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ON THE REVERSE MINKOWSKI'S INTEGRAL INEQUALITY

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Abstract. The aim of this work is to obtain the reverse Minkowski integral inequality. For this aim, we first give a proposition which is important for our main results. Then we establish some reverse Minkowski integral inequalities for parameters $0 < p < 1$ and $p < 0$, respectively.

1. Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics and present a very active and attractive field of research. As example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals ([2, 3]). One of the famous integral inequalities is Minkowski's integral inequality. In particular the following statement was proved for $p \geq 1$ (for details to see [1]).

Theorem 1.1. *Let $1 \leq p \leq +\infty$, $\Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be a measurable sets. Suppose that f is measurable on $\Omega \times A$ and $f(\cdot, y) \in L_p(\Omega)$ for almost all $y \in A$. Then*

$$\left\| \int_A f(\cdot, y) dy \right\|_{L_p(\Omega)} \leq \int_A \|f(\cdot, y)\|_{L_p(\Omega)} dy, \tag{1.1}$$

if the right-hand side is finite.

Remark 1.1. If $0 < p < 1$, $\operatorname{mes} A > 0$ and $\operatorname{mes} \Omega > 0$ inequality (1.1) is not valid (to see [1]).

In this paper we obtain some integral inequalities which are reverse versions of the inequality (1.1).

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2. Preliminaries

2.1. **Reverse Young's and Holder's Inequalities.** The following inequalities are well-known Young inequalities. Let $a>0$, $b>0$ and $\frac{1}{p}+\frac{1}{p'}=1$, then

$$ab \leq \frac{a^p}{p}+\frac{b^{p'}}{p'}, \quad \text{for } p \geq 1, \tag{2.1}$$

$$ab \geq \frac{a^p}{p}+\frac{b^{p'}}{p'}, \quad \text{for } 0<p<1. \tag{2.2}$$

Corollary 2.1 (Reverse Young's inequality)**.** *Let $a>0$, $b>0$ and $\frac{1}{p}+\frac{1}{p'}=1$, then*

$$ab \geq \frac{a^p}{p}+\frac{b^{p'}}{p'}, \quad \textit{for } p<0. \tag{2.3}$$

Proof. We have $\frac{p'-1}{p'}=\frac{1}{p}$, $(p-1)(p'-1)=1$ and inequality (2.3) is equivalent to

$$\frac{a^{p-1}}{bp}+\frac{b^{p'-1}}{ap'} \leq 1.$$

We take $t=\frac{a^{p-1}}{b}$, then

$$\frac{b^{p'-1}}{ap'}=\frac{a^{(p-1)(p'-1)}}{t^{(p'-1)}ap'}=\frac{1}{t^{(p'-1)}p'}=\frac{t^{-(p'-1)}}{p'}.$$

We obtain

$$\frac{a^{p-1}}{bp}+\frac{b^{p'-1}}{ap'}=\frac{t}{p}+\frac{t^{-(p'-1)}}{p'}=f(t), \quad t>0.$$

For all $t>0$, we have

$$f'(t)=\frac{1}{p}-\frac{p'-1}{p'}t^{-p'}=\frac{1}{p}-\frac{1}{p}t^{-p'}=\frac{1}{p}(1-t^{-p'}),$$

for all $p<0$ and $0<p'<1$, we get

$$\begin{aligned} f'(t)=&0 \Leftrightarrow 1-t^{-p'}=0 \Leftrightarrow t=1,\\ f'(t)>&0 \Leftrightarrow 1-t^{-p'}<0 \Leftrightarrow 0<t<1. \end{aligned}$$

Hence, the function f is majored with $f(1)=1$ for all $t \in (0,\infty)$.

We deduce that

$$\frac{a^{p-1}}{bp}+\frac{b^{p'-1}}{ap'} \leq 1 \Leftrightarrow ab \geq \frac{a^p}{p}+\frac{b^{p'}}{p'}, \quad \text{for } p<0.$$ □

Corollary 2.2 (Reverse Hölder's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $p<0$, we suppose that f, g are measurable on Ω.*

If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$ (p' is the conjugate parameter), then

$$\int_\Omega |fg|dt \geq \|f\|_{L_p}\|g\|_{L_{p'}}. \tag{2.4}$$

Proof. Choose $a = \frac{|f|}{\|f\|_{L_p}}$, $b = \frac{|g|}{\|g\|_{L_{p'}}}$ and by using reverse Young's inequality (2.3), we write

$$\frac{|fg|}{\|f\|_{L_p} \cdot \|g\|_{L_{p'}}} \geq \frac{|f|^p}{p\|f\|_{L_p}^p} + \frac{|g|^{p'}}{p'\|g\|_{L_{p'}}^{p'}},$$

by integrand the above inequality we obtain

$$\int_\Omega \frac{|f(t)g(t)|}{\|f\|_{L_p} \cdot \|g\|_{L_{p'}}} dt \geq \int_\Omega \frac{|f(t)|^p}{p\|f\|_{L_p}^p} dt + \int_\Omega \frac{|g(t)|^{p'}}{p'\|g\|_{L_{p'}}^{p'}} dt = 1,$$

and thus

$$\int_\Omega |f(t)g(t)| dt \geq \|f\|_{L_p} \|g\|_{L_{p'}}, \quad \text{for } p < 0.$$

□

Remark 2.1. We can write

$$\int_\Omega |f(t)g(t)| dt \geq \left(\int_\Omega |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_\Omega |g(t)|^{p'} dt\right)^{\frac{1}{p'}},$$

hence

$$\left(\int_\Omega |f(t)g(t)| dt\right)^p \leq \left(\int_\Omega |f(t)|^p dt\right) \left(\int_\Omega |g(t)|^{p'} dt\right)^{p-1}$$

(see [4]).

Now we give a proposition which will be used frequently in the proof of main theorems.

Let $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$ and we defined the set $\mathbb{E}$ by

$$\mathbb{E} = \{f \mid f : (a,b) \times (c,d) \to \mathbb{R},\ f \geq 0 \text{ or } f \leq 0\}.$$

Suppose $H : (a,b) \times (c,d) \to \mathbb{C}$ a measurable function defined by

$$H(x,y) = f_1(x,y) + i\, f_2(x,y),$$

where $f_1,\ f_2 \in \mathbb{E}$.

Proposition 2.1. *(i) If $f_1 = 0$ or $f_2 = 0$, then*

$$\left|\int_c^d |H(x,y)|\, dy\right| = \left|\int_c^d H(x,y) dy\right|. \tag{2.5}$$

(ii) If $f_1 \neq 0$ and $f_2 \neq 0$, then

$$\left|\int_c^d |H(x,y)|\, dy\right| \leq \sqrt{2} \left|\int_c^d H(x,y) dy\right|. \tag{2.6}$$

Proof. (*i*) If $f_2 = 0$, then

$$\left|\int_c^d |H(x,y)|\, dy\right| = \left|\int_c^d |f_1(x,y)|\, dy\right| = \left|\int_c^d f_1(x,y) dy\right| = \left|\int_c^d H(x,y) dy\right|.$$

If $f_1 = 0$, then

$$\left|\int_c^d |H(x,y)|\, dy\right| = \left|\int_c^d |i f_2(x,y)|\, dy\right| = \left|\int_c^d |f_2(x,y)| dy\right|$$

$$= \left| \int_c^d f_2(x,y)dy \right| = \left| \int_c^d i f_2(x,y)dy \right|$$
$$= \left| \int_c^d H(x,y)dy \right|.$$

(ii) If $f_1 \neq 0$ and $f_2 \neq 0$, then

$$\left| \int_c^d |H(x,y)|dy \right|^2 = \left| \int_c^d \left[f_1^2(x,y) + f_2^2(x,y) \right]^{\frac{1}{2}} dy \right|^2$$
$$= \left(\int_c^d \left| f_1^2 + f_2^2 \right|^{\frac{1}{2}} (x,y)dy \right)^2$$
$$= \left\| f_1^2 + f_2^2 \right\|_{L_p(c,d)}, \qquad \text{with } p = \frac{1}{2},$$
$$\left| \int_c^d H(x,y)dy \right|^2 = \left| \int_c^d f_1(x,y)dy + i \int_c^d f_2(x,y)dy \right|^2$$
$$= \left(\int_c^d f_1(x,y)dy \right)^2 + \left(\int_c^d f_2(x,y)dy \right)^2$$
$$= \left(\int_c^d |f_1(x,y)|dy \right)^2 + \left(\int_c^d |f_2(x,y)|dy \right)^2$$
$$= \left\| f_1^2 \right\|_{L_p(c,d)} + \left\| f_2^2 \right\|_{L_p(c,d)}, \qquad \text{with } p = \frac{1}{2}.$$

For all $0 < p < 1$ we have

$$\left\| f_1^2 + f_2^2 \right\|_{L_p(c,d)} \leq 2^{\frac{1}{p}-1} \left(\left\| f_1^2 \right\|_{L_p(c,d)} + \left\| f_2^2 \right\|_{L_p(c,d)} \right),$$

for $p = \frac{1}{2}$ we obtain

$$\left| \int_c^d |H(x,y)|\, dy \right|^2 \leq 2 \left| \int_c^d H(x,y)dy \right|^2.$$

Then

$$\left| \int_c^d |H(x,y)|\, dy \right| \leq \sqrt{2} \left| \int_c^d H(x,y)dy \right|. \qquad \square$$

In this work we consider the reverse inequality of (1.1), with $0 < p < 1$ and $p < 0$ for $f : (a,b) \times (c,d) \to \mathbb{K}$, with $\mathbb{K}$ is $\mathbb{C}$, $\mathbb{E}$ or $i\mathbb{E}$.

3. Main Results

In this section we obtain some reverse Minkowski type inequalities.

Theorem 3.1. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b) \times (c,d) \to \mathbb{C}$ *is measurable with* $\mathrm{Re}(H), \mathrm{Im}(H) \in \mathbb{E}$, $\mathrm{Re}(H)\mathrm{Im}(H) \neq 0$ *and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\| \int_c^d H(\cdot, y) dy \right\|_{L_p(a,b)} \geq (\sqrt{2})^{p-2} \int_c^d \|H(\cdot, y)\|_{L_p(a,b)} dy, \tag{3.1}$$

if left-hand side is finite.

Proof. We have

$$\left| \int_c^d H(x,y) dy \right| \leq \int_c^d |H(x,y)| \, dy.$$

Then for $p - 1 < 0$ we get

$$\left| \int_c^d H(x,y) dy \right|^{p-1} \geq \left(\int_c^d |H(x,y)| \, dy \right)^{p-1}.$$

By Proposition 2.1, we obtain

$$\begin{aligned}
\left| \int_c^d H(x,y) dy \right|^p &= \left| \int_c^d H(x,y) dy \right|^{p-1} \left| \int_c^d H(x,y) dy \right| \\
&\geq \left(\int_c^d |H(x,y)| \, dy \right)^{p-1} \left| \int_c^d H(x,y) dy \right| \\
&\geq \left(\int_c^d |H(x,y)| \, dy \right)^{p-1} (\sqrt{2})^{-1} \left| \int_c^d |H(x,y)| dy \right| \\
&= (\sqrt{2})^{-1} \left(\int_c^d |H(x,y)| \, dy \right)^{p-1} \left| \int_c^d |H(x,y)| dy \right|.
\end{aligned}$$

By integrating the last inequality, we establish

$$\begin{aligned}
\int_a^b \left| \int_c^d H(x,y) dy \right|^p dx &\geq (\sqrt{2})^{-1} \int_a^b \left(\int_c^d |H(x,t)| \, dt \right)^{p-1} \left| \int_c^d |H(x,y)| dy \right| dx \\
&= (\sqrt{2})^{-1} \int_a^b \left| \int_c^d \left(\int_c^d |H(x,t)| \, dt \right)^{p-1} |H(x,y)| dy \right| dx \\
&\geq (\sqrt{2})^{-1} \left| \int_a^b \left\{ \int_c^d \left(\int_c^d |H(x,t)| \, dt \right)^{p-1} |H(x,y)| dy \right\} dx \right| \\
&= (\sqrt{2})^{-1} \left| \int_c^d \left\{ \int_a^b \left(\int_c^d |H(x,t)| \, dt \right)^{p-1} |H(x,y)| dx \right\} dy \right|.
\end{aligned}$$

Let

$$R_1 = \int_a^b \left(\int_c^d |H(x,t)| \, dt \right)^{p-1} |H(x,y)| dx$$

and suppose that $G(x) = \left(\int_c^d |H(x, y)| \, dy \right)^{p-1}$.

Therefore, we get

$$\begin{aligned}
\|G(x)\|_{L_{p'}((a,b))} &= \left(\int_a^b \left| \int_c^d |H(x, y)| \, dy \right|^{p'(p-1)} dx \right)^{\frac{1}{p'}} \\
&= \left(\int_a^b \left| \int_c^d |H(x, y)| \, dy \right|^{p} dx \right)^{\frac{p-1}{p}} \\
&= \left\{ \left(\int_a^b \left| \int_c^d |H(x, y)| \, dy \right|^{p} dx \right)^{\frac{1}{p}} \right\}^{p-1} \\
&= \left\| \int_c^d |H(x, y)| \, dy \right\|_{L_p((a,b))}^{p-1}.
\end{aligned}$$

The last expression is finite (see hypothoses of theorem) then $G(x) \in L_{p'}((a, b))$. By applying the reverse Hölder's inequality and using Proposition 2.1, we obtain

$$\begin{aligned}
R_1 &\geq \left(\int_a^b \left| \int_c^d |H(x, t)| \, dt \right|^{p'(p-1)} dx \right)^{\frac{1}{p'}} \left(\int_a^b |H(x, y)|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left| \int_c^d |H(x, t)| \, dt \right|^{p} dx \right)^{\frac{1}{p'}} \left(\int_a^b |H(x, y)|^p dx \right)^{\frac{1}{p}} \\
&\geq \left(\int_a^b \left(\sqrt{2}\right)^p \left| \int_c^d H(x, t) dt \right|^{p} dx \right)^{\frac{1}{p'}} \left(\int_a^b |H(x, y)|^p dx \right)^{\frac{1}{p}} \\
&= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left| \int_c^d H(x, t) dt \right|^{p} dx \right)^{\frac{1}{p'}} \left(\int_a^b |H(x, y)|^p dx \right)^{\frac{1}{p}} = R_2.
\end{aligned}$$

Then we get

$$\int_c^d R_1 dy \geq \int_c^d R_2 dy,$$

$$R_2 > 0 \rightarrow \left| \int_c^d R_1 dy \right| \geq \left| \int_c^d R_2 dy \right| = \int_c^d R_2 dy.$$

Thus, we conclude that

$$\begin{aligned}
\int_a^b \left| \int_c^d H(x, y) dy \right|^p dx &\geq \left(\sqrt{2}\right)^{-1} \left| \int_c^d R_1 dy \right| \\
&\geq \left(\sqrt{2}\right)^{-1} \int_c^d R_2 dy \\
&= \left(\sqrt{2}\right)^{p-2} \left(\int_a^b \left| \int_c^d H(x, t) dt \right|^{p} dx \right)^{\frac{1}{p'}} \left(\int_a^b |H(x, y)|^p dx \right)^{\frac{1}{p}} dy.
\end{aligned}$$

Therefore, we get

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)\left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{-\frac{1}{p'}}$$
$$\geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy,$$

then

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{1-\frac{1}{p'}} \geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Finally, we conclude that

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \geq \left(\sqrt{2}\right)^{p-2}\int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy,$$

which completes the proof. □

Theorem 3.2. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b)\times(c,d) \to \mathbb{E}$ *is measurable and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.2}$$

if left-hand side is finite.

Theorem 3.3. *Let* $0 < p < 1$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b)\times(c,d) \to i\mathbb{E}$ *is measurable and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.3}$$

if left-hand side is finite.

Proof. The proof of Theorem 3.2 and Theorem 3.3 is similar to Theorem 3.1. □

Theorem 3.4. *Let* $p < 0$, $-\infty < a < b < +\infty$ *and* $-\infty < c < d < +\infty$. *Suppose that* $H : (a,b)\times(c,d) \to \mathbb{C}$ *is measurable with* $\mathrm{Re}(H), \mathrm{Im}(H) \in \mathbb{E}$, $\mathrm{Re}(H)\mathrm{Im}(H) \neq 0$ *and* $H(x,y) \in L_{p,x}(a,b)$ *for almost all* $y \in (c,d)$. *Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq (\sqrt{2})^{p-2}\int_c^d \|H(\cdot,y)\|_{L_p(a,b)}dy, \tag{3.4}$$

if left-hand side is finite.

Proof. By using the inequality

$$\left|\int_c^d H(x,y)dy\right| \leq \int_c^d |H(x,y)|\, dy,$$

we get

$$\left|\int_c^d H(x,y)dy\right|^p \geq \left(\int_c^d |H(x,y)|\, dy\right)^p, \quad \text{for } p<0.$$

By integrating the last inequality, we get

$$\begin{aligned}
\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx &\geq \int_a^b \left(\int_c^d |H(x,y)|\, dy\right)^p dx \\
&= \int_a^b \left[\left(\int_c^d |H(x,t)|\, dt\right)^{p-1} \left(\int_c^d |H(x,y)|\, dy\right)\right] dx \\
&= \int_a^b \left[\int_c^d \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dy\right] dx \\
&= \int_c^d \left\{\int_a^b \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dx\right\} dy.
\end{aligned}$$

Let

$$R_3 = \int_a^b \left(\int_c^d |H(x,t)|\, dt\right)^{p-1} |H(x,y)|\, dx.$$

By the reverse Hölder's inequality and Proposition 2.1, we obtain

$$\begin{aligned}
R_3 &\geq \left(\int_a^b \left|\int_c^d |H(x,t)|\, dt\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left|\int_c^d |H(x,t)|\, dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_a^b \left(\sqrt{2}\right)^p \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} = R_4.
\end{aligned}$$

That is, we get

$$\int_c^d R_3 dy \geq \int_c^d R_4 dy.$$

Therefore, we obtain

$$\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx \geq \int_c^d R_3 dy \geq \int_c^d R_4 dy$$

and

$$\int_c^d R_4 dy = \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{\frac{1}{p'}} \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy$$
$$= \left(\sqrt{2}\right)^{p-1} \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{\frac{1}{p'}} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

It follows that

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right) \left(\int_a^b \left|\int_c^d H(x,t)dt\right|^p dx\right)^{-\frac{1}{p'}}$$
$$\geq \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Consequently, we get

$$\left(\int_a^b \left|\int_c^d H(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \geq \left(\sqrt{2}\right)^{p-1} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}}$$
$$\geq \left(\sqrt{2}\right)^{p-2} \int_c^d \left(\int_a^b |H(x,y)|^p dx\right)^{\frac{1}{p}}.$$

This completes the proof. □

Theorem 3.5. *Let $p < 0$, $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$. Suppose that $H : (a,b) \times (c,d) \to \mathbb{E}$ is measurable and $H(x,y) \in L_{p,x}(a,b)$ for almost all $y \in (c,d)$. Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)} dy, \tag{3.5}$$

if left-hand side is finite.

Theorem 3.6. *Let $p < 0$, $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$. Suppose that $H : (a,b) \times (c,d) \to i\mathbb{E}$ is measurable and $H(x,y) \in L_{p,x}(a,b)$ for almost all $y \in (c,d)$. Then*

$$\left\|\int_c^d H(\cdot,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|H(\cdot,y)\|_{L_p(a,b)} dy, \tag{3.6}$$

if left-hand side is finite.

Proof. The proof of Theorem 3.5 and Theorem 3.6 is similar to Theorem 3.4. □

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