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CHAOS AND SHADOWING IN GENERAL SYSTEMS

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Abstract. In this paper we describe some basic notions of topological dynamical systems for maps of type $f : X \times X \to X$ named general systems. This is proved that every uniformly expansive general system has the shadowing property and every uniformly contractive general system has the (asymptotic) average shadowing and shadowing properties. In the rest, Devaney chaos for general systems is considered. Also, we show that topological transitivity and density of periodic points of a general systems imply topological ergodicity. We also obtain some results on the topological mixing and sensitivity for general systems.

1. Introduction

Shadowing and ergodic properties in discrete dynamical systems have received increasing attention in recent years [4–7]. Many authors investigated the relation between shadowing properties and other ergodic properties such as mixing and transitivity [10, 12, 14]. In [2] Blank introduced the notion of average-shadowing property and Gu [9] followed the same scheme to introduce the notion of the asymptotic average shadowing property. In [14] Sakai considered various shadowing properties for positively expansive maps on compact metric spaces and prove that for a positively expansive map; Lipschitz shadowing property, the s-limit shadowing property and the strong shadowing property are all equivalent to the shadowing property. He also prove that average shadowing property and topological transitivity are equivalent for every positively expansive map on a compact metric space. Theorem B in [3] shows that the two-sided limit shadowing property implies topological mixing. In [5, 6] the author introduce uniformly contractive (expansive) iterated function systems (IFS)

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and prove that every uniformly expansive IFS has shadowing property and every uniformly contractive IFS has shadowing and (asymptotic) average shadowing properties. R. Gu [9,11] prove that every onto continuous map on a compact metric space with (asymptotic) average shadowing property is chain transitive. Also, in [5,6] the author prove similar results for iterated function systems.

The relationship between chaos and shadowing is an interesting topic for many researchers in the recent years. There are different definitions of chaos. One of the popular definition is Devaney chaos. Indeed a map f is chaotic in the case of Devaney if the periodic points of f is dense, f is topologically transitive and is sensitive. This is well known that the density of periodic points and topological transitivity imply sensitivity. Sanz-Serna [15] devised a method to simulate chaos by use of shadowing lemma. In [1], the authors introduced the notion of P-chaos by changing the condition of transitivity in the definition of Devaney chaos to the shadowing property, and they proved that every P-chaotic systems on a connected space is Devaney chaotic with positive topological entropy.

In this paper we consider a generalization for discrete dynamical systems which introduced in [13]. The main idea of this generalization is based on considering maps $f : X \times X \to X$ instead of maps $f : X \to X$, as discrete dynamical systems. Firstly, we define basic notions, such as, orbit, periodic orbit, shadowing and ergodic properties which we need in the following. Section 3 is devoted to shadowing properties, the main result of this section is Theorem 3.1 which shows that in generalized dynamics uniformly expansivity implies shadowing property. Then two examples of general systems on symbolic space and unit circle are given which have shadowing properties. In section 4, we study the chaotic properties of a general dynamical system. We show that similar original maps and non-autonomous discrete systems [16], the density of periodic points and topological transitivity imply sensitivity in general systems. Finally, we obtain some notions such as topological ergodicity, topological mixing and sensitivity for general systems.

2. PRELIMINARIES

Let (X, d) be a complete metric space and $f : X \times X \to X$ be a continuous map. For $x \in X$, define the orbit of x as follows: $O(x) = \{x_n\}_{n=0}^{\infty}$, where $x_1 = x_0 = x$ and $x_{n+1} = f(x_{n-1}, x_n)$ for all $n \geq 1$.

We say that $x \in X$ is a periodic point of period m if $x_{km+i} = x_i$ for every $k \in \mathbb{N}$ and $0 \leq i \leq n$.

The map f is called to be sensitive if there is $e > 0$ such that for every $x \in X$ and every open subset U of X containing x, there is a point $y \in U$ and $n \in \mathbb{N}$ such that $d(x_n, y_n) > e$.

We say that f is topologically transitive if for every nonempty open sets U, V, if there is $z \in U$ such that for some $m \in \mathbb{N}$, $z_m \in V$. We say that f is chaotic in the sense of Devaney on X if:

1. f is topologically transitive in X;

2. the set of all periodic point of f is dense in X;
3. f is sensitive.

Definition 2.1. The map $f : X \times X \to X$ is said to be contractive if there is a constant $0 < \alpha < 1$, called a contractive constant, such that for every disjoint points $(x, y), (z, w) \in X \times X$ then $d(f(x, y), f(z, w)) < \alpha \max\{d(x, z), d(y, w)\}$.

3. Shadowing and Expanding

For given $\delta > 0$, a sequence $\{x_n\}_{n\geq 0}$ in X is said to be a δ-pseudo orbit of $f : X \times X \to X$ if $x_1 = x_0$ and for every $n \geq 1$ we have $d(x_{n+1}, f(x_{n-1}, x_n)) < \delta$.

One says that the map $f : X \times X \to X$ has the *shadowing property* if for given $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\geq 0}$ there exists $y_0 \in X$ such that $d(x_0, y_0) < \epsilon$ and $d(x_n, f(y_{n-2}, y_{n-1})) \leq \epsilon$ for all $n \geq 2$. In this case one says that the orbit $\{y_n\}_{n\geq 0}$ or the point y_0, ϵ-shadows the δ-pseudo orbit $\{x_n\}_{n\geq 0}$.

Definition 3.1. The map $f : X \times X \to X$ is said to be uniformly expansive if there exists constants $0 < \lambda < 1$ such that for $\mathrm{x}, \mathrm{y} \in X \times X$

$$d(f(\mathrm{x}), f(\mathrm{y})) > \lambda^{-1} d'(\mathrm{x}, \mathrm{y}),$$

where $\mathrm{x} = (x_1, x_2)$, $\mathrm{y} = (y_1, y_2)$ and $d'((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$.

Definition 3.2. A sequence $\{x_i\}_{i\geq 0}$ of points in X is called an asymptotic average pseudo orbit of f if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} d(f(x_{i-1}, x_i), x_{i+1}) = 0.$$

A sequence $\{x_i\}_{i\geq 0}$ in X is said to be asymptotically shadowed in average by a point z in X if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(z_i, x_i) = 0,$$

where $\{z_i\}_{i\geq 0}$ is orbit of the point z.

Definition 3.3. Let $f : X \times X \to X$ be a continuous map. For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ of points in X is called a δ-average-pseudo-orbit of f if there is a number $N = N(\delta)$ such that for all $n \geq N$

$$\frac{1}{n} \sum_{i=1}^{n-1} d(f(x_{i-1}, x_i), x_{i+1}) < \delta.$$

We say that f has the average shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ϵ-shadowed in average by some point $y \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(y_i, x_i) < \epsilon,$$

where $\{y_i\}_{i\geq 0}$ is orbit of the point y.

In the next theorem whose proof is based on [8, Theorem 2.2], we provide some coefficient conditions for a general system to have the shadowing property.

Theorem 3.1. *Let $f : X \times X \to X$ be an uniformly expansive map and for every $x \in X$ the restricted functions $f : \{x\} \times X \to X$ and $f : X \times \{x\} \to X$ be surjective, then f has the shadowing property.*

Proof. The main idea of the proof is to find a Cauchy sequence which converges to a point that ϵ-traced our considered δ-pseudo orbit. Assume that for every $x \in X$ the orbit of x, denoted by $\{x^{f,n}\}_{n\geq 0}$, as $x^{f,0} = x$, $x^{f,1} = x$ and $x^{f,n+1} = f(x^{f,n-1}, x^{f,n})$ for all $n \geq 1$. For given $\epsilon > 0$ take $\delta = (\lambda - 1)\epsilon$, where $0 < \lambda < 1$ is expansivity constant and let $\{x_n\}$ be a δ-pseudo orbit of f. Consider the sequence $\{z_n\}_{n\geq 0}$ in X defined as follows: $z_0 = x_0$, $z_1 = x_1 = x_0$ and z_2 be a point that $x_2 = f(z_1, z_2)$ and for every $n > 2$, z_n be a point that $x_n = z_n^{f,n}$. Given $n \geq 1$ and $0 \leq k \leq n-1$, denote

$$z_{n,k} = z_n^{f,k}. \tag{3.1}$$

This implies that for any $n \geq 1$ and $2 \leq k \leq n-1$ we have:

$$z_{n,k} = f(z_n^{f,k-2}, z_n^{f,k-1}), \quad x_n = f(z_{n,n-2}, z_{n,n-1}). \tag{3.2}$$

Claim. The sequence $\{z_n\}_{n\geq 0}$ in X is a Cauchy sequence.

Proof of Claim. Consider the function $\varphi : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\varphi(s,t) = \begin{cases} \lambda, & s = t, \\ \dfrac{d(f(s), f(t))}{d'(s,t)}, & s \neq t, \end{cases}$$

where $0 < \lambda < 1$ is the expansivity ratio number. This implies that for every $(a,b) \neq (c,d) \in X \times X$, we have that

$$d(a,c) \leq \frac{d(f(a,b), f(c,d))}{\lambda} \quad \text{and} \quad d(b,d) \leq \frac{d(f(a,b), f(c,d))}{\lambda}. \tag{3.3}$$

Firstly, fixing $n \geq 1$ and $m \geq 1$, by using (3.1), (3.2) and above inequalities we obtain:

$$d(z_n, z_{n+m}) \leq \frac{d(z_{n,1}, z_{n+m,1})}{\lambda} \leq \frac{d(z_{n,2}, z_{n+m,2})}{\lambda^2} \leq \cdots \leq \frac{d(x_n, z_{n+m,n-1})}{\lambda^{n-1}}.$$

Secondly, by induction on $m \geq 1$ we show that the following inequality holds uniformly with respect to $n \geq 1$:

$$d(x_n, z_{n+m,n-1}) \leq \delta \sum_{k=1}^{m} \lambda^{-k}. \tag{3.4}$$

Indeed, for $m = 1$ the inequality (3.4) follows from (3.2) and (3.3):

$$d(x_n, z_{n+1,n-1}) \leq \frac{d(f(x_{n-1}, x_n), f(z_{n+1,n-2}, z_{n,n-1}))}{\lambda} = \frac{d(f(x_{n-1}, x_n), x_{n+1})}{\lambda} \leq \frac{\delta}{\lambda}.$$

Assume that (3.4) holds for some $m = p \geq 1$ uniformly on $n \geq 1$. Taking into account this assumption, as well as (3.2), (3.3) and (3.4) for $m = p + 1$:

$$\begin{aligned} d(x_n, z_{n+p+1,n-1}) &\leq \frac{d(f(x_{n-1}, x_n), f(z_{n+p+1,n-2}, z_{n+p,n-1}))}{\lambda} \\ &= \frac{d(f(x_{n-1}, x_n), z_{n+p+1,n})}{\lambda} \\ &\leq \frac{d(f(x_{n-1}, x_n), x_{n+1}) + d(x_{n+1}, z_{n+p+1,n})}{\lambda} \\ &\leq \frac{1}{\lambda}\left(\delta + \delta \sum_{k=1}^{p} \lambda^{-k}\right) \\ &\leq \delta \sum_{k=1}^{p+1} \lambda^{-k}. \end{aligned}$$

Then (3.4) holds for any $m \geq 1$ and any $n \geq 1$.

So, we have the following relation:

$$d(z_n, z_{n+m}) \leq \frac{1}{\lambda^n}\delta \sum_{k=1}^{m} \lambda^{-k} \leq \frac{1}{\lambda^n} \cdot \frac{\delta}{\lambda - 1} = \frac{\epsilon}{\lambda^n} \cdot \frac{\delta}{\lambda - 1} \leq \epsilon\lambda^{-n}. \tag{3.5}$$

Hence, $\{z_n\}_{n \geq 0}$ in X is a Cauchy sequence.

Now, we continue the proof of the theorem.

Let y denote its limit and consider the sequence $\{y^{f,n}\}$ as orbit of y. From (3.1) one has for any $k \geq 0$

$$\lim_{n \to \infty} z_{n,k} = y^{f,k}.$$

Letting $m \to \infty$ in (3.5) implies $d(z_n, y) \leq \epsilon\lambda^{-n}$, and consequently

$$d(x_n, y^{f,n}) \leq \lambda^n(\lambda^{-n}\epsilon) = \epsilon.$$

Therefore, the orbit $\{y^{f,n}\}_{n \geq 0}$ ϵ-shadows the δ-pseudo orbit $\{x_n\}_{n \geq 0}$. □

Theorem 3.2. *If $f : X \times X \to X$ is uniformly contracting, then it has shadowing property.*

Proof. Assume that $0 < \beta < 1$ is the contracting ratio of f. Given $\epsilon > 0$ take $\delta = \frac{(1-\alpha)\epsilon}{2}$ and suppose that $\{x_i\}_{i \geq 0}$ is a δ-pseudo orbit for f. So, $d(f(x_{i-1}, x_i), x_{i+1}) < \delta$ for all $i \geq 1$. Put $\beta_i = d(f(x_{i-1}, x_i), x_{i+1})$ for all $i \geq 1$. Consider an orbit $\{y_i\}_{i \geq 0}$ such that $d(y_0, x_0) < \frac{\epsilon}{2}$ and $y_{i+1} = f(y_{i-1}, y_i)$ for all $i \geq 1$.

Now we will show that $d(y_i, x_i) < \epsilon$ for all $i \geq 0$. Put $M = d(x_0, y_0)$. Obviously,

$$d(x_1, y_1) \leq d(x_1, f(x_0, x_0)) + d(f(x_0, x_0), f(y_0, y_0)) \leq \beta_0 + \alpha M.$$

Similarly,

$$\begin{aligned} d(x_2, y_2) \leq& d(x_2, f(x_0, x_1)) + d(f(x_0, x_1), f(y_0, y_1)) \\ \leq& \beta_1 + \alpha d(x_1, y_1) \\ \leq& \beta_1 + \alpha(\beta_0 + \alpha M) \end{aligned}$$

and

$$\begin{aligned} d(x_3, y_3) \leq& d(x_3, f(x_1, x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \beta_2 + \alpha d(x_2, y_2) \\ \leq& \beta_2 + \alpha(\beta_1 + \alpha d(x_1, y_1)) \\ \leq& \beta_2 + \alpha(\beta_1 + \alpha(\beta_0 + \alpha M)) \\ =& \beta_2 + \alpha\beta_1 + \alpha^2\beta_0 + \alpha^3 M. \end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \beta_{i-1} + \alpha\beta_{i-2} + \cdots + \alpha^{i-1}\beta_0 + \alpha^i M.$$

This implies that any

$$d(x_n, y_n) \leq \delta(1 + \alpha + \cdots + \alpha^{n-1}) \leq \frac{1}{1-\beta} + M < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

and so, the proof is complete. □

In [5,6], Fatehi Nia proved that every uniformly contractive IFS has average shadowing property and asymptotic average shadowing property. The next theorems show that similar results are established for general systems.

Theorem 3.3. *If $f : X \times X \to X$ is contracting, then it has the average shadowing property.*

Proof. Assume that $\beta < 1$ is the contracting ratio of f. For given $\epsilon > 0$, take $\delta = \frac{(1-\beta)\epsilon}{2} \leq \frac{\epsilon}{2}$ and suppose $\{x_i\}_{i\geq 0}$ is a δ-pseudo orbit for f. So, there exists a natural number $N = N(\delta)$ such that $\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i, x_{i+1}), x_{i+2}) < \delta$ for all $n \geq N(\delta)$. Put $\alpha_i = d(f(x_i, x_{i+1}), , x_{i+2})$ for all $i \geq 0$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $d(x_0, y_0) < \delta \leq \frac{\epsilon}{2}$ and $y_{i+2} = f(y_i, y_{i+1})$ for all $i \geq 0$.

Now we will show that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) < \epsilon$.

Take $M = d(x_0, y_0)$. Similarly,

$$\begin{aligned} d(x_2, y_2) \leq& d(x_2, f(x_0, x_1)) + d(f(x_0, x_1), f(y_0, y_1)) \\ \leq& \alpha_1 + \beta d(x_1, y_1) \\ \leq& \alpha_1 + \beta(\alpha_0 + \beta M) \end{aligned}$$

and

$$\begin{aligned} d(x_3, y_3) \leq& d(x_3, f(x_1 x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \alpha_2 + \beta d(x_2, y_2) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta d(x_1, y_1)) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta(\alpha_0 + \beta M)) \\ =& \alpha_2 + \beta\alpha_1 + \beta^2\alpha_0 + \beta^3 M. \end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \alpha_{i-1} + \beta\alpha_{i-2} + \cdots + \beta^{i-1}\alpha_0 + \beta^i M.$$

This implies that

$$\begin{aligned}\sum_{i=0}^{n-1} d(y_i, x_i) =& M(1+\beta+\cdots+\beta^{n-1}) + \alpha_0(1+\beta+\cdots+\beta^{n-2}) \\ &+ \alpha_1(1+\beta+\cdots+\beta^{n-3}) + \cdots + \alpha_{n-2} \\ \leq& \frac{1}{1-\beta}\left(M + \sum_{i=0}^{n-2}\alpha_i\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) \leq& \frac{1}{1-\beta}\left(M + \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-2}\alpha_i\right) \\ <& \frac{1}{1-\beta}(M+\delta) \\ \leq& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So, the proof is complete. □

Theorem 3.4. *If a map $f : X \times X \to X$ is uniformly contracting, then it has the asymptotic average shadowing property.*

Proof. Assume that $0 < \beta < 1$ is the contracting ratio of f and suppose $\{x_i\}_{i\geq 0}$ is an asymptotic average pseudo orbit for f. So, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i, x_{i+1}), x_{i+2}) = 0$. Put $\alpha_i = d(f(x_i, x_{i+1}), , x_{i+2})$, for all $i \geq 0$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $y_0 \in X$, $y_1 = f(y_0, y_0)$ and $y_{i+2} = f(y_i, y_{i+1})$, for all $i \geq 0$.

Now, we will show that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) = 0$.

Put $M = d(x_0, y_0)$. Obviously,

$$d(x_2, y_2) \leq d(x_2, f(x_0, x_0)) + d(f(x_0, x_0), f(y_0, y_1)) \leq \alpha_0 + \beta M.$$

Similarly,

$$\begin{aligned}d(x_3, y_3) \leq& d(x_3, f(x_1, x_2)) + d(f(x_1, x_2), f(y_1, y_2)) \\ \leq& \alpha_2 + \beta d(x_2, y_2) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta d(x_1, y_1)) \\ \leq& \alpha_2 + \beta(\alpha_1 + \beta(\alpha_0 + \beta M)) \\ =& \alpha_2 + \beta\alpha_1 + \beta^2\alpha_0 + \beta^3 M.\end{aligned}$$

By induction, one can prove that for each $i > 2$

$$d(x_i, y_i) \leq \alpha_{i-1} + \beta\alpha_{i-2} + \cdots + \beta^{i-1}\alpha_0 + \beta^i M.$$

This implies that

$$\begin{aligned}\sum_{i=0}^{n-1} d(y_i, x_i) \leq & M(1+\beta+\cdots+\beta^{n-1}) \\ & + \alpha_0(1+\beta+\cdots+\beta^{n-2}) \\ & + \alpha_1(1+\beta+\cdots+\beta^{n-3}) + \cdots + \alpha_{n-2} \\ \leq & \frac{1}{1-\beta}\left(M + \sum_{i=0}^{n-2} \alpha_i\right).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(y_i, x_i) \leq \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1-\beta}\left(M + \sum_{i=0}^{n-2} \alpha_i\right)\right) = 0,$$

and so, the proof is complete. □

In the following, we introduce some non trivial examples of general systems on real line, symbolic space and unit circle, that have shadowing properties.

Example 3.1. Consider the following maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = \frac{1}{2}x, \quad f_2(x) = 2x.$$

Take the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = \frac{f_1(x)+f_2(y)}{3}$. So, for every disjoint points $(x, y), (z, w) \in \mathbb{R} \times \mathbb{R}$ then $d(f(x, y), f(z, w)) < \frac{2}{3} \max\{d(x, z), d(y, w)\}$. Then this general system is contracting and has the shadowing properties.

Example 3.2. Let Σ denote the set of all infinite sequence $x = (x_0, x_1, x_2, \dots)$, where $x_n = 0$ or 1. The set Σ becomes a compact metric space if we define the distance between two points x, y by $\rho(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$.

Now, consider the map $f : \Sigma \times \Sigma \to \Sigma$ defined by

$$f(\{x_i\}_{i\geq 0}\{y_i\}_{i\geq 0}) = (x_0, y_0, x_1, y_1, \dots).$$

Please note that if the sequences $\{x_i\}_{i\geq 0}$ and $\{z_i\}_{i\geq 0}$ are equal in n initial elements and $\{y_i\}_{i\geq 0}$ and $\{w_i\}_{i\geq 0}$ are equal in m initial elements, then $f(\{x_i\}_{i\geq 0}, \{y_i\}_{i\geq 0})$ and $f(\{z_i\}_{i\geq 0}\{w_i\}_{i\geq 0})$ are equal in $m + n$ initial elements. This implies that

$$\begin{aligned}& \rho(f(\{x_i\}_{i\geq 0}, \{y_i\}_{i\geq 0}), f(\{z_i\}_{i\geq 0}, \{w_i\}_{i\geq 0})) \\ < & \frac{1}{2} \max\left\{\rho(\{x_i\}_{i\geq 0}, \{z_i\}_{i\geq 0}), \rho(\{y_i\}_{i\geq 0}, \{w_i\}_{i\geq 0})\right\}.\end{aligned}$$

Consequently, the map $f : \Sigma \times \Sigma \to \Sigma$ is contracting and has the shadowing properties mentioned above.

Example 3.3. Consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. The natural distance on $\mathbb{R}$ induces a distance, d, on S^1. Let $f : S^1 \times S^1 \to S^1$ be a map defined by $f(x, y) = (2x + 3y) \pmod 1$. This is clear that this is an uniformly expanding map and for every $x, y \in S^1$

the maps $f:\{x\}\times S^1\to S^1$ and $f:S^1\times\{y\}\to S^1$ are surjective. Then, Theorem 3.1 implies that the function $f:S^1\times S^1\to S^1$ as a general system has the shadowing property.

4. Chaos

In this section, we consider the notion of Devaney's chaos for general systems and prove some results about the relations between this notion and some main properties in general systems.

Theorem 4.1. *Let X be an unbounded metric space with no isolated points. If $f:X\times X\to X$ is topologically transitive and the set of all periodic points is dense in X, then it is sensitive.*

Proof. Let $x\in X$ be an arbitrary point and U be any neighborhood of x. We will show that there exist $z\in U$ and $m>0$ such that $d(x_m,z_m)>\frac{1}{4}$. Since there are not isolated points and by density of the periodic points, there exists a periodic point $y\in U$ such that $y\neq x$. Put $c:=\max\{d(x,z)>0 : z\in O(y)\}$. Let $c>\frac{1}{2}$. Since X is unbounded, $X\setminus\overline{B_{2c}(x)}$ is a nonempty open subset. Topological transitivity of f implies that there is $y'\in U$ and $m'>0$ such that $y'_{m'}\in X\setminus\overline{B_{2c}(x)}$.

On the other hand $O(y)\subset\overline{B_c(x)}$, therefore

$$d(y_{m'},y'_{m'})\geq d(x,y'_{m'})-d(x,y_{m'})>2c-c=c>\frac{1}{2}.$$

So, we have either $d(x_{m'},y'_{m'})>\frac{1}{4}$ or $d(x_{m'},y_{m'})>\frac{1}{4}$.

The above result is once $c>\frac{1}{2}$. Now, suppose that $c\leq\frac{1}{2}$. By transitivity, there exists $y''\in U$ and $m''>0$ such that $y''_{m''}\in X\setminus\overline{B_1(x)}$. Also we have that $y_{m''}\in\overline{B_c(x)}\subset\overline{B_{\frac{1}{2}}(x)}$. Hence,

$$d(y_{m''},y''_{m''})\geq d(x,y''_{m''})-d(x,y_{m''})>1-\frac{1}{2}=\frac{1}{2}.$$

Thus, either $d(x_{m''},y''_{m''})>\frac{1}{4}$ or $d(x_{m''},y_{m''})>\frac{1}{4}$.

So, the proof is complete. ☐

Corollary 4.1. *Let X be an unbounded metric space with no isolated points. If $f:X\times X\to X$ is topologically transitive and the set of all periodic points is dense in X, then it is chaotic in the sense of Devaney.*

Remark 4.1. If $f:X\to X$ (X is a complete metric space) and $O(x)=\{x_n\}_{n=0}^{\infty}$, where $x_{n+1}=f(x_n)$, then we have $O(x_k)\subseteq O(x)$ for every $k\geq 1$. In this case f is topological transitive if and only if it is transitive (f has a dense orbit). But, for a general system $f:X\times X\to X$ may the above fact is not true. For example for $x\in X$

$$\begin{aligned}O(x)=&\{x=x_0,x_1=f(x,x),x_2=f(x_0,x_1),\dots\},\\ O(x_1)=&\{x_1=(x_1)_0,(x_1)_1=f(x_1,x_1),(x_1)_2=f((x_1)_0,(x_1)_1),\dots\},\end{aligned}$$

and may $f(x_1, x_1) \notin O(x)$. In this case the density of an orbit of a point may be does not show topological transitivity. Indeed if U and V are two nonempty open subsets of X, then the density of an orbit of a point z implies there are positive integers $n > m$ such that $z_m \in U$ and $z_n \in V$. But this does not show the topological transitivity, because z_n may be not in the orbit of z_m.

The above remark motivated us to define "*strong dense orbit*" of x as follows.

We say that the orbit of $x \in X$ is strong dense orbit if the orbit of x is dense and every element of the orbit of x is also dense in X. We say that the map $f : X \times X \to X$ is strong transitive if it has a strong dense orbit.

Theorem 4.2. *Let X be a complete metric space. If the map $f : X \times X \to X$ is strong transitive, then it is topological transitive. If the map $f : X \times X \to X$ is topological transitive, then it is transitive* (f *has a dense orbit*).

Proof. Let the orbit of z be strong dense orbit and U and V be two nonempty open subsets of X. Then the density of the orbit of point z implies there is a positive integer n such that $z_n \in U$. The strong density of the orbit z implies the orbit of z_n meets V. This shows that f is topological transitive. Suppose that f is topological transitive and U_i, $i = 1, 2, \ldots$, are a countable basis of X. Put $O^-(U_i) = \{x \in X : O(x) \cap U_i \neq \emptyset\}$. Since f is continuous and topological transitive, so $O^-(U_i)$ is open and dense in X. Since X is complete, so $\bigcap U_i \neq \emptyset$. The orbit of every $x \in \bigcap U_i$ is dense in X. This implies f is transitive. $\square$

We say that the map $f : X \times X \to X$ is topologically ergodic if for every two nonempty open sets $U, V \subset X$ there exist an increasing sequence of positive integers $\{n_k\}_{k=0}^{\infty}$ and an integer $l \geq 1$ such that for every $k \geq 1$, $n_{k+1} - n_k \leq l$, there is $z \in U$ such that $z_{n_k} \in V$.

Theorem 4.3. *Let X be a compact metric space and $f : X \times X \to X$ be a continuous map. If f is topologically transitive and the periodic points of f are dense in X, then f is topologically ergodic.*

Proof. Let U and V be two nonempty open subsets of X. Since f is topologically transitive, there is $x \in U$ and $n > 0$ such that $x_n \in V$. Consider $\epsilon > 0$ such that $B_\epsilon(x_n) \subset V$. By continuity of f, there exists open neighborhood W of x such that $W_n \subset V$ is as follows:

$$W = W_0, \quad W_1 = f(W_0, W_0), \quad W_2 = f(W_0, W_1), \ldots, W_n = f(W_{n-2}, W_{n-1}).$$

We can see that $x_n \in W_n$. Since the set of all periodic points is dense in X, there exists a periodic point $q \in W$ with period m. Therefore, $q_n \in W_n \subset V$. So, for each $k \geq 0$ we have $q_{n+km} = q_n \in V$. Hence, for each $k \geq 0$, $q_{km} = q \in U$ and $q_{n+km} = q_n \in V$. So, f is topologically ergodic. $\square$

Let $f : X \times X \to X$ be a continuous map. For $x, y \in X$ and $\epsilon \geq 0$ given, an ϵ-chain from x o y of length $n + 1$ is a sequence $\{x = x_0, x_1, x_2, \ldots, x_n = y\}$ for which

$d(x_{i+1}, f(x_{i-1}, x_i)) < \epsilon$ for each $1 \leq i \leq n-1$. f is said to be topologically chain transitive if for every $x, y \in X$, there exists an ϵ-chain from x to y for every $\epsilon > 0$.

We say that f is topologically chain mixing if for every $\epsilon > 0$ and $x, y \in X$ there is $N \in \mathbb{N}$ such that for each $n \geq N$, there exists an ϵ-chain from x to y of length n.

Lemma 4.1. *If f is topologically chain mixing and has the shadowing property then f is topologically mixing.*

Proof. The proof is clear. □

Theorem 4.4. *Let $f : X \times X \to X$ be an open continuous map with a fixed point a, $f(a, a) = a$. If f is topologically transitive, then f is chain mixing.*

Proof. Let $x, y \in X$ and $\epsilon \geq 0$ be given. Since f is topologically transitive there exist $z, z' \in X$ and $m, m' \in \mathbb{N}$ such that

$$\begin{aligned} d(x_1 = f(x,x), z) &< \epsilon, \\ d(z_m, a) &< \epsilon, \\ d(z', a) &< \epsilon, \\ d(z'_m, y) &< \epsilon. \end{aligned}$$

Put $N = m + m' + 1$. So, for each $n \geq N$ sequence $\{x = x_0, z, \ldots, z_{m-1}, \underbrace{a, a, \ldots, a}, z', \ldots, z'_{m'-1}, y\}$ is an ϵ-chain of length n. Hence, f is chain mixing. □

Theorem 4.5. *By assumption of previous theorem, if f has the shadowing property, then f is topologically mixing.*

Proof. By previous theorem and lemma proof is complete. □

Definition 4.1. We say that $f : X \times X \to X$ is n-sensitive if there is integer $e > 0$ such that for every non empty open subset $U \subset X$, there exist pairwise disjoint points $x_1, \ldots, x_n \in U$ and $k \in \mathbb{N}$ such that

$$\min_{1 \leq i \neq j \leq n} d((x_i)_k, (x_j)_k) > e.$$

Theorem 4.6. *Let $f : X \times X \to X$ be a continuous transitive map with n fixed points $p_1, \ldots, p_n$. If f has the shadowing property, then f is n-sensitive.*

Proof. Suppose $e = \frac{1}{2} \min\{d(p_i, p_j) : i \neq j\}$ and U be an open subset of X. Let $x_0 \in U$ and $0 < \epsilon < \frac{e}{2}$ such that $B_\epsilon(x_0) \subset U$. By assumption of theorem and previous theorem, f is topologically mixing. So for every $1 \leq i \leq n$, there exists k_i such that there is δ-chain of length l from x_0 to p_i for every $l \geq k_i$. Where $\delta > 0$ is in the definition of shadowing property for $\epsilon > 0$.

Hence, for every $1 \leq i \leq n$ there exists $z_i \in U$ such that $d(z_i, x_0) < \epsilon$ and $d((z_i)_l, p_i) < \epsilon$. Put $k = \max\{k_i : 1 \leq i \leq n\}$. Therefore, $\{z_1, z_2, \ldots, z_n\} \subset U$ and $d((z_i)_k, p_i) < \epsilon$.

Hence, we have

$$\min_{1\leq i\neq j\leq n} d((z_i)_k,(z_j)_k) > \frac{e}{4}.$$

This prove the theorem. $\square$

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