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BASES OF THE PERTURBED SYSTEM OF EXPONENTS IN WEIGHTED LEBESGUE SPACE WITH A GENERAL WEIGHT

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Abstract. The weighted Lebesgue and Hardy spaces with a general weight are considered. Basicity of a part of exponential system is proved in Hardy classes, where the weight satisfies the Muckenhoupt condition. Using these results the basicity of the perturbed system of exponents in the weighted Lebesgue space is studied. Some special cases are considered.

1. Introduction

When solving many problems for equations of mixed type by Fourier method (see e.g., [11–13, 18]), there appear perturbed systems of sines and cosines of the following form

$$(1.1)\qquad \{\sin(nt+\alpha(t))\}_{n\in\mathbb{N}}, \quad \{\cos(nt+\alpha(t))\}_{n\in\mathbb{N}},$$

where $\alpha : [0,\pi] \to \mathbb{R}$ is some real function. Verification of the Fourier method requires to study basis properties (completeness, minimality, basicity and etc.) of system (1.1) in the appropriate spaces of functions (usually in Lebesgue or Sobolev spaces). In Lebesgue space $L_p \equiv L_p(0,\pi)$, $1 < p < +\infty$, these problems have been well studied for a wide class of functions $\alpha(\cdot)$ in [1–6, 14, 15, 21, 22]. Basis properties of system (1.1) are closely related to the analogous properties of the system of exponents of the form

$$(1.2)\qquad \left\{e^{i(nt+\beta(t)\operatorname{sgn} n)}\right\}_{n\in\mathbb{Z}},$$

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where $\beta : [-\pi, \pi] \to \mathbb{R}$ is some function. These problems with respect to the systems (1.1), (1.2) in weighted Lebesgue spaces with a power weight have been studied in [9, 16, 17, 20].

In this work, the basicity of system (1.2) is studied in weighted Lebesgue space $L_{p,\nu} \equiv L_{p,\nu}(-\pi, \pi)$, with a general weight $\nu(\cdot)$. For the basicity of this system in $L_{p,\nu}$ the sufficient conditions on the function $\beta(\cdot)$ and weight $\nu(\cdot)$ are obtained. For this, firstly, the weighted Hardy classes $H^{\pm}_{p,d\rho}$ are defined, the basicity of a part of exponential system is studied in these classes, and these results are applied to the basicity of the system (1.2) in $L_{p,\nu}$.

2. Necessary Facts

Let $\mathbb{C}$ be the complex plane and $\omega = \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle on $\mathbb{C}$. The expression $f \sim g$, in M, means that the following inequality is true

$$\text{exists } \delta > 0 : \quad \delta \le \frac{f(x)}{g(x)} \le \delta^{-1} \quad \text{for all } x \in M.$$

Let us consider the basicity of the following parts of exponential system

$$\left\{e^{int}\right\}_{n \in \mathbb{Z}_+}, \tag{2.1}$$

$$\left\{e^{-int}\right\}_{n \ge m}, \tag{2.2}$$

in weighted spaces $H^{+}_{p,d\rho}$ and ${}_mH^{-}_{p,d\rho}$, respectively. These facts are needed in the study of basicity of the perturbed system of exponents with a phase in weighted Lebesgue spaces $L_{p,d\rho}$. It should be noted that for the basicity of the exponential system $\{e^{int}\}_{n\in\mathbb{Z}}$ in $L_{p,d\rho}$ it is necessary that the weight function $\rho(\cdot)$ be absolutely continuous on $[-\pi, \pi]$. Indeed, in the case of basicity the following conditions (biorthogonality condition)

$$\int_{-\pi}^{\pi} e^{int} d\rho = 0, \quad \text{for all } n \ge 1,$$

should be fulfilled. And these conditions imply that (see, e.g., I. I. Privalov [19] or I. I. Danilyuk [7]) $\rho(\cdot)$ is absolutely continuous on $[-\pi, \pi]$, and let

$$\nu(t) = \rho'(t), \quad t \in (-\pi, \pi).$$

Therefore, in the sequel, we will consider the weighted Lebesgue space $L_{p,\nu} \equiv L_{p,\nu}(-\pi, \pi)$, with a norm

$$\|f\|_{p,\nu} = \left(\int_{-\pi}^{\pi} |f(t)|^p \nu(t)\, dt\right)^{\frac{1}{p}}, \quad \text{for all } f \in L_{p,\nu}.$$

Based on this norm, weighted Hardy classes $H^{+}_{p,\nu}$:

$$H^{+}_{p,\nu} \equiv \left\{f \in H^{+}_{1} : f^{+} \in L_{p;\nu}\right\},$$

are defined, endowed with the norm

$$\|f\|_{H^{+}_{p,\nu}} = \left\|f^{+}\right\|_{p,\nu},$$

where $f^+ = f/_{\partial\omega}$ is a restriction of the function f on $\partial\omega$.

Similarly we define the weighted Hardy class ${}_mH^-_{p,\nu}$ of functions which are analytic outside the unit circle ω. Let ${}_mH^-_p$ be a usual Hardy class of functions that are analytic outside the unit circle ω and have a pole of order $k \leq m$ at infinity. Assume

$$ {}_mH^-_{p,\nu} \equiv \left\{ f \in_m H^-_1 \, : \, f^- \in L_{p,\nu} \right\}. $$

The norm is defined by the expression

$$ \|f\|_{{}_mH^-_{p,\nu}} = \left\| f^- \right\|_{p,\nu}, $$

where $f^- = f/_{\partial\omega}$ is the restriction of function $f \in_m H^-_{p,\nu}$ on $\partial\omega$.

3. Basicity of System of Exponents in $L_{p,\nu}$

Consider the weighted space $L_{p,\nu}$, $1 < p < +\infty$, where $\nu(\cdot)$ is some weight. Assume that $\nu(\cdot)$ satisfies Muckenhoupt condition (see, e.g., [8])

$$ \sup_{I \subset [-\pi,\pi]} \left(\frac{1}{|I|} \int_I \nu(t) \, dt \right) \left(\frac{1}{|I|} \int_I |\nu(t)|^{-\frac{1}{p-1}} \, dt \right)^{p-1} < +\infty. \tag{3.1} $$

Since $\nu \in L_1$, it is clear that $\{e^{int}\}_{n\in\mathbb{Z}} \subset L_{p,\nu}$. Consider the functionals $\{\vartheta_n\}_{n\in\mathbb{Z}}$

$$ \vartheta_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, e^{-int} dt, \quad \text{for all } n \in \mathbb{Z}. $$

We have

$$ |\vartheta_n(f)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(t) \, \nu^{\frac{1}{p}}(t) \, e^{-int} \nu^{-\frac{1}{p}}(t) \, dt \right| \leq \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t) \, dt \right)^{\frac{1}{q}} \|f\|_{p,\nu}, \tag{3.2} $$

where q is the conjugate of a number p, $\frac{1}{p} + \frac{1}{q} = 1$. From the condition (3.1) it directly follows

$$ \int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t) \, dt < +\infty. $$

Then from the inequality (3.2) we obtain that the functionals ϑ_n are continuous in $L_{p,\nu}$, for all $n \in \mathbb{Z}$, and moreover $\vartheta_n\left(e^{ikt}\right) = \delta_{nk}$ for all $n, k \in \mathbb{Z}$. As a result the system $\{e^{int}\}_{n\in\mathbb{Z}}$ is minimal in $L_{p,\nu}$. Consider the partial sums

$$ S_m(f) = \sum_{n=-m}^{m} \vartheta_n(f) \, e^{int}, \quad f \in L_{p,\nu}. $$

We have

$$ S_m(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, D_m(x-t) \, dt, \quad m \in \mathbb{N}, $$

where $D_m(\cdot)$ denotes the Dirichlet kernel

$$ D_m(t) = \frac{\sin\left(m + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}, \quad m \in \mathbb{N}. $$

As it known (see, e.g., Dj. Garnett [8]) if $\nu \in A_p$, then the Hilbert transformation is bounded in $L_{p,\nu}$. Hence, it directly follows that

$$\|S_m(f)\|_{p,\nu} \le M \|f\|_{p,\nu}, \quad \text{for all } m \in \mathbb{N},$$

holds, where $M > 0$ is an absolute constant. As a result, it follows from the basis criterion that the system $\{e^{int}\}_{n\in\mathbb{Z}}$ forms a basis for $L_{p,\nu}$. It is easy to see that for $p = 2$ it forms a Riesz basis in $L_{2,\nu}$ if and only if $\nu \sim 1$.

Statement 1. Let $\nu \in A_p$. Then the system of exponents $\{e^{int}\}_{n\in\mathbb{Z}}$ forms a basis (for $p = 2$ it forms a Riesz basis if and only if $\nu \sim 1$) in $L_{p,\nu}$.

In the case $p = 2$ this fact has been previously established by K. S. Lizorkin, P. I. Ghazarian [10].

Take an arbitrary $f \in H^+_{p,\nu}$ and let $\nu \in A_p$. As $f \in H^+_1$, it is clear that

$$\int_{-\pi}^{\pi} f^+\left(e^{it}\right) e^{int} dt = 0, \quad \text{for all } n \in \mathbb{N},$$

holds. Then by Statement 1 the function f^+ has the following representation in $L_{p,\nu}$

$$f^+\left(e^{it}\right) = \sum_{n=0}^{\infty} f_n e^{int},$$

where

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+\left(e^{it}\right) e^{-int} dt, \quad \text{for all } n \in \mathbb{Z}_+.$$

Consider the functionals $\{H^+_n\}_{n\in\mathbb{Z}_+}$

$$H^+_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+\left(e^{it}\right) e^{-int} dt, \quad n \in \mathbb{Z}_+.$$

Following (3.2), we have

$$\left|H^+_n(f)\right| \le \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t)\, dt\right)^{\frac{1}{q}} \left\|f^+\right\|_{p,\nu} = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \nu^{-\frac{q}{p}}(t)\, dt\right)^{\frac{1}{q}} \|f\|_{H^+_{p,\nu}},$$

for all $f \in H^+_{p,\nu}$. This implies the inclusion $\{H^+_n\}_{n\in\mathbb{Z}_+} \subset \left(H^+_{p,\nu}\right)^*$ and moreover it is evident that

$$H^+_n\left(z^k\right) = \delta_{nk}, \quad \text{for all } n, k \in \mathbb{Z}_+,$$

i.e., the system $\{z^n\}_{n\in\mathbb{Z}_+}$ is minimal in $H^+_{p,\nu}$. It is absolutely clear that the expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

is true in $H^+_{p,\nu}$. As a result, we obtain the basicity of the system $\{z^n\}_{n\in\mathbb{Z}_+}$ in $H^+_{p,\nu}$.

Restrictions of class $H^+_{p,\nu}$ $({}_mH^-_{p,\nu})$ to the unit circle $\partial\omega$ will be denoted by $L^+_{p,\nu}$ $({}_mL^-_{p,\nu})$. It is easy to see that the system $\{e^{int}\}_{n\in\mathbb{Z}_+}$ forms a basis for $L^+_{p,\nu}$.

Similarly, we prove the basicity of the system $\{z^{-n}\}_{n\ge m}$ $(\{e^{-int}\}_{n\ge m})$ in ${}_mH^-_{p,\nu}$ $({}_mL^-_{p,\nu})$. Thus, the following theorem holds true.

Theorem 3.1. *Let $\nu \in A_p$, $1 < p < +\infty$. Then*
i) the system $\{z^n\}_{n\in\mathbb{Z}_+}$ ($\{e^{int}\}_{n\in\mathbb{Z}_+}$) forms a basis for $H^+_{p,\nu}$ (for $L^+_{p,\nu}$);
ii) the system $\{z^{-n}\}_{n\geq m}$ ($\{e^{-int}\}_{n\geq m}$) forms a basis for ${}_mH^-_{p,\nu}$ (${}_mL^-_{p,\nu}$).
For $p = 2$ these bases are Riesz bases if and only if $\nu \sim 1$ on $[-\pi, \pi]$.

4. Bases from the Perturbed System of Exponents in $L_{p,\nu}$

Consider the following system of exponents

$$\left\{e^{i\left(nt-\frac{1}{2}\theta(t)\,\mathrm{sgn}\,n\right)}\right\}_{n\in\mathbb{Z}}, \tag{4.1}$$

where $\theta(\cdot)$ is a piecewise Hölder function on $[-\pi, \pi]$. Let $\{s_k\}_1^r : -\pi < s_1 < \cdots < s_r < \pi$ be the points of discontinuity of the function $\theta(\cdot)$ and

$$h_k = \theta(s_k + 0) - \theta(s_k - 0), \quad k = \overline{1, r},$$

be the corresponding jumps of $\theta(\cdot)$ at these points. Assume $h_0 = \theta(-\pi) - \theta(\pi)$. By $\omega(\cdot)$ denote the following weight function

$$\omega(t) \equiv \left|\sin\frac{t-\pi}{2}\right|^{\frac{h_0}{2\pi}} \prod_{k=1}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}.$$

Consider the following non-homogeneous Riemann problem in classes $H^+_{p,\nu} \times_{-1} H^-_{p,\nu}$

$$e^{-i\frac{1}{2}\theta(t)}F^+\left(e^{it}\right) - e^{i\frac{1}{2}\theta(t)}F^-\left(e^{it}\right) = f(t), \quad t \in (-\pi, \pi), \tag{4.2}$$

where $f \in L_{p,\nu}$ is an arbitrary function. Suppose that the following conditions hold

$$\omega^{-p}\nu \in A_p, \quad \frac{h_k}{2\pi} < 1, \quad k = \overline{0, r}. \tag{4.3}$$

Then the problem (4.2) has a unique solution of the form

$$F(z) = \frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z^+\left(e^{it}\right)}\cdot\frac{dt}{1-ze^{-it}}, \tag{4.4}$$

where $Z(\cdot)$ is the canonical solution of the corresponding homogeneous problem

$$e^{-i\frac{1}{2}\theta(t)}Z^+\left(e^{it}\right) - e^{i\frac{1}{2}\theta(t)}Z^-\left(e^{it}\right) = 0, \quad t \in (-\pi, \pi),$$

which is defined by the following expressions

$$Z(z) \equiv \begin{cases} X(z), & |z| < 1, \\ X^{-1}(z), & |z| > 1, \end{cases}$$

$$X(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Applying Sokhotski-Plemelj formulas to (4.4) we get

$$F^+\left(e^{it}\right) = \frac{1}{2} f(t) - \frac{Z^+(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{f(s)}{Z^+(e^{is})} \cdot \frac{ds}{1 - e^{i(t-s)}} \equiv \left(S^+ f\right)(t),$$

$$F^-\left(e^{it}\right) = -\frac{1}{2} \cdot \frac{Z^-(e^{it})}{Z^+(e^{it})} f(t) - \frac{Z^-(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{f(s)}{Z^+(e^{is})} \cdot \frac{ds}{1 - e^{i(t-s)}}$$

$$= -\frac{e^{-i\theta(t)}}{2} f(t) - \frac{Z^-(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{f(s)}{Z^+(e^{is})} \cdot \frac{ds}{1 - e^{i(t-s)}} \equiv \left(S^- f\right)(t).$$

Let us assume that $\nu \in A_p$. Then, by Theorem 3.1, the system $\{e^{int}\}_{n\in\mathbb{Z}_+}$ ($\{e^{-int}\}_{n\in\mathbb{N}}$) forms a basis for $L^+_{p,\nu}$ (for ${}_{-1}L^-_{p,\nu}$). As

$$|Z(0)|^{\pm 1} < +\infty, \quad |Z(\infty)|^{\pm 1} < +\infty,$$

it is clear that the inclusions

$$F^+\left(e^{it}\right) \in L^+_{p,\nu}, \quad F^-\left(e^{it}\right) \in_{-1} L^-_{p,\nu},$$

hold. By $\{H_n^-\}_{n\in\mathbb{N}} \subset \left({}_{-1}L^-_{p,\nu}\right)^*$ ($\{H_n^+\}_{n\in\mathbb{Z}_+} \subset \left(L^+_{p,\nu}\right)^*$) denote the system biorthogonal to $\{e^{-int}\}_{n\in\mathbb{N}}$ ($\{e^{int}\}_{n\in\mathbb{Z}_+}$). We expand the functions $F^{\pm}(e^{it})$ with respect to these bases

$$F^+\left(e^{it}\right) = \sum_{n=0}^{\infty} H_n^+\left(S^+ f\right) e^{int}, \tag{4.5}$$

$$F^-\left(e^{it}\right) = \sum_{n=1}^{\infty} H_n^-\left(S^- f\right) e^{-int}.$$

Substituting these expansions into (4.2) we obtain that the function $f(\cdot)$ can be expanded in series by the system (4.1) in $L_{p,\nu}$. Let us show that such a decomposition is unique. Take the function $f(t) = e^{-i\frac{1}{2}\theta(t)} e^{ikt}$ in (4.2), where $k \in \mathbb{Z}_+$ is some number. The following functions

$$F^+(z) \equiv z^k, \quad F^-(z) \equiv 0,$$

are also the solutions of this problem. Comparing this solution with (4.5), from the uniqueness of the solution of the problem (4.2) we obtain

$$H_n^+\left[S^+\left(e^{i\left(kt - \frac{1}{2}\theta(t)\right)}\right)\right] = \delta_{nk}, \quad \text{for all } n, k \in \mathbb{Z}_+, \tag{4.6}$$

$$H_n^-\left[S^-\left(e^{i\left(kt - \frac{1}{2}\theta(t)\right)}\right)\right] = 0, \quad \text{for all } n \in \mathbb{N}, k \in \mathbb{Z}_+. \tag{4.7}$$

From similar considerations it follows

$$H_n^+\left[S^+\left(e^{-i\left(kt - \frac{1}{2}\theta(t)\right)}\right)\right] = 0, \quad \text{for all } n \in \mathbb{Z}_+, k \in \mathbb{N}, \tag{4.8}$$

$$H_n^-\left[S^-\left(e^{-i\left(kt - \frac{1}{2}\theta(t)\right)}\right)\right] = \delta_{nk}, \quad \text{for all } n, k \in \mathbb{N}.$$

Operators $S^{\pm}$ boundedly act in $L_{p,\nu}$. Indeed, it suffices to prove that the integral operator

$$(Sf)(t) = Z^+\left(e^{it}\right)\int_{-\pi}^{\pi}\frac{f(s)}{Z^+(e^{is})}\cdot\frac{ds}{1-e^{i(t-s)}}$$

is bounded in $L_{p,\nu}$. The condition $f \in L_{p,\nu}$ implies the inclusion $g = f\nu^{\frac{1}{p}} \in L_p$. We have

$$\begin{aligned}(Sf)(t) =& Z^+\left(e^{it}\right)\int_{-\pi}^{\pi}\frac{g(s)}{Z^+(e^{is})\,\nu^{\frac{1}{p}}(s)}\cdot\frac{ds}{1-e^{i(t-s)}}\\ =&\nu^{-\frac{1}{p}}(t)\left[Z^+\left(e^{it}\right)\nu^{\frac{1}{p}}(t)\right]\int_{-\pi}^{\pi}\frac{g(s)}{Z^+(e^{is})\,\nu^{\frac{1}{p}}(s)}\cdot\frac{ds}{1-e^{i(t-s)}}.\end{aligned}\tag{4.9}$$

It is easy to see that

$$\left|Z^+\left(e^{is}\right)\right| \sim \omega(s), \quad s \in [-\pi,\pi].$$

So, $\omega^{-p}\nu \in A_p$, it follows from (4.9) that

$$\left\|(Sf)\,\nu^{\frac{1}{p}}\right\|_p \le M\,\|g\|_p = M\left\|f\nu^{\frac{1}{p}}\right\|_p = M\,\|f\|_{p,\nu},$$

i.e.,

$$\|Sf\|_{p,\nu} \le M\,\|f\|_{p,\nu},$$

where $M > 0$ is a constant independent of f. This means that the operator S is acting boundedly in $L_{p,\nu}$.

Thus, we have proved that $\{H_n^{\pm}\circ S^{\pm}\} \subset (L_{p,\nu})^*$. Then (4.6), (4.8) imply the minimality of the system (4.1) in $L_{p,\nu}$. The following theorem is true.

Theorem 4.1. *Let the following inequalities be satisfied*

$$\left\{\nu;\omega^{-p}\nu\right\} \subset A_p, \quad h_k < 2\pi, \quad k = \overline{0,r},\tag{4.10}$$

where the weight function $\omega(\cdot)$ is defined by the expression

$$\omega(t) = \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}, \quad s_0 = \pi,$$

h_k, $k = \overline{1,r}$ are jumps of the function $\theta(\cdot)$ at points $-\pi < s_1 < \cdots < s_r < \pi$, $h_0 = \theta(-\pi) - \theta(\pi)$. Then the system of exponents (4.1) forms a basis for $L_{p,\nu}$, $1 < p < +\infty$. For $p = 2$ it forms a Riesz basis for $L_{2,\nu}$ if and only if $\nu \sim 1$ on $[-\pi,\pi]$.

Let us consider some special cases of this theorem. Let the weight function $\nu(\cdot)$ have the form

$$\nu(t) = \prod_{k=1}^{m}|t-t_k|^{\alpha_k},\tag{4.11}$$

where $\{t_k\}_1^m \subset (-\pi,\pi)$ are distinct points. Suppose that the condition

$$\{s_k\}_1^r \bigcap \{t_k\}_1^m = \emptyset\tag{4.12}$$

holds. In this case, the product $\omega^{-p}\nu$ has the representation

$$\omega^{-p}(t)\,\nu(t) = \prod_{k=1}^{m} |t-t_k|^{\alpha_k} \prod_{k=0}^{r} \left|\sin\frac{t-s_k}{2}\right|^{-\frac{ph_k}{2\pi}}.$$

It is easy to see that $\omega^{-p}\nu \in A_p$ is true if and only if the following inequalities are satisfied (see, e.g., J. Garnett [20])

$$-1 < -\frac{ph_k}{2\pi} < p-1, \quad k=\overline{0,r},$$
$$-1 < \alpha_k < p-1, \quad k=\overline{1,m}.$$

Thus, the following corollary is true.

Corollary 4.1. *Let the condition* (4.12) *hold and the inequalities*

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p}, \quad k=\overline{0,r},$$
$$-1 < \alpha_k < p-1, \quad k=\overline{1,m},$$

be fulfilled. Then the system of exponents (4.1) *forms a basis in* $L_{p,\nu}$, $1<p<+\infty$.

Consider the particular case for the functions $\theta(\cdot)$ and $\nu(\cdot)$

$$\theta(t) = \alpha\, t + \beta \operatorname{sgn} t, \quad t\in[-\pi,\pi], \quad \nu(t) = |t|^{\gamma}.$$

The function $\theta(\cdot)$ has a unique point of discontinuity $s_1=0$. We have

$$h_1 = \theta(+0)-\theta(-0) = 2\beta, \quad h_0 = \theta(-\pi)-\theta(\pi) = -2\alpha\pi - 2\beta.$$

As a result $\omega(\cdot)$ is of the form

$$\omega(t) = \left|\sin\frac{t-\pi}{2}\right|^{-\frac{\alpha\pi+\beta}{\pi}} \left|\sin\frac{t}{2}\right|^{\frac{\beta}{\pi}}$$
$$\sim |t|^{\frac{\beta}{\pi}} |t-\pi|^{-\left(\alpha+\frac{\beta}{\pi}\right)} |t+\pi|^{-\left(\alpha+\frac{\beta}{\pi}\right)}, \quad t\in[-\pi,\pi].$$

Consequently,

$$\omega^{-p}(t)\,\nu(t) \sim |t|^{\frac{-p\beta}{\pi}+\gamma} |t-\pi|^{\left(\alpha+\frac{\beta}{\pi}\right)p} |t+\pi|^{\left(\alpha+\frac{\beta}{\pi}\right)p}.$$

Applying Theorem 4.1, we obtain the following.

Corollary 4.2. *Let the inequality*

$$-1<\gamma<p-1, \quad -1<\gamma-\frac{p\beta}{\pi}<p-1, \quad -\frac{1}{p} < \alpha+\frac{\beta}{\pi} < \frac{1}{q},$$

be fulfilled. Then the system of exponents

$$\left\{e^{i[(n+\alpha\operatorname{sgn} n)t+\beta\operatorname{sgn} t\operatorname{sgn} n]}\right\}_{n\in\mathbb{Z}},$$

forms a basis for $L_{p,|t|^{\gamma}}$, $1<p<+\infty$.

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