p-1+2K}^2}\right)\right]\right\},$$

where $\varphi_K = \varphi\left(\sigma^2 \chi_n^2, \sigma^2 \chi_{p-1+2K}^2\right)$.

3.1. **Limit of risks ratios.**

Proposition 3.3. *Assume that* $\delta^{\varphi}\left(X,S^2,T^2\right)$ *is given in* (3.5), *such that* $p\geq 3$ *and* φ *satisfies*

(H) $\left|\frac{p-3}{n+2}-\varphi\left(S^2,T^2\right)\right|\leq g\left(S^2\right)$ *a.s., where* $E\left\{g^2\left(\sigma^2\chi^2_{n+4}\right)\right\}=O\left(\frac{1}{n^2}\right)$.

If $\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2=c$, *then*

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\varphi}\left(X,S^2,T^2\right),\theta\right)}{R\left(X,\theta\right)}=\frac{c}{1+c}.$$

Proof. We follow the same steps of the proof of Theorem 2.1, endeed we write $\alpha=\frac{p-3}{n+2}$ and $\varphi\left(S^2,T^2\right)=\varphi$. As

$$R\left(\delta^{\varphi}(X,S^2,T^2),\theta\right)=E\left\{\sum_{i=1}^{p}\left[\left(1-\varphi\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]^2\right\}$$

and

$$R\left(\delta^{JS}(X,S^2,T^2),\theta\right)=E\left\{\sum_{i=1}^{p}\left[\left(1-\alpha\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]^2\right\},$$

we have

$$\begin{aligned}\Delta_{JS}=&R\left(\delta^{\varphi}(X,S^2,T^2),\theta\right)-R\left(\delta^{JS}(X,S^2,T^2),\theta\right)\\=&2E\left\{\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\right]\left[\left(1-\frac{\left(\alpha+\varphi\right)}{2}\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]\right\}\\=&2E\left\{\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\left(1-\frac{\left(\alpha+\varphi\right)}{2}\frac{S^2}{T^2}\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)^2\right]\right.\\&\left.-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \theta_i\right]\right\}\\=&2E\left\{\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]+\frac{1}{2}\left[\left(\alpha-\varphi\right)^2\frac{S^4}{T^4}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]\right.\\&\left.-\alpha\left[\left(\alpha-\varphi\right)\frac{S^4}{T^4}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \left(\theta_i-\overline{X}\right)\right]\right\}\\=&2E\left\{\left[\left(\alpha-\varphi\right)S^2\right]+\frac{1}{2}\left[\left(\alpha-\varphi\right)^2\frac{S^4}{T^2}\right]-\alpha\left[\left(\alpha-\varphi\right)\frac{S^4}{T^2}\right]\right.\\&\left.-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \left(\theta_i-\overline{X}\right)\right]\right\}.\end{aligned}$$

As $\sum\limits_{i=1}^{p}\left(X_i-\overline{X}\right)=0$, then

$$E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{X})\right\}$$

$$
\begin{aligned}
&=E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{\theta})+(\overline{\theta}-\overline{X})(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})\right\}\\
&=E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{\theta})\right\}\\
&=2\sigma^2E\left\{K\left(\alpha-\varphi(\sigma^2\chi^2_n,\sigma^2\chi^2_{p-1+2K})\right)\frac{\chi^2_n}{\chi^2_{p-1+2K}}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Delta_{JS}\leq&2E\left\{(|\alpha-\varphi|)\,S^2+\frac{1}{2}(\alpha-\varphi)^2\frac{S^4}{T^2}+\alpha\,(|\alpha-\varphi|)\frac{S^4}{T^2}\right.\\
&\left.+2\sigma^2K\left(\left|\alpha-\varphi\left(\sigma^2\chi^2_n,\sigma^2\chi^2_{p-1+2K}\right)\right|\right)\frac{\chi^2_n}{\chi^2_{p-1+2K}}\right\}.
\end{aligned}
$$

From hypothesis (H) and the independence to two variables S^2 and T^2, we have

$$
\begin{aligned}
\Delta_{JS}\leq&2\left\{E\left[S^2g\left(S^2\right)\right]+\frac{1}{2}E\left[S^4g^2\left(S^2\right)\right]\;E\left(\frac{1}{T^2}\right)\right.\\
&\left.+\alpha E\left[S^4g\left(S^2\right)\right]\;E\left(\frac{1}{T^2}\right)\right\}+2\sigma^2E\left[S^2g\left(S^2\right)\frac{K}{\chi^2_{p-1+2K}}\right]\\
=&2E\left[S^4\frac{g\left(S^2\right)}{S^2}\right]+E\left[S^4g^2\left(S^2\right)\right]E\left(\frac{1}{T^2}\right)\\
&+2\alpha E\left[S^4g\left(S^2\right)\right]E\left(\frac{1}{T^2}\right)+2\sigma^2E\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{K}{\chi^2_{p-1+2K}}\right].
\end{aligned}
$$

Using the conditional expectation, we have

$$
\begin{aligned}
E\left[S^4\frac{g\left(S^2\right)}{S^2}\frac{K}{\chi^2_{p-1+2K}}\right]&=E\left\{E\left(\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{K}{\chi^2_{p-1+2K}}\right]\middle|S^2\right)\right\}\\
&=\frac{1}{2}E\left\{E\left(\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{2K}{p-3+2K}\right]\middle|S^2\right)\right\}\\
&\leq\frac{1}{2}E\left[S^4\frac{g\left(S^2\right)}{S^2}\right].
\end{aligned}
$$

From the Lemma 5.1 of the Appendix, the independence of two variables χ^2_{n+4} and χ^2_{p-1+2K} and the fact that $E\left(\frac{1}{\chi^2_{p-1+2K}}\right)=E\left(\frac{1}{p-3+2K}\right)\leq\frac{1}{p-3}$, we obtain

$$
\begin{aligned}
\Delta_{JS}\leq&2n\,(n+2)\,\sigma^2\left\{E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]+\frac{1}{2}E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]\;E\left(\frac{1}{\chi^2_{p-1+2K}}\right)\right\}\\
&+2n\,(n+2)\,\sigma^2\left(\alpha E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right]+E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]\right)\;E\left(\frac{1}{\chi^2_{p-1+2K}}\right)
\end{aligned}
$$

$$\begin{aligned}&\leq 2n\left(n+2\right)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]+\frac{n\left(n+2\right)}{p-3}\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]\\&+2n\sigma^2 E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right]+2\sigma^2\frac{n\left(n+2\right)}{p-3}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\Delta_{JS}}{p\sigma^2}\leq&\frac{2n\left(n+2\right)}{p}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]+\frac{n\left(n+2\right)}{p\left(p-3\right)}E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]\\&+\frac{2n}{p}E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right]+\frac{2\lambda}{p\sigma^2}\frac{n\left(n+2\right)}{p-3}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right],\end{aligned}$$

where $\lambda=\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/\sigma^2$.

From the condition $E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]=O\left(\frac{1}{n^2}\right)$, when n is in the neighborhood of $+\infty$, we have

$$\frac{\Delta_{JS}}{p\sigma^2}\leq\frac{2\left(n+2\right)}{np}\sqrt{M}+\frac{n+2}{np\left(p-3\right)}M+\frac{2}{p}\sqrt{M}+\frac{2\lambda}{p\sigma^2}\cdot\frac{\left(n+2\right)}{np}M,$$

where M is real strictly positive.

As $\lim\limits_{p\to+\infty}\frac{\lambda}{p}=\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2=c$, hence

$$\lim_{n,p\to+\infty}\frac{\Delta_{JS}}{p\sigma^2}\leq 0.$$

Thus, from Propositions 3.1 and 3.2, we have

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\varphi}\left(X,S^2,T^2\right),\theta\right)}{R\left(X,\theta\right)}=\frac{c}{1+c}.$$ □

3.2. **Minimaxity.**

Proposition 3.4. *Assume that $\delta^{\varphi}\left(X,S^2,T^2\right)$ is given in (3.5), such that $p\geq 4$. If*

(a) *$\varphi\left(S^2,T^2\right)$ is monotone non-decreasing in T^2;*

(b) $0\leq\varphi\left(S^2,T^2\right)\leq\dfrac{2\left(p-3\right)}{n+2}$.

A sufficient condition so that the estimator $\delta^{\varphi}\left(X,S^2,T^2\right)$ is minimax is, for any k, $k=0,1,2,\ldots$, and for each fixed T^2

$$E\left\{\varphi\left(\sigma^2\chi^2_{n+4}\ ,\sigma^2\chi^2_{p-1+2k}\right)\right\}\leq E\left\{\varphi\left(\sigma^2\chi^2_{n+2}\ ,\sigma^2\chi^2_{p-1+2k}\right)\right\}.$$

Proof. The proof is similar to proof of Theorem 2.2. Endeed, from condition (b), we obtain

$$\Delta_{\varphi}=\sigma^2 E\left\{\chi^2_n\ \varphi_K\left[\frac{\chi^2_n\ \varphi_K}{\chi^2_{p-1+2K}}-2\left(1-\frac{2K}{\chi^2_{p-1+2K}}\right)\right]\right\}$$

$$\leq \sigma^2 E\left\{\chi_n^2\ \varphi_K\left[\frac{\frac{2(p-3)}{n+2}\chi_n^2}{\chi_{p-1+2K}^2}-2\left(1-\frac{2K}{\chi_{p-1+2K}^2}\right)\right]\right\}.$$

We will prove that the expectation on the right hand side being non-positive for any $K=k,\ k=0,1,2,\dots$

By using the conditional expectation, we have

$$\begin{aligned}\Delta_\varphi &\leq \sigma^2 E\left[E\left\{\chi_n^2\ \varphi_k\left[\frac{\frac{2(p-3)}{n+2}\chi_n^2}{\chi_{p-1+2k}^2}-2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right]\middle|\ \chi_n^2\right\}\right]\\ &\leq \sigma^2 E\left\{\chi_n^2\ E\left(\varphi_k\middle|\ \chi_n^2\right)\ E\left[\left(\frac{\frac{2(p-3)}{n+2}\ \chi_n^2}{\chi_{p-1+2k}^2}-2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right)\middle|\ \chi_n^2\right]\right\},\end{aligned}$$

the last inequality according to the condition (a) and the fact that the covariance of two functions one increasing and the other decreasing is non-positive.

As

$$\begin{aligned}E\left[\left(\frac{\frac{2(p-3)}{n+2}\ \chi_n^2}{\chi_{p-1+2k}^2}-2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right)\mid \chi_n^2\right] &= E\left[\frac{2\,(p-3)\left(\frac{\chi_n^2}{n+2}-1\right)}{p-3+2k}\middle|\ \chi_n^2\right]\\ &= \frac{2\,(p-3)\left(\frac{\chi_n^2}{n+2}-1\right)}{p-3+2k},\end{aligned}$$

then

$$\begin{aligned}\Delta_\varphi &\leq \sigma^2 E\left\{\chi_n^2\ \frac{2\,(p-3)\left(\dfrac{\chi_n^2}{n+2}-1\right)}{p-3+2k}\ E\left(\varphi_k\mid \chi_n^2\right)\right\}\\ &= \frac{2\,(p-3)\,\sigma^2}{p-3+2k}\ E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\varphi_k\right\}.\end{aligned}$$

Using the sufficient condition

$$E\left\{\varphi\left(\sigma^2\chi_{n+4}^2\ ,\sigma^2\chi_{p-1+2k}^2\right)\right\}\leq E\left\{\varphi\left(\sigma^2\chi_{n+2}^2\,\sigma^2\chi_{p-1+2k}^2\right)\right\},$$

we have

$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\varphi_k\right\}\leq 0,$$

hence $\Delta_\varphi\leq 0$. □

Remark 3.2. Note that the James-Stein estimator given in (3.3) satisfies the conditions of the Proposition 3.4, thus the James-Stein estimator is minimax.

4. SIMULATION

We illustrate the graph of the upper bound given by the formula (2.4) for the risk difference Δ_ψ of the estimator $\delta^{\psi_2}(X, S^2)$ given in the Example 2.2 and the maximum likelihood estimator, divided by the risk of the maximum likelihood estimator $R(X, \theta) = p\sigma^2$, as a function of $d = \|\theta\|^2$ and $s = \sigma^2$, for various values of n and p.

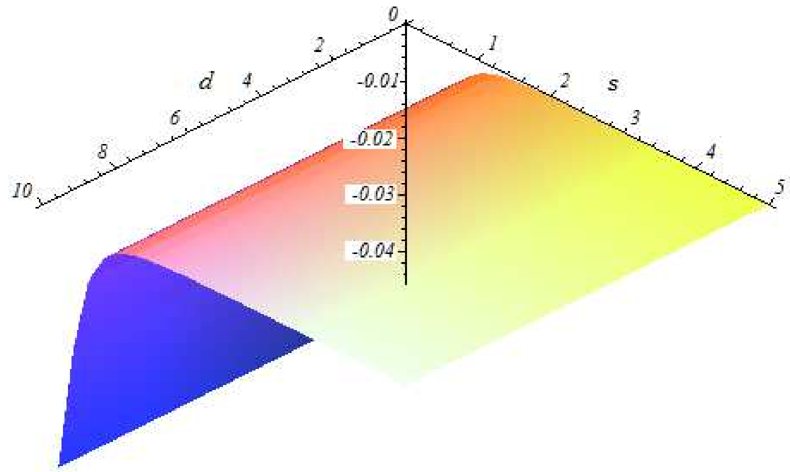

FIGURE 1. $n = 10$ and $p = 4$

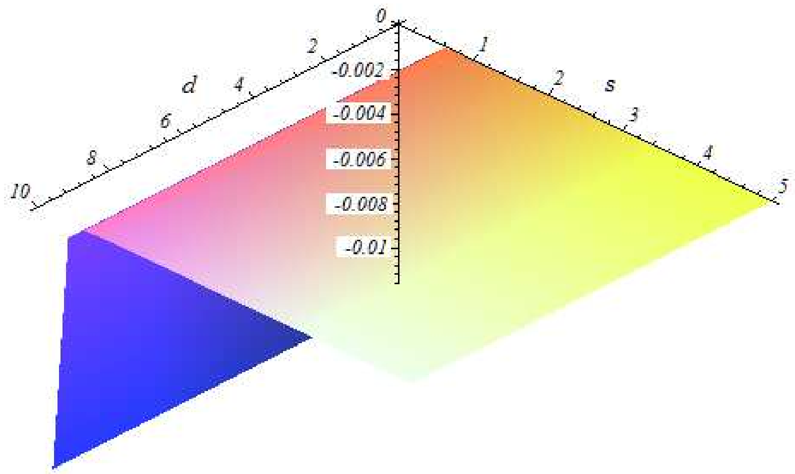

FIGURE 2. $n = 25$ and $p = 10$

In Figure 1 and Figure 2, we note that an upper bound of risks difference of the estimator $\delta^{\psi_2}(X, S^2)$ given in the Example 2.2 and the maximum likelihood estimator X, divided by the risk of the maximum likelihood estimator is negative, thus the estimator $\delta^{\psi_2}(X, S^2)$ is minimax for $n = 10$ and $p = 4$ and for $n = 25$ and $p = 10$.

5. Appendix

Lemma 5.1 (Casella and Hwang [4]). *For any real function h such that $E\left(h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right)$ exists, we have*

$$E\left\{h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right\} = qE\left\{h\left(\chi_{q+2}^2(\lambda)\right)\right\} + 2\lambda E\left\{h\left(\chi_{q+4}^2(\lambda)\right)\right\}.$$

Lemma 5.2 (Benmansour and Hamdaoui [2]). *Let f be a real function. If for $p \geq 3$, $E_{\chi_p^2(\lambda)}[(f(U)]$ exists, then*

(a) *if f is monotone non-increasing, we have*

$$E_{\chi_{p+2}^2(\lambda)}[(f(U)] \leq E_{\chi_p^2(\lambda)}[(f(U)];$$

(b) *if f is monotone non-decreasing, we have*

$$E_{\chi_{p+2}^2(\lambda)}[(f(U)] \geq E_{\chi_p^2(\lambda)}[(f(U)].$$

Proof. (Proposition 3.1) (i)

$$\begin{aligned}
R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right) =& E\left[\sum_{i=1}^{p}\left[\left(1-\phi\left(S^2,T^2\right)\right)\left(X_i-\overline{X}\right)+\overline{X}\ -\theta_i\right]^2\right] \\
=& E\left[\left[1-\phi\left(S^2,T^2\right)\right]^2\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]+E\left[\sum_{i=1}^{p}\left(\overline{X}\ -\theta_i\right)^2\right] \\
&+2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\overline{X}-\theta_i\right)\right].
\end{aligned}$$

As

$$\begin{aligned}
E\left[\left[1-\phi\left(S^2,T^2\right)\right]^2\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right] =& E\left[\left(1-\phi_K\right)^2 T^2\right] \\
=& \sigma^2 E\left[\left(1-\phi_K\right)^2\ \chi_{p-1+2K}^2\right] \qquad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
E\left[\sum_{i=1}^{p}\left(\overline{X}-\theta_i\right)^2\right] =& E\left[\sum_{i=1}^{p}\left(\overline{X}-\overline{\theta}+\overline{\theta}-\theta_i\right)^2\right] \\
=& E\left[\sum_{i=1}^{p}\left(\overline{X}-\overline{\theta}\right)^2\right]+\sum_{i=1}^{p}\left(\overline{\theta}-\theta_i\right)^2+2\left(\sum_{i=1}^{p}\left(\overline{\theta}-\theta_i\right)\right)E\left(\overline{X}-\overline{\theta}\right) \\
=& \sigma^2+\sum_{i=1}^{p}\left(\overline{\theta}-\theta_i\right)^2. \qquad (5.2)
\end{aligned}$$

The last equality comes from the distribution of $\overline{X}$, $\overline{X} \sim N_p\left(\overline{\theta}, \frac{\sigma^2}{p}\right)$ and the fact that $\sum_{i=1}^{p}\left(\overline{\theta}-\theta_i\right) = 0$.

Furthermore, we have

$$
\begin{aligned}
&2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\overline{X}-\theta_i\right)\right]\\
&=-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}+\overline{\theta}-\overline{X}\right)\right]\\
&=-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right]\\
&\quad-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\left(\overline{\theta}-\overline{X}\ \right)\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\right]\\
&=-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right].
\end{aligned}
$$

The last equality follows from the fact that $\sum_{i=1}^{p}\left(X_i-\overline{X}\right)=0$.

Using (b) of Lemma 3.1, we have

$$
-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right]=-4\sigma^2E\left[K\left(1-\phi_K\right)\right]. \tag{5.3}
$$

From formulas (5.1), (5.2) and (5.3) and the fact that $E\left(K\right)=\frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{2\sigma^2}$, we have

$$
\begin{aligned}
R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)&=E\left\{\sigma^2\left(1-\phi_K\right)^2\ \chi^2_{p-1+2K}+\sigma^2+2\sigma^2K-4\sigma^2K\left(1-\phi_K\right)\right\}\\
&=\sigma^2E\left\{\phi_K^2\ \chi^2_{p-1+2K}-2\phi_K\left(\chi^2_{p-1+2K}-2K\right)+p\right\}.
\end{aligned}
$$

(ii) We note that $R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)$ can be written as

$$
\begin{aligned}
R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)=&\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\}\\
&+\sigma^2E\left\{\chi^2_{p-1+2K}\left(\phi_K-1+\frac{2K}{\chi^2_{p-1+2K}}\right)^2\right\}\\
\geq&\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\}=B_p\left(\theta\right).
\end{aligned}
$$

(iii)

$$
B_p\left(\theta\right)=\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\}
$$

$$= \sigma^2 \left\{ p - E \left\{ E \left[\left(\chi^2_{p-1+2K} + \frac{4K^2}{\chi^2_{p-1+2K}} - 4K \right) \mid K \right] \right\} \right\}$$
$$= \sigma^2 \left\{ p - E \left(p - 1 + 2K + \frac{4K^2}{p-3+2K} - 4K \right) \right\}$$
$$= \sigma^2 \left\{ p - 2 - E \left[\frac{(p-3)^2}{p-3+2K} \right] \right\}.$$

Thus, from Lemma 3.1 given in Sun [19], we obtain

$$\lim_{p\to+\infty} b_p(\theta) = \lim_{p\to+\infty} \frac{B_p(\theta)}{p\sigma^2} = \frac{c}{1+c}. \qquad \square$$

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ON (m, h_1, h_2)-G-CONVEX DOMINATED STOCHASTIC PROCESSES

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Abstract. In this paper is introduced the concept of (m, h_1, h_2)-convexity for stochastic processes dominated by other stochastic processes with the same property, some mean square integral Hermite-Hadamard type inequalities for this kind of generalized convexity are established and from the founded results, other mean square integral inequalities for the classical convex, s-convex in the first and second sense, P-convex and MT-convex stochastic processes are deduced.

1. Introduction

In 1974, B. Nagy applied a characterization of measurable stochastic processes to solve a generalization of the (additive) Cauchy functional equation [15]. Later, in 1980 K. Nikodem [17] considered convex stochastic processes, and in 1995 A. Skowronski [27] obtained some further results on Wright convex stochastic processes, which generalize some known properties of convex stochastic processes. For a detailed study about this topic the following references are helpful [2, 3, 13, 24, 25].

Convexity is one of the hypotheses often used in optimization theory. It is generally used to give global validity for certain propositions, which otherwise would only be true locally. A function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is said to be a convex function on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1.1) holds, then f is concave.

Key words and phrases. (m, h_1, h_2)-convexity, dominated convexity, mean square integral inequalities, stochastic processes.

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The convexity of functions and their generalized forms play an important role in many fields such as Economic Science, Biology, Optimization and other [6, 21].

About the concept of convexity, its evolution has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity, h-convexity, MT-convexity, log-convexity, P-convexity, η-convexity, quasi convexity and others, as well as combinations of these new concepts have been introduced. The following references give more information about the research in this area [1, 5, 11, 14, 16, 18, 22, 29].

Similarly, some recent studies have been introduced the following concepts: J-convex [26], Wright-convex [27], strongly convex [9], strongly Wright [10], p-convex [20], harmonically convex [19], s-convex in the first and second sense [12,23] stochastic process.

The well-known Hermite-Hadamard inequality establish that for every convex function $f : I \subset \mathbb{R} \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds for every $a, b \in I$, with $a < b$.

In 2012, D. Kotrys presented the Hermite-Hadamard inequality for convex stochastic processes [8].

Theorem 1.1. *If $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{u-v}\int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot)+X(v, \cdot)}{2} \tag{1.3}$$

almost everywhere for all $u, v \in I$.

Many researchers have developed works where they relate the concepts of generalized convexity and stochastic processes using the inequality (1.3). For example, E. Set et al. in [23] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense, and M. J. Vivas-Cortez and J. E. Hernández Hernández in [30] studied about (h_1, h_2, m)-GA-convexity for stochastic processes.

Following this line of research, this paper introduces the concept of (m, h_1, h_2)-convexity for stochastic processes dominated by other stochastic processes with the same property, some mean square integral Hermite-Hadamard type inequalities for this kind of generalized convexity are established, and from the founded results, other integral inequalities for stochastic processes with other types of convexity are deduced.

2. Preliminaries

The following references [8,13,27,28] contain the basic notions of stochastic processes used in this work.

Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable and $P\{w \in \Omega : X(w) \notin \mathbb{R}\} = 0$. Let $I \subset \mathbb{R}$

be time. A function $X : I \times \Omega \to \mathbb{R}$ is called a stochastic process if for all $t \in I$ the function $X(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable.

In this work I is an interval and $X(t, \cdot)$ is called a stochastic process with continuous time.

It is said that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

(a) continuous in probability on the interval I if for all $t_0 \in I$ it follows that

$$\mu - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability;

(b) mean-square continuous in the interval I if for all $t_0 \in I$

$$\mu - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$

where $\mathbb{E}(X(t, \cdot))$ denote the expectation value of the random variable $X(t, \cdot)$;

(c) increasing (decreasing) if for all $u, v \in I$ such that $t < s$,

$$X(u, \cdot) \le X(v, \cdot), \quad (X(u, \cdot) \ge X(v, \cdot)) \quad \text{(a.e.)};$$

(d) monotonic if it is increasing or decreasing;

(e) differentiable at a point $t \in I$ if there exists a random variable $X'(t, \cdot) : I \times \Omega \to \mathbb{R}$ such that

$$X'(t, \cdot) = \mu - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I.

Definition 2.1. Let (Ω, A, P) be a probability space, $I \subset \mathbb{R}$ be an interval with $E(X(t, \cdot)^2) < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \dots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds

$$\lim_{n \to \infty} E \left[\sum_{k=0}^{n} X(\theta_k, \cdot)(t_k - t_{k-1}) - Y \right]^2 = 0,$$

then it can be written

$$\int_a^b X(t, \cdot) dt = Y(\cdot) \quad \text{(a.e.)}.$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot) dt \le \int_a^b Z(t, \cdot) dt \quad \text{(a.e.)},$$

where $X(t, \cdot) \le Z(t, \cdot)$ in $[a, b]$ ([26]).

In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.

In 1980, K. Nickoden introduced an important definition in which the property of convexity for stochastic processes is established [17].

Definition 2.2. Set $(\Omega, \mathcal{A}, P)$ to be a probability space and $I \subset \mathbb{R}$ be an interval. It is said that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if the following inequality holds almost everywhere

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot), \tag{2.1}$$

for all $u, v \in I$ and $\lambda \in [0,1]$.

In the work of J. E. Hernández Hernández and J. F. Gómez [7] the following definition is introduced.

Definition 2.3. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non negative functions, $m \in (0,1]$ and $I \subset \mathbb{R}$ an interval. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is (m, h_1, h_2)-convex if the following inequality holds almost everywhere

$$X\left(ta + m(1-t)b, \cdot\right) \leq h_1(t)X(a, \cdot) + mh_2(t)X\left(b, \cdot\right), \tag{2.2}$$

for all $a, b \in I$ and $t \in [0,1]$.

Some other kinds of generalized convexity for stochastic process, as s-convexity in the second sense and P-convexity are presented in the same work.

With the notion of dominated convexity introduced by S. S. Dragomir et al. in [4], the following definitions for stochastic processes are introduced.

Definition 2.4. Let $I \subset \mathbb{R}$ be an interval and $G : I \times \Omega \to \mathbb{R}$ be a non negative convex stochastic process. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called a convex dominated by G if the following inequality holds almost everywhere

$$\begin{aligned} &\left|tX(a, \cdot) + (1-t)X\left(b, \cdot\right) - X\left(ta + (1-t)b, \cdot\right)\right| \\ \leq &t(t)G(a, \cdot) + (1-t)G\left(b, \cdot\right) - G\left(ta + (1-t)b, \cdot\right), \end{aligned} \tag{2.3}$$

for all $a, b \in I$ and $t \in [0,1]$.

Definition 2.5. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non negative functions, $m \in (0,1]$, $I \subset \mathbb{R}$ an interval and $G : I \times \Omega \to \mathbb{R}$ be a non negative (m, h_1, h_2)-convex stochastic process. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called a (m, h_1, h_2)-convex dominated by G if the following inequality holds almost everywhere

$$\begin{aligned} &\left|h_1(t)X(a, \cdot) + mh_2(t)X\left(b, \cdot\right) - X\left(ta + m(1-t)b, \cdot\right)\right| \\ \leq &h_1(t)G(a, \cdot) + mh_2(t)G\left(b, \cdot\right) - G\left(ta + m(1-t)b, \cdot\right), \end{aligned} \tag{2.4}$$

for all $a, b \in I$ and $t \in [0,1]$.

Note that if $m = 1$, $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0,1]$ the Definition 2.4 is obtained, if $m = 1$, $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0,1]$ and some $s \in (0,1]$ we have the definition of s-convex stochastic process in the first sense [12]; if $m = 1$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and some $s \in (0,1]$ we have the definition of s-convex stochastic process in the second sense [23]; if $m = 1$, $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ then the definition of P-convex stochastic process follows [7] and also, if $m = 1$, $h_1(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $h_2(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ for all $t \in (0,1)$ the definition of MT-convex stochastic process is obtained.

3. Main Results

Henceforth, $h_1, h_2 : [0, 1] \to \mathbb{R}$ are considered non-negative functions and $m \in (0, 1]$.

Proposition 3.1. *Let $G : I \times \Omega \to \mathbb{R}$ and $X : I \times \Omega \to \mathbb{R}$ be a non negative (m, h_1, h_2)-convex stochastic processes. The following statements are equivalent:*

i) X is a (m, h_1, h_2)-convex dominated by G;
ii) the stochastic processes $(G - X)$ and $(G + X)$ are (m, h_1, h_2)-convex;
iii) there exist two (m, h_1, h_2)-convex stochastic processes $H, K : I \times \Omega \to \mathbb{R}$ such that $X = \frac{1}{2}(H - K)$ and $G = \frac{1}{2}(H + K)$.

Proof. $i) \Leftrightarrow ii)$ The condition (2.4) is equivalent to

$$\begin{aligned} &G(ta + m(1-t)b, \cdot) - h_1(t)G(a, \cdot) - mh_2(t)G(b, \cdot) \\ \leq &h_1(t)X(a, \cdot) + mh_2(t)X(b, \cdot) - X(ta + m(1-t)b, \cdot) \\ \leq &h_1(t)G(a, \cdot) + mh_2(t)G(b, \cdot) - G(ta + m(1-t)b, \cdot), \end{aligned}$$

and, from this double inequality, making a correct rearrange it follows that

$$(G + X)(ta + m(1-t)b, \cdot) \leq h_1(t)(G + X)(a, \cdot) + mh_2(t)(G + X)(b, \cdot)$$

and

$$(G - X)(ta + m(1-t)b, \cdot) \leq h_1(t)(G - X)(a, \cdot) + mh_2(t)(G - X)(b, \cdot).$$

$iii) \Rightarrow ii)$ Lets define $X = \frac{1}{2}(H - K)$ and $G = \frac{1}{2}(H + K)$. Adding and subtracting we have $(G + X) = H$ and $(G - X) = K$, so, both are (m, h_1, h_2)-convex stochastic processes.

$ii) \Rightarrow iii)$ By condition $ii)$, $(G + X)$ and $(G - X)$ are (m, h_1, h_2)-convex stochastic processes, so $H = G + K$ and $K = G - X$ are (m, h_1, h_2)-convex stochastic processes. □

Proposition 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a (m, h_1, h_2)-convex stochastic process and $A : \Omega \to \mathbb{R}$ a random variable, then the stochastic process defined by $A(\cdot)X(t, \cdot)$ is (m, h_1, h_2)-convex.*

Proof. Using Definition 2.3 we have the desired result. □

Proposition 3.3. *Let $G : I \times \Omega \to \mathbb{R}$ be a (m, h_1, h_2)-convex stochastic process and $X, Y : I \times \Omega \to \mathbb{R}$ two (m, h_1, h_2)-convex stochastic process dominated by G, then we have that $X + Y$ is a (m, h_1, h_2)-convex stochastic process dominated by $2G$. Also, if $A : \Omega \to \mathbb{R}$ is a random variable, then the (m, h_1, h_2)-convex stochastic process defined by $A(\cdot)X(t, \cdot)$ is dominated by $|A(\cdot)| G$.*

Proof. With the help of Definition 2.5 and the triangular inequality we have that

$$\begin{aligned}
&|h_1(t)(X+Y)(u,\cdot)+mh_2(t)(X+Y)(v,\cdot)-(X+Y)(tu+(1-t)v,\cdot)| \\
=&|h_1X(u,\cdot)+mh_2(t)X(v,\cdot)-X(tu+(1-t)v,\cdot) \\
&+h_1Y(u,\cdot)+mh_2(t)Y(v,\cdot)-Y(tu+(1-t)v,\cdot)| \\
\leq&|h_1X(u,\cdot)+mh_2(t)X(v,\cdot)-X(tu+(1-t)v,\cdot)| \\
&+|h_1Y(u,\cdot)+mh_2(t)Y(v,\cdot)-Y(tu+(1-t)v,\cdot)| \\
\leq&2(h_1(t)G(u,\cdot)+mh_2(t)G(v,\cdot)-G(tu+(1-t)v,\cdot))
\end{aligned}$$

and

$$\begin{aligned}
&|h_1(t)A(\cdot)X(u,\cdot)+mh_2(t)A(\cdot)X(v,\cdot)-A(\cdot)X(tu+(1-t)v,\cdot)| \\
\leq&|A(\cdot)|\,(h_1(t)G(u,\cdot)+mh_2(t)G(v,\cdot)-G(tu+(1-t)v,\cdot)).
\end{aligned}$$

The proof is complete. □

Remark 3.1. The previous proposition is also valid for the case of subtraction of stochastic processes, and it is easily proved that the algebraic sum of n (m,h_1,h_2)-convex stochastic processes, each one dominated by the same (m,h_1,h_2)-convex stochastic process G is a (m,h_1,h_2)-convex stochastic process dominated by nG.

Proposition 3.4. *Let $G: I\times\Omega\to\mathbb{R}$ be a (m,h_1,h_2)-convex stochastic process, $\{X_k\}_{k=1}^n$ be a finite collection of (m,h_1,h_2)-convex stochastic process dominated by G, and $\{A_k\}_{k=1}^n$ a finite collection of random variables. Then $\sum_{k=1}^n A_k(\cdot)X_k(t,\cdot)$ is dominated by $\sum_{k=1}^n |A_k|\,G$.*

Theorem 3.1. *Let $X: I\times\Omega\to\mathbb{R}$ be a mean square integrable stochastic process on the interval $[0,b/m]$ and (m,h_1,h_2)-convex. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{h_1(1/2)}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}X(u,\cdot)du \tag{3.1}$$

and

$$\frac{1}{b-a}\int_a^b X(u,\cdot)\,dt \leq\frac{(X(a,\cdot)+X(b,\cdot))}{2}I(h_1) \tag{3.2}$$

$$+\frac{m\left(X\left(\frac{a}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)\right)}{2}I(h_2), \tag{3.3}$$

where

$$I(h_1)=\int_0^1 h_1(t)dt \quad \textit{and} \quad I(h_2)=\int_0^1 h_2(t)dt.$$

Proof. Let $a,b\in I$ and $m\in(0,1]$. Then for $t\in[0,1]$ we have

$$X\left(\frac{a+b}{2},\cdot\right)=X\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right),$$

and using the (m,h_1,h_2)-convexity of X we obtain

$$X\left(\frac{a+b}{2},\cdot\right) \leq h_1(1/2)X\left(ta+(1-t)b,\cdot\right)+mh_2(1/2)X\left(t\frac{a}{m}+(1-t)\frac{b}{m},\cdot\right).$$

Integrating over $t\in[0,1]$ it follows that

$$\begin{aligned}X\left(\frac{a+b}{2},\cdot\right) \leq & h_1(1/2)\int_0^1 X\left(ta+(1-t)b,\cdot\right)dt\\ &+mh_2(1/2)\int_0^1 X\left(t\frac{a}{m}+(1-t)\frac{b}{m},\cdot\right)dt,\end{aligned}$$

and with the change of variable $u=ta+(1-t)b$ and $v=t\frac{a}{m}+(1-t)\frac{b}{m}$ we achieve the inequality (3.1).

Now, using the $(m,h_1,h_2)-$convexity of X we have

$$X\left(ta+(1-t)b,\cdot\right)\leq h_1(t)X\left(a,\cdot\right)+mh_2(t)X\left(\frac{b}{m},\cdot\right) \tag{3.4}$$

and

$$X\left((1-t)a+tb,\cdot\right)\leq h_1(t)X\left(b,\cdot\right)+mh_2(t)X\left(\frac{a}{m},\cdot\right). \tag{3.5}$$

Adding (3.4) and (3.5) and integrating over $t\in[0,1]$ it follows that

$$\begin{aligned}&\int_0^1 X\left(ta+(1-t)b,\cdot\right)dt+\int_0^1 X\left((1-t)a+tb,\cdot\right)dt\\ &\leq\left(X\left(a,\cdot\right)+X\left(b,\cdot\right)\right)\int_0^1 h_1(t)dt+m\left(X\left(\frac{a}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)\right)\int_0^1 h_2(t)dt.\end{aligned}$$

So, with the above change of variable and doing

$$I(h_1)=\int_0^1 h_1(t)dt \quad\text{and}\quad I(h_2)=\int_0^1 h_2(t)dt,$$

the inequality (3.3) is attained.

The proof is complete. □

Corollary 3.1. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{b-a}\int_a^b X(u,\cdot)du\leq\frac{X\left(a,\cdot\right)+X\left(b,\cdot\right)}{2}.$$

Proof. Letting $m=1$, $h_1(t)=t$ and $h_2(t)=1-t$, $t\in[0,1]$, in Theorem 3.1, we obtain the desired result. □

Corollary 3.2. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and s-convex stochastic process in the second sense. Then the following inequalities* *hold almost everywhere*

$$2^{s-1}X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\leq\frac{X(a,\cdot)+X(b,\cdot)}{(s+1)}.$$

Proof. Let $s\in(0,1]$. Making $m=1$, $h_1(t)=t^s$ and $h_2(t)=(1-t)^s$ for $t\in[0,1]$ in Theorem 3.1, it follows the desired result. □

Corollary 3.3. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and s-convex stochastic process in the first sense. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\leq\frac{X(a,\cdot)+X(b,\cdot)}{2}.$$

Proof. Let $s\in(0,1]$. Making $m=1$, $h_1(t)=t^s$ and $h_2(t)=1-t^s$ for $t\in[0,1]$ in Theorem 3.1, we have the desired result. □

Corollary 3.4. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and P-convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{2}{b-a}\int_a^b X(u,\cdot)du\leq 2(X(a,\cdot)+X(b,\cdot)).$$

Proof. Letting $m=1$, $h_1(t)=h_2(t)=1$ for $t\in[0,1]$ in Theorem 3.1 we obtain the desired result. □

Corollary 3.5. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and MT-convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{2(b-a)}\int_a^b X(u,\cdot)du\leq\frac{\pi(X(a,\cdot)+X(b,\cdot))}{4}.$$

Proof. Letting $m=1$, $h_1(t)=\sqrt{t}/2\sqrt{1-t}$ and $h_2(t)=\sqrt{1-t}/2\sqrt{t}$ for $t\in[0,1]$ in Theorem 3.1 we have the desired result. □

Remark 3.2. The inequality found in Corollary 3.1 coincides with that presnted in [8], the result found in Corollary 3.2 coincides with that presented in Theorem 6 in [23].

Theorem 3.2. *Let $X,G:I\times\Omega\to\mathbb{R}$ be a mean square integrable stochastic process on the interval $[0,b/m]$ and (m,h_1,h_2)-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\begin{aligned}&\left|h_1(1/2)\frac{1}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}X(u,\cdot)-X\left(\frac{a+b}{2},\cdot\right)\right|\\ \leq&h_1(1/2)\frac{1}{b-a}\int_a^b G(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}G(u,\cdot)-G\left(\frac{a+b}{2},\cdot\right)\end{aligned}$$

and

$$\left|\frac{(X(a,\cdot)+X(b,\cdot))}{2}I(h_1)+\frac{m}{2}\left(X\left(\frac{a}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)\right)I(h_2)-\frac{1}{b-a}\int_a^b X\left(u,\cdot\right)du\right|$$
$$\leq\frac{(G(a,\cdot)+G(b,\cdot))}{2}I(h_1)+\frac{m}{2}\left(G\left(\frac{a}{m},\cdot\right)+G\left(\frac{b}{m},\cdot\right)\right)I(h_2)-\frac{1}{b-a}\int_a^b G\left(u,\cdot\right)du,$$

where

$$I(h_1)=\int_0^1 h_1(t)dt \quad \text{and} \quad I(h_2)=\int_0^1 h_2(t)dt.$$

Proof. Let $a, b \in I$ and $m \in (0,1]$. Then, for $t \in [0,1]$ we have

$$X\left(\frac{a+b}{2},\cdot\right)=X\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right)$$

and

$$G\left(\frac{a+b}{2},\cdot\right)=G\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right).$$

Using definition of (m, h_1, h_2)-convexity dominated by G we obtain that

$$\left|h_1(t)X(ta+(1-t)b,\cdot)+mh_2(t)X\left((1-t)\frac{a}{m}+t\frac{b}{m},\cdot\right)-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq h_1(1/2)G(ta+(1-t)b,\cdot)+mh_2(1/2)G\left((1-t)\left(\frac{a}{m}\right)+t\left(\frac{b}{m}\right),\cdot\right)-G\left(\frac{a+b}{2},\cdot\right).$$

Integrating over $t \in [0,1]$ it follows that

$$\left|h_1(1/2)\frac{1}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m} X\left(u,\cdot\right)-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq h_1(1/2)\frac{1}{b-a}\int_a^b G(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m} G\left(u,\cdot\right)-G\left(\frac{a+b}{2},\cdot\right).$$

So, the first inequality is obtained.

Now, also we have

$$\left|h_1(t)X(a,\cdot)+mh_2(t)X\left(\frac{b}{m},\cdot\right)-X\left(ta+(1-t)b,\cdot\right)\right|$$
$$\leq h_1(1/2)G(a,\cdot)+mh_2(1/2)G\left(\frac{b}{m},\cdot\right)-G\left(ta+(1-t)b,\cdot\right)$$

and

$$\left|h_1(t)X(b,\cdot)+mh_2(t)X\left(\frac{a}{m},\cdot\right)-X\left((1-t)a+tb,\cdot\right)\right|$$
$$\leq h_1(1/2)G(b,\cdot)+mh_2(1/2)G\left(\frac{a}{m},\cdot\right)-G\left((1-t)a+tb,\cdot\right).$$

Adding these inequalities, integrating over $t \in [0,1]$ and taking the notation

$$I(h_1) = \int_0^1 h_1(t)dt \quad \text{and} \quad I(h_2) = \int_0^1 h_2(t)dt,$$

we obtain the desired result. $\square$

Corollary 3.6. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left| \frac{1}{b-a} \int_a^b X(u,\cdot)du - X\left(\frac{a+b}{2},\cdot\right) \right| \le \frac{1}{b-a} \int_a^b G(u,\cdot)du - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left| \frac{(X(a,\cdot)+X(b,\cdot))}{2} - \frac{1}{b-a} \int_a^b X(u,\cdot)\,du \right| \le \frac{G(a,\cdot)+G(b,\cdot)}{2} - \frac{1}{b-a} \int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m=1$, $h_1(t)=t$ and $h_2(t)=1-t$, $t \in [0,1]$, in Theorem 3.2 we achieve the desired result. $\square$

Corollary 3.7. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and s-convex in the second sense. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left| \frac{1}{b-a} \int_a^b X(u,\cdot)du - 2^{s-1}X\left(\frac{a+b}{2},\cdot\right) \right| \le \frac{1}{b-a} \int_a^b G(u,\cdot)du - 2^{s-1}G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left| \frac{(X(a,\cdot)+X(b,\cdot))}{s+1} - \frac{1}{b-a} \int_a^b X(u,\cdot)\,du \right| \le \frac{(G(a,\cdot)+G(b,\cdot))}{s+1} - \frac{1}{b-a} \int_a^b G(u,\cdot)\,du.$$

Proof. Let $s \in (0,1]$. Making $m=1$, $h_1(t)=t^s$ and $h_2(t)=(1-t)^s$, $t \in [0,1]$, in Theorem 3.2 we have the desired result. $\square$

Corollary 3.8. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable on the interval I and s-convex stochastic process in the first sense. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left| \frac{1}{2^{s-1}(b-a)} \int_a^b X(u,\cdot) - X\left(\frac{a+b}{2},\cdot\right) \right| \le \frac{1}{2^{s-1}(b-a)} \int_a^b G(u,\cdot) - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left| \frac{(X(a,\cdot)+X(b,\cdot))}{2} - \frac{1}{b-a} \int_a^b X(u,\cdot)\,du \right| \le \frac{(G(a,\cdot)+G(b,\cdot))}{2} - \frac{1}{b-a} \int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m=1$, $h_1(t)=t^s$ and $h_2(t)=1-t^s$, $t \in [0,1]$, in Theorem 3.2 it follows the desired result. $\square$

Corollary 3.9. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes and P-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{2}{b-a}\int_a^b X(u,\cdot)du - X\left(\frac{a+b}{2},\cdot\right)\right| \le \frac{2}{b-a}\int_a^b G(u,\cdot)du - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|(X(a,\cdot)+X(b,\cdot)) - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \le (G(a,\cdot)+G(b,\cdot)) - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m=1$, $h_1(t)=h_2(t)=1$ for all $t\in[0,1]$, in Theorem 3.2 we have the desired result □

Corollary 3.10. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and MT-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{1}{b-a}\int_a^b X(u,\cdot)du - 2X\left(\frac{a+b}{2},\cdot\right)\right| \le \frac{1}{b-a}\int_a^b G(u,\cdot)du - 2G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|\frac{\pi\,(X(a,\cdot)+X(b,\cdot))}{4} - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \le \frac{\pi\,(G(a,\cdot)+G(b,\cdot))}{4} - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m=1$, $h_1(t)=\sqrt{t}/2\sqrt{1-t}$ and $h_2(t)=\sqrt{1-t}/2\sqrt{t}$ for $t\in[0,1]$ in Theorem 3.2 we obtain the desired result. □

4. Conclusions

In the development of the present work it was introduced the concept of (m,h_1,h_2)-convex stochastic process dominated by another stochastic process of the same type, also some properties associated with them were found (Definition 2.5, Propositions 3.1, 3.2 and 3.3). From the aforementioned definition the Hermite-Hadamard inequality for stochastic processes (Theorem 3.1) was found and some Corollaries that involve the same inequality for classical convex stochastic process and other types of generalized convex stochastic process (Corollaries 3.1–3.5). Also it was studied the absolute value of the difference of the extremes of right and left side of the Hermite-Hadamard inequality for the generalized convex stochastic process under study, similarly some corollaries for other types of convexity were found (Theorem 3.2 and Corollaries 3.6–3.10).

The author hopes that the results presented will stimulate the study of the relationship between generalized convexity and stochastic processes, thus providing a path to possible applications.

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$\alpha\beta$-WEIGHTED d_g-STATISTICAL CONVERGENCE IN PROBABILITY

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Abstract. In this paper we consider the notion of generalized density, namely, the natural density of weight g was introduced by Balcerzak et al. (Acta Math. Hungar. **147**(1) (2015) 97–115) and the entire investigation is performed in the setting of probability space extending the recent results of Ghosal (Appl. Math. Comput. **249** (2014) 502–509) and Das et al. (Filomat **31**(5) (2017) 1463–1473).

1. Introduction

In the year 1932, Agnew [1] defined the *deferred Cesàrro mean* of sequences of real numbers such as

$$(D_{p,q}x)_n = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k,$$

where $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ are sequences of non-negative integers satisfying

$$p_n < q_n, \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n\to\infty} q_n = +\infty.$$

In 2016, the concept of *deferred statistical convergence* (similar concept has been discussed by Aktuğlu [3] in 2014 which was named as *$\alpha\beta$-statistical convergence*) were given by Küçükaslan and Yilmaztürk [21] such as (earlier this concept has been defined by the same authors and submitted as a thesis to Mersin University/Turkey).

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Let $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ be two sequences as above. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be deferred statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k - L| \geq \varepsilon\}| = 0.$$

After then some work has been carried out with related to this concept [13, 21, 26].

In [4], the notion of natural density [14, 15, 23] (and also the natural density of order α [5, 7]) was further extended as follows: Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [4] by the formula

$$\bar{d}_g(A) = \limsup_{n\to\infty} \frac{\operatorname{card}(A \cap [1, n])}{g(n)},$$

for $A \subset \mathbb{N}$. Then the family $\mathcal{I}_g = \{A \subset \mathbb{N} : \bar{d}_g(A) = 0\}$ forms an ideal. It was also observed in [4] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$. Hence, we additionally assume that $\frac{n}{g(n)} \nrightarrow 0$. So that $\mathbb{N} \notin \mathcal{I}_g$ and it was observed in [4, 10], that $\mathcal{I}_g$ is a proper admissible P-ideal of $\mathbb{N}$. The collection of all functions g of this kind satisfying the above-mentioned property is denoted by G.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, ρ) is said to be d_g-statistically convergent to $a \in X$ if for any $\varepsilon > 0$ we have $d_g(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, a) \geq \varepsilon\}$.

Another generalization of the statistical convergence is known as weighted statistical convergence which was established by Karakaya and Chishti [20] in 2009 and gradually improved by Aizpuru et al. [2], Cinar and Et [6, 12], Das et al. [9], Ghosal [16–18], Işik and Altin [19], Mursaleen et al. [22] and Som [25].

In this paper the idea of four types of convergences of a sequence of random variables, namely,

(a) $\alpha\beta$-weighted d_g-statistically convergent sequence in probability;
(b) $\alpha\beta$-weighted d_g-strongly Cesàrro convergence in probability;
(c) g-weighted $S_{\alpha\beta}$-convergence in probability;
(d) g-weighted $N_{\alpha\beta}$-convergence in probability all have been introduced and the interrelations among them have been investigated. Also their certain basic properties are analyzed.

The main object of this paper is to improve all the existing results in this direction [4, 9, 11, 16, 17] which could be effectively extended. Moreover, we intend to establish the relations among these four summability notions. It is important to note that the methods of proofs and in particular the examples are not analogous to the real case.

2. Definitions and Notations

The following definitions and notions will be needed in sequel.

Definition 2.1 (see [3])**.** Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ and $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that

(i) α and β are both non-decreasing;

(ii) $\beta_n \geq \alpha_n$ for all $n \in \mathbb{N}$;
(iii) $(\beta_n - \alpha_n) \to \infty$ as $n \to \infty$.
Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) to a real number x if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^\gamma} \left|\{k \in [\alpha_n, \beta_n] : |x_k - x| \geq \varepsilon\}\right| = 0.$$

In this case we write $x_n \xrightarrow{S^\gamma_{\alpha\beta}} x$ and the set of all sequences which are $\alpha\beta$-statistically convergent of order γ is denoted by $S^\gamma_{\alpha\beta}$.

Definition 2.2 (see [9]). A sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ is said to be a weighted sequence if there exists a positive real number δ such that $t_n > \delta$ for all $n \in \mathbb{N}$.

Definition 2.3 (see [17]). Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf\limits_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted $\alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) to x if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \left|\{k \leq T_{\alpha\beta(n)} : t_k|x_k - x| \geq \varepsilon\}\right| = 0.$$

In this case we write $x_n \xrightarrow{(S^\gamma_{\alpha\beta}, t_n)} x$. The class of all weighted $\alpha\beta$-statistically convergent sequences of order γ is denoted by $(S^\gamma_{\alpha\beta}, t_n)$.

Definition 2.4 (see [17]). Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $\alpha\beta$-statistical convergence of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X (where $X : \mathcal{W} \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \left|\{k \leq T_{\alpha\beta(n)} : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}\right| = 0.$$

In this case, $X_n \xrightarrow{(S^\gamma_{\alpha\beta}, P^\phi, t_n)} X$ and the class of all weighted modulus statistically convergent sequences of order γ in probability is denoted by $(S^\gamma_{\alpha\beta}, P^\phi, t_n)$.

Definition 2.5 (see [17]). Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $\alpha\beta$-strongly Cesàrro convergent of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^\gamma_{\alpha\beta(n)}} \sum_{k\in[\alpha_n,\beta_n]} t_k\phi(P(|X_k - X| \geq \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(N^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$ and the class of all sequences of random variables which are weighted modulus $\alpha\beta$-strong Cesàrro convergent of order γ in probability, is denoted by $(N^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

Definition 2.6 (see [17]). Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $S_{\alpha\beta}$-convergence of order γ in probability (where $0 < \gamma \le 1$) to a random variable X if for every $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{T^{\gamma}_{\alpha\beta(n)}} \left|\{k \in I_{\alpha\beta(n)} : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}\right| = 0,$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)]}, T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not grater than x. In this case we write $X_n \xrightarrow{(WS^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$. The class of all weighted modulus $S_{\alpha\beta}$-convergent sequences of order γ in probability is denoted by $(WS^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

Definition 2.7 ([17]). Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$, as $n \to \infty$ and ϕ be a modulus function. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus $N_{\alpha\beta}$-convergence of order γ in probability (where $0 < \gamma \le 1$) to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{T^{\gamma}_{\alpha\beta(n)}} \sum_{k\in I_{\alpha\beta(n)}} t_k\phi(P(|X_k - X| \ge \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(WN^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)} X$ and the class of all weighted modulus $N_{\alpha\beta}$-convergent sequences of order γ in probability is denoted by $(WN^{\gamma}_{\alpha\beta}, P^{\phi}, t_n)$.

3. Main Results

First we introduce the definition of $\alpha\beta$-weighted d_g-statistical convergence in probability of random variables as follows.

Definition 3.1. Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-weighted d_g-statistically convergent in probability to a random variable X (where $X : \mathcal{W} \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \left|\{k \le T_{\alpha\beta(n)} : t_kP(|X_k - X| \ge \varepsilon) \ge \delta\}\right| = 0.$$

Hence, we assume that $g : (0,\infty) \to (0,\infty)$ is a continuous function such that $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{n\to\infty} \frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \ne 0$ and we write $X_n \xrightarrow{\alpha\beta WS^{p}_{d_g}} X$ and the class of all $\alpha\beta$-weighted d_g-statistically convergent sequences in probability is denoted by $\alpha\beta WS^{p}_{d_g}$.

Throughout the paper we assume that $g:(0,\infty)\to(0,\infty)$ is a continuous function such that $\lim_{x\to\infty} g(x)=\infty$ and $\lim_{n\to\infty}\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})}\neq 0$.

Theorem 3.1. *If* $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ *and* $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} Y$, *then* $P\{X=Y\}=1$.

Proof. If possible let $P\{X=Y\}\neq 1$. Then there exist two positive real numbers ε,δ such that $P(|X-Y|\geq\varepsilon)>\delta$ and $t_n>\delta$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned}\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})}=&\frac{1}{g(T_{\alpha\beta(n)})}\ |\{k\leq T_{\alpha\beta(n)}:t_kP(|X-Y|\geq\varepsilon)\geq\delta^2\}|\\ \leq&\frac{1}{g(T_{\alpha\beta(n)})}\ \left|\left\{k\leq T_{\alpha\beta(n)}:t_kP\left(|X_k-X|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta^2}{2}\right\}\right|\\ &+\frac{1}{g(T_{\alpha\beta(n)})}\ \left|\left\{k\leq T_{\alpha\beta(n)}:t_kP\left(|X_k-Y|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta^2}{2}\right\}\right|,\end{aligned}$$

which is impossible because the right hand side tends to zero as $n\to\infty$ but not the left hand side. Hence, the result follows. $\square$

The following example shows that weighted $\alpha\beta$-statistical convergence in probability [17] and $\alpha\beta$-weighted d_g-statistical convergence in probability are totally different.

Example 3.1. Let the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n\in\begin{cases}\{-1,1\}\text{ with p.m.f } P(X_n=-1)=P(X_n=0), & \text{if } n\in\{1^2,2^2,3^2,\dots\},\\ \{0,1\}\text{ with p.m.f } P(X_n=0)=1-\frac{1}{n^4}, & \\ P(X_n=1)=\frac{1}{n^4}, & \text{otherwise.}\end{cases}$$

Let $t_n=2n$, $\alpha_n=n$, $\beta_n=n^2$ for all $n\in\mathbb{N}$ and $g(x)=\sqrt[4]{x}$ for all $x\in(0,\infty)$. Then $T_{\alpha\beta(n)}=n^4+n$ for all $n\in\mathbb{N}$ and $\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})}\nrightarrow 0$ as $n\to\infty$.

For $0<\varepsilon<1$, we get

$$P(|X_n-0|\geq\varepsilon)=\begin{cases}1, & \text{if } n=m^2,\ \text{where } m\in\mathbb{N},\\ \frac{1}{n^4}, & \text{if } n\neq m^2,\ \text{where } m\in\mathbb{N}.\end{cases}$$

Now, let $0<\delta<1$. Then

$$\frac{1}{T_{\alpha\beta(n)}}|\{k\leq T_{\alpha\beta(n)}:t_kP(|X_k-0|\geq\varepsilon)\geq\delta\}|\leq\frac{2}{n^2}$$

and

$$\frac{1}{g(T_{\alpha\beta(n)})}|\{k\leq T_{\alpha\beta(n)}:t_kP(|X_k-0|\geq\varepsilon)\geq\delta\}|\geq\frac{\sqrt{n^4+n}-1}{\sqrt[4]{n^4+n}}\geq n.$$

This shows that $\{X_n\}_{n\in\mathbb{N}}$ is weighted $\alpha\beta$-statistically convergent in probability to a random variable 0 but it is not $\alpha\beta$-weighted d_g-statistically convergent in probability to 0.

Therefore we come to a conclusion that Definition 3.1 is the non-trivial extension of the notions obtained by different authors in the past, because if we take $g(x) = x^\gamma$ for all $x \in (0, \infty)$ and $0 < \gamma \leq 1$ then Definition 3.1 reduces to the Definition 2.1 [9] and Definition 2.1 [17].

The proof of the following two theorems are straightforward, so we choose to state these results without proof.

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. If $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ and $P(|X| \geq \alpha) = 0$ for some positive real number α, then $f(X_n) \xrightarrow{\alpha\beta WS^p_{d_g}} f(X)$.*

Theorem 3.3. *Let $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} x$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f(X_n) \xrightarrow{\alpha\beta WS^p_{d_g}} f(x)$.*

Definition 3.2. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-weighted d_g-strongly Cesàrro convergent in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) = 0.$$

In this case, $X_n \xrightarrow{\alpha\beta WN^p_{d_g}} X$ and the class of all $\alpha\beta$-weighted d_g-strongly Cesàrro convergent sequences in probability is denoted by $\alpha\beta WN^p_{d_g}$.

In the following, the relationship between $\alpha\beta WS^p_{d_g}$ and $\alpha\beta WN^p_{d_g}$ is investigated.

Theorem 3.4. *Let ζ be a positive real number such that $t_n > \zeta$ for all $n \in \mathbb{N}$. If $\{\alpha_n\}_{n\in\mathbb{N}}$ be a bounded sequence and $\liminf_{n\to\infty} \frac{\beta_n}{T_{\alpha\beta(n)}} \geq 1$, then $\alpha\beta WN^p_{d_g} \subset \alpha\beta WS^p_{d_g}$.*

Proof. Let $X_n \xrightarrow{\alpha\beta WN^p_g} X$ and $\varepsilon > 0$. Then

$$\begin{aligned} &\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) \\ \geq &\frac{\zeta}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}|. \end{aligned}$$

Hence, the result follows. □

The following example shows that, the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ in $\alpha\beta WS^p_{d_g}$ converges to X but not in $\alpha\beta WN^p_{d_g}$ converges to X.

Example 3.2. Let $t_n = n$, $\alpha_n = 1$, $\beta_n = n$ for all $n \in \mathbb{N}$ and $g(x) = \sqrt[4]{x}$ for all $x \in (0, \infty)$. Then $T_{\alpha\beta(n)} = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$ and $\frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \nrightarrow 0$ as $n \to \infty$.

Consider the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1,1\} \text{ with probability } \frac{1}{2}, & \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \{0,1\} \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^{\frac{3}{2}}}, & \\ P(X_n = 1) = \frac{1}{n^{\frac{3}{2}}}, & \text{if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$, then,

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1, & \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}, \\ \frac{1}{n^{\frac{3}{2}}}, & \text{if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

This implies $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} 0$.

Let $H = \{n \in \mathbb{N} : \ n \neq \{T_m\}^{T_m} \text{ where } m \in \mathbb{N}\}$.

Now we have the inequality

$$\begin{aligned} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - 0| \geq \varepsilon) &= \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\in H}} t_k P(|X_k - 0| \geq \varepsilon) + \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\notin H}} t_k P(|X_k - 0| \geq \varepsilon) \\ &> \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\in H}} \frac{1}{\sqrt{k}} + \sum_{\substack{k\in[\alpha_n,\beta_n] \\ k\notin H}} \\ &> \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \quad \left(\text{since } \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \text{ for all } n \geq 2\right) \end{aligned}$$

This implies $\frac{1}{g(T_{\alpha\beta(n)})}\sum_{k=1}^{n} t_k P(|X_k - 0| \geq \varepsilon) > \frac{\sqrt{n}}{\sqrt[4]{\frac{n(n+1)}{2}}} \geq 1$. This inequality shows that $\{X_n\}_{n\in\mathbb{N}}$ is not $\alpha\beta WN^p_{d_g}$ summable to 0.

Theorem 3.5. *Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be bounded such that*

$$\limsup_{n\to\infty} \frac{\beta_n - \alpha_n}{g(T_{\alpha\beta(n)})} < \infty.$$

Then $\alpha\beta WS^p_{d_g} \subset \alpha\beta WN^p_{d_g}$.

Proof. Let $X_n \xrightarrow{\alpha\beta WS^p_{d_g}} X$ and $t_n \leq M_1$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \frac{\beta_n-\alpha_n}{g(T_{\alpha\beta(n)})} < M_2$, where M_1 and M_2 are positive real numbers. For any $\varepsilon, \delta > 0$ setting $H = \{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}$. Then

$$\begin{aligned} &\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - X| \geq \varepsilon) \\ =&\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]\cap H} t_k P(|X_k - X| \geq \varepsilon) + \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]\cap H^c} t_k P(|X_k - X| \geq \varepsilon) \\ \leq&\frac{M_1}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}| + M_2\delta. \end{aligned}$$

Since δ is arbitrary, the result follows. $\square$

The following example shows that the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ in $\alpha\beta WN^p_{d_g}$ converges to X but not in $\alpha\beta WS^p_{d_g}$ converges to X.

Example 3.3. Let $c \in (0,1)$, $\gamma \in (2c, 4c) \cap \mathbb{Q}$ and a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by

$$X_n \in \begin{cases} \{-1,0\}, \text{ with p.m.f } P(X_n = -1) = \frac{1}{n}, & \\ P(X_n = 0) = 1 - \frac{1}{n}, & \text{if } n = [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}, \\ \{0,1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^8}, & \\ P(X_n = 1) = \frac{1}{n^8}, & \text{if } n \neq [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $t_n = 2n$, $\alpha_n = n$, $\beta_n = n^2$ for all $n \in \mathbb{N}$ and $g(x) = x^{\frac{\gamma}{4}}$ for all $x \in (0, \infty)$. Then $T_{\alpha\beta(n)} = n^4 + n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{T_{\alpha\beta(n)}}{g(T_{\alpha\beta(n)})} \neq 0$.

For $0 < \varepsilon$, $\delta < 1$, we get

$$\begin{aligned} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in[\alpha_n,\beta_n]} t_k P(|X_k - 0| \geq \varepsilon) \leq & \frac{2}{n^\gamma} \left\{ (n^{2c} - n^c + 1) + \left(\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{(n^2)^3} \right) \right\} \\ \leq & \frac{M}{n^{\gamma - 2c}} \quad \text{(where } M \text{ is a positive constant)} \end{aligned}$$

and

$$\frac{1}{g(T_{\alpha\beta(n)})} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - 0| \geq \epsilon) \geq \delta\}| \geq \frac{(n^4+n)^c}{n^\gamma} > \frac{1}{2} n^{4c-\gamma}.$$

So, $\{X_n\}_{n\in\mathbb{N}} \in \alpha\beta WN^p_{d_g}$ but not in $\alpha\beta WS^p_{d_g}$.

Now we would like to introduce the definitions of g-weighted $S_{\alpha\beta}$-convergence in probability and g-weighted $N_{\alpha\beta}$-convergence in probability for a sequence of random variables as follows.

Definition 3.3. Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k$ for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be g-weighted $S_{\alpha\beta}$-convergence in probability to X if for every $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0,$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)]}, T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not greater than x. In this case we write $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} X$. The class of all g-weighted $S_{\alpha\beta}$-convergent sequences in probability is denoted by $WS^{d_g}_{\alpha\beta}$.

It is very obvious that if $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} X$ and $X_n \xrightarrow{WS^{d_g}_{\alpha\beta}} Y$, then $P\{X = Y\} = 1$.

Definition 3.4. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k\in[\alpha_n,\beta_n]} t_k \to \infty$ as $n \to \infty$. The sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be g-weighted $N_{\alpha\beta}$-convergence in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - X| \geq \varepsilon) = 0.$$

In this case $X_n \xrightarrow{WN_{\alpha\beta}^{d_g}} X$ and the class of all g-weighted $N_{\alpha\beta}$-convergent sequences in probability is denoted by $WN_{\alpha\beta}^{d_g}$.

In the following, the relationship between $WS_{\alpha\beta}^{d_g}$ and $WN_{\alpha\beta}^{d_g}$ is investigated.

Theorem 3.6. *Let $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. Then $WN_{\alpha\beta}^{d_g} \subset WS_{\alpha\beta}^{d_g}$ and this inclusion is strict.*

Proof. For the first part of this theorem, let $\varepsilon, \delta > 0$, then

$$\begin{aligned}
&\sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - X| \geq \varepsilon) \\
=& \sum_{k\in I_{\alpha\beta(n)}, t_k P(|X_k - X|\geq\varepsilon)\geq\delta} t_k P(|X_k - X| \geq \varepsilon) + \sum_{k\in I_{\alpha\beta(n)}, t_k P(|X_k - X|\geq\varepsilon)<\delta} t_k P(|X_k - X| \geq \varepsilon) \\
\geq& \delta |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - X| \geq \varepsilon) \geq \delta\}|.
\end{aligned}$$

For the second part we will give an example. Let $t_n = n$, $\alpha(n) = n!$, $\beta(n) = (n+1)!$ for all $n \in \mathbb{N}$ and $g(x) = \sqrt{x}$ for all $x \in (0, \infty)$ and a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\}, \text{ with p.m.f } P(X_n = 1) = P(X_n = -1), \quad \text{if } n \text{ is the first} \\ [\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}] \text{ integer in the interval } (T_{[\alpha(n)]}, T_{[\beta(n)]}], \\ \{0, 1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^3}, P(X_n = 1) = \frac{1}{n^3}, \quad \text{otherwise.} \end{cases}$$

For $0 < \varepsilon, \delta < 1$, we get

$$\frac{1}{g(T_{\alpha\beta(n)})} |\{k \in I_{\alpha\beta(n)} : t_k P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} \to 0, \quad \text{as } n \to 0.$$

For next

$$\begin{aligned}
&\frac{1}{g(T_{\alpha\beta(n)})} \sum_{k\in I_{\alpha\beta(n)}} t_k P(|X_k - 0| \geq \varepsilon) \\
\geq& \frac{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}]\{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}] + 1\}}{2\sqrt{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} > \frac{1}{3} > 0.
\end{aligned}$$

Hence, the result. ☐

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SOME REMARKS ON VARIOUS SCHUR CONVEXITY

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Abstract. The aim of this work is to investigate the Schur convexity, Schur geometrically convexity, Schur harmonically convexity and Schur power convexity of some special functions. Some sufficient conditions are obtained to guarantee the above-mentioned properties satisfy. We attain some special inequalities. Also, we obtain some applications of main results.

1. introduction

Throughout this work, we denote $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) : \ x_i > 0, \ i = 1, 2, \ldots, n\}$. For the convenience of the readers, we recall the relevant material.

Definition 1.1 ([5]). Let $n \geq 2$ and $x, y \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We say that x is majorized by y and denoted by $x \prec y$, if

$$\begin{aligned} &\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad \text{for } 1 \leq k \leq n-1, \\ &\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{aligned}$$

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in decreasing order.

Let $E \subseteq \mathbb{R}^n$ be a set with nonempty interior. We say $\varphi : E \to \mathbb{R}$ is Schur convex if $x \prec y$ implies $\varphi(x) \leq \varphi(y)$ and φ is said to be Schur concave if $-\varphi$ is Schur convex.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a symmetric function, if $f(Px) = f(x)$ for any $x \in \mathbb{R}^n$ and any $n \times n$ permutation matrix P. A set $E \subseteq \mathbb{R}^n$ is called symmetric, if

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$x \in E$ implies $xP \in E$ for any $n \times n$ permutation matrix P. Also, a set $E \subseteq \mathbb{R}^n$ is called a convex set if for any $x, y \in E$ and $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in E$.

In this work, we need the following three lemmas.

Lemma 1.1 ([5]). *Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior and $\varphi : E \to \mathbb{R}$ is a continuous symmetric function on E. If φ is differentiable on* $\operatorname{int} E$, *then φ is Schur convex (Schur concave) on E if and only if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \textit{or } (\leq 0)$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$.

Lemma 1.2 ([2,7]). *Let $E \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior and $\varphi : E \to \mathbb{R}_+$ be continuous on E and differentiable on* $\operatorname{int} E$. *Then φ is Schur geometrically convex (Schur geometrically concave) if and only if φ is symmetric on E and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \textit{or } (\leq 0) \tag{1.1}$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$, where E is a geometrically convex set, if for any $x, y \in E$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$, we have $x^\alpha y^\beta \in E$.

Since for any $x_1, x_2 \in \mathbb{R}$, we have

$$(x_1 - x_2)(\log x_1 - \log x_2) \geq 0,$$

we can reduce (1.1) *to the following inequality*

$$(x_1 - x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \textit{or } (\leq 0). \tag{1.2}$$

Lemma 1.3. ([6, Lemma 2.2]). *Let $E \subseteq \mathbb{R}^n_+$ be a symmetric harmonic convex set with nonempty interior and $\varphi : E \to \mathbb{R}_+$ be a continuous symmetric function on E. If φ is differentiable on* $\operatorname{int} E$, *then φ is Schur harmonic convex (Schur harmonic concave) on E if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \textit{or } (\leq 0)$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$, where E is a harmonic convex set, if for any $x, y \in E$, we have $\frac{2xy}{x+y} \in E$.

In 1923, the Schur convexity was discovered by I. Schur. It has many interested applications of symmetric functions in Hadamard's inequality, analytic inequalities, stochastic ordering and some other branches of graphs and matrices, see for example [1,3,4].

We organize this paper as follow. We establish the integral mean of fg is Schur convex, Schur geometrical convex, Schur harmonic convex, and Schur power convex on $[0,\infty)\times[0,\infty)$, for convex, continuous and similarly ordered functions f and g. In Section 3, we obtain some applications of results in Section 2.

2. Main Results

In this section, we obtain some results for special functions to be Schur convex (Schur concave), Schur geometrically convex, Schur harmonically convex, and Schur power convex.

We say that $f,g:\mathbb{R}\to\mathbb{R}$ are similarly ordered function if for all $x,y\in\mathbb{R}$, we have

$$(f(x)-f(y))(g(x)-g(y))\geq 0,$$

if the above inequality reversed, we say that f and g have oppositely ordered.

Lemma 2.1. *Let $f,g:\mathbb{R}\to[0,\infty)$ be convex, continuous and similarly ordered functions. Then for $x,y\in\mathbb{R}$, we have*

$$\frac{1}{y-x}\int_x^y f(t)g(t)dt\leq\frac{f(x)g(x)+f(y)g(y)}{2}.$$

Proof. Since f and g have similarly ordered, for any $x,y\in\mathbb{R}$ we have

$$(f(x)-f(y))(g(x)-g(y))\geq 0.$$

It follows that

$$f(x)g(y)+f(y)g(x)\leq f(x)g(x)+f(y)g(y). \tag{2.1}$$

On the other hand, f and g are convex functions, so for $x,y\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\begin{aligned}f(tx+(1-t)y)&\leq tf(x)+(1-t)f(y),\\ g(tx+(1-t)y)&\leq tg(x)+(1-t)g(y).\end{aligned}$$

By multiplying both sides of the latter inequalities together and integrating on $[0,1]$, we get

$$\begin{aligned}&\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dt\\ &\leq\int_0^1[t^2f(x)g(x)+t(1-t)[f(x)g(y)+g(x)f(y)]+(1-t)^2f(y)g(y)]dt,\end{aligned}$$

with change of variable $u=tx+(1-t)y=t(x-y)+y$, it follows

$$\begin{aligned}\frac{1}{y-x}\int_x^y f(u)g(u)du&\leq\frac{f(x)g(x)+f(y)g(y)}{3}+\frac{f(x)g(y)+f(y)g(x)}{6}\\ &\leq\frac{f(x)g(x)+f(y)g(y)}{2}.\end{aligned}$$

Now, (2.1) follows from the last inequality. □

Theorem 2.1. *Let $f, g : \mathbb{R} \to [0, \infty)$ be convex, continuous and similarly ordered functions. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur convex on $\mathbb{R}^2$.

Proof. By Lemma 2.1, we have

$$\begin{aligned}\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y-x) =& \left[-\frac{1}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{f(y)g(y)}{y-x}\right.\\ &\left.-\frac{1}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{f(x)g(x)}{y-x}\right](y-x)\\ =& f(x)g(x) + f(y)g(y) - \frac{2}{y-x}\int_x^y f(t)g(t)dt \geq 0.\end{aligned}$$

Now Lemma 1.1 implies that F is Schur convex. ☐

Corollary 2.1. *Let $\alpha \geq 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha e^t dt, & x \neq y, \\ x^\alpha e^x, & x = y, \end{cases}$$

is Schur convex on $[0, \infty) \times [0, \infty)$.

Proof. Suppose that $f, g : [0, \infty) \to [0, \infty)$ are defined by $f(t) = t^\alpha$ and $g(t) = e^t$. Since $\alpha \geq 1$, the function f is increasing and convex, according to Theorem 2.1, F is Schur convex. ☐

The next two corollaries are results of Theorem 2.1.

Corollary 2.2. *Let $f : \mathbb{R} \to [0, \infty)$ be increasing, continuous and convex function. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y e^t f(t) dt, & x \neq y, \\ e^x f(x), & x = y, \end{cases}$$

is Schur convex on $\mathbb{R}^2$.

Corollary 2.3. *Let $f : [0, \infty) \to [0, \infty)$ be increasing, continuous and convex function and $\alpha \geq 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha f(t) dt, & x \neq y, \\ x^\alpha f(x), & x = y, \end{cases}$$

is Schur convex on $[0, \infty) \times [0, \infty)$.

Similar to Lemma 2.1, we have the following lemma for concave and oppositely ordered functions.

Lemma 2.2. *Let $f, g : \mathbb{R} \to [0, \infty)$ be concave, continuous and oppositely ordered functions. Then for $x, y \in \mathbb{R}$ we have*

$$\frac{1}{y-x}\int_x^y f(t)g(t)dt \geq \frac{f(x)g(x)+f(y)g(y)}{2}.$$

Theorem 2.2. *Let $f, g : \mathbb{R} \to [0, \infty)$ be concave, continuous and oppositely ordered functions. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur concave on $\mathbb{R}^2$.

Proof. The result follows by similar arguments to the proof of Theorem 2.1 and using Lemma 2.2. □

Theorem 2.2 implies next two corollaries.

Corollary 2.4. *Let $f : [0, \infty) \to [0, \infty)$ be decreasing and concave function and $0 < \alpha < 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha f(t)dt, & x \neq y, \\ x^\alpha f(x), & x = y, \end{cases}$$

is Schur concave on $[0, \infty) \times [0, \infty)$.

Corollary 2.5. *The function*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y \operatorname{sech} t \ln t dt, & x \neq y, \\ \operatorname{sech} x \ln x, & x = y, \end{cases}$$

is Schur concave on $[0, \infty) \times [0, \infty)$.

By Lemmas 1.1, 1.2 and 1.3, we have the following theorem.

Theorem 2.3. *Let f and g be two real continuous functions defined on $\mathbb{R}$, then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur convex (concave) on $[0, \infty) \times [0, \infty)$ if and only if

$$F(x,y) \leq (\geq) \frac{f(x)g(x)+f(y)g(y)}{2}, \tag{2.2}$$

is Schur geometrically convex (*concave*) *on* $[0,\infty)\times[0,\infty)$ *if and only if*

$$F(x,y) \leq (\geq) \frac{xf(x)g(x)+yf(y)g(y)}{x+y}, \tag{2.3}$$

and is Schur harmonically convex (*concave*) *on* $\mathbb{R}^2_+$ *if and only if*

$$F(x,y) \leq (\geq) \frac{x^2f(x)g(x)+y^2f(y)g(y)}{x^2+y^2}. \tag{2.4}$$

Proof. From Lemma 1.1 it follows that F is Schur convex (concave) on $[0,\infty)\times[0,\infty)$ if and only if

$$(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right) \geq 0\,(\leq 0).$$

On the other hand, as in the proof of Theorem 2.1, we have

$$(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right) = f(x)g(x)+f(y)g(y)-\frac{2}{y-x}\int_x^y f(t)g(t)dt.$$

This implies (2.2).

From Lemma 1.2 it follows that F is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$ if and only if

$$(y-x)\left(y\frac{\partial F}{\partial y}-x\frac{\partial F}{\partial x}\right) \geq 0\,(\leq 0).$$

But

$$\begin{aligned}(y-x)\left(y\frac{\partial F}{\partial y}-x\frac{\partial F}{\partial x}\right) &= (y-x)\left[-\frac{y}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{yf(y)g(y)}{y-x}\right.\\ &\quad\left.-\frac{x}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{xf(x)g(x)}{y-x}\right]\\ &= xf(x)g(x)+yf(y)g(y)-\frac{x+y}{y-x}\int_x^y f(t)g(t)dt,\end{aligned}$$

hence (2.3) follows.

From Lemma 1.3 it follows that F is Schur harmonically convex (concave) on $\mathbb{R}^2_+$ if and only if

$$(y-x)\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right) \geq 0\,(\leq 0).$$

On the other hand, we have

$$\begin{aligned}(y-x)\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right) &= (y-x)\left[-\frac{y^2}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{y^2f(y)g(y)}{y-x}\right.\\ &\quad\left.-\frac{x^2}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{x^2f(x)g(x)}{y-x}\right]\\ &= x^2f(x)g(x)+y^2f(y)g(y)-\frac{x^2+y^2}{y-x}\int_x^y f(t)g(t)dt,\end{aligned}$$

Therefore, (2.4) holds. □

In [8, Definition 2.3], we put $f(x) = x^{\alpha}$, then the following definition follows.

Definition 2.1. Let α be a positive real number and $E \subseteq \mathbb{R}_+^n$ be such that $x \in E$ implies $x^{\frac{1}{\alpha}} = (x_1^{\frac{1}{\alpha}}, \dots, x_n^{\frac{1}{\alpha}}) \in E$. A real-valued function $F : E \to \mathbb{R}$ is said to be Schur power convex if

$$F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n),$$

holds for each pair of n-tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in E such that

$$(x_1^{\alpha}, \dots, x_n^{\alpha}) \prec (y_1^{\alpha}, \dots, y_n^{\alpha}),$$

and F is Schur power concave if $-F$ is Schur power convex.

Remark 2.1. Let $E \subseteq \mathbb{R}_+^n$ and α be a positive real number. Then $F : E \to (0, \infty)$ is Schur power convex on E if and only if $F(x^{\frac{1}{\alpha}})$ is Schur convex function.

Lemma 2.3. *Let $E \in \mathbb{R}_+^n$ be a symmetric convex set with nonempty interior and $F : E \to \mathbb{R}$ be a continuous symmetric function on E. If F is differentiable on* $\operatorname{int} E$, *then F is Schur power convex* (*Schur power concave*) *on E if and only if*

$$(x_1^{\alpha} - x_2^{\alpha}) \left(x_1^{1-\alpha} \frac{\partial F}{\partial x_1} - x_2^{1-\alpha} \frac{\partial F}{\partial x_2} \right) \geq 0 \, (\leq 0),$$

for all $x = (x_1, \dots, x_n) \in \operatorname{int} E$ and $\alpha \in \mathbb{R}_+$.

Proof. The result follows by using Definition 2.1 and Remark 2.1 and Lemma 1.1. □

Theorem 2.4. *Let $\alpha \in \mathbb{R}_+$. Let f and g be two real continuous functions defined on* $\mathbb{R}$, *then*

$$F(x, y) = \begin{cases} \dfrac{1}{y - x} \displaystyle\int_x^y f(t) g(t) dt, & x \neq y, \\ f(x) g(x), & x = y, \end{cases}$$

is Schur power convex (*concave*) *on* $[0, \infty) \times [0, \infty)$ *if and only if*

$$F(x, y) \leq (\geq) \frac{x^{1-\alpha} f(x) g(x) + y^{1-\alpha} f(y) g(y)}{x^{1-\alpha} + y^{1-\alpha}}.$$

Proof. Let $x, y \in [0, \infty)$ and $x \neq y$. According to Lemma 2.3, $F(x, y)$ is Schur power convex (concave) if and only if

$$(y^{\alpha} - x^{\alpha}) \left(y^{1-\alpha} \frac{\partial F}{\partial y} - x^{1-\alpha} \frac{\partial F}{\partial x} \right) \geq 0 \, (\leq 0).$$

But we have

$$(y^{\alpha} - x^{\alpha}) \left(y^{1-\alpha} \frac{\partial F}{\partial y} - x^{1-\alpha} \frac{\partial F}{\partial x} \right)$$

$$
\begin{aligned}
&=(y^\alpha - x^\alpha)\left[-\frac{y^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{y^{1-\alpha}f(y)g(y)}{y-x}\right.\\
&\quad\left.-\frac{x^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{x^{1-\alpha}f(x)g(x)}{y-x}\right]\\
&=\frac{x^{1-\alpha}f(x)g(x)+y^{1-\alpha}f(y)g(y)}{y-x} - \frac{x^{1-\alpha}+y^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt.
\end{aligned}
$$

As F is symmetric, that is $F(x,y)=F(y,x)$, we get the conclusion. □

Corollary 2.6. *Let $\alpha,\beta\in(0,\infty)$ and f be a real continuous function defined on $\mathbb{R}$, then*

$$
F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y t^\beta f(t)dt, & x\neq y,\\[2mm] x^\beta f(x), & x=y,\end{cases}
$$

is Schur power convex on $[0,\infty)\times[0,\infty)$ if and only if

$$
F(x,y)\leq \frac{x^{1-\alpha+\beta}f(x)+y^{1-\alpha+\beta}f(y)}{x^{1-\alpha}+y^{1-\alpha}}.
$$

Proof. In Theorem 2.4, put $g(x)=x^\beta$. □

Theorem 2.5. *Let $f,g:[0,\infty)\to[0,\infty)$ be convex (concave), continuous and similarly (oppositely) ordered functions on $[0,\infty)$. Then*

$$
F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x\neq y,\\[2mm] f(x)g(x), & x=y,\end{cases}
$$

(i) *is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$;*
(ii) *is Schur harmonically convex (concave) on $[0,\infty)\times[0,\infty)$;*
(iii) *is Schur power convex (concave) on $[0,\infty)\times[0,\infty)$, if $0<\alpha<1$.*

Proof. (i) As f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$, then for all $x,y\in[0,\infty)$, we have

$$
(y-x)(f(y)g(y)-f(x)g(x))\geq 0\,(\leq 0). \tag{2.5}
$$

This implies that

$$
xf(y)g(y)+yf(x)g(x)\leq(\geq)xf(x)g(x)+yf(y)g(y),
$$

and it follows that

$$
\frac{f(y)g(y)+f(x)g(x)}{2}\leq(\geq)\frac{xf(x)g(x)+yf(y)g(y)}{x+y}. \tag{2.6}
$$

Now, from (2.6) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x,y)$ is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$.

(ii) Since f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$, then for all $x, y \in [0,\infty)$, we have (2.5). It follows that

$$(y^2 - x^2)(f(y)g(y) - f(x)g(x)) \geq 0\,(\leq 0).$$

This implies that

$$x^2 f(y)g(y) + y^2 f(x)g(x) \leq (\geq) x^2 f(x)g(x) + y^2 f(y)g(y),$$

and it follows that

$$\frac{f(y)g(y) + f(x)g(x)}{2} \leq (\geq) \frac{x^2 f(x)g(x) + y^2 f(y)g(y)}{x^2 + y^2}. \tag{2.7}$$

From (2.7) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x, y)$ is Schur harmonically convex (concave) on $[0,\infty) \times [0,\infty)$.

(iii) Since f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$ and $0 < \alpha < 1$, then for all $x, y \in [0,\infty)$ we have

$$(y^{1-\alpha} - x^{1-\alpha})(f(y)g(y) - f(x)g(x)) \geq 0\,(\leq 0).$$

It follows that

$$x^{1-\alpha} f(y)g(y) + y^{1-\alpha} f(x)g(x) \leq (\geq) x^{1-\alpha} f(x)g(x) + y^{1-\alpha} f(y)g(y).$$

This yields

$$\frac{f(y)g(y) + f(x)g(x)}{2} \leq (\geq) \frac{x^{1-\alpha} f(x)g(x) + y^{1-\alpha} f(y)g(y)}{x^{1-\alpha} + y^{1-\alpha}}. \tag{2.8}$$

Now, from the inequality (2.8) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x, y)$ is Schur power convex (concave) on $[0,\infty) \times [0,\infty)$. ☐

Corollary 2.7. *Let $\alpha, \beta \in (1, 2)$. Then*

$$F(x, y) = \begin{cases} \dfrac{1}{y - x} \displaystyle\int_x^y t^{\alpha-1}(1 - t)^{\beta-1} dt, & x \neq y, \\ x^{\alpha-1}(1 - x)^{\beta-1}, & x = y, \end{cases}$$

is Schur concave, geometrically Schur concave and harmonically Schur concave on $[0, 1] \times [0, 1]$. Also, for all $x, y \in [0, 1]$ such that $x \neq y$ the following inequalities hold

$$\frac{1}{y - x} \int_x^y t^{\alpha-1}(1 - t)^{\beta-1} dt \geq \frac{x^{\alpha-1}(1 - x)^{\beta-1} + y^{\alpha-1}(1 - y)^{\beta-1}}{2},$$

$$\frac{1}{y - x} \int_x^y t^{\alpha-1}(1 - t)^{\beta-1} dt \geq \frac{x^{\alpha}(1 - x)^{\beta-1} + y^{\alpha}(1 - y)^{\beta-1}}{x + y},$$

$$\frac{1}{y - x} \int_x^y t^{\alpha-1}(1 - t)^{\beta-1} dt \geq \frac{x^{\alpha+1}(1 - x)^{\beta-1} + y^{\alpha+1}(1 - y)^{\beta-1}}{x^2 + y^2}.$$

Proof. In Theorems 2.2, 2.5, we put $f(x) = x^{\alpha-1}$ and $g(x) = (1-x)^{\beta-1}$. Since $\alpha, \beta \in (1,2)$ on $[0,1]$ the function f is increasing and concave and g is decreasing and concave. It follows that on $[0,1]$ the functions f and g are concave, continuous and oppositely ordered. Now, Theorem 2.3 implies the results. ☐

Theorem 2.6. *Let α be a positive real number and $f : (0,\infty) \to (0,\infty)$ be a log-concave function. Then $t^\alpha f(t)$ is log-concave and the following inequality holds*

$$\frac{1}{y-x}\int_x^y t^\alpha f(t)dt \geq \frac{x^\alpha f(x) - y^\alpha f(y)}{\ln(x^\alpha f(x)) - \ln(y^\alpha f(y))}.$$

Proof. For $\alpha > 0$, function t^α is log-concave. Since $\ln t$ is concave and $\alpha > 0$, we have

$$\lambda\alpha(\ln x) + (1-\lambda)\alpha \ln y \leq \alpha \ln(\lambda x + (1-\lambda)y),$$

so

$$\lambda(\ln x^\alpha) + (1-\lambda)\ln y^\alpha \leq \ln(\lambda x + (1-\lambda)y)^\alpha.$$

Thus, t^α is log-concave. Put $g(x) = x^\alpha f(x)$, then for $t \in [0,1]$, we have

$$\begin{aligned} g(tx+(1-t)y) &= (tx+(1-t)y)^\alpha f(tx+(1-t)y) \\ &\geq (x^\alpha)^t (y^\alpha)^{1-t} (f(x))^t (f(y))^{1-t} \\ &= (x^\alpha f(x))^t (y^\alpha f(y))^{1-t} \\ &= (g(x))^t (g(y))^{1-t} \\ &= \left(\frac{x^\alpha f(x)}{y^\alpha f(y)}\right)^t (y^\alpha f(y)), \end{aligned}$$

that is, $g(x) = x^\alpha f(x)$ is log-concave. By integrating both sides of the above inequality on $[0,1]$ and change of variable $u = tx + (1-t)y$, getting $w = \dfrac{x^\alpha f(x)}{y^\alpha f(y)}$, then we have

$$\int_0^1 (tx+(1-t)y)^\alpha f(tx+(1-t)y)dt \geq y^\alpha f(y) \int_0^1 \left(\frac{x^\alpha f(x)}{y^\alpha f(y)}\right)^t dt,$$

$$\begin{aligned} \frac{1}{y-x}\int_x^y u^\alpha f(u)du &\geq y^\alpha f(y)\int_0^1 w^t dt \\ &= \frac{y^\alpha f(y)}{\ln\left(\dfrac{x^\alpha f(x)}{y^\alpha f(y)}\right)}\left(\frac{x^\alpha f(x)}{y^\alpha f(y)} - 1\right) \\ &= \frac{x^\alpha f(x) - y^\alpha f(y)}{\ln\left(x^\alpha f(x)\right) - \ln(y^\alpha f(y))}. \end{aligned}$$

☐

Lemma 2.4. *Let I be an interval in $\mathbb{R}$ and $f, g : I \to [0, \infty)$ be continuous functions. Then for $x \in I^n \subseteq \mathbb{R}^n$ the function*

$$F(x) = \sum_{i=1}^{n} \int_0^{x_i} f(t)g(t)dt$$

is Schur convex if and only if f and g are similarly ordered functions (is Schur concave if and only if f and g are oppositely ordered functions).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) = (x_1 - x_2)\left(f(x_1)g(x_1) - f(x_2)g(x_2)\right) \geq 0,$$

if and only if f and g are similarly ordered functions. □

Lemma 2.5. *Let I be an interval in $\mathbb{R}$ and $f : I \to (0, \infty)$ be differentiable on* $\operatorname{int} I$. *Then for $x \in I^n \subseteq \mathbb{R}^n$ the function*

$$F(x) = \prod_{i=1}^{n} f(x_i)$$

is Schur convex if and only if $\frac{f'}{f}$ is increasing on I (is Schur concave if and only if $\frac{f'}{f}$ is decreasing on I).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$\begin{aligned}(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) &= (x_1 - x_2)\left(f'(x_1)\prod_{i=2}^{n} f(x_i) - f'(x_2)\prod_{i=1, i\neq 2}^{n} f(x_i)\right) \\ &= (x_1 - x_2)\prod_{i=3}^{n} f(x_i)(f'(x_1)f(x_2) - f(x_1)f'(x_2)) \geq 0,\end{aligned}$$

if and only if $\frac{f'}{f}$ is increasing on I. □

Remark 2.2. As in the literature, the infinite decreasing sequence $x = (x_n)$ majorized by the infinite decreasing sequence $y = (y_n)$ and denoted by $x \prec y$, if there exists an infinite doubly stochastic square matrix $P = (p_{ij})$ (i.e., $p_{ij} \geq 0$ for all $i, j \in \mathbb{N}$, and all rows sum and all columns sum are equal one) such that $x = y.P$. If (α_n) be a sequence in the interval $[0, 1]$, we take $x_1 = \alpha_1 y_1 + (1 - \alpha_1)y_2$, $x_2 = (1 - \alpha_1)y_1 + \alpha_1 y_2$, and $x_3 = \alpha_2 y_3 + (1 - \alpha_2)y_4$, $x_4 = (1 - \alpha_2)y_3 + \alpha_2 y_4, \ldots$, where $y = (y_n)$ is an infinite decreasing real sequence. If we put

$$P = \begin{bmatrix} \alpha_1 & 1-\alpha_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-\alpha_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_2 & 1-\alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1-\alpha_2 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \cdots \end{bmatrix},$$

then $x = yP$ and $x \prec y$.

Example 2.1. In Lemma 2.5, set $f(x) = \sin x$ and $I = (0, \pi)$. The function $f'(x) = \cos x$ and $\frac{f'(x)}{f(x)} = \cot x$ is decreasing on I. So, $F(x) = \prod_{i=1}^n \sin x_i$ is Schur concave. Let $x = (x_n)$ and $y = (y_n)$ be two decreasing sequence in $I = (0, \pi)$, such that $x \prec y$ as in Remark 2.2. Since F is Schur concave, we have $F(x) \geq F(y)$ and so

$$\begin{aligned} &\sin(\alpha_1 y_1 + (1-\alpha_1)y_2)\sin((1-\alpha_1)y_1 + \alpha_1 y_2)\sin(\alpha_2 y_3 + (1-\alpha_2)y_4) \\ &\times \sin((1-\alpha_2)y_3 + \alpha_2 y_4)\cdots \geq \prod_{i=1}^{\infty} \sin y_i. \end{aligned}$$

In the special case, $\alpha_i = \frac{1}{2}$ for all $i \in \mathbb{N}$, we have

$$\sin\left(\frac{y_1+y_2}{2}\right)\sin\left(\frac{y_3+y_4}{2}\right)\cdots \geq \left(\prod_{i=1}^{\infty} \sin y_i\right)^{\frac{1}{2}}.$$

Example 2.2. In Lemma 2.5, put $f(x) = \cos x$ and $I = (0, \frac{\pi}{2})$. The function $f'(x) = -\sin x$ and $\frac{f'(x)}{f(x)} = -\tan x$ is decreasing on I. So $F(x) = \prod_{i=1}^n \cos x_i$ is Schur concave. Let $x = (x_n)$ and $y = (y_n)$ be two decreasing sequence in $I = (0, \frac{\pi}{2})$, such that $x \prec y$ as in Remark 2.2. Since F is Schur concave, we have $F(x) \geq F(y)$ and so

$$\begin{aligned} &\cos(\alpha_1 y_1 + (1-\alpha_1)y_2)\cos((1-\alpha_1)y_1 + \alpha_1 y_2)\cos(\alpha_2 y_3 + (1-\alpha_2)y_4) \\ &\times \cos((1-\alpha_2)y_3 + \alpha_2 y_4)\cdots \geq \prod_{i=1}^{\infty} \cos y_i. \end{aligned}$$

In the special case, $\alpha_i = \frac{1}{2}$ for all $i \in \mathbb{N}$, we have

$$\cos\left(\frac{y_1+y_2}{2}\right)\cos\left(\frac{y_3+y_4}{2}\right)\cdots \geq \left(\prod_{i=1}^{\infty} \cos y_i\right)^{\frac{1}{2}}.$$

As in [9], let $I = (0, l)$ and $L_n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} x_i = ml\right\}$ for some $0 < m < n$, $D_n = I^n \cap L_n$ and $\Omega = (y, \ldots, y)$, where $y = \frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{ml}{n}$.

Lemma 2.6. ([9, Lemma 2.1]). *If $f : I^n \to \mathbb{R}$ is a Schur-convex function, then $f(\Omega)$ is a global minimum in D_n. If f is strictly Schur-convex on I^n, then $f(\Omega)$ is the unique global mimimum in D_n.*

Remark 2.3. In Example 2.2 and Lemma 2.6, put $l = \frac{\pi}{2}$ and $x_i \in (0, \frac{\pi}{2})$, for $i = 1, 2, \ldots, n$, and $\sum\limits_{i=1}^{n} x_i = \pi$. Then $\Omega = (\frac{\pi}{n}, \ldots, \frac{\pi}{n})$ and we have $F(x) \leq F(\Omega)$, that is

$$\prod_{i=1}^{n} \cos x_i \leq \left(\cos \frac{\pi}{n}\right)^n.$$

Similarly in Example 2.1, for $l = \frac{\pi}{2}$, we have

$$\prod_{i=1}^{n} \sin x_i \leq \left(\sin \frac{\pi}{n}\right)^n.$$

Lemma 2.7. *Let I be an interval in $\mathbb{R}$ and $f : I \to (0, \infty)$ be continuous, then for each $x \in I^n \subset \mathbb{R}^n$, the function*

$$F(x) = \prod_{i=1}^{n} \int_0^{x_i} f(t)dt,$$

is Schur convex if and only if $\dfrac{\int_0^x f(t)dt}{f(x)}$ is decreasing on I (is Schur concave if and only if $\frac{\int_0^x f(t)dt}{f(x)}$ is increasing on I).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$\begin{aligned}
(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) =&(x_1 - x_2)\left(f(x_1)\prod_{i=2}^{n}\int_0^{x_i} f(t)dt - f(x_2)\prod_{i=1, i\neq 2}^{n}\int_0^{x_i} f(t)dt\right)\\
=&(x_1 - x_2)\prod_{i=3}^{n}\int_0^{x_i} f(t)dt\\
&\times\left(f(x_1)\int_0^{x_2} f(t)dt - f(x_2)\int_0^{x_1} f(t)dt\right)\\
\geq&0,
\end{aligned}$$

if and only if $\frac{\int_0^x f(t)dt}{f(x)}$ is decreasing on I. □

3. Applications

In this section, we obtain some inequalities, which are the applications of the results in Section 2.

The next two examples are the applications of Lemma 2.1 and Theorems 2.1, 2.3 and 2.5.

Example 3.1. Let $\alpha \geq 1$ and $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$ be the Mittage-Leffler function. Let

$$F(x, y) = \begin{cases} \dfrac{1}{y - x}\displaystyle\int_x^y t^\alpha E_\alpha(t^\alpha)dt, & x \neq y, \\ x^\alpha E_\alpha(x^\alpha), & x = y. \end{cases}$$

Since t^α and $E_\alpha(t)$ are convex, continuous and similarly ordered on $[0,\infty)$, then Lemma 2.1 and Theorems 2.1, 2.3 and 2.5 imply that F is Schur convex, Schur geometrically convex and Schur harmonically convex on $[0,\infty)\times[0,\infty)$ and for $x,y\in[0,\infty)$, the following inequalities hold

$$\begin{aligned}\frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq& \frac{x^\alpha E_\alpha(x^\alpha)+y^\alpha E_\alpha(y^\alpha)}{2},\\ \frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq& \frac{x^{\alpha+1} E_\alpha(x^\alpha)+y^{\alpha+1} E_\alpha(y^\alpha)}{x+y},\\ \frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq& \frac{x^{\alpha+2} E_\alpha(x^\alpha)+y^{\alpha+2} E_\alpha(y^\alpha)}{x^2+y^2}.\end{aligned}$$

Example 3.2. Let $\alpha>0$ and

$$F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt, & x\neq y,\\ \Gamma(x)E_\alpha(x^\alpha), & x=y.\end{cases}$$

Since $\Gamma(t)$ and $E_\alpha(t)$ are convex, continuous and similarly ordered on $\left[\frac{3}{2},\infty\right)$, then Lemma 2.1 and Theorems 2.1, 2.3 and 2.5 imply that F is Schur convex, Schur geometrically convex and Schur harmonically convex on $\left[\frac{3}{2},\infty\right)\times\left[\frac{3}{2},\infty\right)$ and for $x,y\in\left[\frac{3}{2},\infty\right)$, the following inequalities hold

$$\begin{aligned}\frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq& \frac{\Gamma(x)E_\alpha(x^\alpha)+\Gamma(y)E_\alpha(y^\alpha)}{2},\\ \frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq& \frac{x\Gamma(x)E_\alpha(x^\alpha)+y\Gamma(y)E_\alpha(y^\alpha)}{x+y},\\ \frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq& \frac{x^2\Gamma(x)E_\alpha(x^\alpha)+y^2\Gamma(y)E_\alpha(y^\alpha)}{x^2+y^2}.\end{aligned}$$

Remark 3.1. For $x,y\in[0,\infty)$, the following majorizations hold

$$(1+x,1+y)\prec(1+x+y,1), \tag{3.1}$$

$$\left(\frac{1}{H_2(x,y)},\frac{1}{H_2(x,y)}\right)\prec\left(\frac{1}{x},\frac{1}{y}\right), \tag{3.2}$$

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y), \tag{3.3}$$

where $H_2(x,y)=\frac{2}{\frac{1}{x}+\frac{1}{y}}$.

Example 3.3. Let $f,g:[0,\infty)\to[0,\infty)$ be convex, continuous and similarly ordered functions. Then for $x,y\in[0,\infty)$ with $x\neq y$, (3.1), (3.2) and (3.3) and Theorem 2.1 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} f(t)g(t)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} f(t)g(t)dt,$$

$$f\left(\frac{1}{H_2(x,y)}\right)g\left(\frac{1}{H_2(x,y)}\right) \le \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} f(t)g(t)dt,$$
$$f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)g(t)dt.$$

For increasing, continuous and convex function $f:[0,\infty)\to[0,\infty)$ and $\alpha \ge 1$, Remark 3.1 and Corollaries 2.1, 2.3 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^\alpha e^t dt \le \frac{-1}{y+x}\int_{1+x+y}^{1} t^\alpha e^t dt,$$
$$\left(\frac{1}{H_2(x,y)}\right)^\alpha e^{\frac{1}{H_2(x,y)}} \le \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^\alpha e^t dt,$$
$$\left(\frac{x+y}{2}\right)^\alpha e^{\frac{x+y}{2}} \le \frac{1}{y-x}\int_x^y t^\alpha e^t dt,$$
$$\frac{1}{y-x}\int_{1+x}^{1+y} t^\alpha f(t)dt \le \frac{-1}{x+y}\int_{1+x+y}^{1} t^\alpha f(t)dt,$$
$$\left(\frac{1}{H_2(x,y)}\right)^\alpha f\left(\frac{1}{H_2(x,y)}\right) \le \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^\alpha f(t)dt,$$
$$\left(\frac{x+y}{2}\right)^\alpha f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y t^\alpha f(t)dt.$$

For increasing, continuous and convex function $f:[0,\infty)\to[0,\infty)$, Remark 3.1 and Corollary 2.2 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} e^t f(t)dt \le \frac{-1}{x+y}\int_{1+x+y}^{1} e^t f(t)dt,$$
$$e^{\frac{1}{H_2(x,y)}} f\left(\frac{1}{H_2(x,y)}\right) \le \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} e^t f(t)dt,$$
$$e^{\frac{x+y}{2}} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y e^t f(t)dt.$$

Remark 3.1 and Corollary 2.5 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} \operatorname{sech} t \ln t dt \ge \frac{-1}{x+y}\int_{1+x+y}^{1} \operatorname{sech} t \ln t dt,$$
$$\operatorname{sech}\left(\frac{1}{H_2(x,y)}\right)\ln\left(\frac{1}{H_2(x,y)}\right) \ge \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} \operatorname{sech} t \ln t dt,$$
$$\operatorname{sech}\left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right) \ge \frac{1}{y-x}\int_x^y \operatorname{sech} t \ln t dt.$$

Remark 3.1 and Corollary 2.7 imply the following inequalities, for $\alpha,\beta\in(1,2)$,

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^{\alpha-1}(1-t)^{\beta-1}dt \ge \frac{-1}{x+y}\int_{1+x+y}^{1} t^{\alpha-1}(1-t)^{\beta-1}dt,$$

$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha-1}\left(1-\frac{1}{H_2(x,y)}\right)^{\beta-1} \geq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^{\alpha-1}(1-t)^{\beta-1}dt,$$
$$\left(\frac{x+y}{2}\right)^{\alpha-1}\left(1-\frac{x+y}{2}\right)^{\beta-1} \geq \frac{1}{y-x}\int_{x}^{y} t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Remark 3.1 and Example 3.1 imply the following inequalities, for $\alpha \geq 1$ and the Mittage-Leffler function $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^\alpha E_\alpha(t^\alpha)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} t^\alpha E_\alpha\left(t^\alpha\right)dt,$$
$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha} E_\alpha\left(\left(\frac{1}{H_2(x,y)}\right)^{\alpha}\right) \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^\alpha E_\alpha(t^\alpha)dt,$$
$$\left(\frac{x+y}{2}\right)^{\alpha} E_\alpha\left(\left(\frac{x+y}{2}\right)^{\alpha}\right) \leq \frac{1}{y-x}\int_{x}^{y} t^\alpha E_\alpha(t^\alpha)dt.$$

Remark 3.2. Let $\alpha \geq 1$ and $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$ be the Mittage-Leffler function on $(0,\infty)$. In Lemma 2.4, set $f(t) = E_\alpha(t)$ and $g(t) = 1$. Then the function

$$\begin{aligned} F(x) =& \sum_{i=1}^{n}\sum_{k=0}^{\infty}\int_0^{x_i} \frac{t^k}{\Gamma(\alpha k+1)}dt = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{x_i^{k+1}}{(k+1)\Gamma(\alpha k+1)} \\ =& \sum_{k=0}^{\infty}\sum_{i=1}^{n} \frac{x_i^{k+1}}{(k+1)\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{\frac{\sum_{i=1}^{n} x_i^{k+1}}{k+1}}{\Gamma(\alpha k+1)} \end{aligned}$$

is Schur convex on $\mathbb{R}_+^n$.

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ON RAPID EQUIVALENCE AND TRANSLATIONAL RAPID EQUIVALENCE

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Abstract. In this paper we will prove some properties of the rapid equivalence and consider some selection principles and games related to rapidly varying sequences.

1. Introduction

Let $\mathbb{S}$ be the set of sequences of positive real numbers, and $\mathbb{S}_1$ be the set of nondecreasing sequences from $\mathbb{S}$ [5]. Let $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. A sequence c is said to be *rapidly varying* in the sense of de Haan, if

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty \tag{1.1}$$

holds for each $\lambda > 1$. The set of all these sequences is denoted by $R_{\infty,s}$. These sequences are objects in rapid variation theory in the sense of de Haan, which is very important in asymptotic analysis and applications (see, e.g., [1–3, 8, 10, 15]). The theory of rapid variation is an important modification of Karamata's theory of regular variation [13], and its relation can be seen on example of slow and rapid variation within generalized inverse (see, e.g., [7]). Elements of the class $R_{\infty,s}$ are important objects in dynamic systems theory [10, 11, 15], infinite topological games theory and selection principles theory [3–6].

A sequence c is *translationally slowly varying* (in the sense of Karamata) if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{c_n} = 1 \tag{1.2}$$

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holds for each $\lambda \geqslant 1$. Translationally slowly varying sequences form the class $Tr(SV_s)$ (see, e.g., [4–6]), and it holds $R_{\infty,s} \cap Tr(SV_s) \neq \varnothing$, $R_{\infty,s} \setminus Tr(SV_s) \neq \varnothing$ and $Tr(SV_s) \setminus R_{\infty,s} \neq \varnothing$.

A sequence c is *translationally rapidly varying* (in the sense of de Haan) if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{c_n} = +\infty \tag{1.3}$$

holds for each $\lambda \geqslant 1$.

The class of translationally rapidly varying sequences is denoted by $Tr(R_{\infty,s})$. It holds $Tr(R_{\infty,s}) \subsetneq R_{\infty,s}$ for each $\lambda \geqslant 1$ (see, e.g., [5]).

The classes of sequences mentioned above have nice and deep connections with selection principles theory and infinitely long two-person game theory (see, for example, [2, 3, 5, 6]).

Motivated by the study of some equivalence relations on classes of functions and sequences given in [7, 8, 14], in this paper we define a relation on the class of translationally rapidly varying sequences and investigate some properties of this relation. In particular, we study relationships of this relation with selection principles and game theory complementing the research in [2, 3, 5, 6]. We also obtain some additional information on the classes of rapidly varying and translationally rapidly varying sequences.

Definition 1.1. Sequences c and d of positive real numbers are *mutually translationally rapidly equivalent*, denoted by

$$c \overset{tr}{\sim} d \quad \text{as} \quad n \to +\infty,$$

if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{d_n} = +\infty \tag{1.4}$$

and

$$\lim_{n\to+\infty} \frac{d_{[\lambda+n]}}{c_n} = +\infty \tag{1.5}$$

hold for each $\lambda \geqslant 1$.

The previous relation is a modification of the rapid equivalence relation between sequences c and d given by

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = +\infty \tag{1.6}$$

and

$$\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = +\infty, \tag{1.7}$$

for each $\lambda > 1$. We denote it by $c \overset{r}{\sim} d$ as $n \to +\infty$ (see, e.g., [8, 14]).

Let c be a nondecreasing sequence from a subset $\mathcal{V}$ of $\mathbb{S}$. The capacity of c with respect to $\mathcal{V}$ is the subfamily of $\mathbb{S}$ given by $\mathcal{M}_c^{\mathcal{V}} = \{x = (x_n) \in \mathbb{S} \mid c_n \leqslant x_n \leqslant c_{n+1} \text{ for each } n \in \mathbb{N}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of $\mathbb{S}$. Let us adduce two selection principles which we need in this paper:

(a) (Rotberger, see, e.g.,[12]) $S_1(\mathcal{A},\mathcal{B})$: for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $b\in\mathcal{B}$, so that $b_n\in\ A^n$ for each $n\in\mathbb{N}$;

(b) (Kočinac, see, e.g., [9]) $\alpha_2(\mathcal{A},\mathcal{B})$: for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$, there is an element $b\in\mathcal{B}$, so that $b\cap A^n$ is infinite for each $n\in\mathbb{N}$.

Games associated to the previous two selection principles are the following.

$G_1(\mathcal{A},\mathcal{B})$. Two players, I and II, play a round for each positive integer. In m^{th} round, $m\in\mathbb{N}$, the player I plays a sequence $A^m\in\mathcal{A}$, and the player II plays an element $b_m\in A^m$. II wins the play $A^1,b_1;A^2,b_2;\ldots$ if and only if $b=(b_n)\in\mathcal{B}$.

The symbol $G_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round the player I plays an arbitrary element $A^1\in\mathcal{A}$, and the player II chooses a subsequence $A^{r_1(j)}, j\in\mathbb{N}$, of the sequence A^1. At the k^{th} round, $k\geqslant 2$, the player I plays an arbitrary element $A^k\in\mathcal{A}$ and the player II chooses a subsequence $A^{r_k(j)}$ of the sequence A^k, such that $A^{r_k(j)}\cap A^{r_p(j)}=\emptyset$ is satisfied, for each $p\leqslant k-1$. The player II wins the play $A^1,A^{r_1(j)};\ldots;A^k,A^{r_k(j)};\ldots$ if and only if all elements from $Y=\cup_{k\in\mathbb{N}}\cup_{j\in\mathbb{N}}A^{r_k(j)}$ form a subsequence $y\in\mathcal{B}$.

Note that if II has a winning strategy (even if I does not have a winning strategy) in a game defined above, then the corresponding selection principle holds.

Note that in the paper [5] it is proven that the player II does not have a winning strategy in the game $G_1(Tr(SV_s),Tr(SV_s))$.

2. Results

Proposition 2.1. *If $c\in\mathbb{S}$, $d\in\mathbb{S}$ and $c\overset{tr}{\sim}d$ as $n\to+\infty$ holds, then $c\in Tr(R_{\infty,s})$ and $d\in Tr(R_{\infty,s})$.*

Proof. Let $c,d\in\mathbb{S}$ and $c\overset{tr}{\sim}d$ as $n\to+\infty$ hold. Therefore, for $\lambda=1$, it holds $\lim_{n\to+\infty}\frac{c_{n+1}}{d_n}=+\infty$ and $\lim_{n\to+\infty}\frac{d_{n+1}}{c_n}=+\infty$. For $\lambda\geqslant 1$ it holds $\lim_{n\to+\infty}\frac{c_{[\lambda+n]}}{c_n}=\lim_{n\to+\infty}\left(\frac{c_{[\lambda]+n}}{d_{[\lambda]+n-1}}\cdot\frac{d_{[\lambda]+n-1}}{c_{[\lambda]+n-2}}\cdots\frac{d_{n+1}}{c_n}\right)=+\infty$ for each $\lambda\in[k,k+1)$, $k=2s$, $s\in\mathbb{N}$. It means, for $\lambda=2$, $\lim_{n\to+\infty}\frac{c_{n+2}}{c_n}=\lim_{n\to+\infty}\left(\frac{c_{n+2}}{d_{n+1}}\cdot\frac{d_{n+1}}{c_n}\right)=+\infty$. Therefore, $+\infty=\lim_{n\to+\infty}\left(\frac{c_{n+2}}{c_{n+1}}\cdot\frac{c_{n+1}}{c_n}\right)=\lim_{s\to+\infty}\left(\frac{c_{s+1}}{c_s}\right)^2=\left(\lim_{s\to+\infty}\frac{c_{s+1}}{c_s}\right)^2$. Thus, $\lim_{s\to+\infty}\frac{c_{s+1}}{c_s}=+\infty$, so for each $\lambda\geqslant 1$, $\lim_{s\to+\infty}\frac{c_{[\lambda+s]}}{c_s}=+\infty$ holds. Therefore, $c\in Tr(R_{\infty,s})$. Analogously we prove that $d\in Tr(R_{\infty,s})$. □

Proposition 2.2. *The relation $\overset{tr}{\sim}$ is a reflexive, symmetric and nontransitive relation in $Tr(R_{\infty,s})$.*

Proof. 1. (Reflexivity) According to Proposition 2.1, from $c\overset{tr}{\sim}d$ as $n\to+\infty$ it follows $c,d\in Tr(R_{\infty,s})$. The asymptotic relation $\lim_{n\to+\infty}\frac{c_{[\lambda+n]}}{c_n}=+\infty$ holds for each $\lambda\geqslant 1$ in the class $Tr(R_{\infty,s})$, thus $c\overset{tr}{\sim}c$ as $n\to+\infty$.

2. (Symmetry) According to the definition of $\overset{tr}{\sim}$, symmetry holds.

3. (Nontransitivity) The following example shows that the relation is not transitive. Consider the sequences $c_n = (n-1)!\ln(n+1)$, $d_n = n!$ and $e_n = \frac{(n+1)!}{\ln(n+1)}$, $n \in \mathbb{N}$. It holds $c \overset{tr}{\sim} d$, $d \overset{tr}{\sim} e$ as $n \to +\infty$, but $c \overset{tr}{\sim} e$ does not hold as $n \to +\infty$. ☐

Proposition 2.3. *Let $c, d \in \mathbb{S}$. If $c \overset{tr}{\sim} d$, then $c \overset{r}{\sim} d$ as $n \to +\infty$.*

Proof. Let $c, d \in \mathbb{S}$ and $c \overset{tr}{\sim} d$ as $n \to +\infty$. According to Proposition 2.1 it follows $c, d \in Tr(R_{\infty,s}) \subsetneq R_{\infty,s}$. Therefore, $\lim_{n\to+\infty} \frac{c_{n+1}}{d_n} = \lim_{n\to+\infty} \frac{d_{n+1}}{c_n} = +\infty$ holds. It follows $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = \lim_{n\to+\infty} \left(\frac{c_{[\lambda n]}}{c_{[\lambda n]-1}} \cdot \frac{c_{[\lambda n]-1}}{c_{[\lambda n]-2}} \cdots \frac{c_{n+1}}{d_n} \right) = +\infty$ for $\lambda > 1$. Analogously it can be proved that $\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = +\infty$ holds for each $\lambda > 1$, thus $c \overset{r}{\sim} d$ as $n \to +\infty$ holds. ☐

Proposition 2.4. *Let $TS = Tr(SV_s)$, $x \in \mathcal{M}_c^{TS}$. Then it holds $x \sim c$ as $n \to +\infty$ ($\sim$ is the relation defined by $\lim_{n\to+\infty} \frac{x_n}{c_n} = 1$). Also, $\mathcal{M}_c^{TS} \subsetneq Tr(SV_s)$ holds.*

Proof. Let $x \in \mathcal{M}_c^{TS}$. Therefore, it holds $c_n \leqslant x_n \leqslant c_{n+1}$ for each $n \in \mathbb{N}$. It means that $1 \leqslant \lim_{n\to+\infty} \frac{x_n}{c_n} \leqslant \lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = 1$, thus $c \sim x$ as $n \to +\infty$. Thus, $\mathcal{M}_c^{TS} \subsetneq [c]_\sim$ ($[c]_\sim$ is the class of strong asymptotic equivalence, generated by the sequence c). It follows $c \in \mathcal{M}_c^{TS}$ holds ($c \in Tr(SV_s)$). So, if $x \in \mathcal{M}_c^{TS}$, then $x \in [c]_\sim$ and thus $x_n = h_n \cdot c_n$, where for the sequence $h = (h_n)$, $n \in \mathbb{N}$, $h \to 1$ holds as $n \to +\infty$. Therefore, it holds $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = 1$, which means $x \in Tr(SV_s)$.

The sequence $d = (d_n)$, $n \in \mathbb{N}$, given by $d_n = c_{n+1} + \frac{1}{n}$ as $n \to +\infty$, belongs to the class $Tr(SV_s)$ and it does not belong to the class $\mathcal{M}_c^{TS}$. It holds also $d \in [c]_\sim$. It means that $\mathcal{M}_c^{TS} \subsetneq [c]_\sim \subsetneq Tr(SV_s)$ holds. ☐

Proposition 2.5. *The player* II *has a winning strategy in the game $G_1(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$.*

Proof. Let $m \in \mathbb{N}$. In m^{th} round the player I chooses an element $A^m \in \mathcal{M}_c^{TS}$. Then II chooses an element $y_m \in A^m$, $m \in \mathbb{N}$. It holds $c_m \leqslant y_m \leqslant c_{m+1} \leqslant y_{m+1} \leqslant c_{m+2}$, for $m \in \mathbb{N}$. Therefore, $1 \leqslant \frac{y_{m+1}}{y_m} \leqslant \frac{c_{m+2}}{c_m} = \frac{c_{m+2}}{c_{m+1}} \cdot \frac{c_{m+1}}{c_m}$ and $\lim_{n\to+\infty} \frac{y_{m+1}}{y_m} = 1$ hold. Hence, $y \in Tr(SV_s)$ and it holds $c_m \leqslant y_m \leqslant c_{m+1}$, so $y \in \mathcal{M}_c^{TS}$. ☐

Corollary 2.1. *The selection principle $S_1(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$ holds.*

Proposition 2.6. *The player* II *has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$.*

Proof. (m^{th} round, $m \geqslant 1$) Take a sequence $p_1 < p_2 < \cdots$ of prime numbers. In m^{th} round the player I chooses the sequence $A^m \in \mathcal{M}_c^{TS}$ and the player II chooses a subsequence $A^{k_m(n)}$ of the sequence A^m, so that $k_m(n) = p_m^n$ for $n \in \mathbb{N}$. Consider the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A^{k_m(n)}$ of positive real numbers. We can consider this set as the subsequence of the sequence $y = (y_i)$, $i \in \mathbb{N}$, given by

$$y_i = \begin{cases} A^{k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}, \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y, we have that $y \in \mathbb{S}$, $y \sim c$ as $i \to +\infty$, $c_i \leqslant y_i \leqslant c_{i+1}$ for $i \in \mathbb{N}$. Therefore, $y \in \mathcal{M}_c^{TS}$ Also, $y \cap A^m$ has infinitely many elements for each $m \in \mathbb{N}$. This means that II wins the play $A^1, A^{k_1(n)}; A^2, A^{k_2(n)}, \ldots$, i.e., II has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$. $\square$

Corollary 2.2. *The selection principle $\alpha_2(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$ holds.*

Consider now an important subclass of $R_{\infty,s}$.

Let $c \in R_{\infty,s}$. Therefore, it holds $\underline{\lim}_{n\to+\infty} \frac{c_{n+1}}{c_n} = A \geqslant 1$. It follows from (1.1), because $\frac{c_{[\lambda n]}}{c_n} = \frac{c_{[\lambda n]}}{c_{[\lambda n]-1}} \cdots \frac{c_{n+1}}{c_n}$ holds for $n \in \mathbb{N}$ large enough. On the right side there are $[\lambda n] - n$, $n \in \mathbb{N}$, factors which tend to $+\infty$ as $n \to +\infty$.

The class of rapidly varying sequences which satisfy the relation $\lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = A > 1$, $A \in \mathbb{R}$, we will denote by $R_{\infty,s}^{TR}$ and the class of rapidly varying sequences which satisfy the relation $\lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = 1$ by $R_{\infty,s}^{TS}$. We see that

$$R_{\infty,s}^{TR} \cup R_{\infty,s}^{TS} \subsetneq R_{\infty,s}, \quad R_{\infty,s}^{TS} \subsetneq Tr(SV_s) \quad \text{and} \quad R_{\infty,s}^{TR} \subsetneq Tr(RV_s),$$

where $Tr(RV_s)$ is the class of translationally regularly varying sequences in the sense of Karamata (see, e.g., [5]).

Example 2.1. The sequence $(c_n) = (e^n)$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}^{TR}$, and the sequence $(d_n) = (e^{\sqrt{n}})$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}^{TS}$.

Proposition 2.7. *Let $TRV = R_{\infty,s}^{TR}$, $x = (x_n)$, $n \in \mathbb{N}$, and $x \in \mathcal{M}_c^{TRV}$. Then $x_n \asymp c_n$ as $n \to +\infty$ ($\asymp$ is the relation defined by $0 < \liminf_{n\to+\infty} \frac{x_n}{c_n} \leqslant \limsup \frac{x_n}{c_n} < +\infty$). Also, $\mathcal{M}_c^{TRV} \subsetneq R_{\infty,s}$.*

Proof. Let $c \in R_{\infty,s}^{TR} = TRV$, $c \in \mathcal{M}_c^{TRV}$ and for the sequence x it holds $c_n \leqslant x_n \leqslant c_{n+1}$ for $n \in \mathbb{N}$. It means that for some $A \in \mathbb{R}$, it holds

$$1 \leqslant \underline{\lim}_{n\to+\infty} \frac{x_n}{c_n} \leqslant \overline{\lim}_{n\to+\infty} \frac{x_n}{c_n} \leqslant \lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = A < +\infty.$$

Hence, $c \asymp x$ as $n \to +\infty$. Thus, $\mathcal{M}_c^{TRV} \subsetneq [c]_\asymp$ ($[c]_\asymp$ is the class of weak asymptotic equivalence generated by the sequence c). It holds that $c \in \mathcal{M}_c^{TRV}$, $c \in R_{\infty,s}$. If $x \in \mathcal{M}_c^{TRV}$, then $x \in [c]_\asymp$ and $x_n = h_n \cdot c_n$, and for the sequence $h = (h_n)$, $n \in \mathbb{N}$, it holds $1 \leqslant \underline{\lim}_{n\to+\infty} h_n \leqslant \overline{\lim}_{n\to+\infty} h_n \leqslant A < +\infty$. Thus, for $\lambda > 1$,

$$\underline{\lim}_{n\to+\infty} \frac{x_{[\lambda n]}}{x_n} \geqslant \underline{\lim}_{n\to+\infty} \frac{h_{[\lambda n]}}{h_n} \cdot \underline{\lim}_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = \frac{1}{A} \cdot (+\infty) = +\infty$$

holds. The last means that $x \in R_{\infty,s}$ so $\mathcal{M}_c^{TRV} \subsetneq \{c\}_\asymp \subsetneq R_{\infty,s}$. $\square$

Proposition 2.8. *The player* II *has a winning strategy in the game $G_1(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$.*

Proof. Let $m \in \mathbb{N}$. In m^{th} round I chooses an element $A^m \in \mathcal{M}_c^{TRV}$. II chooses an element $y_m \in A^m$, $m \in \mathbb{N}$. Thus, we get the sequence (y_m). Therefore, for each $m \in \mathbb{N}$, $c_m \leqslant y_m \leqslant c_{m+1} \leqslant y_{m+1} \leqslant c_{m+2}$, so $1 \leqslant \frac{y_{m+1}}{y_m} \leqslant \frac{c_{m+2}}{c_m}$. It follows $1 \leqslant \underline{\lim}_{n\to+\infty} \frac{y_{m+1}}{y_m} \leqslant \overline{\lim}_{n\to+\infty} \frac{y_{m+1}}{y_m} \leqslant \overline{\lim}_{n\to+\infty} \frac{c_{m+2}}{c_{m+1}} \cdot \overline{\lim}_{n\to+\infty} \frac{c_{m+1}}{c_m} = A \cdot A = A^2$ and for each $m \in \mathbb{N}$, $c_m \leqslant y_m \leqslant c_{m+1}$. Hence, $y \in \mathcal{M}_c^{TRV}$. $\square$

Corollary 2.3. *The selection principle* $S_1(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$ *holds.*

Proposition 2.9. *The player* II *has a winning strategy in the game* $G_{\alpha_2}(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$.

Proof. (m^{th} round, $m \geqslant 1$) Let $p_1 < p_2 < \cdots$ be a sequence of prime numbers. In m^{th} round I chooses the sequence $A^m \in \mathcal{M}_c^{TRV}$, and II chooses a subsequence $A^{k_m(n)}$ of the sequence A^m, so that $k_m(n) = p_m^n$ for $n \in \mathbb{N}$. Consider the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A^{k_m(n)}$ of positive real numbers. This set we can consider as the subsequence of the sequence $y = (y_i)$, $i \in \mathbb{N}$, given by

$$y_i = \begin{cases} A^{k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}, \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y, we have that $y \in \mathbb{S}$, $y_i \asymp c_i$ as $i \to +\infty$, $c_i \leqslant y_i \leqslant c_{i+1}$ for $i \in \mathbb{N}$. Therefore, $y \in \mathcal{M}_c^{TRV}$. Also, $y \cap A^m$ has infinitely many elements for each $m \in \mathbb{N}$. This means that II wins the play $A^1, A^{k_1(n)}$; $A^2, A^{k_2(n)}; \ldots; A^m, A^{k_m(n)}; \ldots$ In other words, II has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$. □

Corollary 2.4. *The selection principle* $\alpha_2(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$ *holds.*

Remark 2.1. In Propositions 2.8 and 2.9, and in Corollaries 2.3 and 2.4, improvements of some results from [3] are given.

Remark 2.2. Propositions 2.5, 2.6, Corollaries 2.1 and 2.2 hold also for the class $R_{\infty,s}^{TS} \subsetneq Tr(SV_s)$.

A sequence $x = (x_n) \in \mathbb{S}$ is said to be *logarithmic rapidly varying, with base* 2, if $(\log_2 x_n)$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}$ (see, e.g., [6]). The class of all logarithmic rapidly varying sequences is denoted by $L_2(R_{\infty,s})$. It holds $L_2(R_{\infty,s}) \subsetneq R_{\infty,s}$.

Proposition 2.10. *Let* $x, y \in \mathbb{S}_1$ *and* $x \overset{r}{\sim} y$ *as* $n \to +\infty$. *If* $x \in L_2(R_{\infty,s})$ *holds, then* $y \in L_2(R_{\infty,s})$.

Proof. Let sequences $x, y \in \mathbb{S}_1$ be given, and let the sequence $(\log_2 x_n)$, $n \in \mathbb{N}$, be rapidly varying. Define the functions $f(t) = x_{[t]}$ and $g(t) = y_{[t]}$, $t \geqslant 1$. Therefore, it holds $f(t) \overset{r}{\sim} g(t)$ as $t \to +\infty$, and $\log_2 f(t)$ is rapidly varying function. The functions f and g are also nondecreasing. It holds $\frac{\log_2 g(\lambda t)}{\log_2 g(t)} \geqslant \frac{\log_2(f(\lambda^{\frac{2}{3}} \cdot t))}{\log_2(f(\lambda^{\frac{1}{3}} \cdot t))} \to +\infty$ as $t \to +\infty$, for each $\lambda > 1$. For t large enough, $g(t) < f(\lambda^{\frac{1}{3}} \cdot t)$ and $f(\lambda^{\frac{2}{3}} \cdot t) < g(\lambda t)$ hold for $\lambda > 1$. Therefore, $\log_2 g(t) = h(t)$, $t \geqslant 1$, belongs to the class $R_{\infty,f}$ and hence $(\log_2 y_n) \in R_{\infty,s}$. □

Corollary 2.5. *Proposition* 2.10 *holds when* $x_n \overset{tr}{\sim} y_n$ *as* $n \to +\infty$.

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WELL-POSEDNESS AND GENERAL DECAY OF SOLUTIONS FOR THE HEAT EQUATION WITH A TIME VARYING DELAY TERM

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Abstract. We consider the nonlinear heat equation in a bounded domain with a time varying delay term

$$u_t + \Delta^2 u - J(t)\int_0^t g(t-s)\Delta^2 u(s)ds + \alpha K(t)u + \beta K(t)u\left(t-\tau(t)\right) = 0,$$

with initial conditions. By introducing suitable energy and Lyapunov functionals, under some assumptions, we then prove a general decay result of the energy associated of this system under some conditions.

1. Introduction and Statement

Let us consider the following problem

$$\begin{cases} u_t + \Delta^2 u - J(t)\displaystyle\int_0^t g(t-s)\Delta^2 u(s)ds + \alpha K(t)u & \\ \;+\beta K(t)u\left(t-\tau(t)\right) = 0, & \text{in } \Omega\times\,]0,+\infty[\,, \\ u = 0, & \text{on } \partial\Omega\times\,]0,+\infty[\,, \\ u(0) = u_0, & \text{in } \Omega, \\ u\left(t-\tau(0)\right) = h_0\left(t-\tau(0)\right), & \text{in } \Omega\times\,]0,\tau(0)[\,, \end{cases} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, Ω be a bounded open domain in $\mathbb{R}^n, n\in\mathbb{N}^*$ of regular boundary $\partial\Omega$, the function $\tau :]0,+\infty[\longrightarrow]0,+\infty[$, $\tau(t)$ is a time varying delay, α and β are positive real numbers, and the initial data (u_0,h_0) belongs to a suitable function space.

Key words and phrases. Heat equation, time varying delay, energy decay, Lyapunov functional, global existence, viscoelastic term.

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Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [12]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In, physical, chemical, biological, electrical, mechanical and economic phenomena.

in recent years, the stability of partial differential equations with time-varying delays has been studied in [8, 15, 20] via the Lyapunov method.

In the constant delay case the exponential stability was proved in [11, 18] by using the observability inequality which can not be applicable in the time-varying case (since the system is not invariant by translation).

In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 19, 23] and the references therein. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see, e.g., [5, 9, 11, 15]). Hence the stability issue of systems with delay is of theoretical and practical importance.

There are more works on the Lyapunov-based technique for delayed PDEs. Most of these studies analyze the case of constant delays. Thus, the conditions of stability and the exponential limits have been derived for some heat equations and scalar waves with constant delays and boundary conditions of Dirichlet without delay in [21].

S. Bernard, J. Belair and M. C. Mackey [16] studied the stability of the following linear differential equation

$$x' = -\alpha x(t) - \beta \int_0^{+\infty} x(t-s) f(s) ds,$$

where α and β are constants.

Chengming Huang and Stefan Vandewalle [2] considered a more general equation,

$$y'(t) = \alpha y(t) + \beta y(t-\tau) + \gamma \int_{t-\tau}^{t} y(s) ds, \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $u(t) = \phi(t)$ on $[-\tau, 0]$, and proved that the repeated trapezium rule retains the asymptotic stability of (1.2). Wu and Gan in [22] further extended the above study to the case of neutral equations.

In Section 3, page 16, Chengming Huang and Stefan Vandewalle [3] considered the asymptotic stability of multi-dimensional equations of the form

$$y'(t) = Ly(t) + My(t-\tau) + K \int_{t-\tau}^{t} y(\nu) d\nu, \quad t > 0, \tag{1.3}$$

where $L, M, K \in C^{d\times d}$ and $y(t) = \phi(t)$ on $[-\tau, 0]$. The characteristic equation equals

$$\det \left[\lambda I_d - L - Me^{-\tau\lambda} - K \int_{-\tau}^{0} e^{-\tau\nu} d\nu \right] = 0, \tag{1.4}$$

where I_d is the $d \times d$ identity matrix. The zero solution of (1.3) is asymptotically stable if and only if all the roots λ of (1.4) have negative real parts.

Recently the stability of PDEs with time-varying delays was analyzed in [8, 20].

Later, Mohamed Ferhat and Ali Hakem in [7] studied the decay properties of solutions of the folowing system for the initial boundary value problem of a nonlinear wave equation

$$\begin{cases} (|u'|^{\gamma-2}\, u')' - \Delta_x u - \displaystyle\int_0^t g(t-s)\Delta u(s)ds + \mu_1 \Psi(u'(x,t)) & \\ \ + \mu_2 \Psi(u'(x,t-\tau(t))) = 0, & \textit{in } \Omega \times (0,+\infty), \\ u = 0, & \textit{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \textit{in } \Omega, \\ u'(t-\tau(0)) = f_0(t-\tau(0)), & \textit{on } \Omega \times (0,\tau(0)), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $\tau(t) > 0$ is a time varying delay, μ_1 and μ_2 are positive real numbers.

Recently, the case of time-varying delay has been studied in [13, 18]. For example, in Nicaise et al. [18] in one space dimension studied

$$\begin{cases} u' - a u_{xx} = 0, & 0 < x < \pi,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = \mu_0 u(\pi,t) - \mu_1 u(\pi, t-\tau(t)), & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \pi, \\ u(\pi, t-\tau(0)) = f_0(t-\tau(0)), & 0 < t < \tau(0), \end{cases}$$

where $\mu_0, \mu_1 \geq 0$ and $a > 0$. They proved the exponential stability result under the conditions

$$\begin{aligned} &\tau' < 1, \quad \text{for all } t > 0, \\ &\text{exists } M > 0, \quad 0 < \tau_0 \leq \tau \leq M, \quad \text{for all } t > 0, \\ &\tau \in W^{2,\infty}([0,T]), \quad \text{for all } T > 0. \end{aligned}$$

And in 2011 S. Nicaise and C. Pignotti in [13] considered an problem of the form

$$\begin{cases} u'' - \Delta u - a\Delta u' = 0, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \Gamma \times (0,+\infty), \\ \mu u'' = \dfrac{\partial(u + au')}{\partial \nu} - k u'(t-\tau(t)), & \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), & \text{in } \Omega, \\ u' = f_0, & \text{on } \Gamma_1 \times (-\tau(0), 0). \end{cases}$$

We also recall the result by Xu, Yung and Li [4], where the authors proved a result similar to the one in [11] for the one-space dimension by adopting the spectral analysis approach. The case of time-varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [18]) in one-space dimension. They proved an exponential stability result under the condition $\mu_2 \leq \sqrt{1-d}\mu_1$, where the fuction τ satisfies $\tau'(t) \leq d < 1$ for all $t > 0$.

In [14], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

The paper is organized as follows. In Section 2 we present some assumptions and state the main result. The general decay result is proved in Sections 3.

We use the ideas given by G. Li, B. Zhu and Wenjun Liu in [10], and the multiplier technique to prove our result.

2. Preliminaries and Main Results

Firstly we assume the following hypotheses.

(H1) $k : \mathbb{R}_+ \longrightarrow]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$k'(t) \leq -ck(t), \quad \text{for all } t \geq 0, \tag{2.1}$$

where c is a positive constant.

(H2) $J, g, \psi : \mathbb{R}_+ \longrightarrow]0, +\infty[$ are non-increasing differentiable functions satisfying

$$\int_0^{+\infty} g(s)ds < +\infty, \quad 1 - J(t) \int_0^t g(s)ds \geq l > 0, \tag{2.2}$$

and

$$g'(t) < -\psi(t)g(t), \quad \text{for all } t \geq 0, \quad \lim_{t \longrightarrow +\infty} \frac{J'(t)}{\psi(t)J(t)} = 0. \tag{2.3}$$

(H3) For the time-varying delay τ, it is varying betwin two positive constants τ_0, τ_1, and

$$\tau \in W^{2,\infty}([0, T]), \quad \text{for all } T > 0, \tag{2.4}$$

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \text{for all } t > 0, \tag{2.5}$$

$$\tau'(t) \leq d < 1, \quad \text{for all } t > 0. \tag{2.6}$$

(H4) α, β and δ are three positive constants satisfy,

$$\alpha \geq \beta\delta \tag{2.7}$$

and

$$\beta \leq \frac{1}{2\delta k(0)}, \tag{2.8}$$

for some $\delta > 0$.

We now state some lemmas needed later.

Lemma 2.1 (Sobolev-Poincare's inequality)**.** *There exists a constant $C_p = C(\Omega)$ such that*

$$\int_\Omega |w|^2 \, dx \leq C_p \int_\Omega |\Delta w|^2 \, dx, \quad \textit{for all } w \in H_0^1(\Omega). \tag{2.9}$$

We introduce, as in [11], the new variable

$$z(x,\rho,t) = u(x, t-\rho\tau(t)), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty). \tag{2.10}$$

Then, we have

$$\tau(t)z'(x,\rho,t) = (\tau'(t)\rho - 1)z_\rho(x,\rho,t), \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \tag{2.11}$$

where $z' := \frac{\partial z}{\partial t}$ and $z_\rho := \frac{\partial z}{\partial \rho}$. Then problem (1.1) may be rewritten as

$$\begin{cases} u_t + \Delta^2 u - J(t)\int_0^t g(t-s)\Delta^2 u(s)ds + \alpha k(t)u & \\ \quad + \beta k(t) z(1,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ \tau(t)z'(x,\rho,t) = (\rho\tau'(t) - 1)z_\rho(x,\rho,t), & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u = 0, & \text{on } \partial\Omega \times]0,+\infty[, \\ u(0) = u_0, & \text{in } \Omega, \\ z(0,t) = u(t), & \text{in } \Omega(0,+\infty), \\ z(\rho,0) = h_0(-\rho\tau(0)), & \text{in } \Omega \times (0,1). \end{cases} \tag{2.12}$$

We define the energy of solution of problem (2.12) by

$$\begin{aligned} E(t) =& \frac{1}{2}\Big[\alpha k(t)\|u\|_2^2 + \Big(1 - J(t)\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + J(t)(g\circ\Delta u)(t) \\ & + \xi k(t)\tau(t)\int_\Omega\int_0^1 |z(\rho,t)|^2 d\rho dx\Big], \end{aligned} \tag{2.13}$$

where ξ is a positive constant, and

$$(g\circ\Delta w)(t) = \int_0^t g(t-\nu)\|\Delta w(t) - \Delta w(\nu)\|^2 d\nu.$$

Now we will establish a general decay rate estimate for the energy.

3. Decay of Solutions

We firstly give the global existence of solutions of the system, which has been proved in [10].

Proposition 3.1. ([10, Lemma 2.1]). *Let* **(H1)-(H4)** *hold. Then given* $u_0 \in H_1^0(\Omega)$, $h_0 \in L^2(\Omega \times (0,1))$ *and* $T > 0$, *there exists a unique weak solution* (u,z) *of the problem* (2.12) *on* $(0,T)$ *such that*

$$u \in C(0,T; H_1^0(\Omega)) \cap C_1(0,T; L^2(\Omega)).$$

Lemma 3.1. *Let* (2.6) *and* (2.7) *be satisfied,* ξ *be a positive constant and* δ *sufficiently small such that*

$$\frac{\beta}{2\delta(1-d)} \le \xi \le \alpha c, \tag{3.1}$$

and (u,z) *the solution of the problem* (2.12)*. Then, the energy functional defined by* (2.13) *it may be non-increasing function and satisfies*

$$\begin{aligned} E'(t) \leq & \frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t) - \frac{1}{2}J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ & + \frac{\xi}{2}k'(t)\tau(t)\int_\Omega\int_0^1 |z(\rho,t)|^2\, d\rho dx \\ \leq & \frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t) - \frac{1}{2}J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 . \end{aligned} \tag{3.2}$$

Proof. At first, multiplying the first equation in (2.12) by u', integrating over Ω and using integration by parts, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_2^2 + \alpha k(t)\|u\|_2^2\right) + \|u'\|_2^2 - \frac{1}{2}\alpha k'(t)\|u\|_2^2 + \beta k(t)\int_\Omega u' z(1,t)dx \\ & - J(t)\int_0^t g(t-s)\int_\Omega \Delta u'\Delta u(s)dxds = 0. \end{aligned} \tag{3.3}$$

We denote by $I_1(t)$ to the last term on the left side of (3.3) for $I_1(t)$ we have

$$\begin{aligned} I_1(t) = & J(t)\int_0^t g(t-s)\int_\Omega \Delta u'(t)\left(\Delta u(t)-\Delta u(s)\right)dxds \\ & - J(t)\int_0^t g(t-s)\int_\Omega \Delta u'(t)\Delta u(t)dxds \\ = & \frac{1}{2}\frac{d}{dt}\left[\int_0^t J(t)g(t-s)\int_\Omega |\Delta u(t)-\Delta u(s)|^2\, dxds \right. \\ & \left. - J(t)\int_0^t g(s)\int_\Omega |\Delta u(t)|^2\, dxds\right] + \frac{1}{2}\left(J(t)\int_0^t g(s)ds\right)'\int_\Omega |\Delta u(t)|^2\, dxds \\ & - \frac{1}{2}\left(\int_0^t (J(t)g(t-s))'\int_\Omega |\Delta u(t)-\Delta u(s)|^2\right)dxds \\ = & \frac{1}{2}\frac{d}{dt}\left[\int_0^t J(t)g(t-s)\int_\Omega |\Delta u(t)-\Delta u(s)|^2\, dxds \right. \\ & \left. - J(t)\int_0^t g(s)ds\int_\Omega |\Delta u(t)|^2\, dx\right] + \frac{1}{2}J(t)g(t)\|\Delta u\|_2^2 \\ & + \frac{1}{2}J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 - \frac{1}{2}J'(t)\left(g\circ\Delta u\right)(t) - \frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t). \end{aligned} \tag{3.4}$$

Inserting (3.4) into (3.3) and using Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\alpha k(t)\|u\|_2^2 + \left(1 - J(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + J(t)\left(g\circ\Delta u\right)(t)\right) \\ \leq & \frac{1}{2}\alpha k'(t)\|u\|_2^2 - (1-\delta\beta k(t))\|u'\|_2^2 + \frac{\beta k(t)}{4\delta}\|z(1,t)\|_2^2 + \frac{1}{2}J'(t)\left(g\circ\Delta u\right)(t) \\ & + \frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t) - \frac{1}{2}\left(J'(t)\left(\int_0^t g(s)ds\right) + J(t)g(t)\right)\|\Delta u\|_2^2 . \end{aligned} \tag{3.5}$$

Secondly, we multiply the second equation in (2.12) by $\xi k(t)z(x,\rho,t)$ and integrate over $\Omega\times(0,1)$, to get

$$\frac{\xi}{2}k(t)\tau(t)\int_\Omega\int_0^1 \frac{d}{dt}\left|z(\rho,t)\right|^2 d\rho dx = -\frac{\xi}{2}k(t)\int_\Omega\int_0^1 (1-\rho\tau'(t))\frac{\partial}{\partial\rho}\left|z(\rho,t)\right|^2 d\rho dx.$$

And from there we find

$$\begin{aligned}\frac{d}{dt}\left(\frac{\xi}{2}k(t)\tau(t)\int_\Omega\int_0^1 \left|z(\rho,t)\right|^2 d\rho dx\right) =&\frac{\xi}{2}\left(k(t)\tau(t)\right)'\int_\Omega\int_0^1 \left|z(\rho,t)\right|^2 d\rho dx\\ &-\frac{\xi}{2}k(t)\int_\Omega\left[(1-\rho\tau'(t))\left|z(\rho,t)\right|^2\right]_0^1 dx\\ &-\frac{\xi}{2}k(t)\tau'(t)\int_\Omega\int_0^1 \left|z(\rho,t)\right|^2 d\rho dx.\end{aligned}\tag{3.6}$$

Taking the sum of (3.5) and (3.6), we obtain that

$$\begin{aligned}E'(t) \leq&\frac{1}{2}\left(\alpha k'(t)+\xi k(t)\right)\|u\|_2^2-\left(1-\delta\beta k(t)\right)\|u'\|_2^2+\frac{1}{2}J'(t)\left(g\circ\Delta u\right)(t)\\ &-\frac{k(t)}{2}\left(\xi\left(1-\tau'(t)\right)-\frac{\beta}{2\delta}\right)\|z(1,t)\|_2^2+\frac{1}{2}J(t)\left(g'\circ\Delta u\right)(t)\\ &-\frac{1}{2}\left(J'(t)\left(\int_0^t g(s)ds\right)+J(t)g(t)\right)\|\Delta u\|_2^2+\frac{\xi}{2}k'(t)\tau(t)\int_\Omega\int_0^1\left|z(\rho,t)\right|^2 d\rho dx.\end{aligned}\tag{3.7}$$

Combining (3.1), (3.7) and hypotheses (**H1**)-(**H4**), the proof of Lemma 3.1 is complete. □

Theorem 3.1. *Assume* **(H1)-(H4)**. *Then there exist positive constants C and K_0 such that for any solution of problem* (2.12)*, the energy satisfies the following estimate*

$$E(t)\leq Ce^{-K_0\int_0^t \psi(t)J(t)dt},\tag{3.8}$$

for every $t\geq 0$.

Now, we define the functional $F(t)$ as follows

$$F(t)=\frac{1}{2}\int_\Omega u^2dx.\tag{3.9}$$

Lemma 3.2. *The functional F satisfies the following estimate*

$$\begin{aligned}F'(t)\leq\left[\delta-1+\left(\int_0^t g(s)ds\right)J(t)\right]\|\Delta u\|_2^2+\frac{1-l}{4\delta}J(t)\left(g\circ u\right)(t)\\ +\left(\delta\beta-\alpha\right)k(t)\|u\|_2^2+\frac{\beta}{4\delta}\|z(t,1)\|_2^2.\end{aligned}\tag{3.10}$$

Proof. Differentiating and integrating by parts, we get

$$F'(t)=-\|\Delta u\|_2^2+J(t)\int_\Omega\int_0^t g(t-s)\Delta u(t)\Delta u(s)dsdx-k(t)\int_\Omega\left(\alpha u^2+\beta uz(t,1)\right)dx.\tag{3.11}$$

We denote by $F_1(t)$ the second term on the right-hand side of above equality. By using Young's and Cauchy-Schwarz inequalities, we have

(3.12)
$$\begin{aligned}
F_1(t) =& J(t)\int_\Omega\int_0^t g(t-s)\Delta u(t)\left[\Delta u(s)-\Delta u(t)\right]dsdx + J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2\\
\leq& J(t)\int_\Omega\int_0^t g(t-s)\left|\Delta u(t)\right|\left|\Delta u(s)-\Delta u(t)\right|dsdx + J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2\\
\leq& \delta\|\Delta u\|_2^2 + \frac{J^2(t)}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|ds\right)^2 dx\\
&+ J(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2\\
\leq& \left(\int_0^t g(s)ds\right)\frac{J^2(t)}{4\delta}\int_\Omega\int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 dsdx\\
&+ \left(\delta + J(t)\left(\int_0^t g(s)ds\right)\right)\|\Delta u\|_2^2\\
\leq& \left(\delta + J(t)\left(\int_0^t g(s)ds\right)\right)\|\Delta u\|_2^2 + \frac{1-l}{4\delta}J(t)\left(g\circ\Delta u\right)(t).
\end{aligned}$$

Inserting (3.12) into (3.11), we obtain the required proof. □

Lemma 3.3. *Let $G(t)$ be the function defined by*

$$G(t) = \int_\Omega\int_0^t g(t-s)u(t)\left[u(s)-u(t)\right]dsdx. \tag{3.13}$$

satisfies the estimate

(3.14)
$$\begin{aligned}
G'(t) \leq& \left[\delta + 2\delta\left(1-l\right)^2 + \left(1-l\right)\left(\delta_1 - \left(\int_0^t g(s)ds\right)\right)\right]\|\Delta u\|_2^2\\
&+ \left[2\delta + \left(\alpha k(0)+\delta\beta\right)\left(\int_0^t g(s)ds\right)\right]\|u\|_2^2\\
&+ \left(\int_0^t g(s)ds\right)\left[\frac{1}{2\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \left(\frac{1-l}{4\delta_1}\right)\right]\left(g\circ\Delta u\right)(t)\\
&- \frac{g(0)}{4\delta}C_p^2\left(g'\circ\Delta u\right)(t) + \left(\frac{\beta}{4\delta}k(0)+\delta\right)\|z(t,1)\|_2^2.
\end{aligned}$$

Proof. We take the derivative of $G(t)$ to get,

$$\begin{aligned}
G'(t) =& \int_\Omega\int_0^t g(t-s)u'(t)\left[u(s)-u(t)\right]dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u.u'dx\\
&+ \int_\Omega\int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right]dsdx,
\end{aligned} \tag{3.15}$$

using the problem (2.12) we obtain

$$
\begin{aligned}
G'(t) =& \int_\Omega \int_0^t g(t-s)\left[u(s)-u(t)\right] ds \Big[-\Delta^2 u + J(t)\int_0^t g(t-s)\Delta^2 u(s)ds \\
& -\alpha K(t)u - \beta K(t) z\left(1,t\right)\Big] dx + \int_\Omega \int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right] dsdx \\
& - \left(\int_0^t g(s)ds\right)\int_\Omega u \Big[-\Delta^2 u + J(t)\int_0^t g(t-s)\Delta^2 u(s)ds \\
& -\alpha K(t)u - \beta K(t) z\left(1,t\right)\Big] dx \\
=& -\int_\Omega \Delta u \int_0^t g(t-s)\left[\Delta u(s) - \Delta u(t)\right] dsdx \\
& + J(t)\int_\Omega \int_0^t g(t-s)\Delta u(s)ds \int_0^t g(t-s)\left[\Delta u(s)-\Delta u(t)\right] dsdx \\
& + \int_\Omega \int_0^t g'(t-s)u(t)\left[u(s)-u(t)\right] dsdx + \left(\int_0^t g(s)ds\right)\int_\Omega \left|\Delta u\right|^2 dx \\
& - \alpha K(t)\int_\Omega u \int_0^t g(t-s)\left[u(s)-u(t)\right] dsdx \\
& - \beta K(t)\int_\Omega z\left(t,1\right)\int_0^t g(t-s)\left[u(s)-u(t)\right] dsdx \\
& + \left(\int_0^t g(s)ds\right)\int_\Omega u\left(\alpha K(t)u + \beta K(t) z\left(t,1\right)\right) dx \\
& - J(t)\left(\int_0^t g(s)ds\right)\int_\Omega \Delta u \int_0^t g(t-s)\Delta u(s) dsdx \\
=& \sum_{i=1}^{8} G_i(t),
\end{aligned} \tag{3.16}
$$

where $G_i(t)$, $i = \overline{1,8}$, denote the terms on the right side of the above equality in order. $G_1(t)$, $G_2(t)$ and $G_3(t)$ can be estimated as in [17] as follows, for any $\delta > 0$. By Young's and Cauchy-Schwartz, we obtain

$$
G_1(t) \le \delta \left\|\Delta u\right\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)\left(g\circ\Delta u\right)(t) \tag{3.17}
$$

and

$$
\begin{aligned}
G_2(t) \le& \delta J^2(t)\int_\Omega \left(\int_0^t g(t-s)\left(\left|\Delta u(t)\right| + \left|\Delta u(s)-\Delta u(t)\right|\right) ds\right)^2 dx \\
& + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g(t-s)\left(\left|\Delta u(s)-\Delta u(t)\right|\right) ds\right)^2 dx \\
\le& \delta J^2(t)\left(\int_0^t g(s)ds\right)\Big[2\int_\Omega \int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 dsdx \\
& + 2\int_\Omega \int_0^t g(t-s)\left|\Delta u(t)\right|^2 dsdx\Big] + \frac{1}{4\delta}\left(\int_0^t g(s)ds\right)\left(g\circ\Delta u\right)(t) \\
\le& \left(2\delta J^2(t) + \frac{1}{4\delta}\right)\left(\int_0^t g(s)ds\right)\left(g\circ\Delta u\right)(t) + 2\delta\left(1-l\right)^2\left\|\Delta u\right\|_2^2 .
\end{aligned} \tag{3.18}
$$

For $G_3(t)$ and $G_5(t)$, we use Cauchy-Schwartz, Young's and Poincare's inequalities, we get

(3.19)
$$\begin{aligned}
G_3(t) &\leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left(\int_0^t g'(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g'(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx \\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t -g'(t-s)ds\right)\left(\int_0^t -g'(t-s)\left|u(s)-u(t)\right|^2 ds\right) dx \\
&\leq \delta \left\|u\right\|_2^2 + \frac{1}{4\delta}C_p^2 \left(\int_0^t -g'(t-s)ds\right)\left(\int_0^t -g'(t-s)\left|\Delta u(s)-\Delta u(t)\right|^2 ds\right) dx \\
&\leq \delta \left\|u\right\|_2^2 - \frac{1}{4\delta}C_p^2 \left(\int_0^t -g'(t-s)ds\right)\left(g'\circ \Delta u\right)(t) \\
&\leq \delta \left\|u\right\|_2^2 - \frac{g(0)}{4\delta}C_p^2 \left(g'\circ \Delta u\right)(t)
\end{aligned}$$

and

(3.20)
$$\begin{aligned}
G_5(t) &\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k(^2 t)}{4\delta}\int_\Omega \left(\int_0^t g(t-s)\left(u(s)-u(t)\right) ds\right)^2 dx \\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(t)}{4\delta}\left(\int_0^t g(s)ds\right)\int_\Omega \int_0^t g(t-s)\left(u(s)-u(t)\right)^2 dsdx \\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(t)}{4\delta}\left(\int_0^t g(s)ds\right) C_p^2 \int_\Omega \int_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right)^2 dsdx \\
&\leq \delta \left\|u\right\|_2^2 + \frac{\alpha^2 k^2(0)}{4\delta}C_p^2 \left(\int_0^t g(s)ds\right)\left(g\circ \Delta u\right)(t).
\end{aligned}$$

Similarly, we have

$$G_6(t) \leq \delta \left\|z\left(t,1\right)\right\|_2^2 + \frac{\alpha^2 k^2(0)}{4\delta}C_p^2 \left(\int_0^t g(s)ds\right)\left(g\circ \Delta u\right)(t), \tag{3.21}$$

$$G_7(t) \leq \left(\int_0^t g(s)ds\right)\left[\left(\alpha k(0)+\delta\beta\right)\left\|u\right\|_2^2 + \frac{\beta}{4\delta}k^2(0)\left\|z(t,1)\right\|_2^2\right] \tag{3.22}$$

and

$$\begin{aligned}
G_8(t) \leq &- \left(\int_0^t g(s)ds\right) J(t)\left[\int_\Omega \Delta u \int_0^t g(t-s)\left(\Delta u(s)-\Delta u(t)\right) dsdx \right.\\
&\left. + \left(\int_0^t g(s)ds\right)\left\|\Delta u\right\|_2^2\right] \\
\leq &- \left(\int_0^t g(s)ds\right) J(t)\int_\Omega \left|\Delta u\right| \int_0^t g(t-s)\left|\Delta u(s)-\Delta u(t)\right| dsdx \\
&- \left(\int_0^t g(s)ds\right)^2 J(t)\left\|\Delta u\right\|_2^2
\end{aligned}$$

$$\leq \left(\int_0^t g(s)ds\right) J(t) \left[\delta_1 \|\Delta u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega \left(\int_0^t g(t-s)\left|\Delta u(s) - \Delta u(t)\right| ds\right)^2 dx\right]$$
$$- \left(\int_0^t g(s)ds\right)^2 J(t) \|\Delta u\|_2^2,$$

(3.23)

$$G_8(t) \leq \left(\int_0^t g(s)ds\right) J(t) \left[\delta_1 \|\Delta u\|_2^2 + \frac{\left(\int_0^t g(s)ds\right)}{4\delta_1}(g\circ \Delta u)(t)\right]$$
$$- \left(\int_0^t g(s)ds\right)^2 J(t) \|\Delta u\|_2^2$$
$$\leq \left(\int_0^t g(s)ds\right) J(t) \left[\left(\delta_1 - \int_0^t g(s)ds\right) \|\Delta u\|_2^2 + \frac{\left(\int_0^t g(s)ds\right)}{4\delta_1}(g\circ \Delta u)\right]$$
$$\leq \left(\int_0^t g(s)ds\right)\left(\delta_1 - \int_0^t g(s)ds\right) J(t) \|\Delta u\|_2^2 + \frac{1-l}{4\delta_1}\left(\int_0^t g(s)ds\right)(g\circ \Delta u).$$

Summarizing these estimates with (3.16), we get (3.14). □

Lemma 3.4. *Now, as in* [7, Lemma 3.4]*, we introduce the folowing functional*

$$\Phi(t) = \int_0^1 e^{-2\rho\tau(t)}\int_\Omega z^2(t,\rho)dxd\rho. \tag{3.24}$$

Then

$$\Phi'(t) \leq \frac{d-1}{\tau_1}e^{-2\tau_1}\|z(t,1)\|_2^2 + \frac{1}{\tau_0}\|u\|_2^2 - \left(\frac{\tau'(t)}{\tau_1}+2\right)e^{-2\tau_1}\int_0^1 \|z(t,\rho)\|_2^2 d\rho. \tag{3.25}$$

Proof. By differentiating, using the second equation in (2.12) and integrating by parts over $(0,1)$, we get

(3.26)

$$\Phi'(t) = -2\tau'(t)\int_0^1 \rho e^{-2\rho\tau(t)}\int_\Omega z^2(t,\rho)dxd\rho + 2\int_0^1 e^{-2\rho\tau(t)}\int_\Omega z'(t,\rho)z(t,\rho)dxd\rho$$
$$= -2\tau'(t)\int_0^1 \rho e^{-2\rho\tau(t)}\int_\Omega z^2(t,\rho)dxd\rho$$
$$+ 2\int_0^1 e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z_\rho(t,\rho)z(t,\rho)dxd\rho.$$

We denote by $\Phi_1(t)$ the last term in the right-hand side of the equality above

$$\Phi_1(t) = \int_0^1 e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)}\frac{d}{d\rho}z^2(t,\rho)dxd\rho$$
$$= \left[e^{-2\rho\tau(t)}\int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z^2(t,\rho)dx\right]_0^1$$

$$- \int_\Omega \int_0^1 z^2(t,\rho) \frac{d}{d\rho} \left(e^{-2\rho\tau(t)} \frac{\rho\tau'(t)-1}{\tau(t)} \right) d\rho dx$$

(3.27)

$$\begin{aligned}
\Phi_1(t) =& \left[e^{-2\rho\tau(t)} \int_\Omega \frac{\rho\tau'(t)-1}{\tau(t)} z^2(t,\rho) dx \right]_0^1 + 2\tau'(t) \int_\Omega \int_0^1 \rho e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx \\
&- \left(\frac{\tau'(t)}{\tau(t)} + 2 \right) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx \\
\leq& \frac{\tau'(t)-1}{\tau(t)} e^{-2\tau(t)} \left\| z^2(t,1) \right\|_2^2 + \frac{1}{\tau(t)} \left\| z^2(t,0) \right\|_2^2 \\
&+ 2\tau'(t) \int_\Omega \int_0^1 \rho e^{-2\rho\tau(t)} z^2(t,\rho) d\rho dx - \left(\frac{\tau'(t)}{\tau(t)} + 2 \right) e^{-2\tau(t)} \int_\Omega \int_0^1 z^2(t,\rho) d\rho dx.
\end{aligned}$$

Since

$$e^{-2\tau_1} \leq e^{-2\tau(t)} \leq e^{-2\rho\tau(t)} \leq 1, \quad \text{for all } \rho \in (0,1),\ t > 0,$$

inserting (3.27) in (3.26), we obtain (3.25). □

Now, we are ready to prove the general decay result. For this, we define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) = NE(t) + J(t) \left(\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right).$$

Taking the derivative of $\mathcal{L}(t)$ with respect to t we have

(3.28)

$$\mathcal{L}'(t) = NE'(t) + J(t) \left(\epsilon F'(t) + \epsilon_1 G'(t) + \epsilon_2 \Phi'(t) \right) + J'(t) \left(\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right).$$

By using (3.9), (3.13), (3.24), Young's and Poincare's inequalities, we obtain

(3.29)

$$\begin{aligned}
J'(t) \left[\epsilon F(t) + \epsilon_1 G(t) + \epsilon_2 \Phi(t) \right] \leq& \left(\epsilon - \frac{\epsilon_1}{2} \right) J'(t) \|u\|_2^2 + \epsilon_2 J'(t) e^{-2\tau_0} \int_0^1 \|z(t,\rho)\|_2^2 d\rho \\
&- \frac{\epsilon_1}{2} \left(\int_0^t g(s) ds \right) C_p^2 J'(t) \left(g \circ \Delta u \right)(t).
\end{aligned}$$

Exploiting (3.29) in (3.28) and using (3.7), (3.10), (3.14) and **(H2)**, we arrive at

(3.30)

$$\begin{aligned}
\mathcal{L}'(t) \leq& - J(t) \left[\left(\epsilon - \frac{\epsilon_1}{2} \right) \frac{J'(t)}{J(t)} + \epsilon \left(\alpha - \beta \delta_0' \right) M \right. \\
&\left. - \epsilon_1 \left(2\delta + (k(0)\alpha + \delta\beta) \left(\int_0^t g(s) ds \right) J(t) \right) - \frac{\epsilon_2}{\tau_0} \right] \|u\|_2^2 \\
&- \left[N \frac{M}{2} \left(\xi (1-d) - \frac{\beta}{2\delta} \right) - J(0) \left(\frac{\epsilon\beta}{4\delta} + \epsilon_1 \left(\frac{\beta}{4\delta} k^2(0) + \delta \right) \right) \right] \|z(1,t)\|_2^2 \\
&+ J(t) \left[\frac{N}{2} - \epsilon_1 \frac{g(0)}{4\delta} C_p^2 \right] \left(g' \circ \Delta u \right)(t)
\end{aligned}$$

$$
\begin{aligned}
&- J(t)\left[\frac{N}{2}\left(\int_0^t g(s)ds\right)\frac{J'(t)}{J(t)} + \epsilon\,(1-\delta) - \epsilon_1\left(\delta + 2\delta\,(1-l)^2\right)\right.\\
&\left.-\left(\int_0^t g(s)ds\right)J(t)\left((\epsilon+\epsilon_1\delta_1) - \epsilon_1\left(\int_0^t g(s)ds\right)\right)\right]\|\Delta u\|_2^2\\
&- J(t)\left[\epsilon_2 e^{-2\tau_1} + \frac{J'(t)}{J(t)}e^{-2\tau_0}\right]\int_0^1 \|z(\rho,t)\|_2^2\,d\rho\\
&+\left[\epsilon_1\left(\int_0^t g(s)ds\right)\left(\frac{1}{2\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \frac{1-l}{4\delta_1}\right)\right.\\
&\left.+\epsilon\frac{1-l}{4\delta}J(0) - \frac{\epsilon_1}{2}\left(\int_0^t g(s)ds\right)C_p^2\frac{J'(t)}{J(t)}\right]J(t)\,(g\circ\Delta u)\,(t).
\end{aligned}
$$

At this point, choose ϵ_1, ϵ_2 small enough such that $0<\epsilon_2<\epsilon_1<\epsilon$ and δ_1 sufficiently small such that

$$
\epsilon\,(l-\delta) > \epsilon_1\left(\delta + 2\delta\,(1-l)^2 - (1-l)\left(\delta_1 - \int_0^t g(s)ds\right)\right) = C\,(\epsilon_1,\delta) > 0
$$

and

$$
C_0\,(\epsilon_1,\epsilon_2) = \epsilon\,(\alpha-\beta\delta)\,M - \epsilon_1\left(2\delta + (k(0)\alpha+\delta\beta)\left(\int_0^t g(s)ds\right)\right) - \frac{\epsilon_2}{\tau_0} > 0.
$$

Since (3.1), once ϵ_1 and δ are fixed, we want to choose N large enough such that

$$
N\frac{M}{2}\left(\xi\,(1-d) - \frac{\beta}{2\delta}\right) - J(0)\left(\frac{\epsilon\beta}{4\delta} + \epsilon_1\left(\frac{\beta}{4\delta}k^2(0) + \delta\right)\right) > 0
$$

and

$$
\frac{N}{2} - \epsilon_1\frac{g(0)}{4\delta}C_p^2 > 0.
$$

For this, (3.30) becomes

$$
\begin{aligned}
\mathcal{L}'(t) \leq &- J(t)\left[\frac{N}{2}\left(\int_0^t g(s)ds\right)\frac{J'(t)}{J(t)} + \epsilon\,(1-\delta) - C\,(\epsilon_1,\delta)\right]\|\Delta u\|_2^2\\
&- J(t)\left[\epsilon_2 e^{-2\tau_1} + \frac{J'(t)}{J(t)}e^{-2\tau_0}\right]\int_0^1 \|z(\rho,t)\|_2^2\,d\rho\\
&+\left[C_1 - \frac{\epsilon_1}{2}\left(\int_0^t g(s)ds\right)C_p^2\frac{J'(t)}{J(t)}\right]J(t)\,(g\circ\Delta u)\,(t)\\
&- J(t)\left[\epsilon - C_0\,(\epsilon_1,\epsilon_2)\right]\|u\|_2^2,
\end{aligned}
\tag{3.31}
$$

where

$$
C_1 = \epsilon_1\left(\int_0^t g(s)ds\right)\left(\frac{1}{4\delta} + \frac{1}{4\delta} + 2\delta J^2(0) + \frac{\alpha^2+\beta^2}{4\delta}k^2(0) + \frac{1-l}{4\delta_1}\right) + \epsilon\frac{1-l}{4\delta}J(0).
\tag{3.32}
$$

We then use **(H2)** and choose $t_1 \geq t_0$ so that there exist two positive constants C_2 and C_3, such that (3.31) takes the form

$$\mathcal{L}'(t) \leq -C_2 J(t)E(t) + C_3 J(t) (g \circ \Delta u)(t), \quad \text{for all } t > t_1. \tag{3.33}$$

On the other hand, as in [6], multiplying (3.33) by $\psi(t)$ and using (3.31) and (3.2), we have

$$\begin{aligned} \psi(t)\mathcal{L}'(t) &\leq -C_2\psi(t)J(t)E(t) + C_3\psi(t)J(t)(g \circ \Delta u)(t) \\ &\leq -C_2\psi(t)J(t)E(t) - C_3 J(t)(g' \circ \Delta u)(t) \\ &\leq -C_2\psi(t)J(t)E(t) - 2C_3E'(t) - C_3J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 . \end{aligned} \tag{3.34}$$

By (2.13), we have

$$\begin{aligned} (\psi(t)\mathfrak{L}(t) + 2C_3E(t))' &\leq -C_2\psi(t)J(t)E(t) - C_3J'(t)\left(\int_0^t g(s)ds\right)\|\Delta u\|_2^2 \\ &\leq -\psi(t)J(t)\left[C_2 + \frac{2}{l\psi(t)J(t)}C_3J'(t)\left(\int_0^t g(s)ds\right)\right]E(t). \end{aligned} \tag{3.35}$$

From $\lim_{t\to+\infty}\frac{J'(t)}{\psi(t)J(t)} = 0$, we can choose $t_2 \geq t_1$ and then (3.35) gives

$$(\psi(t)\mathcal{L}(t) + 2C_3E(t))' \leq -\frac{C_2}{2}\psi(t)J(t)E(t), \quad \text{for all } t > t_2. \tag{3.36}$$

We define here, the function $\mathfrak{L}$ by

$$\mathfrak{L}(t) = \psi(t)\mathcal{L}(t) + 2C_3E(t). \tag{3.37}$$

By the definition of the functionals $F(t)$, $G(t)$, $\Phi(t)$ and $E(t)$, since $\psi'(t) \leq 0$, we can prove $\mathfrak{L}(t)$ equivalent to $E(t)$ and there exists a positive constant λ such that

$$\mathfrak{L}'(t) \leq -\lambda\psi(t)J(t)\mathfrak{L}(t), \quad \text{for all } t \geq t_2. \tag{3.38}$$

By simple integration of 3.38 over $[t_2, t]$ and use the equivalence of $\mathfrak{L}(t)$ and $E(t)$ we obtain

$$E(t) \leq Ce^{-K_0\int_{t_2}^t \psi(t)J(t)dt}, \quad \text{for all } t \geq t_2.$$

By the continuity and boundedness of $E(t)$ in the interval $[0, t_2]$, we have

$$E(t) \leq Ce^{-K_0\int_0^t \psi(t)J(t)dt}, \quad \text{for all } t \geq 0.$$

The proof of Theorem 3.1 is completed.

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RIESZ LACUNARY SEQUENCE SPACES OF FRACTIONAL DIFFERENCE OPERATOR

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Abstract. In this paper, we intend to make new approach to introduce and study some fractional difference sequence spaces by Riesz mean associated with infinite matrix and a sequence of modulus functions over n-normed spaces. Various algebraic and topological properties of these newly formed sequence spaces have been explored and some inclusion relations concerning these spaces are also establish. Finally, we make an effort to study the statistical convergence through fractional difference operator.

1. Introduction and Preliminaries

Baliarsingh and Dutta [1] introduced fractional difference operators $\Delta^{\tilde{\gamma}}$, $\Delta^{(\tilde{\gamma})}$, $\Delta^{-\tilde{\gamma}}$, $\Delta^{(-\tilde{\gamma})}$ and discussed some topological results among these operators. Meng and Mei [17] introduced binomial fractional difference sequence spaces by clubbing binomial matrix and fractional difference operator. Recently, Baliarsingh et al. [4] studied approximation theorems and statistical convergence in fractional difference sequence spaces. Also, double difference fractional order sequence spaces has been introduced by Baliarsingh in [5]. In [23] Nayak et al. introduced some weighted mean fractional difference sequence spaces. Kirişci and Kadak [15] proposed almost convergent fractional order difference sequence spaces. The reader can refer to the textbooks Başar [6] and Mursaleen [20] for relevant terminology and required details on the domain of triangles, sequence spaces and related topics. By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of natural, real and complex numbers respectively. Let w be the space of all real or

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complex sequences. For a proper fraction $\tilde{\gamma}$, defined the fractional difference operators $\Delta^{\tilde{\gamma}}: w \to w$, $\Delta^{(\tilde{\gamma})}: w \to w$ and their inverses are as follows:

$$\Delta^{\tilde{\gamma}}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\gamma}+1)}{i!\Gamma(\tilde{\gamma}+1-i)} x_{\nu+i}, \tag{1.1}$$

$$\Delta^{(\tilde{\gamma})}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\gamma}+1)}{i!\Gamma(\tilde{\gamma}+1-i)} x_{\nu-i}, \tag{1.2}$$

$$\Delta^{-\tilde{\gamma}}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\tilde{\gamma})}{i!\Gamma(1-\tilde{\gamma}-i)} x_{\nu+i}, \tag{1.3}$$

$$\Delta^{(-\tilde{\gamma})}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\tilde{\gamma})}{i!\Gamma(1-\tilde{\gamma}-i)} x_{\nu-i}. \tag{1.4}$$

We suppose that the series defined in (1.1)–(1.4) are convergent. For $\tilde{\gamma} = \frac{1}{2}$, we have

- $\Delta^{\frac{1}{2}} x_\nu = x_\nu - \frac{1}{2}x_{\nu+1} - \frac{1}{8}x_{\nu+2} - \frac{1}{16}x_{\nu+3} - \frac{5}{128}x_{\nu+4} - \frac{7}{256}x_{\nu+5} - \cdots;$
- $\Delta^{-\frac{1}{2}} x_\nu = x_\nu + \frac{1}{2}x_{\nu+1} + \frac{3}{8}x_{\nu+2} + \frac{5}{16}x_{\nu+3} + \frac{35}{128}x_{\nu+4} + \frac{63}{256}x_{\nu+5} + \cdots;$
- $\Delta^{(\frac{1}{2})} x_\nu = x_\nu - \frac{1}{2}x_{\nu-1} - \frac{1}{8}x_{\nu-2} - \frac{1}{16}x_{\nu-3} - \frac{5}{128}x_{\nu-4} - \frac{7}{256}x_{\nu-5} - \cdots;$
- $\Delta^{(-\frac{1}{2})} x_\nu = x_\nu + \frac{1}{2}x_{\nu-1} + \frac{3}{8}x_{\nu-2} + \frac{5}{16}x_{\nu-3} + \frac{35}{128}x_{\nu-4} + \frac{63}{256}x_{\nu-5} + \cdots$

For more details about fractional difference operator (see [3]). By $\Gamma(m)$, we denote the Gamma function of a real number m and $m \notin \{0, -1, -2, -3, \dots\}$. Now, by the definition it will be expressed as associate improper integral, i.e.,

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt. \tag{1.5}$$

It is clear from (1.5) if $m \in \mathbb{N}$, the set of nonnegative integers, then $\Gamma(m+1) = m!$. For this reason, Gamma function is considered to be a generalization of elementary factorial function. Currently, we tend to state some properties of Gamma function that are as follows:

(i) if $m \in \mathbb{N}$, then we have $\Gamma(m+1) = m!$;
(ii) if $m \in \mathbb{R} \setminus \{0, -1, -2, -3, \dots\}$, then we have $\Gamma(m+1) = m\Gamma(m)$;
(iii) for particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!, \dots$

Let U and V be two sequence spaces and $\mathcal{A} = (a_{n\nu})$ be an infinite matrix of real or complex numbers. Then we say that $\mathcal{A}$ defines a matrix transformation from U into V if for every sequence $x = (x_\nu) \in U$, the sequence $\mathcal{A}x = \{\mathcal{A}_n(x)\}$, the $\mathcal{A}$-transform of x is in V, where

$$\mathcal{A}_n(x) = \sum_\nu a_{n\nu} x_\nu, \quad n \in \mathbb{N}.$$

The idea of n-normed spaces was introduced by Misiak [19]. Let X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$ and $n \in \mathbb{N}$. A real valued function $||\cdot, \dots, \cdot||$ on X^n satisfying the following four conditions:

(i) $||\vartheta_1, \vartheta_2, \dots, \vartheta_n|| = 0$ if and only if $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are linearly dependent in X;
(ii) $||\vartheta_1, \vartheta_2, \dots, \vartheta_n||$ is invariant under permutation;
(iii) $||\beta\vartheta_1, \vartheta_2, \dots, \vartheta_n|| = |\beta| \ ||\vartheta_1, \vartheta_2, \dots, \vartheta_n||$ for any $\beta \in \mathbb{R}$;

(iv) $||\vartheta+\vartheta',\vartheta_2,\ldots,\vartheta_n|| \leq ||\vartheta,\vartheta_2,\ldots,\vartheta_n|| + ||\vartheta',\vartheta_2,\ldots,\vartheta_n||$ is called an n-norm on X and the pair $(X, ||\cdot,\ldots,\cdot||)$ is called a n-normed space over the field $\mathbb{R}$. For more definition and results on n-normed spaces see [13, 14, 22]. A sequence (x_ν) in a n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{\nu\to\infty} \|(x_\nu - L, z_1,\ldots,z_{n-1})\| = 0, \quad \text{for every } z_1,\ldots,z_{n-1} \in X.$$

A sequence (x_ν) in a n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{\nu,p\to\infty} \|(x_\nu - x_p, z_1,\ldots,z_{n-1})\| = 0, \quad \text{for every } z_1,\ldots,z_{n-1} \in X.$$

In a n-normed space $(X, \|\cdot,\ldots,\cdot\|)$, a sequence (x_ν) is said to be bounded if for a positive constant M, $\|(x_\nu, z_1,\ldots,z_{n-1})\| \leq M$ for all $z_1,\ldots,z_{n-1} \in X$.

Let (p_v) be a sequence of positive real numbers and $P_n = p_1 + p_2 + \cdots + p_n$ for all $n \in \mathbb{N}$. Thus, the Riesz transformation of $x = (x_\nu)$ is defined as

$$t_n = \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu x_\nu. \tag{1.6}$$

If the sequence (t_n) contains a finite limit L, then the sequence (x_ν) is said to be Riesz convergent to L. The set of all Riesz convergent sequence is denoted by (R, P_n). Let us note that if $P_n \to \infty$, as $n \to \infty$. Then Riesz mean is regular. If $p_\nu = 1$ for every natural number ν in (1.6), then Riesz mean reduces to Cesàro mean of order one.

An increasing non-negative integer sequence $\theta = (\nu_r)$ with $\nu_0 = 0$ and $\nu_r - \nu_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (\nu_{r-1}, \nu_r]$. We write $h_r = \nu_r - \nu_{r-1}$ and q_r denotes the ratio $\frac{\nu_r}{\nu_{r-1}}$. The space of lacunary strongly convergence was defined by Freedman et al. [10] as follows:

$$N_\theta = \Big\{ x = (x_\nu) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu\in I_r} |x_\nu - L| = 0 \text{ for some } L \Big\}.$$

The space N_θ is a BK-space with the norm

$$\|x\| = \sup\bigg(\frac{1}{h_r} \sum_{\nu\in I_r} |x_\nu| \bigg).$$

Let $\theta = (\nu_r)$ be a lacunary sequence and (p_ν) be a sequence of positive real numbers such that $H_r = \sum_{\nu\in I_r} p_\nu$, $P_{\nu_r} = \sum_{\nu\in(0,\nu_r]} p_\nu$, $P_{\nu_{r-1}} = \sum_{\nu\in(0,\nu_{r-1}]} p_\nu$, $Q_r = \frac{P_{\nu_r}}{P_{\nu_{r-1}}}$, $P_0 = 0$. Clearly, $H_r = P_{\nu_r} - P_{\nu_{r-1}}$ and the intervals determine by θ and (p_ν) are denoted by $I'_r = (P_{\nu_{r-1}}, P_{\nu_r}]$. If we take $p_\nu = 1$ for all $\nu \in \mathbb{N}$, then H_r, P_{ν_r}, $P_{\nu_{r-1}}$, Q_r and I'_r reduce to h_r, ν_r, ν_{r-1}, q_r and I_r, respectively.

A function $\psi : X \to \mathbb{R}$ is termed as paranorm, where X be a linear metric space, if following conditions are satisfied

(i) $\psi(x) \geq 0$ for all $x \in X$;

(ii) $\psi(-x) = \psi(x)$ for all $x \in X$;

(iii) $\psi(x+y) \leq \psi(x) + \psi(y)$ for all $x, y \in X$;

(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $\psi(x_n - x) \to 0$ as $n \to \infty$, then $\psi(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A function $\mathfrak{f} : [0, \infty) \to [0, \infty)$ is said to be modulus function if

(i) $\mathfrak{f}(\upsilon) = 0$ if and only if $\upsilon = 0$;

(ii) $\mathfrak{f}(\upsilon_1 + \upsilon_2) \leq f(\upsilon_1) + f(\upsilon_2)$ for all υ_1, υ_2;

(iii) $\mathfrak{f}$ is increasing;

(iv) $\mathfrak{f}$ is continuous from the right at 0.

The modulus function may be bounded or unbounded. Later, modulus function has been discussed in [21, 25–27, 29] and references therein.

Lemma 1.1. *Consider $\mathfrak{f} = (\mathfrak{f}_\nu)$ be a sequence of modulus functions and $0 < \rho < 1$. Then for each $x > \rho$, we have*

$$\mathfrak{f}_\nu(x) \leq \frac{2\mathfrak{f}_\nu(1)(x)}{\rho}.$$

For a proper fraction $\tilde{\gamma}$, let $\mathfrak{f} = (\mathfrak{f}_\nu)$ be a sequence of modulus functions, $q = (q_\nu)$ be a bounded sequence of strictly positive real numbers, $\mu = (\mu_\nu)$ be a sequence of strictly positive real numbers and θ be a lacunary sequence. In this paper we define the following sequence spaces as follows:

$$\begin{aligned} &[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\ =&\left\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu\left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] = 0\right\}, \\ &[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\ =&\left\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu\left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] = 0,\right. \\ &\left.\text{for some } L > 0\right\} \end{aligned}$$

and

$$\begin{aligned} &[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\ =&\left\{x = (x_\nu) \in w : \sup_r \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu\left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] < \infty\right\}. \end{aligned}$$

If the sequence $x = (x_\nu)$ is convergent to the limit L in

$$[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$$

we denote it by $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = L$.

Suppose $\mathfrak{f}(x) = x$. Then above spaces reduces to $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ and $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$.

By taking $q = (q_\nu) = 1$ for all $\nu \in \mathbb{N}$, then we get the spaces $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ and $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$.

Suppose $p_\nu = 1$ for all $\nu \in \mathbb{N}$, then we get the spaces as follows:

$$\begin{aligned}
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\
=&\Big\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} = 0\Big\}, \\
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\
=&\Big\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} = 0\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\
=&\Big\{x = (x_\nu) \in w : \sup_r \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} < \infty\Big\}.
\end{aligned}$$

Suppose (p_ν) be a sequence of positive numbers and $P_n = p_1 + p_2 + \ldots + p_n$. Now, we define the sequence spaces as follows:

$$\begin{aligned}
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\
=&\Big\{x = (x_\nu) \in w : \lim_{n\to\infty} \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] = 0\Big\}, \\
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\
=&\Big\{x = (x_\nu) \in w : \lim_{n\to\infty} \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] = 0\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\
=&\Big\{x = (x_\nu) \in w : \sup_n \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] < \infty\Big\}.
\end{aligned}$$

If $0 < q_\nu \leq \sup q_\nu = D$, $C = \max\{1, 2^{D-1}\}$. Then

$$|c_\nu + d_\nu|^{q_\nu} \leq C(|c_\nu|^{q_\nu} + |d_\nu|^{q_\nu}), \tag{1.7}$$

for every natural number ν and $c_\nu, d_\nu \in \mathbb{R}$.

The main purpose of this paper is to introduce and study some lacunary convergent sequence spaces defined by Riesz mean via modulus functions over $n-$normed spaces. We shall make an effort to study some interesting algebraic and topological properties of concerning sequence spaces. Also, we examine some interrelations between these sequence spaces.

2. Main Results

Theorem 2.1. *Suppose* $\mathfrak{f} = (\mathfrak{f}_\nu)$ *be a sequence of modulus functions,* $\Delta^{(\tilde{\gamma})}$ *be a fractional difference operator,* $\mu = (\mu_\nu)$ *be a sequence of positive real numbers and* $q = (q_\nu)$ *be a bounded sequence of positive real numbers. Then the sequence spaces* $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ *and* $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$ *are linear spaces over the field* $\mathbb{R}$ *of real numbers.*

Proof. Consider $x = (x_\nu), y = (y_\nu) \in [\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ and $\alpha, \beta \in \mathbb{R}$. Since f is additive and by using inequality (1.7), we have

$$\begin{aligned}
&\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} (\alpha x_\nu + \beta y_\nu), z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \Big\} \\
\leq &\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(|\alpha| \Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
&+ \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(|\beta| \Big\| \mu_\nu \Delta^{(\tilde{\gamma})} y_\nu, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
\leq &C \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
&+ C \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} y_\nu, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
\to &0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Hence, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ is a linear space. Similarly, we can prove others. ☐

Theorem 2.2. *Let* $\mathfrak{f} = (\mathfrak{f}_\nu)$ *be a sequence of modulus functions and* $q = (q_\nu)$ *be a bounded sequence of strictly positive real numbers. Then the sequence space* $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ *is a paranormed space with respect to the paranorm*

$$\psi(x) = \sup_r \left(\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \right)^{\frac{1}{M}},$$

where $M = \max\{1, \sup_\nu q_\nu < \infty\}$.

Proof. Consider $x = (x_\nu), y = (y_\nu) \in [\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$. Clearly, $\psi(x) \geq 0$ and $\psi(0) = 0$. Now, by using Minkowski's inequality, we get

$$\begin{aligned}
&\left(\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} (x_\nu + y_\nu), z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \right)^{\frac{1}{M}} \\
\leq &\left(\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \right)^{\frac{1}{M}}
\end{aligned}$$

$$+\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}y_\nu, z_1,\cdots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}.$$

Hence, $\psi(x+y)\leq\psi(x)+\psi(y)$.

Finally, we prove that the scalar multiplication is continuous. Let γ be any complex number. Then

$$\psi(\gamma x)=\sup_r\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\cdots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}\leq K_\gamma^{\frac{D}{M}}\psi(x),$$

where K_γ is a positive integer such that $\gamma\leq K_\gamma$. Now, let $\gamma\to 0$ for any fixed x with $\psi(x)\neq 0$. So, by using definition of $\mathfrak{f}$ for $|\gamma|<1$, we have

$$\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}<\epsilon,\quad\text{for } r>r_0(\epsilon). \tag{2.1}$$

Since $\mathfrak{f}$ is continuous and taking γ small enough, for $1\leq r\leq r_0$, we have

$$\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}<\epsilon. \tag{2.2}$$

Now, by combining (2.1) and (2.2) implies that $\psi(\gamma x)\to 0$ as $\gamma\to 0$. Thus, the space $[\mathfrak{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]_0$ is a paranormed space with respect to the paranorm $\psi(\cdot)$. $\square$

Theorem 2.3. *Suppose $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a sequence of modulus functions, $q=(q_\nu)$ be a bounded sequence of positive real numbers, $\mu=(\mu_\nu)$ be a sequence of positive real numbers and $\theta=(\nu_r)$ be a lacunary sequence such that* $\limsup_r Q_r<\infty$. *Then* $[\mathfrak{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]\subseteq[\mathfrak{R},\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.

Proof. Let $x=(x_\nu)\in[\mathfrak{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$. Then for every $\epsilon>0$ there exists i_0 such that for every $i>i_0$

$$\mathbf{A}_i=\frac{1}{H_i}\sum_{\nu\in I_i} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu-L, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]<\epsilon. \tag{2.3}$$

Then, there is some positive constant N such that

$$\mathbf{A}_i\leq N,\quad\text{for all } i. \tag{2.4}$$

Now, $\limsup_r Q_r<\infty$. Then, there exists some positive number K such that

$$Q_r\leq K,\quad\text{for all } r\geq 1. \tag{2.5}$$

Therefore, for $\nu_{r-1}<n\leq\nu_r$ and by (2.3), (2.4) and (2.5), we have

$$\begin{aligned}\frac{1}{P_n}\sum_{\nu=1}^{n}p_\nu y_\nu\leq&\frac{1}{P_{\nu_{r-1}}}\sum_{\nu=1}^{\nu_r}p_\nu y_\nu\\ =&\frac{1}{P_{\nu_{r-1}}}\left(\sum_{\nu\in I_1}p_\nu y_\nu+\sum_{\nu\in I_2}p_\nu y_\nu+\cdots+\sum_{\nu\in I_{i_0}}p_\nu y_\nu+\sum_{\nu\in I_{i_0+1}}p_\nu y_\nu+\right.\end{aligned}$$

$$\begin{aligned}
&\cdots + \sum_{\nu\in I_r} p_\nu y_\nu\Bigg)\\
=&\frac{1}{P_{\nu_{r-1}}}\left(\mathbf{A}_1 H_1 + \mathbf{A}_2 H_2 + \cdots + \mathbf{A}_{i_0} H_{i_0} + \mathbf{A}_{i_0+1} H_{i_0+1} + \cdots + \mathbf{A}_r H_r\right)\\
\leq&\frac{N}{P_{\nu_{r-1}}}\left(H_1 + H_2 + \cdots + H_{i_0}\right) + \frac{\epsilon}{P_{\nu_{r-1}}}\left(H_{i_0+1} + H_{i_0+2} + \cdots + H_r\right)\\
=&\frac{N}{P_{\nu_{r-1}}}\Big(P_{\nu_1} - P_{\nu_0} + P_{\nu_2} - P_{\nu_1} + \cdots + P_{\nu_{i_0}} - P_{\nu_{i_0-1}}\Big)\\
&+\frac{\epsilon}{P_{\nu_{r-1}}}\Big(P_{\nu_{i_0+1}} - P_{\nu_{i_0}} + P_{\nu_{i_0+2}} - P_{\nu_{i_0+1}} \cdots + P_{\nu_r} - P_{\nu_{r-1}}\Big)\\
=&\frac{NP_{\nu_{i_0}}}{P_{\nu_{r-1}}} + \frac{\epsilon(P_{\nu_r} - P_{\nu_{i_0}})}{P_{\nu_{r-1}}}\\
\leq&\frac{NP_{\nu_{i_0}}}{P_{\nu_{r-1}}} + \epsilon K,
\end{aligned}$$

where $y_\nu = a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}$. Now, $P_{\nu_{r-1}} \to \infty$ as $r \to \infty$, then we have $x \in [\mathfrak{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. This completes the proof. □

Corollary 2.1. *Let (p_ν) be sequence of positive numbers. If $1 < \liminf_r Q_r \leq \limsup_r Q_r < \infty$. Then*

$$[\mathfrak{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] = [\mathfrak{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|].$$

Theorem 2.4. *The following inclusions are true.*

(i) If $p_\nu < 1$ for all $\nu \in \mathbb{N}$, then

$$[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset [\mathfrak{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|],$$

with $[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]-\lim x = [\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]-\lim x = L.$

(ii) If $p_\nu > 1$ for all $\nu \in \mathbb{N}$ and $\frac{H_r}{h_r}$ be upper bounded. Then

$$[\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset [\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|],$$

with $[\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]-\lim x = [\mathfrak{C}_{\theta_r}, f, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]-\lim x = L.$

Proof. (i) Let $p_\nu < 1$ for all $\nu \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$. So, there exists a constant M_1 such that $0 < M_1 \leq \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$. Let $x = (x_\nu)$ be a sequence which converges to the limit L in $[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. Then for $\epsilon > 0$ we get

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
<&\frac{1}{M_1 h_r}\sum_{\nu\in I_r} a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}.
\end{aligned}$$

Now, we get the desired result by taking the limit as $r \to \infty$.

(ii) It is easy so we omit it. □

Theorem 2.5. *Suppose $\mathfrak{f}$ and $\mathfrak{f}'$ be two sequences of modulus functions. Then the following inclusions hold:*

(i) $[\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset [\mathcal{R}, \mathfrak{f} \circ \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$;

(ii) $[\mathcal{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \cap [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset \mathcal{R}, \mathfrak{f} + \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$.

Proof. Suppose $x = (x_\nu) \in [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. For given $\epsilon > 0$, choose $\rho \in (0, 1)$ such that $\mathfrak{f}_\nu(t) < \epsilon$ for all $0 < t < \rho$. Then we have

$$\begin{aligned}
&\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
=&\frac{1}{H_r} \sum_{\nu \in I_r, \left[\mathfrak{f}'_\nu \left(\left\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{q_\nu} < \rho} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, \\
&z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
&+ \frac{1}{H_r} \sum_{\nu \in I_r, \left[\mathfrak{f}'_\nu \left(\left\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{q_\nu} \geq \rho} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, \\
&z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
\leq&(\epsilon)^D + \max \left\{ 1, \left(\frac{2\mathfrak{f}_\nu(1)}{\rho} \right) \right\} \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big].
\end{aligned}$$

Thus, we get $x = (x_\nu) \in [\mathcal{R}, \mathfrak{f} \circ \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. This completes the proof.

(ii) Let

$$x = (x_\nu) \in [\mathcal{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \cap [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|].$$

Then, we have

$$\begin{aligned}
&\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu + \mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
\leq&C \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
&+ C \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}'_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
\to&0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Therefore, $(x_\nu) \in [\mathcal{R}, \mathfrak{f} + \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. This completes the proof. □

3. Statistical Convergence

The concept of statistical convergence was introduced independently by Fast [9] and Steinhaus [28]. Statistical convergence has been further studied by Connor [8], Fridy ([11], [12]), Miller [18], Balcerzak et al. [2], Y. Q. Cao and Xiaofei Qu [7] and others. In this section, we introduce some inclusion relation between $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$ and $[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.

Definition 3.1. A sequence $x=(x_\nu)$ is said to be $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$-convergent to L if for every $\epsilon>0$,

$$\frac{1}{H_r}\Big|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\|)\geq\epsilon\}\Big| = 0.$$

In this case, we write $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}-\lim x = L$ or $x_\nu\to LS_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$.

Theorem 3.1. *Let $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a sequence of modulus functions and $0<\inf_\nu q_\nu\leq q_\nu\leq\sup_\nu q_\nu = D<\infty$. Then $[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]\subset S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$.*

Proof. Consider $x=(x_\nu)\in[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$ and given $\epsilon>0$. Then for each $z_1,\ldots,z_{n-1}$, we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
=&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu-L,z_1,\ldots,z_{n-1}\|\geq\epsilon} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu-L,z_1,\ldots,z_{n-1}\|<\epsilon} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu-L,z_1,\ldots,z_{n-1}\|\geq\epsilon} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r}[\mathfrak{f}_\nu(\epsilon)]^{q_\nu}\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r}\min\Big\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu},[\mathfrak{f}_\nu(\epsilon)]^D\Big\}\\
\geq&R\frac{1}{H_r}\Big|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\|)\geq\epsilon\}\Big|,
\end{aligned}$$

where $R=\min\Big\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu},[\mathfrak{f}_\nu(\epsilon)]^D\Big\}$. Thus, $(x_\nu)\in S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$. □

Theorem 3.2. *Let $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a bounded sequence of modulus functions and $q=(q_\nu)$ be a bounded sequence of positive real numbers. If $0<\inf_\nu q_\nu\leq q_\nu\leq\sup_\nu q_\nu = D<\infty$, then $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}\subset[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.*

Proof. Suppose $x = (x_\nu) \in S_{[\mathfrak{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\dots,\cdot\|]}$ and $\epsilon > 0$ be given. Since $\mathfrak{f}$ is bounded, then there exists an integer J such that $\mathfrak{f}(x) < J$ for all $x > 0$, then we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
=&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|\ge\epsilon} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|<\epsilon} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
\le&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|\ge\epsilon} \max\left\{J^{\inf q_\nu}, J^D\right\}\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|<\epsilon} [\mathfrak{f}(\epsilon)]^{q_\nu}\\
\le&\max\left\{J^{\inf q_\nu}, J^D\right\}\frac{1}{H_r}\left|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\|)\ge\epsilon\}\right|\\
&+\max\left\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu}, [\mathfrak{f}_\nu(\epsilon)]^D\right\}.
\end{aligned}$$

Thus, $(x_\nu) \in [\mathfrak{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\dots,\cdot\|]$. This completes the proof. □

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GENERALIZED AVERAGED GAUSSIAN FORMULAS FOR CERTAIN WEIGHT FUNCTIONS

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Abstract. In this paper we analyze the generalized averaged Gaussian quadrature formulas and the simplest truncated variant for one of them for some weight functions on the interval $[0, 1]$ considered by Milovanović in [10]. We shall investigate internality of these formulas for the equivalents of the Jacobi polynomials on this interval and, in some special cases, show the existence of the Gauss-Kronrod quadrature formula. We also include some examples showing the corresponding error estimates for some non-classical orthogonal polynomials.

1. Introduction

Consider the l-point Gauss quadrature formula

$$Q_l^G(f) = \sum_{i=1}^{l} w_i^{(l)} f(x_i^{(l)})$$

on the interval $[a, b]$ with respect to a weight function w for the integral

$$I(f) = \int_a^b f(x)w(x)dx.$$

It has the highest possible degree of exactness, $2l - 1$, and

$$Q_l^G(p) = I(p), \quad p \in \mathcal{P}^{2l-1},$$

where $\mathcal{P}^m$ denotes the space of polynomials of degree up to m.

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To estimate the error $(I-Q_l^G)(f)$, one can use the difference $(A-Q_l^G)(f)$, where A is some quadrature formula of degree greater than $2l-1$. Any such quadrature formula A requires at least $l+1$ additional nodes, so it will have at least $2l+1$ nodes. One classical way for constructing a $(2l+1)$-node formula A for certain weight functions is Gauss-Kronrod quadrature formula with degree of exactness at least $3l+1$. The Gauss-Kronrod formulas are of optimal degree, given that the nodes of $G_w^{(l)}$ are included. For some weight functions on compact intervals, such as the Legendre weight function $w(x)=1$ on $[-1,1]$, the Gauss-Kronrod formulas have real zeros inside the interval that interlace with the nodes of the Gauss formula and have positive weights. The polynomials of degree $l+1$ that vanish in the $l+1$ additional nodes are called Stieltjes polynomials. However, a real Gauss-Kronrod extension of a Gauss formula may not exist in general. This happens e.g. for the Gauss-Laguerre and Gauss-Hermite cases (see [6]), as well as for the Jacobi weights $w^{\alpha,\beta}(t)=(1-t)^\alpha(1+t)^\beta$ for $\min\{\alpha,\beta\}\geqslant 0$ and $\max\{\alpha,\beta\}>5/2$ if l is large enough (see [13]).

Another approach (see [7, 8, 11]) is to construct a new quadrature formula Q_{l+1} for the functional

$$I_\theta(f)=\int_a^b f(x)w(x)dx-\theta Q_l^G(f),$$

for a given $\theta\in\mathbb{R}$, and then use the stratified quadrature formulas $Q_{2l+1}=\theta Q_n^G+Q_{l+1}$ to estimate the error Q_n^G. As a special case, Laurie in [8] introduced the anti-Gaussian quadrature formula Q_{l+1}^A

$$(I-Q_{l+1}^A)(p)=-(I-Q_l^G)(p),\quad p\in\mathcal{P}^{2l+1}.$$

The averaged formula

$$Q_{2l+1}^L=\frac{1}{2}(Q_l^G+Q_{l+1}^A),$$

also introduced in [8], is of the stratified type and has the degree of exactness at least $2l+1$. In the case of the Laguerre and Hermite weight functions, more general averaged formulas $\frac{1}{2+\gamma}((1+\gamma)Q_n^G+Q_{l+1}^A)$ with $\gamma>-1$ were considered in [4]. Here γ is chosen so that the degree of exactness is as large as possible. These modified formulas, denoted by Q_{2l+1}^{GF}, are also stratified extensions. Moreover, among all stratified extensions, these are the unique formulas with the maximum degree of exactness.

Recently, by following the results in [12] which characterize positive quadrature formulas, Spalević [16] introduced a new $(2l+1)$-node quadrature formula, called generalized averaged Gaussian quadrature formula. Here we denote it by Q_{2l+1}^S. In the cases of Laguerre and Hermite weight functions, this formula turns out to coincide with Q_{2l+1}^{GF}. The generalized averaged Gaussian formula has a degree of exactness at least $2l+2$, but for one class of weight functions the degree of exactness is $3n+1$ and hence the formula coincides with Gauss-Kronrod formula (see [18]). Further, the truncated generalized averaged Gauss formulas $Q_{2l-r+1}^{(l-r)}$ are introduced in [14], where $l\geqslant 2$ and $r=1,2,\ldots,l-1$. These formulas have fewer nodes and the same degree of exactness as the generalized averaged Gauss formulas. Hence, the truncated generalized averaged Gauss formulas can be useful as substitutes when (real) Gauss-Kronrod formula do not exist.

According to [8, 16] and [1], the generalized averaged Gaussian formulas and truncated variant for one of them have real nodes with positive weights, and only the two outermost nodes may be exterior. Thus it remains to analyze when these formulas are internal, i.e., all nodes are interior. This property is important when the integrand f is defined only on the interval $[a, b]$ and has also been investigated in [1, 2] and [3].

In this paper, we are analyzing mentioned averaged formulas for some weight functions recently considered by Milovanović in [10]. In two of these cases the orthogonal polynomials can be expressed in terms of the Jacobi polynomials on $[0, 1]$. For these, we will consider internality of the averaged formulas. In some simple cases of these polynomials, the generalized averaged Gaussian formulas coincide with the Gauss-Kronrod formula. The other two cases yield non-classical polynomials on $[0, 1]$, and in these cases we will give examples showing the error estimates for the Gauss formula.

2. The Extraction of Orthogonal Polynomials from Generating Function for Reciprocal of Odd Numbers

Let $\{\pi_l(x)\}_{l=0}^{\infty}$ be a sequence of monic polynomials orthogonal on $[a, b]$ with respect to the weight function $w(x)$. These polynomials satisfy the three-term recurrence relation

$$\pi_{l+1}(x) = (x - \alpha_l)\pi_l(x) - \beta_l \pi_{l-1}(x), \quad l = 0, 1, \dots, \tag{2.1}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$. Here α_l and β_l are the recurrence coefficients and it is convenient to define $\beta_0 = \int_a^b w(x)dx$. The same recurrence coefficients occur in the Jacobi continued fraction associated with the weight function $w(x)$,

$$F(x) = \int_a^b \frac{w(t)}{x - t} dt \sim \frac{\beta_0}{x - \alpha_0 -} \frac{\beta_1}{x - \alpha_1 -} \cdots,$$

which is known as the Stieltjes transform of the weight function $w(x)$. The l-th convergent of this continued fraction is

$$\frac{\beta_0}{x - \alpha_0 -} \frac{\beta_1}{x - \alpha_1 -} \cdots \frac{\beta_{l-1}}{x - \alpha_{l-1}} = \frac{\sigma_l(x)}{\pi_l(x)},$$

where $\sigma_l(x)$ are the associated polynomials,

$$\sigma_l(x) = \int_a^b \frac{\pi_l(x) - \pi_l(t)}{x - t} w(t) dt, \quad l \geqslant 0.$$

These polynomials satisfy the same recurrence relation (2.1), where $\sigma_0 = 0$ and $\sigma_{-1} = -1$ (see [9, pp. 111–114]).

Recently Shashikala [15] considered the series

$$T(x) = 1 + \frac{1}{3}x + \frac{1}{5}x^2 + \cdots + \frac{1}{2l+1}x^l + \cdots .$$

Using the regular continued fraction,

$$T(x) = \frac{1}{1+} \frac{-\frac{1}{3}x}{1+} \frac{-\frac{4}{15}x}{1+} \cdots \frac{-\frac{l^2}{4l^2-1}x}{1+} \cdots, \tag{2.2}$$

and taking even and odd convergents, he obtained four sequences of monic orthogonal polynomials $\{Q_l^{(\nu)}(x)\}_{l=0}^{\infty}$, $\nu = 1,2,3,4$. These polynomials satisfy the three-term recurrence relation (2.1), with $Q_0^{(\nu)}(x) = 1$ and $Q_1^{(1)}(x) = x - \frac{1}{3}$, $Q_1^{(2)}(x) = x - \frac{3}{5}$, $Q_1^{(3)}(x) = x - \frac{4}{15}$, $Q_1^{(4)}(x) = x - \frac{11}{21}$. The first two polynomials, extracted from the denominators of (2.2), are classical orthogonal polynomials (cf. [9, pp. 121–146]), whereas the other two, extracted from the numerators, are non-classical polynomials.

Let us consider the polynomials $p_l^{(1)}(x)$ and $p_l^{(2)}(x)$ orthogonal on $[0,1]$ with respect to the weight functions

$$w^{(1)}(x) = (1-x)^{\lambda-1/2}/\sqrt{x} \quad \text{and} \quad w^{(2)}(x) = \sqrt{x}(1-x)^{\lambda-1/2}, \quad \lambda > -1/2. \tag{2.3}$$

These polynomials satisfy the relation (2.1) with the recurrence coefficients (see [10])

$$\begin{aligned} a_0^{(1)} &= \frac{1}{2(\lambda+1)}, & a_l^{(1)} &= \frac{4l^2+4\lambda l+\lambda-1}{2(\lambda+2l-1)(\lambda+2l+1)}, \\ b_0^{(1)} &= \frac{\sqrt{\pi}\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}, & b_l^{(1)} &= \frac{l(2l-1)(\lambda+l-1)(2\lambda+2l-1)}{4(\lambda+2l-2)(\lambda+2l-1)^2(\lambda+2l)}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} a_0^{(2)} &= \frac{3}{2(\lambda+2)}, & a_l^{(2)} &= \frac{3\lambda+4l^2+4(\lambda+1)l}{2(\lambda+2l)(\lambda+2l+2)}, \\ b_0^{(2)} &= \frac{\sqrt{\pi}\Gamma(\lambda+1/2)}{2\Gamma(\lambda+2)}, & b_l^{(2)} &= \frac{l(2l+1)(\lambda+l)(2\lambda+2l-1)}{4(\lambda+2l-1)(\lambda+2l)^2(\lambda+2l+1)}. \end{aligned} \tag{2.5}$$

Actually, these polynomials are the (monic) Jacobi polynomials transformed to the interval $[0,1]$, with parameters $(\lambda-1/2, \mp 1/2)$, i.e.,

$$p_l^{(1)}(x) = \frac{1}{2^l} p_l^{(\lambda-1/2,-1/2)}(2x-1), \quad p_l^{(2)}(x) = \frac{1}{2^l} p_l^{(\lambda-1/2,1/2)}(2x-1), \tag{2.6}$$

where $p_l^{(\alpha,\beta)}$ are the monic Jacobi polynomials with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$ (see [9, pp. 131–140]).

Milovanović in [10] showed that, for $\lambda = 1/2$, the coefficients (2.4) and (2.5) reduce to the ones for the polynomials $Q^{(1)}(x)$ and $Q^{(2)}(x)$, respectively.

Let $a_l^{(\alpha,\beta)}$ and $b_l^{(\alpha,\beta)}$ be the recurrence coefficients for the monic Jacobi polynomials $p_l^{(\alpha,\beta)}$. It is easy to see that for $l \geqslant 1$ we have

$$a_l^{(1)} = \frac{a_l^{(\lambda-1/2,-1/2)}+1}{2}, \quad b_l^{(1)} = \frac{b_l^{(\lambda-1/2,-1/2)}}{4}, \tag{2.7}$$

$$a_l^{(2)} = \frac{a_l^{(\lambda-1/2,1/2)}+1}{2}, \quad b_l^{(2)} = \frac{b_l^{(\lambda-1/2,1/2)}}{4}. \tag{2.8}$$

We may also be interested in the cases $\lambda = 0$ and $\lambda = 1$. Let $T_l(x)$, $U_l(x)$, $V_l(x)$ and $W_l(x)$ be the Chebyshev polynomials of first, second, third and fourth kinds, respectively. For $\lambda = 0$ we get $p_l^{(1)}(x) = \frac{1}{2^l}T_l(2x-1)$ and $p_l^{(2)}(x) = \frac{1}{2^l}V_l(2x-1)$. Similarly, for $\lambda = 1$ we obtain $p_l^{(1)}(x) = \frac{1}{2^l}W_l(2x-1)$ and $p_l^{(2)}(x) = \frac{1}{2^l}U_l(2x-1)$. In each of these cases, the generalized averaged Gaussian quadrature formula coincides with the Gauss-Kronrod quadrature formula.

It was also proved in [10] that the polynomials $Q^{(3)}(x)$ and $Q^{(4)}(x)$ are orthogonal on $[0,1]$ with respect to the weight functions

$$w^{(3)}(x) = \frac{2/\sqrt{x}}{4(\tanh^{-1}\sqrt{x})^2 + \pi^2} \quad \text{and} \quad w^{(4)}(x) = \frac{2\sqrt{x}}{4(\tanh^{-1}\sqrt{x})^2 + \pi^2}, \tag{2.9}$$

respectively. The corresponding orthogonal polynomials are non-classical on $[0,1]$ and their respective recurrence coefficients are

$$a_0^{(3)} = \frac{4}{15}, \quad a_l^{(3)} = \frac{8l^2+12l+3}{(4l+1)(4l+5)}, \quad b_l^{(3)} = \frac{(2l)^2(2l+1)^2}{(4l-1)(4l+1)^2(4l+3)},$$

and

$$a_0^{(4)} = \frac{11}{21}, \quad a_l^{(4)} = \frac{8l^2+20l+11}{(4l+3)(4l+7)}, \quad b_l^{(4)} = \frac{(2l+1)^2(2l+2)^2}{(4l+1)(4l+3)^2(4l+5)}.$$

Later on we will present some examples showing the error estimates for the Gauss quadrature with respect to these weights using the mentioned averaged formulas.

3. The Generalized Averaged Gaussian Formula Q_{2l+1}^L

The generalized averaged Gaussian formula Q_{2l+1}^L, introduced in [8], is internal if the smallest zero x_1^π and the largest zero x_{l+1}^π of the polynomial

$$\pi_{l+1}(x) = p_{l+1}(x) - \beta_l p_{l-1}(x)$$

belong to the interval $[0,1]$ (see [8]). Here p_j, $j = 0,1,\ldots$, are the orthogonal polynomials and β_j, $j = 1,2,\ldots$, the recurrence coefficients corresponding to the original weight function. The largest zero x_{l+1}^π belongs to $[0,1]$ if and only if

$$\frac{p_{l+1}(1)}{\beta_l p_{l-1}(1)} \geqslant 1.$$

Similarly, the smallest zero x_1^π belongs to $[0,1]$ if and only if

$$\frac{p_{l+1}(0)}{\beta_l p_{l-1}(0)} \geqslant 1.$$

Obviously, the previous conditions are equivalent to the conditions for the Jacobi polynomials with the same parameters. Indeed, using (2.6)–(2.8), these conditions reduce to

$$\frac{p_{l+1}^{(\lambda-1/2,\mp 1/2)}(x)}{\beta_l^{(\lambda-1/2,\mp 1/2)} p_{l-1}^{(\lambda-1/2,\mp 1/2)}(x)} \geqslant 1,$$

where $x \in \{-1, 1\}$. Hence, Theorem 3 from [7] can be applied.

For the weight function $w^{(1)}(x)$, the conditions (18) and (19) from [7] reduce to

$$2\lambda^3 + (8l-1)\lambda^2 + (8l^2-1)\lambda \geqslant 0 \quad \text{and} \quad \lambda^2 - \lambda \geqslant 0,$$

respectively. The first condition obviously holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l (the leading coefficient in l in the latter case is negative). The second condition holds for $\lambda \in (-1/2, 0] \cup [1, \infty)$.

Similarly, for the weight function $w^{(2)}(x)$, the conditions (18) and (19) from [7] reduce to

$$2\lambda^3 + (8l+3)\lambda^2 + (8l^2+8l+1)\lambda \geqslant 0 \quad \text{and} \quad 8l^2 + (8\lambda+8)l + 3\lambda^2 + 3\lambda \geqslant 0,$$

respectively. The first condition holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. The second condition holds for $\lambda > -1/2$.

Thus we have the following result.

Theorem 3.1. *The generalized averaged Gaussian formula Q^L_{2l+1} for the weight functions $w^{(1)}(x)$ and $w^{(2)}(x)$ is internal when $\lambda \geqslant 1$ and $\lambda \geqslant 0$, respectively.*

4. The Generalized Averaged Gaussian Formula Q^S_{2l+1}

Consider the generalized averaged formula Q^S_{2l+1} introduced in [16]. This formula is internal if the smallest zero x^F_1 and the largest zero x^F_{l+1} of the polynomial

$$F_{l+1}(x) = p_{l+1}(x) - \beta_{l+1}p_{l-1}(x)$$

belong to the interval $[0, 1]$ (see [16]). Here p_j, $j = 0, 1, \ldots$, are the orthogonal polynomials and β_j, $j = 2, 3, \ldots$, the recurrence coefficients corresponding to the original weight function. The largest zero x^F_{l+1} belongs to $[0, 1]$ if and only if

$$\frac{p_{l+1}(1)}{\beta_{l+1}p_{l-1}(1)} \geqslant 1.$$

Similarly, the smallest zero x^F_1 belongs to $[0, 1]$ if and only if

$$\frac{p_{l+1}(0)}{\beta_{l+1}p_{l-1}(0)} \geqslant 1.$$

As for the formula Q^L_{2l+1}, the previous conditions reduce to ones for the corresponding Jacobi polynomials. So we use Theorem 3.1 from [17].

For the weight function $w^{(1)}(x)$, the conditions (3.5) and (3.6) from [17] reduce to

$$2\lambda^3 + (8l+3)\lambda^2 + (8l^2-5)\lambda \geqslant 0 \quad \text{and} \quad \lambda - \lambda^2 \geqslant 0.$$

The first condition holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. On the other hand, the second condition holds for $\lambda \in [0, 1]$.

For the weight function $w^{(2)}(x)$, the conditions (3.5) and (3.6) from [17] reduce to

$$2\lambda^3 + (8l+7)\lambda^2 + (8l^2+8l-3)\lambda \geqslant 0 \quad \text{and} \quad 8l^2 + (8\lambda+8)l + 7\lambda - \lambda^2 \geqslant 0.$$

The first condition obviously holds for $\lambda \geqslant 0$, but not for $\lambda \in (-1/2, 0)$ and sufficiently large l. The second condition holds for $\lambda \in (-1/2, 7)$, whereas for $\lambda \geqslant 7$ we have

$$8l^2 + (8\lambda + 8)l + 7\lambda - \lambda^2 > 8l^2 + 8\lambda l - \lambda^2 \geqslant 0, \quad \text{for } l \geqslant \frac{\sqrt{6}-2}{4}\lambda.$$

Hence, we have the following result.

Theorem 4.1. *The generalized averaged Gaussian formula Q^S_{2l+1} for the weight function $w^{(1)}(x)$ is internal when $\lambda \in [0, 1]$. In the case of the weight function $w^{(2)}(x)$, that formula is internal when $\lambda \in [0, 7)$. For $\lambda \geqslant 7$, internality occurs when $l \geqslant \frac{\sqrt{6}-2}{4}\lambda$.*

Now let us consider the cases $\lambda = 0$ and $\lambda = 1$, i.e., the polynomials $\frac{1}{2^l}T_l(2x-1)$, $\frac{1}{2^l}V_l(2x-1)$, $\frac{1}{2^l}W_l(2x-1)$ and $\frac{1}{2^l}U_l(2x-1)$. We have $\alpha_l = \alpha$ and $\beta_l = \beta > 0$ for $l \geqslant r$, where $r = 2$ for the polynomial $\frac{1}{2^l}T_l(2x-1)$ and $r = 1$ for the polynomials $\frac{1}{2^l}V_l(2x-1)$, $\frac{1}{2^l}W_l(2x-1)$ and $\frac{1}{2^l}U_l(2x-1)$. Hence, Theorem 3.1 from [18] can be applied and we have the following result.

Theorem 4.2. *For the weight function $w^{(1)}(x)$ with $\lambda = 0$ and $l \geqslant 3$, the quadrature formulas Q^L_{2l+1} and Q^S_{2l+1} have the algebraic degree of exactness at least $3l+1$. Hence, these formulas coincide with the corresponding Gauss-Kronrod quadrature formula and the monic polynomials $\pi_{l+1} \equiv F_{l+1}$ coincide with the corresponding monic Stieltjes polynomials. The same results hold for the the weight function $w^{(1)}(x)$, when $\lambda = 1$ and weight function $w^{(2)}(x)$ when $\lambda \in \{0, 1\}$ and $l \geqslant 1$.*

Using the previous fact, one has a simple method to compute the Gauss-Kronrod quadrature formula. The computation of the latter formula is more complicated in general (see [5]).

5. Truncated Generalized Averaged Gaussian Formulas

Let us consider the truncated generalized averaged Gaussian formulas $Q^{(l-r)}_{2l-r+1}$ ($l \geqslant 2$) introduced in [14] for $r = l-1$. This formula is internal if the smallest zero τ_1 and the largest zero τ_{l+2} of the polynomial

$$t_{l+2}(x) = (x - \alpha_{l-1})p_{l+1}(x) - \beta_{l+1}p_l(x) \tag{5.1}$$

belong to the interval $[0, 1]$ (see [1]). Here p_j, $j = 2, 3, \ldots$, are the orthogonal polynomials and α_j, $j = 1, 2, \ldots$, and β_j, $j = 3, 4, \ldots$, the recurrence coefficients corresponding to the original weight function.

Obviously, in the case of the weight functions given in (2.3), the polynomials (5.1) have two outermost zeros inside the interval $[0, 1]$ if and only if the corresponding polynomials for the Jacobi weight functions with the same parameters have two outermost zeros inside the interval $[-1, 1]$. Using Theorem 3.4 from [1], we have that internality holds for $l \geqslant 3$.

Let $l = 2$. In the case of the weight function $w^{(1)}(x)$, the conditions (3.12) and (3.13) from [1] reduce to

$$-\lambda^3 + 19\lambda^2 + 105\lambda + 45 \geqslant 0 \quad \text{and} \quad 2\lambda^4 + 25\lambda^3 + 81\lambda + 63\lambda + 45 \geqslant 0.$$

The first condition holds for $\lambda \in [\lambda_1, \lambda_2]$, where $\lambda_1 \approx -0.46943$ and $\lambda_2 \approx 23.54142$ are the largest two zeros of the polynomial $-x^3+19x^2+105x+45$. The second condition holds for $\lambda > -1/2$.

Similarly, for the weight function $w^{(2)}(x)$, this formula is internal if and only if

$$\lambda^3 + 48\lambda^2 + 260\lambda + 216 \geqslant 0 \quad \text{and} \quad 2\lambda^4 + 31\lambda^3 + 136\lambda^2 + 188\lambda + 168 \geqslant 0.$$

These conditions hold for $\lambda > -1/2$.

Theorem 5.1. *The truncated generalized averaged Gaussian formula for the weight function $w^{(1)}(x)$ is internal when $\lambda > -1/2$ and $l \geqslant 3$. For $l = 2$ internality holds when $\lambda \in [\lambda_1, \lambda_2]$, where $\lambda_1 \approx -0.46943$ and $\lambda_2 \approx 23.54142$ are the largest two zeros of the polynomial $-x^3+19x^2+105x+45$. For the weight function $w^{(2)}(x)$ this formula is internal when $\lambda > -1/2$.*

6. Numerical Results

Example 6.1. We illustrate Theorems 3.1, 4.1 and 5.1 through some computations in the case of the weight function $w^{(2)}$ for some values of l and λ. In the considered cases, the corresponding averaged formulas are internal.

Table 1 displays the values of the nodes x_1^π and x_{l+1}^π for the formula Q_{2l+1}^L.

Table 2 displays the values of the nodes x_1^F and x_{l+1}^F for the formula Q_{2l+1}^S. Note that for $\lambda = 1$ this formula coincides with the previous one, and also with the Gauss-Kronrod quadrature formula (see Theorem 4.2).

Table 3 displays the values of the nodes τ_1 and τ_{l+2} for the formula $Q_{l+2}^{(1)}$.

Table 1: The values of x_1^π and x_{l+1}^π for $w^{(2)}$ and some l and λ.

λ	l	x_1^π	x_{l+1}^π
0.5	5	1.84918630347802(−2)	9.93315648803352(−1)
	10	5.32426071493249(−3)	9.98085997371715(−1)
	15	2.48373203616388(−3)	9.99108179903793(−1)
	20	1.43168514326074(−3)	9.99486155846300(−1)
1	5	1.70370868554659(−2)	9.82962913144534(−1)
	10	5.08927905953363(−3)	9.94910720940466(−1)
	15	2.40763666390156(−3)	9.97592363336098(−1)
	20	1.39810140940993(−3)	9.98601898590590(−1)

Table 2: The values of x_1^F and x_{l+1}^F for $w^{(2)}$ and some l and λ.

λ	l	x_1^F	x_{l+1}^F
0.5	5	1.85485046684558(−2)	9.93270563061661(−1)
	10	5.32892821283948(−3)	9.98082336550544(−1)
	15	2.48474645373049(−3)	9.99107386760496(−1)
	20	1.43202203935648(−3)	9.99485892741121(−1)
1	5	1.70370868554659(−2)	9.82962913144534(−1)
	10	5.08927905953363(−3)	9.94910720940466(−1)
	15	2.40763666390156(−3)	9.97592363336098(−1)
	20	1.39810140940993(−3)	9.98601898590590(−1)

Table 3: The values of τ_1 and τ_{l+2} for $w^{(2)}$ and some l and λ.

λ	l	τ_1	τ_{l+2}
0.5	5	4.05074383379349(−2)	9.76146311190531(−1)
	10	1.50966909367400(−2)	9.91134246875255(−1)
	15	7.80960712033176(−3)	9.95418464436467(−1)
	20	4.75922686471797(−3)	9.97209253940011(−1)
1	5	3.80602337443566(−2)	9.61939766255643(−1)
	10	1.45290912869740(−2)	9.85470908713026(−1)
	15	7.59612349389597(−3)	9.92403876506104(−1)
	20	4.65702698183462(−3)	9.95342973018165(−1)

Example 6.2. We find the outermost nodes in the case of the weight function $w^{(1)}$ for the formula Q_{2l+1}^L with $\lambda = 0.5$ (Table 4) and for the formula Q_{2l+1}^S with $\lambda = -0.2$ (Table 5) for some l. Here these formulas have exterior node(s).

Table 4: The values of x_1^π and x_{l+1}^π for $w^{(1)}$, $\lambda = 0.5$ and some l.

λ	l	x_1^π	x_{l+1}^π
0.5	5	−1.03583467673738(−5)	9.91983668229218(−1)
	10	−7.09110640371522(−7)	9.97894782375997(−1)
	15	−1.44570778097492(−7)	9.99048751274800(−1)
	20	−4.64835853269242(−8)	9.99460470025489(−1)

Table 5: The values of x_1^F and x_{l+1}^F for $w^{(2)}$, $\lambda = -0.2$ and some l.

λ	l	x_1^F	x_{l+1}^F
−0.2	5	−4.13229856738924(−5)	1.00140197341566
	10	−2.37471751038235(−6)	1.00033417984287
	15	−4.59266799101858(−7)	1.00014681665572
	20	−1.43959966526914(−7)	1.00008217031089

Example 6.3. Consider the integral

$$I(f) = \int_0^1 f(t)w(t)dt,$$

where $f(t) = 999.1^{\log_{10}(\varepsilon+t)}$, $\varepsilon = 10^{-6}$ and $w(t) = w^{(2)}(t)$. In Table 6, the estimation of the errors $|I(f) - Q_l^G(f)|$ for Gauss quadrature formula are obtained by means of the quantities $E_{LG} = \left|Q_{2l+1}^L(f) - Q_l^G(f)\right|$, $E_{SG} = |Q_{2l+1}^S(f) - Q_l^G(f)|$ and $E_{TSG} = |Q_{l+2}^{(1)}(f) - Q_l^G(f)|$, for some l and λ. As in the previous example, $Q_{2l+1}^L \equiv Q_{2l+1}^S$ for $\lambda = 1$. The sharp errors are denoted by Error.

Table 6: The estimates E_{LG}, E_{SG}, E_{TSG} and the sharp errors Error for some l and λ.

λ	l	E_{LG}	E_{SG}	E_{TSG}	Error
0.5	5	1.5198(−10)	1.5192(−10)	1.4323(−10)	1.5219(−10)
	10	4.3114(−13)	4.3106(−13)	3.4123(−13)	4.3190(−13)
	15	1.3219(−14)	1.3218(−14)	8.7665(−15)	1.3244(−14)
	20	1.0866(−15)	1.0865(−15)	6.1493(−16)	1.0886(−15)
1	5	1.1092(−10)	1.1092(−10)	1.0410(−10)	1.1108(−10)
	10	3.5846(−13)	3.5846(−13)	2.8175(−13)	3.5911(−13)
	15	1.1599(−14)	1.1599(−14)	7.6384(−15)	1.1621(−14)
	20	9.8190(−16)	9.8190(−16)	5.5211(−16)	9.8378(−16)

Note that the integrand in the previous example is not defined for some nodes in Example 6.2.

Example 6.4. The next table displays the same estimations as in the previous example for the integrand $f(t) = e^{3t}\sin 10t$ and the weight function $w(t) = w^{(3)}(t)$ from (2.9). Note that for the weight functions given in (2.9), the corresponding orthogonal polynomials are non-classical. Thus there is no analytical expression for the orthogonal polynomials. Consequently, there is no general claim for internality of the averaged formulas.

Table 7: The estimates E_{LG}, E_{SG}, E_{TSG} and the sharp errors Error for some l.

l	E_{LG}	E_{SG}	E_{TSG}	Error
5	3.4273(−3)	3.4276(−3)	3.4209(−3)	3.4276(−3)
10	8.4359(−11)	8.4359(−11)	8.4340(−11)	8.4359(−11)
15	9.6941(−21)	9.6941(−21)	9.6934(−21)	9.6941(−21)
20	3.1798(−32)	3.1798(−32)	3.1797(−32)	3.1798(−32)

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(m, n)-HYPERFILTERS IN ORDERED SEMIHYPERGROUPS

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Abstract. First, we generalize concepts of left hyperfilters, right hyperfilters and hyperfilters of an ordered semihypergroup by introducing concepts of left-m-hyperfilters, right-n-hyperfilters and (m, n)-hyperfilters of an ordered semihypergroup. Then, some properties of these generalized hyperfilters have been studied. Finally, left-m-hyperfilters (resp. right-n-hyperfilters, (m, n)-hyperfilters) of $(m, 0)$-regular (resp. $(0, n)$-regular, (m, n)-regular) ordered semihypergroups characterize in terms of their completely prime generalized $(m, 0)$-hyperideals (resp. $(0, n)$-hyperideals, (m, n)-hyperideals).

1. Introduction and Preliminaries

In 1934, Marty [12] introduced the concept of hyperstructure and defined the notion of hypergroup. The beauty of hyperstructure is that in hyperstructures multiplication of two elements is a set while in classical algebraic structures, the multiplication of two elements is an element which is the main reason for the researcher to attract towards such type of algebraic structures. Thus, the notion of algebraic hyperstructures is a generalization of classical notion of algebraic structures. The concept of ordered semihypergroup is a generalization of the concept of ordered semigroup and was introduced by Heidari and Davvaz in [6]. Thereafter it was studied by several authors. Davvaz et al. [1, 2, 6, 13] studied some properties of hyperideals, bi-hyperideals and quasi-hyperideals in ordered semihypergroups. The notion of (m, n)-ideals of semigroups was introduced by Lajos [10] as a generalization of the notion of bi-ideals in semigroups. In [9], authors introduced the notion of an (m, n)-quasi-hyperideal

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and proved different characterizations of (m,n)-quasi-hyperideals and minimal (m,n)-quasi-hyperideals in semihypergroups.

In 1987, Kehayopulu [7] introduced the concept of filter on *poe*-semigroups. Later on in 1990, Kehayopulu [8] defined the relation $\mathcal{N}$ on *po*-semigroup. The study of left(right)-filter on *po*-semigroup initiated by S. K. Lee and S. S. Lee [11], and gave some characterizations of the left(right)-filter of *po*-semigroup in term of the right(left) prime ideals. In 2015, the notion of left hyperfilters, right hyperfilters and hyperfilters of ordered semihypergroups introduced by Tang et al. [14] and also investigated their related properties and characterized hyperfilters in terms of completely prime hyperideals in ordered semihypergroups. In 2016, Omidi and Davvaz [13] defined an equivalence relation $\mathcal{N}$ as follows. Let H be an ordered semihypergroup. Then, $\mathcal{N} = \{(a,b) \in H \times H \mid N(a) = N(b)\}$, where $N(a)$ denote the hyperfiter of H generated by an element a of H, and also shown that $\mathcal{N}$ is the intersection of the semilattice equivalence relation $\sigma_P = \{(a,b) \in H \times H \mid a,b \in P \text{ or } a,b \notin P\}$, where P is completely prime hyperideal of H. Recently, Gu and Tang [4] constructed a strongly ordered regular equivalence relation on an ordered semihypergroup by using the concept of hyperfilter and shown that the corresponding quotient structure is a semilattice.

A hyperoperation on a non-empty set H is a map $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ (the set of all non-empty subsets of H). In such case, the H is called a hypergroupoid. Let H be a hypergroupoid, A and B be any non-empty subsets of H. Then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

We shall write, in whatever follows, $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ instead of $\{x\} \circ A$ for any $x \in H$. Also, for simplicity, throughout the paper, we denote $a \circ a \circ \cdots \circ a$ (m-copies of a) with a^m for all $a \in H$ and $m \in \mathbb{Z}$. Moreover, the hypergroupoid H is called a semihypergroup if, for all $x, y, z \in H$,

$$(x \circ y) \circ z = x \circ (y \circ z),$$

i.e.,

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset T of semihypergroup H is called a subsemihypergroup of H if $T \circ T \subseteq T$.

Definition 1.1 ([14])**.** Let H be a non-empty set. The triplet $(H, \circ, \leq)$ is called an ordered semihypergroup if $(H, \circ)$ is a semihypergroup and $(H, \leq)$ is a partially ordered set such that

$$x \leq y \Rightarrow x \circ z \leq y \circ z \quad \text{and} \quad z \circ x \leq z \circ y,$$

for all $x, y, z \in H$. Here, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Let H be an ordered semihypergroup. For a non-empty subset T of H, we denote $(T] = \{x \in H \mid x \leq a \text{ for some } a \in T\}$.

Definition 1.2. Let H be an ordered semihypergroup and A be a non-empty subset of H. Then, A is called a left (resp. right) hyperideal [2] of H if

(1) $H \circ A \subseteq A$ (resp. $A \circ H \subseteq A$);
(2) $(A] \subseteq A$.

A is called hyperideal of H if A is both left hyperideal and right hyperideal of H.

A subsemihypergroup F of ordered semihypergroup H is called left hyperflter (resp. right hyperfilter) [14] if for any $a, b \in H$, $a \circ b \cap F \neq \emptyset$ implies $a \in F$ (resp. $b \in F$) and for any $a \in F, b \in H$ such that $a \leq b$ implies $b \in F$. If F is both left-hyperflter and right-hyperflter of H, then F is said to be hyperfilter of H.

An ordered semihypergroup H is called regular (left regular, right regular) [2] if for each $x \in H$, $x \in (x \circ H \circ x]$ ($x \in (H \circ x \circ x]$, $x \in (x \circ x \circ H]$).

Lemma 1.1 ([2])**.** *Let H be an ordered semihypergroup and A, B be any non-empty subsets of H. Then the following hold:*

(1) $A \subseteq (A]$;
(2) $A \subseteq B$ $(A] \subseteq (B]$;
(3) $(A] \circ (B] \subseteq (A \circ B]$;
(4) $((A] \circ (B]] = (A \circ B]$;
(5) $(A] \cup (B] = (A \cup B]$.

Throughout this paper, H always denotes an ordered semihypergroup and m, n denote positive integers, unless otherwise specified.

2. Main Results

Definition 2.1. A subsemihypergroup F of ordered semihypergroup H is called left-m-hyperfilter (resp. right-n-hyperfilter) if

(1) for any $a, b \in S$, $a \circ b \cap F \neq \emptyset$ implies $a^m \subseteq F$ (resp. $b^n \subseteq F$);
(2) $a \in F$, $a \leq b \in S$ implies $b \in F$.

If F is both left-m-hyperfilter and right-n-hyperfilter of H, then F is called (m, n)-hyperfilter.

Remark 2.1. In particular for $m = 1$ (resp. $n = 1$), F is a left hyperfilter (resp. right hyperfilter). Clearly, each left hyperfilter (resp. right hyperfilter, hyperfilter) of an ordered semihypergroup H is left-m-hyperfilter (resp. right-n-hyperfilter, (m, n)-hyperfilter) for each positive integers m and n. Indeed let F be any hyperfilter of H and $a, b \in H$ such that $a \circ b \cap F \neq \emptyset$. As F is left hyperfilter, $a \in F$. Since F is left hyperfilter, F is subsemihypergroup, and thus $a^m \subseteq F$. Therefore, the concept of a left-m-hyperfilter (resp. right-n-hyperfilter, (m, n)-hyperfilter) is the generalization of a left hyperfilter (resp. right hyperfilter, hyperfilter). Conversely, each left-m-hyperfilter (resp. right-n-hyperfilter, (m,n)-hyperfilter) need not be a left hyperfilter (resp. right hyperfilter, hyperfilter).

Example 2.1. Let $H=\{a,b,c,d\}$. Define hyperoperation $\circ$ and order $\leq$ on H as follows:

$$\begin{array}{c|cccc}
\circ & a & b & c & d \\
\hline
a & \{a,b\} & \{a,b\} & \{a,b\} & \{a,b\} \\
b & \{a,b\} & \{a,b\} & \{a,b\} & \{a,b\} \\
c & \{a,b\} & \{a,b\} & \{a,b\} & \{b\} \\
d & \{a,b\} & \{a,b\} & \{b\} & \{c\}
\end{array},$$

$$\leq:=\{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Then H is an ordered semihypergroup. Let $F=\{a,b\}$. Since $d\circ c\cap F\neq\emptyset$ but $d\notin F$ while $d^3\subseteq F$. Therefore, F is a left-3-hyperfilter of H but not a left hyperfilter of H.

Lemma 2.1. *Let H be an ordered semihypergroup and T be a subsemihypergroup of H. Then, for every left-m-hyperfilter* (*resp. right-n-hyperfilter*) *F of H, either $F\cap T=\emptyset$ or $F\cap T$ is a left-m-hyperfilter* (*resp. right-n-hyperfilter*) *of T.*

Proof. Let $F\cap T\neq\emptyset$ and $x,y\in F\cap T$. Then, $x,y\in F,T$. As F and T are left-m-hyperfilter and subsemihypergroup of H, respectively. So $x\circ y\subseteq F$ and $x\circ y\subseteq T$. Thus, $x\circ y\subseteq F\cap T$. Next, we assume that for any $x,y\in T$, $x\circ y\cap(F\cap T)\neq\emptyset$. Therefore, $x\circ y\cap F\neq\emptyset$. Since $x,y\in H$ and F is left-m-hyperfilter of H, $x^m\subseteq F$. Also $x^m\subseteq T$. Thus, $x^m\subseteq(F\cap T)$. Finally, take an element $x\in T\cap F$ and $y\in T$ such that $x\leq y$. As F is left-m-hyperfilter of H and $F\ni x\leq y\in H$, $y\in F$. Therefore, $y\in T\cap F$. Hence, $F\cap T$ is a left-m-hyperfilter of T. □

Corollary 2.1. *Let H be an ordered semihypergroup and T be a subsemihypergroup of H. Then for every (m,n)-hyperfilter F of H, either $F\cap T=\emptyset$ or $F\cap T$ is an (m,n)-hyperfilter of T.*

Lemma 2.2. *Let H be an ordered semihypergroup and $\{F_i\mid i\in I\}$ be a family of left-m-hyperfilters* (*resp. right-n-hyperfilters*) *of H. If $\bigcap_{i\in I}F_i\neq\emptyset$, then $\bigcap_{i\in I}F_i$ is a left-m-hyperfilter* (*resp. right-n-hyperfilter*) *of H.*

Proof. Assume that $\bigcap_{i\in I}F_i\neq\emptyset$ and $x,y\in\bigcap_{i\in I}F_i$. Then $x,y\in F_i$ for each $i\in I$. As for each $i\in I$, F_i is left-m-hyperfilter, $x\circ y\subseteq F_i$. Therefore, $x\circ y\subseteq\bigcap_{i\in I}F_i$. Thus, $\bigcap_{i\in I}F_i$ is a subsemihypergroup of H. Now, let $x,y\in H$ and $x\circ y\subseteq\bigcap_{i\in I}F_i$. Therefore, $x\circ y\subseteq F_i$ for each $i\in I$. As F_i's are left-m-hyperfilters, $x^m\subseteq F_i$ for each $i\in I$. So, $x^m\subseteq\bigcap_{i\in I}F_i$. Now take an element $a\in\bigcap_{i\in I}F_i$ and $b\in H$ such that $a\leq b$. Then $a\in F_i$ for each $i\in I$. Since F_i's are left-m-hyperfilters, $b\in\bigcap_{i\in I}F_i$. Hence, $\bigcap_{i\in I}F_i$ is a left-m-hyperfilter. □

Corollary 2.2. *Let H be an ordered semihypergroup and $\{F_i\mid i\in I\}$ be a family of (m,n)-hyperfilters of H. If $\bigcap_{i\in I}F_i\neq\emptyset$, then $\bigcap_{i\in I}F_i$ is an (m,n)-hyperfilter of H.*

Remark 2.2. Union of any family of left-m-hyperfilters (resp. right-n-hyperfilters, (m, n)-hyperfilters) of ordered semihypergroup H is not a left-m-hyperfilter (resp. right-n-hyperfilter, (m, n)-hyperfilter) in general.

Following example shows that in general union of any family of left-m-hyperfilters (resp. right-n-hyperfilters, (m, n)-hyperfilters) of ordered semihypergroup H is not a left-m-hyperfilter (resp. right-n-hyperfilter, (m, n)-hyperfilter).

Example 2.2. Let $H = \{a, b, c, d, e\}$. Define hyperoperation $\circ$ and an order $\leq$ on H as follows:

$$\begin{array}{c|ccccc} \circ & a & b & c & d & e \\ \hline a & \{b\} & \{b\} & \{d\} & \{d\} & \{d\} \\ b & \{b\} & \{b\} & \{d\} & \{d\} & \{d\} \\ c & \{d\} & \{d\} & \{c,e\} & \{d\} & \{c,e\} \\ d & \{d\} & \{d\} & \{d\} & \{d\} & \{d\} \\ e & \{d\} & \{d\} & \{c,e\} & \{d\} & \{c,e\} \end{array},$$

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (c, e)\}.$$

Then H is an ordered semihypergroup. Here $F_1 = \{b\}$ is left-2-hyperfilter because $F_1 \circ F_1 \subseteq F_1$ and $a \circ a \cap F_1 \neq \emptyset$ implies $a^2 \subseteq F_1$. Thus F_1 is left-2-hyperfilter but not a hyperfilter. Similarly, $F_2 = \{c, e\}$ is left-2-hyperfilter. Now $F_1 \cup F_2 = \{b, c, e\}$, since $b \circ c = \{d\} \nsubseteq F_1 \cup F_2$, therefore $F_1 \cup F_2$ is not a subsemihypergroup of H, and hence $F_1 \cup F_2$ is not a left-2-hyperfilter.

Let $(H, \bullet, \leq_H)$ and $(T, \circ, \leq_T)$ be two ordered semihypergroups. Under the coordinatewise multiplication

$$(h_1, t_1) \diamond (h_2, t_2) = h_1 \bullet h_2 \times t_1 \circ t_2,$$

where $(s_1, t_1), (s_2, t_2) \in H \times T$ the cartesian product $H \times T$ of H and T forms a semihypergroup. Define a partial order $\leq$ on $H \times T$ by $(h_1, t_1) \leq (h_2, t_2)$ if and only if $h_1 \leq_H h_2$ and $t_1 \leq_T t_2$, where $(h_1, t_1), (h_2, t_2) \in H \times T$. Then, $(H \times T, \diamond, \leq)$ is an ordered semihypergroup [4].

Lemma 2.3. *Let $(H, \bullet, \leq_H)$ and $(T, \circ, \leq_T)$ be two ordered semihypergroups, F_1 and F_2 be two left-m-hyperfilter (resp. right-n-hyperfilter) of H and T, respectively. Then $F_1 \times F_2$ is a left-m-hyperfilter (resp. right-n-hyperfilter) of $H \times T$.*

Proof. Let $(a, b), (c, d) \in F_1 \times F_2$. Now $(a, b) \diamond (c, d) = a \bullet c \times b \circ d$. As $a, c \in F_1$, $b, d \in F_2$ and F_1, F_2 are left-m-hyperfilters of H and T respectively, $a \bullet c \subseteq F_1, b \circ d \subseteq F_2$. Therefore, $a \bullet c \times b \circ d \subseteq F_1 \times F_2$, it follows that $F_1 \times F_2$ is a subsemihypergroup of $H_1 \times H_2$. Next, we assume that $(a, b), (c, d) \in H_1 \times H_2$ such that $(a, b) \diamond (c, d) \cap F_1 \times F_2 \neq \emptyset$.

Now, we have

$$
\begin{aligned}
&(a,b)\diamond(c,d)\cap F_1\times F_2\neq\emptyset\\
\Rightarrow &a\bullet c\times b\circ d\cap F_1\times F_2\neq\emptyset\\
\Rightarrow &a\bullet c\cap F_1\neq\emptyset \text{ and } b\circ d\cap F_2\neq\emptyset\\
\Rightarrow &a^m\subseteq F_1 \text{ and } b^m\subseteq F_2\\
\Rightarrow &(a^m,b^m)\subseteq F_1\times F_2\\
\Rightarrow &(a,b)^m\subseteq F_1\times F_2.
\end{aligned}
$$

Finally, we consider an element $(a,b)\in F_1\times F_2$ and $(c,d)\in H\times T$ such that $(a,b)\leq(c,d)$. Therefore, $a\leq_H c$ and $b\leq_T d$. Since F_1 and F_2 are left-m-hyperfilters of H and T, $c\in F_1$ and $d\in F_2$. Thus, $(c,d)\in F_1\times F_2$. Hence, $F_1\times F_2$ is a left-m-hyperfilter of $H\times T$. □

Corollary 2.3. *Let $(H,\bullet,\leq_H)$ and $(T,\circ,\leq_T)$ be two ordered semihypergroups, F_1 and F_2 be two (m,n)-hyperfilters of H and T, respectively. Then $F_1\times F_2$ is an (m,n)-hyperfilter of $H\times T$.*

Definition 2.2 ([9])**.** Let H be an ordered semihypergroup, m and n be the positive integers. Then, a subsemihypergroup A of H is called an (m,n)-hyperideal of H if

(1) $A^m\circ H\circ A^n\subseteq A$ and
(2) $(A]\subseteq A$.

Dually, we may define $(m,0)$-hyperideal and $(0,n)$-hyperideal of H.

If we drop the subsemihypergroup condition from the above definition, then A is called a generalized (m,n)-hyperideal of H. Similarly, a generalized $(m,0)$-hyperideal and a generalized $(0,n)$-hyperideal are defined.

Remark 2.3. It is easy to check that each (m,n)-hyperideal (resp. $(m,0)$-hyperideal, $(0,n)$-hyperideal) of any ordered semihypergroup is always a generalized (m,n)-hyperideal (resp. $(m,0)$-hyerideal, $(0,n)$-hyperideal), but the converse is not true in general. This has been shown by the following example.

Example 2.3. Let $H=\{a,b,c,d\}$. Define hyperoperation $\circ$ and order $\leq$ on H as follows:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$
d	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$

,

$$\leq:=\{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Then H is an ordered semihypergroup. The subset $\{a,d\}$ of H is a generalized (m,n)-hyperideal of H, for all integers $m,n\geq 2$, which is not an (m,n)-hyperideal of H.

A generalized $(m,0)$-hyperideal (resp. generalized $(0,n)$-hyperideal, generalized (m,n)-hyperideal) A of an ordered semihypergroup H is called completely prime if for any two elements $a,b \in H$ such that $a \circ b \cap A \neq \emptyset$, then $a \in A$ or $b \in A$.

Let H be an ordered semihypergroup and m,n positive integers. Then H is called an (m,n)-regular (resp. $(m,0)$-regular, $(0,n)$-regular) if for any $a \in H$ there exists $x \in H$ such that $a \leq a^m \circ x \circ a^n$ (resp. $a \leq a^m \circ x, a \leq x \circ a^n$) i.e., if $a \in (a^m \circ H \circ a^n]$ (resp. $a \in (a^m \circ H], a \in (H \circ a^n]$) equivalently for each non-empty subset A of H, $A \subseteq (A^m \circ H \circ A^n]$ (resp. $A \subseteq (A^m \circ H]$, $A \subseteq (H \circ A^n]$).

Lemma 2.4. *Let H be an $(m,0)$-regular (resp. $(0,n)$-regular) ordered semihypergroup and F be a non-empty subset of H. Then the following statements are equivalent:*

(1) *F is left-m-hyperfilter (resp. right-n-hyperfilter) of H;*
(2) *$H \setminus F = \emptyset$ or $H \setminus F$ is completely prime generalized $(m,0)$-hyperideal ($(0,n)$ -hyperideal) of H, where $H \setminus F$ is the complement of F in H.*

Proof. (1) $\Rightarrow$ (2). Assume that $H \setminus F \neq \emptyset$. If $(H \setminus F)^m \circ H \subseteq F$, then $H \setminus F \subseteq (H \setminus F)^m \circ H \subseteq F$, which is a contradiction. Therefore, $(H \setminus F)^m \circ H \subseteq H \setminus F$. Let $H \ni a \leq b \in H \setminus F$. If $a \in F$, then, as F is a left-m-hyperfilter, we have $b \in F$, which is a contradiction. Thus, $a \in H \setminus F$. To show that $H \setminus F$ is completely prime $(m,0)$-hyperideal of H, let $a,b \in H$, $a \circ b \cap H \setminus F \neq \emptyset$. If $a \in F$ and $b \in F$, $a \circ b \subseteq F$. Thus, either $a \in H \setminus F$ or $b \in H \setminus F$.

(2) $\Rightarrow$ (1). Let $H \setminus F$ is completely prime generalized $(m,0)$-hyperideal of H. Let $a,b \in F$. If $a \circ b \subseteq H \setminus F$, by hypothesis $a \in H \setminus F$ or $b \in H \setminus F$, a contradiction. Thus $a \circ b \subseteq F$ it follows that F is subsemihypergroup. Now consider for any $a,b \in H, a \circ b \cap F \neq \emptyset$. If $a^m \subseteq H \setminus F$, then since H is $(m,0)$-regular there exist $s_1, s_2 \in H$ such that $a \circ b \leq a^m \circ s_1 \circ b \leq (a^m)^m \circ s_2 \circ s_1 \circ b \subseteq (H \setminus F)^m \circ H \subseteq H \setminus F$. So, $a \circ b \subseteq H \setminus F$, a contradiction. Therefore, $a^m \subseteq F$. Now take any element $a \in F$ and $b \in H$ such that $a \leq b$. If $b \in H \setminus F$, then $a \in H \setminus F$ which is a contradiction. Thus, $b \in F$. Hence, F is a left-m-filter of H. ☐

Corollary 2.4. *Let H be an (m,n)-regular ordered semihypergroup and F be a non-empty subset of H. Then the following statements are equivalent:*

(1) *F is (m,n)-hyperfilter of H;*
(2) *$H \setminus F = \emptyset$ or $H \setminus F$ is completely prime generalized (m,n)-hyperideal of H, where $H \setminus F$ is the complement of F in H.*

Lemma 2.5. *An $(m,0)$-regular ($(0,n)$-regular) ordered semihypergroup H does not contain proper left-m-hyperfilters (right-n-hyperfilters) if and only if H does not contain proper completely prime generalized $(m,0)$-hyperideals ($(0,n)$-hyperideals).*

Proof. Assume that H does not contain a proper left-m-hyperfilter. Let A be any proper completely prime generalized $(m,0)$-hyperideal of H. Then, by Lemma 2.4, $H \setminus A$ is proper left-m-hyperfilter of H which is a contradiction. Therefore, H does not contain any left-m-hyperfilter.

Conversely, assume that H does not contain proper completely prime $(m,0)$-hyperideals. Let F be any proper left-m-hyperfilter of H. Then by Lemma 2.4, $H \setminus F$ is a proper completely prime generalized $(m,0)$-hyperideal of H which is a contradiction. Hence, H does not contain proper left-m-hyperfilters. $\square$

Corollary 2.5. *An (m,n)-regular ordered semihypergroup H does not contain proper (m,n)-hyperfilters if and only if H does not contain proper completely prime generalized (m,n)-hyperideals.*

Let $(H,\diamond,\leq_H)$ and $(T,\star,\leq_T)$ be two ordered semihypergroups. A mapping $\phi : H \to T$ is called a normal homomorphism if for each $a,b \in H$, $\phi(x\diamond y) = \phi(x)\star\phi(y)$ and ϕ is isotone, i.e., for each $x,y \in H$, $x \leq_H y$ implies $\phi(x) \leq_T \phi(y)$. Further, ϕ is called reverse isotone if for all $x,y \in H$, $\phi(x) \leq_T \phi(y)$ implies $x \leq_H y$.

Lemma 2.6. *Let $(H,\star,\leq_H)$ and $(T,\diamond,\leq_T)$ be two ordered semihypergroups and $\phi : H \to T$ normal homomorphism. If F a left-m-hyperfilter (right-n-hyperfilter) of T, then $\phi^{-1}(F)$ is a left-m-hyperfilter (right-n-hyperfilter) of H.*

Proof. First, we show that $\phi^{-1}(F)$ is a subsemihypergroup of H. Let $a,b \in \phi^{-1}(F)$, then $\phi(a),\phi(b) \in F$. As ϕ is normal homomorphism and F is left-m-hyperfilter of T, $\phi(a\star b) = \phi(a)\diamond\phi(b) \subseteq F$. So $a\star b \subseteq \phi^{-1}(F)$. Next, take any $a,b \in H$ such that

$$\begin{aligned}
(a\star b)\cap\phi^{-1}(F) \neq \emptyset &\Rightarrow \phi(a\star b)\cap F \neq \emptyset\\
&\Rightarrow (\phi(a)\diamond\phi(b))\cap F \neq \emptyset\\
&\Rightarrow (\phi(a))^m \subseteq F\\
&\Rightarrow \phi(a)\diamond\phi(a)\diamond\cdots\diamond\phi(a) \subseteq F\\
&\Rightarrow \phi(a\star a\star\cdots\star a) \subseteq F\\
&\Rightarrow a^m \subseteq \phi^{-1}(F).
\end{aligned}$$

If $a \in \phi^{-1}(F), b \in H$ such that $a \leq_H b$, then $\phi(a) \in F$ and $\phi(a) \leq_T \phi(b)$. Therefore, $\phi(b) \in F$ implies $b \in \phi^{-1}(F)$. Hence, $\phi^{-1}(F)$ is an left-m-hyperfilter of H. $\square$

Corollary 2.6. *Let $(H,\star,\leq_H)$ and $(T,\diamond,\leq_T)$ be two ordered semihypergroups and $\phi : H \to T$ normal homomorphism. If F an (m,n)-hyperfilter of T, then $\phi^{-1}(F)$ is an (m,n)-hyperfilter of H.*

3. Conclusion

When we take $m = 1 = n$, in all results of this paper, then we obtain all results for left hyperfilters, right hyprerfilters and hyperfilters in an ordered semihypergroup and some characterizations of regular ordered semihypergroups which is the main application of results presented in this paper. Also we can extend all the results of this paper in the setting of ordered Γ-semihypergroup.

4. Problems

(1) Under what condition a left-m-hyperfilter (right-n-hyperfilter, (m,n)-hyperfilter) of an ordered semihypergroup coincides with a left hyperfilter (right hyperfilter, hyperfilter)?

(2) Under what conditions arbitrary union of left-m-hyperfilters (right-n-hyperfilters, (m,n)-hyperfilters) of an ordered semihypergroup is a left-m-hyperfilter (right-n-hyperfilter, (m,n)-hyperfilter)?

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EXISTENCE THEOREMS FOR A COUPLED SYSTEM OF NONLINEAR MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. We discuss the existence and uniqueness of solutions for a coupled system of nonlinear multi-term fractional differential equations complemented with coupled nonlocal boundary conditions by applying the methods of modern functional analysis. An example illustrating the uniqueness result is presented. Some interesting observations are also described.

1. Introduction

The topic of boundary value problems has been fascinating due to its extensive applications in applied and technical sciences. In recent years, an overwhelming interest has been shown in the study of fractional differential equations and inclusions equipped with a variety of boundary conditions, for instance, see [1,2,25,26,28,30] and the references cited therein. Coupled systems of fractional-order differential equations also constitute an important area of investigation in view of occurrence of such systems in disease models [9,10], chaos [31], ecology [16] and so forth. Some recent theoretical work on the topic can be found in the articles [3,4,6,7,12,29].

On the other hand, coupled systems involving more than one fractional order differential operators need to be addressed further to strengthen the hot topic of boundary value problems. Examples include Bagley-Torvik [27] and Basset [20]

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equations. For some recent results on multi-term (sequential) fractional differential equations, see [5, 8, 18, 19].

The nonlocal nature of fractional order operators is the key factor in the popularity of fractional calculus, which has extended the scope of the existing integer-order models by providing their fractional order counterparts. Examples include fractional reaction-diffusion systems [13], anomalous diffusion [15], chaotic neuron model [23], groundwater hydrology [24] and so forth. For more details, we refer the reader to texts [14, 21, 22].

Motivated by recent work on fractional order coupled systems, we introduce and study a coupled system of multi-term fractional differential equations:

$$\begin{cases} L_{a_i}^r u(t) = f(t, u(t), v(t)), & 0 < r < 1, \\ L_{b_i}^p v(t) = g(t, u(t), v(t)), & 0 < p < 1, \end{cases} \tag{1.1}$$

complemented with nonlocal multi-point coupled boundary conditions:

$$\begin{cases} u(0) = 0, \quad u'(0) = 0, \quad u(1) = \sum_{i=1}^{\rho} \alpha_i v(\eta_i), \\ v(0) = 0, \quad v'(0) = 0, \quad v(1) = \sum_{j=1}^{h} \beta_j u(\xi_j), \quad \eta_i < \xi_j, \quad \text{for all } i, j, \end{cases} \tag{1.2}$$

where

$$L_{a_i}^r = a_2\ {}^cD^{r+2} + a_1\ {}^cD^{r+1} + a_0\ {}^cD^r, \quad L_{b_i}^p = b_2\ {}^cD^{p+2} + b_1\ {}^cD^{p+1} + b_0{}^cD^p,$$

${}^cD^q$ is the Caputo-type fractional derivative of order $q = r, p$, $f, g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and a_i, b_i, $i = 0, 1, 2$, are real constants such that $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$ with $a_2 \neq 0 \neq b_2$. The existence and uniqueness results for the problem (1.1)–(1.2) are derived via Leray-Schauder alternative and Banach fixed point theorem respectively.

The rest of the paper is arranged as follows. In Section 2, we recall some preliminary concepts of fractional calculus and present an auxiliary lemma. The main results and an illustrative are presented in Section 3. The paper concludes with some interesting observations.

2. Basic Results

We begin this section with some preliminary concepts of fractional calculus [17, 32].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}$, $\alpha > 0$, for a locally integrable real-valued function χ on $-\infty \leq a < t < b \leq +\infty$ is defined by

$$I_a^\alpha \chi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s) ds,$$

where Γ is the Euler gamma function.

Definition 2.2. Let χ, $\chi^{(m)} \in L^1[a,b]$ for $-\infty \leq a < t < b \leq +\infty$. The Riemann-Liouville fractional derivative of χ of order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$, is defined as

$$D_a^\alpha \chi(t) = \frac{d^m}{dt^m} I_a^{1-\alpha} \chi(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int\limits_a^t (t-s)^{m-1-\alpha} \chi(s) ds,$$

while the Caputo fractional derivative χ of order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$, is defined by

$${}^cD_a^\alpha \chi(t) = D_a^\alpha \left[\chi(t) - \chi(a) - \chi'(a) \frac{(t-a)}{1!} - \cdots - \chi^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.1. If $\chi \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, is defined as

$${}^cD_a^\alpha \chi(t) = I_a^{1-\alpha} \chi^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^t (t-s)^{m-1-\alpha} \chi^{(m)}(s) ds.$$

In our analysis, I^α and ${}^cD^\alpha$ respectively denote Riemann-Liouville fractional integral and Caputo fractional derivative, with $a = 0$.

Lemma 2.1 ([17])**.** *For $\phi \in C(0,1) \cap L(0,1)$ holds:*

$$I^\alpha({}^cD^\alpha \varphi(t)) = \varphi(t) - c_0 - c_1 t - \cdots - c_{n-1} t^{n-1}, \quad t > 0,\ n-1 < \alpha < n,$$

where c_i, $i = 1, \ldots, n-1$, are arbitrary constants.

Definition 2.3. A pair of functions $u, v \in C([0,1], \mathbb{R})$ satisfying the equations (1.1) and the boundary conditions (1.2) is called a solution of the problem (1.1)–(1.2), where it is assumed that u, v possess the Caputo fractional derivative of order $r+2$ and $p+2$ respectively on $(0,1)$.

We need the following auxiliary lemma, which concerns the linear variant of problem (1.1)–(1.2).

Lemma 2.2. *Let $a_1^2 - 4a_2 a_0 = 0$, $b_1^2 - 4b_2 b_0 = 0$, $a_2 \neq 0$, $b_2 \neq 0$ and $w, z \in C([0,1], \mathbb{R})$. Then the solution (u,v) (in the sense of Definition* 2.3*) of the system of linear fractional differential equations*

$$\begin{cases} L_{a_i}^r u(t) = w(t), & 0 < r < 1, \\ L_{b_i}^p v(t) = z(t), & 0 < p < 1, \end{cases} \tag{2.1}$$

supplemented with the boundary conditions (1.2) *is given by*

$$\begin{aligned} u(t) = & \frac{1}{a_2} \int_0^t \int_0^s \phi(t) \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta) d\theta ds \\ & + \lambda_1(t) \left[\frac{\sum_{i=1}^r \alpha_i}{b_2} \int_0^{\eta_j} \int_0^s \zeta(\eta_i) \frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta) d\theta ds \right. \\ & \left. - \frac{1}{a_2} \int_0^1 \int_0^s \phi(1) \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta) d\theta ds \right] \end{aligned}$$

$$
\begin{aligned}
&+\lambda_2(t)\left[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right.\\
&\left.-\frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right]
\end{aligned}\tag{2.2}
$$

and

$$
\begin{aligned}
v(t)=&\frac{1}{b_2}\int_0^t\int_0^s \zeta(t)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\\
&+\mu_1(t)\left[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right.\\
&\left.-\frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right]\\
&+\mu_2(t)\left[\frac{\sum_{i=1}^{r}\alpha_i}{b_2}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right.\\
&\left.-\frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right],
\end{aligned}\tag{2.3}
$$

where

$$
\begin{aligned}
\phi(\kappa)=&(\kappa-s)e^{m(\kappa-s)},\quad \zeta(\kappa)=(\kappa-s)e^{n(\kappa-s)},\quad \kappa=t,1,\eta_i \text{ and } \xi_j,\\
m=&\frac{-a_1}{2a_2},\quad n=\frac{-b_1}{2b_2},\quad \lambda_1(t)=\frac{(mte^{mt}-e^{mt}+1)(ne^n-e^n+1)}{\mu},\\
\lambda_2(t)=&\frac{(mte^{mt}-e^{mt}+1)(n\sum_{i=1}^{\rho}\alpha_i\eta_i e^{n\eta_i}-\sum_{i=1}^{\rho}\alpha_i e^{\eta_i}+\sum_{i=1}^{\rho}\alpha_i)}{\mu},\\
\mu_1(t)=&\frac{(nte^{nt}-e^{nt}+1)(me^m-e^m+1)}{\mu},\\
\mu_2(t)=&\frac{(nte^{nt}-e^{nt}+1)(m\sum_{j=1}^{h}\beta_j\xi_j e^{m\xi_j}-\sum_{j=1}^{h}\beta_j e^{m\xi_j}+\sum_{j=1}^{h}\beta_j)}{\mu},\\
\mu=&(me^m-e^m+1)(ne^n-e^n+1)\\
&-\left(n\sum_{i=1}^{\rho}\alpha_i\eta_i e^{n\eta_i}-\sum_{i=1}^{\rho}\alpha_i e^{\eta_i}+\sum_{i=1}^{\rho}\alpha_i)(m\sum_{j=1}^{h}\beta_j\xi_j e^{m\xi_j}-\sum_{j=1}^{h}\beta_j e^{m\xi_j}+\sum_{j=1}^{h}\beta_j\right)\neq 0.
\end{aligned}\tag{2.4}
$$

Proof. Applying the integral operators I^r and I^p respectively on the first and second equations of (2.1) and then using Lemma 2.1, we get

$$
(a_2D^2+a_1D+a_0)u(t)=\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}w(s)ds+c_1,\tag{2.5}
$$

$$
(b_2D^2+b_1D+b_0)v(t)=\int_0^t\frac{(t-s)^{p-1}}{\Gamma(p)}z(s)ds+d_1,\tag{2.6}
$$

where c_1 and d_1 are arbitrary constants. Using the method of variation of parameters to solve (2.5) and (2.6), we get

$$u(t) = c_2 e^{mt} + c_3 t e^{mt} + \frac{1}{a_2}\int_0^t (t-s)e^{m(t-s)}\left(\int_0^s \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta)d\theta + c_1\right) ds \tag{2.7}$$

and

$$v(t) = d_2 e^{nt} + d_3 t e^{nt} + \frac{1}{b_2}\int_0^t (t-s)e^{n(t-s)}\left(\int_0^s \frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta)d\theta + d_1\right) ds, \tag{2.8}$$

where m and n are given by (2.4). Using $u(0)=0$, $u'(0)=0$ and $v(0)=0$, $v'(0)=0$ in (2.7) and (2.8) respectively, we find that $c_2 = c_3 = 0$ and $d_2 = d_3 = 0$ and consequently, we have

$$u(t) = c_1\left[\frac{mte^{mt} - e^{mt} + 1}{a_2 m^2}\right] + \frac{1}{a_2}\left[\int_0^t (t-s)e^{m(t-s)}\left(\int_0^s \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta)d\theta\right) ds\right], \tag{2.9}$$

$$v(t) = d_1\left[\frac{nte^{nt} - e^{nt} + 1}{b_2 n^2}\right] + \frac{1}{b_2}\left[\int_0^t (t-s)e^{n(t-s)}\left(\int_0^s \frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta)d\theta\right) ds\right]. \tag{2.10}$$

On the other hand, using the conditions $u(1) = \sum_{i=1}^{r} \alpha_i v(\eta_i)$ and $v(1) = \sum_{j=1}^{h} \beta_j u(\xi_j)$ in (2.9) and (2.10), respectively, we get the system:

$$A_1 c_1 - B_1 d_1 = V_1, \quad -B_2 c_1 + A_2 d_1 = V_2, \tag{2.11}$$

where

$$\begin{aligned}
A_1 &= \frac{me^m - e^m + 1}{a_2 m^2}, \quad A_2 = \frac{ne^n - e^n + 1}{b_2 n^2},\\
B_1 &= \frac{\sum_{i=1}^{\rho} \alpha_i (n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1)}{b_2 n^2}, \quad B_2 = \frac{\sum_{j=1}^{h} \beta_j (m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1)}{a_2 m^2},
\end{aligned}$$

$$\begin{aligned}
V_1 &= \frac{\sum_{i=1}^{\rho}\alpha_i}{b_2}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta)d\theta ds - \frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta)d\theta ds,\\
V_2 &= \frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta)d\theta ds - \frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta)d\theta ds.
\end{aligned} \tag{2.12}$$

Solving the system (2.11), we find that

$$c_1 = \frac{A_2 V_1 + B_1 V_2}{A_1 A_2 - B_1 B_2}, \quad d_1 = \frac{B_2 V_1 + A_1 V_2}{A_1 A_2 - B_1 B_2}.$$

Substituting the values of c_1 and d_1 in (2.9) and (2.10) respectively together with the notations (2.12) leads to the solution (2.2) and (2.3). The converse can be proven by direct computation. The proof is completed. □

3. Existence and Uniqueness Results

Define by $\mathcal{M} = \{u \mid u \in C([0,1], \mathbb{R})\}$ the Banach space endowed with norm $\|u\|_{\mathcal{M}} = \sup_{t\in[0,1]} |u(t)|$. Then the product space $(\mathcal{M} \times \mathcal{M}, \|\cdot\|_{\mathcal{M}\times\mathcal{M}})$ is a Banach space equipped with the norm $\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} = \|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}}$ for $(u,v) \in \mathcal{M} \times \mathcal{M}$.

In view of Lemma 2.2, we introduce an operator $\mathcal{Q} : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ as

$$\mathcal{Q}(u,v)(t) := (\mathcal{Q}_1(u,v)(t), \mathcal{Q}_2(u,v)(t)),$$

where

$$\begin{aligned}
\mathcal{Q}_1(u,v)(t) =& \frac{1}{a_2}\int_0^t \int_0^s \phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \\
&+ \lambda_1(t)\Bigg[\frac{\sum_{i=1}^{r}\alpha_i}{b_2}\int_0^{\eta_j}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \\
&- \frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds\Bigg] \\
&+ \lambda_2(t)\Bigg[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \\
&- \frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_2(u,v)(t) =& \frac{1}{b_2}\int_0^t \int_0^s \zeta(t)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \\
&+ \mu_1(t)\Bigg[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \\
&- \frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds\Bigg] \\
&+ \mu_2(t)\Bigg[\frac{\sum_{i=1}^{r}\alpha_i}{b_2}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \\
&- \frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds\Bigg].
\end{aligned}$$

For the sake of brevity, we set the following notations:

(3.1)
$$\Delta_1 = \frac{1}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m - e^m + 1| + \widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\right\},$$

(3.2)
$$\Delta_2 = \frac{1}{|b_2|n^2\Gamma(p+1)}\left\{(1+\widehat{\mu}_1)|ne^n - e^n + 1| + \widehat{\mu}_2\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\},$$

(3.3)
$$\Lambda_1 = \frac{1}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n - e^n + 1| + \widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\},$$
(3.4)
$$\Lambda_2 = \frac{1}{|a_2|m^2\Gamma(r+1)}\left\{\widehat{\mu}_2|me^m - e^m + 1| + \widehat{\mu}_1\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\right\},$$
$$\widehat{\lambda}_1 = \max_{t\in[0,1]}|\lambda_1(t)|, \quad \widehat{\lambda}_2 = \max_{t\in[0,1]}|\lambda_2(t)|, \quad \widehat{\mu}_1 = \max_{t\in[0,1]}|\mu_1(t)|, \quad \widehat{\mu}_2 = \max_{t\in[0,1]}|\mu_2(t)|.$$

Our first result, dealing with the existence of solutions for the problem (1.1)–(1.2), is based on Leray-Schauder alternative.

Lemma 3.1. (Leray-Schauder alternative [11]). *Let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator and that $\mathcal{A}(\mathcal{F}) = \{x \in \mathcal{E} \mid x = \nu\mathcal{F}(x)$ for some $0 < \nu < 1\}$. Then either the set $\mathcal{A}(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fixed point.*

Theorem 3.1. *Assume that*

(H_1) *there exist real constants $\delta_i, \gamma_i > 0$, $i = 1, 2$, and $\delta_0 > 0$, $\gamma_0 > 0$ such that*
$$|f(t, u_1, u_2)| \le \delta_0 + \delta_1|u_1| + \delta_2|u_2|$$
and
$$|g(t, u_1, u_2)| \le \gamma_0 + \gamma_1|u_1| + \gamma_2|u_2|, \quad \text{for all } u_i \in \mathbb{R},\ i = 1, 2.$$

(H_2) $\max\{\omega_1, \omega_2\} < 1$, *where*

(3.5) $$\omega_1 = \delta_1(\Delta_1 + \Lambda_2) + \gamma_1(\Delta_2 + \Lambda_1), \quad \omega_2 = \delta_2(\Delta_1 + \Lambda_2) + \gamma_2(\Delta_2 + \Lambda_1),$$

Δ_1, Δ_2, Λ_1, Λ_2 are respectively given by (3.1), (3.2), (3.3) *and* (3.4).

Then the problem (1.1)–(1.2) *has at least one solution on* $[0, 1]$.

Proof. We first show that the operator $\mathcal{Q} : \mathcal{M}\times\mathcal{M} \to \mathcal{M}\times\mathcal{M}$ is completely continuous. The operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are continuous since the functions f and g are continuous, and thus the operator $\mathcal{Q}$ is continuous. Let $\Omega \subset \mathcal{M}\times\mathcal{M}$ be a bounded set. Then $|f(t, u(t), v(t))| \le L_1$, $|g(t, u(t), v(t))| \le L_2$ for all $(u, v) \in \Omega$, where L_1 and L_2 are positive constants. In consequence for any $(u, v) \in \Omega$, we get
$$\begin{aligned}\|\mathcal{Q}_1(u,v)\|_{\mathcal{M}} &= \sup_{t\in[0,1]}|\mathcal{Q}_1(u,v)(t)| \\ &\le \frac{L_1}{|a_2|m^2\Gamma(r+1)}\Bigg\{(1+\widehat{\lambda}_1)|me^m - e^m + 1| \\ &\quad + \widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\Bigg\} \\ &\quad + \frac{L_2}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n - e^n + 1| + \widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\}\end{aligned}$$

$$=L_1\Delta_1 + L_2\Lambda_1. \tag{3.6}$$

Similarly, it can be shown that

$$\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}} \leq L_2\Delta_2 + L_1\Lambda_2. \tag{3.7}$$

From (3.6) and (3.7), we deduce that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are uniformly bounded, and hence the operator $\mathcal{Q}$ is uniformly bounded.

Next, we show that $\mathcal{Q}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|\mathcal{Q}_1(u,v)(t_2) - \mathcal{Q}_1(u,v)(t_1)|\\
\leq&\frac{L_1}{|a_2|}\left\{\int_0^{t_1}\int_0^s |\phi(t_2)-\phi(t_1)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds + \int_{t_1}^{t_2}\int_0^s |\phi(t_2)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right\}\\
&+|\lambda_1(t_2)-\lambda_1(t_1)|\left\{\frac{L_2\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s |\zeta(\eta_i)|\frac{(s-\theta)^{p-1}}{\Gamma(p)}d\theta ds\right.\\
&\left.+\frac{L_1}{|a_2|}\int_0^1\int_0^s |\phi(1)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right\}\\
&+|\lambda_2(t_2)-\lambda_2(t_1)|\left\{\frac{L_1\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s |\phi(\xi_j)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right.\\
&\left.+\frac{L_2}{|b_2|}\int_0^1\int_0^s |\zeta(1)|\frac{(s-\theta)^{p-1}}{\Gamma(p)}d\theta ds\right\}\\
\leq&\left[\frac{L_1}{|a_2|m^2\Gamma(r+1)}\left\{(t_1^r-t_2^r)|m(t_1-t_2)e^{m(t_1-t_2)}-e^{m(t_1-t_2)}+1|\right.\right.\\
&\left.+t_1^r|mt_1e^{mt_1}-e^{mt_1}+1|\right\}\\
&+|\lambda_1(t_2)-\lambda_1(t_1)|\left\{\frac{L_2\sum_{i=1}^{\rho}\alpha_i\eta_i^p}{|b_2|n^2\Gamma(p+1)}|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\right.\\
&\left.+\frac{L_1}{|a_2|m^2\Gamma(r+1)}|me^m-e^m+1|\right\}\\
&+|\lambda_2(t_2)-\lambda_2(t_1)|\left\{\frac{L_1\sum_{j=1}^{h}\beta_j\xi_j^r}{|a_2|m^2\Gamma(r+1)}||m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\right.\\
&\left.\left.+\frac{L_2}{|b_2|n^2\Gamma(p+1)}|ne^n-e^+1|\right\}\right] \to 0 \quad \text{as} \quad t_2-t_1\to 0,
\end{aligned}$$

independently of $(u,v)\in\Omega$. Analogously, we have

$$|\mathcal{Q}_2(u,v)(t_2) - \mathcal{Q}_2(u,v)(t_1)| \to 0 \quad \text{as} \quad t_2-t_1\to 0,$$

independently of $(u,v)\in\Omega$. Hence, the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are equicontinuous and thus the operator $\mathcal{Q}$ is equicontinuous. By Arzelá-Ascoli's theorem, we deduce that the operator $\mathcal{Q}$ is completely continuous.

Lastly, we consider a set $\Theta(\mathcal{Q}) = \{(u,v) \in \mathcal{M} \times \mathcal{M} \mid (u,v) = \nu\mathcal{Q}(u,v);\ 0 \le \nu \le 1\}$ and show that it is bounded. Let $(u,v) \in \Theta$. Then $(u,v) = \nu\mathcal{Q}(u,v)$. For any $t \in [0,1]$, we have $u(t) = \nu\mathcal{Q}_1(u,v)(t)$, $v(t) = \nu\mathcal{Q}_2(u,v)(t)$. Thus,

$$
\begin{aligned}
|u(t)| =& |\nu\mathcal{Q}_1(u,v)(t)| \le |\mathcal{Q}_1(u,v)(t)| \\
\le& \frac{1}{|a_2|}\int_0^t\int_0^s \phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds \\
&+ |\lambda_1(t)|\left[\frac{\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right. \\
&\left.+\frac{1}{|a_2|}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right] \\
&+ |\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right. \\
&\left.+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right] \\
\le& \frac{\delta_0+\delta_1|u|+\delta_2|v|}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m - e^m + 1|\right. \\
&\left.+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j\xi_j^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\right\} \\
&+ \frac{(\gamma_0+\gamma_1|u|+\gamma_2|v|)}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n - e^n + 1| + \widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i\eta_i^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\} \\
=& (\delta_0+\delta_1|u|+\delta_2|v|)\Delta_1 + (\gamma_0+\gamma_1|u|+\gamma_2|v|)\Lambda_1,
\end{aligned}
$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|u\|_{\mathcal{M}} \le (\delta_0 + \delta_1\|u\|_{\mathcal{M}} + \delta_2\|v\|_{\mathcal{M}})\Delta_1 + (\gamma_0 + \gamma_1\|u\|_{\mathcal{M}} + \gamma_2\|v\|_{\mathcal{M}})\Lambda_1. \tag{3.8}$$

Likewise, we can obtain

$$\|v\|_{\mathcal{M}} \le (\gamma_0 + \gamma_1\|u\|_{\mathcal{M}} + \gamma_2\|v\|_{\mathcal{M}})\Delta_2 + (\delta_0 + \delta_1\|u\|_{\mathcal{M}} + \delta_2\|v\|_{\mathcal{M}})\Lambda_2. \tag{3.9}$$

From (3.8) and (3.9), we find that

$$
\begin{aligned}
\|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}} \le& \delta_0(\Delta_1+\Lambda_2) + \gamma_0(\Delta_2+\Lambda_1) \\
&+ \|u\|_{\mathcal{M}}\left(\delta_1(\Delta_1+\Lambda_2) + \gamma_1(\Delta_2+\Lambda_1)\right) \\
&+ \|v\|_{\mathcal{M}}\left(\delta_2(\Delta_1+\Lambda_2) + \gamma_2(\Delta_2+\Lambda_1)\right) \\
\le& \omega_0 + \max\{\omega_1,\omega_2\}\|(u,v)\|_{\mathcal{M}\times\mathcal{M}},
\end{aligned} \tag{3.10}
$$

where $\omega_0 = \delta_0(\Delta_1+\Lambda_2) + \gamma_0(\Delta_2+\Lambda_1)$ and ω_1, ω_2 are given by (3.5).

In view of the definition $\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} = \|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}}$, (3.10) leads to

$$\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} \le \frac{\omega_0}{1-\max\{\omega_1,\omega_2\}}.$$

Consequently, the set $\Theta(\mathcal{Q})$ is bounded. By Lemma 3.1, the operator $\mathcal{Q}$ has at least one fixed point. Therefore, the problem (1.1)–(1.2) has at least one solution on $[0,1]$, which finish the proof. $\square$

In the following result, we prove the uniqueness of solutions for the problem at hand by means of Banach fixed point theorem.

Theorem 3.2. *Assume that:*

(H_3) *for all $t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1,2$, there exist positive constants ℓ_1 and ℓ_2 such that*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq \ell_1(|u_1 - v_1| + |u_2 - v_2|),$$
$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq \ell_2(|u_1 - v_1| + |u_2 - v_2|).$$

Then there exists a unique solution for the problem (1.1)–(1.2) *on* $[0,1]$ *if*

$$\ell_1(\Delta_1 + \Lambda_2) + \ell_2(\Delta_2 + \Lambda_1) < 1, \tag{3.11}$$

where Δ_1, Δ_2, Λ_1, Λ_2 are given by (3.1)–(3.4).

Proof. Let us consider a closed ball $B_{r^*} = \{(u,v) \in \mathcal{M} \times \mathcal{M} \mid \|(u,v)\|_{\mathcal{M}\times\mathcal{M}} \leq r^*\}$ and show that $\mathcal{Q}B_{r^*} \subset B_{r^*}$, where

$$r^* \geq \frac{M_1(\Delta_1 + \Lambda_2) + M_2(\Delta_2 + \Lambda_1)}{1 - \ell_1(\Delta_1 + \Lambda_2) - \ell_2(\Delta_2 + \Lambda_1)}, \quad M_1 = \sup_{t\in[0,1]} |f(t,0,0)|,$$
$$M_2 = \sup_{t\in[0,1]} |g(t,0,0)|.$$

For $(u,v) \in B_r$, $t \in [0,1]$, using (H_3), we get

$$\begin{aligned}
|f(t,u(t),v(t))| &\leq |f(t,u(t),v(t)) - f(t,0,0) + f(t,0,0)| \\
&\leq \ell_1\left(|u(t)| + |v(t)|\right) + M_1 \\
&\leq \ell_1\left(\|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}}\right) + M_1 \\
&\leq \ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} + M_1 \leq \ell_1 r^* + M_1.
\end{aligned} \tag{3.12}$$

In a similar manner, we can find that

$$|g(t,u(t),v(t))| \leq \ell_2 r^* + M_2. \tag{3.13}$$

Then, using (3.12) and (3.13), we obtain

$$\begin{aligned}
\|\mathcal{Q}_1(u,v)\|_{\mathcal{M}} &= \sup_{t\in[0,1]} |\mathcal{Q}_1(u,v)(t)| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{1}{|a_2|} \int_0^t \int_0^s \phi(t) \frac{(s-\theta)^{r-1}}{\Gamma(r)} |f(\theta,u(\theta),v(\theta))| d\theta ds \right. \\
&\quad + |\lambda_1(t)| \left[\frac{\sum_{i=1}^r \alpha_i}{|b_2|} \int_0^{\eta_i} \int_0^s \zeta(\eta_i) \frac{(s-\theta)^{p-1}}{\Gamma(p)} |g(\theta,u(\theta),v(\theta))| d\theta ds \right. \\
&\quad \left. + \frac{1}{|a_2|} \int_0^1 \int_0^s \phi(1) \frac{(s-\theta)^{r-1}}{\Gamma(r)} |f(\theta,u(\theta),v(\theta))| d\theta ds \right]
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right.\\
&\left.\left.+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right]\right\}\\
\le&(\ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_1)\left[\frac{1}{|a_2|}\int_0^t|(t-s)e^{m(t-s)}|\frac{s^r}{\Gamma(r+1)}ds\right.\\
&+\frac{|\lambda_1(t)|}{|a_2|}\int_0^1|(1-s)e^{m(1-s)}|\frac{s^r}{\Gamma(r+1)}ds\\
&\left.+\frac{|\lambda_2(t)|}{|a_2|}\sum_{j=1}^{h}\beta_j\int_0^{\xi_j}|(\xi_j-s)e^{m(\xi_j-s)}|\frac{s^r}{\Gamma(r+1)}ds\right]\\
&+(\ell_2\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_2)\left[\frac{|\lambda_1(t)|}{|b_2|}\sum_{i=1}^{\rho}\alpha_i\int_0^{\eta_i}|(\eta-s)e^{n(\eta_i-s)}|\frac{s^p}{\Gamma(p+1)}ds\right.\\
&\left.+\frac{|\lambda_2(t)|}{|b_2|}\int_0^1|(1-s)e^{n(1-s)}|\frac{s^p}{\Gamma(p+1)}ds\right]\\
\le&\frac{\ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_1}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m-e^m+1|\right.\\
&\left.+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\right\}\\
&+\frac{(\ell_2\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_2)}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n-e^n+1|\right.\\
&\left.+\widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\right\}
\end{aligned}
$$

$$
\le(\ell_1r^*+M_1)\Delta_1+(\ell_2r^*+M_2)\Lambda_1. \tag{3.14}
$$

Similarly, we have

$$
\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}}\le(\ell_2r^*+M_2)\Delta_2+(\ell_1r^*+M_1)\Lambda_2. \tag{3.15}
$$

From the inequalities (3.14) and (3.15), we get

$$
\|\mathcal{Q}(u,v)\|_{\mathcal{M}\times\mathcal{M}}=\|\mathcal{Q}_1(u,v)\|_{\mathcal{M}}+\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}}\le r^*,
$$

which implies that $\mathcal{Q}B_{r^*}\subset B_{r^*}$. Now we will prove that the operator $\mathcal{Q}$ is a contraction. For $u_i,v_i\in B_{r^*}$, $i=1,2$, and for each $t\in[0,1]$, we have

$$
\begin{aligned}
&\|\mathcal{Q}_1(u_1,v_1)-\mathcal{Q}_1(u_2,v_2)\|_{\mathcal{M}}\\
=&\sup_{t\in[0,1]}|\mathcal{Q}_1(u_1,v_1)(t)-\mathcal{Q}_1(u_2,v_2)(t)|\\
\le&\sup_{t\in[0,1]}\left\{\frac{1}{|a_2|}\int_0^t\int_0^s\phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda_1(t)|\left[\frac{\sum_{i=1}^{r}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u_1(\theta),v_1(\theta))-g(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right.\\
&\left.+\frac{1}{|a_2|}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right]\\
&+|\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right.\\
&\left.\left.+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u_1(\theta),v_1(\theta))-g(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right]\right\}\\
\leq&\frac{\ell_1}{|a_2|}\int_0^t|(t-s)e^{m(t-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\\
&+|\lambda_1(t)|\left[\frac{\ell_2\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}|(\eta_i-s)e^{n(\eta_i-s)}|\frac{s^p}{\Gamma(p+1)}(|u_1-v_1|+|u_2-v_2|)ds\right.\\
&\left.+\frac{1}{|a_2|}\int_0^1|(1-s)e^{m(1-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\right]\\
&+|\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}|(\xi_j-s)e^{m(\xi_j-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\right.\\
&\left.+\frac{\ell_2}{|b_2|}\int_0^1|(1-s)e^{n(1-s)}|\frac{s^p}{\Gamma(p+1)}(|u_1-v_1|+|u_2-v_2|)ds\right]\\
\leq&\frac{\ell_1}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m-e^m+1|\right.\\
&\left.+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\right\}(|u_1-v_1|+|u_2-v_2|)\\
&+\frac{\ell_2}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n-e^n+1|\right.\\
&\left.+\widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\right\}(|u_1-v_1|+|u_2-v_2|)\\
\leq&(\ell_1\Delta_1+\ell_2\Lambda_1)(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}).
\end{aligned}
$$

Similarly, one can find that

$$
\begin{aligned}
\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\|_{\mathcal{M}}=&\sup_{t\in[0,1]}|\mathcal{Q}_2(u_1,v_1)(t)-\mathcal{Q}_2(u_2,v_2)(t)|\\
&\leq(\ell_2\Delta_2+\ell_1\Lambda_2)(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\|_{\mathcal{M}\times\mathcal{M}}\\
=&\|\mathcal{Q}_1(u_1,v_1)-\mathcal{Q}_1(u_2,v_2)\|_{\mathcal{M}}+\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\|_{\mathcal{M}}\\
\leq&\left(\ell_1(\Delta_1+\Lambda_2)+\ell_2(\Delta_2+\Lambda_1)\right)\left(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}\right),
\end{aligned}
$$

which, in view of the assumption (3.11), implies that $\mathcal{Q}$ is a contraction. Consequently, by Banach's contraction mapping principle, the operator $\mathcal{Q}$ has a unique fixed point, which is indeed the unique solution of the problem (1.1)–(1.2). This completes the proof. $\square$

Example 3.1. Consider the coupled system of multi-term fractional differential equations:

$$\begin{cases} \left(2\ {}^cD^{12/5} + 4\ {}^cD^{7/5} + 2\ {}^cD^{2/5}\right) u(t) = \dfrac{1}{\sqrt{t^2+25}}\{\cos u(t) + |v(t)| + \tan^{-1} t\}, \\ \left(\ {}^cD^{17/7} + 2\ {}^cD^{10/7} +\ {}^cD^{3/7}\right) v(t) = \dfrac{t^2}{t+6}\left\{|u(t)| + \dfrac{|v(t)|^3}{1+|v(t)|^3} + \sin t\right\}, \end{cases} \tag{3.16}$$

equipped with boundary conditions

$$\begin{cases} u(0) = 0, \quad u'(0) = 0, \quad u(1) = 2v(1/6) + v(1/5) + 2v(1/4), \\ v(0) = 0, \quad v'(0) = 0, \quad v(1) = 3u(1/2) + u(3/4). \end{cases} \tag{3.17}$$

Here, $q = 2/5$, $p = 3/7$, $\eta_1 = 1/6$, $\eta_2 = 1/5$, $\eta_3 = 1/4$, $\xi_1 = 1/2$, $\xi_2 = 3/4$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\beta_1 = 3$, $\beta_2 = 1$, $a_1^2 - 4a_2a_0 = 0$, $b_1^2 - 4b_2b_0 = 0$ and

$$f(t, u(t), v(t)) = \frac{1}{\sqrt{t^2+25}}\{\cos u(t) + |v(t)| + \tan^{-1} t\}$$

and

$$g(t, u(t), v(t)) = \frac{t^2}{t+6}\left\{|u(t)| + \frac{|v(t)|^3}{1+|v(t)|^3} + \sin t\right\}.$$

Clearly $\ell_1 = 1/5$ and $\ell_2 = 1/6$ as

$$|f(t, u_1(t), v_1(t)) - f(t, u_2(t), v_2(t))| \le \frac{1}{5}\{|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|\},$$
$$|g(t, u_1(t), v_1(t)) - g(t, u_2(t), v_2(t))| \le \frac{1}{6}\{|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|\}.$$

Using the given data, we find that $\Delta_1 \approx 0.71336$, $\Delta_2 \approx 1.3058$, $\Lambda_1 \approx 0.70297$, and $\Lambda_2 \approx 1.2161$. Further

$$\ell_1(\Delta_1 + \Lambda_2) + \ell_2(\Delta_2 + \Lambda_1) \approx 0.72069 < 1.$$

Hence we deduce by Theorem 3.2 that the problem (3.16)–(3.17) has a unique solution on $[0, 1]$.

4. Conclusions

We have analyzed a fully coupled boundary value problem of nonlinear multi-term fractional differential equations and nonlocal multi-point boundary conditions under the assumption that $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$. Though the tools of fixed point theory employed in the present analysis are the standard ones, yet their exposition to the problem at hand enhances the scope of the literature on fractional order boundary value problems. The cases $a_1^2 > 4a_0a_2$, $b_1^2 > 4b_0b_2$ and $a_1^2 < 4a_0a_2$, $b_1^2 < 4b_0b_2$ for the problem (1.1)–(1.2) can be handled in a manner similar to that of $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$.

As a special case, the results for a coupled system of nonlinear multi-term fractional differential equations equipped with the two-point boundary conditions: $u(0) = 0$, $u'(0) = 0$, $u(1) = 0$, $v(0) = 0$, $v'(0) = 0$, $v(1) = 0$ follow by taking all $\alpha_i = 0$, $i = 1, \ldots, \rho$, and $\beta_j = 0$, $j = 1, \ldots, h$, in the results of this paper.

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