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PARANORMED RIESZ DIFFERENCE SEQUENCE SPACES OF FRACTIONAL ORDER

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Abstract. In this article we introduce paranormed Riesz difference sequence spaces of fractional order α, $r_0^t\left(p,\Delta^{(\alpha)}\right)$, $r_c^t\left(p,\Delta^{(\alpha)}\right)$ and $r_\infty^t\left(p,\Delta^{(\alpha)}\right)$ defined by the composition of fractional difference operator $\Delta^{(\alpha)}$, defined by $(\Delta^{(\alpha)}x)_k = \sum\limits_{i=0}^{\infty}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k-i}$, and Riesz mean matrix R^t. We give some topological properties, obtain the Schauder basis and determine the α-, β- and γ- duals of the new spaces. Finally, we characterize certain matrix classes related to these new spaces.

1. Introduction

Throughout the paper $\Gamma(m)$ will denote the gamma function of all real numbers $m \notin \{0,-1,-2,\ldots\}$. $\Gamma(m)$ can be expressed as an improper integral given by

$$\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx. \tag{1.1}$$

Using (1.1), we state some properties of gamma function which are used throughout the text:

1. for $m \in \mathbb{N}$, $\Gamma(m+1) = m!$;
2. for any real number $m \notin \{0,-1,-2,\ldots\}$, $\Gamma(m+1) = m\Gamma(m)$;
3. for particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!, \ldots$

Throughout the paper $\mathbb{N} = \{0,1,2,3,\ldots\}$ and let w be the space of all real valued sequences. By ℓ_∞, c_0 and c we mean the spaces all bounded, null and convergent

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sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_1, cs and bs, we mean the spaces of absolutely summable, convergent series and bounded series, respectively. The space ℓ_1 is normed by $\sum_k |x_k|$ and the spaces cs and bs are normed by $\sup_n |\sum_{k=0}^n x_k|$. Here and henceforth, the summation without limit runs from zero to ∞. Also, let $e = \{1, 1, 1 \dots\}$ and $e^{(k)}$ be the sequences whose only non-zero term is 1 in the k^{th} place for each $k \in \mathbb{N}$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $M = \max\{1, H\}$, where $H = \sup_k p_k$. Then, Maddox [43, 44] defined the sequence spaces $\ell_\infty(p)$, $c_0(p)$, $c(p)$ and $\ell(p)$ as follows:

$$\begin{aligned} \ell_\infty(p) &= \left\{ x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty \right\}, \\ c_0(p) &= \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\}, \\ c(p) &= \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R} \right\} \end{aligned}$$

and

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\},$$

which are complete spaces paranormed by

$$g(x) = \sup_{k\in\mathbb{N}} |x_k|^{\frac{p_k}{M}} \quad \text{and} \quad h(x) = \left(\sum_k |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex entries. Then A defines a matrix mapping from X to Y if for every sequence $x = (x_k)$, the A-transform of x, i.e., $Ax = \{(Ax)_n\} \in Y$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad n \in \mathbb{N}. \tag{1.2}$$

The sequence space X_A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.3}$$

is called the domain of matrix A.

By (X, Y), we denote the class of all matrices A from X to Y. Thus $A \in (X, Y)$ if and only if the series on the R.H.S. of the (1.2) converges for each $n \in \mathbb{N}$ and $x \in X$ such that $Ax \in Y$ for all $x \in X$.

The notion of difference sequence space $X(\Delta)$ for $X = \{\ell_\infty, c, c_0\}$ was introduced by Kızmaz [40]. Since then several authors [15–19, 21–24] generalized the notion of difference operator Δ and studied various sequence spaces of integer order. However, for a positive proper fraction α, Baliarsingh and Dutta [10] (see also [11, 12, 20]) have defined a generalized fractional difference operator $\Delta^{(\alpha)}$ and its inverse as

$$(\Delta^{(\alpha)}x)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k-i}, \tag{1.4}$$

$$(\Delta^{(-\alpha)}x)_k = \sum_i (-1)^i \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} x_{k-i}. \tag{1.5}$$

Throughout the paper it is assumed that the series on the R.H.S. of (1.4) and (1.5) are convergent for $x = (x_k) \in w$. It is more convenient to express $\Delta^{(\alpha)}$ as a triangle

$$(\Delta^{(\alpha)})_{nk} = \begin{cases} (-1)^{n-k} \dfrac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Moreover, Dutta and Baliarsingh [20] also studied the paranormed difference sequence spaces of fractional order $X(\Gamma, \Delta^{\tilde{\alpha}}, u, p)$ for $X = \{c_0, c, \ell_\infty\}$, where

$$(\Delta^{\tilde{\alpha}}x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}.$$

Furthermore, Baliarsingh and Dutta [11] studied the sequence spaces $X(\Gamma, \Delta^{\tilde{\alpha}}, p)$ for $X = \{c_0, c, \ell_\infty\}$. For some nice papers on fractional difference operator and related sequence spaces, one may refer to [10–13, 20, 25–34] and the references mentioned therein.

Let (t_k) be a sequence of positive numbers and let

$$T_n = \sum_{k=0}^{n} t_k, \quad n \in \mathbb{N}.$$

The Riesz mean matrix $R^t = (r^t_{nk})$ was defined in [1, 3] as

$$r^t_{nk} = \begin{cases} \dfrac{t_k}{T_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

The Riesz sequence spaces r^t_∞, r^t_0 and r^t_c were introduced by Malkowsky [3] as follows:

$$r^t_\infty = (\ell_\infty)_{R^t}, \quad r^t_0 = (c_0)_{R^t} \quad \text{and} \quad r^t_c = (c)_{R^t}.$$

Altay and Başar [1] introduced the paranormed Riesz sequence spaces $r^t(p)$ as

$$r^t(p) = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k \right|^{p_n} < \infty \right\}.$$

The paranormed Riesz sequence spaces $r^t_\infty(p)$, $r^t_0(p)$ and $r^t_c(p)$ were studied by Altay and Başar [2] as follows:

$$r^t_\infty(p) = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{T_n} \sum_{k=0}^{n} t_k x_k \right|^{p_n} < \infty \right\},$$

$$r_0^t(p)=\left\{x=(x_k)\in w:\lim_{n\to\infty}\left|\frac{1}{T_n}\sum_{k=0}^{n}t_kx_k\right|^{p_n}=0\right\}\text{ and}$$
$$r_c^t(p)=\left\{x=(x_k)\in w:\lim_{n\to\infty}\left|\frac{1}{T_n}\sum_{k=0}^{n}t_kx_k-l\right|^{p_n}=0\text{ for some }l\in\mathbb{R}\right\}.$$

Since then various authors studied Riesz sequence spaces. One may refer to [1–7] and the references cited therein for more studies on Riesz sequence spaces. Following Altay and Başar [1, 2] and Baliarsingh [12], we construct a more generalised Riesz paranormed difference sequence spaces of fractional order and study in detail the related problems.

2. Riesz Difference Operator of Fractional Order and Sequence Spaces

In this section, we define the product matrix $R^t(\Delta^{(\alpha)})$, obtain its inverse, introduce paranormed Riesz difference sequence spaces of fractional order $r_\infty^t\left(p,\Delta^{(\alpha)}\right)$, $r_c^t\left(p,\Delta^{(\alpha)}\right)$ and $r_0^t\left(p,\Delta^{(\alpha)}\right)$ and give some topological properties of the spaces.

Combining the Riesz mean matrix R^t and the difference operator $\Delta^{(\alpha)}$, we obtain a new product matrix $R^t(\Delta^{(\alpha)})=(\tilde{r}_{nk}^t)$ given by

$$\tilde{r}_{nk}^t=\begin{cases}\sum\limits_{i=k}^{n}(-1)^{i-k}\dfrac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)}\dfrac{t_i}{T_n}, & \text{if } 0\le k\le n,\\ 0, & \text{if } k>n.\end{cases}$$

Equivalently,

$$R^t(\Delta^{(\alpha)})=\begin{bmatrix}1 & 0 & 0 & \dots\\ \dfrac{t_0}{T_1}-\alpha\dfrac{t_1}{T_1} & \dfrac{t_1}{T_1} & 0 & \dots\\ \dfrac{t_0}{T_2}-\alpha\dfrac{t_1}{T_2}+\dfrac{\alpha(\alpha-1)}{2!}\dfrac{t_2}{T_2} & \dfrac{t_1}{T_2}-\alpha\dfrac{t_2}{T_2} & \dfrac{t_2}{T_2} & \dots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

Now, by simple calculation, one may obtain the inverse of the matrix $R^t(\Delta^{(\alpha)})$ as given in the following lemma.

Lemma 2.1. *The inverse of the product matrix $R^t(\Delta^{(\alpha)})$ is given by*

$$(R^t(\Delta^{(\alpha)}))^{-1}_{nk}=\begin{cases}(-1)^{n-k}\sum\limits_{j=k}^{k+1}\dfrac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)}\dfrac{T_k}{t_j}, & \textit{if } 0\le k<n,\\ \dfrac{T_n}{t_n}, & \textit{if } k=n,\\ 0, & \textit{if } k>n.\end{cases}$$

Let us define a sequence $y = (y_n)$ which will be frequently used as the $R^t(\Delta^{(\alpha)})$-transform of the sequence $x = (x_k)$ as follows:

$$y_n = \sum_{k=0}^{n-1} \left[\sum_{i=k}^{n} (-1)^{i-k} \frac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)} \frac{t_i}{T_n} \right] x_k + \frac{t_n}{T_n} x_n, \quad n \in \mathbb{N}. \tag{2.1}$$

Now, we define the paranormed Riesz difference sequence spaces of fractional order α, $r^t_\infty \left(p, \Delta^{(\alpha)}\right)$, $r^t_c \left(p, \Delta^{(\alpha)}\right)$ and $r^t_0 \left(p, \Delta^{(\alpha)}\right)$ as follows:

$$\begin{aligned} r^t_\infty \left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in \ell_\infty(p) \right\}, \\ r^t_c \left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in c(p) \right\}, \\ r^t_0 \left(p, \Delta^{(\alpha)}\right) &= \left\{ x = (x_n) \in w : R^t(\Delta^{(\alpha)})x \in c_0(p) \right\}. \end{aligned}$$

Using the notation (1.3), the above sequence spaces may be rewritten as:

$$\begin{aligned} r^t_\infty \left(p, \Delta^{(\alpha)}\right) &= (\ell_\infty(p))_{R^t(\Delta^{(\alpha)})}, \\ r^t_c \left(p, \Delta^{(\alpha)}\right) &= (c(p))_{R^t(\Delta^{(\alpha)})}, \\ r^t_0 \left(p, \Delta^{(\alpha)}\right) &= (c_0(p))_{R^t(\Delta^{(\alpha)})}. \end{aligned}$$

The above sequence spaces reduce to the following classes of sequence spaces in the special cases of α and $p = (p_k)$:

1. if $\alpha = 0$ then above classes reduce to $X(p)$ for $X = \{r^t_\infty, r^t_c, r^t_0\}$ as studied by Altay and Başar [2], which further reduce to X in the case of $p = (p_k) = e$ as studied by Malkowsky [3];
2. if $\alpha = 1$ then above classes reduce to $X(p, \Delta^{(1)})$ for $X = \{r^t_\infty, r^t_c, r^t_0\}$, where $(\Delta^{(1)}x)_k = x_k - x_{k-1}$;
3. if $\alpha = m$ then above classes reduce to $X(p, \Delta^{(m)})$ for $X = \{r^t_\infty, r^t_c, r^t_0\}$, where $(\Delta^{(m)}x)_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{m-j}$.

We begin with the following result.

Lemma 2.2. *The operator $R^t(\Delta^{(\alpha)}) : w \to w$ is linear.*

Proof. The proof is a routine verification and hence omitted. □

Theorem 2.1. *The sequence space $r^t_0 \left(\Delta^{(\alpha)}\right)$ is a linear metric space paranormed by*

$$g_{\Delta^{(\alpha)}}(x) = \sup_{k \in \mathbb{N}} \left| \left(R^t(\Delta^{(\alpha)})x \right)_k \right|^{\frac{p_k}{M}}. \tag{2.2}$$

$g_{\Delta^{(\alpha)}}$ is paranorm for the spaces $r^t_\infty(p, \Delta^{(\alpha)})$ and $r^t_c(p, \Delta^{(\alpha)})$ only in the trivial case, with $\inf p_k > 0$ when $r^t_\infty(p, \Delta^{(\alpha)}) = r^t_\infty(\Delta^{(\alpha)})$ and $r^t_c(p, \Delta^{(\alpha)}) = r^t_0(\Delta^{(\alpha)})$.

Proof. We prove the theorem for the space $r^t_0(\Delta^{(\alpha)})$.

Clearly, $g_{\Delta^{(\alpha)}}(\theta) = 0$ and $g_{\Delta^{(\alpha)}}(-x) = g_{\Delta^{(\alpha)}}(x)$ for all $x \in r^t_0 \left(\Delta^{(\alpha)}\right)$. To show the linearity of $g_{\Delta^{(\alpha)}}$ with respect to coordinate wise addition and scalar multiplication, we take any two sequences $u, v \in r_0^t(p, \Delta^{(\alpha)})$ and scalars α_1 and α_2 in $\mathbb{R}$. Since $R^t(\Delta^{(\alpha)})$ is linear and using Maddox [45], we get

$$\begin{aligned}
&g_{\Delta^{(\alpha)}}(\alpha_1 u + \alpha_2 v)\\
&= \sup_k \left| \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)} \frac{t_i}{T_k} \right] (\alpha_1 u_j + \alpha_2 v_j) + \frac{t_k}{T_k}(\alpha_1 u_k + \alpha_2 v_k) \right|^{\frac{p_k}{M}}\\
&\leq \max\{1, |\alpha_1|\} \sup_k \left| \left(R^t(\Delta^{(\alpha)}) u \right)_k \right|^{\frac{p_k}{M}} + \max\{1, |\alpha_2|\} \sup_k \left| \left(R^t(\Delta^{(\alpha)}) v \right)_k \right|^{\frac{p_k}{M}}\\
&= \max\{1, |\alpha_1|\} g_{\Delta^{(\alpha)}}(u) + \max\{1, |\alpha_2|\} g_{\Delta^{(\alpha)}}(v).
\end{aligned}$$

This follows the subadditivity of $g_{\Delta^{(\alpha)}}$, i.e.,

$$g_{\Delta^{(\alpha)}}(x+y) \leq g_{\Delta^{(\alpha)}}(x) + g_{\Delta^{(\alpha)}}(y), \quad \text{for all } x, y \in r_0^t\left(p, \Delta^{(\alpha)}\right).$$

Let $\{x^n\}$ be any sequence of points in $r_0^t(p, \Delta^{(\alpha)})$ such that $g_{\Delta^{(\alpha)}}(x^n - x) \to 0$ and also (β_n) be any sequence of scalars such that $\beta_n \to \beta$ as $n \to \infty$. Then by using the subadditivity of $g_{\Delta^{(\alpha)}}$, we get

$$g_{\Delta^{(\alpha)}}(x^n) \leq g_{\Delta^{(\alpha)}}(x) + g_{\Delta^{(\alpha)}}(x^n - x).$$

Now, since $\{g_{\Delta^{(\alpha)}}(x^n)\}$ is bounded, we have

$$\begin{aligned}
g_{\Delta^{(\alpha)}}(\beta_n x^n - \beta x) = \sup_k &\Bigg| \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)} \frac{t_i}{T_k} \right] (\beta_n x_j^n - \beta x_j)\\
&+ \frac{t_k}{T_k}(\beta_n x_k^n - \beta x_k) \Bigg|^{\frac{p_k}{M}}\\
\leq\ & |\beta_n - \beta|^{\frac{p_k}{M}} g_{\Delta^{(\alpha)}}(x^n) + |\beta|^{\frac{p_k}{M}} g_{\Delta^{(\alpha)}}(x^n - x)\\
\to\ & 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, scalar multiplication for $g_{\Delta^{(\alpha)}}$ is continuous. Consequently, $g_{\Delta^{(\alpha)}}$ is a paranorm on the sequence space $r_0^t(p, \Delta^{(\alpha)})$. This completes the proof of the theorem. □

Theorem 2.2. *The sequence space $r_0^t(p, \Delta^{(\alpha)})$ is a complete linear metric space paranormed by $g_{\Delta^{(\alpha)}}$ defined in* (2.2).

Proof. Let $x^i = \{x_k^{(i)}\}$ be any Cauchy sequence in $r_0^t(p, \Delta^{(\alpha)})$. Then for $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$ such that

$$g_{\Delta^{(\alpha)}}(x^i - x^j) < \varepsilon,$$

for all $i, j \geq N_0(\varepsilon)$. This implies that $\{(R^t(\Delta^{(\alpha)})x^0)_k, (R^t(\Delta^{(\alpha)})x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for each fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, the sequence $((R^t(\Delta^{(\alpha)})x^i)_k)$ converges. We assume that $(R^t(\Delta^{(\alpha)})x^i)_k \to (R^t(\Delta^{(\alpha)})x)_k$ as $i \to \infty$. Now, for each $k \in \mathbb{N}$, $j \to \infty$ and $i \geq N_0(\varepsilon)$, it is clear that

$$\left| (R^t(\Delta^{(\alpha)})x^i)_k - (R^t(\Delta^{(\alpha)})x)_k \right| < \frac{\varepsilon}{2}. \tag{2.3}$$

Again, $x^i = \{x_k^{(i)}\} \in r_0^t(p, \Delta^{(\alpha)})$. This implies that

$$\left|(R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}, \tag{2.4}$$

for all $k \in \mathbb{N}$. Therefore, using (2.3) and (2.4), we obtain

$$\begin{aligned}\left|(R^t(\Delta^{(\alpha)})x)_k\right|^{\frac{p_k}{M}} &\leq \left|(R^t(\Delta^{(\alpha)})x)_k - (R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} + \left|(R^t(\Delta^{(\alpha)})x^i)_k\right|^{\frac{p_k}{M}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for all $i \geq N_0(\varepsilon)$. This shows that the sequence $((R^t(\Delta^{(\alpha)})x)_k)$ belongs to the space $c_0(p)$. Since (x^i) is any arbitrary Cauchy sequence, the space $r_0^t(p, \Delta^{(\alpha)})$ is complete. □

Theorem 2.3. *The paranormed Riesz difference sequence spaces* $r_0^t(p, \Delta^{(\alpha)})$, $r_c^t(p, \Delta^{(\alpha)})$ *and* $r_\infty^t(p, \Delta^{(\alpha)})$ *are linearly isomorphic to* $c_0(p)$, $c(p)$ *and* $\ell_\infty(p)$, *respectively, where* $0 < p_k \leq H < \infty$.

Proof. We prove the result for the space $r_\infty^t(p, \Delta^{(\alpha)})$. Using the notation (2.1), we define a mapping $\varphi : r_\infty^t(p, \Delta^{(\alpha)}) \to \ell_\infty(p)$ by $x \mapsto y = \varphi x$. Clearly, φ is linear and $x = 0$ whenever $\varphi x = 0$. Thus, φ is injective.

Let $y = (y_k) \in \ell_\infty(p)$ and using (2.1) define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1}\left[\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{T_j}{t_i}y_j\right] + \frac{T_k}{t_k}y_k, \quad k \in \mathbb{N}. \tag{2.5}$$

Then

$$\begin{aligned} g_{\Delta^{(\alpha)}}(x) &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}(-1)^{i-j}\frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)}\frac{t_i}{T_k}\right]x_j + \frac{t_k}{T_k}x_k\right|^{\frac{p_k}{M}} \\ &= \sup_{k\in\mathbb{N}}\left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{\frac{p_k}{M}} \\ &= \sup_{k\in\mathbb{N}}|y_k|^{\frac{p_k}{M}} < \infty, \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, $x \in r_\infty^t(p, \Delta^{(\alpha)})$. Consequently, φ is surjective and paranorm preserving. Thus, $r_\infty^t(p, \Delta^{(\alpha)}) \cong \ell_\infty(p)$. □

3. Schauder Basis

In this section, we shall construct the Schauder basis for the sequence spaces $r_0^t(p, \Delta^{(\alpha)})$ and $r_c^t(p, \Delta^{(\alpha)})$.

We recall that a sequence (x_k) of a normed space $(X, \|\cdot\|)$ is called a Schauder basis for X if for every $u \in X$ there exist a unique sequence of scalars (a_k) such that

$$\lim_{n\to\infty} \left\| u - \sum_{k=0}^{n} a_k x_k \right\| = 0.$$

Theorem 3.1. *Let $\lambda_k(t) = (R^t(\Delta^{(\alpha)})x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(t) = (b_n^{(k)}(t))$ of the elements of the space $r_0^t(p, \Delta^{(\alpha)})$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(t) = \begin{cases} \sum_{i=j}^{j+1} (-1)^{k-j} \dfrac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)} \dfrac{T_j}{t_i}, & \text{if } k < n, \\ \dfrac{T_n}{t_n}, & \text{if } k = n, \\ 0, & k > n. \end{cases}$$

Then

(a) *the sequence $(b^{(k)}(t))$ is basis for the space $r_0^t(p, \Delta^{(\alpha)})$ and every $x \in r_0^t(p, \Delta^{(\alpha)})$ has a unique representation of the form*

$$x = \sum_k \lambda_k(t) b^{(k)}(t); \tag{3.1}$$

(b) *the set $\{(R^t(\Delta^{(\alpha)}))^{-1}e, b^{(k)}(t)\}$ is a basis for the space $r_c^t(p, \Delta^{(\alpha)})$ and every $x \in r_c^t(p, \Delta^{(\alpha)})$ has a unique representation of the form*

$$x = le + \sum_k |\lambda_k(t) - l| \, b^{(k)}(t),$$

where $l = \lim_{k\to\infty}(R^t(\Delta^{(\alpha)})x)_k$.

Proof. (a) By the definition of $R^t(\Delta^{(\alpha)})$ and $b^{(k)}(t)$, it is clear that

$$\left(R^t(\Delta^{(\alpha)})b^{(k)}(t)\right) = e^{(k)} \in c_0(p), \tag{3.2}$$

for $0 < p_k \leq H < \infty$. Let $x \in r_0^t(p, \Delta^{(\alpha)})$ and for every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(t) b^{(k)}(t). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$R^t(\Delta^{(\alpha)})x^{[m]} = \sum_{k=0}^{m} \lambda_k(t) R^t(\Delta^{(\alpha)}) b^{(k)}(t) = (R^t(\Delta^{(\alpha)})x)_k e^{(k)},$$

and

$$\left(R^t(\Delta^{(\alpha)})(x - x^{[m]})\right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (R^t(\Delta^{(\alpha)})x)_i, & \text{if } i > m. \end{cases}$$

Now, for $\varepsilon > 0$ there exists an integer m_0 such that

$$\sup_{i \geq m} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$\begin{aligned} g_{\Delta^{(\alpha)}}\left(x - x^{[m]}\right) &= \sup_{i \geq m} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} \\ &\leq \sup_{i \geq m_0} \left|(R^t(\Delta^{(\alpha)})x)_i\right|^{\frac{p_k}{M}} < \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

for all $m \geq m_0$.

To show the uniqueness of the representation, we suppose that

$$x = \sum_k \mu_k(t) b^{(k)}(t).$$

Then, we have

$$\begin{aligned} (R^t(\Delta^{(\alpha)})x)_n &= \sum_k \mu_k(t) \left(R^t(\Delta^{(\alpha)}) b^{(k)}(t)\right)_n \\ &= \sum_k \mu_k(t) e_n^{(k)} = \mu_n(t), \quad n \in \mathbb{N}. \end{aligned}$$

This contradicts the fact that $(R^t(\Delta^{(\alpha)})x)_k = \lambda_k(t)$, $k \in \mathbb{N}$. Thus, the representation (3.1) is unique.

(b) The proof is analogous to the previous theorem and hence omitted. $\square$

4. α-, β- and γ-duals

In this section we shall compute α-, β- and γ-duals of $r_0^t(\Delta^{(\alpha)})$, $r_c^t(\Delta^{(\alpha)})$ and $r_\infty^t(\Delta^{(\alpha)})$. Note that the notation α used for α-dual has different meaning to that of the operator $\Delta^{(\alpha)}$.

For the sequence spaces X and Y, define multiplier sequence space $M(X, Y)$ by

$$M(X, Y) = \{p = (p_k) \in w : px = (p_k x_k) \in Y, \text{ for all } x = (x_k) \in X\}.$$

Then the α-, β- and γ-duals of X are given by

$$X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, cs), \quad X^\gamma = M(X, bs),$$

respectively. Now, we give the following lemmas given in [41] which will be used to obtain the duals. Throughout $\mathcal{F}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 4.1. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statement hold:*

(a) $A \in (\ell_\infty(p), \ell(q))$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} B^{\frac{1}{p_k}} \right|^{q_n} < \infty, \quad \text{for all integers } B > 1 \text{ and } q_n \geq 1 \text{ for all } n;$$

(b) $A \in (\ell_\infty(p), \ell_\infty(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| B^{\frac{1}{p_k}} \right)^{q_n} < \infty, \quad \text{for all integers } B > 1;$$

(c) $A \in (\ell_\infty(p), c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{\frac{1}{p_k}} < \infty, \quad \text{for all integers } B > 1,$$

$$\text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} \left(\sum_k |a_{nk} - \alpha_k| B^{\frac{1}{p_k}} \right)^{q_n} = 0, \quad \text{for all } B > 1;$$

(d) $A \in (\ell_\infty(p), c_0(q))$ *if and only if*

$$\lim_{n\to\infty} \left(\sum_k |a_{nk}| B^{\frac{1}{p_k}} \right)^{q_n} = 0, \quad \text{for all integers } B > 1.$$

Lemma 4.2. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statement hold:*

(a) $A \in (c_0(p), \ell_\infty(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| B^{\frac{-1}{p_k}} \right)^{q_n} < \infty, \quad \text{for all integers } B > 1; \tag{4.1}$$

(b) $A \in (c_0(p), c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{\frac{-1}{p_k}} < \infty \ , \quad \text{for all integers } B > 1, \tag{4.2}$$

$$\text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \sup_{n\in\mathbb{N}} \sum_k |a_{nk} - \alpha_k| M^{\frac{1}{q_n}} B^{\frac{-1}{p_k}} < \infty, \tag{4.3}$$

for all integers $M, B > 1$,

$$\text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0, \quad \text{for all } k \in \mathbb{N}; \tag{4.4}$$

(c) $A \in (c_0(p), c_0(q))$ *if and only if*

$$\text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{\frac{1}{q_n}} B^{\frac{-1}{p_k}} < \infty, \quad \text{for all integers } M, B > 1, \tag{4.5}$$

$$\text{exists } (\alpha_k) \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} |a_{nk}|^{q_n} = 0, \quad \text{for all } k \in \mathbb{N}. \tag{4.6}$$

Lemma 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statement hold:*

(a) $A \in (c(p), \ell_\infty(q))$ *if and only if* (4.1) *holds and*

$$\sup_{n\in\mathbb{N}} \left|\sum_k a_{nk}\right|^{q_n} < \infty;$$

(b) $A \in (c(p), c(q))$ *if and only if* (4.2), (4.3) *and* (4.4) *hold and*

$$\text{exists } \alpha \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \left|\sum_k a_{nk} - \alpha\right|^{q_n} = 0;$$

(c) $A \in (c(p), c_0(q))$ *if and only if* (4.5) *and* (4.6) *hold and*

$$\lim_{n\to\infty} \left|\sum_k a_{nk}\right|^{q_n} = 0.$$

Theorem 4.1. *Define the sets* $\nu_1(p)$, $\nu_2(p)$, $\nu_3(p)$, $\nu_4(p)$, $\nu_5(p)$ *and* $\nu_6(p)$ *as follows:*

$$\nu_1(p) = \bigcap_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| B^{\frac{1}{p_k}} < \infty \Bigg\},$$

$$\nu_2(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty \text{ and } \left(\frac{a_k T_k}{t_k} B^{\frac{1}{p_k}}\right) \in c_0 \right\},$$

$$\nu_3(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{1}{p_k}} < \infty \text{ and } \left\{\Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k\right\} \in \ell_\infty \right\},$$

$$\nu_4(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| B^{\frac{-1}{p_k}} < \infty \Bigg\},$$

$$\nu_5(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in w :$$
$$\sum_n \left| \sum_k \left[\sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \frac{T_k}{t_j} a_k + \frac{T_n}{t_n} a_n \right] \right| < \infty \Bigg\},$$

$$\nu_6(p) = \bigcap_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) T_k \right| B^{\frac{-1}{p_k}} < \infty \right\},$$

where

$$\Delta^{(\alpha)} \left(\frac{a_k}{t_k}\right) = \frac{a_k}{t_k} + \sum_{j=k+1}^{n} (-1)^{j-k} a_j \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}. \tag{4.7}$$

Then

$$\left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\alpha = \nu_1(p), \quad \left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\beta = \nu_2(p), \quad \left[r_\infty^t(p, \Delta^{(\alpha)})\right]^\gamma = \nu_3(p),$$

$$\left[r_c^t(p,\Delta^{(\alpha)})\right]^\alpha = \nu_4(p)\cap\nu_5(p), \quad \left[r_c^t(p,\Delta^{(\alpha)})\right]^\beta = \nu_6(p)\cap cs,$$
$$\left[r_c^t(p,\Delta^{(\alpha)})\right]^\gamma = \nu_6(p)\cap bs, \quad \left[r_0^t(p,\Delta^{(\alpha)})\right]^\alpha = \nu_4(p),$$
$$\left[r_0^t(p,\Delta^{(\alpha)})\right]^\beta = \left[r_0^t(p,\Delta^{(\alpha)})\right]^\gamma = \nu_6(p).$$

Proof. We prove the theorem for the space $r_\infty^t(p,\Delta^{(\alpha)})$. Consider the sequence $a=(a_k)\in w$ and $x=(x_k)$ is as defined in (2.5), then we have

$$\begin{aligned} a_nx_n &= \sum_{j=0}^{n-1}\left[\sum_{i=j}^{j+1}(-1)^{n-j}\frac{\Gamma(-\alpha+1)}{(n-i)!\Gamma(-\alpha-n+i+1)}\frac{T_j}{t_i}a_ny_j\right]+\frac{T_n}{t_n}a_ny_n \\ &= (Gy)_n, \quad \text{for each } n\in\mathbb{N}, \end{aligned} \tag{4.8}$$

where $G=(g_{nk})$ is a matrix defined by

$$g_{nk}=\begin{cases} \displaystyle\sum_{j=k}^{k+1}(-1)^{n-k}\frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)}\frac{T_k}{t_j}a_n, & \text{if } 0\le k<n,\\ \dfrac{T_n}{t_n}a_n, & \text{if } k=n,\\ 0, & \text{if } k>n.\end{cases}$$

Thus, we deduce from (4.8) that $ax=(a_nx_n)\in\ell_1$ whenever $x=(x_k)\in r_\infty^t(p,\Delta^{(\alpha)})$ if and only if $Gy\in\ell_1$ whenever $y=(y_k)\in\ell_\infty(p)$. This yields that $a=(a_n)\in\left[r_\infty^t(p,\Delta^{(\alpha)})\right]^\alpha$ if and only if $G\in(\ell_\infty(p),\ell_1)$.

Thus, by using Lemma 4.1(a) with $q_n=1$ for all n, we conclude that

$$\left[r_\infty^t(p,\Delta^{(\alpha)})\right]^\alpha=\nu_1(p).$$

Now, consider the following equation

$$\begin{aligned} \sum_{k=0}^{n}a_kx_k &= \sum_{k=0}^{n}a_k\left[\sum_{j=0}^{k-1}\left(\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{T_j}{t_i}y_j\right)+\frac{T_k}{t_k}y_k\right] \\ &= \sum_{k=0}^{n-1}y_kT_k\left(\frac{a_k}{t_k}+\sum_{j=k+1}^{n}(-1)^{j-k}a_j\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}\right)+\frac{T_n}{t_n}a_ny_n \\ &= \sum_{k=0}^{n-1}y_kT_k\Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)+\frac{T_n}{t_n}a_ny_n \quad (4.9) \\ &= (Hy)_n, \quad \text{for each } n\in\mathbb{N}, \quad (4.10) \end{aligned}$$

where $H = (h_{nk})$ is a matrix defined by

$$h_{nk} = \begin{cases} \Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)T_k, & \text{if } 0 \le k < n, \\ \frac{T_n}{t_n}a_n, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

and $\Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)$ is as defined in (4.7). Thus, we deduce from (4.10) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in r^t_\infty(p, \Delta^{(\alpha)})$ if and only if $Hy \in c$ whenever $y = (y_k) \in \ell_\infty(p)$. Therefore, by using Lemma 4.1 (c) with $q = (q_n) = 1$, we get that

$$\sum_k \left|\Delta^{(\alpha)}\left(\frac{a_k}{t_k}\right)T_k\right| B^{\frac{1}{p_k}} < \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{T_k}{t_k}a_k B^{\frac{1}{p_k}} = 0.$$

Thus, $\left[r^t_\infty(p, \Delta^{(\alpha)})\right]^\beta = \nu_2(p)$.

Similarly, by using Lemma 4.1 (b), with $q_n = 1$ for all n, we can deduce that $\left[r^t_\infty(p, \Delta^{(\alpha)})\right]^\gamma = \nu_3(p)$. This completes the proof of the theorem. The duals of the other spaces can be obtained by the similar proceedings and using Lemma 4.2 and 4.3. □

5. Matrix Transformations

In this section, we give certain results regarding matrix transformation of the Riesz sequence spaces of fractional order to $X(p)$ where $X = \{\ell_\infty, c, c_0\}$. Let $q = (q_n)$ be a non-decreasing bounded sequence of positive real numbers. For brevity, we write

$$\Delta^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right) = \frac{a_{nk}}{t_k} + \sum_{j=k+1}^{n}(-1)^{j-k}a_{nj}\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}$$

and

$$\Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right) = \frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty}(-1)^{j-k}a_{nj}\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i},$$

for all $n, k \in \mathbb{N}$. Let $x, y \in w$ be connected by the relation $y = R^t(\Delta^{(\alpha)})x$. Then we have by (4.9)

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1}\Delta^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k y_k + a_{nm}\frac{t_m}{T_m}y_m, \qquad n, m \in \mathbb{N}. \tag{5.1}$$

Now, let us consider the following conditions before we proceed:

$$\lim_{k\to\infty}\frac{a_{nk}}{t_k}T_k B^{\frac{1}{p_k}} = 0, \qquad \text{for all } n, B \in \mathbb{N}, \tag{5.2}$$

$$\sup_{n\in\mathbb{N}}\left[\sum_k\left|\Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right)T_k\right|B^{\frac{1}{p_k}}\right]^{q_n} < \infty, \qquad \text{for all } B \in \mathbb{N}, \tag{5.3}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\Delta^{(\alpha)}_\infty\left(\frac{a_{nk}}{t_k}\right)T_k\right|B^{\frac{1}{p_k}} < \infty, \qquad \text{for all } B \in \mathbb{N}, \tag{5.4}$$

(5.5) exists $(\alpha_k) \subset \mathbb{R}$ such that $\lim_{n\to\infty} \left[\sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right| B^{\frac{1}{p_k}}\right]^{q_n} = 0,$

for all $B \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}} \left[\sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k\right| B^{\frac{-1}{p_k}}\right]^{q_n} < \infty, \quad \text{for all } B \in \mathbb{N}, \tag{5.6}$$

$$\sup_{n\in\mathbb{N}} \left|\sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k\right|^{q_n} < \infty, \quad \text{for all } n \in \mathbb{N}, \tag{5.7}$$

(5.8) exists $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} \left|\sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha\right|^{q_n} = 0,$

(5.9) exists $(\alpha_k) \subset \mathbb{R}$ such that $\lim_{n\to\infty} \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right|^{q_n} = 0, \quad$ for all $k \in \mathbb{N}$,

(5.10) exists $(\alpha_k) \subset \mathbb{R}$ such that $\sup_{n\in\mathbb{N}} L^{\frac{1}{q_n}} \sum_k \left|\Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k - \alpha_k\right| B^{\frac{-1}{p_k}} < \infty,$

for all L exists $B \in \mathbb{N}$.

Theorem 5.1. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) *$A \in (r_\infty^t(p,\Delta^{(\alpha)}), \ell_\infty(q))$ if and only if* (5.2) *and* (5.3) *hold;*
(b) *$A \in (r_\infty^t(p,\Delta^{(\alpha)}), c(q))$ if and only if* (5.2), (5.4) *and* (5.5) *hold;*
(c) *$A \in (r_\infty^t(p,\Delta^{(\alpha)}), c_0(q))$ if and only if* (5.2) *holds and* (5.5) *holds, with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Proof. We give the proof of (a) as the rest can be obtained in the similar manner. Let $A = (a_{nk}) \in (r_\infty^t(p,\Delta^{(\alpha)}), \ell_\infty(q))$ and $x = (x_k) \in r_\infty^t(p,\Delta^{(\alpha)})$. Consider equation (5.1). Since Ax exists and belongs to the space $\ell_\infty(q)$, therefore the necessity of the condition (5.2) is obvious. Now, letting $m \to \infty$ in equation (5.1), we straightly get

$$\begin{aligned} Ax &= \sum_k \left(\frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)t_i}\right) T_k y_k \\ &= \sum_k \Delta_\infty^{(\alpha)}\left(\frac{a_{nk}}{t_k}\right)T_k y_k. \end{aligned} \tag{5.11}$$

This implies that $A(R^t(\Delta^{(\alpha)}))^{-1}y \in \ell_\infty(q)$. That is, $A(R^t(\Delta^{(\alpha)}))^{-1} \in (\ell_\infty(p), \ell_\infty(q))$. Therefore, $A(R^t(\Delta^{(\alpha)}))^{-1}$ satisfies the lemma 4.1(b) which is equivalent to the condition (5.3). This shows the necessity of the condition (5.3).

Conversely, let the conditions (5.2) and (5.3) hold and $x \in r_\infty^t(p,\Delta^{(\alpha)})$. Then it is clear that Ax exists. Now, using equation (5.11) and the condition (5.3) with $B > \max\{1, \sup_k |y_k|^{p_k}\}$, we get

$$\|Ax\|_{\ell_\infty(q)} = \sup_{n\in\mathbb{N}} \left|\sum_k a_{nk}x_k\right|^{q_n}$$

$$
\begin{aligned}
&= \sup_{n\in\mathbb{N}} \left| \sum_k \left(\frac{a_{nk}}{t_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)} \right) T_k y_k \right|^{q_n} \\
&\le \sup_{n\in\mathbb{N}} \left(\sum_k \left| \Delta_{\infty}^{(\alpha)} \left(\frac{a_{nk}}{t_k} \right) T_k y_k \right| \right)^{q_n} \\
&\le \sup_{n\in\mathbb{N}} \left(\sum_k \left| \Delta_{\infty}^{(\alpha)} \left(\frac{a_{nk}}{t_k} \right) T_k \right| B^{\frac{1}{p_k}} \right)^{q_n} < \infty.
\end{aligned}
$$

This concludes that $A \in (r_{\infty}^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$. □

By the similar proceedings, we can derive the following results.

Theorem 5.2. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) $A \in (r_c^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$ *if and only if* (5.2), (5.6) *and* (5.7) *hold;*
(b) $A \in (r_c^t(p, \Delta^{(\alpha)}), c(q))$ *if and only if* (5.2), (5.8), (5.9) *and* (5.10) *hold and* (5.6) *also holds, with $q_n = 1$ for all $n \in \mathbb{N}$;*
(c) $A \in (r_c^t(p, \Delta^{(\alpha)}), c_0(q))$ *if and only if* (5.2) *holds and* (5.8), (5.9) *and* (5.10) *also hold, with $\alpha = 0$, $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 5.3. *Let $A = (a_{nk})$ be an infinite matrix. Then the following hold:*

(a) $A \in (r_0^t(p, \Delta^{(\alpha)}), \ell_{\infty}(q))$ *if and only if* (5.2) *and* (5.6) *hold;*
(b) $A \in (r_0^t(p, \Delta^{(\alpha)}), c(q))$ *if and only if* (5.2), (5.9) *and* (5.10) *hold and* (5.6) *also holds, with $q_n = 1$ for all $n \in \mathbb{N}$;*
(c) $A \in (r_0^t(p, \Delta^{(\alpha)}), c_0(q))$ *if and only if* (5.2) *holds and* (5.9) *and* (5.10) *also hold, with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Conclusion

In this article, we introduce paranormed difference sequence spaces $r_{\infty}^t(\Delta^{(\alpha)})$, $r_c^t(\Delta^{(\alpha)})$ and $r_0^t(\Delta^{(\alpha)})$ of fractional order α, investigate their topological properties, Schauder basis, α-, β- and γ- duals and characterize the matrix classes related to these spaces. We conclude that the results obtained from the matrix domain of the product matrix $R^t(\Delta^{(\alpha)})$ are more general and extensive than the existent results of the previous authors. We expect that our results might be a reference for further studies in this field. In our next paper, we will investigate the results obtained from the matrix domain $R^t(\Delta^{(\alpha)})$ in the spaces ℓ_p of absolutely p-summable sequences, $1 \le p < \infty$.

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