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NON-CONFORMABLE FRACTIONAL LAPLACE TRANSFORM

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Abstract. In this paper we present an extension of Fractional Laplace Transform in the framework of the non-conformable local fractional derivative. Its main properties are studied and it is applied to the resolution of fractional differential equations.

1. Preliminaries

In mathematics, the Laplace transform is an integral transform n, it takes a function of a real variable t (often time) to a function of a complex variable s (complex frequency). Laplace transforms are usually restricted to functions of t with $t \geq 0$, consequently of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. The Laplace transform is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable s (often frequency) and yields a function of a real variable t (often time). Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications. So, for example, Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and convolution into multiplication.

Regarding the birth of the fractional calculus, all historians agree on the dating of the date and how it was produced. This fact took place after a publication of Leibniz where he introduced the notation of the differential calculus, in particular of

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the expression known today as $\frac{d^n y}{dx^n}$ that makes reference to the derivative of order n of the function and, with $n \in \mathbb{N}$. But did it make sense to extend the values of n to the set of rational, irrational, or complex numbers in that expression?

We know that the fractional derivative of a non-integer function can be conceived in two branches: global (classical) and local. The former are often defined by means of integral transforms, Fourier or Mellin, which means in particular that their nature is not local, has "memory", in the second case, they are defined locally by a certain incremental quotients. The first are associated with the emergence of the Fractional Calculation itself, with the pioneering works of Euler, Laplace, Lacroix, Fourier, Abel, Liouville,... until the establishment of the classical definitions of Riemann-Liouville and Caputo. Recent extensions and applications of these notions to various fields can be found in [2–4, 7, 13, 18, 21, 21]. There are some attempts to extend the classical notion of Laplace Transform to the non-integer case, we recommend consult [20].

Recently, in [8] Khalil et al. defined a new local fractional derivative called the conformable fractional derivative, based on the limit definition of the derivative. Namely, for a function $h : [0, \infty) \to \mathbb{R}$, the non-conformable fractional derivative of h of order α of h at t is defined by

$$D_\alpha(h)(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon t^{1-\alpha}\right) - h(t)}{\epsilon}, \quad \alpha \in (0, 1), t > 0.$$

In [1], Abdeljaward improve this new theory. For instance, definitions of left and right conformable derivatives and fractional integrals of higher order (i.e., of order $\alpha > 1$), Taylor power series, fractional integration by parts formulas and chain rule are provided by him.

Now, we give the definition of the non-conformable fractional derivative with its important properties which are useful in order to obtain our main results, which is explained in the following definition [5].

Definition 1.1. Given a function $h : [0, \infty) \to \mathbb{R}$. Then, the non-conformable fractional derivative $N_3^\alpha(h)(t)$ of order α of h at t is defined by

$$N_3^\alpha(h)(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon t^{-\alpha}\right) - h(t)}{\epsilon}, \quad \alpha \in (0, 1), t > 0.$$

If h is α-differentiable in some $(0, \alpha)$, $\alpha > 0$, $\lim_{t \to 0^+} h^{(\alpha)}(t)$ exist, then define

$$h^{(\alpha)}(0) = \lim_{t \to 0^+} h^{(\alpha)}(t).$$

Remark 1.1. Additionally, note that if h is differentiable, then

$$N_3^\alpha(h)(t) = t^{-\alpha} h'(t), \quad \text{where } h'(t) = \lim_{\epsilon \to 0} \frac{h\left(t + \epsilon\right) - h(t)}{\epsilon}.$$

We can write $h^{(\alpha)}(t)$ for $D_\alpha(h)(t)$ or $\frac{d_\alpha}{d_\alpha t}(h(t))$ to denote the non-conformable fractional derivatives of h of order α at t. In addition, if the non-conformable fractional derivative N_3^α of h of order α exists, then we simply say h is N-differentiable.

In [5, 14], we can see that the chain rule is valid for non-conformable fractional derivatives.

Theorem 1.1. *Let $\alpha \in (0,1]$, g a N-differentiable function at $t > 0$, f be differentiable in the range of $g(t)$. Then*

$$N_3^\alpha(f \circ g)(t) = f'(g(t))N_3^\alpha(g(t)).$$

Proof. We prove the result following a standard limit-approach. First case, if the function g is constant in a neighborhood of $a > 0$ then $N_3^\alpha(f \circ g)(t) = 0$. If g is not a constant in a neighborhood of $a > 0$ we can find and $\varepsilon_0 > 0$ such that $g(x_1) \neq g(x_2)$ for any $x_1, x_2 \in (a - t_0, a + t_0)$. Now, since g is continuous at a, for ε sufficiently small, we have

$$\begin{aligned} N_3^\alpha(f \circ g)(a) =& \lim_{\varepsilon \to 0} \frac{fg((t + \varepsilon a^{-\alpha})) - f(g(a))}{\varepsilon} \\ =& \lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\ =& \lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\ =& \lim_{k \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon}. \end{aligned}$$

Making $\varepsilon_1 = g(a + \varepsilon a^{-\alpha}) - g(a)$ in the first factor we have

$$\lim_{\varepsilon \to 0} \frac{f(g(a + \varepsilon a^{-\alpha})) - f(g(a))}{g(a + \varepsilon a^{-\alpha}) - g(a)} = \lim_{\varepsilon_1 \to 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1},$$

and from here

$$\begin{aligned} N_3^\alpha(f \circ g)(a) =& \lim_{\varepsilon_1 \to 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1} \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon a^{-\alpha}) - g(a)}{\varepsilon} \\ =& f'(g(a))N_3^\alpha g(a). \end{aligned}$$

□

The following function will play an important role in our work.

Definition 1.2. Let $\alpha \in (0,1)$ and c a real number. We define the fractional exponential in the following way

$$E_\alpha^{n_3}(c,t) = \exp\left(c \frac{t^{\alpha+1}}{\alpha+1}\right).$$

Following the ideas presented in [5, 14] we can easily prove the next result.

Theorem 1.2. *Let $\alpha \in (0,1]$ and h, g be α-differentiable at a point $t > 0$. Then*

(a) $N_3^\alpha(uf + vg) = uN_3^\alpha(h) + vN_3^\alpha(g)$ *for all* $u, v \in \mathbb{R}$;
(b) $N_3^\alpha(hg) = N_3^\alpha(g) + gN_3^\alpha(h)$;
(c) $N_3^\alpha\left(\frac{h}{g}\right) = \frac{hN_3^\alpha(g) - gN_3^\alpha(h)}{g^2}$;
(d) $N_3^\alpha(c) = 0$ *for all constant function* $h(t) = c$;

(e) $N_3^\alpha(1) = 0$;
(f) $N_3^\alpha\left(\frac{1}{1+\alpha}t^{1+\alpha}\right) = 1$;
(g) $N_3^\alpha\left(E_\alpha^{n_3}(c,t)\right) = cE_\alpha^{n_3}(c,t)$;
(h) $N_3^\alpha\left(\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)\right) = c\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)$;
(i) $N_3^\alpha\left(\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)\right) = -c\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)$.

Proof. (a) Let $H(t) = (af+bg)(t)$. Then $N_3^\alpha H(t) = \lim_{\varepsilon\to 0}\frac{H(t+\varepsilon t^{-\alpha})-H(t)}{\varepsilon}$ and from this we have the desired result.

(b) From definition we have

$$\begin{aligned}
N_3^\alpha(fg)(t) =& \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{-\alpha})g(t+\varepsilon t^{-\alpha})-f(t)g(t)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{-\alpha})g(t+\varepsilon t^{-\alpha})-f(t)g(t+\varepsilon t^{-\alpha})+f(t)g(t+\varepsilon t^{-\alpha})-f(t)g(t)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\frac{\left(f(t+\varepsilon t^{-\alpha})-f(t)\right)g(t+\varepsilon t^{-\alpha})}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{\left(g(t+\varepsilon t^{-\alpha})-g(t)\right)f(t)}{\varepsilon}\\
=& fN_3^\alpha(g)(t)+gN_3^\alpha(f)(t).
\end{aligned}$$

(c) In a similar way to the previous one we have

$$N_3^\alpha\left(\frac{f}{g}\right)(t) = \lim_{\varepsilon\to 0}\frac{\frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})}-\frac{f(t)}{g(t)}}{\varepsilon}.$$

But

$$\begin{aligned}
\frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})}-\frac{f(t)}{g(t)} =& \frac{f(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})}-\frac{f(t)}{g(t)}\frac{g(t+\varepsilon t^{-\alpha})}{g(t+\varepsilon t^{-\alpha})}\\
=& \frac{f(t+\varepsilon t^{-\alpha})g(t)-f(t)g(t+\varepsilon t^{-\alpha})}{g(t)g(t+\varepsilon t^{-\alpha})}\\
=& \frac{f(t+\varepsilon t^{-\alpha})g(t)-f(t)g(t+\varepsilon t^{-\alpha})-f(t)g(t)+f(t)g(t)}{g(t)g(t+\varepsilon t^{-\alpha})}\\
=& \frac{\left(f(t+\varepsilon t^{-\alpha})-f(t)\right)g(t)-\left(g(t+\varepsilon t^{-\alpha})-g(t)\right)f(t)}{g(t)g(t+\varepsilon t^{-\alpha})}.
\end{aligned}$$

From this last expression we obtain the expected result.
(d) Easily follows from definition.
(e) Is a particular case of the previous one.
(f) From Remark 1.1 we have

$$N_3^\alpha\left(\frac{1}{1+\alpha}t^{1+\alpha}\right) = t^{-\alpha}\frac{1}{1+\alpha}\left((1+\alpha)t^\alpha\right) = 1.$$

g) From Remark 1.1 and the chain rule we have

$$N_3^\alpha\left(E_\alpha^{n_3}(c,t)\right) = N_3^\alpha\left[\exp\left(c\frac{t^{\alpha+1}}{\alpha+1}\right)\right] = t^{-\alpha}\left[\exp\left(c\frac{t^{\alpha+1}}{\alpha+1}\right)\right]\left(c\frac{(\alpha+1)t^\alpha}{(\alpha+1)}\right)$$

$$=cE_\alpha^{n_3}(c,t).$$

To prove cases (h) and (i) it is sufficient to proceed as in the previous case, taking into account the Remark 1.1 and using the chain rule. $\square$

Now, we give the definition of non-conformable fractional integral.

Definition 1.3. Let $\alpha \in (0,1]$ and $0 \leq u \leq v$. We say that a function $h : [u,v] \to \mathbb{R}$ is α-fractional integrable on $[u,v]$, if the integral

$$_{N_3}J_u^\alpha h(x) = \int_u^x \frac{h(t)}{t^{-\alpha}}dt$$

exists and is finite.

The following statement is analogous to the one known from the ordinary calculus (see [15]).

Theorem 1.3. *Let f be N-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

a) *if f is differentiable, then $_{N_3}J_{t_0}^\alpha \left(N_3^\alpha f(t)\right) = f(t) - f(t_0)$;*

b) $N_3^\alpha \left(_{N_3}J_{t_0}^\alpha f(t)\right) = f(t)$.

Proof. a) From definition we have

$$_{N_3}J_{t_0}^\alpha \left(N_3^\alpha f(t)\right) = \int_{t_0}^t \frac{N_3^\alpha f(s)}{s^{-\alpha}}ds = \int_{t_0}^t \frac{f'(s)s^{-\alpha}}{s^{-\alpha}}ds = f(t) - f(t_0).$$

b) Analogously we have

$$N_3^\alpha \left(_{N_3}J_{t_0}^\alpha f(t)\right) = t^{-\alpha}\frac{d}{dt}\left[\int_{t_0}^t \frac{f(s)}{s^{-\alpha}}ds\right] = f(t). \qquad \square$$

An important property, and necessary, in our work is that established in the following result.

Theorem 1.4 (Integration by parts)**.** *Let functions u,v be N-differentiable functions in (t_0,∞), with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

$$_{N_3}J_{t_0}^\alpha \left((uN_3^\alpha v)(t)\right) = [uv(t) - uv(t_0)] -_{N_3} J_{t_0}^\alpha \left((vN_3^\alpha u)(t)\right).$$

Proof. It is sufficient to use Theorem 1.2 and Theorem 1.3. $\square$

In short time, many studies about theory and applications of the fractional differential equations which based on these new fractional derivative definitions [6, 11, 15, 16, 19].

In this paper we establish the first results to formalize a new version of a Laplace transform, in this case non-conformable, which will allow its application to a wide class of fractional differential equations. In the conformable case, there are some attempts that can be consulted in [6, 9–12, 19].

2. Results

Definition 2.1 (Exponential order). A function f is said to be of generalized exponential order if there exist constants M and a such that $|f(t)| \leq ME_{\alpha}^{n_3}(a,t)$ for sufficiently large t.

We are now in a position to define the non-conformable fractional Laplace transform.

Definition 2.2. Let $\alpha \in (0,1)$ and c a real number. Let f be a real function defined for $t \geq 0$ and consider $s \in \mathbb{C}$, if the integral

$$ {}_{N_3}J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) = \int_0^{+\infty} E_{\alpha}^{n_3}(-s,t)f(t)d_{\alpha}t = \int_0^{+\infty} \frac{E_{\alpha}^{n_3}(-s,t)f(t)}{t^{-\alpha}}dt $$

converge for the given value of s, you can define the function F given by the expression

$$ F(s) = {}_{N_3}J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty), \tag{2.1} $$

and we will write $F = \mathcal{L}_N(f)$.

To the operator $\mathcal{L}_N$ we will call it the N-transformed of Laplace and we will say that F is the N-transformed of f. In turn, f is the N-inverse transform function of F and we will write it as $f = \mathcal{L}_N^{-1}\{F\}$, where $\mathcal{L}_N^{-1}$ is the N-transformed inverse Laplace operator.

As in the classic case, we must impose conditions to (2.1), so that the previous definition makes sense. If f satisfies the following two conditions:

(a) f is a piecewise continuous in the interval $(0,T]$ for any $T \in (0,+\infty)$;
(b) f is of generalized exponential order; that is, there are positive constants M and a, satisfying Definition 2.1 with $\operatorname{Re}(a-c) < 0$ and $|f(t)| \leq ME_{\alpha}^{n_3}(a,t)$ for all t and $\alpha \in (0,1]$.

Then the N-transformed of Laplace $F(s)$ of f exists for $s > a$. In effect, since f is of generalized exponential order, there exists constants $T > 0$, $K > 0$ and $a \in \mathbb{R}$ such that $|f(t)| \leq KE_{\alpha}^{n_3}(a,t)$ for all $t \geq T$ and $\alpha \in (0,1]$. Now we write $I = {}_{N_3}J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) = {}_{N_3}J_0^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(T) + {}_{N_3}J_T^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) = I_1 + I_2$. Since f is a piecewise continuous, I_1 exists. For the second integral I_1, we note that for $t \geq T$ we have $|E_{\alpha}^{n_3}(-s,t)f(t)| \leq KE_{\alpha}^{n_3}(-(s-a),t)$. Thus,

$$ {}_{N_3}J_T^{\alpha} E_{\alpha}^{n_3}(-s,t)f(t)(+\infty) \leq K {}_{N_3}J_T^{\alpha} E_{\alpha}^{n_3}(-(s-a),t)(+\infty) = \frac{K}{s-a}, \quad s > a. $$

Since the integral I_2 converges absolutely for $s > a$, I_2 converges for $s > a$. Thus, both I_1 and I_2 exist and hence I exists for $s > a$. Then we have that f is an N-transformable function.

Theorem 2.1. *Let $\alpha \in (0,1]$. So, we have*

(a) $\mathcal{L}_N(1) = \frac{1}{s}$, *from here we have* $\mathcal{L}_N(c) = cL_N(1)$ *for any* $c \in \mathbb{R}$;
(b) $\mathcal{L}_N(t^b) = \frac{(1+\alpha)^{\frac{b}{1+\alpha}}\Gamma\left(1+\frac{b}{1+\alpha}\right)}{s^{1+\frac{b}{1+\alpha}}}$, *where the gamma function* Γ *is defined by* $\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}dt$, $\Gamma(a,0) := \Gamma(a)$ *and* $b > -1$;

(c) $\mathcal{L}_N(E_\alpha^{n_3}(c,t)) = \frac{1}{s-c}$, *c any real number and* $s-c>0$*;*
(d) $\mathcal{L}_N(f(t)E_\alpha^{n_3}(c,t) = F(s-c)$, *with* $\mathcal{L}_N(f(t)) = F(s)$, *c any real number and* $s-c>0$*;*
(e) $\mathcal{L}_N(\sin\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{c}{s^2+c^2}$*;*
(f) $\mathcal{L}_N(\cos\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{s}{s^2+c^2}$*;*
(g) $\mathcal{L}_N(\sinh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{c}{s^2-c^2}$*;*
(h) $\mathcal{L}_N(\cosh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{s}{s^2-c^2}$*.*

Proof. (a) From definition directly.
(b) Through a change of variables we have

$$ {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)t^b(+\infty) = \frac{(1+\alpha)^{\frac{b}{1+\alpha}}}{s^{1+\frac{b}{1+\alpha}}}\,{}_{N_3}J_0^\alpha E_\alpha^{n_3}(-u)u^{\frac{b}{1+\alpha}}(+\infty), $$

where the desired result is obtained.
(c) Consider $f(t) = E_\alpha^{n_3}(c,t)$, with $c\in\mathbb{R}$. Then

$$ {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)E_\alpha^{n_3}(c,t)(+\infty) = {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-(s-c),t)(+\infty) = \frac{1}{s-c}. $$

(d) Suppose $\mathcal{L}_N f(t) = F(s)$ for $s>k$. So, we have

$$ \begin{aligned} {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-s,t)E_\alpha^{n_3}(c,t)f(t)(+\infty) =& {}_{N_3}J_0^\alpha E_\alpha^{n_3}(-(s-c),t)f(t)(+\infty) \\ =& F(s-c), \quad s-c>k. \end{aligned} $$

(e) Using ${}_{N_3}J^\alpha E_\alpha^{n_3}(b,t)\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{E_\alpha^{n_3}(b,t)}{a^2+b^2}\left\{b\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) - a\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right)\right\}$ we obtain the expected result.
(f) Similar to previous one, using

$$ {}_{N_3}J^\alpha E_\alpha^{n_3}(b,t)\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) = \frac{E_\alpha^{n_3}(b,t)}{a^2+b^2}\left\{b\cos\left(a\frac{t^{1+\alpha}}{1+\alpha}\right) + a\sin\left(a\frac{t^{1+\alpha}}{1+\alpha}\right)\right\}. $$

(g) As $\mathcal{L}_N(\sinh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)) = \frac{1}{2}\left\{\mathcal{L}_N E_\alpha^{n_3}(c,t) - \mathcal{L}_N E_\alpha^{n_3}(-c,t)\right\}$ it is easy to get the required conclusion.
(h) From $\mathcal{L}_N(\cosh\left(c\frac{t^{1+\alpha}}{1+\alpha}\right)) = \frac{1}{2}\left\{\mathcal{L}_N E_\alpha^{n_3}(c,t) + \mathcal{L}_N E_\alpha^{n_3}(-c,t)\right\}$ it is obtained directly. □

Anallogously, the following propositions can be proved from the definition of N-transformed and the non-conformable integral.

Proposition 2.1. *If the functions f and g are transformable, then there is the transform of the sum and is equal to the sum of the transforms, that is*

$$ \mathcal{L}_N(f+g) = \mathcal{L}_N(f) + \mathcal{L}_N(g). $$

Proposition 2.2. *If the function f is transformable and λ is a real number, then there is the transform of product of λ by f and is equal to product of λ by the transform of f, that is*

$$\mathcal{L}_N(\lambda f) = \lambda \mathcal{L}_N(f).$$

Remark 2.1. Taking into account the two previous propositions, we say that $\mathcal{L}_N$ is a linear operator.

Proposition 2.3. *If f is a transformable function, then so is its N-derivative and you have*

$$\mathcal{L}_N(N_3^\alpha f) = s\mathcal{L}_N(f) - f(0). \tag{2.2}$$

Proof. Already $\mathcal{L}_N(N_3^\alpha f)$ exists, because f is of non-conformable exponential order and continuous. On an interval $[a,b]$ where $N_3^\alpha f$ is continuous, integrating by parts in (2.2), gives

$$\int_a^b \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt = f(b)E_\alpha^{N_3}(-s,b) - f(a)E_\alpha^{N_3}(-s,a) + s\int_a^b \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt.$$

On any interval $[0,K]$ there are finitely many intervals $[a,b]$ on each of which $N_3^\alpha f$ is continuous. Add above equality across these finitely many intervals $[a,b]$. The boundary values on adjacent intervals match and the integrals add to give

$$\int_0^K \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt = f(K)E_\alpha^{N_3}(-s,b) - f(0) + s\int_0^K \frac{E_\alpha^{N_3}(-s,t)N_3^\alpha f(t)}{t^{-\alpha}}dt.$$

Taking the limit $K \to +\infty$ across this equality, we obtain the desired result. ☐

Analogously we have the following.

Proposition 2.4. *If the k consecutive derivatives $N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f)))$ are N-transformable, then we have*

$$\begin{aligned}&\mathcal{L}_N\left[N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f)))\right]\\ =&s^k\mathcal{L}_N(f) - s^{k-1}f(0) - s^{k-2}N_3^\alpha f(0) - s^{k-3}N_3^\alpha(N_3^\alpha f(0)) - \cdots - N_3^\alpha(N_3^\alpha(\cdots(N_3^\alpha f(0)))).\end{aligned}$$

Proposition 2.5. *Let g be of non-conformable exponential order and continuous for $t \geq 0$. Then*

$$\mathcal{L}_N\left(\int_0^x \frac{g(x)}{x^{-\alpha}}dx\right) = \frac{1}{s}\mathcal{L}_N\left\{g(t)\right\}.$$

Proof. Let $f(t) = \left(\int_0^t \frac{g(x)}{x^{-\alpha}}dx\right)$. Then f is of exponential order and continuous then we have $\mathcal{L}_N\left(\int_0^t \frac{g(x)}{x^{-\alpha}}dx\right) = \mathcal{L}_N$ by definition and $\mathcal{L}_N = \frac{1}{s}\mathcal{L}\left(N_3^\alpha f(t)\right)$ because $f(0) = 0$. From here we reach the conclusion without difficulty. ☐

The following result establishes the relationship between the classic Laplace Transform and the N-transform defined above.

Theorem 2.2. *Let $\alpha \in (0,1)$ and f be a N-transformable function, then we have*

$$\mathcal{L}_N(f) = L\left[f\left(((1+\alpha)z)^{\frac{1}{1+\alpha}}\right)\right],$$

where $\mathcal{L}$ is the classical Laplace transform defined by $\mathcal{L}(g) = \int_0^{+\infty} e^{-st}g(t)dt$.

Proof. Simply make the change of the variables $z = \frac{t^{1+\alpha}}{1+\alpha}$. □

One of the most important results of the classic Laplace transform is the convolution product of two $\mathcal{L}$-transformable functions, we are already in a position to provide an analogous result for the N-transform defined in (2.1).

Theorem 2.3. *Let $\alpha \in (0,1]$ and $f, g : [0,+\infty] \to \mathbb{R}$ be real functions. If $F_\alpha(s) = L_N\left[f\left(t^{1+\alpha}\right)\right](s)$ and $G_\alpha(s) = \mathcal{L}_N\left[g\left(t\right)\right](s)$, then the next equality is satisfied*

$$\mathcal{L}_N(f * g)(s) = F_\alpha(s)G_\alpha(s),$$

where

$$(f * g)(t) = \int_0^t \left[f\left(t^{1+\alpha} - \tau^{1+\alpha}\right)\right] g(\tau)d_\alpha\tau.$$

Proof. It is sufficient to change the variables $u^{1+\alpha} = t^{1+\alpha} - \tau^{1+\alpha}$ and apply the properties of the $\mathcal{L}_N$ operator. □

2.1. **Existence of non-conformable Laplace transform.** In this subsection, the bounded and existence of non-conformable Laplace transform are presented.

Theorem 2.4. *Let f be piecewise continuous on $[0,\infty)$ and non-conformable exponentially bounded, then*

$$\lim_{s\to\infty} F_\alpha(s) = 0,$$

where $F_\alpha(s) = \mathcal{L}_\alpha[f(t)](s)$.

Proof. Since f is generalized order exponential, there exist t_0, M_1, c such that $|f(t)| \leq M_1E_\alpha^{n_3}(c,t)$ for $t \geq t_0$. Also, f is piecewise continuous on $[0,t_0]$ and hence f is bounded, so there exists M_2 such that $|f(t)| \leq M_2$ for $t \in [0,t_0]$. Choosing $M = \max\{M_1, M_2\}$, we have $|f(t)| \leq ME_\alpha^{n_3}(c,t)$ for $t \geq 0$. Now, we have‘

$$\begin{aligned}\left|\int_0^\tau E_\alpha^{n_3}(-s,t)f(t)d_\alpha t\right| &\leq \int_0^\tau \left|E_\alpha^{n_3}(-s,t)f(t)\right| d_\alpha t\\ &\leq M\int_0^\tau E_\alpha^{n_3}(-s+c,t)d_\alpha t\\ &= \frac{M}{s-c} - \frac{E_\alpha^{n_3}(-s+c,t)}{s-c}.\end{aligned}$$

This gives

$$\lim_{\tau\to\infty}\left|\int_0^\tau E_\alpha^{n_3}(-s,t)f(t)d_\alpha t\right| \leq \frac{M}{s-c}.$$

This completes the proof. □

3. Examples and Applications

Example 3.1. Consider the non-conformable differential equation:

$$N_3^\alpha x(t) = \lambda x(t), \quad x(0) = x_0, \quad \alpha \in (0, 1]. \tag{3.1}$$

Clearly, if $\alpha = 1$ the equation above is just one of the simplest classical ordinary differential equations which is defined by the hypothesis that the rate of growth of a given function $x(t)$ is proportional to the current value (e.g. Maltius's population model), i.e., $x'(t) = \lambda x(t)$, $x(0) = x_0$ the exact solution of this is $x(t) = x_0 e^{\lambda t}$.

Applying the non-conformable Laplace Transform to both sides of equation (3.1), we get

$$\begin{aligned}\mathcal{L}_N \left(N_3^\alpha x(t)\right) =& \lambda \mathcal{L}_N \left(x(t)\right),\\ s\, X_\alpha(s) - x_0 =& \lambda X_\alpha(s).\end{aligned}$$

Simplifying this we get

$$X_\alpha(s) = \frac{x_0}{s+1}. \tag{3.2}$$

Taking the inverse non-conformable Laplace transform to (3.2), we get

$$x(t) = x_0 E_\alpha^{N_3}(-1, t) = -\frac{x_0}{\alpha+1} t^{\alpha+1}.$$

The solution of (3.1), obtained from non-conformable Laplace transformation method, are shown in Figure 1 for different values of α.

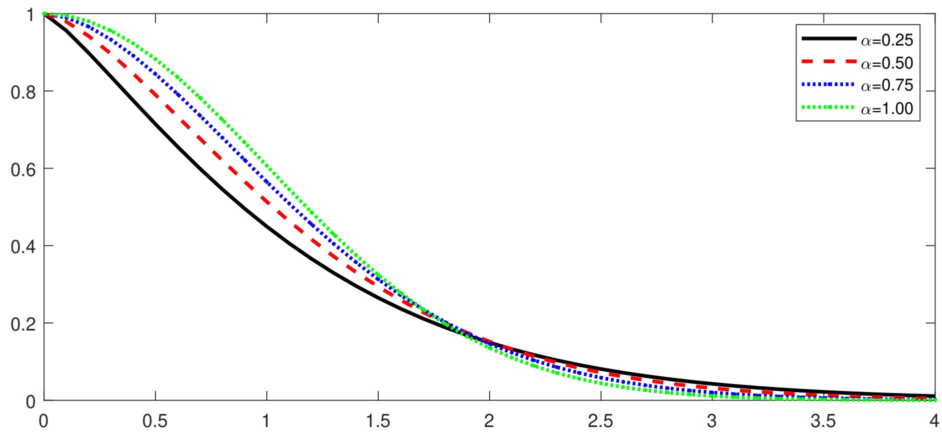

Figure 1. Non-conformable Laplace solution of (3.1) for different values of α.

Example 3.2. Consider the non-conformable fractional Bertalanffy-logistic differential equation

$$N_3^\alpha x(t) = x^{\frac{2}{3}}(t) - x(t), \quad x(0) = x_0, \quad \alpha \in (0, 1). \tag{3.3}$$

The solution of the classic Bertalanffy-logistic differential equation $x'(t) = x^{\frac{2}{3}}(t) - x(t)$, $x(0) = x_0$ is $x(t) = \left[1 + \left(x_0^{\frac{2}{3}} - 1\right)e^{-\frac{t}{3}}\right]^3$. By using the change of variable $z = 3x^{\frac{1}{3}}$ in (3.3), we find

$$N_3^\alpha z(t) = 1 - \frac{2}{3}z(t), \quad z_0 = 3x_0^{\frac{1}{3}}. \tag{3.4}$$

Applying the non-conformable Laplace transform $\mathcal{L}$ to both sides of equation (3.4) we obtain

$$L_N(z(t)) = \frac{3}{s} + \frac{z_0 - 3}{s + \frac{1}{3}}.$$

Finally, applying the inverse Laplace transform we have the solution of (3.3) in the form $x(t) = \left[1 + \left(x_0^{\frac{2}{3}} - 1\right)e^{-\frac{t^{1+\alpha}}{3(1+\alpha)}}\right]^3$.

With $\alpha = 0.25, 0.50, 0.75, 1.00,$ the non-conformable Laplace transformation solution of (3.3) are shown in Figures 2 and 3 for $x_0 = 2$ and $x_0 = 4$, respectively.

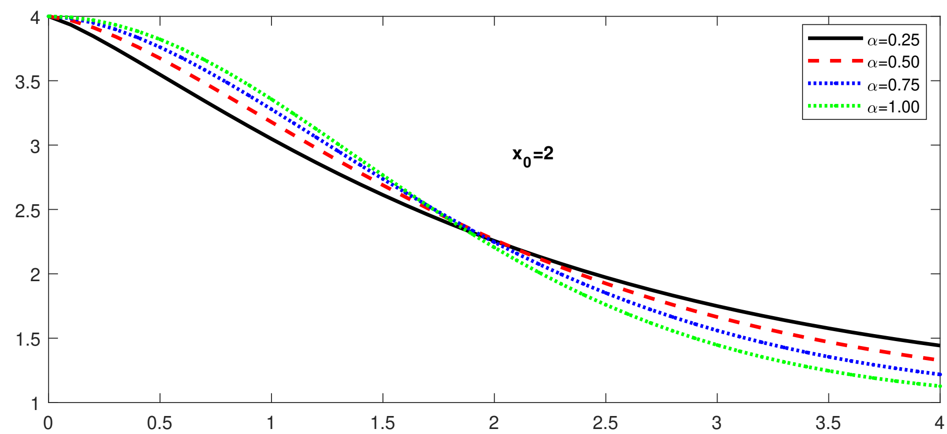

FIGURE 2. Non-conformable Laplace solution of (3.2) for $x_0 = 2$ and different values of α.

Example 3.3. Consider the non-conformable fractional differential equation

$$N_3^\alpha(N_3^\alpha x(t)) + cx(t) = 0, \quad \alpha \in (0, 1], \tag{3.5}$$

with the initial conditions $x(0) = x_0$, $N_3^\alpha x(0) = 0$. Clearly, if $\alpha = 1$ the previous differential equation approximates the characterization of small oscillations of a pendulum, i.e., $x''(t) + cx(t) = 0$, $x(0) = x_0$, $x'(0) = 0$, where $c = \frac{g}{L}$, with g the gravity acceleration and L the length of the pendulum rod. The exact solution to this problem is $x(t) = x_0 \cos \sqrt{c}t = x_0 \cos \sqrt{\frac{g}{L}}t$. Applying the non-conformable Laplace transform to

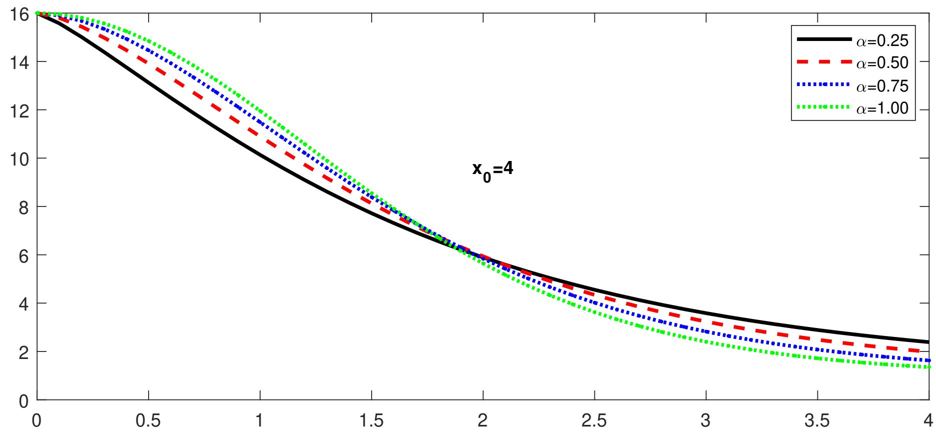

FIGURE 3. Non-conformable Laplace solution of (3.2) for $x_0 = 4$ and different values of α.

the both hand sides of (3.5), we get $(s^2+c)X(s)-sx_0 = 0$, thus $X(s) = \frac{sx_0}{(s^2+c)}$. Taking the inverse non-conformable Laplace transform we obtain $x(t) = x_0 \cos\left(\sqrt{\frac{g}{L}}\frac{t^{\alpha+1}}{\alpha+1}\right)$.

Example 3.4. Now consider the circuit consisting of a voltage source $v(t)$ in series with a resistor (R), a capacitor (C) and an inductor (L), as well as a switch that can be in the open or closed position. The circuit equation in the time domain is $Rx(t) + \frac{1}{c}\int_0^t x(u)du + v_C(0) + Lx'(t) = v(t)$, we assume that $x(0) = 0$ (i.e., the switch is open until $t = 0$, allowing the capacitor to maintain its initial condition $v_C(t)$ before that moment) and $v(t) = A$. The corresponding non-conformable fractional differential equation is

$$Rx(t) + \frac{1}{c}\,{}_NJ_0^{\alpha}(x)(t) + v_C(0) + LN_3^{\alpha}x(t) = A, \quad \alpha \in (0,1].$$

Applying the non-conformable Laplace transform to both sides of above equation, we get $X(s) = \frac{A-v_C(0)}{L(s^2+\frac{R}{L}s+\frac{1}{LC})}$. The poles of the characteristic equation can be obtained as $s = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \left(\frac{R}{2C}\right)^2} = -\sigma \pm iw$ assuming the radicand is positive we have $X(s) = \frac{A-v_C(0)}{L((s+\sigma)^2+w^2)}$. After taking inverse N-transform and reorder you get

$$x(t) = \frac{A - v_C(0)}{wL}\, E_{\alpha}^{N_3}(-\sigma, t)\sin\left(w\frac{t^{\alpha+1}}{\alpha+1}\right).$$

4. EPILOGUE

The fundamental goal of this work has been to generalize the main theorems of the classical Laplace transform into the non-conformable Laplace transform. The goal has been achieved, whereby the non-conformable derivative definition has been used to construct some of these theorems and relations. We calculate the non-conformable

Laplace transform from some elementary functions and establish the non-conformable version of the transform of the successive derivative, the integral of a function and the convolution of the fractional functions. In addition, the bounded and the existence of the non-conformable Laplace transform are presented. The findings of this study indicate that the results obtained in the fractional case are adjusted to the results obtained in the ordinary case. Finally, we show the application of the N-transform to the resolution of fractional differential equations.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] P. Agarwal, S. S. Dragomir, M. Jleli and B. Samet (Eds.), *Advances in Mathematical Inequalities and Applications*, Birkhuser, Basel, 2018.

[3] P. Agarwal, S. Deniz S. Jain, A. A. Alderremy and S. Aly, *A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques*, Phys. A, DOI 10.1016/j.physa.2019.122769.

[4] A. A. El-Sayed and P. Agarwal, *Numerical solution of multiterm variableorder fractional differential equations via shifted Legendre polynomials*, Math. Methods Appl. Sci. **42** (2019), 3978–3991.

[5] P. M. Guzman, G. Langton, L. M. Lugo, J. Medina and J. E. Nápoles Valdés, *A new definition of a fractional derivative of local type*, J. Math. Anal. **9**(2) (2018), 88–98.

[6] C. Hetal, *The fractional Laplace transform solution for fractional differential equation of the oscillator in the presence of an external forces*, International Journal of Advanced Science and Technology **5**(2) (2018), 6–11.

[7] S. Jain and P. Agarwal, *On new applications of fractional calculus*, Bol. Soc. Parana. Mat. (3) **37**(3) (2019), 113–118.

[8] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[9] L. Kexue and P. Jigen, *Laplace transform and fractional differential equations*, Appl. Math. Lett. **24** (2011), 2019–2023.

[10] N. A. Khan, O. A. Razzaq and M. Ayaz, *Some properties and applications of conformable fractional Laplace transform* (*CFLT*), J. Fract. Calc. Appl. **9**(1) (2018), 72–81.

[11] S. D. Lin and C. H. Lu, *Laplace transform for solving some families of fractional differential equations and its applications*, Adv. Difference Equ. **2013** (2013), Paper ID 137, 9 pages.

[12] S. Liang, R. Wu and L. Chen, *Laplace transform of fractional order differential equations*, Electron. J. Differential Equations **2015**(139) (2015), 1–15.

[13] R. R. Nigmatullina and P. Agarwal, *Direct evaluation of the desired correlations: Verication on real data*, Phys. A **534** (2019), Paper ID 121558.

[14] J. E. Nápoles, P. M. Guzman and L. M. Lugo, *Some new results on non conformable fractional calculus*, Adv. Dyn. Syst. Appl. **13**(2) (2018), 167–175.

[15] J. E. Nápoles, P. M. Guzman and L. M. Lugo, *On the stability of solutions of fractional non conformable differential equations*, Stud. Univ. Babes-Bolyai Math. (to appear).

[16] M. Pospísil and L. P. Skripková, *Sturm's theorems for conformable fractional differential equations*, Math. Commun. **21** (2016), 273–281.

[17] M. Ruzhansky, Y. Je Cho, P. Agarwal and I. Area, *Advances in Real and Complex Analysis with Applications*, Springer, Singapore, 2017.

[18] S. Rekhviashvili, A. Pskhu, P. Agarwal and S. Jain, *Application of the fractional oscillator model to describe damped vibrations*, Turkish Journal of Physics **43**(3) (2019), 236–242.

[19] F. S. Silva, D. M. Moreira and M. A. Moret, *Conformable Laplace transform of fractional differential equations*, Axioms **7**(3) (2018), 55.

[20] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu and P. Agarwal, *On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem*, Entropy **17**(2) (2015), 885–902.

[21] X. Zhang, P. Agarwal, Z. Liu and H. Peng, *The general solution for impulsive differential equations with Riemann-Liouville fractional-order* $q \in (1, 2)$, Open Math. **13**(1) (2015), 908-930.

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$A^{\mathcal{I}}$-STATISTICAL APPROXIMATION OF CONTINUOUS FUNCTIONS BY SEQUENCE OF CONVOLUTION OPERATORS

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Abstract. In this paper, following the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences introduced by Savas et al. [22], we deal with Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a, b]$, the space of all real valued continuous functions on $[a, b]$, in the line of Duman [6]. In the Section 3, we study the rate of $A^{\mathcal{I}}$-statistical convergence.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $C[a, b]$ denotes the space of all real valued continuous functions defined on $[a, b]$, endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a, b]$. For a sequence $\{T_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [14] first established the necessary and sufficient conditions for the uniform convergence of $\{T_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1,\ e_2 = x,\ e_3 = x^2$ (see [1]). The study of the Korovkin type approximation theory has a long history and is a well-established area of research (see [4, 5, 7–11]).

Our primary interest, in this paper is to obtain a general Korovkin type approximation theorem for a sequence of positive convolution operators defined on $C[a, b]$, in $A^{\mathcal{I}}$-statistical sense. In the section 3, we study the rate of $A^{\mathcal{I}}$-statistical convergence.

The concept of statistical convergence of a sequence of real numbers was first introduced by Fast [12]. This is a generalization of usual convergence. Further investigations started in this area after the works of Šalát [19] and Fridy [13]. Consequently,

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the notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et al. [17]. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([15, 16]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [2, 3, 15, 16, 18, 20]). In particular, in [21, 22] the very general notion of $A^{\mathcal{I}}$-statistical convergence was introduced.

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (ii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e., $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. The real number sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-convergent to L provided that for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$.

If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

We say that x is A-summable to L if $\lim_{n\to\infty} A_n(x) = L$. A matrix A is called regular if $A \in (c, c)$ and $\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k$ for all $x = \{x_k\}_{k\in\mathbb{N}} \in c$, when c, as usual, stands for the set of all convergent sequences. It is well-known that the necessary and sufficient conditions for A to be regular are

I) $||A|| = \sup_n \sum_k |a_{nk}| < \infty$;

II) $\lim_n a_{nk} = 0,$ for each k;

III) $\lim_n \sum_k a_{nk} = 1$.

For a non-negative regular matrix $A = (a_{nk})$ following [15], a set K is said to have A-density if $\delta_A(K) = \lim_n \sum_{k\in K} a_{nk}$ exists.

The real number sequence $\{x_k\}_{k\in\mathbb{N}}$ is A-statistically convergent to L provided that for every $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has A-density zero (see [15]). Throughout the paper $\mathcal{I}$ will denote the non-trivial admissible ideal on $\mathbb{N}$.

2. $A^{\mathcal{I}}$-Statistical Approximation for a Sequence of Convolution Operators

We first recall the definition.

Definition 2.1 ([21, 22])**.** Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case we write $A^{\mathcal{I}}$-st-$\lim_n x_n = L$.

Note that for $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, $A^{\mathcal{I}}$-statistical convergence becomes A-statistical convergence [15].

We consider the Banach space $C[a, b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a, b]$. Let L be a positive linear operator. Then $L(f) \geq 0$ for any positive function f. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f; x)$.

Theorem 2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f_i) - f_i\| = 0$, with $f_i = t^i$, $i = 0, 1, 2$, then for all $f \in C[a, b]$ we have $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f) - f\| = 0$.*

Proof. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0, C_1, C_2 (depending on $\varepsilon > 0$) such that

$$\|L_n(f) - f\| \leq \varepsilon + C_2\|L_n(f_2) - f_2\| + C_1\|L_n(f_1) - f_1\| + C_0\|L_n(f_0) - f_0\|.$$

If this is done then our hypothesis implies that for $\varepsilon > 0$, $\delta > 0$

$$\left\{ n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in N : \|L_k(f) - f\| \geq \varepsilon\}$.

To this end, start by observing that for each $x \in [a, b]$ the function $0 \leq \Psi \in C[a, b]$ defined by $\Psi(t) = (t - x)^2$. Since each L_n is positive, $L_n(\Psi; x)$ is a positive function. In particular, we have

$$\begin{aligned} 0 \leq L_n(\Psi; x) =& L_n(t^2; x) - 2xL_n(t; x) + x^2 L_n(1; x) \\ =& (L_n(t^2; x) - t^2(x)) - 2x(L_n(t; x) - t(x)) + x^2(L_n(1; x) - 1(x)) \\ \leq& \|L_n(t^2) - t^2\| + 2b\|L_n(t) - t\| + b^2\|L_n(1) - 1\|, \end{aligned}$$

for each $x \in [a, b]$. Let $M = \|f\|$. Since f is bounded on the whole real axis, we can write

$$|f(t) - f(x)| < 2M, \quad -\infty < t, x < \infty.$$

Also, since f is continuous on $[a, b]$, we have

$$|f(t) - f(x)| < \varepsilon,$$

for all t, x satisfying $|t - x| \leq \delta$.

On the other hand, if $|t - x| \geq \delta$, then it follows that

$$-\frac{2M}{\delta^2}(t - x)^2 \leq -2M \leq f(t) - f(x) \leq 2M \leq \frac{2M}{\delta^2}(t - x)^2.$$

Therefore, for all $t \in (-\infty, \infty)$ and all $x \in [a, b]$ we get

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t - x)^2,$$

where δ is a fixed real number.

Since each L_n is positive, we have

$$\begin{aligned}-\varepsilon L_n(f_0;x)-\frac{2M}{\delta^2}L_n(\Psi;x)\leq & L_n(f(t);x)-f(x)L_n(f_0;x)\\ \leq & \varepsilon L_n(f_0;x)+\frac{2M}{\delta^2}L_n(\Psi;x).\end{aligned}$$

Next, let $K=\frac{2M}{\delta^2}$ and we get

$$\begin{aligned}|L_n(f(t);x)-f(x)L_n(f_0;x)|\leq & \varepsilon L_n(f_0;x)+\frac{2M}{\delta^2}L_n(\Psi;x)\\ = & \varepsilon+\varepsilon[L_n(f_0;x)-f_0(x)]+KL_n(\Psi;x)\\ \leq & \varepsilon+\varepsilon|L_n(f_0;x)-f_0(x)|+KL_n(\Psi;x).\end{aligned}$$

In particular,

$$\begin{aligned}|L_n(f(t);x)-f(x)|\leq & |L_n(f(t);x)-f(x)L_n(f_0;x)|+|f(x)||L_n(f_0;x)-f_0(x)|\\ \leq & \varepsilon+KL_n(\Psi;x)+(M+\varepsilon)|L_n(f_0;x)-f_0(x)|,\end{aligned}$$

which implies

$$\|L_n(f)-f\|\leq\varepsilon+C_2\|L_n(f_2)-f_2\|+C_1\|L_n(f_1)-f_1\|+C_0\|L_n(f_0)-f_0\|,$$

where $C_2=K$, $C_1=2bK$ and $C_0=(\varepsilon+b^2K+M)$, i.e.,

$$\|L_n(f)-f\|\leq\varepsilon+C\sum_{i=0}^{2}\|L_n(f_i)-f_i\|,\quad i=0,1,2,$$

where $C=\max\{C_0,C_1,C_2\}$. For a given $\varepsilon'>0$, choose $\varepsilon>0$ such that $\varepsilon<\varepsilon'$ and let us define the following sets

$$\begin{aligned}D= & \{n:\|L_n(f)-f\|\geq\varepsilon'\},\\ D_1= & \left\{n:\|L_n(f_0)-f_0\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\},\\ D_2= & \left\{n:\|L_n(f_1)-f_1\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\},\\ D_3= & \left\{n:\|L_n(f_2)-f_2\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\}.\end{aligned}$$

It follows that $D\subseteq D_1\cup D_2\cup D_3$ and consequently for all $n\in\mathbb{N}$

$$\sum_{k\in D}a_{nk}\leq\sum_{k\in D_1}a_{nk}+\sum_{k\in D_2}a_{nk}+\sum_{k\in D_3}a_{nk},$$

which implies that for any $\sigma>0$

$$\left\{n\in\mathbb{N}:\sum_{k\in D}a_{nk}\geq\sigma\right\}\subseteq\bigcup_{i=1}^{3}\left\{n\in\mathbb{N}:\sum_{k\in D_i}a_{nk}\geq\frac{\sigma}{3}\right\}.$$

Therefore, from hypothesis,

$$\left\{ n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma \right\} \in \mathcal{I}.$$

Hence, we have the proof. □

We now consider the following convolution operators defined on $C[a,b]$ by

$$L_n(f;x) = \int_a^b f(y) K_n(y-x)dy, \quad n \in \mathbb{N},\ x \in [a,b] \text{ and } f \in C[a,b], \tag{2.1}$$

where a and b are two real numbers such that $a < b$. Throughout the paper we assume that K_n is a continuous function on $[a-b, b-a]$ and also that $K_n(u) \geq 0$ for all $n \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Consider the function Ψ on $[a,b]$ defined by $\Psi(y) = (y-x)^2$ for each $x \in [a,b]$.

Theorem 2.2. *Let $A = (a_{ij})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators from $C[a,b]$ into $C[a,b]$. If $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f_0) - f_0\| = 0$, with $f_0(y) = 1$ and $A^{\mathcal{I}}$-st-$\lim_n \|L_n(\Psi)\| = 0$, then for all $f \in C[a,b]$ we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\| = 0.$$

Proof. Let $\Psi(y) := (y-x)^2$ be a function on $[a,b]$, where $x \in [a,b]$ and $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a, b are two real numbers such that $a < b$. Since L_n is a positive linear operator then $L_n(\Psi;x) \geq 0$.

Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f \in C[a,b]$ and $x \in [a,b]$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon, \quad \text{whenever } |y-x| \leq \delta.$$

Let $I_\delta = [x-\delta, x+\delta] \cap [a,b]$. So,

$$\begin{aligned} |f(y) - f(x)| &= |f(y)-f(x)|\Psi_{I_\delta}(y) + |f(y)-f(x)|\Psi_{[a,b]-I_\delta}(y) \\ &\leq \varepsilon + 2M\delta^{-2}(y-x)^2. \end{aligned}$$

Since L_n's are positive and linear so we have,

$$\begin{aligned} |L_n(f;x) - f(x)| &= \left| \int_a^b f(y)K_n(y-x)dy - f(x) \right| \\ &= \left| \int_a^b (f(y)-f(x))K_n(y-x)dy + f(x)\int_a^b K_n(y-x)dy - f(x) \right| \\ &\leq \left| \int_a^b (f(y)-f(x))K_n(y-x)dy \right| + |f(x)| \cdot \left| \int_a^b K_n(y-x)dy - 1 \right| \\ &\leq \int_a^b |f(y)-f(x)| \cdot |K_n(y-x)dy| + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\ &\leq \int_a^b (\varepsilon + 2M\delta^{-2}(y-x)^2)K_n(y-x)dy + M|L_n(f_0;x) - f_0(x)| \end{aligned}$$

$$=\varepsilon + (\varepsilon + M)|L_n(f_0; x) - f_0(x)| + 2M\delta^{-2} \mid L_n(\Psi; x) \mid$$
$$\leq \varepsilon + \alpha\{|L_n(f_0; x) - f_0(x)| + |L_n(\Psi; x)|\},$$

where $\alpha = \max\{\varepsilon + M, \frac{2M}{\delta^2}\}$. Therefore,

$$\|L_n(f) - f\| \leq \varepsilon + \alpha\{\|L_n(f_0) - f_0\| + \|L_n(\Psi)\|\}.$$

For given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$ and define the following sets

$$D = \{n : \|L_n(f) - f\| \geq r\},$$
$$D_1 = \left\{n : \|L_n(f_0) - f_0\| \geq \frac{r - \varepsilon}{2\alpha}\right\},$$
$$D_2 = \left\{n : \|L_n(\Psi)\| \geq \frac{r - \varepsilon}{2\alpha}\right\}.$$

It follows that $D \subseteq D_1 \cup D_2$ and consequently for all $n \in \mathbb{N}$

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk},$$

which implies that for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \subseteq \bigcup_{i=1}^{2} \left\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\right\}.$$

Therefore, from hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \in \mathfrak{I}.$$

Hence, we have the proof. □

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$ and let $\|f\|_\delta = \sup_{a+\delta\leq x\leq b-\delta} |f(x)|$, $f \in C[a, b]$.

In order to give our main result we need the following lemmas.

Lemma 2.1. *Let $A = (a_{ij})$ be a non negative regular summability matrix. Assume that δ is a fixed positive number such that $\delta < \frac{b-a}{2}$. If the conditions*

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1, \tag{2.2}$$

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n (\sup_{|y|\geq\delta} K_n(y)) = 0 \tag{2.3}$$

hold, then for the operators L_n, where $L_n(f; x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a, b]$, $f \in C[a, b]$ and a, b are real numbers $a < b$, we have

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0, \quad \textit{with } f_0(y) = 1.$$

Proof. Let $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then

$$\delta \le x-a \le b-a \Rightarrow -(b-a) \le a-x \le -\delta$$

and

$$\delta \le b-x \le b-a.$$

Now $L_n(f_0;x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy$. Then we have

$$\int_{-\delta}^{\delta} K_n(y)dy \le L_n(f_0;x) \le \int_{-(b-a)}^{b-a} K_n(y)dy.$$

Therefore,

$$\|L_n(f_0) - f_0\|_\delta \le u_n,$$

where $u_n = \max\left\{\left|\int_{-\delta}^{\delta} K_n(y)dy - 1\right|, \left|\int_{-(b-a)}^{b-a} K_n(y)dy - 1\right|\right\}.$

Therefore, $A^{\mathcal{I}}$-st-$\lim_n u_n = 0$ for all $\delta > 0$ such that $\delta < \frac{b-a}{2}$. Now for given $\varepsilon > 0$ define the following sets

$$D := \{n \in \mathbb{N} : \|L_n(f_0) - f_0\|_\delta \ge \varepsilon\},$$
$$D' := \{n \in \mathbb{N} : u_n \ge \varepsilon\}.$$

So $D \subseteq D'$. Then for all $n \in \mathbb{N}$ we have,

$$\sum_{k\in D} a_{nk} \le \sum_{k\in D'} a_{nk}.$$

Then for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \ge \sigma\right\} \subseteq \left\{n \in \mathbb{N} : \sum_{k\in D'} a_{nk} \ge \sigma\right\}.$$

From hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k\in D'} a_{nk} \ge \sigma\right\} \in \mathcal{I}.$$

Hence,

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \ge \sigma\right\} \in \mathcal{I}.$$

So , we have the proof. □

Lemma 2.2. *Let $A = (a_{ij})$ be a non negative regular summability matrix. If conditions (2.2) and (2.3) hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all convolution operators L_n defined by $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a, b are two real numbers such that $a < b$, we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(\Psi)\|_\delta = 0, \qquad with\ \Psi(y) = (y-x)^2.$$

Proof. For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Since $\Psi(y) = y^2 - 2xy + x^2$, then $\Psi \in C[a,b]$ for all $x \in [a+\delta, b-\delta]$. Now $L_n(\Psi; x) = L_n(f_2; x) - 2xL_n(f_1; x) + x^2 L_n(f_0; x)$, with $f_i(y) = y^i,\ i = 0,\ 1\ , 2$. Then for all $n \in \mathbb{N}$

$$L_n(\Psi; x) = \int_a^b (y-x)^2 K_n(y-x)dy = \int_{a-x}^{b-x} y^2 K_n(y)dy \leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy.$$

Since the function f_2 is continuous at $y = 0$ for given $\varepsilon > 0$ exists $\eta > 0$ such that $y^2 < \varepsilon$ for all y satisfying $|y| \leq \eta$. We have two cases such that $\eta \geq b-a$ or $\eta < b-a$.

Case 1. Let $\eta \geq b-a$. Therefore, $0 \leq L_n(\Psi; x) \leq \varepsilon \int_{-(b-a)}^{b-a} K_n(y)dy$. By condition (2.3), $0 \leq L_n(\Psi; x) \leq \varepsilon$ and $A^{\mathcal{I}}\text{-st-}\lim_n \|L_n(\Psi)\|_\delta = 0$ for $\eta \geq b-a$.

Case 2: Let $\eta < b-a$. Therefore, $L_n(\Psi; x) \leq \int_{|y|\geq\eta} y^2 K_n(y)dy + \int_{|y|\leq\eta} y^2 K_n(y)dy$ and hence we obtain

$$\|L_n(\Psi; x)\|_\delta \leq a_n \int_\eta^{b-a} y^2 dy + \varepsilon \int_{|y|\leq\eta} K_n(y)dy = a_n \frac{(b-a)^3 - \eta^3}{3} + \varepsilon b_n,$$

where $a_n = \sup_{|y|\geq\eta} K_n(y)$ and $b_n = \int_{|y|\leq\eta} K_n(y)dy$. Also we have from hypotheses

$$A^{\mathcal{I}}\text{-st-}\lim_n a_n = 0$$

and

$$A^{\mathcal{I}}\text{-st-}\lim_n b_n = 1.$$

Taking, $M = \max\left\{\frac{(b-a)^3-\eta^3}{3}, \varepsilon\right\}$ we have for all $n \in \mathbb{N}$

$$\|L_n(\Psi)\|_\delta \leq \varepsilon + M(a_n + |b_n - 1|).$$

For given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Let

$$\begin{aligned} D &= \{n \in \mathbb{N} : \|L_n(\Psi)\|_\delta \geq r\}, \\ D_1 &= \left\{n \in \mathbb{N} : a_n \geq \frac{r-\varepsilon}{2M}\right\}, \\ D_2 &= \left\{n \in \mathbb{N} : |b_n - 1| \geq \frac{r-\varepsilon}{2M}\right\}. \end{aligned}$$

Therefore, $D \subseteq D_1 \cup D_2$. Hence, for all $n \in \mathbb{N}$ we have,

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk},$$

which implies that for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \subseteq \bigcup_{i=1}^{2} \left\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\right\}.$$

Therefore, from the hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \in \mathcal{I}.$$

Hence, we have the proof. $\square$

Now the following main result follows from Theorem 2.2 and Lemma 2.1, 2.2.

Theorem 2.3. *Let $A = (a_{ij})$ be a non negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by* (2.1)*. If conditions* (2.2) *and* (2.3) *hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_\delta = 0.$$

If we take $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, we get the following result.

Corollary 2.1. ([6, Corollary 2.5]). *Let $A = (a_{ij})$ be a non negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by*

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy,$$

$n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a and b are two real numbers such that $a < b$. If conditions

$$st_A - \lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1$$

and

$$st_A - \lim_n \sup_{|y|\geq\delta} K_n(y) = 0$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$st_A - \lim_n \|L_n(f) - f\|_\delta = 0.$$

Remark 2.1. We now exhibit a sequence of positive convolution operators for which Corollary 2.1 does not apply but Theorem 2.3 does. Let

$$u_n = \begin{cases} 1, & \text{for } n \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Choose an infinite subset $C = \{p_1 < p_2 < p_3 \cdots\}$ from $\mathcal{I} \setminus \mathcal{I}_d$, where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N}$ with natural density zero.

Let $A = (a_{nk})$ be given by

$$a_{nk} = \begin{cases} 1, & \text{if } n = p_i,\ k = 2p_i \text{ for some } i \in \mathbb{N}, \\ 1, & \text{if } n \neq p_i \text{ for any } i, k = 2n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K(\varepsilon) = \{k \in \mathbb{N} : |u_k - 0| \geq \varepsilon\}$ is the set of all even integers. Observe that

$$\sum_{k\in K(\varepsilon)} a_{nk} = \begin{cases} 1, & \text{if } n = p_i \text{ for some } i \in \mathbb{N}, \\ 0, & \text{if } n \neq p_i \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus, for any $\delta > 0$, $\left\{n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta\right\} = C \in \mathcal{I} \setminus \mathcal{I}_d$ which shows that $\{u_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to 0 though x is not A-statistically convergent.

Now let the operators L_n on $C[a,b]$ be defined by

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy.$$

If we choose $K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} e^{-n^2y^2}$, then

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) K_n(y-x) dy.$$

Now for every $\delta > 0$ such that $\delta < \frac{b-a}{2}$ we have

$$\begin{aligned}\int_{-\delta}^{\delta} K_n(y)dy =& \frac{n(1+u_n)}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-n^2y^2} dy - \int_{|y|\geq\delta} e^{-n^2y^2} dy\right)\\ =& \frac{2(1+u_n)}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-y^2} dy - \int_{\delta.n}^{\infty} e^{-y^2} dy\right).\end{aligned}$$

Since $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim\limits_n \int_{\delta.n}^{\infty} e^{-y^2} dy = 0$. Also since $A^{\mathcal{I}}\text{-st-}\lim_n (1+u_n) = 1$, we immediately get

$$A^{\mathcal{I}}\text{-st-}\lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1.$$

On the other hand, we have

$$\sup_{|y|\geq\delta} K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2} \leq \frac{n(1+u_n)}{e^{n^2\delta^2}}.$$

Since $\lim_n \frac{n}{e^{n^2\delta^2}} = 0$ and $A^{\mathcal{I}}\text{-st-}\lim\limits_n (1+u_n) = 1$, we conclude that

$$A^{\mathcal{I}}\text{-st-}\lim_n \sup_{|y|\geq\delta} K_n(y) = 0.$$

Therefore, from Theorem 2.3,

$$A^{\mathcal{I}}\text{-st-}\lim_n \|L_n(f) - f\|_\delta = 0, \quad \text{for all } f \in C[a,b].$$

However note that, as $\{u_k\}_{k\in\mathbb{N}}$ is not A-statistically convergent to zero so K_n do not satisfy the hypotheses of Corollary 2.1.

3. Rate of $A^{\mathcal{I}}$-Statistical Convergence

In this section we study the rates of $A^{\mathcal{I}}$-statistical convergence in Theorem 2.3 using the modulus of continuity. Let $f \in C[a,b]$. The modulus of continuity denoted by $\omega(f,\alpha)$ is defined to be

$$\omega(f,\alpha) = \sup_{|y-x|\leq\alpha} |f(y) - f(x)|.$$

The modulus of continuity of the function f in $C[a,b]$ gives the maximum oscillation of f in any interval of length not exceeding $\alpha > 0$. It is well-known that if $f \in C[a,b]$, then
$$\lim_{\alpha\to 0} \omega(f,\alpha) = \omega(f,0) = 0,$$
and that for any constants $c > 0$, $\alpha > 0$,
$$\omega(f, c\alpha) \leq (1 + [c])\omega(f,\alpha),$$
where $[c]$ is the greatest integer less than or equal to c.

Next we introduce the following definition.

Definition 3.1. Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{c_n\}_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to a number L with the rate of $o(c_n)$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\left\{ j \in \mathbb{N} : \frac{1}{c_j} \sum_{\{n : |x_n - L| \geq \varepsilon\}} a_{jn} \geq \delta \right\} \in \mathcal{I}.$$
In this case we write $A^{\mathcal{I}}$-st-$o(c_n)$- $\lim_n x_n = L$.

We establish the following theorem.

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators given by* (2.1)*. Assume further that $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ are two positive non-increasing sequences. If for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$*
$$A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$
and
$$A^{\mathcal{I}}\text{-}st\text{-}o(d_n)\text{-}\lim_n \omega(f,\alpha_n) = 0,$$
where $\alpha_n := \sqrt{\|L_n(\Psi)\|_\delta}$, then for all $f \in C[a,b]$ we have
$$A^{\mathcal{I}}\text{-}st\text{-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0,$$
where $p_n := \max\{c_n, d_n\}$.

Proof. Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a,b]$ and $x \in [a+\delta, b-\delta]$. By positivity and linearity of the operators L_n and using the inequalities for any $\alpha > 0$ we get
$$\begin{aligned}
|L_n(f;x) - f(x)| \leq & L_n(|f(y) - f(x)|;x) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & L_n\left(\omega\left(f, \alpha\frac{|y-x|}{\alpha}\right);x\right) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & \omega(f,\alpha) L_n\left(1 + \left[\frac{|y-x|}{\alpha}\right];x\right) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & \omega(f,\alpha)\left\{L_n(f_0;x) + \frac{1}{\alpha^2} L_n(\psi;x)\right\} + |f(x)| \cdot |L_n(f_0;x) - f_0(x)|.
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$

$$\|L_n(f) - f\|_\delta \leq \omega(f,\alpha)\left\{\|L_n(f_0)\|_\delta + \frac{1}{\alpha^2}\|L_n(\Psi)\|_\delta\right\} + M_1\|L_n(f_0) - f_0\|_\delta,$$

where $M_1 := \|f\|_\delta$. Now let $\alpha := \alpha_n = \sqrt{\|L_n(\Psi)\|_\delta}$. Then we have

$$\begin{aligned}\|L_n(f) - f\|_\delta \leq& \omega(f,\alpha_n)\{\|L_n(f_0)\|_\delta + 1\} + M_1\|L_n(f_0) - f_0\|_\delta \\ \leq& 2\omega(f,\alpha_n) + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta + M_1\|L_n(f_0) - f_0\|_\delta.\end{aligned}$$

Let $M = \max\{2, M_1\}$. Then we can write for all $n \in \mathbb{N}$ that

$$\|L_n(f) - f\|_\delta \leq M\{\omega(f,\alpha_n) + \|L_n(f_0) - f_0\|_\delta\} + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta.$$

Given $\varepsilon > 0$, define the following sets:

$$\begin{aligned}D :=& \left\{n : \|L_n(f) - f\|_\delta \geq \varepsilon\right\}, \\ D_1 :=& \left\{n : \omega(f,\alpha_n) \geq \frac{\varepsilon}{3M}\right\}, \\ D_2 :=& \left\{n : \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3}\right\}, \\ D_3 :=& \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3M}\right\}.\end{aligned}$$

Then $D \subseteq D_1 \cup D_2 \cup D_3$. Also, we define

$$\begin{aligned}D_2' =& \left\{n : \omega(f,\alpha_n) \geq \sqrt{\frac{\varepsilon}{3}}\right\}, \\ D_2'' =& \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \sqrt{\frac{\varepsilon}{3}}\right\}.\end{aligned}$$

Therefore, $D_2 \subseteq D_2' \cup D_2''$. Hence, we get $D \subseteq D_1 \cup D_2' \cup D_2'' \cup D_3$. Since $p_n = \max\{c_n, d_n\}$ we obtain for all $j \in \mathbb{N}$ that

$$\frac{1}{p_j}\sum_{n\in D} a_{jn} \leq \frac{1}{d_j}\sum_{n\in D_1} a_{jn} + \frac{1}{d_j}\sum_{n\in D_2'} a_{jn} + \frac{1}{c_j}\sum_{n\in D_2''} a_{jn} + \frac{1}{c_j}\sum_{n\in D_3} a_{jn}.$$

As

$$A^{\mathcal{I}}\text{-st-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$

and

$$A^{\mathcal{I}}\text{-st-}o(d_n)\text{-}\lim_n \omega(f,\alpha_n) = 0.$$

Therefore,

$$\left\{j \in \mathbb{N} : \frac{1}{p_j}\sum_{n\in D} a_{jn} \geq \delta\right\} \in \mathcal{I},$$

i.e.,

$$A^{\mathcal{I}}\text{-st-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0, \quad \text{for all } f \in C[a,b],$$

where $p_n := \max\{c_n, d_n\}$. Hence, the result follows. □

4. Conclusions

Following the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences, we have encountered a Korovkin type approximation theory (Theorem 2.3) for a sequence of positive convolution operators defined on $C[a, b]$. We have exhibited an example which shows that Theorem 2.3 is stronger than its A-statistical version [6, Corollary 2.5]. The third section states about the rates of the $A^{\mathcal{I}}$-statistical convergence.

We are very much interested whether the results of this paper are valid for the function f with two variables. Again we are interested whether the results are relevant on infinite interval.

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References

[1] C. D. Aliprantis and O. Burkinshaw, *Principles of Real Analysis*, Academic Press, New York, 1998.

[2] J. Connor, *The statistical and strong p-Cesáro convergence of sequences*, Analysis **8** (1988), 47–63.

[3] K. Demirci, *Strong A-summabilty and A-statistical convergence*, Indian J. Pure Appl. Math. **27** (1996), 589–593.

[4] O. Duman, M. K. Khan and C. Orhan, *A-statistical convergence of approximating operators*, Math. Inequal. Appl. **6**(4) (2003), 689–699.

[5] O. Duman, E. Erkuş and V. Gupta, *Statistical rates on the multivariate approximation theory*, Math. Comp. Model. **44**(9–10) (2006), 763–770.

[6] O. Duman, *A-statistical convergence of sequences of convolution operators*, Taiwanese J. Math. **12**(2) (2008), 523–536.

[7] S. Dutta and P. Das, *Korovkin type approximation theorem in $A_2^{\mathcal{I}}$-statistical sense*, Mat. Vesnik **67**(4) (2015), 288–300.

[8] S. Dutta, S. Akdağ and P. Das, *Korovkin type approximation theorem via $A_2^{\mathcal{I}}$-summability method*, Filomat **30**(10) (2016), 2663–2672.

[9] S. Dutta and R. Ghosh, *Korovkin type approximation on an infinite interval via generalized matrix summability method using ideal*, Stud. Univ. Babes-Bolyai Math. (2019) (to appear).

[10] S. Dutta and R. Ghosh, *Korovkin type approximation on an infinite interval in $A_2^{\mathcal{I}}$-statistical sense*, Acta Math. Univ. Comenianae **LXXXIX**(1) (2020), 131–142.

[11] E. Erkuş and O. Duman, *A-statistical extension of the Korovkin type approximation theorem*, Proc. Indian Acad. Sci. Math. Sci. **115**(4) (2005), 499–508.

[12] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[13] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[14] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, 1960.

[15] E. Kolk, *Matrix summability of statistically convergent sequences*, Analysis **13** (1993), 77–83.

[16] E. Kolk, *The statistical convergence in Banach spaces*, Acta Comment. Univ. Tartu. Math. **928** (1991), 41–52.

[17] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26**(2) (2000/2001), 669–685.

[18] M. Mursaleen and A. Alotaibi, *Statistical summability and approximation by de la Vallée-poussin mean*, Appl. Math. Lett. **24** (2011), 672–680.

[19] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[20] E. Savas and P. Das, *A generalized statistical convergence via ideals*, Appl. Math. Lett. **24** (2011), 826–830.

[21] E. Savas, P. Das and S. Dutta, *A note on some generalized summability methods*, Acta Math. Univ. Comenianae **82**(2) (2013), 297–304.

[22] E. Savas, P. Das and S. Dutta, *A note on strong matrix summability via ideals*, Appl. Math. Lett. **25** (2012), 733–738.

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SPECTRA OF THE LOWER TRIANGULAR MATRIX $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ OVER c_0

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Abstract. The infinite lower triangular matrix $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ is considered over the sequence space c_0, where l and l' are positive integers. The diagonal and sub-diagonal entries of the matrix consist of the oscillatory sequences $r = (r_{k(\text{mod } l)+1})$ and $s = (s_{k(\text{mod } l')+1})$, respectively. The rest of the entries of the matrix are zero. It is shown that the matrix represents a bounded linear operator. Then the spectrum of the matrix is evaluated and partitioned into its fine structures: point spectrum, continuous spectrum, residual spectrum, etc. In particular, the spectra of the matrix $\mathbb{B}(r_1, \dots, r_4; s_1, \dots, s_6)$ are determined. Finally, an example is taken in support of the results.

1. Introduction

The study of the spectrum of a bounded linear operator has received much attention in recent years due to its wide application in functional analysis, classical quantum mechanics, etc. Let A be an infinite matrix that is bounded and linear in a Banach space U. Then many dynamical systems can be reformulated as the system of linear equations $Ax = \lambda x$, where λ is a complex number and x is a nonzero vector in U. The stability of this system can be explained by the spectrum of A. In this course, spectrum localization of an infinite matrix over a sequence space is viewed as an important problem by many authors [10, 14–16, 23, 26]. An extensive study of most of the research done in this direction can be found in the review articles [25] and [17].

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For a sequence $x = (x_k)$, the backward difference operator Δ is defined by $\Delta x = x_k - x_{k-1}$, where $x_{-1} = 0$. The matrix representation of this operator is as follows:

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In short, Δ is an infinite matrix whose diagonal entries and subdiagonal entries are the constant sequences $(1, 1, \dots)$ and $(-1, -1, \dots)$, respectively. Akhmedov and Başar [1] determined the spectral decompositions of this operator over bv_p $(1 \leq p < \infty)$, whereas Altay and Başar [3] evaluated the spectra of the same operator over the spaces c and c_0. Altay and Başar [4] then considered the difference operator $B(r, s)$ over c_0 and c, which is a generalization of the operator Δ. The diagonal and subdiagonal entries of $B(r, s)$ contain the sequences $(r, r, \dots)$ and $(s, s, \dots)$, where r and $s \neq 0$ are real numbers. Furkan and Bilgiç studied $B(r, s)$ in the same direction over ℓ_p and bv_p in [6]. For more study, we refer [2, 7, 8, 12, 13, 18, 19, 22, 24] etc. Now if one considers the more generalized difference matrix whose diagonal and subdiagonal entries are the oscillatry sequences $(r_1, r_2, \dots, r_l, r_1, \dots)$ and $(s_1, s_2, \dots, s_{l'}, s_1, \dots)$, where l and l' are some positive integers, then the number of limit points of both the sequences will be different and it will be interesting to study the spectral property of the matrix.

In this paper, we have determined the spectra and fine spectra of the generalized difference matrix $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ in which the diagonal entries consist of a sequence whose terms are oscillating between the points $r_1, r_2, \dots, r_l$ and the subdiagonal entries consist of an oscillatory sequence whose terms are oscillating between the points $s_1, s_2, \dots, s_{l'}$. Furthermore, the spectra and fine spectra of the matrix $\mathbb{B}(r_1, \dots, r_4; s_1, \dots, s_6)$ are also discussed.

2. Preliminaries

Let U and V be Banach spaces. Then the space of all bounded linear operators from U into V is denoted by $B(U, V)$. If $U = V$, then the space is denoted by $B(U)$. Let $L \in B(U)$ and U^* be dual of U. Then the adjoint $L^* \in B(U^*)$ of L is defined by $(L^* f)(t) = f(Lt)$ for all $f \in U^*$. Let $J : D(J) \to U$ be a linear operator defined over a subset $D(J)$ of U. Then the operator $(J - \lambda I)^{-1}$ is called the resolvent operator of J, where λ is a complex number and I is the identity operator.

A complex number λ is said to be a *regular value* [11] of a linear operator $J : D(J) \to U$ if and only if the operator $(J - \lambda I)^{-1}$ exists, bounded and is defined on a set which is dense in U. The set of all regular values of the linear operator J is called *resolvent set* and is denoted by $\rho(J)$. The complement $\sigma(J) = \mathbb{C} - \rho(J)$ is called the *spectrum* of J. The spectrum $\sigma(J)$ is further partitioned into the following three disjoint sets.

(a) $\sigma_p(J) = \{\lambda \in \mathbb{C} : (J - \lambda I)^{-1}$ does not exist$\}$. This set is called the *point spectrum* (*discrete spectrum*) of the operator J. The members of this set are called *eigenvalues* of J.

(b) $\sigma_c(T)$, which is defined as the set of all complex numbers λ for which $(J-\lambda I)^{-1}$ exists and defined on a set which is dense in U, but it is not a bounded operator in U. This set is called *continuous spectrum* of J.

(c) $\sigma_r(T)$, which contains all those complex numbers for which $(J - \lambda I)^{-1}$ exists, defined on a set which is not dense in U. This set is called the *residual spectrum* of J.

Let $R(J - \lambda I)$ denotes the range of the operator $J - \lambda I$. Goldberg [9] has classified the spectrum using the following six properties of $R(J - \lambda I)$ and $(J - \lambda I)^{-1}$:

(I) $R(J - \lambda I) = U$;
(II) $R(J - \lambda I) \neq U$ but $\overline{R(J - \lambda I)} = U$;
(III) $\overline{R(J - \lambda I)} \neq U$

and

(1) $(J - \lambda I)^{-1}$ exists and it is bounded;
(2) $(J - \lambda I)^{-1}$ exists but it is not bounded;
(3) $(J - \lambda I)^{-1}$ does not exist.

Based on the above six properties, the Goldberg's classification of the spectrum can be given as shown in the Table 1.

Table 1. Subdivisions of spectrum of a bounded linear operator

	(I)	(II)	(III)
1	$\rho(J,U)$	—	$\sigma_r(J,U)$
2	$\sigma_c(J,U)$	$\sigma_c(J,U)$	$\sigma_r(J,U)$
3	$\sigma_p(J,U)$	$\sigma_p(J,U)$	$\sigma_p(J,U)$

Theorem 2.1 ([21]). *Let L be a bounded linear operator on a normed linear space U. Then L has a bounded inverse if and only if L^* is onto.*

Lemma 2.1 ([20]). *An infinite matrix $A = (a_{nk}) \in B(c_0)$ if and only if*

(*a*) $(a_{nk})_k \in \ell_1$ *for all n and* $\sup_n \sum_k |a_{nk}| < \infty$*;*
(*b*) $(a_{nk})_n \in c_0$ *for all k.*

Moreover, the norm $\|A\| = \sup_n \sum_k |a_{nk}|$.

Throughout the paper, we denote the set of natural numbers by $\mathbb{N}$, the set of complex numbers by $\mathbb{C}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We assume that $x_{-n} = 0$ for all $n \in \mathbb{N}$.

3. Main Results

Let l and l' be two natural numbers. Suppose that H is the least common multiple of l and l'. Let r_i, $i = 1, \ldots, l$, and $s_i \neq 0$, $i = 1, \ldots, l'$, be complex numbers. Then the matrix $\mathbb{B}(r_1, \ldots, r_l; s_1, \ldots, s_{l'})$ is defined as $\mathbb{B} = (b_{ij})_{i,j \geq 0}$, where

$$b_{ij} = \begin{cases} r_{j(\text{mod } l)+1}, & \text{when } i = j, \\ s_{j(\text{mod } l')+1}, & \text{when } i = j+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

That is

$$\mathbb{B} = \begin{bmatrix} r_1 & & & & \\ s_1 & \ddots & & \text{\Large 0} & \\ & \ddots & r_l & & \\ & & s_{l'} & \ddots & \\ \text{\Large 0} & & & \ddots & \end{bmatrix}.$$

If the matrix $\mathbb{B}$ transforms a sequence $x = (x_k)$ into $y = (y_k)$, then

$$y_k = \sum_{j=0}^{\infty} b_{kj} x_j = b_{k,k-1} x_{k-1} + b_{kk} x_k = s_{(k-1)(\text{mod } l')+1} x_{k-1} + r_{k(\text{mod } l)+1} x_k, \tag{3.2}$$

for all $k \in \mathbb{N}_0$.

Theorem 3.1. *$\mathbb{B} \in B(c_0)$ and $\|\mathbb{B}\|_{c_0} \leq \max_{i,j}\{|r_i| + |s_j|\colon 1 \leq i \leq l, 1 \leq j \leq l'\}$.*

Suppose that a is an integer and n is a natural number. We denote, by $[a_n]$, the set of all non-negative integers x for which n divides $x - a$. Then $a(\text{mod } n)$ is the least member of $[a_n]$. Let α and β be the mappings which are defined on the set of integers as follows:

$$\alpha(k) = k(\text{mod } l) + 1$$

and

$$\beta(k) = k(\text{mod } l') + 1.$$

Without loss of generality, we assume that $s_{\beta(k)} s_{\beta(k+1)} \cdots s_{\beta(k+j)} = 1$ and $(r_{\alpha(k)} - \lambda)(r_{\alpha(k+1)} - \lambda) \cdots (r_{\alpha(k+j)} - \lambda) = 1$, when $k + j < k$. If λ is a complex number such that $(\mathbb{B} - \lambda I)^{-1}$ exists, then the entries of the matrix $(\mathbb{B} - \lambda I)^{-1} = (z_{nk})$, $n \geq 0$, and $k \geq 0$, are given by

$$z_{nk} = \begin{cases} \dfrac{(-1)^{n-k} s_{\beta(k)} \cdots s_{\beta(k+\zeta''-1)}}{(r_{\alpha(k)} - \lambda) \cdots (r_{\alpha(k+\zeta')} - \lambda)} \cdot \dfrac{(s_1 \ldots s_{l'})^{m''}}{\{(r_1 - \lambda) \cdots (r_l - \lambda)\}^{m'}} & \\ \times \left\{ \dfrac{(s_1 \cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1 - \lambda) \cdots (r_l - \lambda)\}^{\frac{H}{l}}} \right\}^m, & \text{when } n > k, \\ \dfrac{1}{r_{\alpha(k)} - \lambda}, & \text{when } n = k, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

where ζ, ζ' and ζ'' are the least non-negative integers such that

$$\begin{cases} n-k = mH+\zeta, \\ \qquad \zeta = m'l+\zeta', \\ \qquad \zeta = m''l'+\zeta'', \end{cases} \tag{3.4}$$

for some non-negative integers m, m' and m''.

Lemma 3.1. *If* $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l} > (|s_1|\cdots|s_{l'}|)^{1/l'}$, *then* $(\mathbb{B}-\lambda I)^{-1} \in B(c_0)$.

Proof. Since $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l} > (|s_1|\cdots|s_{l'}|)^{1/l'}$ and $s_1, s_2, \ldots, s_{l'}$ are non-zero, therefore $\lambda \neq r_i$ for all $i = 1, 2, \ldots, l$. Then the matrix $\mathbb{B}-\lambda I$ is a triangle and hence $(\mathbb{B}-\lambda I)^{-1} = (z_{nk})$ exists, which is given by (3.3). We first consider a row of $(\mathbb{B}-\lambda I)^{-1}$ which is a multiple of H, that is $n = \widetilde{m}H$ for some $\widetilde{m} \in \mathbb{N}_0$. Now, let $k = \hat{m}H$ for $\hat{m} = 0, 1, \ldots, \widetilde{m}$. Then (3.4) implies that $n-k = (\widetilde{m}-\hat{m})H$ and $m' = m'' = \zeta = \zeta' = \zeta'' = 0$. Thus, from (3.3), we have

$$z_{nk} = \frac{(-1)^{n-k}}{r_{\alpha(k)}-\lambda}\left\{\frac{(s_1\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^{\widetilde{m}-\hat{m}},$$

for all $\hat{m} = 0, 1, \ldots, \widetilde{m}$. Therefore,

$$\sum_{k\in[0_H]} |z_{nk}| = \frac{1}{|r_{\alpha(k)}-\lambda|}\sum_{j=0}^{\tilde{m}}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^{j},$$

where $[0_H]$ denotes the set of all non-negative integers which are multiple of H. For the same row, if we consider $k = \hat{m}H+1$ for $\hat{m} = 0, 1, \ldots, \tilde{m}-1$, then $n-k = (\widetilde{m}-\hat{m}-1)H+H-1$. Let m_1' and m_1'' be quotients and ζ_1' and ζ_1'' be remainders when $H-1$ is divided by l and l' respectively, that is

$$H-1 = m_1'l+\zeta_1',$$
$$H-1 = m_1''l'+\zeta_1''.$$

Then, from (3.3), we obtain that

$$\begin{aligned} z_{nk} =& \frac{(-1)^{n-k}s_{\beta(k)}\cdots s_{\beta(k+\zeta_1''-1)}}{(r_{\alpha(k)}-\lambda)\cdots(r_{\alpha(k+\zeta_1')}-\lambda)}\cdot\frac{(s_1\cdots s_{l'})^{m_1''}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{m_1'}} \\ &\times\left\{\frac{(s_1\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^{\widetilde{m}-\hat{m}-1}, \end{aligned}$$

for all $\hat{m} = 0, 1, \ldots, \widetilde{m}-1$. Hence,

$$\begin{aligned} \sum_{k\in[1_H]} |z_{nk}| =& \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_1''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_1')}-\lambda|}\cdot\frac{(|s_1|\cdots|s_{l'}|)^{m_1''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_1'}} \\ &\times\sum_{j=0}^{\widetilde{m}-1}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^{j}, \end{aligned}$$

where $[1_H]$ denotes the set of all nonnegative integers x such that H divides $x-1$. Similarly, for $k = \hat{m}H+2, \ldots, \hat{m}H+H-1$, we have

$$\sum_{k\in[2_L]} |z_{nk}| = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_2''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_2')}-\lambda|}\cdot\frac{(|s_1|\ldots|s_{l'}|)^{m_2''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_2'}} \times \sum_{j=0}^{\widetilde{m}-1}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^j,$$

$$\vdots$$

$$\sum_{k\in[(H-1)_L]} |z_{nk}| = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_{H-1}''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_{H-1}')}-\lambda|}\cdot\frac{(|s_1|\cdots|s_{l'}|)^{m_{H-1}''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_{H-1}'}} \times \sum_{j=0}^{\widetilde{m}-1}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^j,$$

for some integers ζ_i', ζ_i'', m_i' and m_i'' for all $i\in\{2,3,\ldots,H-1\}$. Thus,

$$\sum_{k=0}^{\infty} |z_{nk}| = \frac{1}{|r_{\alpha(k)}-\lambda|}\sum_{j=0}^{\widetilde{m}}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^j + M\sum_{j=0}^{\widetilde{m}-1}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^j, \tag{3.5}$$

where

$$M = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_1''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_1')}-\lambda|}\cdot\frac{(|s_1|\cdots|s_{l'}|)^{m_1''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_1'}} + \cdots + \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_{L-1}''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_{L-1}')}-\lambda|}\cdot\frac{(|s_1|\cdots|s_{l'}|)^{m_{H-1}''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m'_{H-1}}}.$$

Let $M_0 = \max\left\{\frac{1}{|r_{\alpha(k)}-\lambda|}, M\right\}$. Then

$$\sum_{k=0}^{\infty}|z_{nk}| \le \frac{2M_0(|r_1-\lambda||r_2-\lambda|\cdots|r_l-\lambda|)^{\frac{H}{l}}}{(|r_1-\lambda||r_2-\lambda|\cdots|r_l-\lambda|)^{\frac{H}{l}} - (|s_1||s_2|\cdots|s_{l'}|)^{\frac{H}{l'}}}.$$

Therefore, $\sup_{n\in[0_H]}\sum_{k=0}^{\infty}|z_{nk}| < \infty$. Similarly, we prove that

$$\sup_{n\in[1_H]}\sum_{k=0}^{\infty}|z_{nk}| < \infty,$$
$$\sup_{n\in[2_H]}\sum_{k=0}^{\infty}|z_{nk}| < \infty,$$
$$\vdots$$

$$\sup_{n\in[(H-1)_H]}\sum_{k=0}^{\infty}|z_{nk}|<\infty.$$

Thus,

$$\sup_n\sum_{k=0}^{\infty}|z_{nk}|=\max\left\{\sup_{n\in[0_H]}\sum_{k=0}^{\infty}|z_{nk}|,\ \sup_{n\in[1_H]}\sum_{k=0}^{\infty}|z_{nk}|,\dots,\sup_{n\in[(H-1)_H]}\sum_{k=0}^{\infty}|z_{nk}|\right\}.$$

This implies that $\sup_n\sum_{k=0}^{\infty}|z_{nk}|<\infty$. Likewise, for an arbitrary column of $(\mathbb{B}-\lambda I)^{-1}$, adding the entries separately whose rows n belong to $[0_H],[1_H],\dots,[(H-1)_H]$ respectively, we get $\sum_{n=0}^{\infty}|z_{nk}|<\infty$. Therefore, $\lim_{n\to\infty}|z_{nk}|=0$ for all $k\in\mathbb{N}_0$. Hence, by Lemma 2.1, the matrix $(\mathbb{B}-\lambda I)^{-1}\in B(c_0)$. $\square$

Consider the set $S=\left\{\lambda\in\mathbb{C}:(|\lambda-r_1|\cdots|\lambda-r_l|)^{\frac{1}{l}}\le(|s_1|\cdots|s_{l'}|)^{\frac{1}{l'}}\right\}$. Then we have the following theorem.

Theorem 3.2. *$\sigma(\mathbb{B},c_0)=S$.*

Proof. First, we prove that $\sigma(\mathbb{B},c_0)\subseteq S$. Let λ be a complex number that does not belong to S. Then $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l}>(|s_1|\cdots|s_{l'}|)^{1/l'}$. In that case, from Lemma 3.1, it follows that $(\mathbb{B}-\lambda I)^{-1}\in B(c_0)$. That is, $\lambda\notin\sigma(\mathbb{B},c_0)$. Hence, $\sigma(\mathbb{B},c_0)\subseteq S$.

Next, we show that $S\subseteq\sigma(\mathbb{B},c_0)$. Let $\lambda\in S$. Then, $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l}\le(|s_1|\cdots|s_{l'}|)^{1/l'}$. If λ equals any of the r_i for all $i\in\{1,2,\dots,l\}$, then the range of the operator $\mathbb{B}-\lambda I$ is not dense in c_0, and hence $\lambda\in\sigma(\mathbb{B},c_0)$. Therefore, we assume that $\lambda\ne r_i$ for all $i\in\{1,2,\dots,l\}$. In that case, $\mathbb{B}-\lambda I$ is a triangle and $(\mathbb{B}-\lambda I)^{-1}=(z_{nk})$ exists, which is given by (3.3). Let $y=(1,0,0,\dots)\in c_0$ and let $x=(x_k)$ be the sequence such that $(\mathbb{B}-\lambda I)^{-1}y=x$. It follows, from (3.3), that

$$x_{nH}=z_{nH,0}=\frac{(-1)^{nH}}{r_1-\lambda}\left\{\frac{(s_1s_2\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^n, \tag{3.6}$$

for all $n\in N_0$. Since $\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{1/l}\le(s_1\cdots s_{l'})^{\frac{1}{l'}}$, the subsequence (x_{nH}) of x does not converge to 0. Consequently, the sequence $x=(x_k)\notin c_0$. Therefore, $(B-\lambda I)^{-1}\notin B(c_0)$. Thus, $\lambda\in\sigma(\mathbb{B},c_0)$ and hence $S\subseteq\sigma(\mathbb{B},c_0)$. This proves the theorem. $\square$

Theorem 3.3. *$\sigma_p(\mathbb{B},c_0)=\emptyset$.*

Proof. Let $\lambda\in\sigma_p(\mathbb{B},c_0)$. Then there exists a nonzero sequence $x=(x_k)$ such that $\mathbb{B}x=\lambda x$. This implies that

$$s_{(k-1)(\text{mod } l')+1}x_{k-1}+r_{k(\text{mod } l)+1}x_k=\lambda x_k. \tag{3.7}$$

Let x_{k_0} be the first non-zero term of the sequence $x=(x_k)$. Then from the relation (3.7), we find that $\lambda=r_{k_0(\text{mod } l)+1}$. Next, for $k=k_0+l$, (3.7) becomes

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1}+r_{(k_0+l)(\text{mod } l)+1}x_{k_0+l}=\lambda x_{k_0+l}.$$

That is,

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1} + r_{k_0(\text{mod } l)+1}x_{k_0+l} = \lambda x_{k_0+l}. \tag{3.8}$$

Putting $\lambda = r_{k_0(\text{mod } l)+1}$ in (3.8), we find that

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1} = 0.$$

As $s_{(k_0+l-1)(\text{mod } l')+1} \neq 0$, therefore $x_{k_0+l-1} = 0$. Similarly, using (3.7) for $k = k_0+l-1$ and putting the value $x_{k_0+l-1} = 0$, we obtain $x_{k_0+l-2} = 0$. Repeating the same step for $k = k_0 + l - 2, k_0 + l - 3, \dots, k_0 + 1$, we deduce that $x_{k_0} = 0$, which is a contradiction. Hence, $\sigma_p(\mathbb{B}, c_0) = \emptyset$. □

Let $\mathbb{B}^* = (b^*_{ij})$ denote the adjoint of the operator $\mathbb{B}$. Then the matrix representation of $\mathbb{B}^*$ is equal to the transpose of the matrix $\mathbb{B}$. It follows that

$$b^*_{ij} = \begin{cases} r_{i(\text{mod } l)+1}, & \text{when } i = j, \\ s_{i(\text{mod } l')+1}, & \text{when } i+1 = j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

That is,

$$\mathbb{B}^* = \begin{bmatrix} r_1 & s_1 & & & & \\ & r_2 & \ddots & & 0 & \\ & & \ddots & s_{l'} & & \\ & & & r_1 & \ddots & \\ 0 & & & & \ddots & \ddots \end{bmatrix}.$$

The next theorem gives the point spectrum of the operator B^*.

Theorem 3.4. $\sigma_p(\mathbb{B}^*, c_0^*) = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}\right\}$.

Proof. Let $\lambda \in \sigma_p(\mathbb{B}^*, c_0^* \cong \ell_1)$. Then there exists a nonzero sequence $x = (x_k) \in \ell_1$ such that $\mathbb{B}^*x = \lambda x$. From this relation, the subsequences $(x_{kH}), (x_{kH+1}), \dots, (x_{kH+H-1})$ of $x = (x_k)$ are given by

$$\begin{aligned} x_{kH} &= \left\{ \frac{((\lambda - r_1) \cdots (\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0 \\ x_{kH+1} &= \frac{(\lambda - r_1)}{s_1} \left\{ \frac{((\lambda - r_1) \cdots (\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0 \\ &\vdots \\ x_{kH+H-1} &= \frac{(\lambda - r_1)^{\frac{H}{l}} \cdots (\lambda - r_{l-1})^{\frac{H}{l}} (\lambda - r_l)^{\frac{H}{l}-1}}{s_1^{\frac{H}{l'}} \cdots s_{l'-1}^{\frac{H}{l'}} s_{l'}^{\frac{H}{l'}-1}} \end{aligned}$$

$$\times \left\{ \frac{((\lambda - r_1)\cdots(\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0.$$

Thus,

$$\begin{aligned}\sum_{k=0}^{\infty}|x_k| &= \sum_{k=0}^{\infty}|x_{kH}| + \sum_{k=0}^{\infty}|x_{kH+1}| + \cdots + \sum_{n=0}^{\infty}|x_{kH+H-1}| \\ &= \left(1 + \left|\frac{\lambda - r_1}{s_1}\right| + \cdots + \left|\frac{(\lambda - r_1)^{\frac{H}{l}} \cdots (\lambda - r_{l-1})^{\frac{H}{l}} (\lambda - r_l)^{\frac{H}{l}-1}}{s_1^{\frac{H}{l'}} \cdots s_{l'-1}^{\frac{H}{l'}} s_{l'}^{\frac{H}{l'}-1}}\right|\right) \\ &\quad \times \sum_{k=0}^{\infty} \left| \frac{((\lambda - r_1)\cdots(\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right|^k |x_0|.\end{aligned}$$

Clearly, the sequence $x = (x_k) \in \ell_1$ if and only if $(|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}$. This proves the theorem. □

Theorem 3.5. $\sigma_r(\mathbb{B}, c_0) = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}\right\}$.

Proof. The residual spectrum of a bounded linear operator L on a Banach space U is given by the relation $\sigma_r(L, U) = \sigma_p(L^*, U^*) \setminus \sigma_p(L, U)$. Therefore, $\sigma_r(\mathbb{B}, c_0) = \sigma_p(\mathbb{B}^*, c_0^*) \setminus \sigma_p(\mathbb{B}, c_0)$. Then the proof of this theorem is an easy consequence of the Theorems 3.3 and 3.4. □

Theorem 3.6. $\sigma_c(\mathbb{B}, c_0) = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} = (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}\right\}$.

Proof. Since spectrum of an operator on a Banach space is disjoint union of point, residual and continuous spectrum, therefore from Theorems 3.2, 3.3 and 3.5, we deduce that

$$\sigma_c(\mathbb{B}, c_0) = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} = (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}\right\}. \qquad □$$

Theorem 3.7. $\{r_1, r_2, \ldots, r_l\} \subseteq III_1(\mathbb{B}, c_0)$.

Proof. Theorem 3.5 shows that $r_1 \in \sigma_r(\mathbb{B}, c_0)$. However, $\sigma_r(\mathbb{B}, c_0) = III_1(\mathbb{B}, c_0) \cup III_2(\mathbb{B}, c_0)$. Therefore, to prove $r_1 \in III_1\sigma(\mathbb{B}, c_0)$, we shall show that the matrix $\mathbb{B} - r_1 I$ has bounded inverse and from Theorem 2.1, it will be sufficient to show that $(\mathbb{B} - r_1 I)^*$ is onto. For this, let $y = (y_k) \in \ell_1$. Then $(\mathbb{B} - r_1 I)^* x = y$ implies that

$$(r_{i(\text{mod } l)+1} - r_1)x_i + s_{i(\text{mod } l')+1}x_{i+1} = y_i, \tag{3.10}$$

for all $i \in \mathbb{N}_0$. Solving (3.10) for $x = (x_i)$, we obtain that

$$x_{mH+k} = \sum_{j=0}^{k-2} \frac{1}{s_{j(\text{mod } l')+1}} \prod_{i=j+1}^{k-1} \frac{r_1 - r_{i(\text{mod } l)+1}}{s_{i(\text{mod } l')+1}} y_{mH+j} + \frac{y_{mH+k-1}}{s_{(k-1)(\text{mod } l')+1}}, \tag{3.11}$$

for $k = 1, \ldots, H$, and $m = 0, \ldots, \infty$. Let

$$C_j = \frac{1}{s_{j(\text{mod } l')+1}} \prod_{i=j+1}^{k-1} \frac{r_1 - r_{i(\text{mod } l)+1}}{s_{i(\text{mod } l')+1}},$$

for $j = 0, \ldots, k-2$, and

$$C_{k-1} = \frac{1}{s_{(k-1)(\text{mod } l')+1}}.$$

Then (3.11) can be written as

$$x_{mH+k} = C_0 y_{mH} + C_1 y_{mH+1} + \cdots + C_{k-1} y_{mH+k-1}. \tag{3.12}$$

Taking summation from $m = 0$ to ∞ of the absolute values of x_{mH+k}, we obtain

$$\sum_{m=0}^{\infty} |x_{mH+k}| \leq |C_0| \sum_{m=0}^{\infty} |y_{mH}| + |C_1| \sum_{m=0}^{\infty} |y_{mH+1}| + \cdots + |C_{k-1}| \sum_{m=0}^{\infty} |y_{mH+k-1}|. \tag{3.13}$$

Since $y = (y_k) \in \ell_1$, therefore the right hand side of the inequality (3.13) is a sum of k finite terms. Thus, $\sum_{m=0}^{\infty} |x_{mH+k}| < \infty$ for $k \in \{1, 2, \ldots, H\}$. This implies that the series

$$\sum_i |x_i| = |x_0| + \sum_{m=0}^{\infty} |x_{mH+1}| + \sum_{m=0}^{\infty} |x_{mH+2}| + \cdots + \sum_{m=0}^{\infty} |x_{mH+H}| \tag{3.14}$$

is a sum of $H + 1$ finite terms. Hence, $x = (x_i) \in \ell_1$. We have shown that for every $y = (y_i) \in \ell_1$ there exists a sequence $x = (x_i) \in \ell_1$ such that $(\mathbb{B} - r_1 I)^* x = y$. That is, $(B - r_1 I)^*$ is onto. Similarly, we can show that $r_i \in III_1(\mathbb{B}, c_0)$ for $i = 2, \ldots, l$. This proves the theorem. $\square$

Theorem 3.8. $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \ldots, r_l\} \subseteq III_2(\mathbb{B}, c_0)$.

Proof. Let λ belongs to the set $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \ldots, r_l\}$. Then $(|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \ldots |s_{l'}|)^{\frac{1}{l'}}$ and $\lambda \notin r_i$ for all $i \in \{1, 2, \ldots, l\}$. This inequality shows that the series $\sum_{k=0}^{\infty} |z_{nk}|$ in (3.5) is not convergent when n goes to infinity. In that case, $\mathbb{B} - \lambda I$ does not have bounded inverse. Then from Table 1, we find that $\lambda \in III_2(\mathbb{B}, c_0)$. Hence $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \ldots, r_l\} \subseteq III_2(\mathbb{B}, c_0)$. $\square$

Theorem 3.9. $III_1(\mathbb{B}, c_0) = \{r_1, r_2, \ldots, r_l\}$.

Proof. From Table 1, we have $\sigma_r(\mathbb{B}, c_0) = III_1(\mathbb{B}, c_0) \cup III_2(, c_0)$ and the union is disjoint. Then taking complement of the inclusion of Theorem 3.8 in $\sigma_r(\mathbb{B}, c_0)$, we obtain that $\sigma_r(\mathbb{B}, c_0) \setminus III_2(\mathbb{B}, c_0) \subseteq \{r_1, r_2, \ldots, r_l\}$. That is, $III_1(\mathbb{B}, c_0) \subseteq \{r_1, r_2, \ldots, r_l\}$. This inclusion together with Theorem 3.7 implies that $III_1(\mathbb{B}, c_0) = \{r_1, r_2, \ldots, r_l\}$. $\square$

Theorem 3.10. $III_2(\mathbb{B}, c_0) = \sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \ldots, r_l\}$.

Proof. Taking complement of the result of Theorem 3.9 in $\sigma_r(\mathbb{B}, c_0)$, we obtain that $III_2(\mathbb{B}, c_0) = \sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \ldots, r_l\}$. $\square$

4. Fine Spectra of the Matrix $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)$

We consider the matrix

$$\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6) = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ s_1 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & s_2 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & s_3 & r_4 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & s_4 & r_1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & s_5 & r_2 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & s_6 & r_3 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & s_1 & r_4 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2 & r_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Now, consider the following sets:

$$D = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} \le (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\},$$

$$D_1 = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} < (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\},$$

$$D_2 = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} = (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\}.$$

From the discussion of the previous section, we deduce the following results:

(a) $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6) \in B(c_0)$;
(b) $\|\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)\|_{c_0} \le \max_{i,j}\{|r_i| + |s_j|\colon 1 \le i \le 4, 1 \le j \le 6\}$;
(c) $\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D$;
(d) $\sigma_p(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \emptyset$;
(e) $\sigma_p(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)^*, c_0^* \cong \ell_1) = D_1$;
(f) $\sigma_r(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_1$;
(g) $\sigma_c(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_2$;
(h) $III_1\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \{r_1, r_2, r_3, r_4\}$;
(i) $III_2\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_1 \setminus \{r_1, r_2, r_3, r_4\}$.

In particular, if we take $r_1 = 1 - i$, $r_2 = -i$, $r_3 = -1.5$, $r_4 = -i$ and $s_1 = i$, $s_2 = 1 + i$, $s_3 = -2$, $s_4 = -1.5$, $s_5 = 1 - i$, $s_6 = -1$, then the spectrum is given by

$$\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \left\{\lambda \in \mathbb{C} : (|\lambda - 1 + i||\lambda + i|^2|\lambda + 1.5|)^{\frac{1}{4}} \le 6^{\frac{1}{6}}\right\},$$

which is shown by the shaded region in Figure 1.

5. Conclusions

We have studied the spectral decomposition of the matrix $\mathbb{B}(r_1, \ldots, r_l; s_1, \ldots, s_{l'})$, which generalizes the following matrices.

- The backward difference operator Δ [3] for $l = 1$, $l' = 1$, $r_1 = 1$ and $s_1 = -1$.
- The Right shift operator for $l = 1$, $l' = 1$, $r_1 = 0$ and $s_1 = 1$.

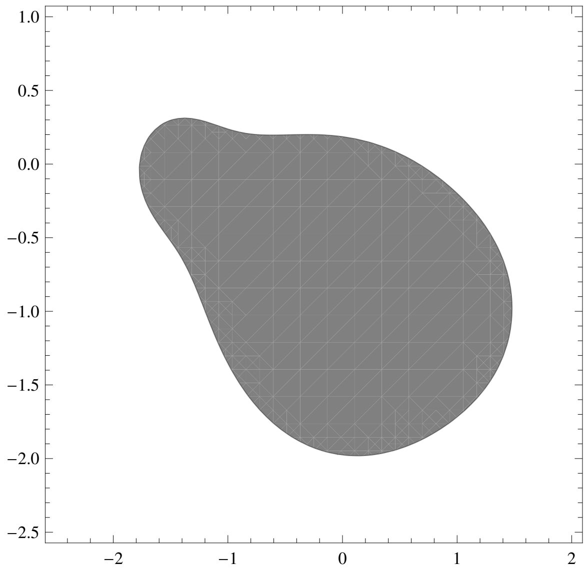

FIGURE 1. Spectrum of $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)$.

- The Zweier matrix [5] for $l = 1$, $l' = 1$, $r_1 = s$ and $s_1 = 1 - s$ for some complex numbers $s \neq 0, 1$.
- The generalized difference operator $B(r, s)$ [4] for $l = 1$, $l' = 1$, $r_1 = r$ and $s_1 = s$ for some complex numbers r and $s \neq 0$.

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REFERENCES

[1] A. M. Akhmedov and F. Başar, *The fine spectra of the difference operator $\triangle$ over the sequence space $bv_p(1 \leq p < \infty)$*, Acta Math. Sin. (Engl. Ser.) **23** (2007), 1757–1768.
[2] A. M. Akhmedov and S. R. El-Shabrawy, *Spectra and fine spectra of lower triangular double-band matrices as operators on ℓ_p $(1 \leq p < \infty)$*, Math. Slovaca **65** (2015), 1137–1152.
[3] B. Altay and F. Başar, *On the fine spectrum of the difference operator $\triangle$ on c_0 and c*, Inform. Sci. **168** (2004), 217–224.
[4] B. Altay and F. Başar, *On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c*, Int. J. Math. Math. Sci. **2005** (2005), 3005–3013.
[5] B. Altay and M. Karakuş, *On the spectrum and the fine spectrum of the zweier matrix as an operator on some sequence spaces*, Thai J. Math. **3** (2005), 153–162.
[6] H. Bilgiç and H. Furkan, *On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$*, Nonlinear Anal. **68** (2008), 499–506.
[7] R. Birbonshi and P. D. Srivastava, *On some study of the fine spectra of n-th band triangular matrices*, Complex Anal. Oper. Theory **11** (2017), 739–753.
[8] E. Dündar and F. Başar, *On the fine spectrum of the upper triangle double band matrix $\triangle^+$ on the sequence space c_0*, Math. Commun. **18** (2013), 337–348.

[9] S. Goldberg, *Unbounded linear operators: Theory and applications*, Courier Corporation, Mineola, New York, 2006.
[10] M. Gonzalez, *The fine spectrum of the Cesàro operator in ℓ_p $(1 < p < \infty)$*, Arch. Math. (Basel) **44** (1985), 355–358.
[11] E. Kreyszig, *Introductory functional analysis with applications*, Wiley, New York, 1989.
[12] H. S. Özarslan and E. Yavuz, *A new note on absolute matrix summability*, J. Inequal. Appl. **2013** (2013), Paper ID 474, 7 pages.
[13] A. Patra, R. Birbonshi and P. D. Srivastava, *On some study of the fine spectra of triangular band matrices*, Complex Anal. Oper. Theory **13** (2019), 615–635.
[14] J. B. Reade, *On the spectrum of the Cesàro operator*, Bull. Lond. Math. Soc. **17** (1985), 263–267.
[15] B. E. Rhoades, *The fine spectra for weighted mean operators*, Pacific J. Math. **104** (1983), 219–230.
[16] B. E. Rhoades, *The fine spectra for weighted mean operators in $B(\ell^p)$*, Integral Equations Operator Theory **12** (1989), 82–98.
[17] P. D. Srivastava, *Spectrum and fine spectrum of the generelized difference operators acting on some Banach sequence spaces - A review*, Ganita **67** (2017), 57–68.
[18] P. D. Srivastava and S. Kumar, *Fine spectrum of the generalized difference operator $\triangle_v$ on sequence space ℓ_1*, Thai J. Math. **8** (2010), 221–233.
[19] P. D. Srivastava and S. Kumar, *Fine spectrum of the generalized difference operator $\triangle_{uv}$ on sequence space ℓ_1*, Appl. Math. Comput. **218** (2012), 6407–6414.
[20] M. Stieglitz and H. Tietz, *Matrixtransformationen von folgenräumen eine ergebnisübersicht*, Math. Z. **154** (1977), 1–16.
[21] A. E. Taylor and C. J. A. Halberg Jr, *General theorems about a bounded linear operator and its conjugate*, J. Reine Angew. Math. **198** (1957), 93–111.
[22] B. C. Tripathy and P. Saikia, *On the spectrum of the Cesàro operator C_1 on $\overline{bv}_0 \cap \ell_\infty$*, Math. Slovaca **63** (2013), 563–572.
[23] R. B. Wenger, *The fine spectra of the Hölder summability operators*, Indian J. Pure Appl. Math. **6** (1975), 695–712.
[24] M. Yeşilkayagil and F. Başar, *On the fine spectrum of the operator defined by the lambda matrix over the spaces of null and convergent sequences*, Abstr. Appl. Anal. **2013** (2013), Article ID 687393, 13 pages.
[25] M. Yeşilkayagil and F. Başar, *A survey for the spectrum of triangles over sequence spaces*, Numer. Funct. Anal. Optim. (2019) (accepted).
[26] M. Yildirim, *On the spectrum and fine spectrum of the compact Rhaly operators*, Indian J. Pure Appl. Math. **27** (1996), 779–784.

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CHAOS AND SHADOWING IN GENERAL SYSTEMS

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Abstract. In this paper we describe some basic notions of topological dynamical systems for maps of type $f : X \times X \to X$ named general systems. This is proved that every uniformly expansive general system has the shadowing property and every uniformly contractive general system has the (asymptotic) average shadowing and shadowing properties. In the rest, Devaney chaos for general systems is considered. Also, we show that topological transitivity and density of periodic points of a general systems imply topological ergodicity. We also obtain some results on the topological mixing and sensitivity for general systems.

1. Introduction

Shadowing and ergodic properties in discrete dynamical systems have received increasing attention in recent years [4–7]. Many authors investigated the relation between shadowing properties and other ergodic properties such as mixing and transitivity [10, 12, 14]. In [2] Blank introduced the notion of average-shadowing property and Gu [9] followed the same scheme to introduce the notion of the asymptotic average shadowing property. In [14] Sakai considered various shadowing properties for positively expansive maps on compact metric spaces and prove that for a positively expansive map; Lipschitz shadowing property, the s-limit shadowing property and the strong shadowing property are all equivalent to the shadowing property. He also prove that average shadowing property and topological transitivity are equivalent for every positively expansive map on a compact metric space. Theorem B in [3] shows that the two-sided limit shadowing property implies topological mixing. In [5, 6] the author introduce uniformly contractive (expansive) iterated function systems (IFS)

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and prove that every uniformly expansive IFS has shadowing property and every uniformly contractive IFS has shadowing and (asymptotic) average shadowing properties. R. Gu [9, 11] prove that every onto continuous map on a compact metric space with (asymptotic) average shadowing property is chain transitive. Also, in [5, 6] the author prove similar results for iterated function systems.

The relationship between chaos and shadowing is an interesting topic for many researchers in the recent years. There are different definitions of chaos. One of the popular definition is Devaney chaos. Indeed a map f is chaotic in the case of Devaney if the periodic points of f is dense, f is topologically transitive and is sensitive. This is well known that the density of periodic points and topological transitivity imply sensitivity. Sanz-Serna [15] devised a method to simulate chaos by use of shadowing lemma. In [1], the authors introduced the notion of P-chaos by changing the condition of transitivity in the definition of Devaney chaos to the shadowing property, and they proved that every P-chaotic systems on a connected space is Devaney chaotic with positive topological entropy.

In this paper we consider a generalization for discrete dynamical systems which introduced in [13]. The main idea of this generalization is based on considering maps $f : X \times X \to X$ instead of maps $f : X \to X$, as discrete dynamical systems. Firstly, we define basic notions, such as, orbit, periodic orbit, shadowing and ergodic properties which we need in the following. Section 3 is devoted to shadowing properties, the main result of this section is Theorem 3.1 which shows that in generalized dynamics uniformly expansivity implies shadowing property. Then two examples of general systems on symbolic space and unit circle are given which have shadowing properties. In section 4, we study the chaotic properties of a general dynamical system. We show that similar original maps and non-autonomous discrete systems [16], the density of periodic points and topological transitivity imply sensitivity in general systems. Finally, we obtain some notions such as topological ergodicity, topological mixing and sensitivity for general systems.

2. Preliminaries

Let (X, d) be a complete metric space and $f : X \times X \to X$ be a continuous map. For $x \in X$, define the orbit of x as follows: $O(x) = \{x_n\}_{n=0}^{\infty}$, where $x_1 = x_0 = x$ and $x_{n+1} = f(x_{n-1}, x_n)$ for all $n \geq 1$.

We say that $x \in X$ is a periodic point of period m if $x_{km+i} = x_i$ for every $k \in \mathbb{N}$ and $0 \leq i \leq n$.

The map f is called to be sensitive if there is $e > 0$ such that for every $x \in X$ and every open subset U of X containing x, there is a point $y \in U$ and $n \in \mathbb{N}$ such that $d(x_n, y_n) > e$.

We say that f is topologically transitive if for every nonempty open sets U, V, if there is $z \in U$ such that for some $m \in \mathbb{N}$, $z_m \in V$. We say that f is chaotic in the sense of Devaney on X if:

1. f is topologically transitive in X;

2. the set of all periodic point of f is dense in X;
3. f is sensitive.

Definition 2.1. The map $f : X \times X \to X$ is said to be contractive if there is a constant $0 < \alpha < 1$, called a contractive constant, such that for every disjoint points $(x, y), (z, w) \in X \times X$ then $d(f(x, y), f(z, w)) < \alpha \max\{d(x, z), d(y, w)\}$.

3. Shadowing and Expanding

For given $\delta > 0$, a sequence $\{x_n\}_{n\geq 0}$ in X is said to be a δ-pseudo orbit of $f : X \times X \to X$ if $x_1 = x_0$ and for every $n \geq 1$ we have $d(x_{n+1}, f(x_{n-1}, x_n)) < \delta$.

One says that the map $f : X \times X \to X$ has the *shadowing property* if for given $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\geq 0}$ there exists $y_0 \in X$ such that $d(x_0, y_0) < \epsilon$ and $d(x_n, f(y_{n-2}, y_{n-1})) \leq \epsilon$ for all $n \geq 2$. In this case one says that the orbit $\{y_n\}_{n\geq 0}$ or the point y_0, ϵ-shadows the δ-pseudo orbit $\{x_n\}_{n\geq 0}$.

Definition 3.1. The map $f : X \times X \to X$ is said to be uniformly expansive if there exists constants $0 < \lambda < 1$ such that for $\mathrm{x}, \mathrm{y} \in X \times X$

$$d(f(\mathrm{x}), f(\mathrm{y})) > \lambda^{-1} d'(\mathrm{x}, \mathrm{y}),$$

where $\mathrm{x} = (x_1, x_2)$, $\mathrm{y} = (y_1, y_2)$ and $d'((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$.

Definition 3.2. A sequence $\{x_i\}_{i\geq 0}$ of points in X is called an asymptotic average pseudo orbit of f if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} d(f(x_{i-1}, x_i), x_{i+1}) = 0.$$

A sequence $\{x_i\}_{i\geq 0}$ in X is said to be asymptotically shadowed in average by a point z in X if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(z_i, x_i) = 0,$$

where $\{z_i\}_{i\geq 0}$ is orbit of the point z.

Definition 3.3. Let $f : X \times X \to X$ be a continuous map. For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ of points in X is called a δ-average-pseudo-orbit of f if there is a number $N = N(\delta)$ such that for all $n \geq N$

$$\frac{1}{n} \sum_{i=1}^{n-1} d(f(x_{i-1}, x_i), x_{i+1}) < \delta.$$

We say that f has the average shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ϵ-shadowed in average by some point $y \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(y_i, x_i) < \epsilon,$$

where $\{y_i\}_{i\geq 0}$ is orbit of the point y.

In the next theorem whose proof is based on [8, Theorem 2.2], we provide some coefficient conditions for a general system to have the shadowing property.

Theorem 3.1. *Let $f : X \times X \to X$ be an uniformly expansive map and for every $x \in X$ the restricted functions $f : \{x\} \times X \to X$ and $f : X \times \{x\} \to X$ be surjective, then f has the shadowing property.*

Proof. The main idea of the proof is to find a Cauchy sequence which converges to a point that ϵ-traced our considered δ-pseudo orbit. Assume that for every $x \in X$ the orbit of x, denoted by $\{x^{f,n}\}_{n\geq 0}$, as $x^{f,0} = x,\ x^{f,1} = x$ and $x^{f,n+1} = f(x^{f,n-1}, x^{f,n})$ for all $n \geq 1$. For given $\epsilon > 0$ take $\delta = (\lambda - 1)\epsilon$, where $0 < \lambda < 1$ is expansivity constant and let $\{x_n\}$ be a δ-pseudo orbit of f. Consider the sequence $\{z_n\}_{n\geq 0}$ in X defined as follows: $z_0 = x_0$, $z_1 = x_1 = x_0$ and z_2 be a point that $x_2 = f(z_1, z_2)$ and for every $n > 2$, z_n be a point that $x_n = z_n^{f,n}$. Given $n \geq 1$ and $0 \leq k \leq n-1$, denote

$$z_{n,k} = z_n^{f,k}. \tag{3.1}$$

This implies that for any $n \geq 1$ and $2 \leq k \leq n-1$ we have:

$$z_{n,k} = f(z_n^{f,k-2}, z_n^{f,k-1}), \quad x_n = f(z_{n,n-2}, z_{n,n-1}). \tag{3.2}$$

Claim. The sequence $\{z_n\}_{n\geq 0}$ in X is a Cauchy sequence.

Proof of Claim. Consider the function $\varphi : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\varphi(s,t) = \begin{cases} \lambda, & s = t, \\ \dfrac{d(f(s), f(t))}{d'(s,t)}, & s \neq t, \end{cases}$$

where $0 < \lambda < 1$ is the expansivity ratio number. This implies that for every $(a,b) \neq (c,d) \in X \times X$, we have that

$$d(a,c) \leq \frac{d(f(a,b), f(c,d))}{\lambda} \quad \text{and} \quad d(b,d) \leq \frac{d(f(a,b), f(c,d))}{\lambda}. \tag{3.3}$$

Firstly, fixing $n \geq 1$ and $m \geq 1$, by using (3.1), (3.2) and above inequalities we obtain:

$$d(z_n, z_{n+m}) \leq \frac{d(z_{n,1}, z_{n+m,1})}{\lambda} \leq \frac{d(z_{n,2}, z_{n+m,2})}{\lambda^2} \leq \cdots \leq \frac{d(x_n, z_{n+m,n-1})}{\lambda^{n-1}}.$$

Secondly, by induction on $m \geq 1$ we show that the following inequality holds uniformly with respect to $n \geq 1$:

$$d(x_n, z_{n+m,n-1}) \leq \delta \sum_{k=1}^{m} \lambda^{-k}. \tag{3.4}$$

Indeed, for $m = 1$ the inequality (3.4) follows from (3.2) and (3.3):

$$d(x_n, z_{n+1,n-1}) \leq \frac{d(f(x_{n-1}, x_n), f(z_{n+1,n-2}, z_{n,n-1}))}{\lambda} = \frac{d(f(x_{n-1}, x_n), x_{n+1})}{\lambda} \leq \frac{\delta}{\lambda}.$$

Assume that (3.4) holds for some $m = p \geq 1$ uniformly on $n \geq 1$. Taking into account this assumption, as well as (3.2), (3.3) and (3.4) for $m = p + 1$:

$$\begin{aligned} d(x_n, z_{n+p+1,n-1}) &\leq \frac{d(f(x_{n-1}, x_n), f(z_{n+p+1,n-2}, z_{n+p,n-1}))}{\lambda} \\ &= \frac{d(f(x_{n-1}, x_n), z_{n+p+1,n})}{\lambda} \\ &\leq \frac{d(f(x_{n-1}, x_n), x_{n+1}) + d(x_{n+1}, z_{n+p+1,n})}{\lambda} \\ &\leq \frac{1}{\lambda}\left(\delta + \delta \sum_{k=1}^{p} \lambda^{-k}\right) \\ &\leq \delta \sum_{k=1}^{p+1} \lambda^{-k}. \end{aligned}$$

Then (3.4) holds for any $m \geq 1$ and any $n \geq 1$.

So, we have the following relation:

$$d(z_n, z_{n+m}) \leq \frac{1}{\lambda^n}\delta \sum_{k=1}^{m} \lambda^{-k} \leq \frac{1}{\lambda^n} \cdot \frac{\delta}{\lambda - 1} = \frac{\epsilon}{\lambda^n} \cdot \frac{\delta}{\lambda - 1} \leq \epsilon\lambda^{-n}. \tag{3.5}$$

Hence, $\{z_n\}_{n\geq 0}$ in X is a Cauchy sequence.

Now, we continue the proof of the theorem.

Let y denote its limit and consider the sequence $\{y^{f,n}\}$ as orbit of y. From (3.1) one has for any $k \geq 0$

$$\lim_{n\to\infty} z_{n,k} = y^{f,k}.$$

Letting $m \to \infty$ in (3.5) implies $d(z_n, y) \leq \epsilon\lambda^{-n}$, and consequently

$$d(x_n, y^{f,n}) \leq \lambda^n(\lambda^{-n}\epsilon) = \epsilon.$$

Therefore, the orbit $\{y^{f,n}\}_{n\geq 0}$ ϵ-shadows the δ-pseudo orbit $\{x_n\}_{n\geq 0}$. ☐

Theorem 3.2. *If $f : X \times X \to X$ is uniformly contracting, then it has shadowing property.*

Proof. Assume that $0 < \beta < 1$ is the contracting ratio of f. Given $\epsilon > 0$ take $\delta = \frac{(1-\alpha)\epsilon}{2}$ and suppose that $\{x_i\}_{i\geq 0}$ is a δ-pseudo orbit for f. So, $d(f(x_{i-1}, x_i), x_{i+1}) < \delta$ for all $i \geq 1$. Put $\beta_i = d(f(x_{i-1}, x_i), x_{i+1})$ for all $i \geq 1$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $d(y_0, x_0) < \frac{\epsilon}{2}$ and $y_{i+1} = f(y_{i-1}, y_i)$ for all $i \geq 1$.

Now we will show that $d(y_i, x_i) < \epsilon$ for all $i \geq 0$. Put $M = d(x_0, y_0)$. Obviously,

$$d(x_1, y_1) \leq d(x_1, f(x_0, x_0)) + d(f(x_0, x_0), f(y_0, y_0)) \leq \beta_0 + \alpha M.$$

Similarly,

$$\begin{aligned} d(x_2, y_2) \leq& d(x_2, f(x_0, x_1)) + d(f(x_0, x_1), f(y_0, y_1)) \\ \leq& \beta_1 + \alpha d(x_1, y_1) \\ \leq& \beta_1 + \alpha(\beta_0 + \alpha M) \end{aligned}$$

and

$$\begin{aligned} d(x_3, y_3) \leq& d(x_3, f(x_1, x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \beta_2 + \alpha d(x_2, y_2) \\ \leq& \beta_2 + \alpha(\beta_1 + \alpha d(x_1, y_1)) \\ \leq& \beta_2 + \alpha(\beta_1 + \alpha(\beta_0 + \alpha M)) \\ =& \beta_2 + \alpha\beta_1 + \alpha^2\beta_0 + \alpha^3 M. \end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \beta_{i-1} + \alpha\beta_{i-2} + \cdots + \alpha^{i-1}\beta_0 + \alpha^i M.$$

This implies that any

$$d(x_n, y_n) \leq \delta(1 + \alpha + \cdots + \alpha^{n-1}) \leq \frac{1}{1-\beta} + M < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

and so, the proof is complete. □

In [5,6], Fatehi Nia proved that every uniformly contractive IFS has average shadowing property and asymptotic average shadowing property. The next theorems show that similar results are established for general systems.

Theorem 3.3. *If $f : X \times X \to X$ is contracting, then it has the average shadowing property.*

Proof. Assume that $\beta < 1$ is the contracting ratio of f. For given $\epsilon > 0$, take $\delta = \frac{(1-\beta)\epsilon}{2} \leq \frac{\epsilon}{2}$ and suppose $\{x_i\}_{i\geq 0}$ is a δ-pseudo orbit for f. So, there exists a natural number $N = N(\delta)$ such that $\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i, x_{i+1}), x_{i+2}) < \delta$ for all $n \geq N(\delta)$. Put $\alpha_i = d(f(x_i, x_{i+1}), , x_{i+2})$ for all $i \geq 0$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $d(x_0, y_0) < \delta \leq \frac{\epsilon}{2}$ and $y_{i+2} = f(y_i, y_{i+1})$ for all $i \geq 0$.

Now we will show that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) < \epsilon$.

Take $M = d(x_0, y_0)$. Similarly,

$$\begin{aligned} d(x_2, y_2) \leq& d(x_2, f(x_0, x_1)) + d(f(x_0, x_1), f(y_0, y_1)) \\ \leq& \alpha_1 + \beta d(x_1, y_1) \\ \leq& \alpha_1 + \beta(\alpha_0 + \beta M) \end{aligned}$$

and

$$\begin{aligned} d(x_3, y_3) \leq& d(x_3, f(x_1 x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \alpha_2 + \beta d(x_2, y_2) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta d(x_1, y_1)) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta(\alpha_0 + \beta M)) \\ =& \alpha_2 + \beta\alpha_1 + \beta^2\alpha_0 + \beta^3 M. \end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \alpha_{i-1} + \beta\alpha_{i-2} + \cdots + \beta^{i-1}\alpha_0 + \beta^i M.$$

This implies that

$$\begin{aligned}\sum_{i=0}^{n-1} d(y_i, x_i) =& M(1+\beta+\cdots+\beta^{n-1}) + \alpha_0(1+\beta+\cdots+\beta^{n-2}) \\ &+ \alpha_1(1+\beta+\cdots+\beta^{n-3}) + \cdots + \alpha_{n-2} \\ \leq& \frac{1}{1-\beta}\left(M + \sum_{i=0}^{n-2}\alpha_i\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) \leq& \frac{1}{1-\beta}\left(M + \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-2}\alpha_i\right) \\ <& \frac{1}{1-\beta}(M+\delta) \\ \leq& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So, the proof is complete. □

Theorem 3.4. *If a map $f : X \times X \to X$ is uniformly contracting, then it has the asymptotic average shadowing property.*

Proof. Assume that $0 < \beta < 1$ is the contracting ratio of f and suppose $\{x_i\}_{i\geq 0}$ is an asymptotic average pseudo orbit for f. So, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i, x_{i+1}), x_{i+2}) = 0$. Put $\alpha_i = d(f(x_i, x_{i+1}), , x_{i+2})$, for all $i \geq 0$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $y_0 \in X$, $y_1 = f(y_0, y_0)$ and $y_{i+2} = f(y_i, y_{i+1})$, for all $i \geq 0$.

Now, we will show that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) = 0$.

Put $M = d(x_0, y_0)$. Obviously,

$$d(x_2, y_2) \leq d(x_2, f(x_0, x_0)) + d(f(x_0, x_0), f(y_0, y_1)) \leq \alpha_0 + \beta M.$$

Similarly,

$$\begin{aligned}d(x_3, y_3) \leq& d(x_3, f(x_1, x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \alpha_2 + \beta d(x_2, y_2) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta d(x_1, y_1)) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta(\alpha_0 + \beta M)) \\ =& \alpha_2 + \beta\alpha_1 + \beta^2\alpha_0 + \beta^3 M.\end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \alpha_{i-1} + \beta\alpha_{i-2} + \cdots + \beta^{i-1}\alpha_0 + \beta^i M.$$

This implies that

$$\begin{aligned}\sum_{i=0}^{n-1} d(y_i, x_i) \leq & M(1+\beta+\cdots+\beta^{n-1}) \\ & + \alpha_0(1+\beta+\cdots+\beta^{n-2}) \\ & + \alpha_1(1+\beta+\cdots+\beta^{n-3}) + \cdots + \alpha_{n-2} \\ \leq & \frac{1}{1-\beta}\left(M + \sum_{i=0}^{n-2}\alpha_i\right).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) \leq \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1-\beta}\left(M+\sum_{i=0}^{n-2}\alpha_i\right)\right) = 0,$$

and so, the proof is complete. □

In the following, we introduce some non trivial examples of general systems on real line, symbolic space and unit circle, that have shadowing properties.

Example 3.1. Consider the following maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = \frac{1}{2}x, \quad f_2(x) = 2x.$$

Take the map $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = \frac{f_1(x)+f_2(y)}{3}$. So, for every disjoint points $(x, y), (z, w) \in \mathbb{R}\times\mathbb{R}$ then $d(f(x, y), f(z, w)) < \frac{2}{3}\max\{d(x, z), d(y, w)\}$. Then this general system is contracting and has the shadowing properties.

Example 3.2. Let Σ denote the set of all infinite sequence $x = (x_0, x_1, x_2, \dots)$, where $x_n = 0$ or 1. The set Σ becomes a compact metric space if we define the distance between two points x, y by $\rho(x, y) = \sum_{i=0}^{\infty}\frac{|x_i-y_i|}{2^{|i|}}$.

Now, consider the map $f : \Sigma\times\Sigma \to \Sigma$ defined by

$$f(\{x_i\}_{i\geq0}\{y_i\}_{i\geq0}) = (x_0, y_0, x_1, y_1, \dots).$$

Please note that if the sequences $\{x_i\}_{i\geq0}$ and $\{z_i\}_{i\geq0}$ are equal in n initial elements and $\{y_i\}_{i\geq0}$ and $\{w_i\}_{i\geq0}$ are equal in m initial elements, then $f(\{x_i\}_{i\geq0}, \{y_i\}_{i\geq0})$ and $f(\{z_i\}_{i\geq0}\{w_i\}_{i\geq0})$ are equal in $m+n$ initial elements. This implies that

$$\begin{aligned}&\rho(f(\{x_i\}_{i\geq0}, \{y_i\}_{i\geq0}), f(\{z_i\}_{i\geq0}, \{w_i\}_{i\geq0})) \\ <&\frac{1}{2}\max\{\rho(\{x_i\}_{i\geq0}, \{z_i\}_{i\geq0}), \rho(\{y_i\}_{i\geq0}, \{w_i\}_{i\geq0})\}.\end{aligned}$$

Consequently, the map $f : \Sigma\times\Sigma \to \Sigma$ is contracting and has the shadowing properties mentioned above.

Example 3.3. Consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. The natural distance on $\mathbb{R}$ induces a distance, d, on S^1. Let $f : S^1\times S^1 \to S^1$ be a map defined by $f(x, y) = (2x+3y)$ (mod 1). This is clear that this is an uniformly expanding map and for every $x, y \in S^1$

the maps $f : \{x\} \times S^1 \to S^1$ and $f : S^1 \times \{y\} \to S^1$ are surjective. Then, Theorem 3.1 implies that the function $f : S^1 \times S^1 \to S^1$ as a general system has the shadowing property.

4. Chaos

In this section, we consider the notion of Devaney's chaos for general systems and prove some results about the relations between this notion and some main properties in general systems.

Theorem 4.1. *Let X be an unbounded metric space with no isolated points. If $f : X \times X \to X$ is topologically transitive and the set of all periodic points is dense in X, then it is sensitive.*

Proof. Let $x \in X$ be an arbitrary point and U be any neighborhood of x. We will show that there exist $z \in U$ and $m > 0$ such that $d(x_m, z_m) > \frac{1}{4}$. Since there are not isolated points and by density of the periodic points, there exists a periodic point $y \in U$ such that $y \neq x$. Put $c := \max\{d(x, z) > 0 : z \in O(y)\}$. Let $c > \frac{1}{2}$. Since X is unbounded, $X \setminus \overline{B_{2c}(x)}$ is a nonempty open subset. Topological transitivity of f implies that there is $y' \in U$ and $m' > 0$ such that $y'_{m'} \in X \setminus \overline{B_{2c}(x)}$.

On the other hand $O(y) \subset \overline{B_c(x)}$, therefore

$$d(y_{m'}, y'_{m'}) \geq d(x, y'_{m'}) - d(x, y_{m'}) > 2c - c = c > \frac{1}{2}.$$

So, we have either $d(x_{m'}, y'_{m'}) > \frac{1}{4}$ or $d(x_{m'}, y_{m'}) > \frac{1}{4}$.

The above result is once $c > \frac{1}{2}$. Now, suppose that $c \leq \frac{1}{2}$. By transitivity, there exists $y'' \in U$ and $m'' > 0$ such that $y''_{m''} \in X \setminus \overline{B_1(x)}$. Also we have that $y_{m''} \in \overline{B_c(x)} \subset \overline{B_{\frac{1}{2}}(x)}$. Hence,

$$d(y_{m''}, y''_{m''}) \geq d(x, y''_{m''}) - d(x, y_{m''}) > 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus, either $d(x_{m''}, y''_{m''}) > \frac{1}{4}$ or $d(x_{m''}, y_{m''}) > \frac{1}{4}$.

So, the proof is complete. □

Corollary 4.1. *Let X be an unbounded metric space with no isolated points. If $f : X \times X \to X$ is topologically transitive and the set of all periodic points is dense in X, then it is chaotic in the sense of Devaney.*

Remark 4.1. If $f : X \to X$ (X is a complete metric space) and $O(x) = \{x_n\}_{n=0}^{\infty}$, where $x_{n+1} = f(x_n)$, then we have $O(x_k) \subseteq O(x)$ for every $k \geq 1$. In this case f is topological transitive if and only if it is transitive (f has a dense orbit). But, for a general system $f : X \times X \to X$ may the above fact is not true. For example for $x \in X$

$$\begin{aligned} O(x) =&\{x = x_0, x_1 = f(x, x), x_2 = f(x_0, x_1), \dots\}, \\ O(x_1) =&\{x_1 = (x_1)_0, (x_1)_1 = f(x_1, x_1), (x_1)_2 = f((x_1)_0, (x_1)_1), \dots\}, \end{aligned}$$

and may $f(x_1, x_1) \notin O(x)$. In this case the density of an orbit of a point may be does not show topological transitivity. Indeed if U and V are two nonempty open subsets of X, then the density of an orbit of a point z implies there are positive integers $n > m$ such that $z_m \in U$ and $z_n \in V$. But this does not show the topological transitivity, because z_n may be not in the orbit of z_m.

The above remark motivated us to define "*strong dense orbit*" of x as follows.

We say that the orbit of $x \in X$ is strong dense orbit if the orbit of x is dense and every element of the orbit of x is also dense in X. We say that the map $f : X \times X \to X$ is strong transitive if it has a strong dense orbit.

Theorem 4.2. *Let X be a complete metric space. If the map $f : X \times X \to X$ is strong transitive, then it is topological transitive. If the map $f : X \times X \to X$ is topological transitive, then it is transitive* (f *has a dense orbit*).

Proof. Let the orbit of z be strong dense orbit and U and V be two nonempty open subsets of X. Then the density of the orbit of point z implies there is a positive integer n such that $z_n \in U$. The strong density of the orbit z implies the orbit of z_n meets V. This shows that f is topological transitive. Suppose that f is topological transitive and U_i, $i = 1, 2, \ldots$, are a countable basis of X. Put $O^-(U_i) = \{x \in X : O(x) \cap U_i \neq \emptyset\}$. Since f is continuous and topological transitive, so $O^-(U_i)$ is open and dense in X. Since X is complete, so $\bigcap U_i \neq \emptyset$. The orbit of every $x \in \bigcap U_i$ is dense in X. This implies f is transitive. □

We say that the map $f : X \times X \to X$ is topologically ergodic if for every two nonempty open sets $U, V \subset X$ there exist an increasing sequence of positive integers $\{n_k\}_{k=0}^{\infty}$ and an integer $l \geq 1$ such that for every $k \geq 1$, $n_{k+1} - n_k \leq l$, there is $z \in U$ such that $z_{n_k} \in V$.

Theorem 4.3. *Let X be a compact metric space and $f : X \times X \to X$ be a continuous map. If f is topologically transitive and the periodic points of f are dense in X, then f is topologically ergodic.*

Proof. Let U and V be two nonempty open subsets of X. Since f is topologically transitive, there is $x \in U$ and $n > 0$ such that $x_n \in V$. Consider $\epsilon > 0$ such that $B_\epsilon(x_n) \subset V$. By continuity of f, there exists open neighborhood W of x such that $W_n \subset V$ is as follows:

$$W = W_0, \quad W_1 = f(W_0, W_0), \quad W_2 = f(W_0, W_1), \ldots, W_n = f(W_{n-2}, W_{n-1}).$$

We can see that $x_n \in W_n$. Since the set of all periodic points is dense in X, there exists a periodic point $q \in W$ with period m. Therefore, $q_n \in W_n \subset V$. So, for each $k \geq 0$ we have $q_{n+km} = q_n \in V$. Hence, for each $k \geq 0$, $q_{km} = q \in U$ and $q_{n+km} = q_n \in V$. So, f is topologically ergodic. □

Let $f : X \times X \to X$ be a continuous map. For $x, y \in X$ and $\epsilon \geq 0$ given, an ϵ-chain from x o y of length $n + 1$ is a sequence $\{x = x_0, x_1, x_2, \ldots, x_n = y\}$ for which

$d(x_{i+1}, f(x_{i-1}, x_i)) < \epsilon$ for each $1 \leq i \leq n-1$. f is said to be topologically chain transitive if for every $x, y \in X$, there exists an ϵ-chain from x to y for every $\epsilon > 0$.

We say that f is topologically chain mixing if for every $\epsilon > 0$ and $x, y \in X$ there is $N \in \mathbb{N}$ such that for each $n \geq N$, there exists an ϵ-chain from x to y of length n.

Lemma 4.1. *If f is topologically chain mixing and has the shadowing property then f is topologically mixing.*

Proof. The proof is clear. □

Theorem 4.4. *Let $f : X \times X \to X$ be an open continuous map with a fixed point a, $f(a, a) = a$. If f is topologically transitive, then f is chain mixing.*

Proof. Let $x, y \in X$ and $\epsilon \geq 0$ be given. Since f is topologically transitive there exist $z, z' \in X$ and $m, m' \in \mathbb{N}$ such that

$$\begin{aligned} d(x_1 = f(x,x), z) &< \epsilon, \\ d(z_m, a) &< \epsilon, \\ d(z', a) &< \epsilon, \\ d(z'_m, y) &< \epsilon. \end{aligned}$$

Put $N = m + m' + 1$. So, for each $n \geq N$ sequence $\{x = x_0, z, \ldots, z_{m-1}, \underbrace{a, a, \ldots, a}, z', \ldots, z'_{m'-1}, y\}$ is an ϵ-chain of length n. Hence, f is chain mixing. □

Theorem 4.5. *By assumption of previous theorem, if f has the shadowing property, then f is topologically mixing.*

Proof. By previous theorem and lemma proof is complete. □

Definition 4.1. We say that $f : X \times X \to X$ is n-sensitive if there is integer $e > 0$ such that for every non empty open subset $U \subset X$, there exist pairwise disjoint points $x_1, \ldots, x_n \in U$ and $k \in \mathbb{N}$ such that

$$\min_{1 \leq i \neq j \leq n} d((x_i)_k, (x_j)_k) > e.$$

Theorem 4.6. *Let $f : X \times X \to X$ be a continuous transitive map with n fixed points $p_1, \ldots, p_n$. If f has the shadowing property, then f is n-sensitive.*

Proof. Suppose $e = \frac{1}{2}\min\{d(p_i, p_j) : i \neq j\}$ and U be an open subset of X. Let $x_0 \in U$ and $0 < \epsilon < \frac{e}{2}$ such that $B_\epsilon(x_0) \subset U$. By assumption of theorem and previous theorem, f is topologically mixing. So for every $1 \leq i \leq n$, there exists k_i such that there is δ-chain of length l from x_0 to p_i for every $l \geq k_i$. Where $\delta > 0$ is in the definition of shadowing property for $\epsilon > 0$.

Hence, for every $1 \leq i \leq n$ there exists $z_i \in U$ such that $d(z_i, x_0) < \epsilon$ and $d((z_i)_l, p_i) < \epsilon$. Put $k = \max\{k_i : 1 \leq i \leq n\}$. Therefore, $\{z_1, z_2, \ldots, z_n\} \subset U$ and $d((z_i)_k, p_i) < \epsilon$.

Hence, we have

$$\min_{1\leq i\neq j\leq n} d((z_i)_k,(z_j)_k) > \frac{e}{4}.$$

This prove the theorem. □

References

[1] T. Arai and N. Chinen, *P-chaos implies distributional chaos and chaos in the sense of devaney with positive topological entropy*, Topology Appl. **154** (2007), 1254–1262.

[2] M. Blank, *Deterministic properties of stochastically perturbed dynamical systems*, Teor. Veroyatn. Primen. **33** (1988), 659–671.

[3] B. Carvalho and D. Kwietniak, *On homeomorphisms with the two-sided limit shadowing property*, J. Math. Anal. Appl. **420** (2014), 801–813.

[4] A. Fakhari and F. Ghane, *On shadowing: ordinary and ergodic*, J. Math. Anal. Appl. **364** (2010), 151–155.

[5] M. Fatehi Nia, *Iterated function systems with the average shadowing property*, Topology Proc. **48** (2016), 261–275.

[6] M. Fatehi Nia, *Parameterized ifs with the asymptotic average shadowing property*, Qual. Theory Dyn. Syst. **15** (2016), 367–381.

[7] M. Fatehi Nia, *Adding machine maps and minimal sets for iterated function systems*, J. Dyn. Syst. Geom. Theor. **15** (2017), 71–83.

[8] V. Glavan and V. Gutu, *Shadowing in parameterized ifs*, Fixed Point Theory **7** (2006), 263–274.

[9] R. Gu, *The asymptotic average shadowing property and transitivity*, Nonlinear Anal. **67** (2007), 1680–1689.

[10] R. Gu, *The average-shadowing property and topological ergodicity*, J. Comput. Appl. Math. **206** (2007), 796–800.

[11] R. Gu, *On ergodicity of systems with the asymptotic average-shadowing property*, J. Dyn. Syst. Geom. Theor. **55** (2008), 1137–1141.

[12] P. Koscielniak and M. Mazur, *Chaos and the shadowing property*, J. Dyn. Syst. Geom. Theor. **154** (2007), 2553–2557.

[13] A. Mihail, *Recurrent iterated function systems*, Rev. Roumaine Math. Pures. Appl. **53** (2008), 43–53.

[14] K. Sakai, *Various shadowing properties for positively expansive maps*, Topology Appl. **131** (1993), 15–31.

[15] J. Sanz-Serna, *Shadows, chaos, and saddles*, Appl. Numer. Math. **13** (1993), 181–190.

[16] H. Zhu, Y. Shi and H. Shao, *Devaney chaos in non-autonomous discrete systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **26**(11) (2016), Paper ID 16501901.

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COEFFICIENT ESTIMATES FOR SUBCLASS OF m-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, a general subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ of the m-Fold symmetric bi-univalent functions is defined. Also, the estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö problems are obtained for functions in this new subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

1. Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (see details in [2,3]).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f\in\mathcal{S}$ has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$ $(z\in\mathbb{U})$ and

$$f(f^{-1}(w))=w \quad \left(|w|<r_0(f),\, r_0(f)\geq\frac{1}{4}\right).$$

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In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$ (see [10]). We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For examples the functions $\frac{z}{1-z}$ and $-\log(1-z)$ belong to the class $\sigma_{\mathcal{B}}$.

The first time in 1967, Lewin [4] introduced the class $\sigma_{\mathcal{B}}$ and proved that the bound for the second coefficients of every $f \in \sigma_{\mathcal{B}}$ satisfies the inequality $|a_2| < 1.51$. Also, Smith [5] showed that $|a_2| < 2/\sqrt{27}$ and $|a_3| < 4/27$ for bi-univalent polynomial $f(z) = z + a_2z^2 + a_3z^3$ with real coefficients.

Recently many researchers introduced subclasses of bi-univalent functions and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For example, we refer the reader to Srivastava et al. [6, 8, 10] and others [13, 14]. The coefficient estimate problem, i.e., bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \sigma_{\mathcal{B}}$, is still an open problem.

Let m be a positive integer. A domain E is known as m-Fold symmetric if a rotation of E around origin with an angle $2\pi/$ maps E on itself. A function $f(z)$ analytic in $\mathbb{U}$ is said to be m-Fold symmetric if

$$f\left(e^{i\frac{2\pi}{m}}z\right) = e^{i\frac{2\pi}{m}}f(z).$$

For each function $f \in \mathcal{S}$, function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.3}$$

is univalent and maps unit disk $\mathbb{U}$ into a region with m-Fold symmetry.

We denote by $\mathcal{S}_m$ the class of m-Fold symmetric univalent functions in $\mathbb{U}$ and clearly $\mathcal{S}_1 = \mathcal{S}$. Every $f \in \mathcal{S}_m$ has a series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{1.4}$$

Srivastava et al. [11], introduced a natural extensions of m-Fold symmetric univalent functions and defined the class Σ_m of symmetric bi-univalent functions. They obtained the series expansion for $g = f^{-1}$ as:

$$\begin{aligned} f^{-1}(w) =& w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots . \end{aligned} \tag{1.5}$$

For $m = 1$ formula (1.5) coincides with formula (1.2) of the class $\sigma_{\mathcal{B}}$.

In fact, this widely-cited work by Srivastava et al. [7] actually revived the study of m-Fold bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [7, 9, 11, 12].

The aim of the this paper is to introduce new subclass $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ of the m-Fold symmetric bi-univalent functions class Σ_m. Moreover, we obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö problems for functions in this subclass.

The results presented in this paper would generalize and improve some recent works of Altinkaya et al. [1] and Li et al. [13].

2. Subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$

In this section, we introduce and consider the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$.

Definition 2.1. Assume that $h:\mathbb{U}\to\mathbb{C}$ and $p:\mathbb{U}\to\mathbb{C}$, are analytic functions of the form

$$\begin{aligned} h(z) =& 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots , \\ p(w) =& 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots , \end{aligned}$$

such that

$$\min\{\mathrm{Re}((h(z)), \mathrm{Re}\,(p(z))\} > 0 \quad (z\in\mathbb{U}).$$

Let $\lambda \geq 0$ and $\gamma \in \mathbb{C}-\{0\}$. We say that a function f given by (1.4) is in the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$, if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\Big[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)-1\Big] \in h(\mathbb{U}) \quad (z\in\mathbb{U}) \tag{2.1}$$

and

$$1+\frac{1}{\gamma}\Big[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)-1\Big] \in p(\mathbb{U}) \quad (w\in\mathbb{U}), \tag{2.2}$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Definition 2.2. A function $f\in\Sigma_m$ given by (1.4) is said to be in the subclass $\mathcal{C}_{\Sigma_m}(\beta)$ $(0\leq\beta<1)$, if two following conditions are satisfied:

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \quad \text{and} \quad \mathrm{Re}\left(1+\frac{wg''(w)}{g'(w)}\right) > \beta \quad (z,w\in\mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Remark 2.1. There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting classes of m-Fold symmetric bi-univalent functions Σ_m. For example, if we let

$$h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1+2\alpha z^m+2\alpha^2 z^{2m}+\cdots \quad (0<\alpha\leq 1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f\in\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$, then

$$\left|\arg\left(1+\frac{1}{\gamma}\Big[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)-1\Big]\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(1+\frac{1}{\gamma}\Big[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)-1\Big]\right)\right| < \frac{\alpha\pi}{2}.$$

In this case we say that f belongs to the subclass $\mathcal{M}_{\Sigma_m}(\alpha,\lambda,\gamma)$.

Also, for $h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^\alpha$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^\alpha_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1].

If we let

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots \quad (0\le\beta<1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f\in\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$, then

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\right)>\beta$$

and

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]\right)>\beta.$$

In this case we say that f belongs to the subclass $\mathcal{M}_{\Sigma_m}(\beta,\lambda,\gamma)$.

Also, for $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^\beta_{\Sigma_m}$ considered by Altinkaya and Yalcin [1].

Furthermore, for $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ and $\lambda=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ reduces to Definition 2.2.

Remark 2.2. For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{M}^{h,p}_{\Sigma_1}(\lambda,\gamma)=\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$. Special cases of this subclass are illustrated below.

(i) By putting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^\alpha$ and $\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $M_\Sigma(\alpha,\lambda)$ studied by Li and Wang [13].

(ii) By putting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^\alpha$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}^\alpha_{\sigma_{\mathrm{B}}}$ of strongly bi-starlike functions of order α $(0<\alpha\le1)$.

(iii) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ and $\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $B_\Sigma(\beta,\lambda)$ studied by Li and Wang [13].

(iv) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\sigma_{\mathrm{B}}}(\beta)$ of bi-starlike functions of order β $(0\le\beta<1)$.

(v) By putting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ and $\lambda=\gamma=1$, the subclass $\mathcal{M}^{h,p}_{\Sigma}(\lambda,\gamma)$ reduces to the subclass $\mathcal{C}_{\sigma_{\mathrm{B}}}(\beta)$ of bi-convev functions of order β $(0\le\beta<1)$.

Theorem 2.1. *Let f given by* (1.4) *be in the subclass* $\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ $(\lambda\ge0,\ \gamma\in\mathbb{C}-\{0\})$. *Then*

$$|a_{m+1}|\le\min\left\{\frac{|\gamma||h_m|}{m(1+\lambda m)},\sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m(1+2\lambda m)}+\frac{(m+1)|\gamma|^2(|h_m|^2+|p_m|^2)}{4m^2(1+\lambda m)^2},\right.$$
$$\left.\frac{(3\lambda m^2+2\lambda m+2m+1)|\gamma||h_{2m}|+(\lambda m^2+2\lambda m+1)|\gamma||p_{2m}|}{4m^2(1+2\lambda m)(1+\lambda m)}\right\}.$$

Proof. The main idea in the proof of Theorem 2.1 is to get the desired bounds for the coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (2.1) and (2.2), we have

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]=h(z)\quad(z\in\mathbb{U}) \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]=p(w)\quad(w\in\mathbb{U}), \tag{2.4}$$

where each of the functions h and p satisfies the conditions of Definition 2.1. For precise comparison of the coefficients of the above equations, in the following we obtain Taylor-Maclaurin series expansions each side of the equations

(2.5)

$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]\\ =&1+\frac{m(1+\lambda m)}{\gamma}a_{m+1}z^m+\left\{\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1}-\frac{m(1+2\lambda m+\lambda m^2)}{\gamma}a_{m+1}^2\right\}z^{2m}\\ &+\cdots,\end{aligned}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)-1\right]\\ =&1-\frac{m(1+\lambda m)}{\gamma}a_{m+1}w^m+\left\{-\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1}\right.\\ &\left.+\frac{m(1+2m+2\lambda m+3\lambda m^2)}{\gamma}a_{m+1}^2\right\}w^{2m}+\cdots.\end{aligned} \tag{2.6}$$

Also from the Definition 2.1, the analytic functions h and p have the following Taylor-Maclaurin series expansions

$$h(z)=1+h_mz^m+h_{2m}z^{2m}+h_{3m}z^{3m}+\cdots \tag{2.7}$$

and

$$p(w)=1+p_mw^m+p_{2m}w^{2m}+p_{3m}w^{3m}+\cdots. \tag{2.8}$$

By comparing the coefficients of the equations (2.5), (2.7), (2.6) and (2.8), respectively, we get

$$\frac{m(1+\lambda m)}{\gamma}a_{m+1} = h_m, \tag{2.9}$$

$$\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1} - \frac{m(1+2\lambda m+\lambda m^2)}{\gamma}a_{m+1}^2 = h_{2m}, \tag{2.10}$$

$$-\frac{m(1+\lambda m)}{\gamma}a_{m+1} = p_m \tag{2.11}$$

and

$$-\frac{2m(1+2\lambda m)}{\gamma}a_{2m+1} + \frac{m(1+2m+2\lambda m+3\lambda m^2)}{\gamma}a_{m+1}^2 = p_{2m}. \tag{2.12}$$

From (2.9) and (2.11), we get

$$h_m = -p_m \tag{2.13}$$

and

$$a_{m+1}^2 = \frac{\gamma^2(h_m^2+p_m^2)}{2m^2(1+\lambda m)^2}. \tag{2.14}$$

Adding (2.10) and (2.12), we get

$$a_{m+1}^2 = \frac{\gamma(h_{2m}+p_{2m})}{2m^2(1+\lambda m)}. \tag{2.15}$$

Therefore, we find from the equations (2.13), (2.14) and (2.15) that

$$|a_{m+1}| \le \frac{|\gamma||h_m|}{m(1+\lambda m)} \quad \text{and} \quad |a_{m+1}| \le \sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2(1+\lambda m)}},$$

respectively. So, we get the desired estimate on the coefficient $|a_{m+1}|$.

The proof is completed by finding the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (2.12) from (2.10), we get

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)}{2}a_{m+1}^2. \tag{2.16}$$

Putting the value of a_{m+1}^2 from (2.14) into (2.16), it follows that

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)\gamma^2(h_m^2+p_m^2)}{4m^2(1+\lambda m)^2}. \tag{2.17}$$

By substituting the value of a_{m+1}^2 from (2.15) into (2.16), we obtain

$$a_{2m+1} = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)\gamma(h_{2m}+p_{2m})}{4m^2(1+\lambda m)}. \tag{2.18}$$

Therefore, from the equations (2.17) and (2.18), we get

$$|a_{2m+1}| \leq \frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m(1+2\lambda m)} + \frac{(m+1)|\gamma|^2(|h_m|^2+|p_m|^2)}{4m^2(1+\lambda m)^2}$$

and

$$|a_{2m+1}| \leq \frac{(3\lambda m^2+2\lambda m+2m+1)|\gamma||h_{2m}|+(\lambda m^2+2\lambda m+1)|\gamma||p_{2m}|}{4m^2(1+2\lambda m)(1+\lambda m)}. \qquad \square$$

Theorem 2.2. *Let f given by* (1.4) *be in the subclass* $\mathcal{M}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ $(\lambda \geq 0,\ \gamma \in \mathbb{C}-\{0\})$. *Also let ρ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2| \leq \begin{cases} \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{(1+T(\rho))\,|h_{2m}|+(1-T(\rho))\,|p_{2m}|\right\}, & |T(\rho)| \leq 1, \\ \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{\left|1+T(\rho)\right||h_{2m}|+\left|T(\rho)-1\right||p_{2m}|\right\}, & |T(\rho)| \geq 1, \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

Proof. From the equation (2.16), we get

$$a_{2m+1}-\rho a_{m+1}^2 = \frac{\gamma(h_{2m}-p_{2m})}{4m(1+2\lambda m)} + \frac{m-2\rho+1}{2}a_{m+1}^2. \tag{2.19}$$

From the equation (2.15) and (2.19), we have

$$\begin{aligned} a_{2m+1}-\rho a_{m+1}^2 =& \frac{|\gamma|}{4m(1+2\lambda m)}\left\{\left[1+\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)}\right]h_{2m}\right. \\ &\left.+\left[\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}-1\right]p_{2m}\right\}. \end{aligned}$$

Next, taking the absolute values we obtain

$$\begin{aligned} |a_{2m+1}-\rho a_{m+1}^2| \leq& \frac{|\gamma|}{4m(1+2\lambda m)}\left\{\left|1+\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)}\right||h_{2m}|\right. \\ &\left.+\left|\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}-1\right||p_{2m}|\right\}. \end{aligned}$$

Then, we conclude that

$$|a_{2m+1}-\rho a_{m+1}^2| \leq \begin{cases} \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{(1+T(\rho))\,|h_{2m}|+(1-T(\rho))\,|p_{2m}|\right\}, & |T(\rho)| \leq 1, \\ \dfrac{|\gamma|}{4m(1+2\lambda m)}\left\{\left|1+T(\rho)\right||h_{2m}|+\left|T(\rho)-1\right||p_{2m}|\right\}, & |T(\rho)| \geq 1. \end{cases}$$

$\square$

3. Corollaries and Consequences

By setting

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \quad (0 < \alpha \leq 1,\, z \in \mathbb{U}),$$

in Theorem 2.1, we conclude the following result.

Corollary 3.1. *Let f given by* (1.4) *be in the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, \gamma)$ $(0 < \alpha \leq 1$, $\lambda \geq 0$, $\gamma \in \mathbb{C} - \{0\})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2\alpha|\gamma|}{m(1+\lambda m)}, \frac{\alpha}{m}\sqrt{\frac{2|\gamma|}{1+\lambda m}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{\alpha^2|\gamma|}{m(1+2\lambda m)} + \frac{2\alpha^2(m+1)|\gamma|^2}{m^2(1+\lambda m)^2}, \frac{\alpha^2|\gamma|(m+1)}{m^2(1+\lambda m)}\right\}.$$

By setting $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$ $(0 < \alpha \leq 1)$ in Theorem 2.2, we conclude the following result.

Corollary 3.2. *Let f given by* (1.4) *be in the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, \gamma)$ $(0 < \alpha \leq 1$, $\lambda \geq 0$, $\gamma \in \mathbb{C} - \{0\})$. Also let ρ be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \dfrac{\alpha^2|\gamma|}{m(1+2\lambda m)}, & |T(\rho)| \leq 1, \\ \dfrac{\alpha^2|T(\rho)||\gamma|}{m(1+2\lambda m)}, & |T(\rho)| \geq 1, \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

By setting $\gamma = 1$ and $\lambda = 0$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. *Let f given by* (1.4) *be in the subclass $\mathcal{S}^{\alpha}_{\Sigma_m}$ $(0 < \alpha \leq 1)$. Then*

$$|a_{m+1}| \leq \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{\alpha^2}{m} + \frac{2(m+1)\alpha^2}{m^2}, \frac{(m+1)\alpha^2}{m^2}\right\} = \frac{(m+1)\alpha^2}{m^2}.$$

Remark 3.1. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.3 are better than those given in [1, Corolary 6], because of

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{\alpha^2}{m} + \frac{2(m+1)\alpha^2}{m^2} \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

By setting $m = 1$ and $\gamma = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.4. *Let f given by* (1.1) *be in the subclass $M_\Sigma(\alpha, \lambda)$ $(0 < \alpha \leq 1,\ \lambda \geq 0)$. Then*

$$|a_2| \leq \begin{cases} \alpha\sqrt{\dfrac{2}{1+\lambda}}, & 0 \leq \lambda \leq 1, \\ \dfrac{2\alpha}{1+\lambda}, & \lambda \geq 1, \end{cases}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2\alpha^2}{1+\lambda}, & 0 \leq \lambda \leq \dfrac{2+\sqrt{13}}{3}, \\ \dfrac{\alpha^2}{1+2\lambda} + \dfrac{4\alpha^2}{(1+\lambda)^2}, & \lambda \geq \dfrac{2+\sqrt{13}}{3}. \end{cases}$$

Remark 3.2. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.4 are better than those given in [13, Theorem 2.2].

By setting $m = 1$ in Corollary 3.3, we conclude the following result.

Corollary 3.5. *Let f given by* (1.1) *be in the subclass $\mathcal{S}^\alpha_{\sigma_{\mathrm{B}}}$ of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \sqrt{2}\alpha \quad \textit{and} \quad |a_3| \leq 2\alpha^2.$$

By setting

$$\begin{aligned} h(z) =& p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} \\ =& 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots \quad (0 \leq \beta < 1, z \in \mathbb{U}), \end{aligned}$$

in Theorem 2.1, we conclude the following result.

Corollary 3.6. *Let f given by* (1.4) *be in the subclass $\mathcal{M}_{\Sigma_m}(\beta, \lambda, \gamma)$ $(0 \leq \beta < 1,$ $\lambda \geq 0,\ \gamma \in \mathbb{C} - \{0\})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2(1-\beta)|\gamma|}{m(1+\lambda m)}, \sqrt{\frac{2(1-\beta)|\gamma|}{m^2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{(1-\beta)|\gamma|}{m(1+2\lambda m)} + \frac{2(1-\beta)^2(m+1)|\gamma|^2}{m^2(1+\lambda m)^2}, \frac{(1-\beta)(m+1)|\gamma|}{m^2(1+\lambda m)}\right\}.$$

By setting $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ $(0 \leq \beta < 1)$ in Theorem 2.2, we conclude the following result.

Corollary 3.7. *Let f given by (1.4) be in the subclass $\mathcal{M}_{\Sigma_m}(\beta,\lambda,\gamma)$ ($0 \leq \beta < 1$, $\lambda \geq 0$, $\gamma \in \mathbb{C} - \{0\}$). Also let ρ be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \dfrac{(1-\beta)|\gamma|}{m(1+2\lambda m)}, & |T(\rho)| \leq 1, \\ \dfrac{(1-\beta)|\gamma||T(\rho)|}{m(1+2\lambda m)}, & |T(\rho)| \geq 1, \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

By setting $\gamma = 1$ and $\lambda = 0$ in Corollary 3.6, we conclude the following result.

Corollary 3.8. *Let f given by (1.4) be in the subclass $\mathcal{S}_{\Sigma_m}^{\beta}$ ($0 \leq \beta < 1$). Then*

$$|a_{m+1}| \leq \begin{cases} \dfrac{\sqrt{2(1-\beta)}}{m}, & 0 \leq \beta \leq \dfrac{1}{2}, \\ \dfrac{2(1-\beta)}{m}, & \dfrac{1}{2} \leq \beta < 1, \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \dfrac{(m+1)(1-\beta)}{m^2}, & 0 \leq \beta \leq \dfrac{1+2m}{2(1+m)}, \\ \dfrac{2(m+1)(1-\beta)^2}{m^2} + \dfrac{1-\beta}{m}, & \dfrac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

Remark 3.3. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.8 are better than those given in [1, Corolary 7].

By setting $\gamma = 1$ and $\lambda = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.9. *Let f given by (1.4) be in the subclass $\mathcal{C}_{\Sigma_m}(\beta)$ ($0 \leq \beta < 1$). Then*

$$|a_{m+1}| \leq \begin{cases} \dfrac{1}{m}\sqrt{\frac{2(1-\beta)}{(1+m)}}, & 2\beta + m \leq 1, \\ \dfrac{2(1-\beta)}{m(1+m)}, & 2\beta + m \geq 1, \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \dfrac{1-\beta}{m^2}, & 0 \leq \beta \leq \dfrac{1+2m-m^2}{2(1+2m)}, \\ \dfrac{1-\beta}{m(1+2m)} + \dfrac{2(1-\beta)^2}{m^2(1+m)}, & \dfrac{1+2m-m^2}{2(1+2m)} \leq \beta < 1. \end{cases}$$

By setting $m = 1$ and $\gamma = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.10. *Let f given by* (1.1) *be in the subclass* $B_\Sigma(\beta,\lambda)$ $(0 \le \beta < 1,\ \lambda \ge 0)$*. Then*

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2(1-\beta)}{1+\lambda}}, & \lambda + 2\beta \le 1, \\ \dfrac{2(1-\beta)}{1+\lambda}, & \lambda + 2\beta \ge 1, \end{cases}$$

and

$$|a_3| \le \begin{cases} \dfrac{2(1-\beta)}{1+\lambda}, & 0 \le \beta \le \dfrac{3+4\lambda-3\lambda^2}{4(1+2\lambda)}, \\ \dfrac{1-\beta}{1+2\lambda} + \dfrac{4(1-\beta)^2}{(1+\lambda)^2}, & \dfrac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \le \beta < 1. \end{cases}$$

Remark 3.4. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.10 are better than those given in [13, Theorem 3.2].

By setting $m = 1$ in Corollary 3.8, we conclude the following result.

Corollary 3.11. *Let f given by* (1.1) *be in the subclass* $\mathcal{S}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-starlike functions of order* β $(0 \le \beta < 1)$*. Then*

$$|a_2| \le \begin{cases} \sqrt{2(1-\beta)}, & 0 \le \beta \le \dfrac{1}{2}, \\ 2(1-\beta), & \dfrac{1}{2} \le \beta < 1, \end{cases}$$

and

$$|a_3| \le \begin{cases} 2(1-\beta), & 0 \le \beta \le \dfrac{3}{4}, \\ 4(1-\beta)^2 + (1-\beta), & \dfrac{3}{4} \le \beta < 1. \end{cases}$$

By setting $m = 1$ in Corollary 3.9, we conclude the following result.

Corollary 3.12. *Let f given by* (1.1) *be in the subclass* $\mathcal{C}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-convex functions of order* β $(0 \le \beta < 1)$*. Then*

$$|a_2| \le 1-\beta \quad \text{and} \quad |a_3| \le \begin{cases} 1-\beta, & 0 \le \beta \le \dfrac{1}{3}, \\ \dfrac{1-\beta}{3} + (1-\beta)^2, & \dfrac{1}{3} \le \beta < 1. \end{cases}$$

References

[1] Ş. Altinkaya and S. Yalçin, *Coefficient bounds for certain subclasses of m-fold symmetric bi-univalent functions*, J. Math.**2015** (2015), Article ID 241683, 5 pages.

[2] D. Breaz, N. Breaz and H. M. Sirvastava, *An extention of the univalent conditions for a family of integral operators*, Appl. Math. Lett. **22** (2009), 41–44.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg. Tokyo, 1983.

[4] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
[5] H. V. Smith, *Bi-univalent polynomials*, Simon Stevin **50**(2) (1976/77), 115–122.
[6] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat **27**(5) (2013), 831–842.
[7] H. M. Srivastava, S. Gaboury and F. Ghanim, *Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions*, Acta Math. Sci. Ser. B **36**(3) (2016), 863–871.
[8] H. M. Srivastava, S. Gaboury and F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afr. Mat. **28** (2017), 693–706.
[9] H. M. Srivastava, S. Gaboury and F. Ghanim, *Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions*, Acta Univ. Apulensis Math. Inform. **41** (2015), 153–164.
[10] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.
[11] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, *Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions*, Tbilisi Math. J. **7**(2) (2014), 1–10.
[12] H. M. Srivastava, A. Zireh and S. Hajiparvaneh, *Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions*, Filomat **32**(9) (2018), 3143–3153.
[13] X. -F. Li and A. -P. Wang, *Two new subclasses of bi-univalent functions*, International Mathematical Forum **7**(30) (2012), 1495–1504.
[14] A. Zireh and S. Salehian, *On the certain subclass of analytic and bi-univalent functions defined by convolution*, Acta Univ. Apulensis Math. Inform. **44** (2015), 9–19.

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ON THE REVERSE MINKOWSKI'S INTEGRAL INEQUALITY

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Abstract. The aim of this work is to obtain the reverse Minkowski integral inequality. For this aim, we first give a proposition which is important for our main results. Then we establish some reverse Minkowski integral inequalities for parameters $0 < p < 1$ and $p < 0$, respectively.

1. Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics and present a very active and attractive field of research. As example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals ([2, 3]). One of the famous integral inequalities is Minkowski's integral inequality. In particular the following statement was proved for $p \geq 1$ (for details to see [1]).

Theorem 1.1. *Let $1 \leq p \leq +\infty$, $\Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be a measurable sets. Suppose that f is measurable on $\Omega \times A$ and $f(\cdot, y) \in L_p(\Omega)$ for almost all $y \in A$. Then*

$$\left\| \int_A f(\cdot, y)dy \right\|_{L_p(\Omega)} \leq \int_A \|f(\cdot, y)\|_{L_p(\Omega)} dy, \tag{1.1}$$

if the right-hand side is finite.

Remark 1.1. If $0 < p < 1$, $\operatorname{mes} A > 0$ and $\operatorname{mes} \Omega > 0$ inequality (1.1) is not valid (to see [1]).

In this paper we obtain some integral inequalities which are reverse versions of the inequality (1.1).

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2. Preliminaries

2.1. **Reverse Young's and Holder's Inequalities.** The following inequalities are well-known Young inequalities. Let $a > 0$, $b > 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{for } p \geq 1, \tag{2.1}$$

$$ab \geq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{for } 0 < p < 1. \tag{2.2}$$

Corollary 2.1 (Reverse Young's inequality)**.** *Let $a > 0$, $b > 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$ab \geq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \textit{for } p < 0. \tag{2.3}$$

Proof. We have $\frac{p'-1}{p'} = \frac{1}{p}$, $(p-1)(p'-1) = 1$ and inequality (2.3) is equivalent to

$$\frac{a^{p-1}}{bp} + \frac{b^{p'-1}}{ap'} \leq 1.$$

We take $t = \frac{a^{p-1}}{b}$, then

$$\frac{b^{p'-1}}{ap'} = \frac{a^{(p-1)(p'-1)}}{t^{(p'-1)}ap'} = \frac{1}{t^{(p'-1)}p'} = \frac{t^{-(p'-1)}}{p'}.$$

We obtain

$$\frac{a^{p-1}}{bp} + \frac{b^{p'-1}}{ap'} = \frac{t}{p} + \frac{t^{-(p'-1)}}{p'} = f(t), \quad t > 0.$$

For all $t > 0$, we have

$$f'(t) = \frac{1}{p} - \frac{p'-1}{p'}t^{-p'} = \frac{1}{p} - \frac{1}{p}t^{-p'} = \frac{1}{p}(1 - t^{-p'}),$$

for all $p < 0$ and $0 < p' < 1$, we get

$$\begin{aligned} f'(t) =& 0 \Leftrightarrow 1 - t^{-p'} = 0 \Leftrightarrow t = 1, \\ f'(t) >& 0 \Leftrightarrow 1 - t^{-p'} < 0 \Leftrightarrow 0 < t < 1. \end{aligned}$$

Hence, the function f is majored with $f(1) = 1$ for all $t \in (0, \infty)$.

We deduce that

$$\frac{a^{p-1}}{bp} + \frac{b^{p'-1}}{ap'} \leq 1 \Leftrightarrow ab \geq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{for } p < 0. \qquad \square$$

Corollary 2.2 (Reverse Hölder's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $p < 0$, we suppose that f, g are measurable on Ω.*

If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$ (p' is the conjugate parameter), then

$$\int_\Omega |fg|dt \geq \|f\|_{L_p}\|g\|_{L_{p'}}. \tag{2.4}$$

Proof. Choose $a = \frac{|f|}{\|f\|_{L_p}}$, $b = \frac{|g|}{\|g\|_{L_{p'}}}$ and by using reverse Young's inequality (2.3), we write

$$\frac{|fg|}{\|f\|_{L_p} \cdot \|g\|_{L_{p'}}} \geq \frac{|f|^p}{p\|f\|_{L_p}^p} + \frac{|g|^{p'}}{p'\|g\|_{L_{p'}}^{p'}},$$

by integrand the above inequality we obtain

$$\int_\Omega \frac{|f(t)g(t)|}{\|f\|_{L_p} \cdot \|g\|_{L_{p'}}} dt \geq \int_\Omega \frac{|f(t)|^p}{p\|f\|_{L_p}^p} dt + \int_\Omega \frac{|g(t)|^{p'}}{p'\|g\|_{L_{p'}}^{p'}} dt = 1,$$

and thus

$$\int_\Omega |f(t)g(t)| dt \geq \|f\|_{L_p} \|g\|_{L_{p'}}, \quad \text{for } p < 0. \qquad \square$$

Remark 2.1. We can write

$$\int_\Omega |f(t)g(t)| dt \geq \left(\int_\Omega |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_\Omega |g(t)|^{p'} dt \right)^{\frac{1}{p'}},$$

hence

$$\left(\int_\Omega |f(t)g(t)| dt \right)^p \leq \left(\int_\Omega |f(t)|^p dt \right) \left(\int_\Omega |g(t)|^{p'} dt \right)^{p-1}$$

(see [4]).

Now we give a proposition which will be used frequently in the proof of main theorems.

Let $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$ and we defined the set $\mathbb{E}$ by

$$\mathbb{E} = \{ f \mid f : (a,b) \times (c,d) \to \mathbb{R},\ f \geq 0 \text{ or } f \leq 0 \}.$$

Suppose $H : (a,b) \times (c,d) \to \mathbb{C}$ a measurable function defined by

$$H(x,y) = f_1(x,y) + i\, f_2(x,y),$$

where $f_1,\ f_2 \in \mathbb{E}$.

Proposition 2.1. *(i) If $f_1 = 0$ or $f_2 = 0$, then*

$$\left| \int_c^d |H(x,y)|\, dy \right| = \left| \int_c^d H(x,y) dy \right|. \tag{2.5}$$

(ii) If $f_1 \neq 0$ and $f_2 \neq 0$, then

$$\left| \int_c^d |H(x,y)|\, dy \right| \leq \sqrt{2} \left| \int_c^d H(x,y) dy \right|. \tag{2.6}$$

Proof. (i) If $f_2 = 0$, then

$$\left| \int_c^d |H(x,y)|\, dy \right| = \left| \int_c^d |f_1(x,y)|\, dy \right| = \left| \int_c^d f_1(x,y) dy \right| = \left| \int_c^d H(x,y) dy \right|.$$

If $f_1 = 0$, then

$$\left| \int_c^d |H(x,y)|\, dy \right| = \left| \int_c^d |i f_2(x,y)|\, dy \right| = \left| \int_c^d |f_2(x,y)| dy \right|$$

$$= \left| \int_c^d f_2(x,y)dy \right| = \left| \int_c^d i f_2(x,y)dy \right|$$
$$= \left| \int_c^d H(x,y)dy \right|.$$

(ii) If $f_1 \neq 0$ and $f_2 \neq 0$, then

$$\begin{aligned}
\left| \int_c^d |H(x,y)|dy \right|^2 &= \left| \int_c^d \left[f_1^2(x,y) + f_2^2(x,y) \right]^{\frac{1}{2}} dy \right|^2 \\
&= \left(\int_c^d \left| f_1^2 + f_2^2 \right|^{\frac{1}{2}} (x,y)dy \right)^2 \\
&= \left\| f_1^2 + f_2^2 \right\|_{L_p(c,d)}, \qquad \text{with } p = \frac{1}{2}, \\
\left| \int_c^d H(x,y)dy \right|^2 &= \left| \int_c^d f_1(x,y)dy + i \int_c^d f_2(x,y)dy \right|^2 \\
&= \left(\int_c^d f_1(x,y)dy \right)^2 + \left(\int_c^d f_2(x,y)dy \right)^2 \\
&= \left(\int_c^d |f_1(x,y)|dy \right)^2 + \left(\int_c^d |f_2(x,y)|dy \right)^2 \\
&= \left\| f_1^2 \right\|_{L_p(c,d)} + \left\| f_2^2 \right\|_{L_p(c,d)}, \qquad \text{with } p = \frac{1}{2}.
\end{aligned}$$

For all $0 < p < 1$ we have

$$\left\| f_1^2 + f_2^2 \right\|_{L_p(c,d)} \leq 2^{\frac{1}{p}-1} \left(\left\| f_1^2 \right\|_{L_p(c,d)} + \left\| f_2^2 \right\|_{L_p(c,d)} \right),$$

for $p = \frac{1}{2}$ we obtain

$$\left| \int_c^d |H(x,y)|\, dy \right|^2 \leq 2 \left| \int_c^d H(x,y)dy \right|^2.$$

Then

$$\left| \int_c^d |H(x,y)|\, dy \right| \leq \sqrt{2} \left| \int_c^d H(x,y)dy \right|. \qquad \square$$

In this work we consider the reverse inequality of (1.1), with $0 < p < 1$ and $p < 0$ for $f : (a,b) \times (c,d) \to \mathbb{K}$, with $\mathbb{K}$ is $\mathbb{C}$, $\mathbb{E}$ or $i\mathbb{E}$.

3. Main Results

In this section we obtain some reverse Minkowski type inequalities.

Theorem 3.1. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b) \times (c,d) \to \mathbb{C}$ *is measurable with* $\mathrm{Re}(H), \mathrm{Im}(H) \in \mathbb{E}$, $\mathrm{Re}(H)\mathrm{Im}(H) \neq 0$ *and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\| \int_c^d H(\cdot,y)dy \right\|_{L_p(a,b)} \geq (\sqrt{2})^{p-2} \int_c^d \|H(\cdot,y)\|_{L_p(a,b)} dy, \tag{3.1}$$

if left-hand side is finite.

Proof. We have

$$\left| \int_c^d H(x,y)dy \right| \leq \int_c^d |H(x,y)|\, dy.$$

Then for $p-1<0$ we get

$$\left| \int_c^d H(x,y)dy \right|^{p-1} \geq \left(\int_c^d |H(x,y)|\, dy \right)^{p-1}.$$

By Proposition 2.1, we obtain

$$\begin{aligned}
\left| \int_c^d H(x,y)dy \right|^p &= \left| \int_c^d H(x,y)dy \right|^{p-1} \left| \int_c^d H(x,y)dy \right| \\
&\geq \left(\int_c^d |H(x,y)|\, dy \right)^{p-1} \left| \int_c^d H(x,y)dy \right| \\
&\geq \left(\int_c^d |H(x,y)|\, dy \right)^{p-1} (\sqrt{2})^{-1} \left| \int_c^d |H(x,y)| dy \right| \\
&= (\sqrt{2})^{-1} \left(\int_c^d |H(x,y)|\, dy \right)^{p-1} \left| \int_c^d |H(x,y)| dy \right|.
\end{aligned}$$

By integrating the last inequality, we establish

$$\begin{aligned}
\int_a^b \left| \int_c^d H(x,y)dy \right|^p dx &\geq (\sqrt{2})^{-1} \int_a^b \left(\int_c^d |H(x,t)|\, dt \right)^{p-1} \left| \int_c^d |H(x,y)| dy \right| dx \\
&= (\sqrt{2})^{-1} \int_a^b \left| \int_c^d \left(\int_c^d |H(x,t)|\, dt \right)^{p-1} |H(x,y)| dy \right| dx \\
&\geq (\sqrt{2})^{-1} \left| \int_a^b \left\{ \int_c^d \left(\int_c^d |H(x,t)|\, dt \right)^{p-1} |H(x,y)| dy \right\} dx \right| \\
&= (\sqrt{2})^{-1} \left| \int_c^d \left\{ \int_a^b \left(\int_c^d |H(x,t)|\, dt \right)^{p-1} |H(x,y)| dx \right\} dy \right|.
\end{aligned}$$

Let

$$R_1 = \int_a^b \left(\int_c^d |H(x,t)|\, dt \right)^{p-1} |H(x,y)| dx$$

and suppose that $G(x) = \left(\int_c^d |H(x,y)|\,dy\right)^{p-1}$.

Therefore, we get

$$\begin{aligned}
\|G(x)\|_{L_{p'}((a,b))} &= \left(\int_a^b \left|\int_c^d |H(x,y)|\,dy\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}} \\
&= \left(\int_a^b \left|\int_c^d |H(x,y)|\,dy\right|^{p} dx\right)^{\frac{p-1}{p}} \\
&= \left\{\left(\int_a^b \left|\int_c^d |H(x,y)|\,dy\right|^{p} dx\right)^{\frac{1}{p}}\right\}^{p-1} \\
&= \left\|\int_c^d |H(x,y)|\,dy\right\|^{p-1}_{L_p((a,b))}.
\end{aligned}$$

The last expression is finite (see hypotheses of theorem) then $G(x) \in L_{p'}((a,b))$. By applying the reverse Hölder's inequality and using Proposition 2.1, we obtain

$$\begin{aligned}
R_1 &\geq \left(\int_a^b \left|\int_c^d |H(x,t)|\,dt\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left|\int_c^d |H(x,t)|\,dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_a^b \left(\sqrt{2}\right)^p \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} = R_2.
\end{aligned}$$

Then we get

$$\int_c^d R_1 dy \geq \int_c^d R_2 dy,$$

$$R_2 > 0 \to \left|\int_c^d R_1 dy\right| \geq \left|\int_c^d R_2 dy\right| = \int_c^d R_2 dy.$$

Thus, we conclude that

$$\begin{aligned}
\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx &\geq \left(\sqrt{2}\right)^{-1} \left|\int_c^d R_1 dy\right| \\
&\geq \left(\sqrt{2}\right)^{-1} \int_c^d R_2 dy \\
&= \left(\sqrt{2}\right)^{p-2} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.
\end{aligned}$$

Therefore, we get

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)\left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{-\frac{1}{p'}}$$
$$\geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy,$$

then

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{1-\frac{1}{p'}} \geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Finally, we conclude that

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy,$$

which completes the proof. □

Theorem 3.2. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b) \times (c,d) \to \mathbb{E}$ *is measurable and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.2}$$

if left-hand side is finite.

Theorem 3.3. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b) \times (c,d) \to i\mathbb{E}$ *is measurable and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.3}$$

if left-hand side is finite.

Proof. The proof of Theorem 3.2 and Theorem 3.3 is similar to Theorem 3.1. □

Theorem 3.4. *Let* $p < 0$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b) \times (c,d) \to \mathbb{C}$ *is measurable with* $\mathrm{Re}(H), \mathrm{Im}(H) \in \mathbb{E}$, $\mathrm{Re}(H)\mathrm{Im}(H) \neq 0$ *and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq (\sqrt{2})^{p-2}\int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.4}$$

if left-hand side is finite.

Proof. By using the inequality
$$\left|\int_c^d H(x,y)dy\right| \leq \int_c^d |H(x,y)|\, dy,$$
we get
$$\left|\int_c^d H(x,y)dy\right|^p \geq \left(\int_c^d |H(x,y)|\, dy\right)^p, \quad \text{for } p<0.$$
By integrating the last inequality, we get
$$\begin{aligned}\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx &\geq \int_a^b \left(\int_c^d |H(x,y)|\, dy\right)^p dx \\ &= \int_a^b \left[\left(\int_c^d |H(x,t)|\, dt\right)^{p-1}\left(\int_c^d |H(x,y)|\, dy\right)\right] dx \\ &= \int_a^b \left[\int_c^d \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dy\right] dx \\ &= \int_c^d \left\{\int_a^b \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dx\right\} dy.\end{aligned}$$
Let
$$R_3 = \int_a^b \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dx.$$
By the reverse Hölder's inequality and Proposition 2.1, we obtain
$$\begin{aligned}R_3 &\geq \left(\int_a^b \left|\int_c^d |H(x,t)|\, dt\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int_a^b \left|\int_c^d |H(x,t)|\, dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\ &\geq \left(\int_a^b \left(\sqrt{2}\right)^p \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} = R_4.\end{aligned}$$
That is, we get
$$\int_c^d R_3 dy \geq \int_c^d R_4 dy.$$
Therefore, we obtain
$$\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx \geq \int_c^d R_3 dy \geq \int_c^d R_4 dy$$

and

$$\begin{aligned}\int_c^d R_4 dy &= \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy \\ &= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{\frac{1}{p'}} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right) \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{-\frac{1}{p'}} \\ &\geq \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{\frac{1}{p}} &\geq \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\ &\geq \left(\sqrt{2}\right)^{p-2} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $p < 0$, $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$. Suppose that $H : (a,b) \times (c,d) \to \mathbb{E}$ is measurable and $H(x,y) \in L_{p,x}(a,b)$ for almost all $y \in (c,d)$. Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)} dy, \tag{3.5}$$

if left-hand side is finite.

Theorem 3.6. *Let $p < 0$, $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$. Suppose that $H : (a,b) \times (c,d) \to i\mathbb{E}$ is measurable and $H(x,y) \in L_{p,x}(a,b)$ for almost all $y \in (c,d)$. Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)} dy, \tag{3.6}$$

if left-hand side is finite.

Proof. The proof of Theorem 3.5 and Theorem 3.6 is similar to Theorem 3.4. □

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References

[1] V. I. Burenkov, *Functional Spaces. Basic Integral Inequalities Connected with L_p-Spaces*, Peoples' Friendship University of Russia, Moscow, 1989, 5–12.

[2] S. Marinković, P. Rajković and M. Stanković, *The inequalities for some types q-integrals*, Comput. Math. Appl. **56** (2008), 2490–2498.

[3] B. G. Pachpatte, *Mathematical Inequalities*, North Holland Mathematical Library **67**, Oxford, 2005.

[4] B. Yang, *On a new Hardy-type integral inequality*, International Mathematical Forum **67**(2) (2007), 3317–3322.

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LIST COLORING UNDER SOME GRAPH OPERATIONS

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Abstract. The list coloring of a graph $G = G(V, E)$ is to color each vertex $v \in V(G)$ from its color set $L(v)$. If any two adjacent vertices have different colors, then G is properly colored. The aim of this paper is to study the list coloring of some graph operations.

1. Introduction

Throughout this paper, our notations are standard and can be taken from the famous book of West [16]. The set of all positive integers is denoted by $\mathbb{N}$, and for a set X, the power set of X is denoted by $P(X)$. All graphs are assumed to be simple and finite, and if G is such a graph, then its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively.

The graph coloring is an important concept in modern graph theory with many applications in computer science. A function $\alpha : V(G) \to \mathbb{N}$ is called a coloring for G. The coloring α is said to be proper, if for each edge $uv \in E(G)$, $\alpha(u) \neq \alpha(v)$. If the coloring α uses only the colors $[k] = \{1, 2, \ldots, k\}$, then α is called a k-coloring for G, and if such a proper k-coloring exists, then the graph G is said to be k-colorable. The smallest possible number k for which the graph G is k-colorable is the chromatic number of G and is denoted by $\chi(G)$.

The list coloring of graphs is a generalization of the classical notion of graph coloring, which was introduced independently by Erdős, Rubin and Taylor [7] and Vizing [15]. In the list coloring of a graph G, a list $L(v)$ of colors is assigned to each vertex $v \in V(G)$, and we have to find a proper coloring c for G in such a way that $c(v) \in L(v)$, for any vertex v in G. Concretely, we assume that there is a

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function $L : V(G) \to P(\mathbb{N})$ that assigns a set of colors to each vertex of G. A coloring $c : V(G) \to \mathbb{N}$ is called an L-coloring if for all $v \in V(G)$, $c(v) \in L(v)$. This coloring is said to be proper if $c(u) \neq c(v)$, when $uv \in E(G)$. The graph G is called L-colorable if such an L-coloring exists. This graph is k-choosable if it is L-colorable for every assignment L that satisfies $|L(v)| \geq k$, for all $v \in V(G)$. The list chromatic number $\chi_L(G)$ of G is the smallest k such that G is k-choosable. In [9], Isaak showed that the list chromatic number of the Cartesian product K_2 and K_n is equal to $n^2 + \left\lceil \frac{5n}{3} \right\rceil$. One year later, Axenovich [2] proved that if each vertex $x \in V(G) \setminus P$ is assigned a list of colors of size Δ and each vertex $x \in P$ is assigned a list of colors of size 1, then it is possible to color $V(G)$ such that adjacent vertices receive different colors and each vertex has a color from its list, where G is a non-complete graph with maximum degree $\Delta \geq 3$ and P is a subset of vertices with pairwise distance $d(P)$ between them at least 8. After that, in 2009, Rackham [12] studied on the list coloring of K_Δ-free graphs. We encourage potential readers to consult the interesting thesis of Lastrina [10] and Tuza's survey [14] for more information on this topic.

By a well-known result of Nordhaus and Gaddum [11], if G is an n-vertex graph, then $\chi(G) + \chi(\overline{G}) \leq n + 1$, where $\overline{G}$ is the complement of a graph G.

Erdös, Rubin and Taylor [7] extended this inequality to the list coloring of graphs and proved that for every n-vertex graph G, $\chi_L(G) + \chi_L(\overline{G}) \leq n + 1$. Thus, it is natural to study the list coloring of graphs under some other graph operations, which is the main topic of this paper.

Suppose $\{G_i = (V_i, E_i)\}_{i=1}^{N}$ is a family of graphs having a root vertex 0. Following Barrière, Comellas, Dalfó, Fiol, and Mitjana [3, 4], the hierarchical product $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is the graph with vertices as N-tuples $x_N \ldots x_3x_2x_1$, for $x_i \in V_i$, and edges defined as follows:

$$x_N \ldots x_3x_2x_1 \sim \begin{cases} x_N \ldots x_3x_2y_1 & \text{if } y_1 \sim x_1 \text{ in } G_1, \\ x_N \ldots x_3y_2x_1, & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\ x_N \ldots y_3x_2x_1, & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\ \quad\vdots & \quad\vdots \\ y_N \ldots x_3x_2x_1, & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = \cdots = x_{N-1} = 0. \end{cases}$$

In [13], Tavakoli, Rahbarnia and Ashrafi obtained exact formulas for some graph invariants under the hierarchical product, and some applications in chemistry were presented by Arezoomand and Taeri in [1].

Suppose G is a connected graph. Following Cvetković, Doob, Sachs, Yan, Yang and Yeh [6, 17], we define four types of graphs resulting from edge subdivision.

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.
(b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.
(c) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(d) The graph $T(G)$ of a graph G has a vertex for each edge and vertex of G and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in G.

The graphs $S(G)$ and $T(G)$ are called the subdivision and total graphs of G, respectively.

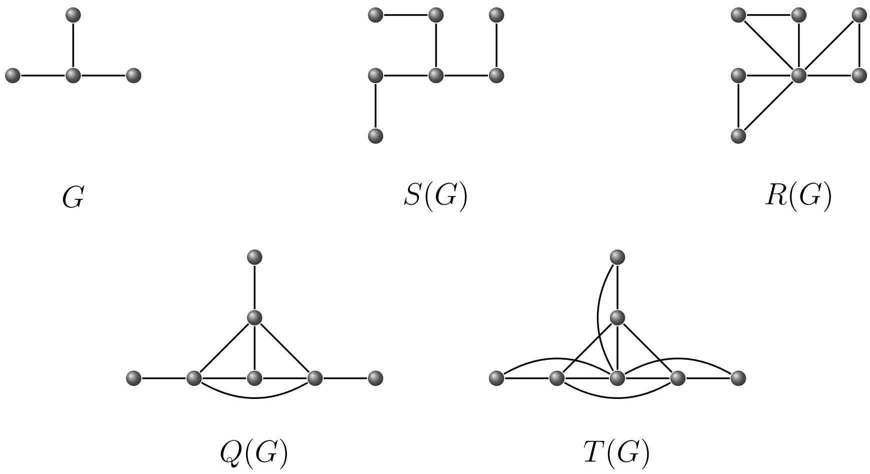


FIGURE 1. Subdivision graphs of G.

Let G and H be two graphs. The corona product GoH is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$, see Yeh and Gutman [19]. In Yarahmadi and Ashrafi [18], the authors obtained exact formulas for some graph invariant under the corona product of graphs. The edge corona product of two graphs G and H, $G\diamondsuit H$, is obtained in a similar way by taking one copy of G and $|E(G)|$ copies of H and joining each end vertices of the i-th edge of G to every vertex in the i-th copy of H, see Chithra, Germina, Sudev, Hou and Shiu [5, 8]. If the graphs G and H have disjoint vertex sets, then $G + H$ will be the graph obtained from G and H by connecting all vertices of G with all vertices of H.

2. Main Results

Suppose G is a simple graph. The suspension of a graph G is another graph G' constructed from G by adding a new vertex u and connecting u to all vertices of G.

2.1. Relationship between the coloring and the list coloring of graphs.

It is clear that the list chromatic number $\chi_L(G)$ of a graph G is at least its chromatic number $\chi(G)$, but it can be strictly larger, in other words $\chi(G) < \chi_L(G)$. We consider the following cases for showing the difference between the list coloring and the coloring of a given graph G.

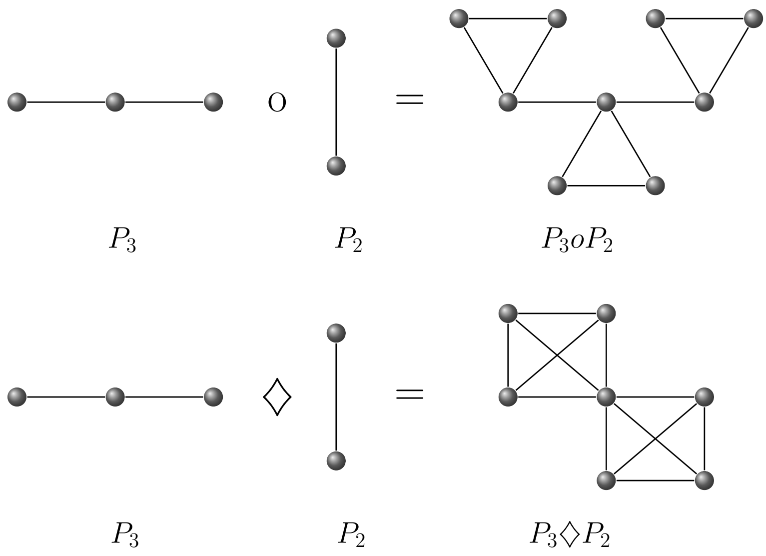


FIGURE 2. The corona and edge corona products of two graphs P_3 and P_2.

- Suppose $\chi_L(G) - \chi(G) = 1$. In this case, if we color the graph with lists of length $\chi(G)$, then in each coloring of this graph there will be at least one vertex x such that all adjacent vertices of x can be colored, and there is no edge that its end vertices cannot be colored.
- Suppose $\chi_L(G) - \chi(G) = 2$. In this case, if we color the graph with lists of length $\chi(G)$, then in each coloring of this graph there will be at least two vertices x and y such that $xy \in E(G)$, all adjacent vertices of x, y can be colored, and there is no triangle in G that its vertices cannot be colored.

Note that the above statements cannot be generalized to the case that $\chi_L(G) - \chi(G) > 2$. To show this, we define $r = \binom{2k-1}{k}$. Then, the complete bipartite graph $K_{r,r}$ is not k-choosable and so $\chi_L(K_{r,r}) > k$. If G has a list coloring of length m in such a way that we can find a coloring in which there is a k-vertex graph without a possible color, then $\chi_L(G) = m + k$. Finally, if the graph G can be colored with lists of length $\chi_L(G) - 1$ then there will be lists of length $\chi_L(G) - 1$, in which for every coloring of these lists there exists a vertex that all its adjacent vertices are colored and there is no edge that its end vertices cannot be colored.

2.2. **List chromatic numbers of the suspension graph and the corona product.** The aim of this section is to compute the list chromatic number of the suspension graph and the corona product of graphs. We start this section by the following crucial result:

Theorem 2.1. *Let G be a graph with $G' = G + K_1$. Then $\chi_L(G') = \chi_L(G)$ or $\chi_L(G) + 1$.*

Proof. Let $V(K_1) = \{u\}$. It is clear that $\chi(G') = \chi(G) + 1$. Suppose $\chi_L(G) = \chi(G)$. Then, $\chi_L(G) + 1 = \chi(G) + 1 = \chi(G') \leq \chi_L(G')$. We claim that $\chi_L(G') = \chi_L(G) + 1$. To prove it, we assign lists of length $\chi_L(G) + 1$ to the vertices of G'. We color u with a color t in $L(u)$. In the worst case, $t \in \bigcap_{v \in V(G')} L(v)$ and since G has a coloring with lists of length $\chi_L(G)$, we will find an appropriate coloring for G'.

We now assume that $\chi_L(G) = \chi(G)+1$. Since $\chi(G') \leq \chi_L(G')$, $\chi(G)+1 = \chi_L(G) \leq \chi_L(G')$. For the list coloring of G' we have the following two cases.

(a) After coloring of G with lists of length $\chi_L(G) - 1$, we will have at most two vertices without a possible color: $\chi_L(G') = \chi_L(G)$. We assign lists of length $\chi_L(G)$ to all vertices of G'. We first consider the case that we cannot color only one vertex of G'. There are two cases for $L(u)$ as follows.

a.1 There is a color $a \in L(u)$ such that for each $v \neq a \in V(G')$, $a \notin L(v)$. In this case, we assign a to the vertex u. By our hypothesis, the problem is changed to the list coloring of G by $\chi_L(G)$ colors, which is possible by definition.

a.2 For each color $a \in L(u)$, there exists a vertex $u \neq v \in V(G')$ such that $a \in L(v)$. Suppose $V(G) = \{v_1, \ldots, v_n\}$ and assign a list L_i to each vertex v_i for $1 \leq i \leq n$. We consider the following two cases.

(i) $L(u) \subseteq \bigcap_{i=1}^n L_i$. In this case, all vertices have the same list of colors. Since $\chi_L(G) = \chi(G) + 1 = \chi(G')$, the vertices of G can be colored with $\chi(G)$ colors and it remains a color for u. Hence, $\chi_L(G') = \chi_L(G)$.

(ii) $L(u) \nsubseteq \bigcap_{i=1}^n L_i$. In this case, there exist a color $a \in L(u)$ and an integer i for $1 \leq i \leq n$, such that $a \in L_i$ and $a \notin \bigcap_{j=1}^n L_j$. We assign the color a to the vertex u and remove a from the list of other vertices. This shows that there exists a list L_j such that $a \notin L_j$. Therefore, the length of some lists is $\chi(G)$ or $\chi(G) + 1$. By the hypothesis, there is only one vertex without a feasible color when a list has length $\chi_L(G) - 1$. It is clear that, in all cases, we will have an appropriate coloring for the graph.

We now assume that after the coloring of the graph with lists of length $\chi_L(G) - 1$ there are two vertices without assigning a color. If we have a color $a \in L(u)$ such that $a \notin \cup_{v \neq u} L(v)$, then by a similar argument as above, we will have an appropriate coloring for the graph. So, we can assume that every color in $L(u)$ will appear in at least one list of colors. We have again the following two cases.

(i) $L(u) \subseteq \bigcap_{i=1}^n L_i$. A similar argument as above shows that we have an appropriate coloring of the graph.

(ii) $L(u) \nsubseteq \bigcap_{i=1}^n L_i$. In this case, there exist a color $a \in L(u)$ and an integer i, for $1 \leq i \leq n$, such that $a \in L_i$ and $a \notin \bigcap_{j=1}^n L_j$. We prove that it is possible to find an appropriate coloring with lists of length $\chi_L(G)$. To do this, we show that there exists at least one color c in $L(u)$, such that c is outside of at least two other lists. On the contrary, we assume that there is at most one list $L(v)$ with $c \notin L(v)$. If c is outside of all the other lists, then clearly we will find an appropriate coloring for the graph. Hence, we can assume that there is a unique v such that $c \notin L(v)$. Therefore, all lists except one of them are equal and we have an appropriate coloring with lists of length $\chi_L(G)-1$, which is impossible. Therefore, G' can be colored with lists of length $\chi_L(G)$.

(b) After the coloring of G with lists of length $\chi_L(G)-1$, we will have more than two vertices without a possible color: In this case, we will prove $\chi_L(G')=\chi_L(G)+1$. Suppose $\chi_L(G)-\chi(G)=m$. We prove that the graph G' does not have a list coloring with lists of length $\chi_L(G)-1$. We assign lists of length $\chi_L(G)-1$ in such a way that there is no appropriate coloring for the graph. Consider the $\chi_L(G)$ copies of the graph G with the same lists and add a_1 to all lists of the first copy of G, a_2 to all lists of the second copy of G, $\ldots$, $a_{\chi_L(G)}$ to all lists of the $\chi_L(G)$-copy of G. We also assign the list $\{1,2,\ldots,\chi_L(G)\}$ to the vertex u. Note that by assigning each of a_i to the vertex u, we will not have an appropriate coloring for the i-th copy of G. Thus, we cannot find a feasible coloring for the graph. Therefore, an appropriate coloring of G' needs lists of length $\chi_L(G)+1$, see Figure 3.

This completes the proof. □

Lemma 2.1. *Suppose G is a graph containing disjoint subgraphs $G_1,\ldots,G_{\chi_L(G)}$ such that for each subgraph we can find lists of length $\chi_L(G)-1$ in which at least one vertex does not have a color. If u is an isolated vertex, then $\chi_L(G')=\chi_L(G)+1$.*

Proof. On the contrary, we assume that $\chi_L(G')=\chi_L(G)$. We define the lists of the graph G' as follows:

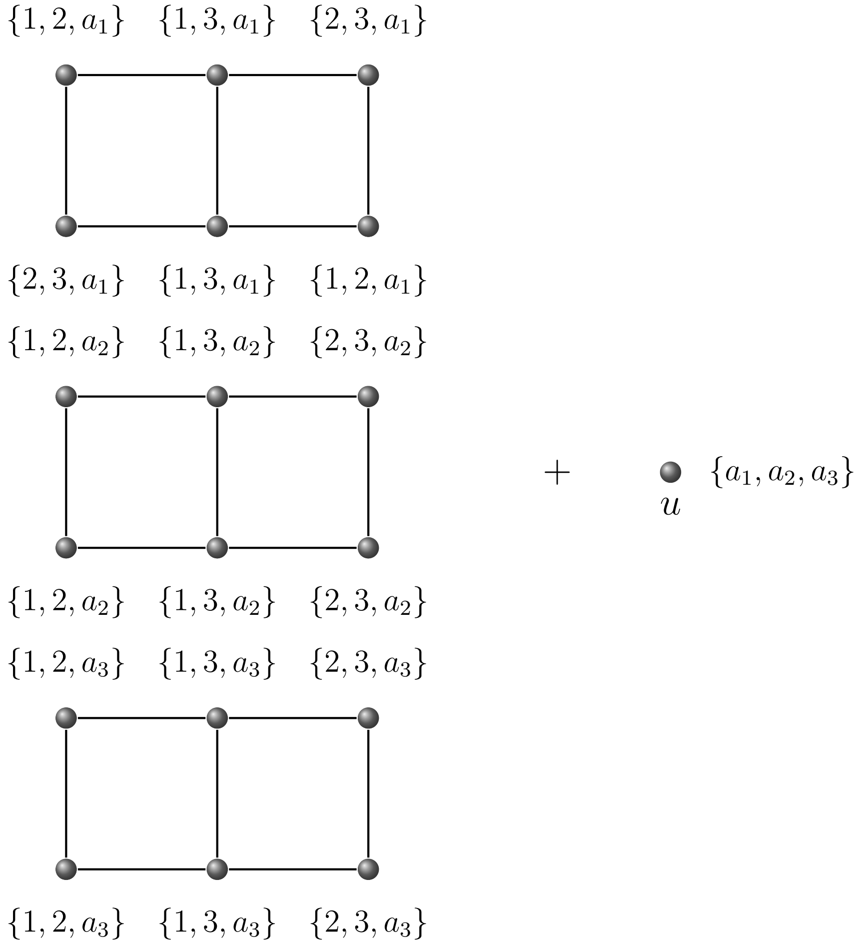
- assign lists of length $\chi_L(G)-1$ to the vertices of G_1 from the set $\{2,3,\ldots,\chi_L(G)+1\}$;
- assign lists of length $\chi_L(G)-1$ to the vertices of G_2 from the set $\{1,3,\ldots,\chi_L(G)+1\}$;
- assign lists of length $\chi_L(G)-1$ to the vertices of G_i from the set $\{1,2,\ldots,i-1,i+1,\ldots,\chi_L(G)+1\}$;
- assign lists of length $\chi_L(G)-1$ to the vertices of $G_{\chi_L(G)}$ from the set $\{1,2,\ldots,\chi_L(G)-1,\chi_L(G)+1\}$.

We now add the color i to all lists corresponding to the subgraph G_i for $1\le i\le\chi_L(G)$, and assign the set $\{1,2,\ldots,\chi_L(G)\}$ to the vertex u. If we assign a color, say i, to the vertex u, then the subgraph G_i cannot be colored, and so G does not have an appropriate coloring, a contradiction. Thus, $\chi_L(G')=\chi_L(G)+1$. □

Corollary 2.1. *Suppose G and H are two graphs. Then,*

$$\chi_L(GoH)\begin{cases} =\max\{\chi_L(G),\chi_L(H)\}, & \chi_L(H)\neq\chi(H) \textit{ and in the coloring of } H \textit{ with lists of length } \chi_L(H)-1 \textit{ at most two vertices cannot be colored,}\\ \le\max\{\chi_L(G),\chi_L(H)+1\}, & \textit{otherwise.}\end{cases}$$

2.3. **List chromatic number of the edge corona product.** Suppose G is a simple graph, $e=uv$ and $u,v\notin V(G)$. Let $G''=G+K_2$, where $V(K_2)=\{u,v\}$ and $E(K_2)=\{e\}$. It is easy to see that $G''=G'+K_1$, where $V(K_1)=\{v\}$ with



Three copies of G

FIGURE 3. Adding the vertex u to a graph G that after coloring with lists of length $\chi_L(G)-1$, the vertex u will be without an assigned color.

$G' = G + K_1$, where $V(K_1) = \{u\}$. It is clear that $\chi(G'') = \chi(G) + 2$. By Corollary 2.1,

$$\chi_L(G'')\begin{cases} = \chi_L(G'), & \chi_L(G') \neq \chi(G') \text{ and in the coloring of the graph with} \\ & \text{llists of ength } \chi_L(G') - 1 \text{ at most two vertices cannot} \\ & \text{be colored,} \\ \leq \chi_L(G'') + 1, & \text{otherwise.} \end{cases}$$

We now apply this inequality to prove the following lemma.

Lemma 2.2. *The list chromatic number of G'' is given by the following formula:*

$$\chi_L(G'') = \begin{cases} \chi_L(G), & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists length } \chi_L(G)-1 \text{ of exactly one vertex cannot be colored,} \\ \chi_L(G)+1, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists length } \chi_L(G)-1 \text{ of exactly two vertices cannot be colored,} \\ \chi_L(G)+2, & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Suppose G and H are two graphs. The list chromatic number of $G\diamondsuit H$ is given by the following formula:*

$$\chi_L(G\diamondsuit H) \begin{cases} = \max\{\chi(G), \chi(H)\}, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists of length } \chi_L(G)-1 \text{exactly one vertex cannot be colored,} \\ \leq \max\{\chi(G), \chi(H)+1\}, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists of length } \chi_L(G)-1 \text{ exactly two vertices cannot be colored,} \\ \leq \max\{\chi(G), \chi(H)+2\}, & \text{otherwise.} \end{cases}$$

2.4. **List chromatic number of the join of two graphs.** The aim of this subsection is to investigate under which conditions $\chi_L(G+H) = \chi_L(G)+\chi_L(H)$. If $\chi_L(G) = \chi(G)$ and $\chi_L(H) = \chi(H)$, then $\chi(G+H) = \chi(G)+\chi(H)$, and so $\chi_L(G+H) = \chi_L(G)+\chi_L(H)$. On the other hand, if one of G or H is a complete graph, then by Corollary 2.1, $\chi_L(G+H) = \chi_L(G)+\chi_L(H)$. In Figures 4 and 5, some examples are given, which show that the quantities $\chi_L(G+H)$ and $\chi_L(G)+\chi_L(H)$ can be non-equal.

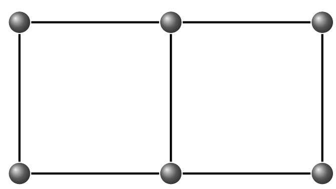
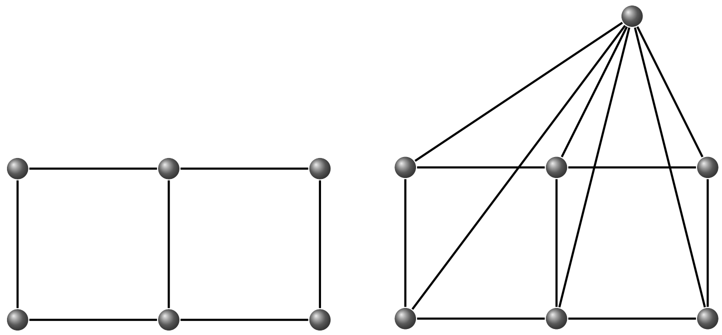


FIGURE 4. Graphs G and $H \cong G$ that $\chi_L(G+H) \neq \chi_L(G)+\chi_L(H)$.

Theorem 2.3. *Suppose G and H are graphs such that the following holds.*

- $\chi_L(H) \leq \chi_L(G)$ *(or* $\chi_L(G) \leq \chi_L(H)$*).*
- *The graph G (H) has subgraphs $G_1, \ldots, G_{\chi_L(G)+1}$ ($H_1, \ldots, H_{\chi_L(H)+1}$) such that for each subgraph G_i for $1 \leq i \leq \chi_L(G)+1$, (or H_i for $1 \leq i \leq \chi_L(H)+1$) there exist lists of length $\chi_L(G)+1$ (or $\chi_L(H)+1$) in such a way that in each subgraph there exists at least one vertex that cannot be colored.*



G $\qquad$ $G+H$

FIGURE 5. Graphs G and $H \cong K_1$ with $\chi_L(G+H) \neq \chi_L(G)+\chi_L(H)$.

Then, $\chi_L(G+H) = \chi_L(G) + \chi_L(H)$.

Proof. On the contrary, we assume that $\chi_L(G+H) = \chi_L(G) + \chi_L(H) - 1$. We assign lists of length $\chi_L(H) - 1$ to the graph H in such a way that H does not have an appropriate coloring related to these lists. Similarly to Lemma 2.1, we assign lists to the subgraphs $G_1, \ldots, G_{\chi_L(G)+1}$ as follows:

- assign lists of length $\chi_L(G) - 1$ to the vertices of G_1 from the set $\{2, 3, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_2 from the set $\{1, 3, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_i from the set $\{1, 2, \ldots, i - 1, i + 1, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of $G_{\chi_L(G)+1}$ from the set $\{1, 2, \ldots, \chi_L(G) - 1, \chi_L(G) + 1\}$.

By our hypothesis, there exists a vertex $x_i \in V(G_i)$, for $1 \leq i \leq \chi_L(G) + 1$, such that in the process of the coloring for vertices of H_i, x_i cannot be colored. We now add the color i to all lists corresponding to the subgraph G_i, for $1 \leq i \leq \chi_L(G) + 1$. We also assign the lists of the graph H to the subgraphs of G in such a way that we assign different lists to at least two vertices of a given subgraph, and at least three lists of each subgraphs are different. Note that the smallest subgraph with these properties has at least six vertices. Next, we assign lists of length $\chi_L(G)$ from the set $\{1, 2, \ldots, \chi_L(G) + 1\}$ to the vertices of H such that at least two vertices of the graph have different lists and if $|V(H)| \geq 3$, then at least three lists of vertices in H are different. We assign numbers to the lists of G and letters to the lists of H. Our main proof will consider the following three separate cases.

(a) *In the coloring of H we use only letters.* By our hypothesis, there will be one vertex that cannot be colored, and we assign the number i to this vertex. So, the subgraph G_i cannot be colored, as desired.

(b) *In the coloring of H we use only numbers.* In this case, we will have a list of letters for a subgraph (vertices of G which are colored with numbers) and since $\chi_L(H) - 1 \leq \chi_L(H) \leq \chi_L(G)$, the graph cannot be colored.

(c) *In the coloring of H we use a combination of letters and numbers.* By our hypothesis, there will be one vertex that cannot be colored, and we assign the number i to this vertex. So, the subgraph G_i cannot be colored, as desired. In this case, we use numbers instead of letters. For example, we use 1 as a. Again, we will have a vertex that cannot be colored by letters and the number 1. We assign the number i to this vertex. Consider a list L in G_i containing number 1. If $a \notin L$, then the graph obviously cannot be colored. If $a \in L$, then we lead to a contradiction with our substitution. So, the graph cannot be colored. In the case that more than one letter is substituted by a number, we lead to a similar contradiction, and so the graph cannot be colored.

This proves that $\chi_L(G + H) = \chi_L(G) + \chi_L(H)$. □

2.5. **List chromatic number of the subdivision graphs.** In this subsection, the list chromatic number of four types of edge subdivision of a graph G containing $R(G), S(G), Q(G)$ and $T(G)$ are computed.

Theorem 2.4. $\chi_L(R(G)) = \max\{\chi_L(G), 3\}$.

Proof. The subdivision graph $R(G)$ is isomorphic to the edge corona product of G and H, where $H = K_1$. Since $\chi_L(H) = \chi(H) = 1$, by Theorem 2.2, $\chi_L(R(G)) = \max\{\chi_L(G), 3\}$. □

Theorem 2.5. *Suppose G has at least one edge. Then $\chi_L(S(G)) = 2$ or 3 and all cases can occur.*

Proof. Suppose $|V(G)| = n$, $|E(G)| = m$ and $V(G) = \{v_1, \ldots, v_n\}$. In the graph $S(G)$, the additional vertices of each edge of G are labeled by $u_1, \ldots, u_m$. It is clear that all cycles of $S(G)$ have even length and so $S(G)$ is a bipartite graph with bipartite classes (U_1, U_2), where $U_1 = V(G)$ and $U_2 = \{u_1, \ldots, u_m\}$. Therefore, $\chi_L(S(G)) \geq \chi(S(G)) = 2$. In Figures 6 and 7, two graphs G_1 and G_2 are presented, such that $\chi_L(S(G_1)) = 2$ and $\chi_L(S(G_2)) = 2$.

To complete the proof, we assign a color to all vertices of $V(G)$ and the other vertices can be colored with two other colors. This proves that $\chi_L(S(G)) \leq 3$, which completes the proof. □

Theorem 2.6. $\chi_L(Q(G)) = \Delta(G) + 1$.

Proof. We use the labeling of the vertices in $S(G)$ given in the proof of Theorem 2.5 for the graph $Q(G)$. By definition of $Q(G)$, each vertex v_i together with all vertices u_j adjacent to v_i constitutes a complete graph of order $\deg(v_i) + 1$ and each u_i is a

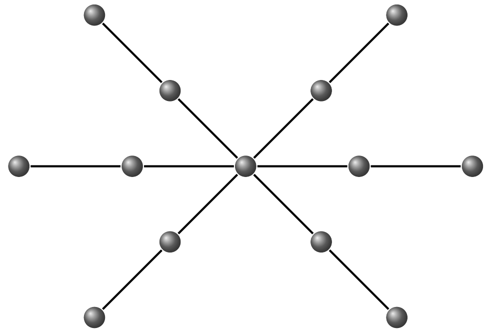


FIGURE 6. The graph G_1 with $\chi_L(S(G_1)) = 2$.

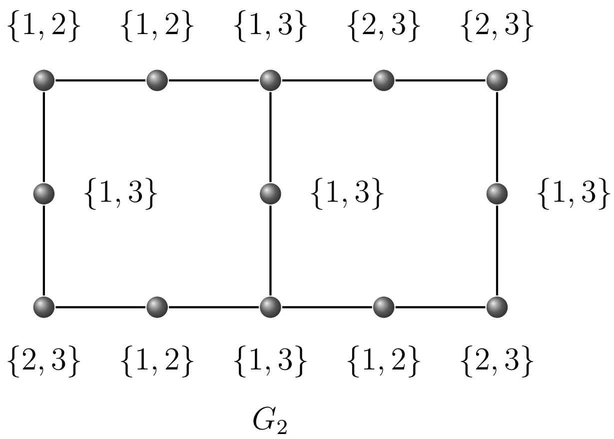


FIGURE 7. The graph G_2 with $\chi_L(S(G_2)) = 3$.

common vertex of exactly two complete subgraphs. So, the graph $Q(G)$ has $|V(G)|$ such complete graphs. It is obvious that for each triangle in G, some of the vertices in $A = \{u_i \mid 1 \leq i \leq m\}$ induces a triangle in $Q(G)$ and in the other case, the vertices in A can not construct a triangle in $Q(G)$. Since G has a vertex of degree $\Delta(G)$, $Q(G)$ has a complete subgraph of order $\Delta(G) + 1$, and so $\chi(Q(G)) \geq \Delta(G) + 1$. We will prove that it is possible to color the graph $Q(G)$ by lists of length $\Delta(G) + 1$. To prove it, we assign lists of length $\Delta(G) + 1$ to all vertices of $Q(G)$. Since $Q(G)$ can be constructed from complete graphs of minimum order 3 and maximum order $\Delta(G) + 1$, each vertex of $V(G)$ is a vertex of exactly one complete graph, each vertex u_i is a common vertex of exactly two complete subgraphs, and each complete graph of order n has n distinct colorings with lists of length n, the graph $Q(G)$ has an appropriate coloring. This proves the theorem. $\square$

Theorem 2.7. $\Delta(G) + 1 \leq \chi_L(T(G)) \leq \Delta(G) + 2$.

Proof. Since the graphs G and $Q(G)$ are subgraphs of $T(G)$, $\max\{\chi_L(G), \chi_L(Q(G))\} \leq \chi_L(T(G))$. On the other hand, $\chi_L(G) \geq \Delta(G)+1$ and so $\Delta(G)+1 \leq \chi_L(T(G))$. To prove $\chi_L(T(G)) \leq \Delta(G)+2$, we assign the lists of length $\Delta(G)+2$ to each vertex of the graph. We first color all vertices of G. Since each vertex of $A = \{u_i \mid 1 \leq i \leq m\}$ are adjacent to two vertices of G, which are adjacent in $T(G)$, the length of lists corresponding to vertices in A is at least $\Delta(G)$. Therefore, $\chi_L(T(G)) \leq \Delta(G)+2$. □

2.6. **List chromatic number of the hierarchical product of graphs.** In this section, the list chromatic number of the hierarchical product of graphs is computed. We first compute this number for the case of two graphs.

Theorem 2.8. *The list chromatic number of the hierarchical product of two graphs G and H is given by the following formula:*

$$\chi_L(G \sqcap H) = \begin{cases} 3, & \chi_L(G) = \chi_L(H) = 2, \ G \text{ has a cycle of even length and the root is a vertex of an even cycle,} \\ 2, & \chi_L(G) = \chi_L(H) = 2, \ G \text{ does not have an even cycle or } G \text{ has an even cycle but the root is not a vertex of an even cycle,} \\ \max\{\chi_L(G), \chi_L(H)\}, & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that $\chi(G \sqcap H) = \max\{\chi(G), \chi(H)\}$. Moreover, if $\chi_L(G) = \chi_L(H) = 2$, G has a cycle of even length and the root is a vertex of an even cycle, then $\chi_L(G \sqcap H) = 3$, see Figure 8. If $\chi_L(G) = \chi_L(H) = 2$, G does not have an even cycle or G has an even cycle, but the root is not a vertex of an even cycle, then $\chi_L(G \sqcap H) = 2$. On the other hand, if $\chi_L(G) > \chi_L(H)$, then clearly the graph $G \sqcap H$ can be colored by lists of length $\chi_L(G)$ and if $\chi_L(G) < \chi_L(H)$, then the graph $G \sqcap H$ can be colored by lists of length $\chi_L(H)$. So, it is enough to consider the case that $\chi_L(G) = \chi_L(H)$. In this case, we first color the graph G by $\chi_L(G)$ colors. In this coloring, for the coloring of each vertex in G, a vertex in H will be colored and if $\chi_L(H) \geq 3$, then the graph will have an appropriate coloring. □

Corollary 2.2. *Suppose $G_1, G_2, \ldots, G_k$ are k simple graphs. Then,*

$$\chi_L(G_k \sqcap \cdots \sqcap G_2 \sqcap G_1) = \begin{cases} 3, & \chi_L(G_1) = \cdots = \chi_L(G_k) = 2, \ G_k \text{ has an even cycle and the root is a vertex of an even cycle,} \\ 2, & \chi_L(G_1) = \cdots = \chi_L(G_k) = 2, \text{ the root is not a vertex of an even cycle or } G_k \text{ does not have a cycle of even length,} \\ \max\{\chi_L(G_1), \ldots, \chi_L(G_k)\}, & \text{otherwise.} \end{cases}$$

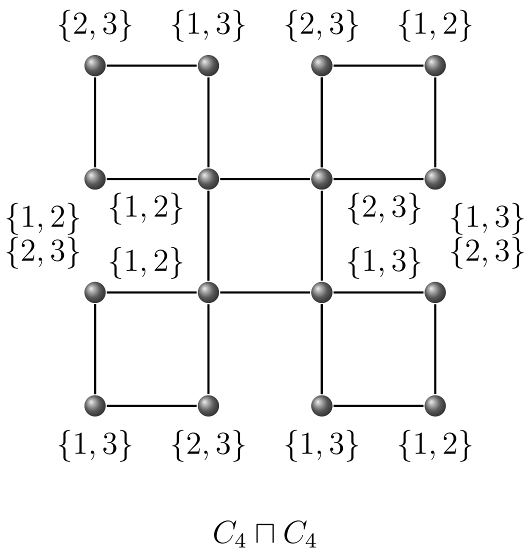


FIGURE 8. The hierarchical product of C_4 and C_4 with the list chromatic number 3.

Proof. We proceed by induction. In Theorem 2.8, we proved the case of $k = 2$. Suppose $k = m - 1$ and $H = G_k \sqcap \cdots \sqcap G_2 \sqcap G_1$. To prove the case of $k = m$, we first assume that $\chi_L(G_1) = \cdots = \chi_L(G_k) = 2$. Then, the following four cases can occur.

(a) *Let G_m be a tree and there are no even cycles in other graphs.* Since the other $m - 1$ graphs do not have even cycles, the graph H does not have an even cycle, and so $\chi_L(G_m \sqcap H) = 2$.

(b) *Let G_m be a tree and there exists at least one even cycle in the other graphs.* Since in the other $m - 1$ graphs we have at least one even cycle, the graph H has an even cycle. If $\chi_L(H) \geq 3$, then $\chi_L(G_m \sqcap H) = \max\{2, 3\} = 3$. If $\chi_L(H) = 2$, then $\chi_L(G_m \sqcap H) = 2$, as desired.

(c) *G_m has an even cycle and there are no even cycles in other graphs.* A similar argument as in the first case shows that $\chi_L(G_m \sqcap H) = 2$.

(d) *G_m has an even cycle and there exists at least one even cycle in the other graphs.* In this case, the graph H has at least one even cycle. If $\chi_L(H) \geq 3$, then $\chi_L(G_m \sqcap H) = \max\{2, 3\} = 3$. Suppose $\chi_L(H) = 2$. If the root vertex is in a cycle, then $\chi_L(H) = 3$, and otherwise $\chi_L(H) = 2$.

Next we assume that there exists i such that $\chi_L(G_i) > 2$. Then

$$\max_{1 \leq i \leq m} \{\chi_L(G_i)\} = \max\left\{\chi_L(G_m), \max_{1 \leq i \leq m-1} \{\chi_L(G_i)\}\right\} = \max\{\chi_L(G_m), \chi_L(H)\}.$$

This shows that the problem for the case of $k = m$ can be reduced to the case of $k = 2$ such that one of the graphs has the list chromatic number greater than 2. By induction hypothesis, this is feasible, and so the proof is complete. $\square$

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References

[1] M. Arezoomand and B. Taeri, *Applications of generalized hierarchical product of graphs in computing the Szeged index of chemical graphs*, MATCH Commun. Math. Comput. Chem. **64** (2010), 591–602.

[2] M. Axenovich, *A note on graph coloring extensions and list-colorings*, Electron. J. Combin. **10** (2003), #N1.

[3] L. Barrière, F. Comellas, C. Dalfó and M. A. Fiol, *The hierarchical product of graphs*, Discrete Appl. Math. **157** (2009), 36–48.

[4] L. Barrière, C. Dalfó, M. A. Fiol and M. Mitjana, *The generalized hierarchical product of graphs*, Discrete Math. **309** (2009), 3871–3881.

[5] K. P. Chithra, K. A. Germina and N. K. Sudev, *On the sparing number of the edge corona of graphs*, International Journal of Computer Applications **118** (1) (2015), 1–5.

[6] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs - Theory and Application*, Academic Press, New York, 1980.

[7] P. Erdős, A. L. Rubin and H. Taylor, *Choosability in graphs*, Congr. Numer. **26** (1980), 122–157.

[8] Y. Hou and W. - C. Shiu, *The spectrum of the edge corona of two graphs*, Electron. J. Linear Algebra **20** (2010), 586–594.

[9] G. Isaak, *Sum list coloring* $2 \times n$ *arrays*, Electron. J. Combin. **9**(8) (2002), #N8.

[10] M. A. Lastrina, *List-coloring and sum-list-coloring problems on graphs*, Ph. D. Thesis, Lowa State University, 2012.

[11] E. A. Nordhaus and J. W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956), 175–177.

[12] T. Rackham, *A note on* $K_{\Delta+1}$*-free precolouring with* Δ *colours*, Electron. J. Combin. **16** (2009), #N28.

[13] M. Tavakoli, F. Rahbarnia and A. R. Ashrafi, *Further results on hierarchical product of graphs*, Discrete Appl. Math. **161** (2013), 1162–1167.

[14] Zs. Tuza, *Graph colorings with local constraints - a survey*, Discuss. Math. Graph Theory **17** (1997), 161–228.

[15] V. G. Vizing, *Colouring the vertices of a graph in prescribed colours* (in Russian), Diskret. Anal. **29** (1976), 3–10.

[16] D. B. West. *Introduction to Graph Theory*, Prentice Hall Inc., Upper Saddle River, New Jersey, 1996.

[17] W. Yan, B. -Y. Yang and Y. -N. Yeh, *The behavior of Wiener indices and polynomials of graphs under five graph decorations*, Appl. Math. Lett. **20** (2007), 290–295.

[18] Z. Yarahmadi and A. R. Ashrafi, *The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs*, Filomat **26**(3) (2012), 467–472.

[19] Y. -N. Yeh and I. Gutman, *On the sum of all distances in composite graphs*, Discrete Math. **135** (1994), 359–365.

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ON THE ENUMERATION OF THE SET OF ELEMENTARY NUMERICAL SEMIGROUPS WITH FIXED MULTIPLICITY, FROBENIUS NUMBER OR GENUS

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Abstract. In this paper we give algorithms that allow to compute the set of every elementary numerical semigroups with given genus, Frobenius number and multiplicity. As a consequence we obtain formulas for the cardinality of these sets.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A numerical semigroup is a subset S of $\mathbb{N}$ which is closed under addition, $0 \in S$ and $\mathbb{N}\backslash S$ has finitely many elements. The cardinality of the set $\mathbb{N}\backslash S$ is called the genus of S and it is denoted by $\mathrm{g}(S)$.

Given a positive integer g, we denote by $\mathcal{S}(g)$ the set of all numerical semigroups with genus g. The problem of determining the cardinality of $\mathcal{S}(g)$ has been widely treated in the literature (see for example [2, 4–7] and [13]). Some of these works were motivated by Amorós's conjecture [5], which says that the sequence of cardinals of $\mathcal{S}(g)$ for $g = 1, 2, \dots$ has a Fibonacci behavior. It is still not known in general if for a fixed positive integer g there are more numerical semigroups with genus $g+1$ than numerical semigroups with genus g.

An algorithm that allows us to compute the set of numerical semigroups with genus g is provided in [3], where elementary numerical semigroups play an important role. In fact, in [3] an equivalence binary relation R is defined over $\mathcal{S}(g)$ such that $\frac{\mathcal{S}(g)}{R} = \{[S] \mid S \text{ is a elementary numerical semigroup with genus } g\}$. Moreover, it is proved that if S and T are elementary numerical semigroups with genus g then $[S] = [T]$ if and only if $S = T$. The main idea of the algorithm in [3] is to compute

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every elementary numerical semigroups S with genus g and, then, to enumerate the elements in $[S]$ for each S.

For any numerical semigroup S, the smallest positive integer belonging to S (respectively, the greatest that does not belong to S) is called the multiplicity (respectively Frobenius number) of S and it is denoted by $\mathrm{m}(S)$ (respectively $\mathrm{F}(S)$) (see [9]).

We say that a numerical semigroup S is elementary if $\mathrm{F}(S) < 2\mathrm{m}(S)$. This type of numerical semigroups were also studied in [8] and [13]. We denote by $\mathcal{E}(m, F, g)$ the set of elementary numerical semigroups with multiplicity m, Frobenius number F and genus g (when one of the parameters to $\mathcal{E}(m, F, g)$ is replaced by the symbol $-$, it represents the set of elementary numerical semigroups in which no restrictions are placed on that parameter).

For any finite set A, $\#A$ denotes the cardinal of A. Given a rational number q we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$.

In Section 2, we review the results of Y. Zhao in [13] which give formulas for $\#\mathcal{E}(m, -, -)$, $\#\mathcal{E}(m, -, g)$ and $\#\mathcal{E}(-, -, g)$, and state that $\#\mathcal{E}(-, -, g+1) = \#\mathcal{E}(-, -, g) + \#\mathcal{E}(-, -, g-1)$. Therefore, we get that $\{\#\mathcal{E}(-, -, g)\}_{g \in \mathbb{N}}$ is a Fibonacci sequence.

In Section 3, we study the sets $\mathcal{E}(m, F, -)$ and $\mathcal{E}(-, F, -)$, find formulas for their cardinality, and describe the behavior of the sequence of cardinals of $\mathcal{E}(-, F, -)$.

In Section 4, we present algorithms for calculating $\mathcal{E}(-, F, g)$ and $\mathcal{E}(m, F, g)$. From these algorithms, we can derive the cardinality of these sets.

Finally, in Section 5 we show that the set of all elementary numerical semigroups $\mathcal{E}$ is a Frobenius variety. This fact, together with the results of [11], allows us to construct recursively the set $\mathcal{E}$.

2. Multiplicity and Genus

Our aim in this section is to see that $\{\#\mathcal{E}(-, -, g)\}_{g \in \mathbb{N}}$ is a Fibonacci sequence. The next result is easy to prove and appears in [13, Proposition 2.1].

Lemma 2.1. *Let m be an integer such that $m \geq 2$ and let A be a subset of $\{m+1, \ldots, 2m-1\}$. Then $\{0, m\} \cup A \cup \{2m, \rightarrow\}$ is an elementary numerical semigroup with multiplicity m. Moreover, every elementary numerical semigroup with multiplicity m is of this form.*

As consequence of the above lemma we have that $\#\mathcal{E}(m, -, -)$ is equal to the number of subsets of a set with $m-1$ elements.

Corollary 2.1. *If m is a positive integer, then $\#\mathcal{E}(m, -, -) = 2^{m-1}$.*

The following result is easy to prove and gives conditions imposed on two positive integers m and g so that there exists at least one elementary numerical semigroup with multiplicity m and genus g.

Proposition 2.1. *Let m and g be nonnegative integers with $m \neq 0$. Then $\mathcal{E}(m, -, g) \neq \emptyset$ if and only if $m - 1 \leq g \leq 2(m-1)$.*

From Lemma 2.1, we know that $S \in \mathcal{E}(m,-,g)$ if and only if $S = \{0, m\} \cup A \cup \{2m, \rightarrow\}$, where A is a subset of $\{m+1, \dots, 2m-1\}$ and $\#A = 2(m-1) - g$. So we have the following result, which is also in [13, Corollary 2.2].

Corollary 2.2. *Let m and g be positive integers such that $m - 1 \leq g \leq 2\,(m-1)$. Then $\#\mathcal{E}(m,-,g) = \binom{m-1}{g-(m-1)}$.*

From the results above we get

$$\mathcal{E}(-,-,g) = \bigcup_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \mathcal{E}(m,-,g).$$

Thus we have the following algorithm.

Algorithm 2.1. Input: g positive integer. Output: $\mathcal{E}(-,-,g)$.

1) For all $m \in \left\{\lceil \frac{g}{2} \rceil + 1, \dots, g+1\right\}$ compute the set $\mathcal{E}(m,-,g)$.
2) Return $\bigcup_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \mathcal{E}(m,-,g)$.

Clearly, we get

$$\#\mathcal{E}(-,-,g) = \sum_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \#\mathcal{E}(m,-,g).$$

By applying Corollary 2.2, we obtain the following result.

Corollary 2.3. *If g is a positive integer, then $\#\mathcal{E}(-,-,g) = \sum_{i=\lceil \frac{g}{2} \rceil}^{g} \binom{i}{g-i}$.*

The Fibonacci sequence is the sequence of positive integers defined by the linear recurrence equation $a_{n+1} = a_n + a_{n-1}$, with $a_0 = a_1 = 1$.

It is clear that $\mathcal{E}(-,-,0) = \{\mathbb{N}\}$ and $\mathcal{E}(-,-,1) = \{\{0, 2, \rightarrow\}\}$ and so $\#\mathcal{E}(-,-,0) = \#\mathcal{E}(-,-,1) = 1$. By using Corollary 2.3, we can obtain [13, Proposition 2.3], which states that $\{\#\mathcal{E}(-,-,g)\}_{g\in\mathbb{N}}$ is a Fibonacci sequence.

Theorem 2.1. *If g is a positive integer, then $\#\mathcal{E}(-,-,g+1) = \#\mathcal{E}(-,-,g) + \#\mathcal{E}(-,-,g-1)$.*

3. Multiplicity and Frobenius Number

Our first goal in this section is to describe sufficient conditions for two positive integers m and F so that there exists at least one elementary numerical semigroups with multiplicity m and Frobenius number F.

Lemma 3.1. *If S is an elementary numerical semigroup such that $S \neq \mathbb{N}$, then $\frac{\mathrm{F}(S)+1}{2} \leq \mathrm{m}(S) \leq \mathrm{F}(S) + 1$ and $\mathrm{m}(S) \neq \mathrm{F}(S)$.*

Proof. Since $S \neq \mathbb{N}$, then $\mathrm{m}(S) \geq 2$ and $\mathrm{m}(S) - 1 \notin S$. Therefore, we have that $\mathrm{m}(S) - 1 \leq \mathrm{F}(S)$. In addition, as S is an elementary numerical semigroup then $\mathrm{F}(S) < 2\,(\mathrm{m}(S))$ and thus $\mathrm{F}(S) + 1 \leq 2\,(\mathrm{m}(S))$. □

From the previous lemma we obtain the following result.

Proposition 3.1. *Let m and F be positive integers. Then $\mathcal{E}(m, F, -) \neq \emptyset$ if and only if $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$.*

It is clear that $\mathcal{E}(F+1, F, -) = \{\{0, F+1, \rightarrow\}\}$ and $\mathcal{E}(F-1, F, -) = \{\{0, F-1, F+1, \rightarrow\}\}$. Hence, we can assume that $F = m+i$, where $i \in \{2, \ldots, m-1\}$. By applying Lemma 2.1, we deduce that $S \in \mathcal{E}(m, F, -)$ if and only if there exists $A \subseteq \{m+1, \ldots, m+i-1\}$ such that $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$. As a consequence we have the following algorithm.

Algorithm 3.1. Input: m and F positive integers such that $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$.

Output: $\mathcal{E}(m, F, -)$.

1) If $m = F+1$, then return $\{\{0, F+1, \rightarrow\}\}$.
2) If $m = F-1$, then return $\{\{0, F-1, F+1, \rightarrow\}\}$.
3) Compute the set $C = \{A \mid A \subseteq \{m+1, \ldots, F-1\}\}$.
4) Return $\{\{0, m\} \cup A \cup \{F+1, \rightarrow\} \mid A \in C\}$.

Gathering all this information, we obtain the following result which can also be deduced from equation (6) of [1].

Corollary 3.1. *Let m and F be positive integers such that $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$. Then*

$$\#\mathcal{E}(m, F, -) = \begin{cases} 1, & \text{if } m = F+1, \\ 2^{F-m-1}, & \text{otherwise.} \end{cases}$$

Next we obtain an algorithm that allows us to compute every elementary numerical semigroup with a given Frobenius number. As a consequence of Proposition 3.1, we have

$$\mathcal{E}(-, F, -) = \bigcup_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \mathcal{E}(m, F, -).$$

Algorithm 3.2. Input: F positive integer.

Output: $\mathcal{E}(-, F, -)$.

1) For all $m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}$ compute (using Algorithm 3.1) the set $\mathcal{E}(m, F, -)$.
2) Return $\mathcal{E}(-, F, -) = \bigcup_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \mathcal{E}(m, F, -)$.

Therefore, we have $\#\mathcal{E}(-, F, -) = \sum_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \#\mathcal{E}(m, F, -)$. From Corollary 3.1 we obtain the following result.

Corollary 3.2. *If F is a positive integer, then $\#\mathcal{E}(-, F-) = 2^{F - \lceil \frac{F+1}{2} \rceil}$.*

We finish this section by describing the behavior of the sequence of cardinalities of $\mathcal{E}(-, F, -)$ for $F = 1, 2, \ldots$ Observe that $\#\mathcal{E}(-, 1, -) = \#\mathcal{E}(-, 2, -) = 1$.

Proposition 3.2. *Let F be an integer greater than or equal to 2.*

1) *If F is odd, then $\#\mathcal{E}(-,F+1,-)=\#\mathcal{E}(-,F,-)$.*
2) *If F is even, then $\#\mathcal{E}(-,F+1,-)=\#\mathcal{E}(-,F,-)+\#\mathcal{E}(-,F-1,-)$.*

Proof. 1) From Corollary 3.2 it is guaranteed that $\#\mathcal{E}(-,F,-)=2^{F-\lceil\frac{F+1}{2}\rceil}=2^{F-\frac{F+1}{2}}=2^{\frac{F-1}{2}}$. By repeating this argument we obtain $\#\mathcal{E}(-,F+1,-)=2^{\frac{F-1}{2}}$. Therefore, we have $\#\mathcal{E}(-,F+1,-)=\#\mathcal{E}(-,F,-)$.

2) Again, by Corollary 3.2, we know that $\#\mathcal{E}(-,F,-)+\#\mathcal{E}(-,F-1,-)=2^{F-\lceil\frac{F+1}{2}\rceil}+2^{F-1-\lceil\frac{F}{2}\rceil}=2^{F-\frac{F+2}{2}}+2^{F-1-\frac{F}{2}}=2^{\frac{F}{2}}$. We obtain $\#\mathcal{E}(-,F+1,-)=2^{F+1-\lceil\frac{F+2}{2}\rceil}=2^{F+1-\frac{F+2}{2}}=2^{\frac{F}{2}}$. Consequently, $\#\mathcal{E}(-,F+1,-)=\#\mathcal{E}(-,F,-)+\#\mathcal{E}(-,F-1,-)$ □

4. Multiplicity, Frobenius Number and Genus

In this section, we aim to find conditions for m, F and g positive integers so that there exists at least one elementary numerical semigroup with a given multiplicity m, Frobenius number F and genus g. The next results are a consequence of the results given in [3, Proposition 2 and Corollary 3].

Lemma 4.1. *Let F and g be two positive integers. Then $g\le F\le 2g-1$ if and only if $\mathcal{E}(-,F,g)\neq\emptyset$.*

Lemma 4.2. *Let F and g be two positive integers such that $g\le F\le 2g-1$, and let $\mathcal{A}_{F,g}=\left\{A\mid A\subseteq\left\{\left\lceil\frac{F+1}{2}\right\rceil,\dots,F-1\right\}\text{ and }\#A=F-g\right\}$. Then $\mathcal{E}(-,F,g)=\{\{0\}\cup A\cup\{F+1\to\}\mid A\in\mathcal{A}_{F,g}\}$.*

As an immediate consequence of Lemmas 4.1 and 4.2 we have the following algorithm.

Algorithm 4.1. Input: F and g positive integers such that $g\le F\le 2g-1$.

Output: $\mathcal{E}(-,F,g)$.

1) Compute the set $C=\left\{A\mid A\subseteq\left\{\left\lceil\frac{F+1}{2}\right\rceil,\dots,\dots,F-1\right\}\text{ and }\#A=F-g\right\}$.
2) Return $\{\{0\}\cup A\cup\{F+1,\to\}\mid A\in C\}$.

As a consequence of the previous algorithm we obtain the following result which also appears in [3, Corollary 4].

Corollary 4.1. *If F and g are positive integers such that $g\le F\le 2g-1$, then $\#\mathcal{E}(-,F,g)=\binom{\lceil\frac{F}{2}\rceil-1}{F-g}$.*

Lemma 4.3. *If m, F and g are three positive integers such that $m\ge 2$ and $\mathcal{E}(m,F,g)\neq\emptyset$, then $m-1\le g\le F<2m$.*

Proof. Since $\mathcal{E}(m,F,g)\neq\emptyset$, then $\mathcal{E}(m,-,g)\neq\emptyset$ and we have that $m-1\le g$. From Lemma 4.1, we deduce that $g\le F$. Finally, by Proposition 3.1, we conclude that $\frac{F+1}{2}\le m$ and thus $F<2m$. □

Finally, we present the main result of this section.

Proposition 4.1. *Let m, F and g be three positive integers such that $m \geq 2$. Then $\mathcal{E}(m, F, g) \neq \emptyset$ if and only if one of the following conditions holds:*

1) $(m, F, g) = (m, m-1, m-1)$;
2) $(m, F, g) = (m, F, m)$ *and* $m < F < 2m$;
3) $m < g < F < 2m$.

Proof. Necessity. If $\mathcal{E}(m, F, g) \neq \emptyset$ then by applying Lemma 4.3, we deduce that $m-1 \leq g \leq F < 2m$. Assume that $S \in \mathcal{E}(m, F, g)$. We distinguish the following four cases.

a) If $g = m-1$, then $S = \{0, m, \rightarrow\}$ and so $F = m-1$. Hence, $(m, F, g) = (m, m-1, m-1)$.
b) If $g = m$, then $m < F < 2m$ and $S = \{0, m, \rightarrow\} \setminus \{F\}$. Whence, $(m, F, g) = (m, F, m)$ and $m < F < 2m$.
c) If $g = F$, then $S = \{0, F+1, \rightarrow\}$ and thus $F + 1 = m$. Once again we have $(m, F, g) = (m, m-1, m-1)$.
d) If $g \notin \{m-1, m, F\}$, then as $m - 1 \leq g \leq F < 2m$ and we deduce that $m < g < F < 2m$.

Sufficiency. It is clear that $\{0, m, \rightarrow\} \in \mathcal{E}(m, m-1, m-1)$ and $\{0, m, \rightarrow\} \setminus \{F\} \in \mathcal{E}(m, F, m)$. Suppose that $m < g < F < 2m$. Let A be a subset of $\{m+1, \dots, F-1\}$, with cardinality $F - g - 1$. Since $\mathrm{g}(S) = m - 1 + F - 1 - m - 1 + 1 - \#A + 1 = F - 1 - F + g + 1 = g$, then $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\} \in \mathcal{E}(m, F, g)$. $\square$

Notice that, by the sufficiency condition of the proof above, we conclude that, if $m < g < F < 2m$, knowing an element in $\mathcal{E}(m, F, g)$ is the same as knowing a subset of $\{m+1, \dots, F-1\}$ with cardinality $F - g - 1$. So we have the following algorithm.

Algorithm 4.2. Input: m, F and g integers such that $2 \leq m < g < F < 2m$.
Output: $\mathcal{E}(m, F, g)$.

1) Compute $C = \{A \mid A \subseteq \{m+1, \dots, F-1\}$ and $\#A = F - g - 1\}$.
2) Return $\{\{0, m\} \cup A \cup \{F+1 \rightarrow\}$ such that $A \in C\}$.

Clearly $\#\mathcal{E}(m, m-1, m-1) = \#\mathcal{E}(m, F, m) = 1$. For the remaining cases the following result gives us the cardinality of $\mathcal{E}(m, F, g)$.

Corollary 4.2. *Let m, F and g be positive integers such that $2 \leq m < g < F \leq 2m$. Then $\#\mathcal{E}(m, F, g) = \binom{F-m-1}{F-g-1}$.*

Proof. As a consequence of Algorithm 4.2 we have that $S \in \mathcal{E}(m, F, g)$ if and only if there exists $A \subseteq \{m+1, \dots, F-1\}$, with cardinality $F - g - 1$ such that $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$. $\square$

We conclude this section by giving an example that illustrates the previous results.

Example 4.1. Let us compute $\mathcal{E}(4,7,5)$. By Corollary 4.2 we have $\#\mathcal{E}(4,7,5) = \binom{7-4-1}{7-5-1} = \binom{2}{1} = 2$. Now by using Algorithm 4.2, with $m = 4$, $F = 7$ and $g = 5$ we can conclude that $C = \{\{5\},\{6\}\}$ and $\mathcal{E}(4,7,5) = \{\{0,4\} \cup \{5\} \cup \{8,\rightarrow\}, \{0,4\} \cup \{6\} \cup \{8,\rightarrow\}\}$.

5. Frobenius Variety

A Frobenius variety (see for example [11]) is a nonempty set V of numerical semigroups fulfilling the following conditions:

1) if S and T are in V, then $S \cap T \in V$;
2) if S is in V and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in V$.

Proposition 5.1. *$\mathcal{E} = \{S \mid S$ is an elementary numerical semigroup$\}$ is a Frobenius variety.*

Proof. If S and T belong to $\mathcal{E}$ it is clear that $S \cap T$ is a numerical semigroup,

$$\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), F(T)\}$$

and

$$\mathrm{m}(S \cap T) \geq \max\{\mathrm{m}(S), \mathrm{m}(T)\}.$$

Therefore, $\mathrm{F}(S \cap T) < 2\mathrm{m}(S \cap T)$ and thus $S \cap T \in \mathcal{E}$.

If S is an element in $\mathcal{E}$ and $S \neq \mathbb{N}$, then clearly $\overline{S} = S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup such that $\mathrm{F}(\overline{S}) < \mathrm{F}(S)$ and $\mathrm{m}(\overline{S})$ is equal to $\mathrm{m}(S)$ or $\mathrm{F}(S)$. Therefore, $\mathrm{F}(\overline{S}) < 2\mathrm{m}(\overline{S})$ and thus $\overline{S} \in \mathcal{E}$. □

We define a directed graph $G(\mathcal{E})$, with edges pointing from T to S, in the following way: the set of vertices is $\mathcal{E}$ and $(T,S) \in \mathcal{E} \times \mathcal{E}$ is an edge of $G(\mathcal{E})$ if and only if $S \cup \{\mathrm{F}(S)\} = T$.

The goal of this section is to see that $G(\mathcal{E})$ is a tree with root equal to $\mathbb{N}$ and to characterize the sons of a vertex. This fact allows us to recursively construct $G(\mathcal{E})$ and consequently $\mathcal{E}$. To this end we need to introduce some concepts and results.

Given a nonempty subset A of $\mathbb{N}$ we will denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is,

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\}, a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1,\ldots,n\}\}.$$

It is well known (see for instance [12]) that every numerical semigroup S is finitely generated, and therefore there exists a finite subset A of $\mathbb{N}$ such that $S = \langle A\rangle$. Furthermore, we say that A is a minimal set of generators of S if no proper subset of A generates S. Every numerical semigroup admits an unique minimal set of generators of S and we denote this set by $\mathrm{msg}(S)$. It is well known (see for instance [12]) that $\mathrm{msg}(S) = (S\backslash\{0\}) \setminus (S\backslash\{0\} + S\backslash\{0\})$ and if $x \in S$ then $S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.

As a consequence of [11, Proposition 24 and Theorem 27] we have the following result.

Theorem 5.1. *The graph $G(\mathcal{E})$ is a tree with root $\mathbb{N}$. Furthermore, the sons of a vertex S of $G(\mathcal{E})$ are in $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S) \text{ and } S\backslash\{x\} \in \mathcal{E}\}$.*

The following result is useful to compute the sons of a vertex of $G(\mathcal{E})$.

Proposition 5.2. *Let S be an elementary numerical semigroup and $x \in msg(S)$ such that $x > \mathrm{F}(S)$. Then $S\backslash\{x\}$ is an elementary numerical semigroup if and only if $x < 2\mathrm{m}(S)$.*

Proof. Suppose that $S = \{0, \mathrm{m}(S), \rightarrow\}$. Then

$$\mathrm{msg}(S) = \{\mathrm{m}(S), \mathrm{m}(S)+1, \ldots, 2\mathrm{m}(S)-1\}$$

and clearly the result is true. If $S \neq \{0, \mathrm{m}(S), \rightarrow\}$ then $\mathrm{m}(S\backslash\{x\}) = \mathrm{m}(S)$ and $\mathrm{F}(S\backslash\{x\}) = x$. Therefore, $S\backslash\{x\}$ is elementary numerical semigroup if and only if $x < 2\mathrm{m}(S)$. $\square$

We illustrate the above results with the following example.

Example 5.1. Let us compute the sons of vertex $S = \{0, 5, 6, 9, \rightarrow\}$ of $G(\mathcal{E})$. We have $\mathrm{msg}(S) = \{5, 6, 9, 13\}$, $\mathrm{F}(S) = 8$ and $\mathrm{m}(S) = 5$. Whence $\{x \in \mathrm{msg}(S) \mid \mathrm{F}(S) < x < 2\mathrm{m}(S)\} = \{9\}$. Using Theorem 5.1 and Proposition 5.2 we conclude that S has an unique son $S\backslash\{9\} = \langle 5, 6, 13, 14\rangle$.

Now, we can recursively construct the tree $G(\mathcal{E})$ starting with $\mathbb{N}$ and connecting each vertex with their sons. First we construct $\mathrm{msg}(S\backslash\{x\})$ from $\mathrm{msg}(S)$, when x is a minimal generator of S greater than $\mathrm{F}(S)$. It is clear that if $\mathrm{msg}(S) = \{m, m+1, \ldots, 2m-1\}$ which is $S = \{0, m, \rightarrow\}$ then $\mathrm{msg}(S\backslash\{m\}) = \{m+1, m+2, \ldots, 2m+1\}$. For the remaining cases, we use the following result that appears in [10, Corollary 18].

Proposition 5.3. *Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1, \ldots, n_p\}$. If $\mathrm{m}(S) = n_1 < n_p$ and $n_p > \mathrm{F}(S)$ then*

$$\mathrm{msg}(S\backslash\{n_p\}) = \begin{cases} \{n_1, \ldots, n_{p-1}\}, & \textit{if exists } i \in \{2, \ldots, p-1\} \textit{ such that} \\ & n_p + n_1 - n_i \in S, \\ \{n_1, \ldots, n_{p-1}, n_p + n_1\}, & \textit{otherwise.} \end{cases}$$

Note that, in the previous proposition, the elements in $\mathrm{msg}(S)$ are not necessarily ordered.

Example 5.2. Let $S = \langle 5, 6, 9, 13\rangle$. Let us compute $\mathrm{msg}(S\backslash\{9\})$. By Proposition 5.3, as $9+5-6 \notin S$ and $9+5-13 \notin S$, we can conclude that $\{5, 6, 13, 14\}$ is the minimal system of generators of $S\backslash\{9\}$.

Using Theorem 5.1 and Proposition 5.2 and 5.3 we obtain the following:

. $\langle 1\rangle$ has only son $\langle 1\rangle\backslash\{1\} = \langle 2, 3\rangle$;
. $\langle 2, 3\rangle$ has two sons $\langle 2, 3\rangle\backslash\{2\} = \langle 3, 4, 5\rangle$ and $\langle 2, 3\rangle\backslash\{3\} = \langle 2, 5\rangle$;
. $\langle 2, 5\rangle$ has no sons;

$$
\begin{tikzcd}
 & & & & & \langle 1\rangle = \mathbb{N} \arrow[d] & \\
 & & & & & \langle 2,3\rangle \arrow[dl] \arrow[dr] & \\
 & & & & \langle 3,4,5\rangle \arrow[dl] \arrow[d] \arrow[dr] & & \langle 2,5\rangle \\
 & & & \langle 4,5,6,7\rangle \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d] & \langle 3,5,7\rangle \arrow[d] & \langle 3,4\rangle & \\
\langle 5,6,7,8,9\rangle & \langle 4,6,7,9\rangle & \langle 4,5,7\rangle & \langle 4,5,6\rangle & \langle 3,7,8\rangle & \ldots &
\end{tikzcd}
$$

. $\langle 3,4,5\rangle$ has three sons $\langle 3,4,5\rangle\backslash\{3\} = \langle 4,5,6,7\rangle$, $\langle 3,4,5\rangle\backslash\{4\} = \langle 3,5,7\rangle$ and $\langle 3,4,5\rangle\backslash\{5\} = \langle 3,4\rangle$;

. $\langle 3,4\rangle$ has no sons;

. $\langle 3,5,7\rangle$ has one son $\langle 3,5,7\rangle\backslash\{5\} = \langle 3,7,8\rangle$;

. $\langle 4,5,6,7\rangle$ has four sons $\langle 4,5,6,7\rangle\backslash\{4\} = \langle 5,6,7,8,9\rangle$, $\langle 4,5,6,7\rangle\backslash\{5\} = \langle 4,6,7,9\rangle$, $\langle 4,5,6,7\rangle\backslash\{6\} = \langle 4,5,7\rangle$ and $\langle 4,5,6,7\rangle\backslash\{7\} = \langle 4,5,6\rangle$;

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References

[1] J. Backelin, *On the number of semigroups of natural numbers*, Math. Scand. **66**(2) (1990), 197–215.

[2] V. Blanco, P. A. García-Sánchez and J. Puerto, *Counting numerical semigroups with short generating functions*, Internat. J. Algebra Comput. **21**(7) (2011), 1217–1235.

[3] V. Blanco and J. C. Rosales, *The set of numerical semigroups of a given genus*, Semigroup Forum **85** (2012), 255–267.

[4] M. Bras-Amorós, *Bounds on the number of numerical semigroups of a given genus*, J. Pure Appl. Algebra **213**(6) (2008), 997–1001.

[5] M. Bras-Amorós, *Fibonacci-like behavior of the number of numerical semigroups of a given genus*, Semigroup Forum **76** (2008), 379–384.

[6] S. Elizalde, *Improved bounds on the number of numerical semigroups of a given genus*, J. Pure Appl. Algebra **214**(10) (2010), 1862–1873.

[7] N. Kaplan, *Couting numerical semigroups by genus and some cases a question of Wilf*, J. Pure Appl. Algebra **216**(5) (2012), 1016–1032.

[8] N. Kaplan and L. Ye, *The proportion of Weierstrass semigroups*, J. Algebra **373** (2013), 377–391.

[9] J. L. Ramirez Alfonsín, *The Diophantine Forbenius Problem*, Oxford University Press, London, 2005.

[10] A. M. Robles-Pérez and J. C. Rosales, *The numerical semigroups of phases'lengths in a simple alphabet*, The Scientific World Journal **2013** (2013), paper ID 459024, 9 pages.
[11] J. C. Rosales, *Families of numerical semigroups closed under finite intersections and for the Frobenius number*, Houston J. Math. **34** (2008), 339–348.
[12] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Developments in Mathematics **20**, Springer, New York, 2009.
[13] Y. Zhao, *Constructing numerical semigroups of a given genus*, Semigroup Forum **80**(2) (2009), 242–254.

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EXISTENCE AND STABILITY OF NONLOCAL INITIAL VALUE PROBLEMS INVOLVING GENERALIZED KATUGAMPOLA DERIVATIVE

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Abstract. In this paper, the existence results for the solutions to nonlocal initial value problems involving generalized Katugampola derivative are established. Some fixed point theorem techniques are used to derive the existence results. In the sequel, we investigate the generalized Ulam-Hyers-Rassias stability corresponding to our problem. Some examples are given to illustrate our main results.

1. Introduction

In recent decades, the theory of continuous fractional calculus and their applications have remains a centre of attraction in many mathematical research. Indeed, fractional differential equations (FDEs) have grabbed desired attention by many authors. One can see [1–5, 7–13, 20, 21, 23, 26, 27, 33, 34] and references therein. Several definitions of fractional derivatives and integrals have been introduced during the theoretical development of fractional calculus. See [1, 2, 5, 7, 8, 16, 20–22, 25, 27] and references therein.

Initially, Hilfer et al. [16,17] have proposed linear differential equations involving new fractional operator. They applied operational method to solve such FDEs. Further, Furati et al. [14, 15] investigated non-linear problems and discussed existence and non-existence results for FDEs with Hilfer derivative operator. Benchohra et al. [6,7], U. N. Katugampola [20, 21], D. B. Dhaigude et al. [8, 9], Kou et al. [23], J. Wang et al. [32, 33] and many more authors, see [1, 2, 5, 19, 29, 31] and references therein, have established the existence results for FDEs with several fractional derivative operators.

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Recently, D. S. Oliveira et al. [27] in their article proposed a new fractional differential operator: Hilfer-Katugampola frational derivative (also known as generalized Katugampola derivative). Further, they established the existence and uniqueness results for the FDEs with generalized Katugampola derivative.

The theory of Ulam stability is also evolved as one of the most interesting field of research. Initially, Ulam [30] established the results on the stability of functional equations. Thereafter, remarkable interest have been shown by authors towards the study of Ulam-Hyers stability and Ulam-Hyers-Rassias stability for various FDEs, see [1,6,7,18,24,31,33] and references therein.

In this paper, we studied the existence and stability of nonlocal initial value problem (IVP) involving generalized Katugampola derivative of the form:

$$ {}^{\rho}D_{a^+}^{\mu,\nu}u(t) = f(t,u(t)), \quad \mu\in(0,1), \nu\in[0,1], t\in(a,b], \tag{1.1}$$

$$ {}^{\rho}I_{a^+}^{1-\beta}u(a) = \sum_{i=1}^{m}\lambda_i u(\kappa_i), \quad \mu\le\beta=\mu+\nu(1-\mu)<1, \kappa_i\in(a,b], \tag{1.2}$$

where f is a given function such that $f:(a,b]\times\mathbb{R}\to\mathbb{R}$, $0<\rho$. The operator ${}^{\rho}D_{a^+}^{\mu,\nu}$ is the generalized Katugampola fractional derivative of order μ and type ν and the operator ${}^{\rho}I_{a^+}^{1-\beta}u(a)$ is the Katugampola fractional integral of order $1-\beta$, with $a>0$, κ_i, $i=1,2,\dots,m$, are prefixed points satisfying $a<\kappa_1\le\kappa_2\le\cdots\le\kappa_m<b$.

Furthermore, the paper is arranged as follows. In Section 2, we recall some basic definitions, important results and preliminary facts. We establish the equivalent mixed type Volterra integral equation for the IVP (1.1)–(1.2). In Section 3, we present existence of solution using the Krasnoselskii fixed point theorem. Further, we present the generalized Ulam-Hyers-Rassias stability to our problem. An illustrative example is given at the end of the main results.

2. Preliminary Results

In this section, we provide some basic definitions of generalized fractional integrals and derivatives, some important results and preliminary facts that are very useful to us in the sequel.

Let $0<a<b<\infty$ be a finite interval on $\mathbb{R}^+$ and $C[a,b]$ be the Banach space of all continuous functions $h:[a,b]\to\mathbb{R}$ with the norm

$$\|h\|_C = \max\{|h(t)| : t\in[a,b]\}.$$

For $0\le\beta<1$ and the parameter $\rho>0$ we define the weighted space of continuous functions h on $(a,b]$ by

$$C_{\beta,\rho}[a,b] = \left\{h:(a,b]\to\mathbb{R} : \left(\frac{t^\rho-a^\rho}{\rho}\right)^\beta h(t)\in C[a,b]\right\},$$

with the norm

$$\|h\|_{C_{\beta,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\beta h(t) \right\|_C = \max_{t\in[a,b]} \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\beta h(t) \right|.$$

It is obvious that $C_{0,\rho}[a,b] = C[a,b]$.

Let $\delta_\rho = \left(t^{\rho-1} \frac{d}{dt} \right)$. We define the Banach space of continuously differentiable functions h on $[a,b]$ by

$$C^1_{\delta_\rho,\beta}[a,b] = \{ h : [a,b] \to \mathbb{R} : \delta_\rho h \in C_{\beta,\rho}[a,b] \},$$

with the norms

$$\|h\|_{C^1_{\delta_\rho,\beta}} = \|h\|_C + \|\delta_\rho h\|_{C_{\beta,\rho}}$$

and

$$\|h\|_{C^1_{\delta_\rho,\beta}} = \max \{ |\delta_\rho h(t)| : t \in [a,b] \}.$$

Note that $C^0_{\delta_\rho,\beta}[a,b] = C_{\beta,\rho}[a,b]$.

Definition 2.1 (Katugampola fractional integral [20, 27]). Let $\mu, c \in \mathbb{R}$, with $\mu > 0$, $u \in Z^p_c(a,b)$, where $Z^p_c(a,b)$ is the space of Lebesgue measurable functions with complex values. The left-sided Katugampola fractional integral of order μ is defined by

$$({}^\rho I^\mu_{a^+} u)(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^t \frac{x^{\rho-1} u(x)}{(t^\rho - x^\rho)^{1-\mu}} dx, \quad t > a. \tag{2.1}$$

Definition 2.2 (Katugampola fractional derivative [21, 27]). Let $\mu, \rho \in \mathbb{R}$ be such that $\mu \notin \mathbb{N}$, $0 < \mu, \rho$. The left-sided Katugampola fractional derivative of order μ is defined by

$$({}^\rho D^\mu_{a^+} u)(t) = \delta^n_\rho \left({}^\rho I^{n-\mu}_{a^+} u \right)(t) = \frac{\rho^{1-n+\mu}}{\Gamma(n-\mu)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{x^{\rho-1} u(x)}{(t^\rho - x^\rho)^{1-n+\mu}} dx, \tag{2.2}$$

where $n = [\mu] + 1$ is such that $[\mu]$ is the integer part of μ.

Definition 2.3 (Generalized Katugampola fractional derivative [27]). Let $0 < \mu \leq 1$ and $0 \leq \nu \leq 1$. The generalized Katugampola fractional derivative (of order μ and type ν) with respect to t is defined by

$$\begin{aligned} ({}^\rho D^{\mu,\nu}_{a^+} u)(t) &= \left\{ \pm {}^\rho I^{\nu(1-\mu)}_{a\pm} \left(t^{\rho-1} \frac{d}{dt} \right)^1 {}^\rho I^{(1-\nu)(1-\mu)}_{a\pm} u \right\}(t) \\ &= \left\{ \pm {}^\rho I^{\nu(1-\mu)}_{a\pm} \delta_\rho {}^\rho I^{(1-\nu)(1-\mu)}_{a\pm} u \right\}(t), \end{aligned} \tag{2.3}$$

where $\rho > 0$, $u \in C_{1-\beta,\rho}[0,1]$ and I is Katugampola fractional integral defined in (2.1).

Remark 2.1. ([27]). For $\beta = \mu + \nu(1-\mu)$, the generalized Katugampola fractional derivative operator ${}^{\rho}D_{a^+}^{\mu,\nu}$ can be expressed as

$$ {}^{\rho}D_{a^+}^{\mu,\nu} = {}^{\rho}I_{a^+}^{\nu(1-\mu)}\delta_{\rho}{}^{\rho}I_{a^+}^{1-\beta} = {}^{\rho}I_{a^+}^{\nu(1-\mu)\rho}D_{a^+}^{\beta}. \tag{2.4} $$

Lemma 2.1 ([27]). *Let $\mu > 0$, $0 \leq \beta < 1$ and $u \in C_{\beta,\rho}[a,b]$. Then*

$$ \left({}^{\rho}D_{a^+}^{\mu}{}^{\rho}I_{a^+}^{\mu}u\right)(t) = u(t), \quad \text{for all } t \in (a,b]. $$

Lemma 2.2 (Semigroup property [27]). *Let $\mu > 0$, $\nu > 0$, $1 \leq q \leq \infty$, $a, b \in (0,\infty)$ such that $a < b$ and $\rho, s \in \mathbb{R}$, $s \leq \rho$. Then the following property holds true*

$$ \left({}^{\rho}I_{a^+}^{\mu}{}^{\rho}I_{a^+}^{\nu}u\right)(t) = \left({}^{\rho}I_{a^+}^{\mu+\nu}u\right)(t), $$

for all $u \in Z_s^q(a,b)$.

Lemma 2.3 ([27]). *Let $t > a$ and for $\mu \geq 0$ and $\nu > 0$, we have*

$$ \left[{}^{\rho}D_{a^+}^{\mu}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1}\right](t) = 0, \quad 0 < \mu < 1, $$
$$ \left[{}^{\rho}I_{a^+}^{\mu}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\nu-1}\right](t) = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\nu-1}. $$

Lemma 2.4 ([27]). *Let $\mu > 0$, $0 \leq \beta < 1$ and $a, b \in (0,\infty)$ such that $a < b$ and $u \in C_{\beta,\rho}[a,b]$. Then*

$$ \left({}^{\rho}I_{a^+}^{\mu}u\right)(a) = \lim_{t \to a^+}\left({}^{\rho}I_{a^+}^{\mu}u\right)(t) = 0, $$

and ${}^{\rho}I_{a^+}^{\mu}u$ is continuous on $[a,b]$ if $\beta < \mu$.

Lemma 2.5 ([27]). *Let $\mu \in (0,1)$, $\nu \in [0,1]$ and $\beta = \mu + \nu - \mu\nu$. If $u \in C_{1-\beta}^{\beta}[a,b]$ then*

$$ {}^{\rho}I_{a^+}^{\beta}{}^{\rho}D_{a^+}^{\beta}u = {}^{\rho}I_{a^+}^{\mu}{}^{\rho}D_{a^+}^{\mu,\nu}u $$

and

$$ {}^{\rho}D_{a^+}^{\beta}{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}D_{a^+}^{\nu(1-\mu)}u. $$

Lemma 2.6 ([27]). *Let $\mu \in (0,1)$, $0 \leq \beta < 1$. If $u \in C_{\beta}[a,b]$ and ${}^{\rho}I_{a^+}^{1-\mu}u \in C_{\beta}^{1}[a,b]$, then for all $t \in (a,b]$*

$$ \left({}^{\rho}I_{a^+}^{\mu}{}^{\rho}D_{a^+}^{\mu}u\right)(t) = -\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1}\frac{\left({}^{\rho}I_{a^+}^{1-\beta}u\right)(a)}{\Gamma(\mu)} + u(t). $$

Lemma 2.7 ([27]). *Let $u \in L^1(a,b)$. If ${}^{\rho}D_{a^+}^{\nu(1-\mu)}u$ exists on $L^1(a,b)$, then*

$$ {}^{\rho}D_{a^+}^{\mu,\nu}{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}I_{a^+}^{\nu(1-\mu)\rho}D_{a^+}^{\nu(1-\mu)}u. $$

Lemma 2.8 ([27])**.** *Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function where $f(\cdot,u(\cdot))\in C_{1-\beta}[a,b]$. A function $u\in C_{1-\beta}^{\beta}[a,b]$ is a solution of fractional IVP*

$$\begin{aligned} D_{a^+}^{\mu,\nu}u(t)&=f(t,u(t)),\quad \mu\in(0,1),\ \nu\in[0,1],\\ I_{a^+}^{1-\beta}u\left(a^+\right)&=u_0,\quad \beta=\mu+\nu-\mu\nu, \end{aligned}$$

if and only if u satisfies the integral equation of Volterra type:

$$u(t)=\frac{u_0(t-a)^{\beta-1}}{\Gamma(\beta)}+\frac{1}{\Gamma(\mu)}\int_a^t(t-x)^{\mu-1}f(x,u(x))\,dx.$$

Definition 2.4 (Volterra integral equation)**.** A linear Volterra integral equation of the second kind has the form of

$$u(t)=u_0(t)+\int_a^t K(t,x)u(x)\,dx,$$

where K is a kernel.

Theorem 2.1 (Krasnoselskii fixed point theorem [28])**.** *Let E be a nonempty closed, bounded and convex subset of a Banach space $(\mathcal{B},\|\cdot\|)$. Further, assume that F and G are two operators defined on E which map E into $\mathcal{B}$ such that*

(a) *$F(x)+G(y)\in E$ for all $x,y\in E$;*
(b) *F is a contraction;*
(c) *G is continuous and compact.*

Then $F+G$ has a fixed point in E.

Using the above fundamental results, the following theorem yields the equivalence between the IVP (1.1)–(1.2) and an improved mixed type Volterra integral equation.

Theorem 2.2. *Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that for any $u\in C_{1-\beta}[a,b]$ $f(\cdot,u(\cdot))\in C_{1-\beta}[a,b]$, where $\beta=\mu+\nu-\mu\nu$, with $0<\mu\le 1$, $0\le\nu\le 1$. Function $u\in C_{1-\beta}^{\beta}[a,b]$ is a solution of IVP (1.1)–(1.2) if and only if it satisfies the following mixed type Volterra integral equation*

$$\begin{aligned} u(t)=&\frac{K}{\Gamma(\mu)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^m\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho-x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\ &+\frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho-x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx, \end{aligned}\tag{2.5}$$

where $K=\left\{\Gamma(\beta)-\sum\limits_{i=1}^m\lambda_i\left(\frac{{\kappa_i}^\rho-a^\rho}{\rho}\right)^{\beta-1}\right\}^{-1}$.

Proof. Let $u \in C^{\beta}_{1-\beta}[a,b]$ be a solution of IVP (1.1)–(1.2). Then by the Lemma 2.8 the solution of IVP (1.1)–(1.2) can be written as

$$u(t) = \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx. \tag{2.6}$$

Now, substitute $t = \kappa_i$ in the above equation

$$u(\kappa_i) = \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{1}{\Gamma(\mu)} \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.$$

Multiplying by λ_i the both hand sides, we get

$$\lambda_i u(\kappa_i) = \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} + \frac{\lambda_i}{\Gamma(\mu)} \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx.$$

Thus, we have

$$\begin{aligned}
{}^\rho I^{1-\beta}_{a^+} u(a) =& \sum_{i=1}^{m} \lambda_i u(\kappa_i), \\
=& \frac{\left({}^\rho I^{1-\beta}_{a^+} u\right)(a)}{\Gamma(\beta)} \sum_{i=1}^{m} \lambda_i \left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\beta-1} \\
&+ \frac{1}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx,
\end{aligned}$$

which implies

$$\left({}^\rho I^{1-\beta}_{a^+} u\right)(a) = \frac{\Gamma(\beta)}{\Gamma(\mu)} K \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx. \tag{2.7}$$

Substituting (2.7) in (2.6) we get (2.5), which proved that u also satisfies integral equation (2.5) when it satisfies IVP (1.1)–(1.2). This proved the necessity. Now, we prove the sufficiency by applying ${}^\rho I^{1-\beta}_{a^+}$ to both hand sides of the integral equation (2.5), we have

$$\begin{aligned}
{}^\rho I^{1-\beta}_{a^+} u(t) =& {}^\rho I^{1-\beta}_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1} \frac{K}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \\
&+ {}^\rho I^{1-\beta}_{a^+} {}^\rho I^{\mu}_{a^+} f(x, u(x)).
\end{aligned}$$

By using Lemma 2.2, Lemma 2.1 and Lemma 2.3, we have

$${}^\rho I^{1-\beta}_{a^+} u(t) = \frac{\Gamma(\beta)}{\Gamma(\mu)} K \sum_{i=1}^{m} \lambda_i \int_a^{\kappa_i} \left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx + {}^\rho I^{1-\nu(1-\mu)}_{a^+} f(t, u(t)).$$

Since $1-\nu(1-\mu)>1-\beta$, by taking the limit as $t\to a$ and using Lemma 2.4, we have

$$\left({}^{\rho}I_{a^+}^{1-\beta}u\right)(a)=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx. \tag{2.8}$$

Now, substituting $t=\kappa_i$ in (2.5), we have

$$\begin{aligned}u(\kappa_i)=&\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&+\frac{1}{\Gamma(\mu)}\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx.\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{i=1}^{m}\lambda_iu(\kappa_i)=&\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&+\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\=&\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\&\times\left\{K\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}+1\right\}\\=&\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx. \qquad (2.9)\end{aligned}$$

It follows from (2.8) and (2.9), that

$${}^{\rho}I_{a^+}^{1-\beta}u(a)=\sum_{i=1}^{m}\lambda_iu(\kappa_i).$$

It follows from Lemma 2.3 and Lemma 2.5 and by applying ${}^{\rho}D_{a^+}^{\beta}$ to both hand sides of (2.5) that

$${}^{\rho}D_{a^+}^{\beta}u(t)={}^{\rho}D_{a^+}^{\nu(1-\mu)}f(t,u(t))\,. \tag{2.10}$$

Since $u\in C_{1-\beta}^{\beta}[a,b]$ and by the definition of $C_{1-\beta}^{\beta}[a,b]$, we have ${}^{\rho}D_{a^+}^{\beta}u\in C_{1-\beta}^{\beta}[a,b]$. Then ${}^{\rho}D_{a^+}^{\nu(1-\mu)}f={}^{\rho}D^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}[a,b]$. It is obvious that for any $f\in C_{1-\beta}[a,b]$, ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}[a,b]$, then ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\beta}^{1}[a,b]$. Thus, f and ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f$ satisfy both the conditions of Lemma 2.6.

Now, it follows from Lemma 2.6, by applying ${}^{\rho}I_{a^+}^{\nu(1-\mu)}$ on both sides of (2.10), that

$$\left({}^{\rho}D_{a^+}^{\mu,\nu}u\right)(t) = -\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\nu(1-\mu)-1}\frac{{}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f(a)}{\Gamma(\nu(1-\mu))} + f(t,u(t)). \tag{2.11}$$

By Lemma 2.4, it implies that ${}^{\rho}I_{a^+}^{1-\nu(1-\mu)}f(a) = 0$. Hence, (2.11) reduces to

$$\left({}^{\rho}D_{a^+}^{\mu,\nu}u\right)(t) = f(t,u(t)).$$

This completes the proof. □

3. Main Result

In the sequel, let us introduce the following hypothesis.

[$Q1$] Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that for any $u\in C_{1-\beta}[a,b]$ $f(\cdot,u(\cdot))\in C_{1-\beta}^{\nu(1-\mu)}[a,b]$. For all $u,v\in\mathbb{R}$ there exists a positive constant $J>0$ such that

$$|f(t,u)-f(t,v)|\le J|u-v|.$$

[$Q2$] The constant

$$\sigma := \frac{JB(\mu,\beta)}{\Gamma(\mu)}\left\{|K|\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1} + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\mu}\right\} < 1, \tag{3.1}$$

where K is defined in the Theorem 2.2.

Now, we will establish our main existence result for IVP (1.1)–(1.2) using Krasnoselskii fixed point theorem.

Theorem 3.1. *Assume that the hypothesis* [$Q1$] *and* [$Q2$] *are satisfied. Then IVP* (1.1)–(1.2) *has at least one solution in* $C_{1-\beta}^{\beta}[a,b]$.

Proof. According to Theorem 2.2, it is sufficient to prove the existence result for the mixed type integral equation (2.5).

Now, define the operator Δ by

$$\begin{aligned}(\Delta u)(t) =& \frac{K}{\Gamma(\mu)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\\ &+\frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx.\end{aligned} \tag{3.2}$$

It is obvious that the operator Δ is well defined and maps $C_{1-\beta}[a,b]$ into $C_{1-\beta}[a,b]$. Let $\hat{f}(x)=f(x,0)$ and

$$\eta := \frac{B(\mu,\beta)}{\Gamma(\mu)}\left\{|K|\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1} + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\mu}\right\}\left\|\hat{f}\right\|_{C_{1-\beta}}. \tag{3.3}$$

Consider a ball $B_s := \left\{u\in C_{1-\beta}[a,b] : \|u\|_{C_{1-\beta}}\le s\right\}$, with $\frac{\eta}{1-\sigma}\le s, \sigma<1$.

Now, let us subdivide the operator Δ into two operators F and G on B_s as follows:

$$(Fu)(t) = \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx$$

and

$$(Gu)(t) = \frac{1}{\Gamma(\mu)}\int_a^{t}\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx.$$

The proof is divided into following steps.

Step I. For every $u, v \in B_s$, $Fu + Gv \in B_s$. For the operator F

$$(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta} = \frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx, \quad t \in (a,b],$$

we have

$$\begin{aligned}\left|(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta}\right| \le & \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\left|f(x,u(x))\right|dx\\ \le & \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\left(\left|f(x,u(x)) - f(x,0)\right|\right.\\ & \left. + \left|f(x,0)\right|\right)dx\\ \le & \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\left(J\left|u(x)\right| + \left|\hat{f}(x)\right|\right)dx.\end{aligned}$$

Here we use the fact that

$$\begin{aligned}\int_a^{t}\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\left|u(x)\right|dx \le & \left\{\int_a^{t}\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\beta-1}x^{\rho-1}dx\right\}\\ & \times \|u(x)\|_{C_{1-\beta}}\\ = & \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1}B(\mu,\beta)\|u(x)\|_{C_{1-\beta}}.\end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}\left|(Fu)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta}\right| \le & \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\left\{\left(\frac{{\kappa_i}^\rho - a^\rho}{\rho}\right)^{\mu+\beta-1}B(\mu,\beta)\right.\\ & \left.\times\left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}}\right)\right\},\end{aligned}$$

which gives
(3.5)

$$\|Fu\|_{C_{1-\beta}} \leq \frac{|K|\, B(\mu,\beta)}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \left\{ \left(\frac{{\kappa_i}^{\rho} - a^{\rho}}{\rho} \right)^{\mu+\beta-1} \left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}} \right) \right\}.$$

For $t \in (a,b]$ and the operator G

$$(Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} = \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx,$$

we have

$$\begin{aligned} \left| (Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \right| &\leq \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} |f(x, u(x))|\, dx \\ &\leq \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \\ &\quad \times \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho} \right)^{\mu-1} x^{\rho-1} \left(J|u(x)| + \left|\hat{f}(x)\right| \right) dx. \end{aligned}$$

Again, by using (3.4), we have

$$\begin{aligned} \left| (Gu)(t) \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \right| &\leq \frac{1}{\Gamma(\mu)} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta} \left\{ \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\mu+\beta-1} \right. \\ &\quad \left. \times B(\mu,\beta) \left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}} \right) \right\} \\ &\leq \frac{B(\mu,\beta)}{\Gamma(\mu)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{\mu} \left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}} \right), \end{aligned}$$

which gives

$$\|(Gu)\|_{C_{1-\beta}} \leq \frac{B(\mu,\beta)}{\Gamma(\mu)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{\mu} \left(J\|u(x)\|_{C_{1-\beta}} + \left\|\hat{f}(x)\right\|_{C_{1-\beta}} \right). \tag{3.6}$$

Combining (3.5) and (3.6) for every $u, v \in B_s$ we have

$$\|Fu + Gv\|_{C_{1-\beta}} \leq \|Fu\|_{C_{1-\beta}} + \|(Gv)\|_{C_{1-\beta}} \leq \sigma s + \eta \leq s,$$

which implies that $Fu + Gv \in B_s$.

Step II. The operator F is contraction mapping.

For any $u, v \in B_s$ and the operator F

$$\{(Fu)(t) - (Fv)(t)\} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{1-\beta}$$

$$=\frac{K}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\left[f(x,u(x))-f(x,v(x))\right]dx$$

we have

$$\begin{aligned}\left|\{(Fu)(t)-(Fv)(t)\}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta}\right| &\leq\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\\ &\quad\times|f(x,u(x))-f(x,v(x))|\,dx\\ &\leq\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}\\ &\quad\times J\,|u(x)-v(x)|\,dx\\ &\leq\frac{J\,|K|}{\Gamma(\mu)}B(\mu,\beta)\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\\ &\quad\times\|u-v\|_{C_{1-\beta}},\end{aligned}$$

which gives

$$\|Fu-Fv\|_{C_{1-\beta}}\leq\frac{J\,|K|}{\Gamma(\mu)}B(\mu,\beta)\sum_{i=1}^{m}\lambda_i\left(\frac{{\kappa_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\|u-v\|_{C_{1-\beta}}\leq\sigma\|u-v\|_{C_{1-\beta}}.$$

Hence, by the hypothesis $[Q2]$ the operator F is a contraction mapping.

Step III. The operator G is compact and continuous.

Since the function $f\in C_{1-\beta}[a,b]$, it is obvious from the definition of $C_{1-\beta}[a,b]$ that the operator G is continuous.

From the equation (3.6) of Step I clearly, G is uniformly bounded on B_s. Next we prove the compactness.

For any $a<t_1<t_2\leq b$ we have

$$\begin{aligned}|(Gu)(t_1)-(Gu)(t_2)|=&\left|\frac{1}{\Gamma(\mu)}\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\right.\\ &\left.-\frac{1}{\Gamma(\mu)}\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx\right|\\ \leq&\frac{\|f\|_{C_{1-\beta}}}{\Gamma(\mu)}\left|\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}x^{\rho-1}dx\right.\\ &\left.-\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}x^{\rho-1}dx\right|\\ \leq&\frac{\|f\|_{C_{1-\beta}}B(\mu,\beta)}{\Gamma(\mu)}\left|\left(\frac{{t_1}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}-\left(\frac{{t_2}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\beta-1}\right|\end{aligned}$$

tending to zero as $t_2 \to t_1$, whether $\mu + \beta - 1 \geq 0$ or $\mu + \beta - 1 < 0$. Thus, G is equicontinuous. Hence, by Arzel-Ascoli Theorem, the operator G is compact on B_s.

It follows from Krasnoselskii fixed point theorem that the IVP (1.1)–(1.2) has at least one solution $u \in C_{1-\beta}[a,b]$. Using the Lemma 2.7 and repeating the process of proof in Theorem 2.2, one can show that this solution is actually in $C^{\beta}_{1-\beta}[a,b]$. This completes the proof. □

3.1. **Ulam-Hyers-Rassias stability.** In this section, we discuss the Ulam stability results for the solution of IVP (1.1)–(1.2).

Definition 3.1 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be Ulam-Hyers stable if there exists a real number $\psi > 0$ such that for every $\varepsilon > 0$ and for each solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu;\nu}_{a^+}u\right)(t) - f(t,u(t))\right| \leq \varepsilon, \quad t \in (a,b], \tag{3.7}$$

there exists $v \in C_{\beta,\rho}$, a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \varepsilon\psi, \quad t \in (a,b].$$

Definition 3.2 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be generalized Ulam-Hyers stable if there exists a continuous function $\psi_f : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi_f(0) = 0$ such that for every solution $u \in C_{\beta,\rho}$ of the inequality (3.7) there exists $v \in C_{\beta,\rho}$, a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \psi_f(\varepsilon), \quad t \in (a,b].$$

Definition 3.3 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be Ulam-Hyers-Rassias stable with respect to $\Psi \in C_{\beta,\rho}((a,b],\mathbb{R}_+)$ if there exists a real number $0 < \psi_\theta$ such that for every $0 < \varepsilon$ and for every solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu,\nu}_{a^+}u\right)(t) - f(t,u(t))\right| \leq \varepsilon\Psi(t), \quad t \in (a,b], \tag{3.8}$$

there exists $v \in C_{\beta,\rho}$ a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \varepsilon\psi_\theta\Psi(t), \quad t \in (a,b].$$

Definition 3.4 ([1])**.** The solution of IVP (1.1)–(1.2) is said to be generalized Ulam-Hyers-Rassias stable with respect to $\Psi \in C_{\beta,\rho}((a,b],\mathbb{R}_+)$ if there exists a real number $0 < \psi_\theta$ such that for every solution $u \in C_{\beta,\rho}$ of the inequality

$$\left|\left({}^{\rho}D^{\mu;\nu}_{a^+}u\right)(t) - f(t,u(t))\right| \leq \Psi(t), \quad t \in (a,b], \tag{3.9}$$

there exists $v \in C_{\beta,\rho}$ a solution of IVP (1.1)–(1.2) satisfying

$$|u(t) - v(t)| \leq \psi_\theta\Psi(t), \quad t \in (a,b].$$

Remark 3.1 ([1]). Clearly

(a) from Definition 3.1 follows Definition 3.2;
(b) from Definition 3.3 follows Definition 3.4;
(c) from Definition 3.3 for $\Psi(\cdot) = 1$ follows Definition 3.2.

Now, we establish the results on generalized Ulam-Hyers-Rassias stability of the IVP (1.1)–(1.2).

Theorem 3.2. *Assume that [Q1] and following hypothesis hold.*

[Q3] *There exists $\omega_\theta > 0$ such that for each $t \in (a, b]$ we have*

$$^{\rho}I^{\mu}_{a^+}\Psi(t) \leq \omega_\theta \Psi(t).$$

[Q4] *There exists a function $p \in C[(a, b], [0, \infty)]$ such that for each $t \in (a, b]$*

$$|f(t, u(t))| \leq \frac{p(t)\Psi(t)}{1+|u|}|u|.$$

Then the solution of IVP (1.1)–(1.2) *satisfies the generalized Ulam-Hyers-Rassias stability with respect to Ψ.*

Proof. Let u be a solution of the inequality (3.9) and let v be a solution of IVP (1.1)–(1.2). Then we have

$$\begin{aligned}v(t) =& \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x, v(x))\,dx\\ &+ \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x, v(x))\,dx\\ =&\Phi_v + \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x, v(x))\,dx,\end{aligned}$$

where

$$\Phi_v = \frac{K}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}f(x, v(x))\,dx.$$

On the other hand, if $\sum\limits_{i=1}^{m}\lambda_i u(\kappa_i) = \sum\limits_{i=1}^{m}\lambda_i v(\kappa_i)$ and $^{\rho}I^{1-\beta}_{a^+}u(a) = {}^{\rho}I^{1-\beta}_{a^+}v(a)$, then $\Phi_u = \Phi_v$. Indeed,

$$\begin{aligned}|\Phi_u - \Phi_v| \leq& \frac{|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\\ &\times|f(x, u(x)) - f(x, v(x))|\,dx\\ \leq& \frac{|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i\int_a^{\kappa_i}\left(\frac{{\kappa_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}J|u - v|\,dx\\ \leq& \frac{J|K|}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\sum_{i=1}^{m}\lambda_i{}^{\rho}I^{1-\beta}_{a^+}|u(\kappa_i) - v(\kappa_i)|\\ =&0.\end{aligned}$$

Hence, $\Phi_u = \Phi_v$. Then we have

$$v(t) = \Phi_u + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} f(x, v(x))\, dx.$$

From inequality (3.9) and $[Q3]$ for each $t \in (a, b]$ we have

$$\left| u(t) - \Phi_u - \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \right| \leq {}^\rho I_{a^+}^\mu \Psi(t) \leq \omega_\theta \Psi(t).$$

Set $\tilde{p} = \sup_{t \in (a,b]} p(t)$. From the hypothesis $[Q3]$ and $[Q4]$ for each $t \in (a, b]$ we have

$$\begin{aligned}
|u(t) - v(t)| \leq & \left| u(t) - \Phi_u - \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} f(x, u(x))\, dx \right| \\
& + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} \left| f(x, u(x)) - f(x, v(x)) \right| dx \\
\leq & \omega_\theta \Psi(t) + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} 2\tilde{p} \Psi(x)\, dx \\
\leq & \omega_\theta \Psi(t) + 2\tilde{p} \left({}^\rho I_{a^+}^\mu \Psi \right)(t) \\
\leq & (1 + 2\tilde{p})\, \omega_\theta \Psi(t) \\
:= & \psi_\theta \Psi(t).
\end{aligned}$$

Thus, the IVP (1.1)–(1.2) is generalized Ulam-Hyers-Rassias stable with respect to Ψ. This completes the proof. □

Following theorem will be useful in the progress of our next result.

Theorem 3.3 ([1]). *Let (Ω, d) be a generalized complete metric space and a strictly contractive operator $\Phi : \Omega \to \Omega$, with a Lipschitz constant $E < 1$. If there exists a non negative integer j such that $d(\Phi^{j+1}u, \Phi^{j+1}u) < \infty$ for some $u \in \Omega$, then the following propositions hold true:*

A: *$\{\Phi^j u\}_{n \in \mathbb{N}}$ converges to a fixed point u^* of Φ;*
B: *u^* is a unique fixed point of Φ in $\Omega^* = \{v \in \Omega : d(\Phi^* u, v) < \infty\}$;*
C: *if $v \in \Omega^*$, then $d(v, u^*) \leq \frac{1}{1-E} d(v, \Phi u)$.*

Let $Z = Z(I, \mathbb{R})$ be the metric space with the metric

$$d(u, v) = \sup_{t \in (a,b]} \frac{\left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\beta} |u(t) - v(t)|}{\Psi(t)}.$$

Theorem 3.4. *Assume that $[Q3]$ and the following assumption hold.*

[Q5] *There exists* $\phi \in C((a,b],[0,\infty))$ *such that for every* $u, v \in \mathbb{R}$ *and for each* $t \in (a,b]$*, we have*

$$\left| f(t,u) - f(t,v) \right| \leq \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\beta} \phi(t) \Psi(t) \left| u - v \right|.$$

If

$$E := \left(\frac{G^\rho - a^\rho}{\rho} \right)^{1-\beta} \phi^* \omega_\theta < 1,$$

where $\phi^* = \sup_{t \in (a,b]} \phi(t)$*, then there exists a unique solution* u_0 *of the IVP* (1.1)–(1.2) *and IVP* (1.1)–(1.2) *is generalized Ulam-Hyers-Rassias stable. Moreover,*

$$\left| u(t) - u_0(t) \right| \leq \frac{\Psi(t)}{1 - E}.$$

Proof. Let the operator $\Delta : C_{\beta,\rho} \to C_{\beta,\rho}$ be defined in (3.2). By using Theorem 3.3, we have

$$\begin{aligned}
\left| (\Delta u)(t) - (\Delta v(t)) \right| \leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} \left| f(x, u(x)) - f(x, v(x)) \right| dx \\
\leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} \phi(x) \Psi(x) \\
& \times \left| \left(\frac{x^\rho - a^\rho}{\rho} \right)^{1-\beta} u(x) - \left(\frac{x^\rho - a^\rho}{\rho} \right)^{1-\beta} v(x) \right| dx \\
\leq & \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} \phi^*(x) \Psi(x) \left\| u - v \right\|_{C_{1-\beta}} dx \\
\leq & \phi^* \left({}^\rho I_{a+}^{\mu} \right) \Psi(t) \left\| u - v \right\|_C \\
\leq & \phi^* \omega_\theta \Psi(t) \left\| u - v \right\|_C .
\end{aligned}$$

Hence,

$$\begin{aligned}
\left| \left(\frac{t^\rho - x^\rho}{\rho} \right)^{1-\beta} (\Delta u)(t) - \left(\frac{t^\rho - x^\rho}{\rho} \right)^{1-\beta} (\Delta v(t)) \right| \leq & \left(\frac{G^\rho - a^\rho}{\rho} \right)^{1-\beta} \phi^* \omega_\theta \\
& \times \Psi(t) \left\| u - v \right\|_C .
\end{aligned}$$

Thus, we have

$$d(\Delta u, \Delta v) = \sup_{t \in (a,b]} \frac{\left\| (\Delta u)(t) - (\Delta v(t)) \right\|_C}{\Psi(t)} \leq E \left\| u - v \right\|_C .$$

This completes the theorem. ☐

3.2. **Examples.**

Example 3.1. Consider the following IVP:

$$ {}^{\rho}D_{0^+}^{\mu,\nu}u(t) = \frac{|u(t)|}{50e^{t+5}(1+|u(t)|)}, \quad t\in(0,1], \tag{3.10} $$

$$ {}^{\rho}I_{0^+}^{1-\beta}u(0) = 5u\left(\frac{1}{2}\right)+3u\left(\frac{3}{4}\right), \quad \beta=\mu+\nu(1-\mu), \tag{3.11} $$

where $\mu=\frac{1}{2}$, $\nu=\frac{2}{3}$ and $\beta=\frac{5}{6}$. Set $f(t,u)=\frac{|u|}{50e^{t+5}(1+|u|)}$, $t\in(0,1]$.

It is obvious that the function f is continuous. For any $u,v\in\mathbb{R}$ and $t\in(0,1]$

$$ |f(t,u)-f(t,v)| \le \frac{1}{50e^5}|u-v|. $$

Thus, the condition $[Q1]$ of Theorem 3.1 is satisfied, with $J=\frac{1}{50e^5}$. Moreover, with some elementary computation for $\rho>0$ we have

$$ |K| = \left|\left\{\Gamma\left(\frac{5}{6}\right) - \left[5\left(\frac{(1/2)^{\rho}-0^{\rho}}{\rho}\right)^{-1/6} + 3\left(\frac{(3/4)^{\rho}-0^{\rho}}{\rho}\right)^{-1/6}\right]\right\}^{-1}\right| < 1 $$

and

$$ \begin{aligned} \sigma =& \frac{1}{50e^5}\cdot\frac{B(1/2,5/6)}{\Gamma(1/2)}\left\{|K|\left[5\left(\frac{(1/2)^{\rho}-0^{\rho}}{\rho}\right)^{1/3} + 3\left(\frac{(3/4)^{\rho}-0^{\rho}}{\rho}\right)^{1/3}\right]\right. \\ &\left. + \left(\frac{1^{\rho}-0^{\rho}}{\rho}\right)^{1/2}\right\} < 1. \end{aligned} $$

Hence, the condition $[Q2]$ of Theorem 3.1 is satisfied.

It follows, from Theorem 3.1, that the IVP (3.10)–(3.11) has at least one solution in $C_{1/6}[0,1]$.

Now, let $\Psi(t)=\frac{1}{t^{2\rho-4}}$ and $p(t)=\frac{1}{50e^{t+5}}$, then

$$ |f(t,u(t))| \leqslant \frac{1}{50e^{t+5}}\cdot\frac{1}{t^{2\rho-4}}\cdot\frac{|u(t)|}{(1+|u(t)|)}. $$

Thus, the condition $[Q4]$ of Theorem 3.2 is satisfied and with the obvious elementary computation, we have

$$ {}^{\rho}I_{0^+}^{\mu}\Psi(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_a^t \frac{x^{\rho-1}\Psi(x)}{(t^{\rho}-x^{\rho})^{1-\mu}}dx \le \frac{1}{\rho^{\mu}\Gamma(\mu)}B\left(\mu,\frac{4}{\rho}-1\right)\Psi(t) \le \omega_{\theta}\Psi(t). $$

Hence, the condition $[Q4]$ of Theorem 3.2 is satisfied with $\omega_{\theta}=\frac{1}{\rho^{\mu}\Gamma(\mu)}B\left(\mu,\frac{4}{\rho}-1\right)$. It follows from the Theorem 3.2 that the IVP (3.10)–(3.11) is generalized Ulam-Hyers-Rassias stable.

4. Conclusion

We have investigated the sufficient conditions for the existence of solutions to the nonlocal initial value problems involving generalized Katugampola derivative. We have used Krasnoselskii fixed point theorem to develop the existence results. Further, we established some conditions for the generalized Ulam-Hyers-Rassias stability corresponding to the considered problem. Finally, as an application, a suitable example is given to demonstrate our main results.

References

[1] S. Abbas, M. Benchohra, J. Lagreg, A. Alsaedi and Y. Zhou, *Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type*, Adv. Difference Equ. **180** (2017), DOI 10.1186/s13662–017–1231–1.

[2] B. Ahmad and S. K. Ntouyas, *Initial value problem of fractional order Hadamard-type functional differential equations*, Electron. J. Differential Equations **2015** (2015), 1–9.

[3] A. O. Akdemir, A. Ekinci and E. Set, *Conformable fractional integrals and related new integral inequalities*, J. Nonlinear Convex Anal. **18** (2017), 661–674.

[4] A. O. Akdemir, E. Set and A. Ekinci, *On new conformable fractional integral inequalities for product of different kinds of convexity*, TWMS Journal of Applied and Engineering Mathematics **9** (2019), 142–150.

[5] R. Almeida, A. B. Malinowska and T. Odzijewicz, *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, Journal of Computational and Nonlinear Dynamics **11** (2016), 11 pages.

[6] M. Benchohra and J. E. Lazreg, *On stability for nonlinear implicit fractional differential equations*, Matematiche (Catania) **70** (2014), 49–61.

[7] M. Benchohra and J. E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative*, Stud. Univ. Babes-Bolyai Math. **62** (2017), 27–38.

[8] D. B. Dhaigude and S. P. Bhairat, *Existence and uniqueness of solution of Cauchy type problem for Hilfer fractional differential equations*, Communications in Applied Analysis **22** (2018), 121–134.

[9] D. B. Dhaigude and S. P. Bhairat, *Existence and continuation of solutions of Hilfer fractional differential equations*, J. Math. Model. **7** (2019), 1–20.

[10] M. A. Dokuyucu, D. Baleanu and E. Celik, *Analysis of the fractional Keller-Segel model*, Filomat **32**(16) (2018), 5633–5643.

[11] M. A. Dokuyucu, E. Celik, H. Bulut and H. M. Baskonus, *Cancer treatment model with the Caputo-Fabrizio fractional derivative*, Eur. Phys. J. Plus **133** (2018), Paper ID 92, DOI 10.1140/epjp/i2018-11950-y.

[12] A. Ekinci and N. Eroğlu, *New generalizations for convex functions via conformable fractional integrals*, Filomat **33**(14) (2019), 4525–4534.

[13] A. Ekinci and M. E. Özdemir, *Some new integral inequalities via Riemann-Liouville integral operators*, Appl. Comput. Math. **3** (2019), 288–295.

[14] K. M. Furati, M. D. Kassim and N. e-. Tatar, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl. **64** (2012), 1616–1626.

[15] K. M. Furati, M. D. Kassim and N. e-. Tatar, *Non-existence of global solutions for a differential equations involving Hilfer fractional derivative*, Electron. J. Differential Equations **2013** (2013), 1–10.

[16] R. Hilfer, *Fractional Time Evolution: Applications in Fractional Calculus in Physics*, World Scientific, London, 2000.

[17] R. Hilfer, Y. Luckho and Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **12** (2009), 289–318.
[18] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
[19] M. D. Kassim, K. M. Furati and N. -E. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative*, Abstr. Appl. Anal. **2012** (2012), Article ID 391062, 17 pages.
[20] U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2011), 860–865.
[21] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014), 1–15.
[22] A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38** (2001), 1191–1204.
[23] C. Kou, H. Zhou and M. Medved, *Existence and continuation theorems of Riemann-Liouville type fractional differential equations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **22** (2011), Article ID 1250077, 12 pages.
[24] K. D. Kucche and S. T. Sutar, *Stability via successive approximation for nonlinear implicit fractional differential equations*, Moroccan Journal of Pure and Applied Analysis **3** (2017), 36–55.
[25] K. S. Miller and B. Ross, *A Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[26] I. Mumcu, E. Set and A. O. Akdemir, *Hermite-Hadamard type inequalities for harmonically convex functions via Katugampola fractional integrals*, Miskolc Math. Notes **20** (2019), 409–424.
[27] D. S. Oliveira and E. C. de Oliveira, *Hilfer-Katugampola fractional derivative*, Comput. Appl. Math. **37** (2018), 3672–3690.
[28] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1973.
[29] R. Subashini, K. Jothimani, S. Saranya and C. Ravichandran, *On the results of Hilfer fractional derivatives with nonlocal conditions*, International Journal of Pure and Applied Mathematics **118** (2018), 277–289.
[30] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1968.
[31] D. Vivek, K. Kanagrajan and E. M. Elsayed, *Nonlocal initial value problems for implicit differential equations with Hilfer-Hadamard fractional derivative*, Nonlinear Analysis: Modelling and Control **23** (2018), 341–360.
[32] J. Wang and Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput. **266** (2015), 850–859.
[33] J. Wang, Y. Zhou and M. Medved, *Existence and stability of fractional differential equations with Hadamard derivative*, Topol. Methods Nonlinear Anal. **41** (2013), 113–133.
[34] H. Yaldiz and A. O. Akdemir, *Katugampola fractional integrals within the class of convex functions*, Turkish Journal of Science **3** (2018), 40–50.

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AN OPERATIONAL APPROACH TO THE GENERALIZED RENCONTRES POLYNOMIALS

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Abstract. In this paper, we study the umbral operators J, M and N associated with the generalized rencontres polynomials $D_n^{(m)}(x)$. We obtain their representations in terms of the differential operator $\mathfrak{D}_x$ and the shift operator E. Then, by using these representations, we obtain some combinatorial and differential identities for the generalized rencontres polynomials. Finally, we extend these results to some related polynomials and, in particular, to the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$.

1. Introduction

1.1. **Generalized rencontres polynomials.** Let $m \in \mathbb{N}$. The generalized rencontres polynomials $D_n^{(m)}(x)$, the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$ are defined by (see [3, 5])

$$D_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} x^k,$$

$$P_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (m+n-k)! x^k,$$

$$A_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}^{(m)} x^k,$$

where the coefficients

$$d_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (m+k)!$$

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are the generalized derangement numbers (see [3, 5]), and the coefficients

$$a_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (m+k)!$$

are the generalized arrangement numbers. For $m = 0$, we have the ordinary derangement numbers d_n [4, page 182] and the ordinary arrangement numbers a_n [4, page 75]. Since

$$\begin{aligned} d^{(m)}(t) &= \sum_{n\geq 0} d_n^{(m)} \frac{t^n}{n!} = \frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, \\ p^{(m)}(t) &= \sum_{n\geq 0} (m+n)! \frac{t^n}{n!} = \frac{m!}{(1-t)^{m+1}}, \\ a^{(m)}(t) &= \sum_{n\geq 0} a_n^{(m)} \frac{t^n}{n!} = \frac{m!\mathrm{e}^{t}}{(1-t)^{m+1}}, \end{aligned}$$

then we have the exponential generating series

$$D^{(m)}(x;t) = \sum_{n\geq 0} D_n^{(m)}(x) \frac{t^n}{n!} = d^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}, \tag{1.1}$$

$$P^{(m)}(x;t) = \sum_{n\geq 0} P_n^{(m)}(x) \frac{t^n}{n!} = p^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{xt}}{(1-t)^{m+1}}, \tag{1.2}$$

$$A^{(m)}(x;t) = \sum_{n\geq 0} A_n^{(m)}(x) \frac{t^n}{n!} = a^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{(x+1)t}}{(1-t)^{m+1}}. \tag{1.3}$$

In particular, from these series, we also have

$$\begin{aligned} D_n^{(m)}(x) &= \sum_{k=0}^{n} \binom{n}{k} (m+k)!(x-1)^{n-k}, \\ A_n^{(m)}(x) &= \sum_{k=0}^{n} \binom{n}{k} (m+k)!(x+1)^{n-k}. \end{aligned} \tag{1.4}$$

Clearly, the polynomials $P_n^{(m)}(x)$ and $A_n^{(m)}(x)$ can be expressed in terms of the polynomials $D_n^{(m)}(x)$, namely

$$\begin{aligned} P_n^{(m)}(x) &= D_n^{(m)}(x+1), \\ A_n^{(m)}(x) &= D_n^{(m)}(x+2). \end{aligned}$$

1.2. **Sheffer sequences and umbral operators.** Given any polynomial sequence $\{p_n(x)\}_{n\geq 0}$, where each $p_n(x)$ is a polynomial with degree n, we can consider the linear operators $J, M, N : \mathbb{Q}[x] \to \mathbb{Q}[x]$ defined for every $n \in \mathbb{N}$, by

$$Jp_n(x) = np_{n-1}(x), \quad Mp_n(x) = p_{n+1}(x) \quad \text{and} \quad Np_n(x) = np_n(x),$$

where J is the umbral derivative (or lowering operator, or annihilation operator), M is the umbral shift (or raising operator or creation operator) and N is the umbral theta operator.

By Sheffer's theorem [11], every linear operator $L : \mathbb{Q}[x] \to \mathbb{Q}[x]$ can be represented by means of an exponential series in the derivative $\mathfrak{D}_x$ with respect to x. More precisely, there exists a unique polynomial sequence $\{L_n(x)\}_{n\geq 0}$, where $L_n(x) \in \mathbb{Q}[x]$ for every $n \in \mathbb{N}$, such that

$$Lp(x) = \sum_{k\geq 0} \frac{L_k(x)}{k!} \mathfrak{D}_x^k \, p(x) = \sum_{k=0}^{n} \frac{L_k(x)}{k!} p^{(k)}(x),$$

for every polynomial $p(x) \in \mathbb{Q}[x]$ of degree n. For instance, the shift operator E^λ, defined by $E^\lambda p(x) = p(x+\lambda)$, is represented by the exponential series $\mathrm{e}^{\lambda \mathfrak{D}_x}$.

A Sheffer sequence [2, 7–11] with spectrum $(g(t), f(t))$ is a polynomial sequence $\{s_n(x)\}_{n\geq 0}$ having exponential generating series

$$s(x;t) = \sum_{n\geq 0} s_n(x) \frac{t^n}{n!} = g(t)\mathrm{e}^{xf(t)},$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series, with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The umbral operators J, M and N associated with a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ with spectrum $(g(t), f(t))$ are given by [9, page 49, 50]

$$\begin{aligned} J &= \widehat{f}(\mathfrak{D}_x), \\ M &= \frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + xf'(\widehat{f}(\mathfrak{D}_x)), \\ N &= MJ = \left(\frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + xf'(\widehat{f}(\mathfrak{D}_x)) \right) \widehat{f}(\mathfrak{D}_x), \end{aligned}$$

where $\widehat{f}(t)$ is the compositional inverse of $f(t)$. In particular, for an Appell sequence [1, 7, 9, 10], i.e., a Sheffer sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ with spectrum $(g(t), t)$ (where $f(t) = \widehat{f}(t) = t$), we have

$$J = \mathfrak{D}_x, \tag{1.5}$$

$$M = \frac{g'(\mathfrak{D}_x)}{g(\mathfrak{D}_x)} + x, \tag{1.6}$$

$$N = MJ = \frac{g'(\mathfrak{D}_x)}{g(\mathfrak{D}_x)} \mathfrak{D}_x + x \mathfrak{D}_x \,. \tag{1.7}$$

By identity (1.5), we have $a_n'(x) = na_{n-1}(x)$ for every $n \in \mathbb{N}$.

By series (1.1), (1.2) and (1.3), the generalized rencontres polynomials $D_n^{(m)}(x)$, the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$ form an Appell sequence, respectively with spectrum

$$\left(\frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, t\right), \quad \left(\frac{m!}{(1-t)^{m+1}}, t\right) \quad \text{and} \quad \left(\frac{m!\mathrm{e}^{t}}{(1-t)^{m+1}}, t\right). \tag{1.8}$$

More generally, the shifted polynomials $D_n^{(m)}(x+\alpha)$ form an Appell sequence with spectrum

$$\left(\frac{m!\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}, t\right). \tag{1.9}$$

In this paper, we will determine the representation of the main umbral operators associated with the generalized rencontres polynomials and then, by using these representations, we obtain some combinatorial and differential identities for the generalized rencontres polynomials. Finally, we extend these results to the shifted polynomials $D_n^{(m)}(x+\alpha)$ and, in particular, to the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$.

2. Operators for the Generalized Rencontres Polynomials

Since the generalized rencontres polynomials form an Appell sequence, by identity (1.5), we have

$$\mathfrak{D}_x\, D_n^{(m)}(x) = nD_{n-1}^{(m)}(x), \quad \text{for all } n \in \mathbb{N},$$

and, more generally,

$$\mathfrak{D}_x^k\, D_n^{(m)}(x) = \binom{n}{k} k! D_{n-1}^{(m)}(x), \quad \text{for all } n, k \in \mathbb{N}.$$

For the second operator M, we have the following result.

Theorem 2.1. *The operator M is given by*

$$M = \frac{m + \mathfrak{D}_x}{1 - \mathfrak{D}_x} + x. \tag{2.1}$$

Proof. The operator M is given by formula (1.6). By the first spectrum in (1.8), we have

$$g(t) = \frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, \quad g'(t) = \frac{m+t}{1-t} g(t) \quad \text{and} \quad \frac{g'(t)}{g(t)} = \frac{m+t}{1-t}.$$

This implies at once formula (2.1). □

From this theorem, we can obtain the following recurrence (already obtained in [3, (10)] by using the exponential series techniques).

Theorem 2.2. *The generalized rencontres polynomials satisfy the recurrence*

$$D_{n+2}^{(m)}(x) = (x + m + n + 1) D_{n+1}^{(m)}(x) - (n+1)(x-1) D_n^{(m)}(x). \tag{2.2}$$

Proof. Since $MD_n^{(m)}(x) = D_{n+1}^{(m)}(x)$, by (2.1), we have
$$D_{n+1}^{(m)}(x) = \frac{m+\mathfrak{D}_x}{1-\mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x),$$
that is
$$(1-\mathfrak{D}_x)D_{n+1}^{(m)}(x) = (m+\mathfrak{D}_x)D_n^{(m)}(x) + (1-\mathfrak{D}_x)xD_n^{(m)}(x).$$
Hence, we have
$$D_{n+1}^{(m)}(x) - \mathfrak{D}_x\, D_{n+1}^{(m)}(x) = mD_n^{(m)}(x) + \mathfrak{D}_x\, D_n^{(m)}(x) + xD_n^{(m)}(x) - D_n^{(m)}(x) - x\,\mathfrak{D}_x\, D_n^{(m)}(x).$$
Now, since the generalized rencontres polynomials form an Appell sequence, we have
$$D_{n+1}^{(m)}(x) - (n+1)D_n^{(m)}(x) = mD_n^{(m)}(x) + nD_{n-1}^{(m)}(x) + xD_n^{(m)}(x) - D_n^{(m)}(x) - nxD_{n-1}^{(m)}(x),$$
that is
$$D_{n+1}^{(m)}(x) = (x+m+n)D_n^{(m)}(x) - n(x-1)D_{n-1}^{(m)}(x).$$
Replacing n by $n+1$, we obtain recurrence (2.2). □

In a similar way, Theorem 2.1 implies the following result.

Theorem 2.3. *The generalized rencontres polynomials satisfy the recurrence*

$$D_{n+1}^{(m)}(x) = (x-1)D_n^{(m)}(x) + (m+1)\sum_{k=0}^{n} \binom{n}{k} k!\, D_{n-k}^{(m)}(x). \tag{2.3}$$

Proof. Since $MD_n^{(m)}(x) = D_{n+1}^{(m)}(x)$, by formula (2.1) we have
$$\begin{aligned}
D_{n+1}^{(m)}(x) &= \frac{m+\mathfrak{D}_x}{1-\mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x)\\
&= m\frac{1}{1-\mathfrak{D}_x} D_n^{(m)}(x) + \frac{\mathfrak{D}_x}{1-\mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x)\\
&= m\sum_{k\geq 0} \mathfrak{D}_x^k\, D_n^{(m)}(x) + \sum_{k\geq 1} \mathfrak{D}_x^k\, D_n^{(m)}(x) + xD_n^{(m)}(x).
\end{aligned}$$
Since $D_n^{(m)}(x)$ is a polynomial of degree n, we have
$$\begin{aligned}
D_{n+1}^{(m)}(x) &= m\sum_{k=0}^{n} \mathfrak{D}_x^k\, D_n^{(m)}(x) + \sum_{k=1}^{n} \mathfrak{D}_x^k\, D_n^{(m)}(x) + xD_n^{(m)}(x)\\
&= (x+m)D_n^{(m)}(x) + (m+1)\sum_{k=1}^{n} \mathfrak{D}_x^k\, D_n^{(m)}(x)\\
&= (x+m)D_n^{(m)}(x) + (m+1)\sum_{k=0}^{n} \mathfrak{D}_x^k\, D_n^{(m)}(x) - (m+1)D_n^{(m)}(x)\\
&= (x-1)D_n^{(m)}(x) + (m+1)\sum_{k=0}^{n} \mathfrak{D}_x^k\, D_n^{(m)}(x).
\end{aligned}$$
Since the generalized rencontres polynomials form an Appell sequence, we have recurrence (2.3). □

Finally, as a direct consequence of (1.7) and Theorem 2.1, we have the following result for the operator N.

Theorem 2.4. *The operator N is given by*

$$N = \frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1-\mathfrak{D}_x} + x\,\mathfrak{D}_x\,. \tag{2.4}$$

Theorem 2.4 immediately implies the following differential equation.

Theorem 2.5. *The generalized rencontres polynomials satisfy the differential equation*

$$(x-1)D_n''(x) - (x+m+n-1)D_n'(x) + nD_n(x) = 0, \tag{2.5}$$

where, for simplicity, we write $D_n(x) = D_n^{(m)}(x)$.

Proof. Since $ND_n^{(m)}(x) = nD_n^{(m)}(x)$, by formula (2.4), we have

$$\frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1-\mathfrak{D}_x} D_n^{(m)}(x) + x\,\mathfrak{D}_x\,D_n^{(m)}(x) = nD_n^{(m)}(x),$$

that is

$$(m\,\mathfrak{D}_x + \mathfrak{D}_x^2)D_n^{(m)}(x) + (1-\mathfrak{D}_x)x\,\mathfrak{D}_x\,D_n^{(m)}(x) = n(1-\mathfrak{D}_x)D_n^{(m)}(x).$$

Hence, setting $D_n(x) = D_n^{(m)}(x)$, we have

$$(m\,\mathfrak{D}_x + \mathfrak{D}_x^2)D_n(x) + (1-\mathfrak{D}_x)xD_n'(x) = n(1-\mathfrak{D}_x)D_n(x)$$

or

$$mD_n'(x) + D_n''(x) + xD_n'(x) - D_n'(x) - xD_n''(x) = nD_n(x) - nD_n'(x).$$

This relation simplifies in the differential equation (2.5). ☐

Notice that, due to the fact that the generalized rencontres polynomials form an Appell sequence, the differential equation (2.5) is equivalent to recurrence (2.2).

Finally, we have the following theorem.

Theorem 2.6. *The generalized rencontres polynomials satisfy the identity*

$$(m+n+1)D_n^{(m)}(x) = (m+1)\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + n(x-1)D_{n-1}^{(m)}(x). \tag{2.6}$$

Proof. Since $ND_n^{(m)}(x) = nD_n^{(m)}(x)$, by (2.4), we have

$$\begin{aligned} nD_n^{(m)}(x) =& \frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1-\mathfrak{D}_x} D_n^{(m)}(x) + x\,\mathfrak{D}_x\,D_n^{(m)}(x) \\ =& m\sum_{k\geq 1}\mathfrak{D}_x^k\,D_n^{(m)}(x) + \sum_{k\geq 2}\mathfrak{D}_x^k\,D_n^{(m)}(x) + x\,\mathfrak{D}_x\,D_n^{(m)}(x) \\ =& m\sum_{k=1}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + \sum_{k=2}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + nxD_{n-1}^{(m)}(x) \\ =& m\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) - mD_n^{(m)}(x) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x)-D_{n}^{(m)}(x)-nD_{n-1}^{(m)}(x)+nxD_{n-1}^{(m)}(x)\\
=&(m+1)\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x)-(m+1)D_{n}^{(m)}(x)+n(x-1)D_{n-1}^{(m)}(x),
\end{aligned}
$$

and this simplifies in identity (2.6). □

Notice that identities (2.3) and (2.6) imply recurrence (2.2).

3. Rodrigues-Like Formulas

In this section, we find a Rodrigues-like formula for the generalized rencontres polynomials. We start by proving the following simple result, generalizing identity (1.4).

Lemma 3.1. *We have the identity*

$$D_{n}^{(m)}(x)=\sum_{k=0}^{n}\binom{n}{k}D_{k}^{(m)}(\alpha)(x-\alpha)^{n-k}. \tag{3.1}$$

Proof. By series (1.1), we have

$$D^{(m)}(x;t)=\frac{m!\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}=\frac{m!\mathrm{e}^{(x-\alpha)t}\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}=\frac{m!\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}\mathrm{e}^{(x-\alpha)t}$$

or

$$D^{(m)}(x;t)=D^{(m)}(\alpha;t)\mathrm{e}^{(x-\alpha)t}.$$

This identity is equivalent to identity (3.1). □

Remark 3.1. Since $D_n^{(m)}(0)=d_n^{(m)}$, $D_n^{(m)}(1)=(m+n)!$ and $D_n^{(m)}(2)=a_n^{(m)}$ for $\alpha=1$ identity (3.1) reduces to identity (1.4), while for $\alpha=0$ and $\alpha=2$ identity (3.1) becomes

$$
\begin{aligned}
D_{n}^{(m)}(x)&=\sum_{k=0}^{n}\binom{n}{k}d_{k}^{(m)}x^{n-k},\\
D_{n}^{(m)}(x)&=\sum_{k=0}^{n}\binom{n}{k}a_{k}^{(m)}(x-2)^{n-k}.
\end{aligned}
$$

Now we can prove the following result.

Theorem 3.1. *For the generalized rencontres polynomials we have the Rodrigues-like formula*

$$D_{n}^{(m)}(x)=D^{(m)}(\alpha,\mathfrak{D}_x)(x-\alpha)^{n}. \tag{3.2}$$

In particular, we have

$$D_{n}^{(m)}(x)=d^{(m)}(\mathfrak{D}_x)x^{n}, \tag{3.3}$$

$$D_{n}^{(m)}(x)=p^{(m)}(\mathfrak{D}_x)(x-1)^{n}, \tag{3.4}$$

$$D_{n}^{(m)}(x)=a^{(m)}(\mathfrak{D}_x)(x-2)^{n}. \tag{3.5}$$

Proof. From identity (3.1), we have

$$D_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} D_k^{(m)}(\alpha)(x-\alpha)^{n-k} = \sum_{k\geq 0} \frac{D_k^{(m)}(\alpha)}{k!} \mathfrak{D}_x^k (x-\alpha)^n$$
$$= \left(\sum_{k\geq 0} D_k^{(m)}(\alpha) \frac{\mathfrak{D}_x^k}{k!} \right) (x-\alpha)^n = D^{(m)}(\alpha; \mathfrak{D}_x)(x-\alpha)^n.$$

This is (3.2). Then, (3.3), (3.4) and (3.5) can be obtained for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, respectively. □

4. Final Remarks

As already noted in the Introduction, the shifted polynomials $D_n^{(m)}(x+\alpha)$ form an Appell sequence with spectrum (1.9). From this simple observation, it is easy to see that the associated umbral operators J_α, M_α and N_α are given by

$$J_\alpha = J, \quad M_\alpha = M + \alpha \quad \text{and} \quad N_\alpha = N + \alpha \, \mathfrak{D}_x \, .$$

All the properties obtained for the generalized rencontres polynomials can be reformulated for the shifted polynomials $D_n^{(m)}(x+\alpha)$, and, in particular, for the polynomials $P_n^{(m)}(x)$ and $A_n^{(m)}(x)$. For instance, from recurrence (2.2), we obtain the recurrences

$$P_{n+2}^{(m)}(x) = (x+m+n+2)P_{n+1}^{(m)}(x) - (n+1)xP_n^{(m)}(x),$$
$$A_{n+2}^{(m)}(x) = (x+m+n+3)A_{n+1}^{(m)}(x) - (n+1)(x+1)A_n^{(m)}(x)$$

and from differential equation (2.5), we obtain the differential equations

$$xP_n''(x) - (x+m+n)P_n'(x) + nP_n(x) = 0,$$
$$(x+1)A_n''(x) - (x+m+n+1)A_n'(x) + nA_n(x) = 0,$$

where, always for simplicity, we write $P_n(x) = P_n^{(m)}(x)$ and $A_n(x) = A_n^{(m)}(x)$. Similarly, from recurrence (2.3), we obtain the recurrences

$$P_{n+1}^{(m)}(x) = xP_n^{(m)}(x) + (m+1) \sum_{k=0}^{n} \binom{n}{k} k! P_{n-k}^{(m)}(x),$$
$$A_{n+1}^{(m)}(x) = (x+1)A_n^{(m)}(x) + (m+1) \sum_{k=0}^{n} \binom{n}{k} k! A_{n-k}^{(m)}(x),$$

and, from identity (2.6), we obtain the identities

$$(m+n+1)P_n^{(m)}(x) = (m+1) \sum_{k=0}^{n} \binom{n}{k} k! P_{n-k}^{(m)}(x) + nxP_{n-1}^{(m)}(x),$$
$$(m+n+1)A_n^{(m)}(x) = (m+1) \sum_{k=0}^{n} \binom{n}{k} k! A_{n-k}^{(m)}(x) + n(x+1)A_{n-1}^{(m)}(x).$$

Finally, from identity (3.1), we obtain the identities

$$P_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} P_k^{(m)}(\alpha)(x-\alpha)^{n-k},$$
$$A_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(m)}(\alpha)(x-\alpha)^{n-k}.$$

and, consequently, we have the Rodrigues-like formulas

$$P_n^{(m)}(x) = P^{(m)}(\alpha, \mathfrak{D}_x)(x-\alpha)^n,$$
$$A_n^{(m)}(x) = A^{(m)}(\alpha, \mathfrak{D}_x)(x-\alpha)^n.$$

As a concluding remark, notice that in [6, 7] we have considered a slight variant of the polynomials considered in this paper, namely the polynomials $D_n^{(\nu)}(x)$ and $A_n^{(\nu)}(x)$ defined by the exponential generating series

$$D^{(\nu)}(x;t) = \sum_{n\geq 0} D_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{v+1}},$$
$$A^{(\nu)}(x;t) = \sum_{n\geq 0} A_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{(x+1)t}}{(1-t)^{\nu+1}},$$

where ν is an arbitrary symbol. Also these polynomials form Appell sequences and the umbral operators J, M and N are the same, except for the fact that m is replaced by ν.

References

[1] P. Appell, *Sur une classe de polynomes*, Ann. Sci. Éc. Norm. Supér. (2) **9** (1880), 119–144.
[2] R. P. Boas and R. C. Buck, *Polynomial Expansions of Analytic Functions*, Academic Press, New York, 1964.
[3] S. Capparelli, M. M. Ferrari, E. Munarini and N. Zagaglia Salvi, *A generalization of the "Problème des Rencontres"*, J. Integer Seq. **21** (2018), Article ID 18.2.8.
[4] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht-Holland, Boston, 1974.
[5] M. M. Ferrari and E. Munarini, *Decomposition of some Hankel matrices generated by the generalized rencontres polynomials*, Linear Algebra Appl. **567** (2019), 180–201.
[6] E. Munarini, *Callan-like identities*, Online J. Anal. Comb. **14** (2019), Article ID 06, 20 pages.
[7] E. Munarini, *Combinatorial identities for Appell polynomials*, Appl. Anal. Discrete Math. **12** (2018), 362–388.
[8] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.
[9] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.
[10] S. M. Roman and G. -C. Rota, *The umbral calculus*, Adv. Math. **27** (1978), 95–188.
[11] I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke Math. J. **5** (1939), 590–622.

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ON THE LIE CENTRALIZERS OF QUATERNION RINGS

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Abstract. In this paper, we investigate the problem of describing the form of Lie centralizers on quaternion rings. We provide some conditions under which a Lie centralizer on a quaternion ring is the sum of a centralizer and a center valued map.

1. Introduction and Preliminaries

Let R be a ring with the center $Z(R)$. For $a, b \in R$ denote the Lie product of a, b by $[a, b] = ab - ba$ and the Jordan product of a, b by $a \circ b = ab + ba$. Let $\phi : R \to R$ be an additive map. Recall that ϕ is said to be a right (left) centralizer map if $\phi(ab) = a\phi(b)(\phi(ab) = \phi(a)b)$ for all $a, b \in R$. It is called a centralizer if ϕ is both a right centralizer and a left centralizer. We say that ϕ is a Jordan centralizer if $\phi(a \circ b) = a \circ \phi(b)$ for all $a, b \in R$. An additive map $\phi : R \to R$ is called a Lie centralizer if

$$\phi[a, b] = [\phi(a), b] \quad (\text{or } \phi[a, b] = [a, \phi(b)]),$$

for each $a, b \in R$. We say that $\phi : R \to R$ is a center valued map if $\phi(R) \subseteq Z(R)$.

In the recently years, the structure of Lie centralizers on rings has been studied by some authors. An important question that naturally arises in this setting is under what conditions on a quaternion ring, a Lie centralizer can be decomposed into the sum of a centralizer and a center valued map. Jing [9] was the first one who introduced the concept of Lie centralizer and showed that every Lie centralizer on some triangular algebras is the sum of a centralizer and a center valued map. The authors [6] proved that a Lie centralizer under some conditions on some trivial extention algebras is the sum of a centralizer and a center valued map. Fošner and Jing [3] studied this result on triangular rings and nest algebras.

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Let S be a ring with identity. Set

$$H(S) = \{s_0 + s_1 i + s_2 j + s_3 k : s_i \in S\} = S \oplus Si \oplus Sj \oplus Sk,$$

where $i^2 = j^2 = k^2 = ijk = -1$ and $ij = -ji$. Then, with the componentwise addition and multiplication subject to the given relations and the conventions that i, j, k commute with S elementwise, $H(S)$ is a ring called the quaternion ring over S.

In this paper, we suppose that S be an unital ring in which 2 is invertible. We describe the Lie centralizers on $H(S)$, we show that if S is commutative or semiprime, then every Lie centralizer on $H(S)$ decomposes into the sum of a centralizer and a center valued map. Among the reasons for studying the mappings on quternion rings, we cite the recently published books and papers ([1,2,8]), in which the authors have considered the important roles of quaternion algebras in other branches of mathematics, such as differential geometry, analysis and quantum fields.

2. Lie Centralizers of Quaternion Rings

Our aim is to study a Lie centralizer map on a quaternion ring. We give conditions under which it is a sum of a centralizer and a center valued map. In the following, we establish a theorem which will be used to prove the fundamental results. From now on, we assume that S is a 2-torsion free ring with identity such that $\frac{1}{2} \in S$ and $R = H(S)$.

Theorem 2.1. *Let $f : R \to R$ be a Lie centralizer. Then there exists a Lie centralizer α on S and a Jordan centralizer β on S such that $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k$ for every element $t = x + yi + zj + wk \in R$.*

Proof. Assume that $f(i) = a + bi + cj + dk$ and $f(j) = a' + b'i + c'j + d'k$ for some suitable coefficients in S. Since f is a Lie centralizer, we have

$$f(k) = \frac{1}{2} f[i, j] = \frac{1}{2}[f(i), j] = bk - di.$$

Furthermore,

$$a + bi + cj + dk = f(i) = \frac{1}{2} f[j, k] = \frac{1}{2}[f(j), k] = -b'j + c'i.$$

Therefore, we get $a = d = 0$, $b' = -c$ and $c' = b$. Hence, $f(i) = bi + cj$ and $f(k) = bk$. Since f is a Lie centralizer, we have

$$f(j) = \frac{1}{2} f[k, i] = \frac{1}{2}[f(k), i] = bj.$$

After renaming the constants, we obtain

$$f(i) = ai + bj, \quad f(j) = aj, \quad f(k) = ak, \tag{2.1}$$

for suitable $a, b, c \in S$. Now, assume that $f(1) = t = x + yi + zj + wk$. We have

$$0 = f[1, i] = ti - it = 2wj - 2zk.$$

Thus, $w = z = 0$. On the other hand, we have

$$0 = f[1, j] = tj - jt = 2yk - 2wi.$$

Hence, $y = w = 0$. Therefore, we have $f(1) = x \in S$. Let $s \in S$, we have

$$0 = f[1, si] = (xs - sx)i.$$

Therefore, we get $xs = sx$. Hence, $f(1) \in Z(S)$. Let $s \in S$ and set $f(si) = x + yi + zj + wk$. Applying f on $[si, i] = 0$, we get $w = z = 0$ and hence $f(si) = x + yi$. Now, applying f on the identities $sk = \frac{1}{2}[si, j]$, $sj = \frac{1}{2}[sk, i]$ and $si = \frac{1}{2}[sj, k]$, and putting $y = \beta(s)$, we obtain

$$f(si) = \beta(s)i, \quad f(sj) = \beta(s)j, \quad f(sk) = \beta(s)k, \tag{2.2}$$

where $\beta : S \to S$ is an additive map uniquely determined by f.

Our next aim is to find $f(s)$ for arbitrary $s \in S$. Set $f(s) = x + yi + zj + wk$. Applying f on $[s, i] = 0$, we obtain $-2zk + 2wj = 0$. So, $z = w = 0$. Now, applying f on $[s, j] = 0$, we obtain that $y = 0$. Therefore, we have $f(s) = x$. Putting $x = \alpha(s)$, we have

$$f(s) = \alpha(s), \tag{2.3}$$

where $\alpha : S \to S$ is a map determined by f. Since f is a Lie centralizer, (2.3) implies that α is a Lie centralizer on S.

Let $s_1, s_2 \in S$. It is obvious that $[s_1 i, s_2 j] = (s_1 \circ s_2)k$, $[s_1 i, s_2 i] = [s_2, s_1]$ and $[s_1, s_2 i] = [s_1, s_2]i$. Now, applying f on this identities and using (2.2) and (2.3), we find, respectively, that

$$\beta(s_1 \circ s_2) = \beta(s_1) \circ s_2, \tag{2.4}$$

$$\alpha[s_1, s_2] = [\beta(s_1), s_2], \tag{2.5}$$

$$\beta[s_1, s_2] = [\alpha(s_1), s_2]. \tag{2.6}$$

(2.4) shows that β is a Jordan centralizer on S. Now, let $t = x + yi + zj + wk$ be an arbitrary element in R. By (2.2) and (2.3), we get $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k$, as desired. $\square$

As a consequence of Theorem 2.1, we have the following results.

Corollary 2.1. *Let S be a 2-torsion free commutative ring with identity such that $\frac{1}{2} \in S$. If $f : H(S) \to H(S)$ be a Lie centralizer, then f is the sum of a centralizer and a center valued map.*

Proof. Since S is 2-torsion free and commutative, the Jordan centralizer β is a centralizer on S. Let $t = x + yi + zj + wk \in H(S)$. Define $\Gamma : H(S) \to H(S)$ by $\Gamma(t) = \beta(x) + \beta(y)i + \beta(z)j + \beta(w)k$. It is easily verified that Γ is a centralizer. By Theorem 2.1, we have $f(t) = \Gamma(t) + \alpha(x) - \beta(x)$. It remains to show that the mapping $\tau : H(S) \to H(S)$ given by $\tau(t) = \alpha(x) - \beta(x)$ is a center valued map. Obviously, τ is a well-defined additive map such that $\tau(H(S)) \subseteq S$. By [4, Lemma 2.1], we have $Z(H(S)) = S$. Therefore, we have $\tau(H(S)) \subseteq Z(H(S))$. This completes the proof. □

Corollary 2.2. *Let S be a 2-torsion free semiprime ring with identity such that $\frac{1}{2} \in S$. If $f : H(S) \to H(S)$ be a Lie centralizer, then f is the sum of a centralizer and a center valued map.*

Proof. Since S is a 2-torsion free semiprime ring, the Jordan centralizer β is a centralizer on S by [10]. Now, let Γ and τ be the mappings defined in Corollary 2.1. It is easily verified that Γ is a centralizer. It remains to show that the mapping τ is a center valued map. Let $s_1, s_2 \in S$. Since β is a centralizer on S, from (2.6), we obtain

$$[\tau(s_1), s_2] = [\alpha(s_1) - \beta(s_1), s_2] = 0. \tag{2.7}$$

Let $t = x + yi + zj + wk$, $t' = x' + y'i + z'j + w'k \in H(S)$. Using (2.7), we have

$$\begin{aligned} [\tau(t), t'] =& [\alpha(x) - \beta(x), t'] \\ =& [\tau(x), x'] + [\tau(x), y']i + [\tau(x), z']j + [\tau(x), w']k \\ =& 0. \end{aligned}$$

Therefore, we have $\tau(H(S)) \subseteq Z(H(S))$. This completes the proof. □

References

[1] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields,* Oxford University Press, New York, 1995.

[2] O. P. Agrawal, *Hamilton operators and dual-number-quaternions in spatial kinematics*, Mechanism and Machine Theory **22**(6) (1987), 569–575.

[3] A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras,* Adv. Oper. Theory **4** (2019), 342–350.

[4] H. Ghahramani, M. N. Ghosseiri and L. Heidari Zade, *On the Lie derivations and generalized Lie derivations of quaternion rings,* Comm. Algebra **47**(3) (2019), 1215–1221.

[5] H. Ghahramani, M. N. Ghosseiri and S. Safari, *Some questions concerning superderivations on $\mathbb{Z}_2$-graded rings,* Aequationes Math. **91**(4) (2017), 725–738.

[6] F. Ghomanjani and M. A. Bahmani, *A note on Lie centralizer maps,* Palest. J. Math. **7**(2) (2018), 468–471.

[7] B. Hvala, *Generalized Lie derivations in prime rings,* Taiwanese J. Math. **11**(5) (2007), 1425–1430.

[8] M. Jafari and Y. Yayli, *Generalized quaternions and their algebraic properties,* Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **64**(1) (2015), 15–27.

[9] W. Jing, *Additivity of Lie centralizers on triangular rings,* Math and Computer Science Working Papers (2011), 1–10.

[10] B. Zalar, *On centralizers of semiprime rings,* Comment. Math. Univ. Carolin. **32** (1991), 609–614.

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BASES OF THE PERTURBED SYSTEM OF EXPONENTS IN WEIGHTED LEBESGUE SPACE WITH A GENERAL WEIGHT

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Abstract. The weighted Lebesgue and Hardy spaces with a general weight are considered. Basicity of a part of exponential system is proved in Hardy classes, where the weight satisfies the Muckenhoupt condition. Using these results the basicity of the perturbed system of exponents in the weighted Lebesgue space is studied. Some special cases are considered.

1. Introduction

When solving many problems for equations of mixed type by Fourier method (see e.g., [11–13, 18]), there appear perturbed systems of sines and cosines of the following form

$$\{\sin(nt + \alpha(t))\}_{n\in\mathbb{N}}, \quad \{\cos(nt + \alpha(t))\}_{n\in\mathbb{N}}, \tag{1.1}$$

where $\alpha : [0, \pi] \to \mathbb{R}$ is some real function. Verification of the Fourier method requires to study basis properties (completeness, minimality, basicity and etc.) of system (1.1) in the appropriate spaces of functions (usually in Lebesgue or Sobolev spaces). In Lebesgue space $L_p \equiv L_p(0, \pi)$, $1 < p < +\infty$, these problems have been well studied for a wide class of functions $\alpha(\cdot)$ in [1–6, 14, 15, 21, 22]. Basis properties of system (1.1) are closely related to the analogous properties of the system of exponents of the form

$$\left\{e^{i(nt+\beta(t)\,\mathrm{sgn}\,n)}\right\}_{n\in\mathbb{Z}}, \tag{1.2}$$

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where $\beta : [-\pi, \pi] \to \mathbb{R}$ is some function. These problems with respect to the systems (1.1), (1.2) in weighted Lebesgue spaces with a power weight have been studied in [9, 16, 17, 20].

In this work, the basicity of system (1.2) is studied in weighted Lebesgue space $L_{p,\nu} \equiv L_{p,\nu}(-\pi, \pi)$, with a general weight $\nu(\cdot)$. For the basicity of this system in $L_{p,\nu}$ the sufficient conditions on the function $\beta(\cdot)$ and weight $\nu(\cdot)$ are obtained. For this, firstly, the weighted Hardy classes $H^{\pm}_{p,d\rho}$ are defined, the basicity of a part of exponential system is studied in these classes, and these results are applied to the basicity of the system (1.2) in $L_{p,\nu}$.

2. Necessary Facts

Let $\mathbb{C}$ be the complex plane and $\omega = \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle on $\mathbb{C}$. The expression $f \sim g$, in M, means that the following inequality is true

$$\text{exists } \delta > 0 : \quad \delta \le \frac{f(x)}{g(x)} \le \delta^{-1} \quad \text{for all } x \in M.$$

Let us consider the basicity of the following parts of exponential system

$$\left\{e^{int}\right\}_{n\in\mathbb{Z}_+}, \tag{2.1}$$

$$\left\{e^{-int}\right\}_{n\ge m}, \tag{2.2}$$

in weighted spaces $H^+_{p,d\rho}$ and ${}_mH^-_{p,d\rho}$, respectively. These facts are needed in the study of basicity of the perturbed system of exponents with a phase in weighted Lebesgue spaces $L_{p,d\rho}$. It should be noted that for the basicity of the exponential system $\{e^{int}\}_{n\in\mathbb{Z}}$ in $L_{p,d\rho}$ it is necessary that the weight function $\rho(\cdot)$ be absolutely continuous on $[-\pi, \pi]$. Indeed, in the case of basicity the following conditions (biorthogonality condition)

$$\int_{-\pi}^{\pi} e^{int} d\rho = 0, \quad \text{for all } n \ge 1,$$

should be fulfilled. And these conditions imply that (see, e.g., I. I. Privalov [19] or I. I. Danilyuk [7]) $\rho(\cdot)$ is absolutely continuous on $[-\pi, \pi]$, and let

$$\nu(t) = \rho'(t), \quad t \in (-\pi, \pi).$$

Therefore, in the sequel, we will consider the weighted Lebesgue space $L_{p,\nu} \equiv L_{p,\nu}(-\pi, \pi)$, with a norm

$$\|f\|_{p,\nu} = \left(\int_{-\pi}^{\pi} |f(t)|^p \nu(t)\, dt\right)^{\frac{1}{p}}, \quad \text{for all } f \in L_{p,\nu}.$$

Based on this norm, weighted Hardy classes $H^+_{p,\nu}$:

$$H^+_{p,\nu} \equiv \left\{f \in H^+_1 : f^+ \in L_{p;\nu}\right\},$$

are defined, endowed with the norm

$$\|f\|_{H^+_{p,\nu}} = \left\|f^+\right\|_{p,\nu},$$

where $f^+ = f/_{\partial\omega}$ is a restriction of the function f on $\partial\omega$.

Similarly we define the weighted Hardy class ${}_mH^-_{p,\nu}$ of functions which are analytic outside the unit circle ω. Let ${}_mH^-_p$ be a usual Hardy class of functions that are analytic outside the unit circle ω and have a pole of order $k \le m$ at infinity. Assume

$$ {}_mH^-_{p,\nu} \equiv \left\{ f \in_m H^-_1 : f^- \in L_{p,\nu} \right\}. $$

The norm is defined by the expression

$$ \|f\|_{{}_mH^-_{p,\nu}} = \left\| f^- \right\|_{p,\nu}, $$

where $f^- = f/_{\partial\omega}$ is the restriction of function $f \in_m H^-_{p,\nu}$ on $\partial\omega$.

3. Basicity of System of Exponents in $L_{p,\nu}$

Consider the weighted space $L_{p,\nu}$, $1 < p < +\infty$, where $\nu(\cdot)$ is some weight. Assume that $\nu(\cdot)$ satisfies Muckenhoupt condition (see, e.g., [8])

$$ \text{(3.1)} \qquad \sup_{I \subset [-\pi,\pi]} \left(\frac{1}{|I|} \int_I \nu(t)\, dt \right) \left(\frac{1}{|I|} \int_I |\nu(t)|^{-\frac{1}{p-1}}\, dt \right)^{p-1} < +\infty. $$

Since $\nu \in L_1$, it is clear that $\{e^{int}\}_{n\in\mathbb{Z}} \subset L_{p,\nu}$. Consider the functionals $\{\vartheta_n\}_{n\in\mathbb{Z}}$

$$ \vartheta_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \quad \text{for all } n \in \mathbb{Z}. $$

We have

$$ \text{(3.2)} \quad |\vartheta_n(f)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(t)\, \nu^{\frac{1}{p}}(t)\, e^{-int} \nu^{-\frac{1}{p}}(t)\, dt \right| \le \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t)\, dt \right)^{\frac{1}{q}} \|f\|_{p,\nu}, $$

where q is the conjugate of a number p, $\frac{1}{p} + \frac{1}{q} = 1$. From the condition (3.1) it directly follows

$$ \int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t)\, dt < +\infty. $$

Then from the inequality (3.2) we obtain that the functionals ϑ_n are continuous in $L_{p,\nu}$, for all $n \in \mathbb{Z}$, and moreover $\vartheta_n\left(e^{ikt}\right) = \delta_{nk}$ for all $n, k \in \mathbb{Z}$. As a result the system $\{e^{int}\}_{n\in\mathbb{Z}}$ is minimal in $L_{p,\nu}$. Consider the partial sums

$$ S_m(f) = \sum_{n=-m}^{m} \vartheta_n(f)\, e^{int}, \quad f \in L_{p,\nu}. $$

We have

$$ S_m(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, D_m(x-t)\, dt, \quad m \in \mathbb{N}, $$

where $D_m(\cdot)$ denotes the Dirichlet kernel

$$ D_m(t) = \frac{\sin\left(m + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}, \quad m \in \mathbb{N}. $$

As it known (see, e.g., Dj. Garnett [8]) if $\nu \in A_p$, then the Hilbert transformation is bounded in $L_{p,\nu}$. Hence, it directly follows that

$$\|S_m(f)\|_{p,\nu} \leq M\|f\|_{p,\nu}, \quad \text{for all } m \in \mathbb{N},$$

holds, where $M > 0$ is an absolute constant. As a result, it follows from the basis criterion that the system $\{e^{int}\}_{n\in\mathbb{Z}}$ forms a basis for $L_{p,\nu}$. It is easy to see that for $p = 2$ it forms a Riesz basis in $L_{2,\nu}$ if and only if $\nu \sim 1$.

Statement 1. Let $\nu \in A_p$. Then the system of exponents $\{e^{int}\}_{n\in\mathbb{Z}}$ forms a basis (for $p = 2$ it forms a Riesz basis if and only if $\nu \sim 1$) in $L_{p,\nu}$.

In the case $p = 2$ this fact has been previously established by K. S. Lizorkin, P. I. Ghazarian [10].

Take an arbitrary $f \in H^+_{p,\nu}$ and let $\nu \in A_p$. As $f \in H^+_1$, it is clear that

$$\int_{-\pi}^{\pi} f^+\left(e^{it}\right)e^{int}dt = 0, \quad \text{for all } n \in \mathbb{N},$$

holds. Then by Statement 1 the function f^+ has the following representation in $L_{p,\nu}$

$$f^+\left(e^{it}\right) = \sum_{n=0}^{\infty} f_n e^{int},$$

where

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^+\left(e^{it}\right)e^{-int}dt, \quad \text{for all } n \in \mathbb{Z}_+.$$

Consider the functionals $\{H^+_n\}_{n\in\mathbb{Z}_+}$

$$H^+_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^+\left(e^{it}\right)e^{-int}dt, \quad n \in \mathbb{Z}_+.$$

Following (3.2), we have

$$\left|H^+_n(f)\right| \leq \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\nu^{-\frac{q}{p}}(t)\,dt\right)^{\frac{1}{q}}\left\|f^+\right\|_{p,\nu} = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\nu^{-\frac{q}{p}}(t)\,dt\right)^{\frac{1}{q}}\|f\|_{H^+_{p,\nu}},$$

for all $f \in H^+_{p,\nu}$. This implies the inclusion $\{H^+_n\}_{n\in\mathbb{Z}_+} \subset \left(H^+_{p,\nu}\right)^*$ and moreover it is evident that

$$H^+_n\left(z^k\right) = \delta_{nk}, \quad \text{for all } n, k \in \mathbb{Z}_+,$$

i.e., the system $\{z^n\}_{n\in\mathbb{Z}_+}$ is minimal in $H^+_{p,\nu}$. It is absolutely clear that the expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

is true in $H^+_{p,\nu}$. As a result, we obtain the basicity of the system $\{z^n\}_{n\in\mathbb{Z}_+}$ in $H^+_{p,\nu}$.

Restrictions of class $H^+_{p,\nu}$ $({}_mH^-_{p,\nu})$ to the unit circle $\partial\omega$ will be denoted by $L^+_{p,\nu}$ $({}_mL^-_{p,\nu})$. It is easy to see that the system $\{e^{int}\}_{n\in\mathbb{Z}_+}$ forms a basis for $L^+_{p,\nu}$.

Similarly, we prove the basicity of the system $\{z^{-n}\}_{n\geq m}$ $(\{e^{-int}\}_{n\geq m})$ in ${}_mH^-_{p,\nu}$ $({}_mL^-_{p,\nu})$. Thus, the following theorem holds true.

Theorem 3.1. *Let $\nu \in A_p$, $1 < p < +\infty$. Then*
i) the system $\{z^n\}_{n\in\mathbb{Z}_+}$ ($\{e^{int}\}_{n\in\mathbb{Z}_+}$) forms a basis for $H^+_{p,\nu}$ (for $L^+_{p,\nu}$);
ii) the system $\{z^{-n}\}_{n\geq m}$ ($\{e^{-int}\}_{n\geq m}$) forms a basis for ${}_mH^-_{p,\nu}$ (${}_mL^-_{p,\nu}$).
For $p = 2$ these bases are Riesz bases if and only if $\nu \sim 1$ on $[-\pi, \pi]$.

4. Bases from the Perturbed System of Exponents in $L_{p,\nu}$

Consider the following system of exponents

$$\left\{e^{i\left(nt-\frac{1}{2}\theta(t)\,\mathrm{sgn}\,n\right)}\right\}_{n\in\mathbb{Z}}, \tag{4.1}$$

where $\theta(\cdot)$ is a piecewise Hölder function on $[-\pi, \pi]$. Let $\{s_k\}_1^r : -\pi < s_1 < \cdots < s_r < \pi$ be the points of discontinuity of the function $\theta(\cdot)$ and

$$h_k = \theta(s_k + 0) - \theta(s_k - 0), \quad k = \overline{1, r},$$

be the corresponding jumps of $\theta(\cdot)$ at these points. Assume $h_0 = \theta(-\pi) - \theta(\pi)$. By $\omega(\cdot)$ denote the following weight function

$$\omega(t) \equiv \left|\sin\frac{t-\pi}{2}\right|^{\frac{h_0}{2\pi}} \prod_{k=1}^{r} \left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}.$$

Consider the following non-homogeneous Riemann problem in classes $H^+_{p,\nu} \times_{-1} H^-_{p,\nu}$

$$e^{-i\frac{1}{2}\theta(t)}F^+\left(e^{it}\right) - e^{i\frac{1}{2}\theta(t)}F^-\left(e^{it}\right) = f(t), \quad t \in (-\pi, \pi), \tag{4.2}$$

where $f \in L_{p,\nu}$ is an arbitrary function. Suppose that the following conditions hold

$$\omega^{-p}\nu \in A_p, \quad \frac{h_k}{2\pi} < 1, \quad k = \overline{0, r}. \tag{4.3}$$

Then the problem (4.2) has a unique solution of the form

$$F(z) = \frac{Z(z)}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} \cdot \frac{dt}{1 - ze^{-it}}, \tag{4.4}$$

where $Z(\cdot)$ is the canonical solution of the corresponding homogeneous problem

$$e^{-i\frac{1}{2}\theta(t)}Z^+\left(e^{it}\right) - e^{i\frac{1}{2}\theta(t)}Z^-\left(e^{it}\right) = 0, \quad t \in (-\pi, \pi),$$

which is defined by the following expressions

$$Z(z) \equiv \begin{cases} X(z), & |z| < 1, \\ X^{-1}(z), & |z| > 1, \end{cases}$$

$$X(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t)\,\frac{e^{it} + z}{e^{it} - z}dt\right\}.$$

Applying Sokhotski-Plemelj formulas to (4.4) we get

$$\begin{aligned}F^{+}\left(e^{it}\right) =&\frac{1}{2}f\left(t\right)-\frac{Z^{+}\left(e^{it}\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f\left(s\right)}{Z^{+}\left(e^{is}\right)}\cdot\frac{ds}{1-e^{i(t-s)}}\equiv\left(S^{+}f\right)\left(t\right),\\ F^{-}\left(e^{it}\right) =&-\frac{1}{2}\cdot\frac{Z^{-}\left(e^{it}\right)}{Z^{+}\left(e^{it}\right)}f\left(t\right)-\frac{Z^{-}\left(e^{it}\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f\left(s\right)}{Z^{+}\left(e^{is}\right)}\cdot\frac{ds}{1-e^{i(t-s)}}\\ =&-\frac{e^{-i\theta(t)}}{2}f\left(t\right)-\frac{Z^{-}\left(e^{it}\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f\left(s\right)}{Z^{+}\left(e^{is}\right)}\cdot\frac{ds}{1-e^{i(t-s)}}\equiv\left(S^{-}f\right)\left(t\right).\end{aligned}$$

Let us assume that $\nu\in A_p$. Then, by Theorem 3.1, the system $\{e^{int}\}_{n\in\mathbb{Z}_+}$ ($\{e^{-int}\}_{n\in\mathbb{N}}$) forms a basis for $L^{+}_{p,\nu}$ (for ${}_{-1}L^{-}_{p,\nu}$). As

$$\left|Z\left(0\right)\right|^{\pm1}<+\infty,\quad\left|Z\left(\infty\right)\right|^{\pm1}<+\infty,$$

it is clear that the inclusions

$$F^{+}\left(e^{it}\right)\in L^{+}_{p,\nu},\quad F^{-}\left(e^{it}\right)\in_{-1}L^{-}_{p,\nu},$$

hold. By $\{H_n^-\}_{n\in\mathbb{N}}\subset\left({}_{-1}L^{-}_{p,\nu}\right)^{*}$ ($\{H_n^+\}_{n\in\mathbb{Z}_+}\subset\left(L^{+}_{p,\nu}\right)^{*}$) denote the system biorthogonal to $\{e^{-int}\}_{n\in\mathbb{N}}$ ($\{e^{int}\}_{n\in\mathbb{Z}_+}$). We expand the functions $F^{\pm}\left(e^{it}\right)$ with respect to these bases

$$F^{+}\left(e^{it}\right)=\sum_{n=0}^{\infty}H_n^{+}\left(S^{+}f\right)e^{int},\tag{4.5}$$
$$F^{-}\left(e^{it}\right)=\sum_{n=1}^{\infty}H_n^{-}\left(S^{-}f\right)e^{-int}.$$

Substituting these expansions into (4.2) we obtain that the function $f\left(\cdot\right)$ can be expanded in series by the system (4.1) in $L_{p,\nu}$. Let us show that such a decomposition is unique. Take the function $f\left(t\right)=e^{-i\frac{1}{2}\theta(t)}e^{ikt}$ in (4.2), where $k\in\mathbb{Z}_+$ is some number. The following functions

$$F^{+}\left(z\right)\equiv z^{k},\quad F^{-}\left(z\right)\equiv0,$$

are also the solutions of this problem. Comparing this solution with (4.5), from the uniqueness of the solution of the problem (4.2) we obtain

$$H_n^{+}\left[S^{+}\left(e^{i\left(kt-\frac{1}{2}\theta(t)\right)}\right)\right]=\delta_{nk},\quad\text{for all } n,k\in\mathbb{Z}_+,\tag{4.6}$$

$$H_n^{-}\left[S^{-}\left(e^{i\left(kt-\frac{1}{2}\theta(t)\right)}\right)\right]=0,\quad\text{for all } n\in\mathbb{N},k\in\mathbb{Z}_+.\tag{4.7}$$

From similar considerations it follows

$$H_n^{+}\left[S^{+}\left(e^{-i\left(kt-\frac{1}{2}\theta(t)\right)}\right)\right]=0,\quad\text{for all } n\in\mathbb{Z}_+,k\in\mathbb{N},\tag{4.8}$$
$$H_n^{-}\left[S^{-}\left(e^{-i\left(kt-\frac{1}{2}\theta(t)\right)}\right)\right]=\delta_{nk},\quad\text{for all } n,k\in\mathbb{N}.$$

Operators $S^{\pm}$ boundedly act in $L_{p,\nu}$. Indeed, it suffices to prove that the integral operator

$$(Sf)(t) = Z^{+}\left(e^{it}\right)\int_{-\pi}^{\pi}\frac{f(s)}{Z^{+}(e^{is})}\cdot\frac{ds}{1-e^{i(t-s)}}$$

is bounded in $L_{p,\nu}$. The condition $f \in L_{p,\nu}$ implies the inclusion $g = f\nu^{\frac{1}{p}} \in L_p$. We have

$$\begin{aligned}(Sf)(t) =& Z^{+}\left(e^{it}\right)\int_{-\pi}^{\pi}\frac{g(s)}{Z^{+}(e^{is})\,\nu^{\frac{1}{p}}(s)}\cdot\frac{ds}{1-e^{i(t-s)}}\\ =& \nu^{-\frac{1}{p}}(t)\left[Z^{+}\left(e^{it}\right)\nu^{\frac{1}{p}}(t)\right]\int_{-\pi}^{\pi}\frac{g(s)}{Z^{+}(e^{is})\,\nu^{\frac{1}{p}}(s)}\cdot\frac{ds}{1-e^{i(t-s)}}.\end{aligned} \tag{4.9}$$

It is easy to see that

$$\left|Z^{+}\left(e^{is}\right)\right| \sim \omega(s), \quad s \in [-\pi,\pi].$$

So, $\omega^{-p}\nu \in A_p$, it follows from (4.9) that

$$\left\|(Sf)\,\nu^{\frac{1}{p}}\right\|_p \le M\,\|g\|_p = M\left\|f\nu^{\frac{1}{p}}\right\|_p = M\,\|f\|_{p,\nu},$$

i.e.,

$$\|Sf\|_{p,\nu} \le M\,\|f\|_{p,\nu},$$

where $M > 0$ is a constant independent of f. This means that the operator S is acting boundedly in $L_{p,\nu}$.

Thus, we have proved that $\{H_n^{\pm}\circ S^{\pm}\} \subset (L_{p,\nu})^{*}$. Then (4.6), (4.8) imply the minimality of the system (4.1) in $L_{p,\nu}$. The following theorem is true.

Theorem 4.1. *Let the following inequalities be satisfied*

$$\left\{\nu;\omega^{-p}\nu\right\} \subset A_p, \quad h_k < 2\pi, \quad k = \overline{0,r}, \tag{4.10}$$

where the weight function $\omega(\cdot)$ is defined by the expression

$$\omega(t) = \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}, \quad s_0 = \pi,$$

h_k, $k = \overline{1,r}$ are jumps of the function $\theta(\cdot)$ at points $-\pi < s_1 < \cdots < s_r < \pi$, $h_0 = \theta(-\pi) - \theta(\pi)$. Then the system of exponents (4.1) forms a basis for $L_{p,\nu}$, $1 < p < +\infty$. For $p = 2$ it forms a Riesz basis for $L_{2,\nu}$ if and only if $\nu \sim 1$ on $[-\pi,\pi]$.

Let us consider some special cases of this theorem. Let the weight function $\nu(\cdot)$ have the form

$$\nu(t) = \prod_{k=1}^{m}|t-t_k|^{\alpha_k}, \tag{4.11}$$

where $\{t_k\}_1^m \subset (-\pi,\pi)$ are distinct points. Suppose that the condition

$$\{s_k\}_1^r \bigcap \{t_k\}_1^m = \emptyset \tag{4.12}$$

holds. In this case, the product $\omega^{-p}\nu$ has the representation

$$\omega^{-p}(t)\,\nu(t)=\prod_{k=1}^{m}|t-t_k|^{\alpha_k}\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{ph_k}{2\pi}}.$$

It is easy to see that $\omega^{-p}\nu\in A_p$ is true if and only if the following inequalities are satisfied (see, e.g., J. Garnett [20])

$$\begin{aligned}-1<-\frac{ph_k}{2\pi}&<p-1,\quad k=\overline{0,r},\\ -1<\alpha_k&<p-1,\quad k=\overline{1,m}.\end{aligned}$$

Thus, the following corollary is true.

Corollary 4.1. *Let the condition* (4.12) *hold and the inequalities*

$$\begin{aligned}-\frac{1}{q}<\frac{h_k}{2\pi}&<\frac{1}{p},\quad k=\overline{0,r},\\ -1<\alpha_k&<p-1,\quad k=\overline{1,m},\end{aligned}$$

be fulfilled. Then the system of exponents (4.1) *forms a basis in* $L_{p,\nu}$, $1<p<+\infty$.

Consider the particular case for the functions $\theta(\cdot)$ and $\nu(\cdot)$

$$\theta(t)=\alpha\,t+\beta\operatorname{sgn}t,\quad t\in[-\pi,\pi],\quad \nu(t)=|t|^{\gamma}.$$

The function $\theta(\cdot)$ has a unique point of discontinuity $s_1=0$. We have

$$h_1=\theta(+0)-\theta(-0)=2\beta,\quad h_0=\theta(-\pi)-\theta(\pi)=-2\alpha\pi-2\beta.$$

As a result $\omega(\cdot)$ is of the form

$$\begin{aligned}\omega(t)&=\left|\sin\frac{t-\pi}{2}\right|^{-\frac{\alpha\pi+\beta}{\pi}}\left|\sin\frac{t}{2}\right|^{\frac{\beta}{\pi}}\\ &\sim|t|^{\frac{\beta}{\pi}}\,|t-\pi|^{-\left(\alpha+\frac{\beta}{\pi}\right)}\,|t+\pi|^{-\left(\alpha+\frac{\beta}{\pi}\right)},\quad t\in[-\pi,\pi].\end{aligned}$$

Consequently,

$$\omega^{-p}(t)\,\nu(t)\sim|t|^{\frac{-p\beta}{\pi}+\gamma}\,|t-\pi|^{\left(\alpha+\frac{\beta}{\pi}\right)p}\,|t+\pi|^{\left(\alpha+\frac{\beta}{\pi}\right)p}.$$

Applying Theorem 4.1, we obtain the following.

Corollary 4.2. *Let the inequality*

$$-1<\gamma<p-1,\quad -1<\gamma-\frac{p\beta}{\pi}<p-1,\quad -\frac{1}{p}<\alpha+\frac{\beta}{\pi}<\frac{1}{q},$$

be fulfilled. Then the system of exponents

$$\left\{e^{i[(n+\alpha\operatorname{sgn}n)t+\beta\operatorname{sgn}t\operatorname{sgn}n]}\right\}_{n\in\mathbb{Z}},$$

forms a basis for $L_{p,|t|^{\gamma}}$, $1<p<+\infty$.

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References

[1] B. T. Bilalov, *On uniform convergence of series with regard to some system of sines*, Differentsial'nye Uravneniya **24**(1) (1988), 175–177.

[2] B. T. Bilalov, *Basicity of some systems of functions*, Differentsial'nye Uravneniya **25** (1989), 163–164.

[3] B. T. Bilalov, *Basicity of some systems of exponents, cosines and sines*, Differentsial'nye Uravneniya **26**(1) (1990), 10–16.

[4] B. T. Bilalov, *On basicity of systems of exponents, cosines and sines in L_p*, Dokl. Math. **365**(1) (1999), 7–8.

[5] B. T. Bilalov, *On basicity of some systems of exponents, cosines and sines in L_p*, Dokl. Math. **379**(2) (2001), 7–9.

[6] B. T. Bilalov, *The basis properties of some systems of exponential functions, cosines, and sines*, Sibirskii Matematicheskii Zhurnal **45**(2) (2004), 264–273.

[7] I. I. Danilyuk, *Non-Regular Boundary Value Problems in the Plane*, Nauka, Moscow, 1975.

[8] G. J. Garnett, *Bounded Analytic Functions*, Mir, Moscow, 1984.

[9] Z. A. Kasumov and A. Basicity, *The basis property of a system of exponents in weighted Lebesgue space*, Estestv. i Tekhnicheskiye Nauki, Moscow **6**(50) (2010), 35–41.

[10] K. S. Kazaryan and P. I. Lizorkin, *Multipliers, bases and unconditional bases in the weighted spaces B and SB*, Proc. Steklov Inst. Math. **187** (1989), 111–130.

[11] E. I. Moiseev, *Some boundary value problems for mixed-type equations*, Differ. Equ. **28**(1) (1992), 105–115.

[12] E. I. Moiseev, *Solution of the Frankl problem in a special domain*, Differentsial'nye Uravneniya **28**(4) (1992), 721–723.

[13] E. I. Moiseev, *On existence and uniqueness of solution a classical problem*, Dokl. Math. **336**(4) (1994), 448–450.

[14] E. I. Moiseev, *On basicity of systems of sines and cosines*, DAN SSSR **275**(4) (1984), 794–798.

[15] E. I. Moiseev, *On basicity of a system of sines*, Differentsial'nye Uravneniya **23**(1) (1987), 177–179.

[16] E. I. Moiseev, *On basicity of systems of cosines and sines in weight space*, Differentsial'nye Uravneniya **34**(1) (1998), 40–44.

[17] E. I. Moiseev, *The basicity in the weight space of a system of eigen functions of a differential operator*, Differentsial'nye Uravneniya **35**(2) (1999), 200–205.

[18] S. M. Ponomarev, *On an eigenvalue problem*, DAN SSSR **249**(5) (1979), 1068–1070.

[19] I. I. Privalov, *Boundary Properties of Analytic Functions*. M-L, Gostekhizdat, 1950.

[20] S. S. Pukhov and A. M. Sedletskii, *Bases of exponents, sines and cosines in weight spaces on finite interval*, Dokl. Math. **425**(4) (2009), 452–455.

[21] S. R. Sadigova, *The general solution of the homogeneous Riemann problem in the weighted Smirnov classes*, Proceedings of the Institute of Mathematics and Mechanics **40**(2) (2014), 115–124.

[22] A. M. Sedletskii, *Biorthogonal expansions of functions in series of exponents on intervals of the real axis*, Uspekhi Matematicheskikh Nauk **37**(5) (1982), 51–95.

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HYPERGROUPS DEFINED ON HYPERGRAPHS AND THEIR REGULAR RELATIONS

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Abstract. The notion of hypergraphs, introduced around 1960, is a generalization of that of graphs and one of the initial concerns was to extend some classical results of graph theory. In this paper, we present some connections between hypergraph theory and hypergroup theory. In this regard, we construct two hypergroupoids by defining two new hyperoperations on $\mathbb{H}$, the set of all hypergraphs. We prove that our defined hypergroupoids are commutative hypergroups and we define hyperrings on $\mathbb{H}$ by using the two defined hyperoperations. Moreover, we study the fundamental group, complete parts, automorphism group and strongly regular relations of one of our hypergroups.

1. Introduction

Hypergraphs generalize standard graphs by defining edges between multiple vertices instead of only two vertices. Hence some properties must be a generalization of graph properties. Formally, a hypergraph is a pair $\Gamma = (X, E)$, where X is a finite set of vertices and $E = \{E_1, \dots, E_n\}$ is a set of hyperedges, which are non-empty subsets of X. The term hypergraph was coined by Berge [2,4], following a remark by Jean-Marie Pal who had used the word hyperedge in a seminar. In 1976, Berge enriched the field once more with his lecture notes [5], also see [3]. The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [16]. Since then, many papers and several books have been written on this topic (see, for instance [6,8–10,18]). Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. After that, many researchers in the field of hyperstructure theory tried to make connections

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between hypergraphs and hyperstructures, for example see [7, 12–14]. Corsini in [7] associated to every hypergraph Γ a commutative quasihypergroup H_Γ and found a necessary and sufficient condition on Γ so that H_Γ is associative. In this paper we continue the study between hypergraphs and algebraic hyperstructures.

Our paper is organized as follows. After an introduction, Section 2 presents some basic definitions concerning hypergroups and hypergraphs that are used throughout this paper. Section 3 defines a new hyperoperation $(\star)$ on $\mathbb{H}$, the set of all hypergraphs and proves some interesting results about $(\mathbb{H}, \star)$. Section 4 presents the fundamental group of our defined hypergroup $(\mathbb{H}, \star)$ and studies its regular relations, complete parts and its automorphism group. Section 5 defines another new hyperoperation $(\circ)$ on $\mathbb{H}$, studies homomorphisms between $(\mathbb{H}, \star)$ and $(\mathbb{H}, \circ)$ and defines hyperrings on $\mathbb{H}$.

2. Basic Definitions

In this section, we present some definitions related to hypergroups and hypergraphs that are used throughout the paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$ for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. An element $x \in H$ is called *idempotent* if $x \circ x = x$. An element $y \in H$ is said to be an *inverse* of $x \in H$ if $e \in x \circ y \cap y \circ x$, where e is an identity in $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it is associative, i.e., if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A subset S of a hypergroup $(H, \circ)$ is called *subhypergroup* of H if it is a hypergroup under $\circ$. A subhypergroup K of a hypergroup $(H, \circ)$ is *normal* if $a \circ K = K \circ a$ for all $a \in H$. A hypergroup $(H, \circ)$ is called a *regular hypergroup* if it has at least one identity and each of its elements admit at least one inverse. A subset I of H is called a *hyperideal* of H if $IH \subseteq H$. A hypergroup H is said to be *simple* if H has no proper hyperideals.

Cyclic semihypergroups have been studied by Desalvo and Freni [11], Vougiouklis [19], Leoreanu [15]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications.

A hypergroup $(H, \circ)$ is cyclic if there exists $h \in H$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots .$$

If there exists $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \cdots \cup h^s$ then H is cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^i = \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ times}}$. It is a *single-power cyclic hypergroup* if there exists $h \in H$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots \quad \text{and} \quad h \cup h^2 \cup \cdots \cup h^{i-1} \subset h^i, \quad \text{for all } i \in \mathbb{N}.$$

Let $(H, \star)$ and $(H', \star')$ be two hypergroups. A function $f : (H, \star) \to (H', \star')$ is said to be a *weak homomorphism* if $f(x_1 \star x_2) \cap f(x_1) \star' f(x_2) \neq \emptyset$ for all $x_1, x_2 \in H$. It is called *homomorphism* if $f(x_1 \star x_2) \subseteq f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$. And it is called a *good homomorphism* if $f(x_1 \star x_2) = f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$.

Two hypergroups are said to be *isomorphic* if there exists a bijective good homomorphism between them. An isomorphism from $(H, \star)$ to itself is called an *automorphism*. The set of all automorphisms of $(H, \star)$ is written as $\mathrm{Aut}(H, \star)$.

3. Hypergroup $(\mathbb{H}, \star)$ Associated to hypergraphs

In this section, we define a new hyperoperation $(\star)$ on the set of all hypergraphs $\mathbb{H}$ and we study some properties of $(\mathbb{H}, \star)$.

A partial hypergraph is a hypergraph with some edges removed.

Definition 3.1. Let $\mathbb{H}$ be the set of all hypergraphs and define $\star$ as follows. For all $H_1, H_2 \in \mathbb{H}$,

$$H_1 \star H_2 = \bigcup \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } H_1 \cup H_2\}.$$

$H_1 \cup H_2$ is the union of all hyperedges from H_1 and H_2. If the same hyperedge corresponding to the same set of vertices occur in both H_1 and H_2 then we consider it once in $H_1 \cup H_2$.

Example 3.1. We present an example on the union of two hypergraphs illustrated in Figures 1, 2 and 3.

Proposition 3.1. *Let $H_1, H_2 \in \mathbb{H}$. Then $\{H_1, H_2\} \subseteq H_1 \star H_2$.*

Proof. The proof results from having H_1, H_2 partial hypergraphs of $H_1 \cup H_2$. □

Proposition 3.2. *Let $H \in \mathbb{H}$. Then $H^m = H^2$ for all $m \geq 2$.*

Proof. For $m \geq 2$, we have that

$$\begin{aligned} H^m &= \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } \underbrace{H \cup H \cdots \cup H}_{m \text{ times}}\} \\ &= \{K \in \mathbb{H} : K \text{ is a partial hypergraph of } H\} \\ &= H^2. \end{aligned}$$

Therefore, $H^m = H^2$ for all $m \geq 2$. □

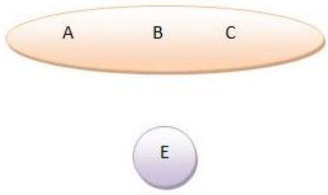

FIGURE 1. Hypergraph H_1

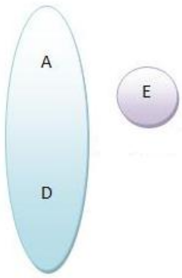

FIGURE 2. Hypergraph H_2

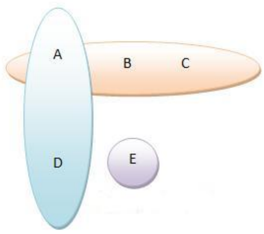

FIGURE 3. Hypergraph $H_1 \cup H_2$

Theorem 3.1. $(\mathbb{H}, \star)$ *is a commutative hypergroup.*

Proof. Let $H_1, H_2, H_3 \in \mathbb{H}$. It is easy to see that $H_1 \star H_2 = H_2 \star H_1$ as $H_1 \cup H_2 = H_2 \cup H_1$. Thus, $\star$ is a commutative hyperoperation.

It is clear that $H_1 \star \mathbb{H} \subseteq \mathbb{H}$. We need to show now that $\mathbb{H} \subseteq H_1 \star \mathbb{H}$. Let $H_2 \in \mathbb{H}$, then $H_2 \in H_1 \star H_2 \subseteq H_1 \star \mathbb{H}$ by Proposition 3.1. Thus, $(\mathbb{H}, \star)$ is a quasihypergroup.

We have that

$$\begin{aligned}
H_1 \star (H_2 \star H_3) =& H_1 \star \bigcup\{K : K \text{ is a partial hypergraph of } H_2 \cup H_3\}\\
=& \bigcup\{H_1 \star K : K \text{ is a partial hypergraph of } H_2 \cup H_3\}\\
=& \bigcup\{M : M \text{ is partial hypergraph of } H_1 \cup K,\\
& K \text{ is partial hypergraph of } H_2 \cup H_3\}\\
=& \bigcup\{M : M \text{ is a partial hypergraph of } H_1 \cup H_2 \cup H_3\}\\
=& \text{partial hypergraphs of } H_1 \cup H_2 \cup H_3.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
(H_1 \star H_2) \star H_3 =& \bigcup\{K : K \text{ is a partial hypergraph of } H_1 \cup H_2\} \star H_3\\
=& \bigcup\{K \star H_3 : K \text{ is a partial hypergraph of } H_1 \cup H_2\}\\
=& \bigcup\{M : M \text{ is partial hypergraph of } K \cup H_3,\\
& K \text{ is partial hypergraph of } H_1 \cup H_2\}\\
=& \bigcup\{M : M \text{ is a partial hypergraph of } H_1 \cup H_2 \cup H_3\}\\
=& \text{partial hypergraphs of } H_1 \cup H_2 \cup H_3.
\end{aligned}$$

Therefore, $(\mathbb{H}, \star)$ is a commutative hypergroup. □

Proposition 3.3. *The only idempotent elements in* $(\mathbb{H}, \star)$ *are hypergraphs with one hyperedge.*

Proof. A hypergraph with exactly one hyperedge has only one partial hypergraph (which is itself) and hence it is idempotent.

If H is an idempotent in $(\mathbb{H}, \star)$, then

$$H \star H = \bigcup\{K : K \text{ is a partial hypergraph of } H\} = H.$$

The latter implies that H has only one partial hypergraph. Thus, H has one hyperedge. □

Proposition 3.4. $(\mathbb{H}, \star)$ *is a regular hypergroup.*

Proof. Proposition 3.1 implies that every element in $\mathbb{H}$ is an identity as $H_1 \in H_1 \star H_2$ for all $H_1, H_2 \in \mathbb{H}$. Let $I(H_1)$ be the set of all inverses of H_1 in $\mathbb{H}$. It is clear that $I(H_1) = \mathbb{H}$. □

Definition 3.2. A nonempty subset M of a hypergroup $(H, \star)$ is linear if $\alpha \star \beta \subseteq M$ and $\alpha/\beta \subseteq M$ for all α, $\beta \in M$. Here, $\alpha/\beta = \{x \in H \mid \alpha \in x \star \beta\}$.

Proposition 3.5. $(\mathbb{H}, \star)$ *has no proper linear subsets.*

Proof. Let M be a linear subset of $(\mathbb{H}, \star)$ and $H_1 \in M$. Having M a linear subset of $(\mathbb{H}, \star)$ implies that $H_1/H_1 \subseteq M$. We have that

$$H_1/H_1 = \{K \in \mathbb{H} : H_1 \in K \star H_1\}.$$

The latter and Proposition 3.1 imply that $H_1/H_1 = \mathbb{H} \subseteq M$. $\square$

Proposition 3.6. $(\mathbb{H}, \star)$ *has no proper normal subhypergroups.*

Proof. For contradiction, suppose that N is a proper normal subhypergroup of $(\mathbb{H}, \star)$. Then there exists $k \in \mathbb{H}$ that is not in N. Having that $k \in k \star N$ (by Proposition 3.1) implies that $N \neq k \star N$. $\square$

Proposition 3.7. $(\mathbb{H}, \star)$ *is a single power cyclic hypergroup with one generator and period two.*

Proof. Let $\alpha = \bigcup_{H_i \in \mathbb{H}} H_i \in \mathbb{H}$. It is clear that α is a generator of $\mathbb{H}$ of period two. Moreover, $\alpha \in \alpha^2 = \mathbb{H}$. $\square$

Proposition 3.8. *Let M be any non-empty set of hypergraphs and*

$$\mathbb{K}_M = \left\{ \lambda : \lambda \text{ is a partial hypergraph of } \bigcup_{K \in M} K \right\}.$$

Then $(\mathbb{K}_M, \star)$ is a cyclic subhypergroup of $(\mathbb{H}, \star)$.

Proof. The proof is straightforward. $\square$

Proposition 3.9. *A subset A of $\mathbb{H}$ is a proper subhypergroup of $(\mathbb{H}, \star)$ if and only if $A = \mathbb{K}_M$ for some non-empty set M of hypergraphs.*

Proof. Let A be a proper subhypergroup of $(\mathbb{H}, \star)$ and suppose, for contradiction, that $A \neq \mathbb{K}_M$. Then there exists K, a partial hypergraph of $\bigcup_{\alpha \in A} \alpha$ that is not in A. The latter implies that K is in the hyperproduct of all elements of A. $\square$

Proposition 3.10. $(\mathbb{H}, \star)$ *is a simple hypergroup.*

Proof. Let $\mathbb{I}$ be a proper hyperideal of $(\mathbb{H}, \star)$. Then $\mathbb{I}\mathbb{H} \subseteq \mathbb{I}$ and there exists $H \in \mathbb{H}$ such that H is not an element in $\mathbb{I}$. Having $H \in \mathbb{I}\mathbb{H}$ implies that $H \in \mathbb{I}$ which contradicts our hypothesis that H is not in $\mathbb{I}$. $\square$

Corollary 3.1. *The only subhypergroups of $(\mathbb{H}, \star)$ are $(\mathbb{K}_M, \star)$ and they are cyclic.*

Proof. The proof results from Propositions 3.8 and 3.9. $\square$

4. Fundamental Relation, Automorphism Group and Complete Parts of $(\mathbb{H}, \star)$

In this section, we present some results related to fundamental relation, automorphism group, strongly regular relations and complete parts of $(\mathbb{H}, \star)$.

Definition 4.1. Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then

(a) $A\overline{R}B$ means that for every $a \in A$ there exists $b \in B$ such that aRb and for every $b' \in B$ there exists $a' \in A$ such that $a'Rb'$;
(b) $A\overline{\overline{R}}B$ means that for every $a \in A$ and $b \in B$, we have aRb.

The equivalence relation R is called

(a) regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively);
(b) strongly regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively);
(c) regular (strongly regular) if it is regular (strongly regular) on the right and on the left.

Theorem 4.1 ([9]). *Let $(H, \circ)$ be a hypergroup and R an equivalence relation on H. Then R is strongly regular if and only if $(H/R, \otimes)$, the set of all equivalence classes, is a group. Here, $\overline{x} \otimes \overline{y} = \{\overline{z} : x \in x \circ y\}$ for all $\overline{x}, \overline{y} \in H/R$.*

The fundamental relation has an important role in the study of semihypergroups and especially of hypergroups.

Definition 4.2 ([9]). For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows: β_1 is the diagonal relation and, if $n > 1$, then

$$a\beta_n b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

$\beta = \bigcup_{n \geq 1} \beta_n$ and $\beta^\star$ is the transitive closure of β.

$\beta^\star$ is called the *fundamental equivalence relation* on H and $H/\beta^\star$ is called the *fundamental group.*

$\beta^\star$ is the smallest strongly regular relation on H and if H is a hypergroup then $\beta = \beta^\star$.

Proposition 4.1. $(\mathbb{H}, \star)$ *has trivial fundamental group.*

Proof. Let $H_1, H_2 \in \mathbb{H}$. Proposition 3.1 asserts that $\{H_1, H_2\} \subset H_1 \star H_2$. The latter implies that $H_1 \beta_2 H_2$. We get now that $H_1 \beta H_2$. Since $(\mathbb{H}, \star)$ is a hypergroup, it follows that $\beta = \beta^\star$. Consequently, $\mathbb{H}/\beta^\star$ has only one equivalence class. □

Proposition 4.2. *Let R be an equivalence relation on $\mathbb{H}$. Then R is strongly regular relation on $\mathbb{H}$ if and only if $\mathbb{H}/R$ is the trivial group.*

Proof. Theorem 4.1 asserts that if $\mathbb{H}/R$ is the trivial group then R is strongly regular relation on $\mathbb{H}$.

Let R be a strongly regular relation on $\mathbb{H}$. For all $x \in \mathbb{H}$, if aRb then $(a \star x)\overline{\overline{R}}(b \star x)$. The latter and having $x \in b \star x$, $a \in a \star x$ imply that aRx. Thus, $\mathbb{H}/R$ contains only one equivalence class. □

Definition 4.3. Let $(H, \circ)$ be an H_v- group and A be a nonempty subset of H. A is a complete part of H if for any natural number n and for all hyperproducts $P \in H_H(n)$, the following implication holds:

$$A \cap P \neq \emptyset \Rightarrow P \subseteq A.$$

Proposition 4.3. *The complete part of* $(\mathbb{H}, \star)$ *is* $\mathbb{H}$.

Proof. Let A be a complete part of $(\mathbb{H}, \star)$ and $a \in A$. Proposition 3.1 asserts that for all $b \in \mathbb{H}$, $a \in A \cap (a \star b) \neq \emptyset$. Having A a complete part of $\mathbb{H}$ implies that $b \in a \star b \subseteq A$. ☐

Proposition 4.4. *Let* $f \in \mathrm{Aut}(\mathbb{H}, \star)$ *and* $\alpha \in \mathbb{H}$. *If* λ *is a partial hypergraph of* α, *then* $f(\lambda)$ *is a partial hypergraph of* $f(\alpha)$. *Moreover,* α *and* $f(\alpha)$ *have same number of partial hypergraphs.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{H}, \star)$ and $\alpha \in \mathbb{H}$. Having $f(\alpha \star \alpha) = f(\alpha) \star f(\alpha)$ implies that $\{f(\lambda) : \lambda \text{ is partial of } \alpha\} = \{\delta : \delta \text{ is partial of } f(\alpha)\}$. The latter implies that if λ is a partial hypergraph of α then $f(\lambda)$ is a partial hypergraph of $f(\alpha)$. Since f is bijective, it follows that α and $f(\alpha)$ have same number of partial hypergraphs. ☐

Theorem 4.2. *Let* f *be a bijective function. Then* $f \in \mathrm{Aut}(\mathbb{H}, \star)$ *if and only if for all* $\alpha, \beta \in \mathbb{H}$ *the following conditions are satisfied:*

1. *if* λ *is a partial hypergraph of* α *then* $f(\lambda)$ *is a partial hypergraph of* $f(\alpha)$, *and*
2. $f(\alpha \star \beta) \subseteq f(\alpha) \star f(\beta)$.

Proof. Let $f \in \mathrm{Aut}(\mathbb{H}, \star)$ and $\alpha \in \mathbb{H}$. Then $f(\alpha \star \beta) = f(\alpha) \star f(\beta)$. The latter and Proposition 4.4 imply that conditions 1. and 2. are satisfied.

Let f be any bijective function satisfying conditions 1. and 2. and let $\alpha, \beta \in \mathbb{H}$. Since α, β are partial hypergraphs of $\alpha \cup \beta$, it follows by condition 1. that $f(\alpha)$, $f(\beta)$ are partial hypergraphs of $f(\alpha \cup \beta)$. The latter implies that $f(\alpha) \cup f(\beta)$ is a partial hypergraph of $f(\alpha \cup \beta)$. Moreover, every partial hypergraph of $f(\alpha) \cup f(\beta)$ is a partial hypergraph of $f(\alpha \cup \beta)$. We get now that

$$\begin{aligned} f(\alpha) \star f(\beta) &= \{\delta \in \mathbb{H} : \delta \text{ is partial hypergraph of } f(\alpha) \cup f(\beta)\} \\ &\subseteq \{\lambda \in \mathbb{H} : \lambda \text{ is partial hypergraph of } f(\alpha \cup \beta)\}. \end{aligned}$$

Consequently, we get that $f(\alpha) \star f(\beta) \subseteq f(\alpha \star \beta)$. Thus, f is a good homomorphism by condition 2. ☐

Remark 4.1. It is easy to see that the identity function satisfies conditions 1. and 2. of Theorem 4.2.

Example 4.1. Let $H \in \mathbb{H}$, α be the hypergraph with vertex v_1 having only one hyperedge and β be the hypergraph with vertex v_2 having only one hyperedge. We define $f : (\mathbb{H}, \star) \to (\mathbb{H}, \star)$ as follows:

$$f(H) = \begin{cases} H, & \text{if } \alpha \cup \beta \text{ is a partial hypergraph of } H; \\ H, & \text{if neither } \alpha \text{ nor } \beta \text{ are partial hypergraphs of } H; \\ \beta \cup (H \setminus \{\alpha\}), & \text{if } \alpha \text{ is a partial hypergraph of } H; \\ \alpha \cup (H \setminus \{\beta\}), & \text{if } \beta \text{ is a partial hypergraph of } H. \end{cases}$$

Then $f \in \mathrm{Aut}(\mathbb{H}, \star)$.

It is clear that f is a bijective function. Also, one can easily show that f satisfies condition 1. and 2. of Theorem 4.2.

5. Relation of $(\mathbb{H}, \star)$ to Another Hypergroup $(\mathbb{H}, \circ)$

In this section, we define a new hyperoperation $(\circ)$ on $\mathbb{H}$ and find some relations between $(\mathbb{H}, \star)$, defined in Section 3, and $(\mathbb{H}, \circ)$.

Definition 5.1. Let $\mathbb{H}$ be the set of all hypergraphs and define $(\mathbb{H}, \circ)$ as follows. For all $H_1, H_2 \in \mathbb{H}$

$$H_1 \circ H_2 = \{H_1, H_2, H_1 \cup H_2\}.$$

We present some results on $(\mathbb{H}, \circ)$ in which their proofs are easy.

Theorem 5.1. $(\mathbb{H}, \circ)$ *is a regular commutative hypergroup.*

Proposition 5.1. *Every element in* $(\mathbb{H}, \circ)$ *is idempotent.*

Proposition 5.2. $(\mathbb{H}, \circ)$ *has no nontrivial cyclic subhypergroup.*

Proof. Proposition 5.1 asserts that $\alpha^k = \alpha$ for all $\alpha \in \mathbb{H}$ and $k \in \mathbb{N}$. □

Definition 5.2. Let $(H, \circ)$ and $(H, \star)$ be two hypergroups. We say that $\circ \leq \star$ if there is $f \in \mathrm{Aut}(H, \star)$ such that $\alpha \circ \beta \subseteq f(\alpha) \star f(\beta)$ for all $\alpha, \beta \in H$.

Proposition 5.3. $\circ \leq \star$.

Proof. Let $i : (\mathbb{H}, \star) \to (\mathbb{H}, \star)$ be the identity map defined by: $i(H) = H$ for all $H \in \mathbb{H}$. It is clear that $i \in \mathrm{Aut}(\mathbb{H}, \star)$.

For all $H_1, H_2 \in \mathbb{H}$, we have each element in $H_1 \circ H_2 = \{H_1, H_2, H_1 \cup H_2\}$ is a partial hypergraph of $H_1 \cup H_2$. On the other hand, we have that $i(H_1) \star i(H_2) = H_1 \star H_2$ is the set of all partial hypergraphs of $H_1 \cup H_2$. Thus, $H_1 \circ H_2 \subseteq i(H_1) \star i(H_2)$. □

Definition 5.3. Let R be a nonempty set with two hyperoperations ($+$ and $\cdot$). We say that $(R, +, \cdot)$ is a hyperring if $(R, +)$ is a commutative hypergroup, $(R, \cdot)$ is a semihypergroup and the hyperoperation $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If the hyperoperation $\cdot$ is weak distributive with respect to $+$, i.e., $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ for all $x, y, z \in R$, we say $(R, +, \cdot)$ that is a weak hyperring.

Proposition 5.4. $(\mathbb{H}, \star, \circ)$ *is a weak commutative hyperring.*

Proof. Propositions 3.1 and 5.1 imply that $(\mathbb{H}, \circ)$ and $(\mathbb{H}, \star)$ are commutative hypergroups. We need to prove that $(\mathbb{H}, \star, \circ)$ is weak distributive. For all $\alpha, \beta, \gamma \in \mathbb{H}$ we have

$$\begin{aligned} \alpha \circ (\beta \star \gamma) &= \bigcup \{\alpha \circ \lambda : \lambda \text{ is a partial hypergraph of } \beta \cup \gamma\} \\ &= \bigcup \{\alpha, \lambda, \alpha \cup \lambda : \lambda \text{ is a partial hypergraph of } \beta \cup \gamma\}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} (\alpha \circ \beta) \star (\alpha \circ \gamma) &= \{\alpha, \beta, \alpha \cup \beta\} \star \{\alpha, \gamma, \alpha \cup \gamma\} \\ &= \text{partial hypergraphs of } \{\alpha, \alpha \cup \gamma, \beta \cup \alpha, \beta \cup \alpha \cup \gamma, \beta \cup \gamma\} \\ &= \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma. \end{aligned}$$

It is easy to see that $\alpha \circ (\beta \star \gamma) \subseteq (\alpha \circ \beta) \star (\alpha \circ \gamma)$. $\square$

Proposition 5.5. $(\mathbb{H}, \circ, \star)$ *is a commutative hyperring.*

Proof. Propositions 3.1 and 5.1 imply that $(\mathbb{H}, \circ)$ and $(\mathbb{H}, \star)$ are commutative hypergroups. We need to prove that $(\mathbb{H}, \circ, \star)$ is distributive. For all $\alpha, \beta, \gamma \in \mathbb{H}$ we have

$$\begin{aligned}\alpha \star (\beta \circ \gamma) &= \alpha \star \{\beta, \gamma, \beta \cup \gamma\} \\ &= \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}(\alpha \star \beta) \circ (\alpha \star \gamma) =& \text{partial hypergraphs of } \alpha \cup \beta \circ \text{ partial hypergraphs of } \alpha \cup \gamma \\ =& \bigcup \{\lambda, \lambda^\star, \lambda \cup \lambda^\star : \lambda \text{ and } \lambda^\star \text{ are partial hypergraphs} \\ & \text{ of } \alpha \cup \beta \text{ and } \alpha \cup \gamma \text{ respectively}\} \\ =& \text{partial hypergraphs of } \alpha \cup \beta \cup \gamma.\end{aligned}$$

Thus, $\alpha \star (\beta \circ \gamma) = (\alpha \star \beta) \circ (\alpha \star \gamma)$. $\square$

Proposition 5.6. *Let* $f : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be any function. Then* f *is a weak homomorphism.*

Proof. Let $\alpha, \beta \in \mathbb{H}$. We have that $f(\alpha \circ \beta) = \{f(\alpha), f(\beta), f(\alpha \cup \beta)\}$. Having $f(\alpha)$, $f(\beta)$ partial hypergraphs of $f(\alpha) \cup f(\beta)$ implies that

$$\{f(\alpha), f(\beta)\} \subseteq f(\alpha \circ \beta) \cap f(\alpha) \star f(\beta) \neq \emptyset.$$

$\square$

Proposition 5.7. *Let* $c : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be the constant function defined by:* $c(H) = K$*, where* K *is the hypergraph defined on any set of vertices with one hyperedge. Then* c *is a good homomorphism.*

Proof. The proof is straightforward by Proposition 3.3. $\square$

Proposition 5.8. *Let* $f : (\mathbb{H}, \circ) \to (\mathbb{H}, \star)$ *be any function that is not equal to that defined in Proposition* 5.7*. Then* f *is not a good homomorphism.*

Proof. Let H be a hypergraph such that $f(H)$ has more than two hyperedges (such an element exists). We have that $f(H \circ H) = f(H)$ and $f(H) \star f(H)$ is the set of all partial hypergraphs of $f(H)$. Since $f(H)$ has more than two hyperedges, it follows that $|f(H) \star f(H)| \geq 2$. Thus, f is not a good homomorphism. $\square$

Proposition 5.9. *Let* $f : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be any function. Then* f *is a weak homomorphism.*

Proof. It is easy to see that $\{f(\alpha), f(\beta)\} \subseteq f(\alpha \star \beta) \cap f(\alpha) \circ f(\beta) \neq \emptyset$. $\square$

Proposition 5.10. *Let* $k : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be the function defined by* $k(\alpha) = H$ *for all* $\alpha \in \mathbb{H}$*. Then* f *is a good homomorphism.*

Proof. The proof is straightforward using Proposition 5.1. □

Proposition 5.11. *Let* $f : (\mathbb{H}, \star) \to (\mathbb{H}, \circ)$ *be any function other than that defined in Proposition* 5.10*. Then* f *is not a homomorphism.*

Proof. Since f is a function other than that defined in Proposition 5.10, it follows that there exist $\alpha, \beta \in \mathbb{H}$ such that $f(\alpha) \neq f(\beta)$. Let $\gamma = \alpha \cup \beta \in \mathbb{H}$. We have that $f(\gamma) \circ f(\gamma) = f(\gamma)$ and $f(\gamma \star \gamma) = \{f(\lambda) : \lambda \text{ is a partial hypergraph of } \gamma\}$. Having that $\alpha \neq \beta$ partial hypergraphs of γ and that $f(\alpha) \neq f(\beta)$ imply that $|f(\gamma \star \gamma)| \geq 2$. The latter implies that $f(\gamma \star \gamma)$ is not a subset of $f(\gamma) \circ f(\gamma)$. □

6. Conclusion

Hypergraph theory, introduced by Berge, is a generalization of graph theory and it has been considered an important topic in Mathematics due to its applications to numerous fields of Science. Our paper studied a connection between hypergraph theory and hypergroup theory. Here we defined hypergroups and hyperrings on the set of all hypergraphs. Also, we studied the fundamental group and regular relations of the defined hypergroups. Several results were obtained.

For future research, one may consider hyperfields associated to hypergraphs and study their properties.

References

[1] M. Al-Tahan and B. Davvaz, *On a special single-power cyclic hypergroup and its automorphisms*, Discrete Mathematics, Algorithms and Applications **7**(4) (2016), 12 pages.

[2] C. Berge, *Graphcs et Hypcrgraphcs*, Dunod, Pum, 1972.

[3] C. Berge, *Hypergraphs-Combinatorics of Finite Sets* (translated from the French), North-Holland Mathematical Library, 45, North-Holland Publishing Co., Amsterdam, 1989.

[4] C. Berge, *Hypergraphs generalizing bipartite graphs*, in: J. Abadie (Ed.), *Integer and Nonlinear Programming*, North-Holland, Amsterdam, 1970, 507–509.

[5] C. Berge, *The multicolorings of graphs and hypergraphs*, in: Y. Alavi, D. R. Lick (Eds.), *Theory and Applications of Graphs*, Lecture Notes in Mathematics **642**, Springer, Berlin, Heidelberg, 2978.

[6] P. Corsini, *Prolegomena of Hypergroup Theory*, Second Edition, Aviani Editore, Italy, 1993.

[7] P. Corsini, *Hypergraphs and hypergroups*, Algebra Universalis **35** (1996), 548–555.

[8] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics **5**, Kluwer Academic Publisher, Boston, London, 2003.

[9] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, New Jersey, 2013.

[10] B. Davvaz, *Semihypergroup Theory*, Elsevier, London, 2016.

[11] M. De Salvo and D. Freni, *Cyclic semihypergroups and hypergroups* (in italian), Atti Sem. Mat. Fis. Univ. Modena **30**(1) (1981), 44–59.

[12] M. Farshi, B. Davvaz and S. Mirvakili, *Degree hypergroupoids associated with hypergraphs*, Filomat **28**(1) (2014), 119–129.

[13] M. Farshi, B. Davvaz and S. Mirvakili, *Hypergraphs and hypergroups based on a special relation*, Comm. Algebra **42** (2014), 3395–3406.

[14] A. Iranmanesh and M. N. Iradmusa, *The combinatorial and algebraic structure of the hypergroup associated to a hypergraph*, J. Mult.-Valued Logic Soft Comput. **11** (2005), 127–136.

[15] V. Leoreanu, *About the simplifiable cyclic semihypergroups*, Italian Journal of Pure and Applied Mathematics **7** (2000), 69–76.
[16] F. Marty, *Sur une generalization de la notion de group*, Congrès des Mathématiciens Scandinaves Tenu à Stockholm, 1934, 45-49.
[17] J. Mittas, *Hypergroups canoniques*, Math. Balkanica **2** (1972), 165–179.
[18] T. Vougiouklis, *Hyperstructures and their Representations*, Hadronic Press, Palm Harbor, USA, 1994.
[19] T. Vougiouklis, *Cyclicity in a special class of hypergroups*, Acta Universitatis Carolinae. Mathematica et Physica **22**(1) (1981), 3–6.

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