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EXISTENCE THEOREMS FOR A COUPLED SYSTEM OF NONLINEAR MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

BASHIR AHMAD[1], AHMED ALSAEDI[1], NAJLA ALGHAMDI[2,1],
AND SOTIRIS K. NTOUYAS[3,1]

Abstract. We discuss the existence and uniqueness of solutions for a coupled system of nonlinear multi-term fractional differential equations complemented with coupled nonlocal boundary conditions by applying the methods of modern functional analysis. An example illustrating the uniqueness result is presented. Some interesting observations are also described.

1. Introduction

The topic of boundary value problems has been fascinating due to its extensive applications in applied and technical sciences. In recent years, an overwhelming interest has been shown in the study of fractional differential equations and inclusions equipped with a variety of boundary conditions, for instance, see [1,2,25,26,28,30] and the references cited therein. Coupled systems of fractional-order differential equations also constitute an important area of investigation in view of occurrence of such systems in disease models [9,10], chaos [31], ecology [16] and so forth. Some recent theoretical work on the topic can be found in the articles [3,4,6,7,12,29].

On the other hand, coupled systems involving more than one fractional order differential operators need to be addressed further to strengthen the hot topic of boundary value problems. Examples include Bagley-Torvik [27] and Basset [20]

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equations. For some recent results on multi-term (sequential) fractional differential equations, see [5, 8, 18, 19].

The nonlocal nature of fractional order operators is the key factor in the popularity of fractional calculus, which has extended the scope of the existing integer-order models by providing their fractional order counterparts. Examples include fractional reaction-diffusion systems [13], anomalous diffusion [15], chaotic neuron model [23], groundwater hydrology [24] and so forth. For more details, we refer the reader to texts [14, 21, 22].

Motivated by recent work on fractional order coupled systems, we introduce and study a coupled system of multi-term fractional differential equations:

$$\begin{cases} L_{a_i}^r u(t) = f(t, u(t), v(t)), & 0 < r < 1, \\ L_{b_i}^p v(t) = g(t, u(t), v(t)), & 0 < p < 1, \end{cases} \tag{1.1}$$

complemented with nonlocal multi-point coupled boundary conditions:

$$\begin{cases} u(0) = 0, \quad u'(0) = 0, \quad u(1) = \sum_{i=1}^{\rho} \alpha_i v(\eta_i), \\ v(0) = 0, \quad v'(0) = 0, \quad v(1) = \sum_{j=1}^{h} \beta_j u(\xi_j), \quad \eta_i < \xi_j, \quad \text{for all } i, j, \end{cases} \tag{1.2}$$

where

$$L_{a_i}^r = a_2 \, {}^cD^{r+2} + a_1 \, {}^cD^{r+1} + a_0 \, {}^cD^r, \quad L_{b_i}^p = b_2 \, {}^cD^{p+2} + b_1 \, {}^cD^{p+1} + b_0 {}^cD^p,$$

${}^cD^q$ is the Caputo-type fractional derivative of order $q = r, p$, $f, g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and a_i, b_i, $i = 0, 1, 2$, are real constants such that $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$ with $a_2 \neq 0 \neq b_2$. The existence and uniqueness results for the problem (1.1)–(1.2) are derived via Leray-Schauder alternative and Banach fixed point theorem respectively.

The rest of the paper is arranged as follows. In Section 2, we recall some preliminary concepts of fractional calculus and present an auxiliary lemma. The main results and an illustrative are presented in Section 3. The paper concludes with some interesting observations.

2. Basic Results

We begin this section with some preliminary concepts of fractional calculus [17, 32].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}$, $\alpha > 0$, for a locally integrable real-valued function χ on $-\infty \leq a < t < b \leq +\infty$ is defined by

$$I_a^\alpha \chi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s) ds,$$

where Γ is the Euler gamma function.

Definition 2.2. Let χ, $\chi^{(m)} \in L^1[a,b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann-Liouville fractional derivative of χ of order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$, is defined as

$$D_a^\alpha \chi(t) = \frac{d^m}{dt^m} I_a^{1-\alpha}\chi(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} \chi(s) ds,$$

while the Caputo fractional derivative χ of order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$, is defined by

$${}^cD_a^\alpha \chi(t) = D_a^\alpha \left[\chi(t) - \chi(a) - \chi'(a)\frac{(t-a)}{1!} - \cdots - \chi^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right].$$

Remark 2.1. If $\chi \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, is defined as

$${}^cD_a^\alpha \chi(t) = I_a^{1-\alpha}\chi^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \chi^{(m)}(s) ds.$$

In our analysis, I^α and ${}^cD^\alpha$ respectively denote Riemann-Liouville fractional integral and Caputo fractional derivative, with $a = 0$.

Lemma 2.1 ([17])**.** *For $\phi \in C(0,1) \cap L(0,1)$ holds:*

$$I^\alpha({}^cD^\alpha \varphi(t)) = \varphi(t) - c_0 - c_1 t - \cdots - c_{n-1}t^{n-1}, \quad t > 0,\ n-1 < \alpha < n,$$

where c_i, $i = 1, \dots, n-1$, are arbitrary constants.

Definition 2.3. A pair of functions $u, v \in C([0,1],\mathbb{R})$ satisfying the equations (1.1) and the boundary conditions (1.2) is called a solution of the problem (1.1)–(1.2), where it is assumed that u, v possess the Caputo fractional derivative of order $r+2$ and $p+2$ respectively on $(0,1)$.

We need the following auxiliary lemma, which concerns the linear variant of problem (1.1)–(1.2).

Lemma 2.2. *Let $a_1^2 - 4a_2a_0 = 0$, $b_1^2 - 4b_2b_0 = 0$, $a_2 \neq 0$, $b_2 \neq 0$ and $w, z \in C([0,1],\mathbb{R})$. Then the solution (u,v) (in the sense of Definition 2.3) of the system of linear fractional differential equations*

$$\begin{cases} L_{a_i}^r u(t) = w(t), & 0 < r < 1, \\ L_{b_i}^p v(t) = z(t), & 0 < p < 1, \end{cases} \tag{2.1}$$

supplemented with the boundary conditions (1.2) is given by

$$\begin{aligned} u(t) =& \frac{1}{a_2}\int_0^t \int_0^s \phi(t) \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta) d\theta ds \\ &+ \lambda_1(t) \left[\frac{\sum_{i=1}^r \alpha_i}{b_2} \int_0^{\eta_j} \int_0^s \zeta(\eta_i) \frac{(s-\theta)^{p-1}}{\Gamma(p)} z(\theta) d\theta ds \right. \\ &\left. - \frac{1}{a_2}\int_0^1 \int_0^s \phi(1) \frac{(s-\theta)^{r-1}}{\Gamma(r)} w(\theta) d\theta ds \right] \end{aligned}$$

$$\begin{aligned}&+\lambda_2(t)\left[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s\phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right.\\&\left.-\frac{1}{b_2}\int_0^1\int_0^s\zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right]\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}v(t)=&\frac{1}{b_2}\int_0^t\int_0^s\zeta(t)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\\&+\mu_1(t)\left[\frac{\sum_{j=1}^{h}\beta_j}{a_2}\int_0^{\xi_j}\int_0^s\phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right.\\&\left.-\frac{1}{b_2}\int_0^1\int_0^s\zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right]\\&+\mu_2(t)\left[\frac{\sum_{i=1}^{r}\alpha_i}{b_2}\int_0^{\eta_i}\int_0^s\zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds\right.\\&\left.-\frac{1}{a_2}\int_0^1\int_0^s\phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds\right],\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}\phi(\kappa)=&(\kappa-s)e^{m(\kappa-s)},\quad\zeta(\kappa)=(\kappa-s)e^{n(\kappa-s)},\quad\kappa=t,1,\eta_i\ \text{and}\ \xi_j,\\m=&\frac{-a_1}{2a_2},\quad n=\frac{-b_1}{2b_2},\quad\lambda_1(t)=\frac{(mte^{mt}-e^{mt}+1)(ne^n-e^n+1)}{\mu},\\\lambda_2(t)=&\frac{(mte^{mt}-e^{mt}+1)(n\sum_{i=1}^{\rho}\alpha_i\eta_ie^{n\eta_i}-\sum_{i=1}^{\rho}\alpha_ie^{\eta_i}+\sum_{i=1}^{\rho}\alpha_i)}{\mu},\\\mu_1(t)=&\frac{(nte^{nt}-e^{nt}+1)(me^m-e^m+1)}{\mu},\\\mu_2(t)=&\frac{(nte^{nt}-e^{nt}+1)(m\sum_{j=1}^{h}\beta_j\xi_je^{m\xi_j}-\sum_{j=1}^{h}\beta_je^{m\xi_j}+\sum_{j=1}^{h}\beta_j)}{\mu},\\\mu=&(me^m-e^m+1)(ne^n-e^n+1)\\&-\left(n\sum_{i=1}^{\rho}\alpha_i\eta_ie^{n\eta_i}-\sum_{i=1}^{\rho}\alpha_ie^{\eta_i}+\sum_{i=1}^{\rho}\alpha_i)(m\sum_{j=1}^{h}\beta_j\xi_je^{m\xi_j}-\sum_{j=1}^{h}\beta_je^{m\xi_j}+\sum_{j=1}^{h}\beta_j\right)\neq0.\end{aligned}\tag{2.4}$$

Proof. Applying the integral operators I^r and I^p respectively on the first and second equations of (2.1) and then using Lemma 2.1, we get

$$(a_2D^2+a_1D+a_0)u(t)=\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}w(s)ds+c_1,\tag{2.5}$$

$$(b_2D^2+b_1D+b_0)v(t)=\int_0^t\frac{(t-s)^{p-1}}{\Gamma(p)}z(s)ds+d_1,\tag{2.6}$$

where c_1 and d_1 are arbitrary constants. Using the method of variation of parameters to solve (2.5) and (2.6), we get

$$u(t) = c_2e^{mt} + c_3te^{mt} + \frac{1}{a_2}\int_0^t (t-s)e^{m(t-s)}\left(\int_0^s \frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta + c_1\right)ds \tag{2.7}$$

and

$$v(t) = d_2e^{nt} + d_3te^{nt} + \frac{1}{b_2}\int_0^t (t-s)e^{n(t-s)}\left(\int_0^s \frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta + d_1\right)ds, \tag{2.8}$$

where m and n are given by (2.4). Using $u(0) = 0$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 0$ in (2.7) and (2.8) respectively, we find that $c_2 = c_3 = 0$ and $d_2 = d_3 = 0$ and consequently, we have

(2.9)
$$u(t) = c_1\left[\frac{mte^{mt} - e^{mt} + 1}{a_2m^2}\right] + \frac{1}{a_2}\left[\int_0^t (t-s)e^{m(t-s)}\left(\int_0^s \frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta\right)ds\right],$$

(2.10)
$$v(t) = d_1\left[\frac{nte^{nt} - e^{nt} + 1}{b_2n^2}\right] + \frac{1}{b_2}\left[\int_0^t (t-s)e^{n(t-s)}\left(\int_0^s \frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta\right)ds\right].$$

On the other hand, using the conditions $u(1) = \sum_{i=1}^{r} \alpha_i v(\eta_i)$ and $v(1) = \sum_{j=1}^{h} \beta_j u(\xi_j)$ in (2.9) and (2.10), respectively, we get the system:

$$A_1c_1 - B_1d_1 = V_1, \quad -B_2c_1 + A_2d_1 = V_2, \tag{2.11}$$

where

$$A_1 = \frac{me^m - e^m + 1}{a_2m^2}, \quad A_2 = \frac{ne^n - e^n + 1}{b_2n^2},$$

$$B_1 = \frac{\sum_{i=1}^{\rho} \alpha_i(n\eta_ie^{n\eta_i} - e^{n\eta_i} + 1)}{b_2n^2}, \quad B_2 = \frac{\sum_{j=1}^{h} \beta_j(m\xi_je^{m\xi_j} - e^{m\xi_j} + 1)}{a_2m^2},$$

(2.12)
$$V_1 = \frac{\sum_{i=1}^{\rho} \alpha_i}{b_2}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds - \frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds,$$

$$V_2 = \frac{\sum_{j=1}^{h} \beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}w(\theta)d\theta ds - \frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}z(\theta)d\theta ds.$$

Solving the system (2.11), we find that

$$c_1 = \frac{A_2V_1 + B_1V_2}{A_1A_2 - B_1B_2}, \quad d_1 = \frac{B_2V_1 + A_1V_2}{A_1A_2 - B_1B_2}.$$

Substituting the values of c_1 and d_1 in (2.9) and (2.10) respectively together with the notations (2.12) leads to the solution (2.2) and (2.3). The converse can be proven by direct computation. The proof is completed. □

3. Existence and Uniqueness Results

Define by $\mathcal{M} = \{u \mid u \in C([0,1], \mathbb{R})\}$ the Banach space endowed with norm $\|u\|_{\mathcal{M}} = \sup_{t\in[0,1]} |u(t)|$. Then the product space $(\mathcal{M} \times \mathcal{M}, \|\cdot\|_{\mathcal{M}\times\mathcal{M}})$ is a Banach space equipped with the norm $\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} = \|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}}$ for $(u,v) \in \mathcal{M}\times\mathcal{M}$.

In view of Lemma 2.2, we introduce an operator $\mathcal{Q} : \mathcal{M}\times\mathcal{M} \to \mathcal{M}\times\mathcal{M}$ as

$$\mathcal{Q}(u,v)(t) := (\mathcal{Q}_1(u,v)(t), \mathcal{Q}_2(u,v)(t)),$$

where

$$\begin{aligned}
\mathcal{Q}_1(u,v)(t) =& \frac{1}{a_2}\int_0^t \int_0^s \phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \\
&+ \lambda_1(t)\left[\frac{\sum_{i=1}^r \alpha_i}{b_2}\int_0^{\eta_j}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \right.\\
&\left. -\frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds\right]\\
&+ \lambda_2(t)\left[\frac{\sum_{j=1}^h \beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \right.\\
&\left. -\frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_2(u,v)(t) =& \frac{1}{b_2}\int_0^t \int_0^s \zeta(t)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \\
&+ \mu_1(t)\left[\frac{\sum_{j=1}^h \beta_j}{a_2}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds \right.\\
&\left. -\frac{1}{b_2}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds\right]\\
&+ \mu_2(t)\left[\frac{\sum_{i=1}^r \alpha_i}{b_2}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)} g(\theta,u(\theta),v(\theta))d\theta ds \right.\\
&\left. -\frac{1}{a_2}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)} f(\theta,u(\theta),v(\theta))d\theta ds\right].
\end{aligned}$$

For the sake of brevity, we set the following notations:

(3.1)
$$\Delta_1 = \frac{1}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m - e^m + 1| + \widehat{\lambda}_2\sum_{j=1}^h \beta_j {\xi_j}^r |m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\right\},$$

(3.2)
$$\Delta_2 = \frac{1}{|b_2|n^2\Gamma(p+1)}\left\{(1+\widehat{\mu}_1)|ne^n - e^n + 1| + \widehat{\mu}_2\sum_{i=1}^\rho \alpha_i {\eta_i}^p |n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\},$$

(3.3)
$$\Lambda_1 = \frac{1}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n - e^n + 1| + \widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\},$$
(3.4)
$$\Lambda_2 = \frac{1}{|a_2|m^2\Gamma(r+1)}\left\{\widehat{\mu}_2|me^m - e^m + 1| + \widehat{\mu}_1\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\right\},$$
$$\widehat{\lambda}_1 = \max_{t\in[0,1]}|\lambda_1(t)|, \quad \widehat{\lambda}_2 = \max_{t\in[0,1]}|\lambda_2(t)|, \quad \widehat{\mu}_1 = \max_{t\in[0,1]}|\mu_1(t)|, \quad \widehat{\mu}_2 = \max_{t\in[0,1]}|\mu_2(t)|.$$

Our first result, dealing with the existence of solutions for the problem (1.1)–(1.2), is based on Leray-Schauder alternative.

Lemma 3.1. (Leray-Schauder alternative [11]). *Let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator and that $\mathcal{A}(\mathcal{F}) = \{x \in \mathcal{E} \mid x = \nu\mathcal{F}(x) \text{ for some } 0 < \nu < 1\}$. Then either the set $\mathcal{A}(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fixed point.*

Theorem 3.1. *Assume that*

(H_1) *there exist real constants $\delta_i, \gamma_i > 0$, $i = 1, 2$, and $\delta_0 > 0$, $\gamma_0 > 0$ such that*
$$|f(t, u_1, u_2)| \leq \delta_0 + \delta_1|u_1| + \delta_2|u_2|$$
and
$$|g(t, u_1, u_2)| \leq \gamma_0 + \gamma_1|u_1| + \gamma_2|u_2|, \quad \text{for all } u_i \in \mathbb{R},\ i = 1, 2.$$

(H_2) $\max\{\omega_1, \omega_2\} < 1$, *where*

(3.5) $$\omega_1 = \delta_1(\Delta_1 + \Lambda_2) + \gamma_1(\Delta_2 + \Lambda_1), \quad \omega_2 = \delta_2(\Delta_1 + \Lambda_2) + \gamma_2(\Delta_2 + \Lambda_1),$$

Δ_1, Δ_2, Λ_1, Λ_2 are respectively given by (3.1), (3.2), (3.3) *and* (3.4).

Then the problem (1.1)–(1.2) *has at least one solution on* $[0, 1]$.

Proof. We first show that the operator $\mathcal{Q} : \mathcal{M}\times\mathcal{M} \to \mathcal{M}\times\mathcal{M}$ is completely continuous. The operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are continuous since the functions f and g are continuous, and thus the operator $\mathcal{Q}$ is continuous. Let $\Omega \subset \mathcal{M}\times\mathcal{M}$ be a bounded set. Then $|f(t, u(t), v(t))| \leq L_1$, $|g(t, u(t), v(t))| \leq L_2$ for all $(u, v) \in \Omega$, where L_1 and L_2 are positive constants. In consequence for any $(u, v) \in \Omega$, we get
$$\begin{aligned}
\|\mathcal{Q}_1(u, v)\|_{\mathcal{M}} = &\sup_{t\in[0,1]}|\mathcal{Q}_1(u, v)(t)| \\
\leq &\frac{L_1}{|a_2|m^2\Gamma(r+1)}\Bigg\{(1 + \widehat{\lambda}_1)|me^m - e^m + 1| \\
&+ \widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_j e^{m\xi_j} - e^{m\xi_j} + 1|\Bigg\} \\
&+ \frac{L_2}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n - e^n + 1| + \widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_i e^{n\eta_i} - e^{n\eta_i} + 1|\right\}
\end{aligned}$$

$$=L_1\Delta_1 + L_2\Lambda_1. \tag{3.6}$$

Similarly, it can be shown that

$$\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}} \le L_2\Delta_2 + L_1\Lambda_2. \tag{3.7}$$

From (3.6) and (3.7), we deduce that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are uniformly bounded, and hence the operator $\mathcal{Q}$ is uniformly bounded.

Next, we show that $\mathcal{Q}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|\mathcal{Q}_1(u,v)(t_2) - \mathcal{Q}_1(u,v)(t_1)|\\
\le&\frac{L_1}{|a_2|}\left\{\int_0^{t_1}\int_0^s |\phi(t_2)-\phi(t_1)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds + \int_{t_1}^{t_2}\int_0^s |\phi(t_2)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right\}\\
&+|\lambda_1(t_2)-\lambda_1(t_1)|\left\{\frac{L_2\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s |\zeta(\eta_i)|\frac{(s-\theta)^{p-1}}{\Gamma(p)}d\theta ds\right.\\
&\left.+\frac{L_1}{|a_2|}\int_0^1\int_0^s |\phi(1)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right\}\\
&+|\lambda_2(t_2)-\lambda_2(t_1)|\left\{\frac{L_1\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s |\phi(\xi_j)|\frac{(s-\theta)^{r-1}}{\Gamma(r)}d\theta ds\right.\\
&\left.+\frac{L_2}{|b_2|}\int_0^1\int_0^s |\zeta(1)|\frac{(s-\theta)^{p-1}}{\Gamma(p)}d\theta ds\right\}\\
\le&\left[\frac{L_1}{|a_2|m^2\Gamma(r+1)}\left\{(t_1^r - t_2^r)|m(t_1-t_2)e^{m(t_1-t_2)} - e^{m(t_1-t_2)} + 1|\right.\right.\\
&\left.+t_1^r|mt_1e^{mt_1} - e^{mt_1} + 1|\right\}\\
&+|\lambda_1(t_2)-\lambda_1(t_1)|\left\{\frac{L_2\sum_{i=1}^{\rho}\alpha_i\eta_i^p}{|b_2|n^2\Gamma(p+1)}|n\eta_ie^{n\eta_i} - e^{n\eta_i} + 1|\right.\\
&\left.+\frac{L_1}{|a_2|m^2\Gamma(r+1)}|me^m - e^m + 1|\right\}\\
&+|\lambda_2(t_2)-\lambda_2(t_1)|\left\{\frac{L_1\sum_{j=1}^{h}\beta_j\xi_j^r}{|a_2|m^2\Gamma(r+1)}||m\xi_je^{m\xi_j} - e^{m\xi_j} + 1|\right.\\
&\left.\left.+\frac{L_2}{|b_2|n^2\Gamma(p+1)}|ne^n - e^+1|\right\}\right] \to 0 \quad \text{as} \quad t_2 - t_1 \to 0,
\end{aligned}$$

independently of $(u,v) \in \Omega$. Analogously, we have

$$|\mathcal{Q}_2(u,v)(t_2) - \mathcal{Q}_2(u,v)(t_1)| \to 0 \quad \text{as} \quad t_2 - t_1 \to 0,$$

independently of $(u,v) \in \Omega$. Hence, the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are equicontinuous and thus the operator $\mathcal{Q}$ is equicontinuous. By Arzelá-Ascoli's theorem, we deduce that the operator $\mathcal{Q}$ is completely continuous.

Lastly, we consider a set $\Theta(\mathcal{Q}) = \{(u,v) \in \mathcal{M}\times\mathcal{M} \mid (u,v) = \nu\mathcal{Q}(u,v);\ 0 \le \nu \le 1\}$ and show that it is bounded. Let $(u,v) \in \Theta$. Then $(u,v) = \nu\mathcal{Q}(u,v)$. For any $t \in [0,1]$, we have $u(t) = \nu\mathcal{Q}_1(u,v)(t)$, $v(t) = \nu\mathcal{Q}_2(u,v)(t)$. Thus,

$$\begin{aligned}
|u(t)| =&|\nu\mathcal{Q}_1(u,v)(t)| \le |\mathcal{Q}_1(u,v)(t)|\\
\le&\frac{1}{|a_2|}\int_0^t\int_0^s \phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\\
&+|\lambda_1(t)|\left[\frac{\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right.\\
&\left.+\frac{1}{|a_2|}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right]\\
&+|\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right.\\
&\left.+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right]\\
\le&\frac{\delta_0+\delta_1|u|+\delta_2|v|}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m-e^m+1|\right.\\
&\left.+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j\xi_j^r|m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\right\}\\
&+\frac{(\gamma_0+\gamma_1|u|+\gamma_2|v|)}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n-e^n+1|+\widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i\eta_i^p|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\right\}\\
=&(\delta_0+\delta_1|u|+\delta_2|v|)\Delta_1+(\gamma_0+\gamma_1|u|+\gamma_2|v|)\Lambda_1,
\end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|u\|_{\mathcal{M}} \le (\delta_0+\delta_1\|u\|_{\mathcal{M}}+\delta_2\|v\|_{\mathcal{M}})\Delta_1+(\gamma_0+\gamma_1\|u\|_{\mathcal{M}}+\gamma_2\|v\|_{\mathcal{M}})\Lambda_1. \tag{3.8}$$

Likewise, we can obtain

$$\|v\|_{\mathcal{M}} \le (\gamma_0+\gamma_1\|u\|_{\mathcal{M}}+\gamma_2\|v\|_{\mathcal{M}})\Delta_2+(\delta_0+\delta_1\|u\|_{\mathcal{M}}+\delta_2\|v\|_{\mathcal{M}})\Lambda_2. \tag{3.9}$$

From (3.8) and (3.9), we find that

$$\begin{aligned}
\|u\|_{\mathcal{M}}+\|v\|_{\mathcal{M}} \le&\delta_0(\Delta_1+\Lambda_2)+\gamma_0(\Delta_2+\Lambda_1)\\
&+\|u\|_{\mathcal{M}}\left(\delta_1(\Delta_1+\Lambda_2)+\gamma_1(\Delta_2+\Lambda_1)\right)\\
&+\|v\|_{\mathcal{M}}\left(\delta_2(\Delta_1+\Lambda_2)+\gamma_2(\Delta_2+\Lambda_1)\right)\\
\le&\omega_0+\max\{\omega_1,\omega_2\}\|(u,v)\|_{\mathcal{M}\times\mathcal{M}},
\end{aligned}\tag{3.10}$$

where $\omega_0 = \delta_0(\Delta_1+\Lambda_2)+\gamma_0(\Delta_2+\Lambda_1)$ and ω_1,ω_2 are given by (3.5).

In view of the definition $\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} = \|u\|_{\mathcal{M}}+\|v\|_{\mathcal{M}}$, (3.10) leads to

$$\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} \le \frac{\omega_0}{1-\max\{\omega_1,\omega_2\}}.$$

Consequently, the set $\Theta(\mathcal{Q})$ is bounded. By Lemma 3.1, the operator $\mathcal{Q}$ has at least one fixed point. Therefore, the problem (1.1)–(1.2) has at least one solution on $[0,1]$, which finish the proof. □

In the following result, we prove the uniqueness of solutions for the problem at hand by means of Banach fixed point theorem.

Theorem 3.2. *Assume that:*

(H_3) *for all* $t \in [0,1]$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1,2$, *there exist positive constants* ℓ_1 *and* ℓ_2 *such that*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq \ell_1(|u_1 - v_1| + |u_2 - v_2|),$$
$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq \ell_2(|u_1 - v_1| + |u_2 - v_2|).$$

Then there exists a unique solution for the problem (1.1)–(1.2) *on* $[0,1]$ *if*

$$\ell_1(\Delta_1 + \Lambda_2) + \ell_2(\Delta_2 + \Lambda_1) < 1, \tag{3.11}$$

where Δ_1, Δ_2, Λ_1, Λ_2 *are given by* (3.1)–(3.4).

Proof. Let us consider a closed ball $B_{r^*} = \{(u,v) \in \mathcal{M} \times \mathcal{M} \mid \|(u,v)\|_{\mathcal{M}\times\mathcal{M}} \leq r^*\}$ and show that $\mathcal{Q}B_{r^*} \subset B_{r^*}$, where

$$r^* \geq \frac{M_1(\Delta_1 + \Lambda_2) + M_2(\Delta_2 + \Lambda_1)}{1 - \ell_1(\Delta_1 + \Lambda_2) - \ell_2(\Delta_2 + \Lambda_1)}, \quad M_1 = \sup_{t\in[0,1]} |f(t,0,0)|,$$
$$M_2 = \sup_{t\in[0,1]} |g(t,0,0)|.$$

For $(u,v) \in B_r$, $t \in [0,1]$, using (H_3), we get

$$\begin{aligned}
|f(t,u(t),v(t))| &\leq |f(t,u(t),v(t)) - f(t,0,0) + f(t,0,0)| \\
&\leq \ell_1\left(|u(t)| + |v(t)|\right) + M_1 \\
&\leq \ell_1\left(\|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}}\right) + M_1 \\
&\leq \ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}} + M_1 \leq \ell_1 r^* + M_1.
\end{aligned} \tag{3.12}$$

In a similar manner, we can find that

$$|g(t,u(t),v(t))| \leq \ell_2 r^* + M_2. \tag{3.13}$$

Then, using (3.12) and (3.13), we obtain

$$\begin{aligned}
\|\mathcal{Q}_1(u,v)\|_{\mathcal{M}} &= \sup_{t\in[0,1]} |\mathcal{Q}_1(u,v)(t)| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \frac{1}{|a_2|}\int_0^t\int_0^s \phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds \\
&\quad + |\lambda_1(t)|\Bigg[\frac{\sum_{i=1}^r \alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds \\
&\quad + \frac{1}{|a_2|}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda_2(t)|\left[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u(\theta),v(\theta))|d\theta ds\right.\\
&\left.\left.+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u(\theta),v(\theta))|d\theta ds\right]\right\}\\
\leq&(\ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_1)\left[\frac{1}{|a_2|}\int_0^t|(t-s)e^{m(t-s)}|\frac{s^r}{\Gamma(r+1)}ds\right.\\
&+\frac{|\lambda_1(t)|}{|a_2|}\int_0^1|(1-s)e^{m(1-s)}|\frac{s^r}{\Gamma(r+1)}ds\\
&\left.+\frac{|\lambda_2(t)|}{|a_2|}\sum_{j=1}^{h}\beta_j\int_0^{\xi_j}|(\xi_j-s)e^{m(\xi_j-s)}|\frac{s^r}{\Gamma(r+1)}ds\right]\\
&+(\ell_2\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_2)\left[\frac{|\lambda_1(t)|}{|b_2|}\sum_{i=1}^{\rho}\alpha_i\int_0^{\eta_i}|(\eta-s)e^{n(\eta_i-s)}|\frac{s^p}{\Gamma(p+1)}ds\right.\\
&\left.+\frac{|\lambda_2(t)|}{|b_2|}\int_0^1|(1-s)e^{n(1-s)}|\frac{s^p}{\Gamma(p+1)}ds\right]\\
\leq&\frac{\ell_1\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_1}{|a_2|m^2\Gamma(r+1)}\left\{(1+\widehat{\lambda}_1)|me^m-e^m+1|\right.\\
&\left.+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\right\}\\
&+\frac{(\ell_2\|(u,v)\|_{\mathcal{M}\times\mathcal{M}}+M_2)}{|b_2|n^2\Gamma(p+1)}\left\{\widehat{\lambda}_2|ne^n-e^n+1|\right.\\
&\left.+\widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\right\}\\
\leq&(\ell_1r^*+M_1)\Delta_1+(\ell_2r^*+M_2)\Lambda_1.
\end{aligned}\tag{3.14}
$$

Similarly, we have

$$\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}}\leq(\ell_2r^*+M_2)\Delta_2+(\ell_1r^*+M_1)\Lambda_2.\tag{3.15}$$

From the inequalities (3.14) and (3.15), we get

$$\|\mathcal{Q}(u,v)\|_{\mathcal{M}\times\mathcal{M}}=\|\mathcal{Q}_1(u,v)\|_{\mathcal{M}}+\|\mathcal{Q}_2(u,v)\|_{\mathcal{M}}\leq r^*,$$

which implies that $\mathcal{Q}B_{r^*}\subset B_{r^*}$. Now we will prove that the operator $\mathcal{Q}$ is a contraction. For $u_i,v_i\in B_{r^*}$, $i=1,2$, and for each $t\in[0,1]$, we have

$$
\begin{aligned}
&\|\mathcal{Q}_1(u_1,v_1)-\mathcal{Q}_1(u_2,v_2)\|_{\mathcal{M}}\\
=&\sup_{t\in[0,1]}|\mathcal{Q}_1(u_1,v_1)(t)-\mathcal{Q}_1(u_2,v_2)(t)|\\
\leq&\sup_{t\in[0,1]}\left\{\frac{1}{|a_2|}\int_0^t\int_0^s\phi(t)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda_1(t)|\Bigg[\frac{\sum_{i=1}^{r}\alpha_i}{|b_2|}\int_0^{\eta_i}\int_0^s \zeta(\eta_i)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u_1(\theta),v_1(\theta))-g(\theta,u_2(\theta),v_2(\theta))|d\theta ds\\
&+\frac{1}{|a_2|}\int_0^1\int_0^s \phi(1)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\Bigg]\\
&+|\lambda_2(t)|\Bigg[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}\int_0^s \phi(\xi_j)\frac{(s-\theta)^{r-1}}{\Gamma(r)}|f(\theta,u_1(\theta),v_1(\theta))-f(\theta,u_2(\theta),v_2(\theta))|d\theta ds\\
&+\frac{1}{|b_2|}\int_0^1\int_0^s \zeta(1)\frac{(s-\theta)^{p-1}}{\Gamma(p)}|g(\theta,u_1(\theta),v_1(\theta))-g(\theta,u_2(\theta),v_2(\theta))|d\theta ds\Bigg]\Bigg\}\\
\leq&\frac{\ell_1}{|a_2|}\int_0^t|(t-s)e^{m(t-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\\
&+|\lambda_1(t)|\Bigg[\frac{\ell_2\sum_{i=1}^{\rho}\alpha_i}{|b_2|}\int_0^{\eta_i}|(\eta_i-s)e^{n(\eta_i-s)}|\frac{s^p}{\Gamma(p+1)}(|u_1-v_1|+|u_2-v_2|)ds\\
&+\frac{1}{|a_2|}\int_0^1|(1-s)e^{m(1-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\Bigg]\\
&+|\lambda_2(t)|\Bigg[\frac{\sum_{j=1}^{h}\beta_j}{|a_2|}\int_0^{\xi_j}|(\xi_j-s)e^{m(\xi_j-s)}|\frac{s^r}{\Gamma(r+1)}(|u_1-v_1|+|u_2-v_2|)ds\\
&+\frac{\ell_2}{|b_2|}\int_0^1|(1-s)e^{n(1-s)}|\frac{s^p}{\Gamma(p+1)}(|u_1-v_1|+|u_2-v_2|)ds\Bigg]\\
\leq&\frac{\ell_1}{|a_2|m^2\Gamma(r+1)}\Bigg\{(1+\widehat{\lambda}_1)|me^m-e^m+1|\\
&+\widehat{\lambda}_2\sum_{j=1}^{h}\beta_j{\xi_j}^r|m\xi_je^{m\xi_j}-e^{m\xi_j}+1|\Bigg\}(|u_1-v_1|+|u_2-v_2|)\\
&+\frac{\ell_2}{|b_2|n^2\Gamma(p+1)}\Bigg\{\widehat{\lambda}_2|ne^n-e^n+1|\\
&+\widehat{\lambda}_1\sum_{i=1}^{\rho}\alpha_i{\eta_i}^p|n\eta_ie^{n\eta_i}-e^{n\eta_i}+1|\Bigg\}(|u_1-v_1|+|u_2-v_2|)\\
\leq&(\ell_1\Delta_1+\ell_2\Lambda_1)(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}).
\end{aligned}
$$

Similarly, one can find that

$$
\begin{aligned}
\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\|_{\mathcal{M}}&=\sup_{t\in[0,1]}|\mathcal{Q}_2(u_1,v_1)(t)-\mathcal{Q}_2(u_2,v_2)(t)|\\
&\leq(\ell_2\Delta_2+\ell_1\Lambda_2)(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\|_{\mathcal{M}\times\mathcal{M}}\\
=&\|\mathcal{Q}_1(u_1,v_1)-\mathcal{Q}_1(u_2,v_2)\|_{\mathcal{M}}+\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\|_{\mathcal{M}}\\
\leq&\left(\ell_1(\Delta_1+\Lambda_2)+\ell_2(\Delta_2+\Lambda_1)\right)\left(\|u_1-u_2\|_{\mathcal{M}}+\|v_1-v_2\|_{\mathcal{M}}\right),
\end{aligned}
$$

which, in view of the assumption (3.11), implies that $\mathcal{Q}$ is a contraction. Consequently, by Banach's contraction mapping principle, the operator $\mathcal{Q}$ has a unique fixed point, which is indeed the unique solution of the problem (1.1)–(1.2). This completes the proof. $\square$

Example 3.1. Consider the coupled system of multi-term fractional differential equations:

$$\begin{cases} \left(2\ {}^cD^{12/5} + 4\ {}^cD^{7/5} + 2\ {}^cD^{2/5}\right) u(t) = \dfrac{1}{\sqrt{t^2+25}}\{\cos u(t) + |v(t)| + \tan^{-1} t\}, \\ \left(\ {}^cD^{17/7} + 2\ {}^cD^{10/7} +\ {}^cD^{3/7}\right) v(t) = \dfrac{t^2}{t+6}\left\{|u(t)| + \dfrac{|v(t)|^3}{1+|v(t)|^3} + \sin t\right\}, \end{cases} \tag{3.16}$$

equipped with boundary conditions

$$\begin{cases} u(0)=0, \quad u'(0)=0, \quad u(1) = 2v(1/6) + v(1/5) + 2v(1/4), \\ v(0)=0, \quad v'(0)=0, \quad v(1) = 3u(1/2) + u(3/4). \end{cases} \tag{3.17}$$

Here, $q = 2/5$, $p = 3/7$, $\eta_1 = 1/6$, $\eta_2 = 1/5$, $\eta_3 = 1/4$, $\xi_1 = 1/2$, $\xi_2 = 3/4$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\beta_1 = 3$, $\beta_2 = 1$, $a_1^2 - 4a_2a_0 = 0$, $b_1^2 - 4b_2b_0 = 0$ and

$$f(t, u(t), v(t)) = \frac{1}{\sqrt{t^2+25}}\{\cos u(t) + |v(t)| + \tan^{-1} t\}$$

and

$$g(t, u(t), v(t)) = \frac{t^2}{t+6}\left\{|u(t)| + \frac{|v(t)|^3}{1+|v(t)|^3} + \sin t\right\}.$$

Clearly $\ell_1 = 1/5$ and $\ell_2 = 1/6$ as

$$|f(t, u_1(t), v_1(t)) - f(t, u_2(t), v_2(t))| \le \frac{1}{5}\{|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|\},$$
$$|g(t, u_1(t), v_1(t)) - g(t, u_2(t), v_2(t))| \le \frac{1}{6}\{|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|\}.$$

Using the given data, we find that $\Delta_1 \approx 0.71336$, $\Delta_2 \approx 1.3058$, $\Lambda_1 \approx 0.70297$, and $\Lambda_2 \approx 1.2161$. Further

$$\ell_1(\Delta_1 + \Lambda_2) + \ell_2(\Delta_2 + \Lambda_1) \approx 0.72069 < 1.$$

Hence we deduce by Theorem 3.2 that the problem (3.16)–(3.17) has a unique solution on $[0, 1]$.

4. Conclusions

We have analyzed a fully coupled boundary value problem of nonlinear multi-term fractional differential equations and nonlocal multi-point boundary conditions under the assumption that $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$. Though the tools of fixed point theory employed in the present analysis are the standard ones, yet their exposition to the problem at hand enhances the scope of the literature on fractional order boundary value problems. The cases $a_1^2 > 4a_0a_2$, $b_1^2 > 4b_0b_2$ and $a_1^2 < 4a_0a_2$, $b_1^2 < 4b_0b_2$ for the problem (1.1)–(1.2) can be handled in a manner similar to that of $a_1^2 = 4a_0a_2$, $b_1^2 = 4b_0b_2$.

As a special case, the results for a coupled system of nonlinear multi-term fractional differential equations equipped with the two-point boundary conditions: $u(0) = 0$, $u'(0) = 0$, $u(1) = 0$, $v(0) = 0$, $v'(0) = 0$, $v(1) = 0$ follow by taking all $\alpha_i = 0$, $i = 1, \ldots, \rho$, and $\beta_j = 0$, $j = 1, \ldots, h$, in the results of this paper.

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[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Email address: bashirahmad_qau@yahoo.com
Email address: aalsaedi@hotmail.com

[2]Department of Mathematics,
Faculty of Science,
University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia
Email address: njl-ghamdi@hotmail.com

[3]Department of Mathematics,
University of Ioannina, 451 10 Ioannina, Greece
Email address: sntouyas@uoi.gr