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ON DISTANCE IRREGULAR LABELING OF DISCONNECTED GRAPHS

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Abstract. A distance irregular k-labeling of a graph G is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that the weights of all vertices are distinct. The weight of a vertex v, denoted by $wt(v)$, is the sum of labels of all vertices adjacent to v (distance 1 from v), that is, $wt(v) = \sum_{u \in N(v)} f(u)$. If the graph G admits a distance irregular labeling then G is called a distance irregular graph. The distance irregularity strength of G is the minimum k for which G has a distance irregular k-labeling and is denoted by $\operatorname{dis}(G)$. In this paper, we derive a new lower bound of distance irregularity strength for graphs with t pendant vertices. We also determine the distance irregularity strength of some families of disconnected graphs namely disjoint union of paths, suns, helms and friendships.

1. Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the set of neighbors of v is denoted by $N(v)$. We write $\deg(v)$ to represent the degree of v. The vertex v is called an *isolated vertex* if $\deg(v) = 0$. Meanwhile, if $\deg(v) = 1$, we then call such a vertex as a *pendant*. Other basic definitions and terminologies about graph theory not mentioned here, we refer the reader to a book [4]. By notation $[a, b]$ with integers a, b we mean the set of all integers x such that $a \leqslant x \leqslant b$.

A graph labeling is a mapping that carries some sets of graph elements to a set of positive integers, called *labels*, such that satisfies certain conditions. If the domain is vertex-set or edge-set, the labelings are called *vertex labelings* or *edge labelings*,

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respectively. If the domain is $V(G) \cup E(G)$, then it is called a *total labeling.* More details about recent results of graph labelings can be found in a great survey by Gallian [5].

One of interesting topics in graph labelings is a *distance irregular labeling.* This labeling is motivated by three concepts in graph labelings, namely a *distance magic labeling* [6], an (a, d)-*distance antimagic labeling* [1] and an *irregular labeling* [3]. For a graph G, a vertex labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is said to be a *distance irregular k-labeling* of G if the weights of all vertices are distinct. The weight of a vertex v, denoted by $wt(v)$, is the sum of labels of all vertices adjacent to v (distance 1 from v), that is, $wt(v) = \sum_{u \in N(v)} f(u)$. If the graph G admits a distance irregular labeling then G is called a *distance irregular graph.* The *distance irregularity strength* of G is the minimum k for which G has a distance irregular k-labeling and is denoted by $\mathrm{dis}(G)$.

The notion of distance irregular labeling was firstly introduced by Slamin in 2017 [8]. In his paper, he showed some particular graphs that admit a distance irregular labeling, such as paths with $\mathrm{dis}(P_n) = \lceil n/2 \rceil$ for $n \geqslant 4$, complete graphs with $\mathrm{dis}(K_n) = n$ for $n \geqslant 3$, cycles with $\mathrm{dis}(C_n) = \lceil (n + 1)/2 \rceil$ for $n \equiv 0, 1, 2, 5 \pmod 8$, and wheels with $\mathrm{dis}(W_n) = \lceil (n + 1)/2 \rceil$ for $n \equiv 0, 1, 2, 5 \pmod 8$. He also proved that for any two different vertices u and v of a graph G, if u and v have the same neighbors, then G has no distance irregular labeling. As a consequence of this property, he showed that some classes of graphs such as complete bipartite graphs, complete multipartite graphs, stars and trees containing vertex with at least two leaves, have no distance irregular labeling. Novindasari, Marjono and Abusini in [7] determined the distance irregularity strength of ladder graph and triangular ladder graph. Recently, in [2], Bong et al. completed the results for the distance irregularity strength of C_n and W_n, for $n \equiv 3, 4, 6, 7 \pmod 8$. In the same paper, they also determined the distance irregularity strength of m-book graphs B_m and $G + K_1$ for any connected graph G admitting a distance irregular labeling.

So far, all papers concerning distance irregular labeling have presented the results only for connected graphs. Meanwhile, determining the distance irregularity strength for disconnected graphs has still never been studied. Motivated by this, in this paper, we study the distance irregular labeling for disconnected graphs. We derive a new lower bound of distance irregularity strength for graphs with t pendant vertices. Also, the distance irregularity strength for some classes of disconnected graphs especially disjoint union of paths, suns, helms and friendships will be determined through this paper.

The following lemma gives the general lower bound for distance irregularity strength of graphs found by Slamin [8].

Lemma 1.1 ([8])**.** *Let G be a graph on p vertices with minimum degree δ and maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. Then*

$$\operatorname{dis}(G) \geqslant \left\lceil \frac{\delta + p - 1}{\Delta} \right\rceil.$$

2. Main Results

Our first result gives a lower bound of distance irregularity strength for a graph having t pendant vertices. We note that the graph is not necessarily connected.

Lemma 2.1. *Let G be a graph on p vertices with maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. If G has t pendant vertices, then*

$$\operatorname{dis}(G) \geqslant \max\left\{t, \left\lceil \frac{p}{\Delta} \right\rceil\right\}.$$

Proof. Let G be a graph on p vertices with maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. For a positive integer t, let $x_1, x_2, \ldots, x_t$ be the pendant vertices of G. Since the weight of every vertex of G must be distinct, then the labels of neighbor of all x_is must be distinct, that is, $f(N(x_1)) \neq f(N(x_2)) \neq \ldots \neq f(N(x_t))$. So, $\operatorname{dis}(G) \geqslant t$. Combining with the lower bound for $\delta = 1$ (since the minimum degree of G is 1) in Lemma 1.1, we have $\operatorname{dis}(G) \geqslant \max\{t, \lceil p/\Delta \rceil\}$. □

The lower bound in Lemma 2.1 is tight as can be seen from Theorem 2.1, 2.2 and 2.3, which present the exact value of distance irregularity strength for disconnected paths, suns and helms, respectively.

2.1. Disjoint union of paths.

In this subsection, we deal with a distance irregular labeling of disconnected paths. Let mP_n be a disjoint union of m identical copies of paths with vertex set $V(mP_n) = \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mP_n) = \{v_i^j v_{i+1}^j : i \in [1, n-1], j \in [1, m]\}$. For $m \geqslant 2$ and $n = 3$, there exist vertices having the same neighbors. Consequently, the graph mP_3 has no distance irregular labeling. However, for $m \geqslant 2$ and $n \geqslant 4$, the graph mP_n admits a distance irregular labeling and its distance irregularity strength will be determined by the following theorem.

Theorem 2.1. *For each $m \geqslant 2$ and $n \geqslant 4$,* $\operatorname{dis}(mP_n) = \lceil mn/2 \rceil$.

Proof. As $n \geqslant 4$, it follows from Lemma 2.1 that $\operatorname{dis}(mP_n) \geqslant \lceil mn/2 \rceil$. To prove the reverse inequality, define a vertex labeling $f : V(mP_n) \to \{1, 2, \ldots, \lceil mn/2 \rceil\}$ as follows.

Case 1. Let $n \equiv 0 \pmod 4$.

For $j \in [1, m]$, label each vertex in the following way:

$$
\begin{aligned}
f(v_i^j) &= \frac{1}{2}(n+1-i)m, && \text{for } i = 1, 5, \dots, n-3,\\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6, \dots, n-2,\\
f(v_i^j) &= \frac{1}{2}(n+1-i)m + j, && \text{for } i = 3, 7, \dots, n-1,\\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8, \dots, n.
\end{aligned}
$$

Hence, for $j \in [1, m]$, the labeling gives the vertex weights as follows:

$$
\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, \dots, n-1,\\
wt(v_i^j) &= (n+1-i)m + j, && \text{for } i = 2, 4, \dots, n.
\end{aligned}
$$

Case 2. Let $n \equiv 1 \pmod 4$.

For $n = 5$, first, label all vertices except v_1^j, $j \in [1, m]$, in the following way:

$$
\begin{aligned}
f(v_2^j) &= \frac{5j}{2} - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
f(v_2^j) &= \left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
f(v_3^j) &= \left\lfloor \frac{5(m+j)}{2} \right\rfloor - \left\lceil \frac{5m}{2} \right\rceil, && \text{for } j \in [1, m],\\
f(v_4^j) &= \left\lfloor \frac{5j}{2} \right\rfloor, && \text{for } j \in [1, m],\\
f(v_5^j) &= \left\lceil \frac{5m}{2} \right\rceil, && \text{for } j \in [1, m].
\end{aligned}
$$

Then, we obtain all vertex weights except $wt(v_2^j)$, $j \in [1, m]$:

$$
\begin{aligned}
wt(v_1^j) &= \frac{5j}{2} - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
wt(v_1^j) &= \left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
wt(v_3^j) &= 5j - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
wt(v_3^j) &= 2\left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
wt(v_4^j) &= \left\lfloor \frac{5(m+j)}{2} \right\rfloor, && \text{for } j \in [1, m],\\
wt(v_5^j) &= \left\lfloor \frac{5j}{2} \right\rfloor, && \text{for } j \in [1, m].
\end{aligned}
$$

Next, for $j \in [1, m]$, the label of v_1^j and the weight of v_2^j will be determined by using the following algorithm.

1. Let $W = \left\{ wt(v_3^j) : j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m \right] \right\}$.

2. For j from 1 up to m, do
   a. $p = f(v_3^j) = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - \left\lceil \frac{5m}{2} \right\rceil$;
   b. $q = wt(v_4^j) = \left\lfloor \frac{5(m+j)}{2} \right\rfloor$.
   c. If $(q-1)$ is contained in W, then
      1) $f(v_1^j) = q - p - 2 = \left\lceil \frac{5m}{2} \right\rceil - 2$;
      2) $wt(v_2^j) = q - 2 = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - 2$;
      3) $W = W \backslash \{q-1\}$.
   d. Else
      1) $f(v_1^j) = q - p - 1 = \left\lceil \frac{5m}{2} \right\rceil - 1$;
      2) $wt(v_2^j) = q - 1 = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - 1$.

For $n \geqslant 9$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_i^j) =& \frac{1}{4}(3n - 4 + i)m - \left\lfloor \frac{mn}{2} \right\rfloor + j - 1, \\
& \text{for } i = 1, 5, \ldots, \frac{n-7}{2} \quad (\text{if } n \equiv 1 \pmod 8), \\
f(v_i^j) =& \frac{1}{4}(3n + i)m - \left\lfloor \frac{mn}{2} \right\rfloor - j + 1, \\
& \text{for } i = 1, 5, \ldots, \frac{n-11}{2} \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_i^j) =& \frac{1}{4}(i-2)m + j, \quad \text{for } i = 2, 6, \ldots, n-3, \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n + 2 - i)m + 1, \\
& \text{for } i = 3, 7, \ldots, \frac{n-11}{2} \quad (\text{if } n \equiv 1 \pmod 8), \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n + 6 - i)m + 2j - 1, \\
& \text{for } i = 3, 7, \ldots, \frac{n-7}{2} \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_i^j) =& \frac{mi}{4}, \quad \text{for } i = 4, 8, \ldots, n-5, \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n + 2 - i)m + j, \\
& \text{for } i = \frac{n-3}{2}, \frac{n+5}{2}, \ldots, n-2 \quad (\text{if } n \equiv 1 \pmod 8) \text{ or} \\
& \text{for } i = \frac{n+1}{2}, \frac{n+9}{2}, \ldots, n-2 \quad (\text{if } n \equiv 5 \pmod 8),
\end{aligned}$$

$$\begin{aligned}
f(v_i^j) =& \frac{1}{4}(3n+i)m - \left\lfloor \frac{mn}{2} \right\rfloor, \\
& \text{for } i = \frac{n+1}{2}, \frac{n+9}{2}, \dots, n-4 \quad (\text{if } n \equiv 1 \pmod 8) \text{ or} \\
& \text{for } i = \frac{n-3}{2}, \frac{n+5}{2}, \dots, n-4 \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_{n-1}^j) =& \frac{1}{2}(n-3)m + j, \\
f(v_n^j) =& \left\lceil \frac{mn}{2} \right\rceil.
\end{aligned}$$

Thus, for $j \in [1, m]$, the labeling provides the following vertex weights:

$$\begin{aligned}
wt(v_i^j) =& \frac{1}{2}(i-1)m + j, && \text{for } i = 1, 3, \dots, n-4, \\
wt(v_i^j) =& \frac{1}{2}(n-3+i)m + j, && \text{for } i = 2, 4, \dots, \frac{n-9}{2}, \\
wt\left(v_{\frac{n-5}{2}}^j\right) =& && \frac{1}{4}(3n-11)m + 2j - 1, \\
wt(v_i^j) =& \frac{1}{2}(n-1+i)m + j, && \text{for } i = \frac{n-1}{2}, \frac{n+3}{2}, \dots, n-1, \\
wt(v_{n-2}^j) =& \frac{1}{4}(3n-11)m + 2j, \\
wt(v_n^j) =& \frac{1}{2}(n-3)m + j.
\end{aligned}$$

Case 3. Let $n \equiv 2 \pmod 4$.
For $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_1^j) &= \frac{1}{2}(n-2)m + j, \\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6, \dots, n-4, \\
f(v_3^j) &= \frac{mn}{2}, \\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8, \dots, n-2, \\
f(v_i^j) &= \frac{1}{2}(n+1-i)m + j, && \text{for } i = 5, 9, \dots, n-1, \\
f(v_i^j) &= \frac{1}{2}(n+1-i)m, && \text{for } i = 7, 11, \dots, n-3, \\
f(v_n^j) &= \frac{1}{2}(n-4)m + j.
\end{aligned}$$

Hence, for $j \in [1, m]$, the labeling provides the following vertex weights:

$$\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, \dots, n-3,\\
wt(v_i^j) &= \frac{1}{2}(2n-i)m + j, && \text{for } i = 2, 4,\\
wt(v_i^j) &= (n+1-i)m + j, && \text{for } i = 6, 8, \dots, n,\\
wt(v_{n-1}^j) &= (n-3)m + j.
\end{aligned}$$

Case 4. Let $n \equiv 3 \pmod 4$.

For $n = 7$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_i^j) &= \frac{1}{4}(i+23)m - \left\lceil \frac{7m}{2} \right\rceil, && \text{for } i = 1, 5,\\
f(v_i^j) &= \frac{1}{4}(i-2)m + j, && \text{for } i = 2, 6,\\
f(v_i^j) &= \frac{1}{4}(i-11)m + \left\lceil \frac{7m}{2} \right\rceil + j, && \text{for } i = 3, 7,\\
f(v_4^j) &= 2m.
\end{aligned}$$

Then, for $j \in [1, m]$, the labeling yields the following vertex weights:

$$\begin{aligned}
wt(v_1^j) &= j,\\
wt(v_i^j) &= \frac{1}{2}(i+6)m + j, && \text{for } i = 2, 4, 6,\\
wt(v_i^j) &= \frac{1}{2}(i+1)m + j, && \text{for } i = 3, 5,\\
wt(v_7^j) &= m + j.
\end{aligned}$$

For $n = 11$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_1^j) &= \left\lfloor \frac{11m}{2} \right\rfloor - 5m,\\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6,\\
f(v_i^j) &= \frac{1}{4}(i-15)m + \left\lceil \frac{11m}{2} \right\rceil + j, && \text{for } i = 3, 7, 11,\\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8,\\
f(v_i^j) &= \frac{1}{4}(i-9)m + \left\lfloor \frac{11m}{2} \right\rfloor, && \text{for } i = 5, 9,\\
f(v_{10}^j) &= m + j.
\end{aligned}$$

So, for $j \in [1, m]$, the labeling gives the following vertex weights:

$$\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, 5, 7,\\
wt(v_2^j) &= 3m + j,\\
wt(v_i^j) &= \frac{1}{2}(i+10)m + j, && \text{for } i = 4, 6, 8, 10,\\
wt(v_i^j) &= (23-2i)m + j, && \text{for } i = 9, 11.
\end{aligned}$$

For $n \geqslant 15$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_i^j) &= \left\lceil \frac{mn}{2} \right\rceil - \frac{1}{4}(i-1)m,\\
&\quad \text{for } i = 1, 5, \ldots, n-2 \quad (\text{if } n \equiv 3 \pmod 8) \text{ or}\\
&\quad \text{for } i = 1, 5, \ldots, \frac{n+11}{2} \quad (\text{if } n \equiv 7 \pmod 8),\\
f(v_i^j) &= \frac{1}{4}(i-2)m + j, \quad \text{for } i = 2, 6, \ldots, n-5,\\
f(v_i^j) &= \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(i+1)m + j,\\
&\quad \text{for } i = 3, 7, \ldots, \frac{n+11}{2} \quad (\text{if } n \equiv 3 \pmod 8) \text{ or}\\
&\quad \text{for } i = 3, 7, \ldots, n \quad (\text{if } n \equiv 7 \pmod 8),\\
f(v_i^j) &= \frac{1}{4}(i+4)m, \quad \text{for } i = 4, 8, \ldots, n-7,\\
f(v_i^j) &= \frac{1}{4}(4n-5-i)m - \left\lceil \frac{mn}{2} \right\rceil + j,\\
&\quad \text{for } i = \frac{n+19}{2}, \frac{n+27}{2}, \ldots, n \quad (\text{if } n \equiv 3 \pmod 8),\\
f(v_i^j) &= \frac{1}{4}(4n-3-i)m - \left\lfloor \frac{mn}{2} \right\rfloor,\\
&\quad \text{for } i = \frac{n+19}{2}, \frac{n+27}{2}, \ldots, n-2 \quad (\text{if } n \equiv 7 \pmod 8),\\
f(v_{n-3}^j) &= \frac{1}{2}(n-5)m,\\
f(v_{n-1}^j) &= m + j.
\end{aligned}$$

Thus, for $j \in [1, m]$, the labeling yields the following vertex weights:

$$\begin{aligned}
wt(v_1^j) &= j, \\
wt(v_i^j) &= \frac{1}{2}(2n-i)m + j, && \text{for } i = 2, 4, \dots, \frac{n+13}{2}, \\
wt(v_i^j) &= \frac{1}{2}(i+1)m + j, && \text{for } i = 3, 5, \dots, n-6, \\
wt(v_i^j) &= \frac{1}{2}(2n-2-i)m + j, && \text{for } i = \frac{n+17}{2}, \frac{n+21}{2}, \dots, n-1, \\
wt(v_{n-4}^j) &= \frac{1}{4}(3n-17)m + j, \\
wt(v_{n-2}^j) &= \frac{1}{2}(n-3)m + j, \\
wt(v_n^j) &= m + j.
\end{aligned}$$

From all cases, it can be checked that the vertex weights form the set $\{1, 2, \dots, mn\}$ and the labels used in the labelings are at most $\lceil mn/2 \rceil$. Thus, $\text{dis}(mP_n) \leqslant \lceil mn/2 \rceil$. As $\lceil mn/2 \rceil \leqslant \text{dis}(mP_n) \leqslant \lceil mn/2 \rceil$, we can conclude that $\text{dis}(mP_n) = \lceil mn/2 \rceil$. $\square$

As an illustration, a distance irregular labeling of $6P_5$ is given in Figure 1, where red numbers show the vertex weights and black numbers represent the label of the vertices.

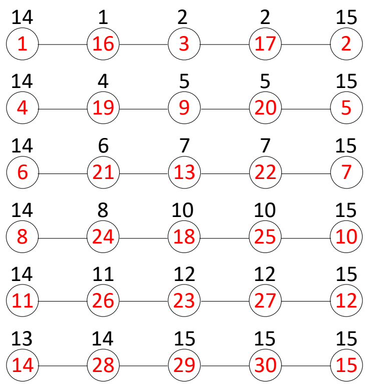

FIGURE 1. A distance irregular 15-labeling of $6P_5$.

2.2. **Disjoint union of suns.** A *sun*, denoted by S_n, is a graph with $2n$ vertices obtained from a cycle by attaching a *pendant vertex* to each cycle's vertex. We then call all vertices adjacent to such pendant vertices as the *rim vertices* of S_n. Now, let us denote by mS_n a disjoint union of m identical copies of sun graphs with vertex set $V(mS_n) = \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mS_n) = \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j u_{i+1}^j : i \in [1, n], j \in [1, m]\}$ where the index i is taken modulo n. Next, we will determine the distance irregularity strength of mS_n in the following theorem.

Theorem 2.2. *For each $m \geqslant 2$ and $n \geqslant 3$,* $\operatorname{dis}(mS_n) = mn$.

Proof. Consider the graph mS_n, with $2mn$ vertices. Since mS_n has mn pendant vertices, according to Lemma 2.1, we have $\operatorname{dis}(mS_n) \geqslant mn$. To prove that mn is the upper bound of $\operatorname{dis}(mS_n)$, it is sufficient to show the existence of a distance irregular mn-labeling of mS_n. To do that, let us define $f : V(mS_n) \to \{1, 2, \ldots, mn\}$ as follows.

For $n = 3$ and $j \in [1, m]$, label every vertex in the following way:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,3], \\ f(v_i^j) &= 2m - j, && \text{for } i \in [1,3]. \end{aligned}$$

Hence, for $j \in [1, m]$, we obtain the following vertex weights:

$$\begin{aligned} wt(u_i^j) &= j - (i-6)m, && \text{for } i \in [1,3], \\ wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1,3]. \end{aligned}$$

For $n = 4$ and $j \in [1, m]$, label every vertex in the following way:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,4], \\ f(v_i^j) &= 3m - j, && \text{for } i = 1, 4, \\ f(v_i^j) &= 2m - j, && \text{for } i = 2, 3. \end{aligned}$$

So, for $j \in [1, m]$, we can get the weight of each vertex as follows:

$$\begin{aligned} wt(u_i^j) &= \frac{1}{2}(15 - i)m + j, && \text{for } i = 1, 3, \\ wt(u_i^j) &= \frac{1}{2}(i + 6)m + j, && \text{for } i = 2, 4, \\ wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1,4]. \end{aligned}$$

For $n \geqslant 5$ and $j \in [1, m]$, label each vertex as follows:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,n], \\ f(v_i^j) &= m \left\lfloor \frac{n+1}{2} \right\rfloor - j, && \text{for } i = 1, n, \\ f(v_i^j) &= (n-i)m - j, && \text{for } i \in \left[2, \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right], \\ f\left(v^j_{\lfloor \frac{n+1}{2} \rfloor}\right) &= \left(n + 1 - \left\lfloor \frac{n+1}{2} \right\rfloor\right) m - j, \\ f(v_i^j) &= (n+2-i)m - j, && \text{for } i \in \left[\left\lfloor \frac{n+1}{2} \right\rfloor + 1, n-1\right]. \end{aligned}$$

Then, for $j \in [1, m]$, we obtain the weight of each vertex as follows:

$$\begin{aligned}
wt(u_i^j) &= \left(n - \frac{2(i-1)}{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor\right) m + j, && \text{for } i = 1, n,\\
wt(u_i^j) &= (n - 2 + i)m + j, && \text{for } i \in \left[2, \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right],\\
wt\left(u^j_{\lfloor \frac{n+1}{2} \rfloor}\right) &= \left(n - 1 + \left\lfloor \frac{n+1}{2} \right\rfloor\right) m + j, &&\\
wt(u_i^j) &= (n + i)m + j, && \text{for } i \in \left[\left\lfloor \frac{n+1}{2} \right\rfloor + 1, n - 1\right],\\
wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1, n].
\end{aligned}$$

Clearly, the largest label appearing on the vertices is mn for each $n \geqslant 3$. Moreover, it can be checked that vertex weights of the pendant vertices and the rim vertices of mS_n constitute the set $\{1, 2, \ldots, mn\}$ and the set $\{mn + 1, mn + 2, \ldots, 2mn\}$, respectively. It means that f is a distance irregular mn-labeling of mS_n. The proof is complete. ☐

In Figure 2, as an illustralion, a distance irregular labeling of $3S_5$ is shown.

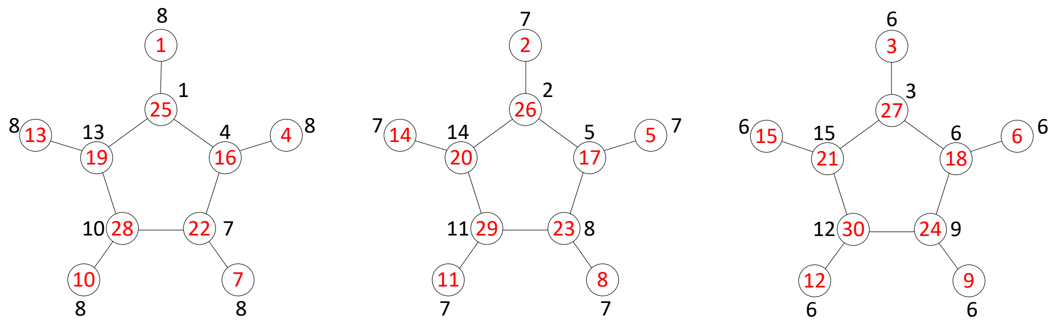

FIGURE 2. A distance irregular 15-labeling of $3S_5$.

2.3. **Disjoint union of helms.** A *helm*, denoted by H_n, is a graph constructed from a sun S_n by joining a new vertex, called *center vertex*, to all the rim vertices of S_n. Next, we focus on a disjoint union of m identical copies of helm graphs mH_n with vertex set $V(mH_n) = \{c^j : j \in [1, m]\} \cup \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mH_n) = \{c^j u_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j u_{i+1}^j : i \in [1, n], j \in [1, m]\}$ where the index i is taken modulo n.

Let us recall the labeling formula of the rim vertices of mS_n defined in the previous theorem, that is, for $m \geqslant 2$, $n \geqslant 3$, $i \in [1, n]$ and $j \in [1, m]$,

$$f(u_i) = j + (i-1)m.$$

The sum of labels of such rim vertices is

$$\sum_{i=1}^{n} f(u_i) = \sum_{i=1}^{n} (j + (i-1)m) = \frac{n}{2}(2j + (n-1)m). \tag{2.1}$$

Next, consider the set of vertex weights of mS_n obtained from Theorem 2.2, namely $\{1, 2, \ldots, 2mn\}$. We want to find all possible n such that the Equation (2.1) is different from all such vertex weights for every $m \geqslant 2$ and $j \in [1, m]$. Therefore,

$$\frac{n}{2}(2j + (n-1)m) > 2mn. \tag{2.2}$$

It is not difficult to show that (2.2) happens if and only if $n \geqslant 5$. Thus, we can use this characteristic to construct a distance irregular labeling of mH_n from the described distance irregular labeling of mS_n for case $n \geqslant 5$.

Next, we will present the distance irregularity strength of mH_n in the following theorem.

Theorem 2.3. *For each $m \geqslant 2$ and $n \geqslant 3$,* $\operatorname{dis}(mH_n) = mn$.

Proof. Consider the graph mH_n on $(2n+1)m$ vertices. Since mH_n has mn pendant vertices, by Lemma 2.1, we get $\operatorname{dis}(mH_n) \geqslant mn$. To prove that mn is the upper bound of $\operatorname{dis}(mH_n)$, it is sufficient to show the existence of an optimal distance irregular mn-labeling of mH_n. Let $f : V(mH_n) \to \{1, 2, \ldots, mn\}$ be a vertex labeling defined as follows.

For $n = 3$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(c^j) &= 1, \\
f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,3], \\
f(v_1^j) &= 3m - j - 1, \\
f(v_2^j) &= \frac{1}{2}(5m-3) - \left\lceil \frac{j}{2} \right\rceil, && \text{if } m \text{ is odd}, \\
f(v_2^j) &= \frac{1}{2}(5m-4) - \left\lfloor \frac{j}{2} \right\rfloor, && \text{if } m \text{ is even}, \\
f(v_3^j) &= 2m - 2 - \left\lceil \frac{j-1}{2} \right\rceil.
\end{aligned}$$

Therefore, for $j \in [1, m]$, we obtain the following vertex weights:

$$\begin{aligned}
wt(c^j) &= 3(m+j), \\
wt(u_1^j) &= 6m + j, \\
wt(u_2^j) &= \frac{1}{2}(9m-1) + \left\lfloor \frac{3j}{2} \right\rfloor, && \text{if } m \text{ is odd}, \\
wt(u_2^j) &= \frac{1}{2}(9m-2) + \left\lceil \frac{3j}{2} \right\rceil, && \text{if } m \text{ is even}, \\
wt(u_3^j) &= 3m - 1 + \left\lfloor \frac{3j+1}{2} \right\rfloor, \\
wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1,3].
\end{aligned}$$

For $n = 4$ and $j \in [1, m]$, label every vertex in the following way:

$$\begin{aligned}
&f(c^j) = 1, \\
&f(u_i^j) = j + (i-1)m, && \text{for } i \in [1,4], \\
&f(v_1^j) = \frac{1}{3}(10m-6) - \left\lceil \frac{2j-2}{3} \right\rceil, && \text{if } m \equiv 0 \pmod 3, \\
&f(v_1^j) = \frac{1}{3}(10m-4) - \left\lceil \frac{2j}{3} \right\rceil, && \text{if } m \equiv 1 \pmod 3, \\
&f(v_1^j) = \frac{1}{3}(10m-5) - \left\lceil \frac{2j-1}{3} \right\rceil, && \text{if } m \equiv 2 \pmod 3, \\
&f(v_2^j) = 2m - j - 1, \\
&f(v_3^j) = 2m - \left\lceil \frac{2j+4}{3} \right\rceil, \\
&f(v_4^j) = 3m - j - 1.
\end{aligned}$$

So, for $j \in [1, m]$, we get the vertex weights as follows:

$$\begin{aligned}
&wt(c^j) = 6m + 4j, \\
&wt(u_1^j) = \frac{1}{3}(22m-3) + \left\lfloor \frac{4j+2}{3} \right\rfloor, && \text{if } m \equiv 0 \pmod 3, \\
&wt(u_1^j) = \frac{1}{3}(22m-1) + \left\lfloor \frac{4j}{3} \right\rfloor, && \text{if } m \equiv 1 \pmod 3, \\
&wt(u_1^j) = \frac{1}{3}(22m-2) + \left\lfloor \frac{4j+1}{3} \right\rfloor, && \text{if } m \equiv 2 \pmod 3, \\
&wt(u_2^j) = 4m + j, \\
&wt(u_3^j) = 6m + \left\lfloor \frac{4j-1}{3} \right\rfloor, \\
&wt(u_4^j) = 5m + j, \\
&wt(v_i^j) = j + (i-1)m, && \text{for } i \in [1,4].
\end{aligned}$$

Now, let $n \geqslant 5$. For the proof purpose only, first, let us denote the described vertex labelings and vertex weights formula of mS_n, $n \geqslant 5$, by f^* and by wt^*, respectively. Next, for $j \in [1, m]$, label every vertex of mH_n such that

$$\begin{aligned}
&f(c^j) = 1, \\
&f(u_i^j) = f^*(u_i^j), \\
&f(v_i^j) = f^*(v_i^j) - 1.
\end{aligned}$$

Then, for $j \in [1, m]$, we obtain the vertex weights as follows:

$$
\begin{aligned}
wt(c^j) &= \frac{n}{2}((n-1)m + 2j),\\
wt(u_i^j) &= wt^*(u_i^j),\\
wt(v_i^j) &= wt^*(v_i^j).
\end{aligned}
$$

It can be verified that all the vertex weights are distinct for all pairs of distinct vertices and the largest label is mn, which lead to $\mathrm{dis}(mH_n) \leqslant mn$. Combining with the lower bound, we have $\mathrm{dis}(mH_n) = mn$. □

We show in Figure 3 a distance irregular labeling of $3H_5$ as an illustration.

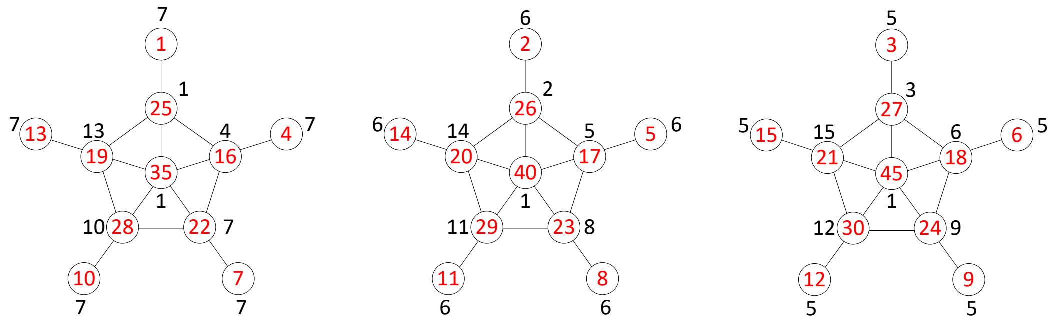

FIGURE 3. A distance irregular 15-labeling of $3H_5$.

2.4. **Disjoint union of friendships.** A *friendship* f_n is a graph obtained by identifying a vertex from n copies of *triangles* K_3. The vertex of degree $2n$ is called the *center vertex* and the remaining vertices are called the *rim vertices*. Now, we focus on a disjoint union of m identical copies of friendships mf_n with vertex set $V(mf_n) = \{c^j : j \in [1, m]\} \cup \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mf_n) = \{c^j u_i^j, c^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\}$.

First, let us consider a single copy of friendship f_n. In the following lemma, we give a necessary condition for f_n to be a distance irregular graph.

Lemma 2.2. *If f_n is a distance irregular graph, then the labels of all rim vertices of f_n must be distinct.*

Proof. Let f be a distance irregular labeling of f_n. Let x, y be any two rim vertices of f_n. We show that $f(x) \neq f(y)$. Let c be the center vertex and let x', y' be rim vertices adjacent to x and y, respectively. We know that $wt(x) = f(c) + f(x')$ and $wt(y) = f(c) + f(y')$. Since $wt(x)$ and $wt(y)$ must be distinct, we get $f(x') \neq f(y')$. Since x, y are arbitrarily two rim vertices in the graph f_n and x', y' are also the rim vertices of f_n, it naturally implies that $f(x) \neq f(y)$. □

It is coherent to say that the property in Lemma 2.2 holds also for disconnected version of friendships. Thus, in any distance irregular labeling of mf_n, the labels of all rim vertices in the j^{th}-copy of f_n are distinct for $j \in [1, m]$. Next, we will determine the distance irregularity strength of mf_n in the following theorem.

Theorem 2.4. *For each $n \geqslant 2$ and $m \in [2, n]$, $\mathrm{dis}(mf_n) = mn + 1$.*

Proof. Firstly, we determine the lower bound of $\mathrm{dis}(mf_n)$. Let k be the largest label of the graph mf_n. The optimal weights of the vertices of mf_n are $2, 3, \ldots, 2mn + 1, wt(c^1), wt(c^2), \ldots, wt(c^m)$. Next, for some $i \in [1, n]$ and some $s \in [1, m]$, let $wt(c^s)$ and $wt(v_i^s)$, be the largest weight of the center vertices of mf_n and the largest weight of the rim vertices of mf_n, respectively. Furthermore, it follows from Lemma 2.2 that the labels of every rim vertex in the j^{th}-copy of f_n, $j \in [1, m]$, must be distinct. Since the center vertex c^s is adjacent to all rim vertices in the s^{th}-copy of f_n, then the largest label used in the computation of $wt(c^s)$ is at most k. On the other hand, we have $wt(v_i^s) \geqslant 2mn + 1$. Since $\deg(v_i^s) = 2$, we obtain $\mathrm{dis}(mf_n) = k \geqslant \lceil (2mn + 1)/2 \rceil = mn + 1$. Next, for the upper bound of $\mathrm{dis}(mf_n)$, construct a vertex labeling $f : V(mf_n) \to \{1, 2, , \ldots, mn + 1\}$ as follows:

$$
\begin{aligned}
f(c^j) &= (2n-1)(j-1)+1, && \text{for } j \in [1,2],\\
f(c^j) &= nj, && \text{for } j \in [3,m],\\
f(u_i^j) &= 2i+j-1, && \text{for } i \in [1,n] \text{ and } j \in [1,2],\\
f(u_i^j) &= 2i+1+(j-2)n, && \text{for } i \in [1,n] \text{ and } j \in [3,m],\\
f(v_i^j) &= 2i+j-2, && \text{for } i \in [1,n] \text{ and } j \in [1,2],\\
f(v_i^j) &= 2i+(j-2)n, && \text{for } i \in [1,n] \text{ and } j \in [3,m].
\end{aligned}
$$

Therefore, we get the vertex weights as follows:

$$
\begin{aligned}
wt(c^j) &= 2n^2+(2j-1)n, && \text{for } j \in [1,2],\\
wt(c^j) &= 2n^2(j-1)+3n, && \text{for } j \in [3,m],\\
wt(u_i^j) &= 2n(j-1)+2i, && \text{for } i \in [1,n] \text{ and } j \in [1,m],\\
wt(v_i^j) &= 2n(j-1)+2i+1, && \text{for } i \in [1,n] \text{ and } j \in [1,m].
\end{aligned}
$$

It can be verified that f is a distance irregular $(mn+1)$-labeling of mf_n as the vertex weights are unique and the labels appearing on the vertices are at most $mn+1$. Thus $\mathrm{dis}(mf_n) \leqslant mn + 1$. This concludes the proof. ☐

An example of distance irregular labeling of mf_n is described in Figure 4.

3. Conclusion

In this paper we initiated to study the distance irregular labeling of disconnected graphs. A new lower bound of the distance irregularity strength for a graph G having

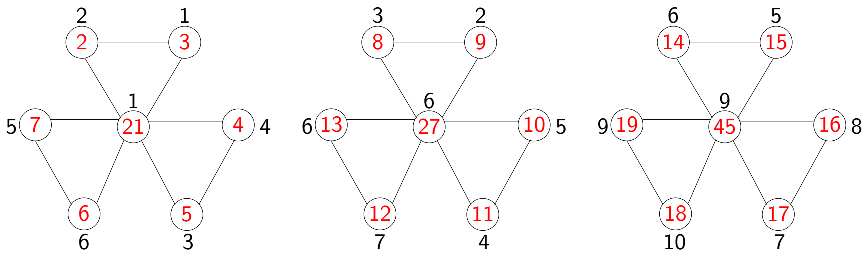

FIGURE 4. A distance irregular 10-labeling of $3f_3$.

t pendant vertices was introduced and we proved that $\operatorname{dis}(G) \geqslant \max\{t, \lceil p/\Delta \rceil\}$. We also showed that this lower bound is sharp for disconnected paths, suns and helms.

Because of the limitation of results we found related to this parameter for disconnected graphs, we propose the open problem below.

Open Problem 1. Determine the distance irregularity strength of other classes of disconnected graphs.

In relation with our lower bound in Lemma 2.1 which works for graphs containing t pendant vertices ($\delta = 1$), the following open problems are also interesting to be studied.

Open Problem 2. Characterize all graphs containing t pendant vertices having distance irregularity strength t. Particularly, characterize all trees with t leaves having distance irregularity strength t.

Open Problem 3. Characterize all graphs containing t pendant vertices having distance irregularity strength $\lceil p/\Delta \rceil$. Specifically, characterize all trees with t leaves having distance irregularity strength $\lceil p/\Delta \rceil$.

In Theorem 2.4, we determined the distance irregularity strength of disconnected friendships mf_n only for $m \leqslant n$. Meanwhile, this parameter is still unsolved for the remaining case of mf_n. Therefore, we also give the following open problem.

Open Problem 4. Determine the distance irregularity strength of mf_n for $m > n$.

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References

[1] S. Arumugam and N. Kamatchi, *On (a, d)-distance antimagic graphs*, Australas. J. Combin. **54** (2012), 279–288.
[2] N. H. Bong, Y. Lin and Slamin, *On distance-irregular labelings of cycles and wheels*, Australas. J. Combin. **69** (2017), 315–322.
[3] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, *Irregular networks*, Congr. Numer. **64** (1988), 197–210.
[4] G. Chartrand and P. Zhang, *Graphs & digraphs*, Sixth Ed., Taylor & Francis Group, 2016.
[5] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. (2018), #DS6.
[6] M. Miller, C. Rodger and R. Simanjuntak, *Distance magic labelings of graphs*, Australas. J. Combin. **28** (2003), 305–315.
[7] S. Novindasari, Marjono and S. Abusini, *On distance irregular labeling of ladder graph and triangular ladder graph*, Pure Math. Sci. **5** (2016), 75–81.
[8] Slamin, *On distance irregular labelling of graphs*, Far East J. Math. Sci. **102** (2017), 919–932.

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MAPS PRESERVING THE SPECTRUM OF SKEW LIE PRODUCT OF OPERATORS

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ABSTRACT. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. In this paper, we show that a surjective map φ on $\mathcal{B}(\mathcal{H})$ satisfies

$$\sigma\left(\varphi(T)\varphi(S)-\varphi(S)\varphi(T)^*\right)=\sigma\left(TS-ST^*\right),\quad T,S\in\mathcal{B}(\mathcal{H}),$$

if and only if there exists a unitary operator $U\in\mathcal{B}(\mathcal{H})$ such that

$$\varphi(T)=\lambda UTU^*,\quad T\in\mathcal{B}(\mathcal{H}),$$

where $\lambda\in\{-1,1\}$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Throughout this paper, $\mathcal{B}(\mathcal{H})$ stands for the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$. Let $\mathcal{B}_s(\mathcal{H})$ (resp. $\mathcal{B}_a(\mathcal{H})$) be the real linear space of all self-adjoint (resp. anti-self-adjoint) operators in $\mathcal{B}(\mathcal{H})$. For every $A\in\mathcal{B}(\mathcal{H})$, the spectrum (resp. the spectral radius) of A is denoted by $\sigma(A)$ (resp. $r(A)$).

The problem of describing maps on operators and matrices that preserve certain functions, subsets and relations has been widely studied in the literature, see [3–6, 9–12, 16, 19–22] and references therein. One of the classical problems in this area of research is to characterize maps preserving the spectra of the product of operators. Molnár in [19] studied maps preserving the spectrum of operator and matrix products. His results have been extended in several directions [1, 2, 7, 8, 13–15, 17] and [18]. In [1], the problem of characterizing maps between matrix algebras preserving the spectrum

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of polynomial products of matrices is considered. In particular, the results obtained therein extend and unify several results obtained in [6] and [8].

Latter in [2], the form of all maps preserving the spectrum and the local spectrum of Skew Lie product of matrices are determined. This paper is a continuation of such recent work, and examines the form of maps preserving the spectrum of skew Lie product of operators on a complex Hilbert space. Mainly, we shall give a characterization of all surjective maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserving the spectrum of the skew Lie product "$[T, S]_* = TS - ST^*$" of operators. Precisely, the following theorem is the main result of this paper.

Theorem 1.1. *A surjective map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$\sigma\left(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*\right) = \sigma\left(TS - ST^*\right), \quad T, S \in \mathcal{B}(\mathcal{H}), \tag{1.1}$$

if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(T) = \pm UTU^*, \tag{1.2}$$

for all $T \in \mathcal{B}(\mathcal{H})$.

Before presenting the proof of the main theorem few comments can be made. Firstly, note that the only restriction on the map φ is surjectivity; no linearity or additivity or continuity is assumed. Also, we point out that the consideration of maps φ from $\mathcal{B}(\mathcal{H})$ onto itself is for the sake of simplicity. Our result and its proof remains valid in the case where φ is a surjective map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are two different Hilbert spaces.

The case of finite dimensional Hilbert spaces was considered in [2] where it is shown that the theorem 1.1 remains valid without the surjectivity assumption of the map φ. The proof given therein is based on a density argument and is completely different from the one presented in the current paper. This paper is divided into three sections. In Section 2, we collect some auxiliary lemmas needed in the proof of the main result. In Section 3, we present the proof of Theorem 1.1.

2. Preliminaries

Given two vectors x and y in $\mathcal{H}$, let $x \otimes y$ be the operator of at most rank one defined by

$$(x \otimes y)(z) := \langle z, y \rangle x, \quad z \in \mathcal{H},$$

and note that $(x \otimes y)^* = y \otimes x$. Let $(e_k)_{k\in I}$ be an orthonormal basis of $\mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, the transpose $A^\top$ of A with respect to the basis $(e_k)_{k\in I}$ is defined as the unique operator such that

$$\langle Ae_k, e_j \rangle = \langle A^\top e_j, e_k \rangle,$$

for any $j, k \in I$.

For any $x = \sum_{k\in I} x_k e_k$, write $\bar{x} = \sum_{k\in I} \overline{x}_k\, e_k$. It is easy to see that

$$(x \otimes y)^\top = \bar{y} \otimes \bar{x},$$

for any $x, y \in \mathcal{H}$.

To prove Theorem 1.1, we need some auxiliary results that we present below. The first lemma describes the spectrum of the skew Lie product $[x \otimes y, A]_*$ for any nonzero vectors $x, y \in \mathcal{H}$ and operator $A \in \mathcal{B}(\mathcal{H})$.

Lemma 2.1. *For any nonzero vectors $x, y \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, set*

$$\Delta_A(x, y) = (\langle Ax, y\rangle + \langle Ay, x\rangle)^2 - 4\|x\|^2\langle A^2y, y\rangle$$

and

$$\Lambda_A(x, y) = (\langle x, Ay\rangle + \langle Ax, y\rangle)^2 - 4\langle x, y\rangle\langle Ax, Ay\rangle.$$

Then

(1) $\sigma\left([x \otimes y, A]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x, y)}\right\}$;
(2) $\sigma\left([A, x \otimes y]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle x, Ay\rangle \pm \sqrt{\Lambda_A(x, y)}\right\}$.

Proof. For the proof of the first item see [2]. The second statement can be proved in a similar way and we therefore omit its proof. □

Corollary 2.1. *For any $x \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have*

$$\sigma(A(x \otimes x) + (x \otimes x)A) \quad = \quad \left\{0, \langle Ax, x\rangle \pm \|x\|\sqrt{\langle A^2x, x\rangle}\right\}.$$

Proof. It suffices to replace x by ix and y by x in Lemma 2.1 (1). □

The second principle gives necessary and sufficient conditions for two operators to be the same.

Lemma 2.2. *For any two operators A and B in $\mathcal{B}(\mathcal{H})$, the following statements are equivalent.*

(1) $A = B$.
(2) $\sigma([X, A]_*) = \sigma([X, B]_*)$ *for every operator* $X \in \mathcal{B}(\mathcal{H})$.
(3) $\sigma([X, A]_*) = \sigma([X, B]_*)$ *for every operator* $X \in \mathcal{B}_a(\mathcal{H})$.

Proof. The proof is the same as that of [2, Corollary 3.2]. □

The next lemma characterizes real scalar operators in terms of skew Lie products.

Lemma 2.3. *For an operator $A \in \mathcal{B}(\mathcal{H})$, we have $\sigma([A, X]_*) = \{0\}$ holds for any operator $X \in \mathcal{B}(\mathcal{H})$ if and only if $A = \alpha I$ for some scalar $\alpha \in \mathbb{R}$.*

Proof. The "*if*" part is obvious. To check the "*only if*" part, assume that

$$\sigma(([A, X]_*)) = \{0\}$$

holds for any operator $X \in \mathcal{B}(\mathcal{H})$. As $A - A^*$ is anti-self-adjoint then

$$\|A - A^*\| = r(A - A^*) = r([A, I]_*) = 0,$$

it follows that $A = A^*$. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\{x, Ax\}$ is a linearly independent set, then by Lemma 2.1 (2) we have

$$\sigma([A, x \otimes x]_*) = \frac{1}{2}\left\{0, \pm\sqrt{\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2}\right\}.$$

This is a contradiction since $\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2 \neq 0$. □

We close this section with the following lemma which gives a characterization of self-adjoint and antiself-adjoint operators in terms of the spectrum of the skew Lie product.

Lemma 2.4. *If $A \in \mathcal{B}(\mathcal{H})$ is nonzero operator, then*

(1) *$A \in \mathcal{B}_s(\mathcal{H})$ if and only if $\sigma([X, A]_*) \subset i\mathbb{R}$, for any $X \in \mathcal{B}(\mathcal{H})$;*
(2) *$A \in \mathcal{B}_a(\mathcal{H})$ if and only if $\sigma([X, A]_*) \subset \mathbb{R}$, for any $X \in \mathcal{B}(\mathcal{H})$.*

Proof. (1) If $A = A^*$, then $\sigma([X, A]_*) \subset i\mathbb{R}$, since $[X, A]_* = XA - AX^* = XA - (XA)^*$. To prove the converse, assume that $\sigma([X, A]_*) \subset i\mathbb{R}$ for any operator $X \in \mathcal{B}(\mathcal{H})$. In particular by Lemma 2.1 (1) we get

$$\sigma([x \otimes y, A]_*) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x, y)}\right\} \subset i\mathbb{R},$$

for any $x, y \in \mathcal{H}$. Which yields that

$$0 \;=\; \Re(\langle Ax, y\rangle - \langle Ay, x\rangle) = \langle (A - A)^* x, y\rangle + \langle y, (A - A^*)x\rangle.$$

Replace x by ix in the above equality, we get

$$\langle (A - A)^* x, y\rangle - \langle y, (A - A^*)x\rangle = 0.$$

Accordingly $\langle (A - A)^* x, y\rangle = 0$ for any $x, y \in \mathcal{H}$. Thus, $A = A^*$.

(2) We have

$$\begin{aligned} A \in \mathcal{B}_a(\mathcal{H}) &\Leftrightarrow iA \in \mathcal{B}_s(\mathcal{H}) \\ &\Leftrightarrow \sigma([X, iA]_*) \subset i\mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}) \quad \text{(by Lemma 2.4 (1))} \\ &\Leftrightarrow i\sigma([X, A]_*) \subset i\mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}) \ \ (\text{since } \sigma([X, iA]_*) = i\sigma([X, A]_*)) \\ &\Leftrightarrow \sigma([X, A]_*) \subset \mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}). \end{aligned}$$

□

3. Proof of Theorem 1.1

The "*if*" part is obvious. We will complete the proof of the "*only if*" part after proving several claims.

Claim 1. φ is injective.

Proof. For $A, B \in \mathcal{B}(\mathcal{H})$, assume that $\varphi(A) = \varphi(B)$. Then, for every $X \in \mathcal{B}(\mathcal{H})$, we have

$$\sigma([X, A]_*) \;=\; \sigma([\varphi(X), \varphi(A)]_*) = \sigma([\varphi(X), \varphi(B)]_* = \sigma([X, B]_*).$$

It then follows from Corollary 2.2 that $A = B$ and φ is injective. □

Claim 2. φ preserves self-adjoint and anti-self adjoint operators in both directions. In particular, we have $\varphi(0) = 0$.

Proof. Pick up an operator $A \in \mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}_s(\mathcal{H})$, then

$$\sigma\left([\varphi(X), \varphi(A)]_*\right) = \sigma\left([X, A]_*\right) \subset i\mathbb{R}.$$

As φ is surjective, then Lemma 2.4 (1) entails that $\phi(A) \in \mathcal{B}_s(\mathcal{H})$. Similarly, if $A \in \mathcal{B}_a(\mathcal{H})$, we have $\sigma([\varphi(X), \varphi(A)]_*) \subset \mathbb{R}$. By Lemma 2.4 (2), we get $\phi(A) \in \mathcal{B}_a(\mathcal{H})$.

For the converse, note that φ is bijective and φ^{-1} satisfies (1.1) A similar discussion entails that if $\varphi^{-1}(A) \in \mathcal{B}_s(\mathcal{H})$ (resp. $\varphi^{-1}(A) \in \mathcal{B}_a(\mathcal{H})$), then so is A. □

Claim 3. φ is homogenous, i.e., $\varphi(\alpha A) = \alpha\varphi(A)$ for any $\alpha \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$.

Proof. For any $\alpha \in \mathbb{C}$ and $A, X \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}\sigma([\varphi(X), \varphi(\alpha A)]_*) =& \sigma([X, \alpha A]_*)\\ =& \alpha\ \sigma([X, A]_*)\\ =& \alpha\ \sigma([\varphi(X), \varphi(A)]_*)\\ =& \sigma([\varphi(X), \alpha\varphi(A)]_*).\end{aligned}$$

Hence,

$$\sigma([\varphi(X), \varphi(\alpha A)]_*) = \sigma([\varphi(X), \alpha\varphi(A)]_*),$$

for any $X \in \mathcal{B}(\mathcal{H})$. Since φ is bijective, we infer from Lemma 2.3 that $\varphi(\alpha A) = \alpha\varphi(A)$. This ends the proof of Claim 3. □

Claim 4. There exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

(i) $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$ or
(ii) $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

Here $A^\top$ is the transpose of A with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$. From Claim 3 and (1.1), we have

$$\begin{aligned}\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) =& -\sigma\left(\varphi(iA)\varphi(iB) - \varphi(iB)\varphi(iA)^*\right)\\ =& -\sigma\left(-AB - BA^*\right)\\ =& \sigma\left(AB + BA^*\right).\end{aligned}$$

Thus,

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{3.1}$$

for any $A, B \in \mathcal{B}(\mathcal{H})$. Now Claim 2 implies that $\varphi(A) \in \mathcal{B}_s(\mathcal{H})$ whenever $A \in \mathcal{B}_s(\mathcal{H})$. This together with Claim 1 entail that the restriction $\varphi_{|\mathcal{B}_s(\mathcal{H})} : \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ is well defined and bijective. Moreover, (3.1) implies that

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)\right) = \sigma\left(AB + BA\right),$$

for any $A, B \in \mathcal{B}_s(\mathcal{H})$. Therefore, by [12, Theorem 3.1] (see also [23, Theorem 2], there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

- $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$ or
- $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

Here $A^\top$ is the transpose of A with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$. $\square$

In particular Claim 4 implies that $\varphi(I) = \pm I$. In the sequel we may and shall assume that $\phi(I) = I$. Define a map $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by putting

$$\psi(A) = U^*\varphi(A)U,$$

for every $A \in \mathcal{B}(\mathcal{H})$. Then ψ is a bijective map satisfying

$$\sigma\left(\psi(A)\psi(B) + \psi(B)\psi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{3.2}$$

for every $A, B \in \mathcal{B}(\mathcal{H})$. Moreover, we have either

$$\psi(A) = A, \quad A \in \mathcal{B}_s(\mathcal{H}), \tag{3.3}$$

or

$$\psi(A) = A^\top, \quad A \in \mathcal{B}_s(\mathcal{H}). \tag{3.4}$$

Claim 5. The form (3.4) cannot occur.

Proof. Assume for the sake of contradiction that $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. Let $\{e_j, j \in I\}$ be the orthonormal basis with respect to which $A^\top$ is computed, for every $A \in \mathcal{B}_s(\mathcal{H})$. To get a contradiction we shall prove that $\langle Ax, x\rangle = \langle \psi(A)x, x\rangle$ for any $x \in \mathcal{H}$ and $A \in \mathcal{B}_s(\mathcal{H})$. To do so it suffices to prove that

$$\langle Ae_k, e_l\rangle = \langle \psi(A)e_k, e_l\rangle, \tag{3.5}$$

for any k and l in I and $A \in \mathcal{B}_s(\mathcal{H})$.

Let $A \in \mathcal{B}_s(\mathcal{H})$ and pick up two elements e_k and e_l in $\{e_j, j \in I\}$. For any $\alpha, \beta \in \mathbb{R}$, set $a = \alpha e_k + \beta e_l$. Note that

$$\psi(a \otimes a) = (a \otimes a)^\top = a \otimes a.$$

Now, by (3.2) we have

$$\begin{aligned}
\sigma\left((a \otimes a)A + A(a \otimes a)\right) =& \sigma\left((a \otimes a)A + A(a \otimes a)^*\right) \\
=& \sigma\left(\psi(a \otimes a)\psi(A) + \psi(A)\psi(a \otimes a)^*\right) \\
=& \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)^*\right) \\
=& \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right).
\end{aligned}$$

Accordingly

$$\sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right) \;=\; \sigma\left((a \otimes a)A + A(a \otimes a)\right). \tag{3.6}$$

Corollary 2.1 together with (3.6) entail that

$$\left\{0, \langle\psi(A)a, a\rangle \pm \|a\|\sqrt{\langle\psi(A)^2a, a\rangle}\right\} = \left\{0, \langle Aa, a\rangle \pm \sqrt{\langle A^2a, a\rangle\|a\|^2}\right\}.$$

Accordingly $\langle\psi(A)a, a\rangle = \langle Aa, a\rangle$. Since α and β are arbitrary, we infer that

$$\langle Ae_k, e_k\rangle = \langle\psi(A)e_k, e_k\rangle$$

and

$$\langle A(e_k + e_l), (e_k + e_l)\rangle = \langle\psi(A)(e_k + e_l), (e_k + e_l)\rangle,$$

for every $k, l \in I$. Since A and $\psi(A)$ are in $\mathcal{B}_s(\mathcal{H})$, we infer that

$$\langle Ae_k, e_l\rangle = \langle\psi(A)e_k, e_l\rangle.$$

This in particular implies that $\psi(A) = A$ for every for any $A \in \mathcal{B}_s(\mathcal{H})$. Which is impossible since $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. □

Claim 6. $\psi(A) = A$ for any $A \in \mathcal{B}(\mathcal{H})$.

Proof. We have $\psi(A) = A$ for any $A \in \mathcal{B}_s(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, using a similar reasoning as above, one can show that $\langle Ax, x\rangle = \langle\psi(A)x, x\rangle$ for any $x \in \mathcal{H}$. Since $\mathcal{H}$ is a complex Hilbert space it yields that $\psi(A) = A$ as desired. The proof is thus complete. □

References

[1] Z. Abdelali, *Maps preserving the spectrum of polynomial products of matrices*, J. Math. Anal. Appl. **480** (2019), Paper ID 123392.

[2] Z. Abdelali, A. Bourhim and M. Mabrouk, *Preservers of spectrum and local spectrum on skew Lie products of matrices*, Contemp. Math. (to appear).

[3] L. Baribeau and T. Ransford, *Non-linear spectrum-preserving maps*, Bull. London Math. Soc. **32** (2000), 8–14.

[4] A. Bourhim and J. Mashreghi, *Maps preserving the local spectrum of triple product of operators*, Linear Multilinear Algebra (2014), 1–9.

[5] A. Bourhim and J. Mashreghi, *Maps preserving the local spectrum of product of operators*, Glasg. Math. J. **57** (2015), 709–718.

[6] A. Bourhim and J. Mashreghi, *A survey on preservers of spectra and local spectra*, in: *CRM Proceedings and Lecture Notes: Invariant Subspaces of the Shift Operator*, R. Amer. Math. Soc., Providence, 2015.

[7] A. Bourhim, J. Mashreghi and A. Stepanyan, *Maps between Banach algebras preserving the spectrum*, Arch. Math. **107** (2016), 609–621.

[8] J.-T. Chan, C.-K. Li and N.-S. Sze, *Mappings preserving spectra of products of matrices*, Proc. Amer. Math. Soc. **135** (2007), 977–986.

[9] C. Costara, *Non-surjective spectral isometries on matrix spaces*, Complex Anal. Oper. Theory **12** (2018), 859–868.

[10] C. Costara and D. Repovš, *Spectral isometries onto algebras having a separating family of finite-dimensional irreducible representations*, J. Math. Anal. Appl. **365** (2010), 605–608.

[11] J. Cui and P. Choonkil, *Maps preserving strong skew Lie product on factor von Neumann algebras*, Acta Math. Sci. **32** (2012), 531–538.

[12] J.-L. Cui and C.-K. Li, *Maps preserving peripheral spectrum of Jordan products of operators*, Oper. Matrices **6** (2012), 129–146.

[13] O. Hatori, T. Miura and H. Takagi, *Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative*, J. Math. Anal. Appl. **326** (2007), 281–296.
[14] J. Hou and K. He, *Non-linear maps on self-adjoint operators preserving numerical radius and numerical range of Lie product*, Linear Multilinear Algebra **64** (2016), 36–57.
[15] J. Hou, C.-K. Li and X. Qi, *Numerical range of Lie product of operators*, Integral Equations Operator Theory **83** (2015), 497–516.
[16] J. Hou, C.-W. Li and N.-C. Wong, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl. **432** (2010), 1049–1069.
[17] J. C. Hou, C. K. Li and N. C. Wong, *Jordan isomorphisms and maps preserving spectra of certain operator products*, Studia Math. **184** (2008), 31–47.
[18] J. C. Hou, C. K. Li and N. C. Wong, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl. **432** (2010), 1049–1069.
[19] L. Molnár, *Some characterizations of the automorphisms of* $\mathcal{B}(\mathcal{H})$ *and* $C(X)$, Proc. Amer. Math. Soc. **130** (2002), 111–120.
[20] P. V. R. Bhatia and A. Sourour, *Maps on matrices that preserve the spectral radius distance*, Studia Math. **134** (1999), 99–110.
[21] A. Taghavi, R. Hosseinzadeh and E. Nasrollahi, *Maps preserving a certain product of operators*, Ric. Mat. (2019), 1–12.
[22] W. Zhang and J. Hou, *Maps preserving peripheral spectrum of Jordan semi-triple products of operators*, Linear Algebra Appl. **435** (2011), 1326–1335.
[23] W. Zhang and J. Hou, *Maps preserving peripheral spectrum of generalized Jordan products of self-adjoint operators*, Abstr. Appl. Anal. **2014** (2014), Article ID 192040, 8 pages.

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EXTENSIONS OF MEIR-KEELER CONTRACTION VIA w-DISTANCES WITH AN APPLICATION

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Abstract. In this article, we conceive the notion of a generalized (α, ψ, q)-Meir-Keeler contractive mapping and then we investigate a fixed point theorem involving such kind of contractions in the setting of a complete metric space via a w-distance. Our obtained result extends and generalizes some of the previously derived fixed point theorems in the literature via w-distances. In addition, to validate the novelty of our findings, we illustrate a couple of constructive numerical examples. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

1. Introduction and Preliminaries

In this paper, we introduce the notion of a generalized (α, ψ, q)-Meir-Keeler contractive mapping and investigate fixed points for such operators in the context of complete metric spaces via a w-distance. For this purpose we first recall the outstanding result of Meir-Keeler [14] (see also [10]).

Theorem 1.1 ([14]). *Let f be a self-map defined on a complete metric space (M, d). Also assume that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that*

$$\varepsilon \leq d(\rho, \varrho) < \varepsilon + \delta \quad \textit{implies} \quad d(f\rho, f\varrho) < \varepsilon,$$

for all $\rho, \varrho \in M$. Then f has a unique fixed point.

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This result is also known as a uniform contraction and it has been studied and extended by a number of researchers in many directions (see [16, 20]). Now we recall the notion of w-distance introduced by Kada et al. [12].

Definition 1.1 ([12]). Let (M, d) be a metric space. A mapping $q : M \times M \to [0, \infty)$ is said to be a w-distance on M if

(i) $q(\rho, \sigma) \leq q(\rho, \varrho) + q(\varrho, \sigma)$ for any $\rho, \varrho, \sigma \in M$;
(ii) q is a lower semi-continuous map in the second variable, that is, when $\rho \in M$ and $\sigma_n \to \sigma$ in M, then we have $q(\rho, \sigma) \leq \liminf_n q(\rho, \sigma_n)$;
(iii) for every $\epsilon > 0$, there is a $\delta > 0$ which $q(\sigma, \rho) \leq \delta$ and $q(\sigma, \varrho) \leq \delta$ imply that $d(\rho, \varrho) \leq \epsilon$.

Let $T : M \to M$ and $\alpha : M \times M \to [0, \infty)$. We say that T is α-orbital admissible (see [17]) if

$$\alpha(p, Tp) \geq 1 \quad \text{implies} \quad \alpha(Tp, T^2p) \geq 1,$$

for all $p \in M$. By using this auxiliary function, it is possible to combine several existing results in the literature, see, e.g. [9, 15, 18, 19] and the related references therein. In particular, Lakzian et al. [13] introduced the concept of (α, ψ, q)-contractive mappings in metric spaces via w-distances and proved fixed point results via this notion.

On the other hand, inspired by the notion of Meir-Keeler contractions, Chen [11] introduced the concept of a weaker Meir-Keeler function as follows.

Definition 1.2 ([11]). A mapping $\psi : [0, \infty) \to [0, \infty)$ is said to be a weaker Meir-Keeler function if, for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $\tau \in [0, \infty)$ with $\epsilon \leq \tau < \epsilon + \delta$, we have an $n_0 \in \mathbb{N}$ satisfying $\psi^{n_0}(\tau) < \epsilon$.

Regarding [11], we also consider the family Ψ of weaker Meir-Keeler functions $\psi : [0, \infty) \to [0, \infty)$ fulfilling the subsequent properties:

(ψ_1) $\psi(\tau) > 0$ whenever $\tau > 0$ and $\psi(0) = 0$;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(\tau) < \infty$, $\tau \in (0, \infty)$;
(ψ_3) for each $y_n \in [0, \infty)$, the following hold:
  (i) when $\lim_{n\to\infty} y_n = \ell > 0$, then $\lim_{n\to\infty} \psi(y_n) < \ell$;
  (ii) whenever $\lim_{n\to\infty} y_n = 0$, we have $\lim_{n\to\infty} \psi(y_n) = 0$.

Along with the aforementioned terminologies, the following lemma is also playing a crucial role in our subsequent studies.

Lemma 1.1 ([12]). *Suppose that (M, d) is a metric space with a w-distance q.*

(i) *For any sequence $\{\rho_n\}$ in M with $\lim_n q(\rho_n, \rho) = \lim_n q(\rho_n, \varrho) = 0$, we have $\rho = \varrho$. Additionally, $q(\sigma, \rho) = q(\sigma, \varrho) = 0$ implies $\rho = \varrho$.*
(ii) *For two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, \infty)$ converging to 0, whenever $q(\rho_n, \varrho_n) \leq \alpha_n$, $q(\rho_n, \varrho) \leq \beta_n$ hold for each $n \in \mathbb{N}$, then the sequence $\{\varrho_n\}$ converges to ϱ.*

(iii) *Suppose that* $\{\rho_n\}$ *is a sequence in* M *such that for every* $\varepsilon > 0$ *there is an* $N_\varepsilon \in \mathbb{N}$ *with* $m > n > N_\varepsilon$ *implies that* $q(\rho_n, \rho_m) < \varepsilon$ *(or* $\lim_{m,n} q(\rho_n, \rho_m) = 0$*). Then* $\{\rho_n\}$ *is a Cauchy sequence.*

In this paper, we define the concept of generalized (α, ψ, q)-Meir-Keeler contractive mappings and by using this new concept, we give some fixed point results. Furthermore, some significant non-trivial numerical examples are investigated to authenticate our findings. Moreover, as an application, the existence of the solution for a non-linear Fredholm integral equation is investigated.

2. (α, ψ, q)-Meir-Keeler Contractions

This section brings the idea of generalized (α, ψ, q)-Meir-Keeler contractive mappings with the help of a weaker Meir-Keeler function. Also, we conceive a fixed point result concerning such kinds of mappings. Now we consider the following expressions:

$$M_q(\rho, \varrho) = \max\left\{q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho), \frac{q(\rho, f\varrho) + q(f\rho, \varrho)}{2}\right\}$$

and

$$m(\rho, \varrho) = \max\left\{d(\rho, \varrho), d(\rho, f\rho), d(\varrho, f\varrho), \frac{d(\rho, f\varrho) + d(f\rho, \varrho)}{2}\right\}.$$

Here, we propose the idea of generalized (α, ψ, q)-Meir-Keeler contractive mappings.

Definition 2.1. Suppose that (M, d) is a metric space with a w-distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0, \infty)$ and an α-orbital admissible map f. Then f is called a generalized (α, ψ, q)-Meir-Keeler contractive mapping if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \le \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho) q(f\rho, f\varrho) < \eta$.

In addition, for $q = d$ and $M_q(\rho, \varrho) = m(\rho, \varrho)$, the mapping f is said to be a generalized (α, ψ)-Meir-Keeler-contractive. Furthermore, f is a (α, ψ, q)-Meir-Keeler contractive map, when $M_q(\rho, \varrho) = q(\rho, \varrho)$ for each $\rho, \varrho \in M$.

The succeeding theorem deals with an interesting fixed point result involving the previously discussed type of maps.

Theorem 2.1. *Suppose that* (M, d) *is a complete metric space with a* w*-distance* q*. Also assume that* f *is a generalized* (α, ψ, q)*-Meir-Keeler contractive map such that there is* $\rho_0 \in M$ *with* $q(f^n\rho_0, f^n\rho_0) = 0$ *for all non-negative integers* n *and* $\alpha(\rho_0, f\rho_0) \ge 1$*. Suppose that one of the following conditions holds.*

(i) *For each* $w \in M$ *satisfying* $w \ne fw$*, we have* $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.

(ii) f *is continuous.*

(iii) *If for some sequence* $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$*, then* $f\rho = \rho$.

Then f *owns a fixed point* $u \in M$*, with* $q(u, u) = 0$.

Proof. We construct a sequence $\{\rho_n\}$ in M such that $\rho_{n+1} = f\rho_n = f^{n+1}\rho_0$ for each $n \in \mathbb{N}$. When $\rho_{n_0} = \rho_{n_0+1}$ for some positive integer n_0, then $u = \rho_{n_0}$ is a fixed point of f. Hence, without loss of generality consider that,

$$\rho_n \neq \rho_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

As f is α-orbital admissible, we have

$$\alpha(\rho_0, \rho_1) = \alpha(\rho_0, f\rho_0) \geq 1 \quad \text{implies} \quad \alpha(f\rho_0, f\rho_1) = \alpha(\rho_1, \rho_2) \geq 1.$$

Using mathematical induction, it follows that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$. Now, we divide the entire proof into four steps and discuss one by one.

Step 1. We first prove that for each $n \in \mathbb{N}$

$$q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n).$$

Note that for every natural number n, we have $q(\rho_n, \rho_{n+1}) > 0$. Since, otherwise by the combination of $q(\rho_n, \rho_{n+1}) = 0$ and the assumption $q(\rho_n, \rho_n) = 0$ and applying Lemma 1.1 we get $\rho_n = \rho_{n+1}$, which is a contradiction. Therefore, we find that

$$\begin{aligned} M_q(\rho_{n-1}, \rho_n) &= \max\left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \\ &> 0. \end{aligned}$$

Hence, we obtain $\psi(M_q(\rho_{n-1}, \rho_n)) > 0$. Now, from the hypothesis and Definition 2.1 for $\eta = \psi(M_q(\rho_{n-1}, \rho_n))$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In particular, since for each $\tau > 0$, $\psi(\tau) < \tau$, we have

$$q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n) q(\rho_n, \rho_{n+1}) < \eta = \psi(M_q(\rho_{n-1}, \rho_n)) < M_q(\rho_{n-1}, \rho_n). \tag{2.1}$$

Since

$$\frac{q(\rho_{n-1}, \rho_{n+1})}{2} \leq \frac{q(\rho_{n-1}, \rho_n) + q(\rho_n, \rho_{n+1})}{2} \leq \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\},$$

we have

$$\begin{aligned} M_q(\rho_{n-1}, \rho_n) &= \max\left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \\ &= \max\left\{ q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1})}{2} \right\} \\ &= \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}. \end{aligned}$$

So, $q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n) = \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}$ and this implies that

$$M_q(\rho_{n-1}, \rho_n) = q(\rho_{n-1}, \rho_n) \quad \text{and} \quad q(\rho_n, \rho_{n+1}) < q(\rho_{n-1}, \rho_n).$$

Now, since $\{q(\rho_{n-1}, \rho_n)\}$ is decreasing and bounded below, it is convergent to $t \geq 0$ such that $q(\rho_n, \rho_{n+1}) \geq t$ for each n. Assume that $t \neq 0$ and $\xi = \lim_n \psi(q(\rho_n, \rho_{n+1}))$. Then by (ψ_3), $0 < \xi < t$ and by Definition 2.1, we can find $\delta > 0$ satisfying

$$\text{when} \quad \xi \leq \psi(M_q(\rho, \varrho)) < \xi + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho) q(f\rho, f\varrho) < \xi, \tag{2.2}$$

for $\rho, \varrho \in M$. Consider $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$ and $\frac{1}{k_0} < \xi$. Then for each $k \geq k_0$ there is $\delta_k \leq \frac{1}{k}$ such that

$$\xi - \frac{1}{k} \leq \psi(M_q(\rho, \varrho)) < \xi - \frac{1}{k} + \delta_k \quad \text{implies} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \xi - \frac{1}{k} < \xi. \tag{2.3}$$

Also there is $k_2 \in \mathbb{N}$ such that for each $n \geq k_2$ one obtains

$$\xi - \frac{1}{k_0} < \psi(q(\rho_{n-1}, \rho_n)) = \psi(M_q(\rho_{n-1}, \rho_n)) < \xi + \frac{1}{k_0} < \xi + \delta.$$

Now, when $\xi \leq \psi(M_q(\rho_{n-1}, \rho_n)) \leq \xi + \frac{1}{k_0}$, by (2.2), we have

$$q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t,$$

and when $\xi - \frac{1}{k_0} \leq \psi(M_q(\rho_{n-1}, \rho_n) < \xi$ by (2.3) and since

$$[\xi - \frac{1}{k_0}, \xi] \subseteq \cup_{k \geq k_0}[\xi - \frac{1}{k}, \xi - \frac{1}{k} + \delta_k),$$

we have $q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t$, which is a contradiction. Therefore, $t = 0$ and so

$$\lim_n q(\rho_n, \rho_{n+1}) = 0. \tag{2.4}$$

Step 2. We prove that $\{\rho_n\}$ is a Cauchy sequence. Alternatively, from the inequality (2.1), we arrive at

$$q(\rho_n, \rho_{n+1}) \leq \psi(q(\rho_{n-1}, \rho_n)), \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Indeed, if there exists some n^* such that

$$q(\rho_{n^*}, \rho_{n^*+1}) \leq \psi(q(\rho_{n^*}, \rho_{n^*+1})) < q(\rho_{n^*}, \rho_{n^*+1}),$$

we get a contradiction. Hence, (2.5) holds. Inductively, we derive, from (2.5), that

$$q(\rho_n, \rho_{n+1}) \leq \psi^n(q(\rho_0, \rho_1)), \quad \text{for all } n \in \mathbb{N}.$$

Fix ε and let $n_\varepsilon \in \mathbb{N}$ such that $\sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)) < \varepsilon$. Furthermore, for $m > n > n_\varepsilon$ we can find that

$$\begin{aligned} q(\rho_n, \rho_m) &\leq q(\rho_n, \rho_{n+1}) + \cdots + q(\rho_{m-1}, \rho_m) \\ &\leq \sum_{k=n}^{m-1} \psi^k(q(\rho_1, \rho_0)) \\ &\leq \sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)). \end{aligned}$$

Hence, we conclude that the sequence $\{\rho_n\}$ is Cauchy. Now, since (M, d) is complete, we can get $u \in M$ with $\rho_n \to u$ in M.

Step 3. u is a fixed point of f.

Case (i). For each $\varrho \in M$ satisfying $\varrho \neq f\varrho$, we have $\inf\{q(\rho, \varrho) + q(\rho, f\rho) : \rho \in M\} > 0$. It implies that for every $\varepsilon > 0$, there is a natural number N such that for

$n > N_\varepsilon$, we have $q(\rho_{N_\varepsilon}, \rho_n) < \varepsilon$. Since, $\rho_n \to u$ and $q(\rho, \cdot)$ is a lower semi-continuous map, we have

$$q(\rho_{N_\varepsilon}, u) \leq \liminf_{n\to\infty} q(\rho_{N_\varepsilon}, \rho_n) \leq \varepsilon.$$

Putting $\varepsilon = \frac{1}{k}$ and $N_\varepsilon = n_k$, we have

$$\lim_{k\to\infty} q(\rho_{n_k}, u) = 0. \tag{2.6}$$

Assume that $u \neq fu$. Then

$$0 < \inf\{q(\sigma, u) + q(\sigma, f\sigma) : \sigma \in M\} \leq \inf\{q(\rho_{n_k}, u) + q(\rho_{n_k}, \rho_{n_k+1}) : k \in \mathbb{N}\}.$$

From (2.4) and (2.6), we derive $\inf\{q(\sigma, u)+q(\sigma, f\sigma) : \sigma \in M\} = 0$, which contradicts the given hypothesis. Therefore, $fu = u$.

Case (ii). Let f be continuous.

Using the triangular inequality, we have

$$q(\rho_n, f^2\rho_n) \leq q(\rho_n, f\rho_n) + q(f\rho_n, f^2\rho_n).$$

Accordingly, letting $n \to \infty$, we obtain $q(\rho_n, f^2\rho_n) \to 0$. Further, Lemma 1.1 confirms that $\{f^2\rho_n\} \to u$ as $n \to \infty$. As f is continuous, we have

$$fu = f(\lim_{n\to\infty} f\rho_n) = \lim_{n\to\infty} f^2\rho_n = u.$$

Hence, u is a fixed point of f.

Case (iii). Here, $\lim_{n\to\infty} q(f\rho_n, u) = \lim_{n\to\infty} q(\rho_{n+1}, u) = \lim_{n\to\infty} q(\rho_n, u)$. Hence, $fu = u$.

Step 4. u is a fixed point with $q(u, u) = 0$.

Conversely, suppose that $q(u, u) > 0$. Then from (2.1), we get

$$0 < q(u, u) = q(fu, fu) \leq \psi(M_q(u, u)) < M_q(u, u) = q(u, u),$$

and this is impossible. Hence, our claim is verified. □

The fixed point obtained in the previous theorem may be not unique. The following examples validate our claim.

Example 2.1. Suppose that G is a locally compact group, $M = L^1(G)$ and

$$q(f, g) = \|g\|_1, \quad f, g \in L^1(G).$$

Then q is a w-distance . Define $\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$ and

$$\alpha(f, g) = \begin{cases} 2, & g = 0 \quad (a.e.), \\ \dfrac{\psi(M_q(f, g))}{2\|g\|_1}, & \text{otherwise}, \end{cases}$$

and for an arbitrary $x \in G$

$$\begin{array}{rcl} T_x : L^1(G) & \to & L^1(G), \\ f & \mapsto & \frac{1}{8}L_x f, \end{array}$$

where $L_x f(y) = f(x^{-1}y)$. Then for each $f \in L^1(G)$ and $x \in G$, since $\|L_x f\|_1 = \|f\|_1$, we conclude that $M_q(f,g) = \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ and so

$$\alpha(f,g) = \frac{\psi(M_q(f,g))}{2\|g\|_1} = \frac{\psi(\max\{\frac{1}{8}\|f\|_1, \|g\|_1\})}{2\|g\|_1} \geq 1.$$

In each of the cases $0 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 1$, $1 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 8$ and $8 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ we conclude that

$$\alpha(T_x f, T_x g) = \frac{\psi(\max\{\frac{1}{64}\|f\|_1, \frac{1}{8}\|g\|_1\})}{\frac{2}{8}\|g\|_1} > 1.$$

So, T_x is α-orbital admissible. Now for each $\eta > 0$ and $\delta = \eta$, if $\eta \leq \psi(M_q(f,g)) < 2\eta$, then for $g \neq 0$ we have

$$\alpha(f,g) q\left(\frac{1}{8}L_x f, \frac{1}{8}L_x g\right) \leq \frac{\psi(M_q(f,g))}{2\|g\|_1}\left(\frac{1}{8}\right)\|g\|_1 \leq \frac{1}{8}\eta < \eta,$$

and for $g = 0$, since $q(T_x f, T_x g) = 0$, we are done. So, T_x is a generalized (α, ψ, q)-Meir-keeler contractive map. Moreover, $\alpha(0, T_x 0) = \alpha(0,0) = 2 > 1$,

$$q(T_x^n 0, T_x^n 0) = q(0,0) = \|0\|_1 = 0,$$

and T_x is continuous. Therefore, all the hypotheses of Theorem 2.1 hold and so, T_x has a fixed point (which is $f = 0$, satisfying $q(0,0) = 0$). Note that for each $f \in L^1(G)$, we have

$$\lim \|T_x^n f\|_1 = \lim \frac{1}{8^n}\|f\|_1 = 0.$$

Therefore, $T_x^n f$ converges to 0 and so 0 is the only fixed point of T_x.

Example 2.2. Suppose that $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ is equipped with the usual metric on $\mathbb{R}$. Consider

$$q(\rho, \varrho) = \begin{cases} \dfrac{1}{n} + \dfrac{1}{m}, & \varrho = \dfrac{1}{2^m},\ \rho = \dfrac{1}{2^n}, \\ 0, & \rho = 0 \text{ or } \varrho = 0, \end{cases}$$

$$\alpha(\rho, \varrho) = \begin{cases} \dfrac{m}{n}, & \varrho = \dfrac{1}{2^m},\ \rho = \dfrac{1}{2^n} \text{ and } 2n \geq m \geq n, \\ 1, & \rho = 0 \text{ or } \varrho = 0, \\ \dfrac{1}{n}, & \text{otherwise}, \end{cases}$$

and $f\rho = \rho^8$. Then $\alpha(0, f0) = 1$, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and f is continuous and also α-orbital admissible. Since if $\alpha(\rho, f\rho) \geq 1$, then $\rho = 0$, since if $\rho = \frac{1}{2^n}$ for some n, then $n \leq 8n \leq 2n$ is impossible. Therefore, $\alpha(f\rho, f^2\rho) \geq 1$. Also if

$$\psi(t) = \begin{cases} \dfrac{t}{2}, & t \in [0,1], \\ \dfrac{1}{2}, & t \in (1, \infty), \end{cases}$$

then, since $0 \leq M_q(\rho, \varrho) \leq 1$, we have $\psi(M_q(\rho, \varrho)) = \frac{1}{2}M_q(\rho, \varrho)$. On the other hand, for each $\eta > 0$ and for $\delta = \eta$, if $\eta \leq \psi(M_q(\rho, \varrho)) < 2\eta$, then ρ or ϱ is non-zero. So if $\rho = \frac{1}{2^n}$ and $\varrho = \frac{1}{2^m}$, then since $\alpha(\rho, \varrho) \leq 2$, we conclude that

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq 2\left(\frac{1}{8n} + \frac{1}{8m}\right) = \frac{1}{4}q(\rho, \varrho) \leq \frac{1}{4}M_p(\rho, \varrho) = \frac{1}{2}\psi(M_p(\rho, \varrho)) < \eta.$$

Also, if one of ρ or ϱ is zero, then $\alpha(\rho, \varrho)q(f\rho, f\varrho) = 0 \leq \eta$. So, f is a generalized (α, ψ, q)-Meir-Keeler contractive map. Therefore, all the conditions of Theorem 2.1 hold. Hence, $\rho = 0$ is the unique fixed point of f.

Example 2.3. Let $M = [0, 1]$ be equipped with the usual metric. Also let us consider the w-distance as $q(\rho, \varrho) = |\rho - \varrho|$ for each $\rho, \varrho \in M$. Further, we define

$$f\rho = \begin{cases} \dfrac{\rho}{20}, & \rho \in [0, 1), \\ 1, & \rho = 1, \end{cases} \qquad \alpha(\rho, \varrho) = \begin{cases} 1, & \rho, \varrho \in [0, 1), \\ 0, & \rho = 1, \end{cases} \qquad \psi(\rho) = \begin{cases} 0, & \rho = 0, \\ \dfrac{1}{3}, & \rho \in \left(0, \dfrac{1}{2}\right), \\ \dfrac{\rho}{2}, & \rho \in \left[\dfrac{1}{2}, 1\right], \\ \dfrac{1}{2}, & \rho \in (1, \infty). \end{cases}$$

Hence, for every $w \in M$ with $fw \neq w$, we obtain $w \neq 0, 1$ and so

$$\lim_{\rho \to w}(|w - \rho| + |\rho - f\rho|) \geq \frac{19}{20}w > 0.$$

Again,

$$\lim_{\rho \to \varrho}(|w - \rho| + |\rho - f\rho|) \geq |w - \varrho| > 0, \quad \varrho \neq w.$$

Therefore, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$ for each $w \in M$ satisfying $w \neq fw$. Besides, for every $\rho \in M$, we obtain $|f^n\rho - f^n\rho| = 0$. Now for each $\eta > 0$, put $\delta = \eta$. Then, $\rho = \varrho$ implies $M_q(\rho, \varrho) = 0$ and when $\rho \neq \varrho$, $M_q(\rho, \varrho) \neq 0$ and further, $\psi(M_q(\rho, \varrho)) \geq \frac{1}{4}$. Therefore, for $\eta > \frac{1}{8}$, there is no $\rho, \varrho \in M$ satisfying

$$\frac{1}{8} \leq \psi(M_q(\rho, \varrho)) < \frac{1}{4}.$$

On the other hand, for $\eta \leq \frac{1}{8}$, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \eta = 2\eta$, we have

$$\alpha(\rho, \varrho)|f\rho - f\varrho| \leq |f\rho - f\varrho| = \left|\frac{\rho}{20} - \frac{\varrho}{20}\right| \leq \frac{2}{20} < \frac{1}{8} < \eta.$$

That is for each ρ, ϱ, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \eta = 2\eta$, then $\alpha(\rho, \varrho)|f\rho - f\varrho| \leq \eta$. Note that $0, 1$ are the fixed points of f.

Remark 2.1. In the case where $q(\rho, \varrho) = \varrho$ for each $\rho, \varrho \in M$, the assumption $q(f^n\rho, f^n\rho) = 0$, for some $\rho \in M$ and for each $n \in \mathbb{N}$, imply that $f^n\rho = 0$ for each n. Therefore, in this case without any another condition, since $\rho_n = 0 = \rho_{n+1}$, the first part of the Theorem 2.1 implies that f possesses a fixed point. For example, let $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$,

$$\alpha(\rho, \varrho) = \begin{cases} 0, & \varrho \in \left\{\dfrac{1}{2^{2k}} : k \in \mathbb{N}\right\}, \\ 1, & \text{otherwise}, \end{cases} \qquad \text{and} \quad f\rho = \begin{cases} \dfrac{\rho}{2}, & \rho \in \left\{\dfrac{1}{2^{2k}} : k \in \mathbb{N}\right\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then f is continuous, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and $\rho, \varrho \in M$ and $\eta, \delta > 0$, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, then we have

$$0 = \alpha(\rho, \varrho) q(f\rho, f\varrho) \leq \eta.$$

Note that 0 is a fixed point of f, since here we require only $q(f^n 0, f^n 0) = 0$.

Now we put down the following additional hypothesis. To attest the uniqueness of the fixed point of f, this condition along with those of Theorem 2.1 is required.
Property U. Let $\alpha(u, v) < 1$, implies that at least one of u or v is not a fixed point of f.

For example if $\alpha(u, v) \geq 1$ for each $u, v \in M$, then the property U is valid.

Theorem 2.2. *Suppose that (M, d) is a metric space with a w-distance q. Also assume that f is a generalized (α,ψ,p)-Meir-Keeler contractive mapping and satisfies all the hypotheses of Theorem* 2.1 *along with the additional property U. Then we can claim the uniqueness of the fixed point of f obtained in Theorem* 2.1.

Proof. We suppose that $u, v \in M$ are two distinct fixed points of f. Then $\alpha(u, v) \geq 1$, $fu = u$, $fv = v$, $q(u, u) = 0$ and $q(v, v) = 0$. Using the aforementioned criteria and (2.1), we obtain

$$q(u, v) = q(fu, fv) \leq \alpha(u, v) q(fu, fv) \leq \psi(M_q(u, v)) = \psi(q(u, v)) < q(u, v),$$

and this is impossible. Hence, f possesses a unique fixed point. □

3. Consequences

This section deals with a few immediate corollaries of our obtained Theorem 2.1. First, we give the following important result for an (α, ψ, q)-Meir-Keeler contractive mapping.

Corollary 3.1. *Suppose that (M, d) is a complete metric space with a w-distance q. Also let f be an (α, ψ, q)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n\rho_0, f^n\rho_0) = 0$ for all non-negative integers n and $\alpha(\rho_0, f\rho_0) \geq 1$. Suppose that one of the following holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.

(ii) *f is continuous.*

(iii) *If for some sequence $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$, with $q(u, u) = 0$.

Putting $\alpha \equiv 1$ in Theorem 2.1, we obtain the trailing important corollary.

Corollary 3.2. *Suppose that (M,d) is a complete metric space with a w-distance q. Also let f be a (ψ,q)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n\rho_0, f^n\rho_0) = 0$ for all non-negative integers n. Suppose that one of the following conditions holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{q(\rho,w)+q(\rho,f\rho) : \rho \in M\} > 0$.

(ii) *f is continuous.*

(iii) *If for some sequence $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n,\rho) = \lim_{n\to\infty} q(f\rho_n,\rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$.

Considering $q = d$ in Theorem 2.1, we deduce the subsequent corollary.

Corollary 3.3. *Suppose that (M,d) is a complete metric space and f be an (α,ψ)-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$ with $\alpha(\rho_0, f\rho_0) \geq 1$ or $\alpha(f\rho_0,\rho_0) \geq 1$. Suppose that one of the following conditions holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have* $\inf\{d(\rho,w)+d(\rho,f\rho) : \rho \in M\} > 0$.

(ii) *f is continuous.*

(iii) *For some sequence $\{\rho_n\}$ in M with $\alpha(\rho_n,\rho_{n+1}) \geq 1$ for all natural numbers n and $\rho_n \to \rho \in M$ as $n \to \infty$, then $\alpha(\rho_n,\rho) \geq 1$ for every $n \in \mathbb{N}$.*

Then f possesses a fixed point $u \in M$.

Taking $\alpha \equiv 1$ in Corollary 3.3, we get the succeeding consequence.

Corollary 3.4. *Suppose that (M,d) is a complete metric space and f be a ψ-Meir-Keeler contractive mapping. Suppose that either f is continuous or* $\inf\{d(\rho,w) + d(\rho,f\rho) : \rho \in M\} > 0$ *for each $w \in M$ with $w \neq fw$. Then f possesses a fixed point $u \in M$.*

Definition 3.1. Suppose that (M,d) is a metric space with a w-distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0,\infty)$ and a self-map f. Then f is said to be a generalized (α,ψ,q)-Meir-Keeler contractive mapping of

(a) Banach type if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho,\varrho \in M$

$$\text{when} \quad \eta \leq \psi(q(\rho,\varrho)) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho,\varrho)p(f\rho,f\varrho) < \eta;$$

(b) Kannan type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho,\varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho,f\rho)+q(\varrho,f\varrho)}{2}\right) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho,\varrho)p(f\rho,f\varrho) < \eta;$$

(c) Kannan type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho,\varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho,f\rho),q(\varrho,f\varrho)\}) < \eta+\delta, \quad \text{we have} \quad \alpha(\rho,\varrho)p(f\rho,f\varrho) < \eta;$$

(d) Chatterjea type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho, f\varrho) + q(\varrho, f\rho)}{2}\right) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(e) Chatterjea type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(f) Reich type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi\left(\frac{q(\rho, \varrho) + q(\rho, f\rho) + q(\varrho, f\varrho)}{3}\right) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(g) Reich type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta;$$

(h) Reich type III if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\text{when} \quad \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta.$$

In addition, for taking $q = d$ in the inequalities above, we can get several other kind of contractions in the context of metric spaces.

If in Theorem 2.1, we change the contraction condition 'generalized (α, ψ, q)-Meir-Keeler contractive mapping' with one of the new contractions defined in Definition 3.1, then we may obtain a similar result as Theorem 2.1. Furthermore, as in Corollary 3.3 and Corollary 3.4, we may get some more results by letting $q = d$. Also, notice that by choosing the auxiliary function α in a proper way in Theorem 2.1, we can deduce more consequences related to cyclic contractions and results in metric spaces endowed with a partially ordered set, see for example [1–8].

4. An Application

In this section, we discuss an application of our obtained fixed point result to a certain kind of non-linear Fredholm integral equations. First of all, we prove a proposition which is going to play a crucial role here.

Proposition 4.1. *Suppose that (M, d) is a metric space with a w-distance q. Also, assume that f is a self-mapping on M satisfying*

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho, \varrho)), \tag{4.1}$$

for all $\rho, \varrho \in M$ and for some $k \in (0, 1)$. Then f is a generalized (α, ψ, q)-Meir-Keeler contractive mapping.

Proof. Consider $\delta = (\frac{1}{k} - 1)\eta$ in Definition 2.1. Accordingly, we derive

$$\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta < \eta + \left(\frac{1}{k} - 1\right)\eta = \frac{\eta}{k},$$

and so, for every $\rho, \varrho \in M$, we obtain $k\eta \leq k\psi(M_q(\rho,\varrho)) < \eta$. Using (4.1), we get

$$\alpha(\rho,\varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho,\varrho)) < \eta.$$

Hence, $\alpha(\rho,\varrho)q(f\rho, f\varrho) < \eta$ and therefore, f is an (α,ψ,q)-Meir-Keeler contractive mapping. □

Now, we try to obtain a criterion to ensure the existence of a solution for a type of non-linear Fredholm integral equation.

Theorem 4.1. *Let us consider the non-linear Fredholm integral equation*

$$(fx)(t) = g(t) + \int_a^b H(t,s,x(s))ds, \tag{4.2}$$

for some $a, b \in \mathbb{R}$, with $a < b$, $g : [a,b] \to \mathbb{R}$ and $H : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ be two continuous maps. Also, assume that the subsequent properties hold:

(i) *$f : C[a,b] \to C[a,b]$ is a continuous mapping;*
(ii) *there exists a weaker Meir-Keeler function ψ and $k \in [0,1)$ satisfying*

$$\begin{aligned}&|H(t,s,x(s))| + |H(t,s,y(s))|\\ \leq&\frac{k\left[\psi\left(\max\left\{\,|x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|,\right.\right.\right.}{b-a}\\ &\frac{\left.\left.\left.\frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2}\right\}\right)\right]-2|g(t)|}{b-a},\end{aligned}$$

for all $t, s \in [a,b]$. Then the non-linear Fredholm integral equation (4.2) *owns a unique solution in $C[a,b]$.*

Proof. Suppose $M = C[a,b]$. Obviously, M is complete with respect to the metric $d : M \times M \to \mathbb{R}^+$ defined as

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

where $x, y \in M$. Now, we consider the map $q : M \times M \to \mathbb{R}^+$ given by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|,$$

where $x, y \in M$. One can easily check that, q is a w-distance on M. Here we have

$$\begin{aligned}&|(fx)(t)| + |(fy)(t)|\\ =&\left|g(t) + \int_a^b H(t,s,x(s))ds\right| + \left|g(t) + \int_a^b H(t,s,y(s))ds\right|\\ \leq&|g(t)| + \left|\int_a^b H(t,s,x(s))ds\right| + |g(t)| + \left|\int_a^b H(t,s,y(s))ds\right|\\ \leq&2\,|g(t)| + \left|\int_a^b H(t,s,x(s))ds\right| + \left|\int_a^b H(t,s,y(s))ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left|g(t)\right| + \int_a^b \left|H(t,s,x(s))\right| ds + \int_a^b \left|H(t,s,y(s))\right| ds \\
&\leq 2\left|g(t)\right| + \int_a^b \left(\left|H(t,s,x(s))\right| + \left|H(t,s,y(s))\right|\right) ds \\
&\leq 2\left|g(t)\right| + \int_a^b \left(\frac{k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|, \right.\right.\right.}{b-a} \right. \\
&\qquad \left. \frac{\left.\left.\left. \frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2} \right\}\right)\right] - 2|g(t)|}{b-a} \right) ds \\
&= 2\left|g(t)\right| + \frac{k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|, \right.\right.\right.}{b-a} \\
&\qquad \frac{\left.\left.\left. \frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2} \right\}\right)\right] - 2|g(t)|}{b-a} \int_a^b ds \\
&= k\left[\psi\left(\max\left\{ |x(t)|+|y(t)|, |x(t)|+|(fx)(t)|, |y(t)|+|(fy)(t)|, \right.\right.\right. \\
&\qquad \left.\left.\left. \frac{(|x(t)|+|(fy)(t)|)+(|(fx)(t)|+|y(t)|)}{2} \right\}\right)\right] \\
&\leq k\left[\psi\left(\max\left\{ q(x,y), q(x,fx), q(y,fy), \frac{q(x,fy)+q(y,fx)}{2} \right\}\right)\right] \\
&= k\left[\psi\left(M_q(x,y)\right)\right],
\end{aligned}
$$

for all $x, y \in M$ and $t \in [0, \infty]$. Thus,

$$
\sup_{t\in[a,b]} |(fx)(t)| + \sup_{t\in[a,b]} |(Ty)(t)| \leq k\left[\psi\left(M_q(x,y)\right)\right],
$$

and therefore for each $x, y \in M$

$$
q(fx, fy) \leq k\left[\psi\left(M_q(x,y)\right)\right].
$$

This implies that f satisfies Proposition 4.1 and hence it is an (α, ψ, q)-Meir-Keeler contractive mapping. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (4.2) has a solution. □

References

[1] Ü. Aksoy, E. Karapınar and İ. M. Erhan, *Fixed points of generalized α-admissible contractions on b-metric spaces with an application to boundary value problems*, J. Nonlinear Convex Anal. **17**(6) (2016), 1095–1108.

[2] A. S. S. Alharbi, H. H. Alsulami and E. Karapınar, *On the power of simulation and admissible functions in metric fixed point theory*, J. Funct. Spaces **2017** (2017), Article ID 2068163.

[3] M. U. Ali, T. Kamran and E. Karapınar, *An approach to existence of fixed points of generalized contractive multivalued mappings of integral type via admissible mapping*, Abstr. Appl. Anal. **2014** (2014), Article ID 141489.

[4] H. Alsulami, S. Gülyaz, E. Karapınar and İ. M. Erhan, *Fixed point theorems for a class of α-admissible contractions and applications to boundary value problem*, Abstr. Appl. Anal. **2014** (2014), Article ID 187031.

[5] S. A. Al-Mezel, C. M. Chen, E. Karapınar and V. Rakočević, *Fixed point results for various α-admissible contractive mappings on metric-like spaces*, Abstr. Appl. Anal. **2014** (2014), Article ID 379358.

[6] H. Aydi, E. Karapınar and H. Yazidi, *Modified F-contractions via α-admissible mappings and application to integral equations*, Filomat **31**(5) (2017), 1141–1148.

[7] H. Aydi, E. Karapınar and D. Zhang, *On common fixed points in the context of Branciari metric spaces*, Results Math. **71**(1) (2017), 73–92.

[8] M. Arshad, E. Ameer and E. Karapınar, *Generalized contractions with triangular α-orbital admissible mapping on Branciari metric spaces*, J. Inequal. Appl. **2016**:63 (2016).

[9] D. Baleanu, S. Rezapour and H. Mohammadi, *Some existence results on nonlinear fractional differential equations*, Philos. Trans. Roy. Soc. A **371**(1990) (2013), Article ID 20120144.

[10] F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72**(3) (1966), 571–575.

[11] C. M. Chen, *Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces*, Fixed Point Theory Appl. **2012**:17 (2012).

[12] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Sci. Math. Jpn. **44**(2) (1996), 381–391.

[13] H. Lakzian, D. Gopal and W. Sintunavarat, *New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations*, J. Fixed Point Theory Appl. **18**(2) (2016), 251–266.

[14] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28**(2) (1969), 326–329.

[15] B. Mohammadi, S. Rezapour and N. Shahzad, *Some results on fixed points of α-ψ-Ćirić generalized multifunctions*, Fixed Point Theory Appl. **2013**:24 (2013).

[16] L. Pasicki, *Some extensions of the Meir-Keeler theorem*, Fixed Point Theory Appl. **2017**:1 (2017).

[17] O. Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2014**:190 (2014).

[18] P. Salimi, A. Latif and N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl. **2013**:151 (2013).

[19] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75**(4) (2012), 2154–2165.

[20] T. Senapati, L. K. Dey, A. Chanda and H. Huang, *Some non-unique fixed point or periodic point results in JS-metric spaces*, J. Fixed Point Theory Appl. **21**:51 (2019).

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INVESTIGATION THE EXISTENCE OF A SOLUTION FOR A MULTI-SINGULAR FRACTIONAL DIFFERENTIAL EQUATION WITH MULTI-POINTS BOUNDARY CONDITIONS

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Abstract. We should try to increase our abilities in solving of complicate differential equations. One type of complicate equations are multi-singular pointwise defined fractional differential equations. We investigate the existence of solutions for a multi-singular pointwise defined fractional differential equation with multi-points boundary conditions. We provide an example to illustrate our main result.

1. Introduction

One possible way that the mathematics has effective role in the various fields the various fields of sciences is to become more powerful and flexible in modeling theory so that different types of phenomena with distinct parameters can be written in mathematical formulas. In this case, different softwares can be developed to allow for more cost-free testing and less material consumption. In this way, a method is working with complicate differential equations. Nowadays, many researchers are studying advanced fractional modelings and its related existence results and qualitative behaviors of solutions for distinct fractional differential equations and inclusions (see for example [1–24, 26–29, 31–34, 36–38]).

In 2013, the existence of solutions for the singular differential equation

$$D^{\alpha}u(t) + f(t, u(t)) = 0,$$

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with boundary conditions $u'(0) = u''(0) = \cdots = u^{n-1}(0) = 0$, $u(1) = \int_0^1 u(s)d\mu(s)$ studied by Vong, where $0 < t < 1$, $n \geq 2$, $\alpha \in (n-1, n)$, μ is a function of bounded variation with $\int_0^1 d\mu(s) < 1$, f may have singularity at $t = 1$ and D^α is the Caputo derivative [39]. In 2014, Jleli et al. proved the existence of positive solutions for the singular fractional problem $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary value conditions $u(0) = u'(0) = 0$ and $u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)$, where $0 < t < 1$, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x)$ is singular at $t = 0$ and D^α is the Caputo derivative [25].

In 2016, Shabibi et al. reviewed the multi-singular pointwise defined fractional integro-differential equation

$$D^\mu x(t) + f(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = 0,$$

with boundary conditions $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$, where $\mu \in [2,3)$, $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$ and $x^{(j)}(0) = 0$ for $j = 2, \ldots, [\mu] - 1$, $0 \leq t \leq 1$, $x \in C^1[0,1]$, $\beta, \xi, \eta \in (0,1)$, $p > 1$, D^μ is the Caputo fractional derivative of order μ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ is a function such that $f(t, \cdot, \cdot, \cdot, \cdot)$ is singular at some points $t \in [0,1]$ [36]. In 2018, Baleanu et al. investigated the pointwise defined problem

$$D^\alpha x(t) + f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)d\xi, \phi(x(t))\right) = 0,$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that

$$\|\phi(x) - \phi(y)\| \leq \theta_0 \|x - y\| + \theta_1 \|x' - y'\|,$$

for some non-negative real numbers θ_0 and $\theta_1 \in [0, \infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α

$$f(t, x_1(t), \ldots, x_5(t)) = f_1(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in [0, \lambda)$,

$$f(t, x_1(t), , x_5(t)) = f_2(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in [\lambda, \mu]$ and

$$f(t, x_1(t), \ldots, x_5(t)) = f(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in (\mu, 1]$, $f_1(t, \cdot, \cdot, \cdot, \cdot)$ and $f_3(t, \cdot, \cdot, \cdot, \cdot)$ are continuous on $[0, \lambda)$ and $(\mu, 1]$ and $f_2(t, \cdot, \cdot, \cdot, \cdot)$ is multi-singular [9].

By using idea of the works, we investigate the existence of solutions for the nonlinear fractional differential pointwise defined equation

$$D^\alpha x(t) = f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)d\xi\right), \tag{1.1}$$

with boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$ while $j \neq k$ for one's $2 \leq k \leq n-1$ and $x(1) = \sum_{i=1}^m \lambda_i D^{\beta_i} x(\gamma_i)$, where $\alpha \geq 2$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \ldots, \beta_m \in (0,1)$, $\lambda_1, \ldots, \lambda_m \in [0, \infty)$, $m \in \mathbb{N}$, D^α is the Caputo fractional derivative of order α, $n = [\alpha] + 1$, $h \in L^1$ and $f \in L^1$ is singular at some points $[0,1]$.

Recall that $D^\alpha x(t) + f(t) = 0$ is a pointwise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E [36]. In this paper, we use $\|\cdot\|_1$ for the norm of $L^1[0,1]$, $\|\cdot\|$ for the sup norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$.

The Riemann-Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by

$$I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s) ds,$$

provided that the right-hand side is pointwise define on (a,∞). We denote $I^p_{0^+} f(t)$ by $I^p f(t)$ [30]. The Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$ and $f : (a,\infty) \to \mathbb{R}$ is a function [30]. Let Ψ be the family of non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$. One can check that $\psi(t) < t$ for all $t > 0$ [35]. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [35]. Let (X,d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever

$$\alpha(x,y) d(Tx, Ty) \leq \psi(d(x,y)),$$

for all $x, y \in X$ [35]. We need next results.

Lemma 1.1 ([35])**.** *Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Lemma 1.2 ([30])**.** *Let $n-1 \leq \alpha < n$ and $x \in C(0,1)$. Then, we have*

$$I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some real constants $c_0, \dots, c_{n-1}$.

2. Main Results

Now, we are ready for preparing our main results.

Lemma 2.1. *Let $\alpha \geq 2$, $[\alpha] = n-1$, $m \in \mathbb{N}$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \dots, \beta_m \in (0,1)$, $\lambda_1, \dots, \lambda_m \in [0,\infty)$ and $f \in L^1[0,1]$, then the solution of the problem $D^\alpha x(t) = f(t)$ with the boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$ while $j \neq k$ for one's $2 \leq k \leq n-1$ such that*

$$\sum_{i=1}^m \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} \neq \frac{1}{k!},$$

and $x(1) = \sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i)$ *is* $x(t) = \int_0^1 G(t,s) f(s) ds$, *where* $G(t,s)$ *is defined by*

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k (1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le s \le t \le 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ *for* $j = 1, 2, \ldots, m$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k (1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le s \le t \le 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$,

$$G(t,s) = \frac{t^k (1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le t \le s \le 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ *for* $j = 1, 2, \ldots, m$, *and*

$$G(t,s) = \frac{t^k (1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le t \le s \le 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$ *and*

$$\Delta := k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1.$$

Proof. By using a similar method in [9], we can show that Lemma 1.1 holds on $L^1[0,1]$. Let $x(t)$ be a solution for the problem. Since $x^{(j)}(0) = 0$ for $j \ge 2$, by using Lemma 1.1, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + c_0 + c_1 t + \cdots + c_n t^n.$$

Since $x(0) = 0$, so $c_0 = 0$. Also since $x^{(j)}(0) = 0$ for $j \ge 2$ and $j \ne k$ so $c_2 = \cdots = c_j = \cdots = c_n = 0$ for $j \ne k$. Thus,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + c_k t^k. \tag{2.1}$$

Hence, we get

$$\begin{aligned} D^{\beta_i} x(t) &= \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} f(s) ds + c_k \frac{\Gamma(k+1)}{\Gamma(k+1-\beta_i)} t^{k-\beta_i} \\ &= \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} f(s) ds + c_k \frac{k!}{\Gamma(k+1-\beta_i)} t^{k-\beta_i}, \end{aligned}$$

and so

$$\lambda_i D^{\beta_i} x(\gamma_i) = \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s) ds + c_k \lambda_i \frac{k!}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i},$$

for all $1 \leq i \leq m$. Therefore, we obtain

$$\sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i) = \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha - \beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds + c_k k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i}.$$

On the other hand, by using (2.1) we have

$$x(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds + c_k.$$

Since $x(1) = \sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i)$, we get

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds + c_k = & \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha - \beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds \\ & + c_k k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i}. \end{aligned}$$

Hence,

$$\begin{aligned} c_k \left[k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1 \right] = & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha - \beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds. \end{aligned}$$

Put $\Delta := k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1$. Then, by using the assumption $\Delta \neq 0$, we have

$$c_k = \frac{1}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds - \frac{1}{\Delta} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha - \beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds$$

and so

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \frac{t^k}{\Delta} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha - \beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \frac{t^k}{\Delta} \frac{\lambda_1}{\Gamma(\alpha - \beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\ & - \cdots - \frac{t^k}{\Delta} \frac{\lambda_m}{\Gamma(\alpha - \beta_m)} \int_0^{\gamma_m} (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds. \end{aligned}$$

If $0 \le t \le \gamma_1 < \cdots < \gamma_m < 1$, then

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^t + \int_t^{\gamma_1} + \cdots + \int_{\gamma_m}^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \left(\int_0^t + \int_t^{\gamma_1} \right) (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k \lambda_m}{\Delta\Gamma(\alpha-\beta_m)} \\
& \times \left(\int_0^t + \int_t^{\gamma_1} + \cdots + \int_{\gamma_{m-1}}^{\gamma_m} \right) (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds.
\end{aligned}$$

If $0 < \gamma_1 \le t \le \gamma_2 < \cdots < \gamma_m < 1$, then

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t \right) (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t + \int_t^{\gamma_2} + \cdots + \int_{\gamma_m}^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k \lambda_m}{\Delta\Gamma(\alpha-\beta_m)} \\
& \times \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t + \int_t^{\gamma_2} + \cdots + \int_{\gamma_{m-1}}^{\gamma_m} \right) (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds.
\end{aligned}$$

By continuing this, finally we get

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^{\gamma_2} + \cdots + \int_{\gamma_m}^t \right) (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^{\gamma_2} + \cdots + \int_{\gamma_m}^t + \int_t^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k}{\Delta} \frac{\lambda_m}{\Gamma(\alpha-\beta_m)} \int_0^{\gamma_m} (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds,
\end{aligned}$$

whenever $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m \le t \le 1$. Hence, $x(t) = \int_0^1 G(t,s) f(s)ds$, where

$$\begin{aligned}
G(t,s) = {} & \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k \lambda_1 (\gamma_1 - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_1)} \\
& - \frac{t^k \lambda_2 (\gamma_2 - s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k \lambda_m (\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},
\end{aligned}$$

when $0 \le s \le t \le 1$ and $s \le \gamma_1 < \gamma_2 < \cdots < \gamma_m$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_2(\gamma_2 - s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 \le s \le \gamma_2 < \cdots < \gamma_m$, in the general case

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_j(\gamma_j - s)^{\alpha-\beta_j-1}}{\Delta\Gamma(\alpha-\beta_j)} - \cdots - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$, for $1 \le j \le m$, thus

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} \le s \le \gamma_m$, and

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \frac{t^k\lambda_1(\gamma_1 - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_1)} - \frac{t^k\lambda_2(\gamma_2 - s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $s \le \gamma_1 < \gamma_2 < \cdots < \gamma_m$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_2(\gamma_2 - s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 \le s \le \gamma_2 < \cdots < \gamma_m$ and in the general case

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_j(\gamma_j - s)^{\alpha-\beta_j-1}}{\Delta\Gamma(\alpha-\beta_j)} - \cdots - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $1 \le j \le m$, thus

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_m(\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} \le s \le \gamma_m$, and finally

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$. Thus,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$, and

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$. $\square$

One can check that

$$\frac{\partial G}{\partial t} = \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{kt^{k-1} \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$,

$$\frac{\partial G}{\partial t} = \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$,

$$\frac{\partial G}{\partial t} = \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{kt^{k-1} \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$, and $\frac{\partial G}{\partial t} = \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}$, when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$.

It is easy to see that G and $\frac{\partial}{\partial t}G$ are continuous with respect to t. Consider the space $X = C^1[0,1]$ with the norm $\|\cdot\|_*$, where $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and $\|\cdot\|$ is the supremum norm on $C[0,1]$. Let f be a map on $[0,1] \times X^4$ such that is singular at some points of $[0,1]$. Define $F : X \to X$ as

$$\begin{aligned} Fx(t) &= \int_0^1 \frac{\partial}{\partial t} G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &\quad + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &\quad - \frac{t^k}{\Delta} \sum_{i=1}^m \frac{\lambda_i(\gamma_i - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\ &\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds, \end{aligned}$$

so

$$\begin{aligned} F'x(t) &= \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &\quad + \frac{kt^{k-1}}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\ &\quad - \frac{kt^{k-1}}{\Delta} \sum_{i=1}^m \frac{\lambda_i(\gamma_i - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\ &\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds. \end{aligned}$$

It is notable that the singular pointwise defined (1.1) has a solution if and only if the map F has a fixed point.

Theorem 2.1. *Let $\alpha \geq 2$, $[\alpha] = n-1$, $m \in \mathbb{N}$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \ldots, \beta_m \in (0,1)$, $\lambda_1, \ldots, \lambda_m \in [0,\infty)$, $h \in L^1[0,1]$ and $m_0 = \int_0^1 |h(s)|ds$. Assume that $f : [0,1] \times X^4 \to \mathbb{R}$ is a singular map on some points $[0,1]$ such that*

$$|f(t, x_1, \ldots, x_4) - f(t, y_1, \ldots, y_4)| \leq \Lambda(t, |x_1 - y_1|, \ldots, |x_4 - y_4|),$$

for all $x_1, \ldots, x_4, y_1, \ldots, y_1 \in X$ and almost all $t \in [0,1]$, where $\Lambda(t, x_1, \ldots, x_4)$ be a real mapping on $[0,1] \times X^4$ such that is non-decreasing with respect to $x_1, \ldots, x_4$,

$$\lim_{z \to 0^+} \frac{\Lambda(t, z, \ldots, z)}{H(z)} = \theta(t),$$

for almost all $t \in [0,1]$ in which $\theta : [0,1] \to \mathbb{R}^+$ is a mapping so that $\hat{\theta} \in L^1[0,1]$, with $\hat{\theta}(s) = (1-s)^{\alpha_i - 2}\theta(s)$, $H : [0,\infty) \to [0,\infty)$ is a linear mapping such that $\lim_{z \to 0^+} H(z) = 0$ and $\lim_{i \to \infty} H^i(t) < \infty$ for all $t \in [0,\infty)$. Here, H^i is the i-th *composition of H with itself. Let*

$$|f(t, x_1, \dots, x_4)| \leq \sum_{k=1}^{n_0} b_j(t) K_j(|x_1|, \dots, |x_4|),$$

almost everywhere on $[0,1]$ and all $x_1, \dots, x_4$, where $n_0 \in \mathbb{N}$, $b_j : [0,1] \to \mathbb{R}^+$, $\hat{b_j} \in L^1[0,1]$, $K_J : X^4 \to \mathbb{R}^+$ is a non-decreasing mapping with respect to all their components with

$$\lim_{z \to 0^+} \frac{K_j(z, \dots, z)}{z} = q_j,$$

for some $q_j \in \mathbb{R}^+$ and $1 \leq j \leq n_0$. If

$$\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \max\left\{\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} q_j, \|\hat{\theta}\|_{[0,1]}\right\} \in \left[0, \frac{1}{M}\right),$$

where $M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\}$, then the pointwise defined equation

$$D^\alpha x(t) = f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi) x(\xi) d\xi\right),$$

with boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$, while $j \neq k$, $2 \leq k \leq n-1$ and $x(1) = \sum_{i=1}^m \lambda_i D^{\beta_i} x(\gamma_i)$, has a solution.

Proof. First we show that F is continuous on X. Let $\epsilon > 0$ be given. Since $H(Mz) \to 0$ as $z \to 0^+$, there exists $\delta_1 > 0$ such that $z \in (0, \delta_1]$ implies that $H(Mz) < \epsilon$. Since

$$\lim_{z \to 0^+} \frac{\Lambda(t, Mz, \dots, Mz)}{H(Mz)} = \theta(t),$$

for almost all $t \in [0,1]$, there exists $\delta_2 > 0$ such that $z \in (0, \delta_2]$ implies that

$$\frac{\Lambda(t, Mz, \dots, Mz)}{H(z)} \leq \theta(t) + \epsilon.$$

Hence, $\Lambda(t, Mz, \dots, Mz) \leq (\theta(t) + \epsilon) H(Mz)$ almost everywhere on $[0,1]$. Let $\delta = \min\{\delta_1, \delta_2, \epsilon\}$ and $z := \|x - y\|_* < \delta$ for $x, y \in X$. Then, we have

$$\Lambda(t, M\|x-y\|_*, \dots, M\|x-y\|_*) \leq (\theta(t) + \epsilon) H(M\|x-y\|_*) < (\theta(t) + \epsilon)\epsilon.$$

So, for all $t \in [0,1]$ and $x, y \in X$ such that $\|x-y\|_* < \delta$ we have

$$
\begin{aligned}
|Fx(t)-Fy(t)| &= \Bigg| \int_0^1 G(t,s)\Bigg[f\Bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\Bigg) \\
&\quad - f\Bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\Bigg)\Bigg] ds \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg| f\Bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\Bigg) \\
&\quad - f\Bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\Bigg)\Bigg| ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \Bigg| f\Bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\Bigg) \\
&\quad - f\Bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\Bigg)\Bigg| ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \Bigg| f\Bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\Bigg) \\
&\quad - f\Bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\Bigg)\Bigg| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Lambda\Bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\Bigg) ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Lambda\Bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\Bigg) ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \Lambda\Bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\Bigg) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Lambda\Bigg(s, \|x-y\|, \|x'-y'\|, \\
&\frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg) ds \\
&+ \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Lambda\Bigg(s, \|x-y\|, \|x'-y'\|, \\
&\frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg) ds \\
&+ \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \Lambda(s, \|x-y\|, \\
&\|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|) ds.
\end{aligned}
$$

Note that $|D^\beta(x-y)(s)| \leq \frac{\|x'-y'\|}{\Gamma(2-\beta)}$ and

$$
\int_0^s h(\xi)|x(\xi)| d\xi \leq \|x\| \int_0^1 h(\xi) d\xi = m_0\|x\|.
$$

Put $M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\}$. Now for each $t \in [0,1]$ and $x, y \in X$, with $\|x-y\|_* < \delta$, we obtain

$$
\begin{aligned}
|Fx(t) - Fy(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\Bigg) ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg) ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\Bigg) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\\
&\times\Lambda(s,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*)ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(\theta(s)+\epsilon)\epsilon ds\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(\theta(s)+\epsilon)\epsilon ds\\
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}(\theta(s)+\epsilon)\epsilon ds\\
\le{}&\frac{1}{\Gamma(\alpha)}[\int_0^1(1-s)^{\alpha_i-2}\theta(s)ds+\epsilon\int_0^t(t-s)^{\alpha_i-1}\theta(s)ds]\epsilon\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}[\int_0^1(1-s)^{\alpha-2}\theta(s)ds+\epsilon\int_0^1(1-s)^{\alpha-1}\theta(s)ds]\epsilon\\
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\int_0^1(1-s)^{\alpha_i-2}\theta(s)ds\right.\\
&\left.+\epsilon\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\theta(s)ds\right]\epsilon\\
={}&\frac{1}{\Gamma(\alpha)}\left[\|\hat\theta\|_{[0,1]}+\frac{\epsilon}{\alpha}t^\alpha\right]\epsilon+\frac{t^k}{|\Delta|\Gamma(\alpha)}\left[\|\hat\theta\|_{[0,1]}+\frac{\epsilon}{\alpha}\right]\epsilon\\
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\|\hat\theta\|_{[0,1]}+\frac{\epsilon}{\alpha-\beta_i}\gamma_i^{\alpha-\beta_i}\right]\epsilon. \qquad (2.2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|Fx-Fy\|\le{}&\left[\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|\Delta|\Gamma(\alpha)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat\theta\|_{[0,1]}\right.\\
&\left.+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
|F'x(t)-F'y(t)|={}&\left|\int_0^1\frac{\partial G}{\partial t}(t,s)\left[f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi\right)\right.\right.\\
&\left.\left.-f\left(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi\right)\right]ds\right|\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\\
&\times\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds\\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\\
&\times \bigg|f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg)\\
&- f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds\\
&+ \frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i} (\gamma_i-s)^{\alpha-\beta_i-1}\\
&\times \bigg|f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg)\\
&- f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds\\
\leq{}& \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\Lambda\bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|,\\
&|D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds\\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Lambda\bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|,\\
&|D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds\\
&+ \frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i} (\gamma_i-s)^{\alpha-\beta_i-1}\Lambda\bigg(s, |x(s)-y(s)|,\\
&|x'(s)-y'(s)|, |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds\\
\leq{}& \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\\
&\times \Lambda\bigg(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\bigg)ds\\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\\
&\times \Lambda\bigg(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\bigg)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\times \Lambda\left(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\right) ds \\
\le\ & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \\
&\times \Lambda\left(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\right) ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\times \Lambda\left(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|\right) ds \\
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\times \Lambda\left(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\right) ds \\
\le\ & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \\
&\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds \\
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds \\
\le\ & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} (\theta(s)+\epsilon)\epsilon ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (\theta(s)+\epsilon)\epsilon ds \\
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} (\theta(s)+\epsilon)\epsilon ds \\
\le\ & \frac{1}{\Gamma(\alpha-1)} \left[\int_0^1 (1-s)^{\alpha_i-2} \theta(s) ds + \epsilon \int_0^t (t-s)^{\alpha_i-1} \theta(s) ds \right] \epsilon \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha_i-2} \theta(s) ds + \epsilon \int_0^1 (1-s)^{\alpha-1} \theta(s) ds \right] \epsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\Bigg[\int_0^1(1-s)^{\alpha_i-2}\theta(s)ds\\
&+\epsilon\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\theta(s)ds\Bigg]\epsilon\\
=&\ \frac{1}{\Gamma(\alpha-1)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha}t^{\alpha}\right]\epsilon+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha}\right]\epsilon\\
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha-\beta_i}\gamma_i^{\alpha-\beta_i}\right]\epsilon.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|F'x-F'y\|\le&\Bigg[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\
&+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\Bigg]\epsilon,
\end{aligned}
$$

and so

$$
\begin{aligned}
\|Fx-Fy\|_*=&\max\left\{\|Fx-Fy\|,\|F'x-F'y\|\right\}\\
\le&\max\Bigg\{\Bigg[\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|\Delta|\Gamma(\alpha)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\
&+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\Bigg]\epsilon,\\
&\Bigg[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\
&+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\Bigg]\epsilon\Bigg\}\\
=&\Bigg[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\
&+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\Bigg]\epsilon.
\end{aligned}
$$

This concludes that $\|Fx-Fy\|_*$ tends to zero as $\|x-y\|_*$ tends to zero and so F is continuous in X. Since for all $1\le j\le n_0$,

$$\lim_{z\to 0^+}\frac{K_j(Mz,\dots,Mz)}{Mz}=q_j,$$

for each $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that

$$K_j(Mz,\dots,Mz)\le(q_j+\epsilon)Mz,$$

for all $0 < z \le \delta$ and $1 \le j \le n_0$. Since

$$M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\left[\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}q_j\right] < 1,$$

there exists $\epsilon_0 > 0$ such that

$$M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\left(\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}(q_j+\epsilon_0)\right) < 1.$$

Let $\delta_0 = \delta(\epsilon_0)$. On the other hand, for almost all $s \in [0,1]$ we have

$$\lim_{z\to 0^+}\frac{\Lambda(s, Mz, \ldots, Mz)}{H(Mz)} = \theta(s).$$

For the given $\epsilon > 0$, there exists $\delta' = \delta'(\epsilon)$ such that for almost everywhere on $[0,1]$, $\Lambda(s, Mz, \ldots, Mz) \le (\theta(s)+\epsilon)H(Mz)$ for $0 < z \le \delta'$. Since

$$M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\|\hat{\theta}\|_{[0,1]} < 1,$$

there exists $\epsilon_1 > 0$ such that

$$\begin{aligned} &M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\|\hat{\theta}\|_{[0,1]} \\ &+ \frac{\epsilon_1 M}{\alpha-1}\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] < 1. \end{aligned}$$

Let $\delta_1 = \delta'(\epsilon_1)$ and $\delta_2 = \min\{\delta_0, \frac{\delta_1}{2}\}$. For each $z \in (0, \delta_2]$ and $1 \le j \le n_0$, we have $K_j(Mz, \ldots, Mz) \le (q_j+\epsilon_0)Mz$ and for each $z \in (0, \delta_1]$ we have

$$\Lambda(s, Mz, \ldots, Mz) \le (\theta(s)+\epsilon_1)H(Mz), \tag{2.3}$$

almost everywhere on $[0,1]$. Let $C = \{x \in X : \|x\|_* \le \delta_2\}$. Define $\alpha : X^2 \to [0,\infty)$ by $\alpha(x,y) = 1$ whenever $x, y \in C$ and $\alpha(x,y) = 0$ otherwise. If $\alpha(x,y) \ge 1$, then

$x, y \in X$ and so $\|x\|_* \leq \delta_2$ and $\|y\|_* \leq \delta_2$. Thus, for each $t \in [0,1]$ we have

$$\begin{aligned}
|Fx(t)| &= \left| \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left| \int_0^s h(\xi)x(\xi)d\xi \right| \right) ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left| \int_0^s h(\xi)x(\xi)d\xi \right| \right) ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left| \int_0^s h(\xi)x(\xi)d\xi \right| \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n_0} \int_0^t (t-s)^{\alpha-1} b_j(s) \\
&\quad \times K_j\left(|x(s)|, |x'(s)|, \frac{\|x'\|}{\Gamma(2-\beta)}, \|x\| \int_0^s |h(\xi)x(\xi)| d\xi\right) ds \\
&\quad + \sum_{j=1}^{n_0} b_j(s) \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times K_j\left(|x(s)|, |x'(s)|, \frac{\|x'\|}{\Gamma(2-\beta)}, \|x\| \int_0^s |h(\xi)x(\xi)| d\xi\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\sum_{j=1}^{n_0}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}b_j(s)\\
&\times K_j\left(|x(s)|,|x'(s)|,\frac{\|x'\|}{\Gamma(2-\beta)},\|x\|\int_0^s|h(\xi)x(\xi)|d\xi\right)ds\\
\le&\ \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}\int_0^1(1-s)^{\alpha-1}b_j(s)\\
&\times K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\\
&+\sum_{j=1}^{n_0}\frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b_j(s)\\
&\times K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\\
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\sum_{j=1}^{n_0}\int_0^{\gamma_i}(1-s)^{\alpha-2}\right.\\
&\left.\times b_j(s)K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\right]\\
\le&\ \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}K_j(M\|x\|_*,\dots,M\|x\|_*)\int_0^1(1-s)^{\alpha-2}b_j(s)ds\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}\sum_{j=1}^{n_0}K_j(M\|x\|_*,\dots,M\|x\|_*)\int_0^1(1-s)^{\alpha-2}b_j(s)ds\\
&+\frac{t^k}{|\Delta|}\sum_{j=1}^{n_0}[K_j(M\|x\|_*,\dots,M\|x\|_*)\\
&\times\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\alpha-2}b_j(s)ds]\\
\le&\ \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\dots,M\delta_2)\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\dots,M\delta_2)\\
&+\frac{t^k}{|\Delta|}\left(\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\dots,M\delta_2)\\
=&\left[\frac{1}{\Gamma(\alpha)}+\frac{t^k}{|\Delta|\Gamma(\alpha)}+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]
\end{aligned}
$$

$$\times \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right].$$

Hence,

$$\begin{aligned}
\|Fx\| &\leq \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \cdot \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right] \\
&\leq \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \cdot \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j + \epsilon_0)\right] M\delta_2 \\
&\leq \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \cdot \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j + \epsilon_0)\right] M\delta_2 \leq \delta_2.
\end{aligned}$$

Similarly, one can concluded that

$$\begin{aligned}
\|F'x\| \leq & \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \\
& \times \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right] \\
\leq & \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \\
& \times \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j + \epsilon_0)\right] M\delta_2 \leq \delta_2,
\end{aligned}$$

and so $\|Fx\|_* = \max\{\|Fx\|, \|F'x\|\} \leq \delta_2$. Thus, $Fx \in C$. Similarly, we can show that $Fy \in C$. Hence, $\alpha(Fx, Fy) \geq 1$. It is obvious that $C \neq \phi$. For $x_0 \in C$, we have $Fx_0 \in C$ and so $\alpha(x_0, Fx_0) \geq 1$. Put

$$\lambda := M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \|\hat{\theta}\|_{[0,1]}.$$

Let $x, y \in C$. Then, $\alpha(x, y) = 1$. On the other hand by using (2.2), for each $x, y \in X$ and $t \in [0, 1]$ we have

$$\begin{aligned}
|Fx(t) - Fy(t)| \leq & \int_0^1 |G(t,s)| \Bigg| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \\
& - f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\right)\Bigg| ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
& \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*) ds \\
& + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}
\end{aligned}$$

$$\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds$$
$$+ \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i - s)^{\alpha-\beta_i-1}$$
$$\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds.$$

If $x, y \in C$, then $\|x\|_* < \delta_1$ and $\|y\|_* < \delta_1$ and so

$$\|x-y\|_* < \|x\|_* + \|y\|_* < 2\delta_* \leq \delta_1.$$

Hence, by using (2.3) we have

$$\Lambda(s, M\|x-y\|_*, \ldots, M\|x-y\|_*) \leq (\theta(s)+\epsilon_1)H(M\|x-y\|_*).$$

Thus, for each $t \in [0,1]$ and $x, y \in C$ we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\quad + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i}(\gamma_i - s)^{\alpha-\beta_i-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\leq \frac{1}{\Gamma(\alpha)}H(M\|x-y\|_*) \\
&\quad \times \left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}H(M\|x-y\|_*) \\
&\quad \times \left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&\quad + \frac{t^k}{|\Delta|}H(M\|x-y\|_*) \\
&\quad \times \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&= H(M\|x-y\|_*)\Bigg[\left(\frac{\|\hat{\theta}\|_{[0,1]}}{\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right) \\
&\quad + \frac{\epsilon_1}{\alpha-1}\left(\frac{\|\hat{\theta}\|_{[0,1]}}{\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\Bigg].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Fx - Fy\| &\leq \Bigg[\bigg(\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\bigg)\|\hat{\theta}\|_{[0,1]} \\
&\quad + \frac{\epsilon_1}{\alpha-1}\bigg(\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\bigg)\Bigg]H(M\|x-y\|_*) \\
&\leq M\Bigg[\bigg(\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\bigg)\|\hat{\theta}\|_{[0,1]} \\
&\quad + \frac{\epsilon_1 M}{\alpha-1}\bigg(\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\bigg)\Bigg]H(\|x-y\|_*) \\
&= \lambda H(\|x-y\|_*).
\end{aligned}$$

Similarly, we conclude that $\|F'x - F'y\| \leq \lambda H(\|x-y\|_*)$. Hence,

$$\begin{aligned}
\|Fx - Fy\|_* &= \max\{\|Fx - Fy\|, \|F'x - F'y\|\} \\
&\leq \lambda H(\|x-y\|_*) = \psi(\|x-y\|_*),
\end{aligned}$$

where $\psi : [0,\infty) \to [0,\infty)$ is defined as $\psi(t) = \lambda H(t)$. Since H is non-decreasing and λ is positive, ψ is non-decreasing. Also,

$$\sum_{i=1}^{\infty}\psi^i(t) = H^{\infty}(t)\frac{\lambda}{1-\lambda},$$

where $H^{\infty}(t) = \lim_{i\to\infty} H^i(t)$. If $x \neq C$ or $y \neq C$, then $\alpha(x,y) = 0$ and so $\alpha(x,y)d(Fx,Fy) \leq \psi(d(x,y))$. Thus, $\alpha(x,y)d(Fx,Fy) \leq \psi(d(x,y))$ for all $x, y \in C$. Now by using Lemma 1.1, F has a fixed point which is the solution of the problem. $\square$

Now, we provide an example to illustrate our main result.

Example 2.1. Consider the pointwise defined problem

$$D^{\frac{7}{2}}x(t) = f\bigg(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t \xi x(\xi)d\xi\bigg), \tag{2.4}$$

with boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$ and $j \neq 3$ and

$$x(1) = \frac{1}{4}D^{\frac{1}{3}}x\left(\frac{1}{10}\right) + \frac{1}{3}D^{\frac{1}{2}}x\left(\frac{1}{5}\right),$$

where

$$f(t, x_1, \dots, x_4) = \frac{t}{4p(t)}(|x_1| + \cdots + |x_4|),$$

$p(t) = 0$ whenever $t \in [0,1] \cap \mathbb{Q}$ and $p(t) = 1$ whenever $t \in [0,1] \cap \mathbb{Q}^c$. Put $h(t) = t$, $\Lambda(t, x_1, \dots, x_4) = f(t, x_1, \dots, x_4)$, $H(z) = z$, $\theta(t) = \frac{t}{p(t)}$, $n_0 = 1$, $b_1(t) = \frac{t}{4p(t)}$, $K_1(x_1, \dots, x_4) = |x_1| + \cdots + |x_4|$ and $q_1 = 4$. Then

$$m_0 = \int_0^1 h(\xi)d(\xi) = \int_0^1 \xi d(\xi) = \frac{1}{2},$$

$\Lambda(t, x_1, \ldots, x_4)$ is a positive and non-decreasing mapping with respect to $x_1, \ldots, x_4$ and

$$\lim_{z\to 0^+} \frac{\Lambda(t, z, \ldots, z)}{H(z)} = \theta(t),$$

for almost all $t \in [0,1]$, $H : [0,\infty) \to [0,\infty)$ is a linear mapping, $\lim_{z\to 0^+} H(z) = 0$ and $\lim_{i\to\infty} H^i(t) = t < \infty$ for all $t \in [0,\infty)$, $\|\hat{\theta}\|_{[0,1]} \leq \frac{2}{5}$,

$$|f(t, x_1, \ldots, x_4)| \leq \sum_{k=1}^{n_0} b_j(t) K_j(|x_1|, \ldots, |x_4|) = b_1(t) K_1(|x_1|, \ldots, |x_4|),$$

almost everywhere on $[0,1]$, $K_1(|x_1|, \ldots, |x_4|)$ is a positive and non-decreasing mapping with respect to $x_1, \ldots, x_4$, $\lim_{z\to 0^+} \frac{K_1(z,\ldots,z)}{z} = 4 = q_1$ and $\|\hat{b_1}\|_{[0,1]} \leq \frac{2}{20}$. Also we have

$$M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\} = \max\left\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{2}\right\} = \frac{2}{\sqrt{\pi}}$$

and

$$\begin{aligned}|\Delta| &:= \left| k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1 \right| \\ &= \left| 3! \left[\frac{\frac{1}{4}}{\Gamma(4-\frac{1}{3})} \left(\frac{1}{10}\right)^{4-\frac{1}{3}} + \frac{\frac{1}{3}}{\Gamma\left(4-\frac{1}{2}\right)} \left(\frac{1}{5}\right)^{4-\frac{1}{2}} \right] - 1 \right| \geq 0.997.\end{aligned}$$

Since

$$\begin{aligned}&\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \max\left\{\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} q_j, \|\hat{\theta}\|_{[0,1]}\right\} \\ \leq &\left[\frac{1}{\Gamma(\frac{7}{2})} + \frac{3}{0.997\Gamma(\frac{7}{2})} + \frac{3}{0.997}\left(\frac{\frac{1}{4}}{\Gamma(\frac{7}{2}-\frac{1}{3})} + \frac{\frac{1}{3}}{\Gamma\left(\frac{7}{2}-\frac{1}{2}\right)}\right)\right] \max\left\{\frac{2}{20} \times 4, \frac{2}{5}\right\} \\ < &\left[\frac{8}{15\sqrt{\pi}} + \frac{8}{0.997 \times 5\sqrt{\pi}} + \frac{3}{0.997}\left(\frac{\frac{1}{4} + \frac{1}{3}}{6}\right)\right] \times \frac{2}{5} \\ < &0.604 < \frac{1}{M}.\end{aligned}$$

By using Theorem 2.1, we conclude that the problem (2.4) has a solution.

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References

[1] E. Akbari Kojabad and S. Rezapour, *Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials*, Adv. Difference Equ. **2017** (2017), Article ID 351, 18 pages.

[2] S. Alizadeh, D. Baleanu and S. Rezapour, *Analyzing transient response of the parallel RCL circuit by using the Caputo-Fabrizio fractional derivative*, Adv. Difference Equ. **2020** (2020), Paper ID 55, 19 pages.

[3] A. Alsaedi, D. Baleanu, S. Etemad and S. Rezapour, *On coupled systems of time-fractional differential problems by using a new fractional derivative*, J. Funct. Spaces **2016** (2015), Article ID 4626940, 8 pages.

[4] M. S. Aydogan, D. Baleanu, A. Mousalou and S. Rezapour, *On high order fractional integro-differential equations including the Caputo-Fabrizio derivative*, Bound. Value Probl. **2018** (2018), Article ID 90, 15 pages.

[5] S. M. Aydogan, D. Baleanu, A. Mousalou and S. Rezapour, *On approximate solutions for two higher-order Caputo-Fabrizio fractional integro-differential equations*, Adv. Difference Equ. **2017** (2017), Paper ID 221, 11 pages.

[6] D. Baleanu, R. Agarwal, H. Mohammadi and S. Rezapour, *Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces*, Bound. Value Probl. **2013** (2013), Paper ID 112, 8 pages.

[7] D. Baleanu, S. Etemad, S. Pourrazi and S. Rezapour, *On the new fractional hybrid boundary value problems with three-point integral hybrid conditions*, Adv. Difference Equ. **2019** (2019), Paper ID 473, 21 pages.

[8] D. Baleanu, K. Ghafarnezhad and S. Rezapour, *On a three steps crisis integro-differential equation*, Adv. Difference Equ. **2018** (2018), Paper ID 153, 19 pages.

[9] D. Baleanu, K. Ghafarnezhad, S. Rezapour and M. Shabibi, *On the existence of solutions of a three steps crisis integro-differential equation*, Adv. Difference Equ. **2018** (2018), Paper ID 135, 20 pages.

[10] D. Baleanu, V. Hedayati, S. Rezapour and M. M. Al-Qurashi, *On two fractional differential inclusions*, Springer Plus **5** (2016), Paper ID 882, 15 pages.

[11] D. Baleanu, H. Khan, H. Jafari, R. A. Khan and M. Alipour, *On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions*, Adv. Difference Equ. **2015** (2015), Paper ID 318, 20 pages.

[12] D. Baleanu, H. Mohammadi and S. Rezapour, *The existence of solutions for a nonlinear mixed problem of singular fractional differential equations*, Adv. Difference Equ. **2013** (2013), Paper ID 359, 14 pages.

[13] D. Baleanu, H. Mohammadi and S. Rezapour, *On a nonlinear fractional differential equation on partially ordered metric spaces*, Adv. Difference Equ. **2013** (2013), Paper ID 83, 12 pages.

[14] D. Baleanu, H. Mohammadi and S. Rezapour, *Analysis of the model of hiv-1 infection of $CD4^+$ T-cell with a new approach of fractional derivative*, Adv. Difference Equ.**2020** (2020), Paper ID 71, 10 pages.

[15] D. Baleanu, A. Mousalou and S. Rezapour, *A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative*, Adv. Difference Equ. **2017** (2017), Paper ID 51, 12 pages.

[16] D. Baleanu, A. Mousalou and S. Rezapour, *On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations*, Bound. Value Probl. **2017** (2017), Paper ID 145, 11 pages.

[17] D. Baleanu, A. Mousalou and S. Rezapour, *The extended fractional Caputo-Fabrizio derivative of order $0 \leq \sigma < 1$ on $C_{\mathbb{R}}[0,1]$ and the existence of solutions for two higher-order series-type differential equations*, Adv. Difference Equ. **2018** (2018), Paper ID 255, 11 pages.

[18] D. Baleanu, S. Rezapour and H. Mohammadi, *Some existence results on nonlinear fractional differential equations*, Philos. Trans. Roy. Soc. A **371** (2013), DOI 10.1098/rsta.2012.0144.

[19] D. Baleanu, S. Rezapour and Z. Saberpour, *On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation*, Bound. Value Probl. **2019** (2019), Paper ID 79, 17 pages.

[20] N. Balkani, S. Rezapour and R. H. Haghi, *Approximate solutions for a fractional q-integro-difference equation*, Journal of Mathematical Extension **13**(3) (2019), 201–214.
[21] M. De La Sena, V. Hedayati, Y. Gholizade Atani and S. Rezapour, *The existence and numerical solution for a k-dimensional system of multi-term fractional integro-differential equations*, Nonlinear Anal. Model. Control **22** (2017), 188–209.
[22] V. Hedayati and S. Rezapour, *On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions*, Kragujevac J. Math. **41 (1)** (2017), 143–158.
[23] V. Hedayati and M. E. Samei, *Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous dirichlet boundary conditions*, Bound. Value Probl. **2019** (2019), Paper ID 141, 23 pages.
[24] S. Hristova and C. Tunc, *Stability of nonlinear Volterra integro-differential equations with Caputo fractional derivative and bounded delays*, Electron. J. Differential Equations **2019** (2019), 1–11.
[25] M. Jleli, E. Karapinar and B. Samet, *Positive solutions for multipoints boundary value problems for singular fractional differential equations*, J. Appl. Math. **2014** (2014), Article ID 596123, 7 pages.
[26] V. Kalvandi and M. E. Samei, *New stability results for a sum-type fractional q-integro-differential equation*, J. Adv. Math. Stud. **12** (2019), 201–209.
[27] H. Khan, C. Tunc, W. Chen and A. Khan, *Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with P-Laplacian operator*, J. Appl. Anal. Comput. **8** (2018), 1211–1226.
[28] S. Liang and M. E. Samei, *New approach to solutions of a class of singular fractional q-differential problem via quantum calculus*, Adv. Difference Equ. **2020** (2020), Paper ID 14, 22 pages.
[29] S. K. Ntouyas and M. E. Samei, *Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus*, Adv. Difference Equ. **2019** (2019), Paper ID 475, 20 pages.
[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[31] M. E. Samei, *Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus*, Adv. Difference Equ. **2020** (2020), Paper ID 23, 23 pages.
[32] M. E. Samei, V. Hedayati and G. K. Ranjbar, *The existence of solution for k-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders*, Mediterr. J. Math. **17** (2020), Paper ID 37, 23 pages.
[33] M. E. Samei, V. Hedayati and S. Rezapour, *Existence results for a fraction hybrid differential inclusion with Caputo-Hadamard type fractional derivative*, Adv. Difference Equ. **2019** (2019), Paper ID 163, 15 pages.
[34] M. E. Samei, G. Khalilzadeh Ranjbar and V. Hedayati, *Existence of solutions for a class of Caputo fractional q-difference inclusion on multifunctions by computational results*, Kragujevac J. Math. **45** (2021), 543–570.
[35] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.
[36] M. Shabibi, M. Postolache, S. Rezapour and S. M. Vaezpour, *Investigation of a multisingular pointwise defined fractional integro-differential equation*, J. Math. Anal. **7** (2016), 61–77.
[37] M. Shabibi, S. Rezapour and S. M. Vaezpour, *A singular fractional integro-differential equation*, Sci. Bull. Univ. Politec. Bush. Series A **79** (2017), 109–118.
[38] M. Talaee, M. Shabibi, A. Gilani and S. Rezapour, *On the existence of solutions for a pointwise defined multi-singular integro-differential equation with integral boundary condition*, Adv. Difference Equ. **2020** (2020), Paper ID 41, 16 pages.
[39] S. W. Vong, *Positive solutions of singular fractional differential equations with integral boundary conditions*, Mathematical and Computer Modelling **57** (2013), 1053–1059.

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$\mathcal{N}$-CUBIC SETS APPLIED TO LINEAR SPACES

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Abstract. The concept of $\mathcal{N}$-fuzzy sets is a good mathematical tool to deal with uncertainties that use the co-domain $[-1, 0]$ for the membership function. The notion of $\mathcal{N}$-cubic sets is defined by combining interval-valued $\mathcal{N}$-fuzzy sets and $\mathcal{N}$-fuzzy sets. Using this $\mathcal{N}$-cubic sets, we initiate a new theory called $\mathcal{N}$-cubic linear spaces. Motivated by the notion of cubic linear spaces we define P-union (resp. R-union), P-intersection (resp. R-intersection) of $\mathcal{N}$-cubic linear spaces. The notion of internal and external $\mathcal{N}$-cubic linear spaces and their properties are investigated.

1. Introduction

The classical set theory failed to handle uncertain, vague and clearly not defined objects because of its limitation to a bivalent condition which is precise in character - an element either belongs or does not belong to the set. As it is well known that Zadeh [19] pioneered the study of fuzzy sets in 1965, which can handle various types of uncertainties successfully in different fields. In contrast to classical set theory fuzzy set theory permits gradual assessment of membership of elements in a set. Fuzzy set theory has rich potential for application in several directions such as topology, analysis, logic, group theory and, semigroup theory. After a decade in 1975, Zadeh [20] introduced interval-valued fuzzy sets as a generalization of a fuzzy set whose members are mapped to the collection of closed subintervals of $[0, 1]$. Attansov $[1, 2]$, further extended the idea of fuzzy sets to intuitionistic fuzzy sets where one can handle membership as well as non-membership of an element. This approach gradually replaced fuzzy sets in dealing with uncertanity and vagueness.

Key words and phrases. $\mathcal{N}$-Interval number, interval-valued $\mathcal{N}$-fuzzy linear space, $\mathcal{N}$-cubic linear space, internal and external $\mathcal{N}$-cubic linear spaces, P-intersection and P-union, R-intersection and R-union.

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Another extension of fuzzy set theory is cubic set theory introduced by Jun et al. [5] in 2010 and examined many properties of cubic sets like internal cubic sets, external cubic sets, P-union, P-intersection, R-union and R-intersection of internal and external cubic sets. Since cubic sets undertake positive part of many physical problems and took no notice of negative aspects wholly. Jun et al [4] brought up a negative valued function and formulated $\mathcal{N}$-structures. Moreover, they applied $\mathcal{N}$ structure theory to subtraction algebra and BCK/BCI algebra [6]. This paved way to the idea of $\mathcal{N}$-cubic sets introduced by Jun [9] combining $\mathcal{N}$-fuzzy sets and interval-valued $\mathcal{N}$-fuzzy sets to cover the negative part of cubic sets along the codomain $[-1, 0]$.

An abundant measure of efforts was executed by researchers in extending fuzzy sets to groups, rings, vector spaces and other branches of mathematics. G. Lubczonok and V. Murali [10] introduced the theory of flags and fuzzy subspaces of vector spaces. Kastras and Liu [7] applied the concept of fuzzy sets to the elementary theory of vector spaces and topological vector spaces. Nanda [11] introduced the concept of fuzzy linear space. Later Gu Wexiang and Lu [18] redefined the concept of fuzzy field and fuzzy linear space and gave some fundamental properties. Vijaybalaji et al. further advanced the theory to cubic linear space combining interval-valued fuzzy linear space and fuzzy linear space and their properties are presented in [17].

In this paper we present the notion of $\mathcal{N}$-cubic linear spaces. After providing essential background on cubic sets, $\mathcal{N}$-cubic linear spaces and their intersection and union properties we confine section 3 to define the concept of $\mathcal{N}$-cubic linear spaces. We introduce the P-union (resp. P-intersection) and R-union (resp. R-intersection) in $\mathcal{N}$-Cubic linear spaces. We show that $\mathcal{N}$-cubic linear space is closed with respect to R-intersection. By giving examples we disprove that R-union, P-union and P-intersection of two $\mathcal{N}$-cubic linear spaces is again a $\mathcal{N}$-cubic linear space. In section 4, we introduce the concept of internal $\mathcal{N}$-cubic linear space and external $\mathcal{N}$-cubic linear space. We also show that internal $\mathcal{N}$-cubic linear space is not closed with respect to P-union, P-intersection and R-union (resp. external) by providing counter examples.

2. Preliminaries

Definition 2.1 ([20])**.** An $\mathcal{N}$-interval number is a closed subinterval of $[-1, 0]$ and the collection of all closed subintervals of $[-1, 0]$ is denoted by $D[-1, 0]$. It is of the form $D[-1, 0] = \{\hat{i} = [i^-, i^+] : i^- \leq i^+, i^-, i^+ \in [0, 1]\}$. Notably the operations "$\geq$", "$\leq$", "$=$", "max", "min" are defined as follows:

(i) $\hat{i_1} \geq \hat{i_2}$ if and only if ${i_1}^- \geq {i_2}^-$ and ${i_1}^+ \geq {i_2}^+$;
(ii) $\hat{i_1} \leq \hat{i_2}$ if and only if ${i_1}^- \leq {i_2}^-$ and ${i_1}^+ \leq {i_2}^+$;
(iii) $\hat{i_1} = \hat{i_2}$ if and only if ${i_1}^- = {i_2}^-$ and ${i_1}^+ = {i_2}^+$;
(iv) $\min\{\hat{i_1}, \hat{i_2}\} = [\min\{{i_1}^-, {i_2}^-\}, \min\{{i_1}^+, {i_2}^+\}]$;
(v) $\max\{\hat{i_1}, \hat{i_2}\} = [\max\{{i_1}^-, {i_2}^-\}, \max\{{i_1}^+, {i_2}^+\}]$.

Definition 2.2 ([20])**.** For an $\mathcal{N}$-interval number $\hat{i_t} \in D[-1, 0]$, where $t \in \Lambda$. We define

$$\inf \hat{i_t} = \left[\inf_{t\in\Lambda} \hat{i_t}^-, \inf_{t\in\Lambda} \hat{i_t}^+\right] \quad \text{and} \quad \sup \hat{i_t} = \left[\sup_{t\in\Lambda} \hat{i_t}^-, \sup_{t\in\Lambda} \hat{i_t}^+\right].$$

Definition 2.3 ([20])**.** An interval valued $\mathcal{N}$-fuzzy set denoted by $\mathcal{I}^{\mathcal{N}}$ on Y is of the form $\mathcal{I}^{\mathcal{N}} = \{< y, \mathcal{I}^{\mathcal{N}}(y) : y \in \mathrm{Y} >\}$, where $\mathcal{I}^{\mathcal{N}} : \mathrm{Y} \to D[0, 1]$ and $\mathcal{I}^{\mathcal{N}}(y) = [\vartheta^-_{\mathcal{I}^{\mathcal{N}}}(y), \vartheta^+_{\mathcal{I}^{\mathcal{N}}}(y)]$ for all $y \in \mathrm{Y}$. Here $\vartheta^-_{\mathcal{I}^{\mathcal{N}}}(y) : \mathrm{Y} \to [0, 1]$ and $\vartheta^+_{\mathcal{I}^{\mathcal{N}}}(y)] : \mathrm{Y} \to [0, 1]$ are fuzzy sets in Y such that $\vartheta^-_{\mathcal{I}^{\mathcal{N}}}(y) \leq \vartheta^+_{\mathcal{I}^{\mathcal{N}}}(y)$.

Definition 2.4 ([5])**.** Let Y be a non-empty set. A cubic set $\mathbf{C}$ of Y is a structure $\mathbf{C} = \{y, \hat{\vartheta}_{\mathbf{C}}(y), \lambda_{\mathbf{C}}(y) | y \in \mathrm{Y}\}$ in which $\hat{\vartheta}_{\mathbf{C}} : \mathrm{Y} \to D[0, 1]$ and $\lambda_{\mathbf{C}} : \mathrm{Y} \to [0, 1]$.

Definition 2.5 ([5])**.** A cubic set $\mathbf{C} = (\hat{\vartheta}_{\mathbf{C}}, \lambda_{\mathbf{C}})$ in a non-empty set Y is said to be an internal cubic set (in brief, ICS) if $\vartheta^-_{\mathbf{C}}(y) \leq \lambda_{\mathbf{C}}(y) \leq \vartheta^+_{\mathbf{C}}(y)$ for all $y \in \mathrm{Y}$. For an external cubic set (in brief, ECS) it is $\lambda_{\mathbf{C}}(y) \notin (\vartheta^-_{\mathbf{C}}(y), \vartheta^+_{\mathbf{C}}(y)$ for all $y \in \mathrm{Y}$.

Definition 2.6 ([17])**.** Let W be a linear space over field F, $(\mathrm{W}, \hat{\vartheta})$ be an interval valued fuzzy linear space, (W, λ) be a fuzzy linear space. A cubic set $\mathbf{C} = (\hat{\vartheta}_{\mathbf{C}}, \lambda_{\mathbf{C}})$ is called a cubic linear space of W if for all $\sigma, \tau \in F$
(i) $\hat{\vartheta}(\sigma a * \tau b) \geq \min\{\hat{\vartheta}(a), \hat{\vartheta}(b)\}$;
(ii) $\lambda(\sigma a * \tau b) \leq \max\{\lambda(a), \lambda(b)\}$.

Definition 2.7 ([9])**.** Let Y be a fixed set. A $\mathcal{N}$-fuzzy set in Y is defined as $\mathbf{N}^F = \{y, \lambda_{\mathbf{N}^F}(y)\} : y \in \mathrm{Y}$ and $\lambda_{\mathbf{N}^F} : \mathrm{Y} \to [-1, 0]$ a membership function for all $y \in \mathrm{Y}$.

Definition 2.8 ([9])**.** Let Y be a non-empty set. A $\mathcal{N}$-cubic set in Y is a structure $\mathbf{N}^{\mathbf{C}} = \Big\{\langle y, \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}(y), \lambda_{\mathbf{N}^{\mathbf{C}}}(y)\rangle | y \in \mathrm{Y}\Big\}$ is briefly denoted by $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}}\rangle$ in which $\hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}} = [\vartheta^-_{\mathbf{N}^{\mathbf{C}}}, \vartheta^+_{\mathbf{N}^{\mathbf{C}}}]$ an interval valued fuzzy set and $\lambda_{\mathbf{N}^{\mathbf{C}}} : \mathrm{Y} \to [-1, 0]$ is a fuzzy set in Y.

Definition 2.9 ([9])**.** Let Y be a non-empty set. An $\mathcal{N}$-cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}}\rangle$ in Y is said to be an internal $\mathcal{N}$-cubic set (INCS) if $\vartheta^-_{\mathbf{N}^{\mathbf{C}}}(y) \leq \lambda_{\mathbf{N}^{\mathbf{C}}} \leq \vartheta^+_{\mathbf{N}^{\mathbf{C}}}(y)$ for all $y \in \mathrm{Y}$. Similarly, external $\mathcal{N}$-cubic set (ENCS) if $\lambda_{\mathbf{N}^{\mathbf{C}}}(y) \notin (\vartheta^-_{\mathbf{N}^{\mathbf{C}}}(y), \vartheta^+_{\mathbf{N}^{\mathbf{C}}}(y))$.

Definition 2.10 ([9])**.** For any $\mathbf{N}^{\mathbf{C}}_i = \Big\{\langle y, \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i}(y), \lambda_{\mathbf{N}^{\mathbf{C}}_i}(y)\rangle : y \in \mathrm{Y}\Big\}$, where $i \in \Lambda$, we define
(a) $\bigcup_{\substack{\mathrm{R} \\ i\in\Lambda}} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (R-union);
(b) $\bigcup_{\substack{\mathrm{P} \\ i\in\Lambda}} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (P-union);
(c) $\bigcap_{\substack{\mathrm{R} \\ i\in\Lambda}} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (P-intersection);
(d) $\bigcap_{\substack{\mathrm{R} \\ i\in\Lambda}} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (R-intersection).

3. Results

In this section, we come across the notion of $\mathcal{N}$-cubic linear space. We also discuss some results in connection with the $\mathcal{N}$-cubic linear space.

3.1. $\mathcal{N}$-cubic linear spaces.

Definition 3.1. For a linear space W over a field F a $\mathcal{N}$-fuzzy set $\mathbf{N}^F$=(W, $\lambda_{\mathbf{N}^F}$) in W is said to be a $\mathcal{N}$-fuzzy linear space $\mathbf{W^F}$=$\{(w, \lambda_{\mathbf{N}^F}(w)) : w \in$ W$, \lambda_{\mathbf{N}^F}(w) \in [-1, 0]\}$ if it satisfies

$$\lambda_{\mathbf{W^F}}(\sigma a * \tau b) \leq \lambda_{\mathbf{W^F}}(a) \cup \lambda_{\mathbf{W^F}}(a),$$

for any $\sigma, \tau \in F$ and $a, b \in$ W.

Definition 3.2. An interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$: W $\to D[-1, 0]$ is said to be an interval-valued $\mathcal{N}$-fuzzy linear space where W over field F if the latter conditions are satisfied

$$\hat{\vartheta}_{\mathbf{N}}(\sigma a * \tau b) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}}(a), \hat{\vartheta}_{\mathbf{N}}(b)\right\},$$

for any for any $\sigma, \tau \in F$ and $a, b \in$ W.

Definition 3.3. Let W be a linear space over field F, (W, $\hat{\vartheta}_{\mathbf{IF}}$) an interval-valued $\mathcal{N}$-fuzzy linear space, (W, $\lambda_{\mathbf{W^F}}$) a $\mathcal{N}$-fuzzy linear space. A **N**-cubic set $\mathbf{N^C} = \langle\hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}}\rangle$ in Y is said to be a **N**-cubic linear space of W if
(i) $\hat{\vartheta}_{\mathbf{N^C}}(\sigma a * \tau b) \leq \max\{\hat{\vartheta}_{\mathbf{N^C}}(a), \hat{\vartheta}_{\mathbf{N^C}}(b)\}$;
(ii) $\lambda_{\mathbf{N^C}}(\sigma a * \tau b) \geq \min\{\lambda_{\mathbf{N^C}}(a), \lambda_{\mathbf{N^C}}(b)\}$,
for all $a, b \in$ W and $\sigma, \tau \in F$.

Example 3.1. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as follows

$$\mathrm{W} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

such that $w_{11} + w_{12} = w_{21}$. Then W is a vector space over the field $GF(2)$.

Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\hat{\vartheta}_{\mathbf{N}}(a) =[-0.9, -0.8],$$
$$\hat{\vartheta}_{\mathbf{N}}(b) =[-0.6, -0.3],$$
$$\hat{\vartheta}_{\mathbf{N}}(c) =[-0.4, -0.1],$$
$$\hat{\vartheta}_{\mathbf{N}}(d) =[-0.8, -0.7].$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space.

Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\lambda_{\mathbf{N}}(a) = -0.4,$$
$$\lambda_{\mathbf{N}}(b) = -0.6,$$
$$\lambda_{\mathbf{N}}(c) = -0.25,$$
$$\lambda_{\mathbf{N}}(d) = -0.9.$$

Here λ is a $\mathcal{N}$-fuzzy linear space of W.

Consequently, the above example satisfied the conditions required for a cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ to be a $\mathcal{N}$-cubic linear space.

Remark 3.1. For any family of real numbers $\{b_j : j \in \Lambda\}$ we define

(i)

$$\bigcup\{b_j : j \in W\} = \begin{cases} \max\{b_j : j \in \Lambda\}, & \text{if } \Lambda \text{ is finite,} \\ \sup\{b_j : j \in \Lambda\}, & \text{otherwise.} \end{cases}$$

(ii)

$$\bigcap\{b_j : j \in W\} = \begin{cases} \min\{b_j : j \in \Lambda\}, & \text{if } \Lambda \text{ is finite,} \\ \inf\{b_j : j \in \Lambda\}, & \text{otherwise.} \end{cases}$$

In the following proposition, we prove that the R-union of a family of **N**-cubic linear spaces is again a **N**-cubic linear space.

Definition 3.4. Let $(\mathrm{W}, \hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}})$ and $(\mathrm{W}, \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})$ be two interval-valued **N**-fuzzy linear spaces. Then the union and intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces can be defined as

$$\hat{\vartheta}_{\mathbf{N_1}} \cap \hat{\vartheta}_{\mathbf{N_2}}(w) = \min\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W},$$
$$\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) = \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W}.$$

Definition 3.5. Let $(\mathrm{W}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ and $(\mathrm{W}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ be two interval-valued $\mathcal{N}$-fuzzy linear spaces. Then the union and intersection of $\mathcal{N}$-fuzzy linear spaces can be defined as

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w) = \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W},$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w) = \max\{\lambda_{\mathbf{N_1}}(w).\lambda_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W}.$$

Proposition 3.1. *Let* $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their R-intersection* $(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_R$=$(\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *is again an* $\mathcal{N}$*-cubic linear space.*

Proof. Since $\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) = \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$. We have

$$\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) = \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\right\},$$

for $w_1, w_1 \in \mathrm{W}$ and $\sigma, \tau \in F$.

From Definition 3.3 we have

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) =& \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\right\},\\ \leq& \max\left\{\max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_1), \hat{\vartheta}_{\mathbf{N_1}}(w_2)\right\}, \max\left\{\hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}\right\},\\ =& \max\left\{\max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_1), \hat{\vartheta}_{\mathbf{N_2}}(w_1)\right\}, \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_2), \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}\right\},\\ =& \max\left\{\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\},\end{aligned}$$

which imply

$$(\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}})(\sigma w_1 * \tau w_2) \leq \max\left\{\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}. \tag{3.1}$$

Hence, $\bigcup\limits_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N_i}}$ is an interval-valued **N**-fuzzy linear space. Since $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N_2}}(w) = \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}$, $w \in$ W. We have

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) = \min\{\lambda_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\},$$

for $w_1, w_1 \in$ W and $\sigma, \tau \in F$.

From Definition 3.3 we have

$$\begin{aligned}\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) =& \min\{\lambda_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\},\\ \geq& \min\{\min\{\lambda_{\mathbf{N_1}}(w_1), \lambda_{\mathbf{N_1}}(w_2)\}, \max\{\lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_2}}(w_2)\}\},\\ =& \min\{\min\{\lambda_{\mathbf{N_1}}(w_1), \lambda_{\mathbf{N_2}}(w_1)\}, \min\{\lambda_{\mathbf{N_1}}(w_2), \lambda_{\mathbf{N_2}}(w_2)\}\},\\ =& \min\{\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_2)\},\end{aligned}$$

which imply

$$(\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}})(\sigma w_1 * \tau w_2) \geq \min\{\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_2)\}. \tag{3.2}$$

Hence, $\bigcap\limits_{i\in\Lambda} \lambda_{\mathbf{N_i}}$ is an interval-valued **N**-fuzzy linear space.

Thus from (3.1) and (3.2) the conditions required for R- intersection to be a $\mathcal{N}$-cubic linear space are satisfied. □

Remark 3.2. By taking an example we prove that the intersection of two interval-valued **N**-fuzzy linear spaces do not satisfy the first condition of **N**-cubic linear space as in Definition 3.3.

Example 3.2. Let W $= M_{2\times2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Consider two interval-valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ in W as given in the Table 1. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W.

From the Definition 3.4

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) =& [-0.7, -0.6], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.5, -0.4],\\ \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) =& [-0.4, -0.2], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.6, -0.4].\end{aligned}$$

TABLE 1. Values of interval-valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.7, −0.5]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.7, −0.6]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−0.4, −0.1]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−0.5, −0.4]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.4, −0.2]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−0.4, −0.1]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−0.3, −0.1]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−0.6, −0.4]

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.7, -0.6], [-0.5, -0.4]\} = [-0.5, -0.4],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.4, -0.2] \leq [-0.5, -0.4]$, which is non-sensical.

From the above example, it is clear that the intersection of two interval-valued **N**-fuzzy linear spaces need not be an interval-valued **N**-fuzzy linear space.

Remark 3.3. Similarly, by taking an example, we prove that the union of two **N**-fuzzy linear spaces does not satisfy the second condition of **N**-cubic linear space as in Definition 3.3.

Example 3.3. Let W = $M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Consider a $\mathcal{N}$-fuzzy set $\lambda_{\mathbf{N}}$ in W as given in the Table 2. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

TABLE 2. Values of $\mathcal{N}$-fuzzy sets $\lambda_{\mathbf{N}}$

$\lambda_{\mathbf{N}_1}(w_{11})$=−0.5	$\lambda_{\mathbf{N}_2}(w_{11})$=−0.2
$\lambda_{\mathbf{N}_1}(w_{12})$=−0.3	$\lambda_{\mathbf{N}_2}(w_{12})$=−0.85
$\lambda_{\mathbf{N}_1}(w_{21})$=−0.4	$\lambda_{\mathbf{N}_2}(w_{21})$=−0.7
$\lambda_{\mathbf{N}_1}(w_{22})$=−0.2	$\lambda_{\mathbf{N}_2}(w_{22})$=−0.6

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.3,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.4, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.2.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.2, -0.3\} = -0.3,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.3$, which is non-sensical.

From the above example, it is clear that the intersection of two $\mathcal{N}$-fuzzy linear spaces need not be a $\mathcal{N}$-fuzzy linear space.

Lemma 3.1. *From the above theorem and examples following statements can be proved.*

(i) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1 C}}, \lambda_{\mathbf{N_1 C}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2 C}}, \lambda_{\mathbf{N_2 C}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their R-union* $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1 C}} \cup \lambda_{\mathbf{N_2 C}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

(ii) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1 C}}, \lambda_{\mathbf{N_1 C}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2 C}}, \lambda_{\mathbf{N_2 C}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their P-union* $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1 C}} \cap \lambda_{\mathbf{N_2 C}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

(iii) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1 C}}, \lambda_{\mathbf{N_1 C}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2 C}}, \lambda_{\mathbf{N_2 C}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their P-intersection* $(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1 C}} \cup \lambda_{\mathbf{N_2 C}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

Proof. (i) From Example 3.2 we can observe that intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ do not satisfy the first condition of $\mathcal{N}$-cubic linear space as in Definition 3.3 and from Example 3.3 union of two $\mathcal{N}$-fuzzy linear spaces $(\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2})$ do not satisfy the second condition of $\mathcal{N}$-cubic linear space as in Definition 3.3. Therefore, the R-union $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1 C}} \cup \lambda_{\mathbf{N_2 C}})$ is not a $\mathcal{N}$-cubic linear space.

(ii) Consider $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ as in Example 3.3. Now by Definition 3.5 $\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w) = \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$. Therefore,

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}) = -0.5, \quad \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12}) = -0.85,$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.7, \quad \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{22}) = -0.6.$$

We note that $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}$ is an $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\begin{aligned}\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) &\geq \min\left\{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\right\}, \\ \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) &\geq \min\left\{-0.5, -0.85\right\} = -0.85,\end{aligned}$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.7 \geq -0.85$. Certainly, $(\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2})$ satisfies the second condition of $\mathcal{N}$-cubic linear spaces. But from Example 3.2 $(\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2})$ is not an interval-valued $\mathbf{N}$-fuzzy linear space. Therefore, P-union $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_P$=$(\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1 C}} \cap \lambda_{\mathbf{N_2 C}})$ is not a $\mathcal{N}$-cubic linear space.

(iii) Consider $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ as in Example 3.2. Now by Definition 3.4 $\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) = \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$, we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.7, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.4, -0.1],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.4, -0.1], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.3, -0.1].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.7, -0.5], [-0.4, -0.1]\} = [-0.4, -0.1],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.4, -0.1] \leq [-0.4, -0.1]$. Certainly, $(\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2})$ satisfies the second condition of $\mathcal{N}$-cubic linear spaces. But from Example 3.3 $(\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2})$ is not a $\mathcal{N}$-fuzzy linear space. Therefore, P-intersection

$$(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N}_1^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$$

need not be a $\mathcal{N}$-cubic linear space. □

4. Internal and External $\mathcal{N}$-Cubic Linear Spaces

In this section, we come out with the notion of internal and external $\mathcal{N}$-cubic linear spaces and confer some of their properties.

Definition 4.1. Suppose W be a linear space over a field F. A $\mathcal{N}$-cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ is said to be an internal $\mathcal{N}$-cubic linear space (shortly, INCLS) if

$$\hat{\vartheta}^{-}_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \leq \hat{\vartheta}^{+}_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$.

Example 4.1. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\hat{\vartheta}_{\mathbf{N}}(w_{11}) = [-0.5, -0.3], \quad \hat{\vartheta}_{\mathbf{N}}(w_{12}) = [-0.4, -0.1],$$
$$\hat{\vartheta}_{\mathbf{N}}(w_{21}) = [-0.8, -0.7], \quad \hat{\vartheta}_{\mathbf{N}}(w_{22}) = [-0.6, -0.4].$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space.

Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\lambda_{\mathbf{N}}(w_{11}) = -\,0.7, \quad \lambda_{\mathbf{N}}(w_{12}) = -0.6,$$
$$\lambda_{\mathbf{N}}(w_{21}) = -\,0.85, \quad \lambda_{\mathbf{N}}(w_{22}) = -0.4.$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^{-}_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^{+}_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^{-}_{\mathbf{N}^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^{+}_{\mathbf{N}^{\mathbf{C}}}(w_{21}),$$

which imply $-0.85 \in [-0.8, -0.7]$. So, $\mathcal{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ is an INCLS.

Definition 4.2. Suppose W be a linear space over a field F. A $\mathcal{N}$-cubic set $\mathbf{N^C} = \langle\hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}}\rangle$ is said to be an external $\mathcal{N}$-cubic linear space (shortly, ENCLS) if

$$\lambda_{\mathbf{N^C}}(\sigma w_1 * \tau w_2) \notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(\sigma w_1 * \tau w_2), \hat{\vartheta}^+_{\mathbf{N^C}}(\sigma w_1 * \tau w_2)\right),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$.

Example 4.2. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\begin{aligned} \hat{\vartheta}_{\mathbf{N}}(w_{11}) =&[-0.5, -0.1], \quad \hat{\vartheta}_{\mathbf{N}}(w_{12}) = [-0.7, -0.4], \\ \hat{\vartheta}_{\mathbf{N}}(w_{21}) =&[-0.8, -0.6], \quad \hat{\vartheta}_{\mathbf{N}}(w_{22}) = [-0.6, -0.3]. \end{aligned}$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space. Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\lambda_{\mathbf{N}}(w) = \begin{cases} -0.9, & \text{when } w = w_{11}, \\ -0.95, & \text{otherwise.} \end{cases}$$

For $\sigma = \tau = 1$ in Definition 4.2 we have

$$\begin{aligned} \lambda_{\mathbf{N^C}}(w_{11} + w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N^C}}(w_{11} + w_{12})\right), \\ \lambda_{\mathbf{N^C}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N^C}}(w_{21})\right), \end{aligned}$$

which imply $-0.95 \notin [-0.8, -0.6]$. So, $\mathcal{N}^{\mathbf{C}} = \langle\hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}}\rangle$ is an ENCLS.

Remark 4.1. In the following proposition, we present that the R-intersection of a family of INCLS's is again an INCLS (resp. ENCLS).

Proposition 4.1. *Let* $\mathcal{N}^I_1 = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}})$ *and* $\mathcal{N}^I_2 = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_2}^\mathbf{C}})$ *be two INCLS. Then their R-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)^I_R = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})$ *is an INCLS.*

Proof. Considering the fact that $\mathcal{N}^I_1$ and $\mathcal{N}^I_2$ are INCLS in W, we have

$$\begin{aligned} \hat{\vartheta}^-_{\mathbf{N_1}^\mathbf{C}}(\sigma w_1 * \tau w_2) \leq \lambda_{\mathbf{N_1}^\mathbf{C}}(\sigma w_1 * \tau w_2) \leq \hat{\vartheta}^+_{\mathbf{N_1}^\mathbf{C}}(\sigma w_1 * \tau w_2), \\ \hat{\vartheta}^-_{\mathbf{N_2}^\mathbf{C}}(\sigma w_1 * \tau w_2) \leq \lambda_{\mathbf{N_2}^\mathbf{C}}(\sigma w_1 * \tau w_2) \leq \hat{\vartheta}^+_{\mathbf{N_2}^\mathbf{C}}(\sigma w_1 * \tau w_2), \end{aligned}$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$. Now since the union of interval-valued fuzzy linear spaces is again an interval-valued $\mathcal{N}$-fuzzy linear space and intersection of $\mathcal{N}$-fuzzy linear space is again a fuzzy linear space. We have

$$\begin{aligned} (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}})^-(\sigma w_1 * \tau w_2) \leq& (\lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})(\sigma w_1 * \tau w_2) \\ \leq& (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}})^+(\sigma w_1 * \tau w_2). \end{aligned}$$

Therefore, $(\mathcal{N}_1 \cap \mathcal{N}_2)^I_R = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})$ is an INCLS. □

Proposition 4.2. *Let* $\mathcal{N}^E_1 = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}})$ *and* $\mathcal{N}^E_2 = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_2}^\mathbf{C}})$ *be two ENCLS. Then their R-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)^E_R = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})$ *is again an ENCLS.*

Proof. Considering the fact that $\mathcal{N}_1^I$ and $\mathcal{N}_2^I$ are ENCLS in W, we have

$$\lambda_{\mathbf{N_1}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \notin \left((\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}})^+(\sigma w_1 * \tau w_2)\right),$$
$$\lambda_{\mathbf{N_2}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \notin \left((\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})^+(\sigma w_1 * \tau w_2)\right),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$. Now since the union of interval-valued fuzzy linear spaces is again an interval-valued $\mathcal{N}$-fuzzy linear space and intersection of $\mathcal{N}$-fuzzy linear space is again a fuzzy linear space. We have

$$\begin{aligned}&(\lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})(\sigma w_1 * \tau w_2)\\ &\notin \left((\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})^+(\sigma w_1 * \tau w_2)\right).\end{aligned}$$

Therefore, $(\mathcal{N}_1 \cap \mathcal{N}_2)_R^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ is an ENCLS. □

Remark 4.2. By taking an example, we disprove the statement that the P-intersection of two interior $\mathcal{N}$-cubic linear spaces is again an interior $\mathcal{N}$-cubic linear space.

Proposition 4.3. *Let $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ and $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ be two INCLS. Then their P-intersection $(\mathcal{N}_1 \cap \mathcal{N}_2)_P^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.3. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 3. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can

Table 3. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.8, −0.5]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.9, −0.8]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−1, −0.9]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−0.6, −0.3]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.85, −0.7]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−1, −0.93]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−0.9, −0.78]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−0.7, −0.4]

check by simple calculation using Definition 3.2. From the Definition 3.4

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.8, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.6, -0.3],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.85, -0.7], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.7, -0.4].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.8, -0.5], [-0.6, -0.3]\} = [-0.6, -0.3],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.85, -0.7] \leq [-0.6, -0.3]$.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 4. We note that

TABLE 4. Value of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=$-0.42$	$\lambda_{\mathbf{N}_2}(w_{11})$=$-0.9$
$\lambda_{\mathbf{N}_1}(w_{12})$=$-0.3$	$\lambda_{\mathbf{N}_2}(w_{12})$=$-0.2$
$\lambda_{\mathbf{N}_1}(w_{21})$=$-0.8$	$\lambda_{\mathbf{N}_2}(w_{21})$=$-0.98$
$\lambda_{\mathbf{N}_1}(w_{22})$=$-0.6$	$\lambda_{\mathbf{N}_2}(w_{22})$=$-0.1$

$\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W.

From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.42, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.2,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.8, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.1.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W.

Since ${\mathcal{N}_1}^I$ and ${\mathcal{N}_2}^I$ are INCLS the example that we have taken will satisfy the condition mentioned in Definition 4.1. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^{-}_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^{-}_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}),$$

which imply $0.8 \in [-0.85, -0.7]$.

Similarly, for ${\mathcal{N}_2}^I = (\hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N}_2^{\mathbf{C}}})$, when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^{-}_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^{-}_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}),$$

which imply $0.98 \in [-1, -0.93]$.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\left\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\right\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.42, -0.2\} = -0.42,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.8 \geq -0.42$, which is non-sensical.

Therefore, the P-intersection of two INCLS need not be an INCLS. $\square$

Remark 4.3. By taking an example, we disprove the statement that the P-intersection of two exterior $\mathcal{N}$-cubic linear spaces is again an exterior $\mathcal{N}$-cubic linear space.

Proposition 4.4. *Let* $\mathcal{N}_1^E = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^E = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two ENCLS. Then their P-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)_P^E = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be an ENCLS.*

Proof. The proof of the above statement follows by the example.

Example 4.4. The proof of the above statement follows by the example. Let W = $M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 5. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can

TABLE 5. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.55, −0.3]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.75, −0.5]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−0.9, −0.8]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−0.4, −0.05]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.6, −0.4]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−0.5, −0.2]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−0.7, −0.5]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−0.38, −0.08]

check by simple calculation using Definition 3.2. From Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.55, -0.3], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.4, -0.05],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.5, -0.2], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.38, -0.08].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.55, -0.3], [-0.4, -0.05]\} = [-0.4, -0.05],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.5, -0.2] \leq [-0.4, -0.05]$.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in Table 6.

TABLE 6. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=−0.2	$\lambda_{\mathbf{N}_2}(w_{11})$=−0.84
$\lambda_{\mathbf{N}_1}(w_{12})$=−0.32	$\lambda_{\mathbf{N}_2}(w_{12})$=−0.4
$\lambda_{\mathbf{N}_1}(w_{21})$=−0.7	$\lambda_{\mathbf{N}_2}(w_{21})$=−0.9
$\lambda_{\mathbf{N}_1}(w_{22})$=−0.25	$\lambda_{\mathbf{N}_2}(w_{22})$=−0.1

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.32,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.7, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.1.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. Since ${\mathcal{N}_1}^I$ and ${\mathcal{N}_2}^I$ are ENCLS the example that we have taken will satisfy the condition mentioned in Definition 4.2. For $\sigma = \tau = 1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12})\right),$$
$$\lambda_{\mathbf{N}_1^\mathbf{C}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{21})\right),$$

which imply $-0.7 \notin [-0.6, -0.4]$.

Similarly, for ${\mathcal{N}_2}^I = (\hat{\vartheta}_{\mathbf{N}_2^\mathbf{C}}, \lambda_{\mathbf{N}_2^\mathbf{C}})$, when $\sigma = \tau = 1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12})\right),$$
$$\lambda_{\mathbf{N}_2^\mathbf{C}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{21})\right),$$

which imply $-0.95 \notin [-0.5, -0.2]$. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.2, -0.32\} = -0.32,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.7 \geq -0.42$, which is non-sensical.

Therefore, the P-intersection of two ENCLS need not be an ENCLS. □

Remark 4.4. By taking an example, we disprove the statement that the P-union of two interior $\mathcal{N}$-cubic linear spaces is again an interior $\mathcal{N}$-cubic linear space.

Proposition 4.5. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_2}^\mathbf{C}})$ *be two INCLS. Then their* P*-union* $(\mathcal{N}_1 \cup \mathcal{N}_2)^I_P = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cap \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})$ *need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.5. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 7. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces

TABLE 7. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.3, −0.1]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.95, −0.85]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−0.5, −0.45]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−1, −0.93]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.4, −0.2]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−0.63, −0.5]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−1, −0.9]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−0.8, −0.7]

in W and that we can check by simple calculation using Definition 3.2. From the

Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.3, -0.1], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-1, -0.93],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.63, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-1, -0.9].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max \left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max \{[-0.95, -0.85], [-1, -0.93]\} = [-0.95, -0.85],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.63, -0.5] \leq [-0.95, -0.85]$, which is non-sensical. Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 8. We note that

TABLE 8. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11}) = -0.65$	$\lambda_{\mathbf{N}_2}(w_{11}) = -0.8$
$\lambda_{\mathbf{N}_1}(w_{12}) = -0.12$	$\lambda_{\mathbf{N}_2}(w_{12}) = -0.25$
$\lambda_{\mathbf{N}_1}(w_{21}) = -0.42$	$\lambda_{\mathbf{N}_2}(w_{21}) = -0.68$
$\lambda_{\mathbf{N}_1}(w_{22}) = -0.3$	$\lambda_{\mathbf{N}_2}(w_{22}) = -0.9$

$\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{11}) = -0.8, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{12}) = -0.25,$$
$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{21}) = -0.68, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{22}) = -0.9.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.1

$$\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}),$$

which imply $0.42 \in [-0.4, -0.2]$.

Similarly, for $\mathcal{N}_2{}^I = (\hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N}_2^{\mathbf{C}}})$ when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}),$$

which imply $0.68 \in [-0.63, -0.5]$.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min \{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) \geq \min \{-0.8, -0.25\} = -0.8,$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.68 \geq -0.8$. Even though the intersection of two $\mathcal{N}$-fuzzy linear spaces satisfies the first condition of Definition 3.3 the intersection of interval-valued $\mathcal{N}$-fuzzy linear spaces failed to satisfy the second condition of Definition 3.3.

Therefore, the P-union of two INCLS need not be an INCLS. $\square$

Remark 4.5. In the latter example, we show that the P-union of two exterior $\mathcal{N}$-cubic linear spaces need not be an exterior $\mathcal{N}$-cubic linear space.

Proposition 4.6. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{c}}, \lambda_{\mathbf{N_1}^\mathbf{c}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{c}}, \lambda_{\mathbf{N_2}^\mathbf{c}})$ *be two ENCLS. Then their P-union* $(\mathcal{N}_1 \cup \mathcal{N}_2)_P^I = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{c}} \cap \hat{\vartheta}_{\mathbf{N_2}^\mathbf{c}}, \lambda_{\mathbf{N_1}^\mathbf{c}} \cap \lambda_{\mathbf{N_2}^\mathbf{c}})$ *need not be an ENCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.6. Let $\mathrm{W} = M_{2\times2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 9.

TABLE 9. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=$[-1, -0.7]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=$[-0.9, -0.5]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=$[-0.75, -0.2]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=$[-1, -0.6]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=$[-0.95, -0.55]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=$[-0.85, -0.4]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=$[-0.6, -0.1]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=$[-0.8, -0.4]$

Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can check by simple calculation using Definition 3.2. From the Definition 3.4 we have

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) &= [-1, -0.7], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-1, -0.6],\\ \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) &= [-0.95, -0.55], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.8, -0.4].\end{aligned}$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) &\leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},\\ \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) &\leq \max\left\{[-0.1, -0.7], [-1, -0.6]\right\} = [-1, -0.6],\end{aligned}$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.95, -0.55] \leq [-1, -0.6]$, which is non-sensical.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 10. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\begin{aligned}\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{11}) &= -0.8, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{12}) = -0.75,\\ \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{21}) &= -0.4, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{22}) = -0.7.\end{aligned}$$

TABLE 10. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=$-0.8$	$\lambda_{\mathbf{N}_2}(w_{11})$=$-0.25$
$\lambda_{\mathbf{N}_1}(w_{12})$=$-0.6$	$\lambda_{\mathbf{N}_2}(w_{12})$=$-0.75$
$\lambda_{\mathbf{N}_1}(w_{21})$=$-0.4$	$\lambda_{\mathbf{N}_2}(w_{21})$=$-0.35$
$\lambda_{\mathbf{N}_1}(w_{22})$=$-0.3$	$\lambda_{\mathbf{N}_2}(w_{22})$=$-0.7$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12})\right),$$
$$\lambda_{\mathbf{N}_1^\mathbf{C}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{21})\right),$$

which imply $-0.4 \notin [-0.95, -0.55]$. Also,

$$\lambda_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12})\right),$$
$$\lambda_{\mathbf{N}_2^\mathbf{C}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{21})\right),$$

which imply $-0.35 \notin [-0.85, -0.4]$. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\left\{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\right\},$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\left\{-0.8, -0.75\right\} = -0.8,$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.8$.

Therefore, the P-union of two ENCLS need not be an ENCLS. □

Remark 4.6. In the latter example, we show that the R-union of two interior $\mathcal{N}$-cubic linear spaces need not be an interior $\mathcal{N}$-cubic linear space.

Proposition 4.7. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_2}^\mathbf{C}})$ *be two INCLS. Then their R-union* $(\mathcal{N}_1 \cap \mathcal{N}_2)_R^I = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cap \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cup \lambda_{\mathbf{N_2}^\mathbf{C}})$ *need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.7. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 11. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can check by simple calculation using Definition 3.2. From the Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.65, -0.45], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.9, -0.8],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.6, -0.4], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-1, -0.95].$$

TABLE 11. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.65, −0.5]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.55, −0.25]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−0.8, −0.7]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−0.9, −0.8]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.5, −0.3]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−0.6, −0.4]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−1, −0.95]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−1, −0.85]

TABLE 12. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=−0.2	$\lambda_{\mathbf{N}_2}(w_{11})$=−0.35
$\lambda_{\mathbf{N}_1}(w_{12})$=−0.7	$\lambda_{\mathbf{N}_2}(w_{12})$=−0.3
$\lambda_{\mathbf{N}_1}(w_{21})$=−0.4	$\lambda_{\mathbf{N}_2}(w_{21})$=−0.5
$\lambda_{\mathbf{N}_1}(w_{22})$=−0.8	$\lambda_{\mathbf{N}_2}(w_{22})$=−0.45

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.65, -0.5], [-0.9, -0.8]\} = [-0.65, -0.5],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.6, -0.4] \leq [-0.65, -0.5]$, which is non-sensical.

Now define two fuzzy sets $\lambda_{\mathbf{N_1}}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 12. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.3,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.4, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.45.$$

We note that $\lambda_{\mathbf{N_1}}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^{-}_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^{-}_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}),$$

which imply $-0.4 \in [-0.5, -0.3]$.

Similarly, for $\mathcal{N}_2{}^I = (\hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N}_2^{\mathbf{C}}})$, when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^{-}_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^{-}_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^{+}_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}),$$

which imply $0.5 \in [-0.6, -0.4]$. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.2, -0.3\} = -0.3,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.3$, which is non-sensical.

Therefore, the R-union of two INCLS need not be an INCLS. □

Remark 4.7. Finally, we show that the R-union of two exterior $\mathcal{N}$-cubic linear spaces need not be an exterior $\mathcal{N}$-cubic linear space. From Example 4.5 we can observe that intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces do not satisfy the first condition of $\mathcal{N}$-cubic linear spaces and in Example 4.7 we can observe that union of two $\mathcal{N}$-fuzzy linear spaces do not satisfy the second condition of $\mathcal{N}$-cubic linear spaces. Hence,

$$(\mathcal{N}_1 \cap \mathcal{N}_2)_R^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{c}}} \cap \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{c}}}, \lambda_{\mathbf{N_1}^{\mathbf{c}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{c}}})$$

need not be an ENCLS.

5. Conclusion

We find huge literature for dealing the uncertain problems like fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, cubic sets, $\mathcal{N}$-fuzzy sets, and $\mathcal{N}$-cubic sets. But in all most all cases we see these sets are properly used for applications in algebra and topology. In order, to extend this idea to linear spaces, we in this paper have introduced the notion of $\mathcal{N}$-cubic linear spaces which also handles the negative features of certain things like side effects of certain medicine. The main rationale of this paper is to extend the idea of $\mathcal{N}$-cubic sets to $\mathcal{N}$-cubic linear spaces and discuss in detail two types of $\mathcal{N}$-cubic linear spaces called ENCLS and INCLS with examples. We also discuss the basic operations like P-union (resp. intersection) and R-union (resp. intersection) intersection of $\mathcal{N}$-cubic linear spaces, ENCLS and INCLS. Sooner, different aggregation operators can be dealt with $\mathcal{N}$-cubic linear spaces. We will define Pythagorean fuzzy linear spaces by using the idea presented in this paper and [16].

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References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96.

[2] K. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems **31** (1989), 343–349.

[3] K. S. Abdukhalikov, M. S. Tulenbaev and U. U. Umirbaev, *On fuzzy bases of vector spaces*, Fuzzy Sets and Systems **63** (1994), 201–206.

[4] M. Gulistan, S. Rashid, Y. B. Jun and S. Kadry, *N-cubic sets and aggregation operators*, Journal of Intelligent and Fuzzy Systems **37**(4) (2019), 5009–5023.

[5] Y. B. Jun, J. Kavikumar and K.-S. So, *N-ideals of subtraction algebras*, Commun. Korean Math. Soc. **25**(2) (2010), 173–184.
[6] Y. B. Jun, C. S. Kim and K. O. Kang, *Cubic sets*, Ann. Fuzzy Math. Inform. **4**(1) (2012), 83–98.
[7] Y. B. Jun, C. S. Kim and M. S. Kang, *Cubic subalgebras and ideals of bck/bci-algebras*, Far East Journal of Mathematical Sciences **44**(2) (2010), 239–250.
[8] A. K. Katsaras and D. B. Liu, *Fuzzy vector spaces and fuzzy topological vector spaces*, J. Math. Anal. Appl. **58** (1977), 135–146.
[9] P. Lubczonok, *Fuzzy vector spaces*, Fuzzy Sets and Systems **38**(3) (1990), 329–343.
[10] G. Lubczonok and V. Murali, *On flags and fuzzy subspaces of vector spaces*, Fuzzy Sets and Systems **125**(2) (2002), 201–207.
[11] S. Nanda, *Fuzzy fields and fuzzy linear spaces*, Fuzzy Sets and Systems **33**(2) (1989), 257–259.
[12] T. Senapati, C. S. Kim, M. Bhowmik and M. Pal, *Cubic subalgebras and cubic closed ideals of b-algebras*, Fuzzy Information and Engineering **7**(2) (2015), 129–149.
[13] T. Senapati and K. P. Shum, *Cubic implicative ideals of bck-algebras*, Missouri J. Math. Sci. **29**(2) (2017), 125–138.
[14] T. Senapati, Y. B. Jun and K. P. Shum, *Cubic set structure applied in up-algebras*, Discrete Math. Algorithms Appl. **10**(4) (2018), Paper ID 1850049.
[15] T. Senapati and K. P. Shum, *Cubic commutative ideals of bck-algebras*, Missouri J. Math. Sci. **30**(1) (2018), 5–19.
[16] T. Senapati and R. R. Yager, *Some new operations over fermatean fuzzy numbers and application of fermatean fuzzy wpm in multiple criteria decision making*, Informatica **30**(2) (2019), 391–412.
[17] S. Vijayabalaji and S. Sivaramakrishnan, *A cubic set theoretical approach to linear space*, Abstr. Appl. Anal. **523**(129) (2015), Article ID 523129, 8 pages.
[18] G. Wenxiang and L. Tu, *Fuzzy linear spaces*, Fuzzy Sets and Systems **49**(3) (1992), 377–380.
[19] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.
[20] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Inform. Sci. **8** (1975), 199–249.

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BOUNDEDNESS OF L-INDEX IN JOINT VARIABLES FOR SUM OF ENTIRE FUNCTIONS

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Abstract. In the paper, we present sufficient conditions of boundedness of **L**-index in joint variables for a sum of entire functions, where $\mathbf{L}:\mathbb{C}^n\to\mathbb{R}^n_+$ is a continuous function, $\mathbb{R}_+=(0,+\infty)$. They are applicable to a very wide class of entire functions because for every entire function F in $\mathbb{C}^n$ with bounded multiplicities of zero points there exists a positive continuous function $\mathbf{L}$ such that F has bounded **L**-index in joint variables. Our propositions are generalizations of Pugh's result obtained for entire functions of one variable of bounded index.

1. Introduction

Let us introduce a main definition. Let $l:\mathbb{C}\to\mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+=(0,+\infty)$. An entire function f is said to be of bounded l-index [15,25] if there exists an integer m, independent of z, such that for all p and all $z\in\mathbb{C}$, $\frac{|f^{(p)}(z)|}{l^p(z)p!}\leq\max\left\{\frac{|f^{(s)}(z)|}{l^s(z)s!}:0\leq s\leq m\right\}$. The least such integer m is called the l-index of $f(z)$ and is denoted by $N(f,l)$. If $l(z)\equiv 1$, then we obtain a definition of function of *bounded index* [16] and in this case we denote $N(f):=N(f,1)$.

In 1970, W. J. Pugh and S. M. Shah [22] posed some questions on properties of entire functions of bounded index. One of these questions is following. II. Classes of functions of bounded index: is the sum (or product) of two functions of bounded index also of bounded index?

Later W. J. Pugh [21] proved that class of entire functions of bounded index is not closed under the operation of addition of the functions. He presented an example of

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two functions, for which its sum is a function of unbounded index. Also, there were deduced conditions providing index boundedness for sum of entire functions, when one addend is a function of bounded index. His example was based on the fact that every entire function with bounded multiplicities of zeros has unbounded index. Moreover, bounded multiplicities of zeros of the entire function $f : \mathbb{C} \to \mathbb{C}$ is necessary and sufficient condition for existence of some positive continuous function $l : \mathbb{C} \to \mathbb{R}_+$ such that f has bounded l-index [13].

There are two approaches to introduce concept of index boundedness in multidimensional complex space. The first approach uses directional derivatives in the definition. It generates a concept of entire function of bounded L-index in direction [4, 7], where $L : \mathbb{C}^n \to \mathbb{R}^n_+$ is a positive continuous function. And the second approach uses all possible partial derivatives in the definition. It leads to a concept of entire function of bounded $\mathbf{L}$-index in joint variables [3, 9], where $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a positive continuous vector-valued function. Pugh's example and his theorem was generalized for entire functions of bounded L-index in direction (see [8, 11]).

Of course, the similar question can be posed for entire functions of bounded $\mathbf{L}$-index in joint variables: What are sufficient conditions that sum of entire functions of bounded $\mathbf{L}$-index in joint variables is also a function of bounded $\mathbf{L}$-index in joint variables?

In [10], there were generalized Pugh's example and the sufficient conditions for this class of functions, if $\mathbf{L} \equiv \mathbf{1}$, i.e., for entire functions of bounded index in joint variables. Here we will formulate and prove theorems which contain sufficient conditions for arbitrary positive continuous vector-function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}_+$.

Note that for every entire function F with bounded multiplicities of zero points [12, 13] there exists a positive continuous function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ such that F is of bounded $\mathbf{L}$-index in joint variables. Thus, the concept of bounded $\mathbf{L}$-index in joint variables allows studying properties of very wide class of entire functions.

The concepts of bounded L-index in a direction and bounded $\mathbf{L}$-index in joint variables have applications in analytic theory of partial differential equations. A connection between these classses of entire functions is partially established in [6, 9]. They allow investigating properties of entire solutions of partial differential equations [4, 7] and their system [19]. Index boundedness of entire solution yields some sharp growth estimates, uniform distribution of zeros, regular behavior of its derivatives, etc. There is also known such a result [23] that if entire functions f and g satisfy differential equations with some additional conditions, then $f + g$ will be of bounded index. Besides, another objects of investigations in theory of bounded index are functions analytic in a polydisc [2], in a ball [5] or in Cartesian product of a disc and a complex plane [1].

2. Notations, Definitions and Auxiliary Results

Let us introduce some standard notations in theory of entire functions of several variables. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be n-dimensional real and complex vector spaces, respectively, $n \in \mathbb{N}$. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$. For $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ let us write $\|K\| = k_1 + \dots + k_n$, $K! = k_1! \cdots k_n!$. For $A = (a_1, \dots, a_n) \in \mathbb{C}^n$, $B = (b_1, \dots, b_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions $A \pm B = (a_1 \pm b_1, \dots, a_n \pm b_n)$, $AB = (a_1 b_1, \dots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}$. For A, $B \in \mathbb{R}^n$ $\max\{A, B\} = (\max\{a_1, b_1\}, \dots, \max\{a_n, b_n\})$, a notation $A < B$ means that $a_j < b_j$ for all $j \in \{1, \dots, n\}$. Similarly, the relation $A \leq B$ is defined.

For $R = (r_1, \dots, r_n)$ we denote by $\mathbb{D}^n(z^0, R) := \{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j \in \{1, \dots, n\}\}$ the polydisc, by $\mathbb{T}^n(z^0, R) := \{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j \in \{1, \dots, n\}\}$ its skeleton and by $\mathbb{D}^n[z^0, R] := \{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j \in \{1, \dots, n\}\}$ the closed polydisc.

For a partial derivative of entire function $F(z) = F(z_1, \dots, z_n)$ we will use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}, \quad \text{where } K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous functions of variable $z \in \mathbb{C}^n$, $j \in \{1, 2, \dots, n\}$.

An entire function $F(z)$ is called a function of bounded $\mathbf{L}$-index in joint variables [3,9], if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}. \tag{2.1}$$

The least integer m for which inequality (2.1) holds is called $\mathbf{L}$-index in joint variables of the function F and is denoted by $N(F, \mathbf{L})$. If $l_j(z_j) \equiv 1$, $j \in \{1, 2, \dots, n\}$, then the entire function is called a function of bounded index (in joint variables) [14, 17, 18, 20, 24].

For $R \in \mathbb{R}_+^n$, $j \in \{1, \dots, n\}$ and $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ we define

$$\begin{aligned}
\lambda_{1,j}(z_0, R) &= \inf \left\{ l_j(z)/l_j(z^0) \colon z \in D^n \left[z^0, R/\mathbf{L}(z^0) \right] \right\}, \quad \lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{C}^n} \lambda_{1,j}(z_0, R), \\
\lambda_{2,j}(z_0, R) &= \sup \left\{ l_j(z)/l_j(z^0) \colon z \in D^n \left[z^0, R/\mathbf{L}(z^0) \right] \right\}, \quad \lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{C}^n} \lambda_{2,j}(z_0, R), \\
\Lambda_k(R) &= (\lambda_{k,j}(R), \dots, \lambda_{k,n}(R)), \quad k \in \{1, 2\}.
\end{aligned}$$

By Q^n we denote a class of functions $\mathbf{L}(z)$ which for some $R^0 \in \mathbb{R}_+^n$ satisfy the condition

$$0 < \Lambda_1(R^0) \leq \Lambda_2(R^0) < +\infty. \tag{2.2}$$

Note that if (2.2) holds for some R_0 then it is valid for all $R \in \mathbb{R}_+^n$.

We need the following proposition.

Theorem 2.1 ([9]). *Let $\mathbf{L} \in Q^n$. An entire function F has bounded $\mathbf{L}$-index in joint variables if and only if for any R', $R'' \in \mathbb{R}^n_+$, $\mathbf{0} < R' < R''$, there exists a number $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$*

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R''}{\mathbf{L}(z^0)}\right)\right\} \leq p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, \frac{R'}{\mathbf{L}(z^0)}\right)\right\}. \tag{2.3}$$

Lemma 2.1 ([3]). *If $\mathbf{L} \in Q^n$, then for every $j \in \{1, \ldots, n\}$ and for every fixed $z^* \in \mathbb{C}^n$ $|z_j| l_j(z^* + z_j \mathbf{1}_j) \to \infty$ as $|z_j| \to \infty$.*

3. Main Result

Theorem 3.1. *Let $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ be a continuous function, F, G be entire functions in $\mathbb{C}^n$, which obey the following conditions:*

a) *$G(z)$ has bounded $\mathbf{L}$-index in joint variables with $N(G, \mathbf{L}) = N < +\infty$;*

b) *there exists $\alpha \in (0, 1)$ such that for all $z \in \mathbb{C}^n$ and for every $\|P\| \geq N + 1$, $P \in \mathbb{N}^n$,*

$$\frac{|G^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq \alpha \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N\right\}; \tag{3.1}$$

c) *for some $z^0 \in \mathbb{C}^n$, $F(z^0) \neq 0$, and every $z \in \mathbb{C}^n$ one has*

$$\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, \frac{2R}{\mathbf{L}(z)}\right)\right\} \leq \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N\right\}, \tag{3.2}$$

where $r_j = |z_j - z_j^0| l_j(z)$, $R = (r_1, \ldots, r_n)$;

d) *one of the following conditions is valid: either exists $c \geq 1$ for all $z \in \mathbb{C}^n$ such that $|z_j - z_j^0| l_j(z) \leq 1$ for some $j \in \{1, \ldots, n\}$ one has*

$$\frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}}{|F(z^0)|} \leq c < +\infty,$$

or $\mathbf{L} \in Q^n$.

Then for every $\varepsilon \in \mathbb{C}$, $|\varepsilon| \leq \frac{1-\alpha}{2c}$ the function

$$H(z) = G(z) + \varepsilon F(z) \tag{3.3}$$

has bounded $\mathbf{L}$-index in joint variables and $N(H, \mathbf{L}) \leq N$.

Proof. The proof uses methods and ideas from [8, 10, 21]. One should observe that for $\mathbf{L} \in Q^n$ by Lemma 2.1 the set $A := \{z \in \mathbb{C}^n\colon |z_j - z_j^0| l_j(z) \leq 1$ for some $j \in \{1, \ldots, n\}\}$ is bounded. Then there exits $c \geq 1$ such that for every $z \in A$ the inequality

$$\frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, 2\max\left\{\frac{\Lambda_2(\mathbf{1})}{\mathbf{L}(z^0)}, \frac{R}{\mathbf{L}(z)}\right\}\right)\right\}}{|F(z^0)|} \leq c < +\infty \tag{3.4}$$

holds.

We write Cauchy's formula for the entire function $F(z)$

$$\frac{F^{(P)}(z)}{P!} = \frac{1}{(2\pi i)^n} \int\limits_{z'\in\mathbb{T}^n(z,R/\mathbf{L}(z))} \frac{F(z')}{(z'-z)^{P+\mathbf{1}}} dz'. \tag{3.5}$$

For the chosen $r_j = |z_j - z_j^0| l_j(z)$ we have

$$\frac{r_j}{l_j(z)} = |z_j' - z_j| \geq |z_j' - z_j^0| - |z_j - z_j^0| = |z_j' - z_j^0| - \frac{r_j}{l_j(z)}.$$

Hence,

$$|z_j' - z_j^0| \leq \frac{2r_j}{l_j(z)}. \tag{3.6}$$

From (3.5) it follows that

$$\begin{aligned} \frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq & \frac{1}{(2\pi)^n \mathbf{L}^P(z)} \cdot \frac{\mathbf{L}^{P+\mathbf{1}}(z)}{R^{P+\mathbf{1}}} \prod_{j=1}^n \frac{2\pi r_j}{l_j(z)} \max\left\{|F(z')| : z' \in \mathbb{T}^n(z, R/\mathbf{L}(z))\right\} \\ \leq & \frac{1}{R^P} \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}. \end{aligned} \tag{3.7}$$

If $R^P > 1$, then (3.7) means that

$$\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}. \tag{3.8}$$

Let $R^P \in (0,1]$. Then for some $j \in \{1,\ldots,n\}$, $r_j = |z_j - z_j^0| l_j(z) \in (0,1]$. Putting $r_j = 1$ for those j in (3.5) and (3.6) and $R' = \max\{\mathbf{1}, R\}$, we similarly deduce

$$\begin{aligned} \frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \leq & \frac{1}{R'^P} \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R'/\mathbf{L}(z))\right\} \\ = & \frac{1}{R'^P} \frac{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R'/\mathbf{L}(z))\}}{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}} \\ & \times \max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\} \\ \leq & \frac{\max\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}}{R'^P |F(z^0)|} \\ & \times \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\} \\ \leq & c \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}, \end{aligned} \tag{3.9}$$

where

$$c = \sup_{\substack{z\in\mathbb{C}^n, \exists j \in\{1,\ldots,n\} \\ |(z_j - z_j^0) l_j(z)| \leq 1}} \frac{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\}}{|F(z^0)|}.$$

If $\mathbf{L} \in Q^n$, then $\sup\left\{\frac{l_j(z^0)}{l_j(z)} : z^0 \in \mathbb{D}^n(z, \frac{\mathbf{1}}{\mathbf{L}(z)})\right\} \le \lambda_{2,j}(\mathbf{1})$. This means that $l_j(z) \ge \frac{l_j(z^0)}{\lambda_{2,j}(\mathbf{1})}$. Using the inequality, we choose

$$c := \sup_{\substack{z\in\mathbb{C}^n, \exists j\in\{1,\dots,n\}\\ |(z_j-z_j^0)l_j(z)|\le 1}} \frac{\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, 2\max\left\{\frac{\Lambda_2(\mathbf{1})}{\mathbf{L}(z^0)}, \frac{R}{\mathbf{L}(z)}\right\}\right)\right\}}{|F(z^0)|} \ge 1$$

in (3.9). In view of (3.8) and (3.9), one has

$$\frac{|F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le c\max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}, \tag{3.10}$$

for all $P \in \mathbb{Z}_+^n$.

We differentiate equality (3.3) p times, $\|P\| = p \ge N+1$, and apply consequently (3.1), (3.10) and (3.2) to the obtained equality

$$\begin{aligned}\frac{|H^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le& \frac{|G^{(P)}(z)|}{P!\mathbf{L}^P(z)} + \frac{|\varepsilon||F^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le \alpha\max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N\right\}\\ &+ c|\varepsilon|\max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\}\\ \le&(\alpha + c|\varepsilon|)\max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N\right\}.\end{aligned} \tag{3.11}$$

If $\|S\| \le N$ then (3.10) is true with $P = S$, but (3.1) does not hold. Therefore, the differentiation of (3.3) give us the lower estimate

$$\begin{aligned}\frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} \ge& \frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} - \frac{|\varepsilon||F^{(S)}(z)|}{S!\mathbf{L}^S(z)}\\ \ge& \frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} - c|\varepsilon|\max\left\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R/\mathbf{L}(z))\right\},\end{aligned} \tag{3.12}$$

where $\|S\| \le N$. From (3.2) and (3.12) we conclude

$$\max\left\{\frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\} \ge (1 - c|\varepsilon|)\max\left\{\frac{|G^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\}. \tag{3.13}$$

If $c|\varepsilon| < 1$, then (3.11) and (3.13) yield

$$\frac{|H^{(P)}(z)|}{P!\mathbf{L}^P(z)} \le \frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|}\max_{0\le s\le N}\left\{\frac{|H^{(S)}(z)|}{S!\mathbf{L}^S(z)} : \|S\| \le N\right\},$$

for $\|P\| \ge N+1$. Assume that $\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|} \le 1$. Then $|\varepsilon| \le \frac{1-\alpha}{2c}$. For this ε the function H has bounded $\mathbf{L}$-index in joint variables with $N(H, \mathbf{L}) \le N$. Proof of Theorem 3.1 is complete. □

Remark 3.1. Every entire function F with $N(F, \mathbf{L}) = 0$ obeys inequality (3.4) (see proof of necessity of Theorem 3 in [9]).

If $\mathbf{L} \in Q^n$, then condition b) in Theorem 3.1 is always satisfied. The next theorem is valid.

Theorem 3.2. *Let $\mathbf{L} \in Q^n$, $\alpha \in (0,1)$ and F, G be entire functions in $\mathbb{C}^n$, which satisfy the following conditions:*

a) $G(z)$ has bounded $\mathbf{L}$-index in joint variables;

b) for some $z^0 \in \mathbb{C}^n$, $F(z^0) \neq 0$, and every $z \in \mathbb{C}^n$ the inequality holds

$$\max\left\{|F(z')| : z' \in \mathbb{T}^n\left(z^0, \frac{2R}{\mathbf{L}(z)}\right)\right\} \leq \max\left\{\frac{|G^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\},$$

where $r_j = |z_j - z_j^0| l_j(z)$, $R = (r_1, \ldots, r_n)$.

If $|\varepsilon| \leq \frac{1-\alpha}{2c}$, then the function

$$H(z) = G(z) + \varepsilon F(z)$$

has bounded $\mathbf{L}$-index in joint variables, with $N(H, L) \leq N(G_\alpha, \mathbf{L}_\alpha)$, where $G_\alpha(z) = G(z/\alpha)$, $\mathbf{L}_\alpha(z) = \mathbf{L}(z/\alpha)$.

Proof. Condition b) in Theorem 3.1 always is obeyed with $N(G_\alpha, \mathbf{L}_\alpha)$ instead $N = N(G, \mathbf{L})$, where $G_\alpha(z) = G(z/\alpha)$, $\mathbf{L}_\alpha(z) = \mathbf{L}(z/\alpha)$, $\alpha \in (0,1)$. Indeed, by Theorem 2.1, inequality (2.3) holds for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{z}{\alpha}$ instead z^0, z respectively in (2.3), we obtain

$$\begin{aligned} &\max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, \frac{R''\alpha}{\mathbf{L}(z^0/\alpha)}\right)\right\} \\ \leq &p_1 \max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, \frac{R'\alpha}{\mathbf{L}(z^0/\alpha)}\right)\right\}. \end{aligned} \tag{3.14}$$

By Theorem 2.1 inequality (3.14) yields that $G_\alpha(z) = G(z/\alpha)$ is of bounded $\mathbf{L}_\alpha$-index in joint variables and vice versa. Therefore, for each $\|P\| \geq N(G_\alpha, \mathbf{L}_\alpha) + 1$ and $\alpha \in (0,1)$ we have

$$\begin{aligned} \frac{|G_\alpha^{(P)}(z)|}{P!\mathbf{L}_\alpha^P(z)} =&\frac{|G^{(P)}(z/\alpha)|}{\alpha^{\|P\|}P!\mathbf{L}^P(z/\alpha)} \leq \max\left\{\frac{|G_\alpha^{(S)}(z)|}{S!\mathbf{L}_\alpha^S(z)} \colon\ \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\} \\ =&\max\left\{\frac{|G^{(S)}(z/\alpha)|}{\alpha^{\|S\|}S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\}. \end{aligned}$$

Hence, for all $z \in \mathbb{C}^n$

$$\begin{aligned} \frac{|G^{(P)}(z/\alpha)|}{P!\mathbf{L}^p(z/\alpha)} \leq &\max\left\{\frac{\alpha^{\|P\|-\|S\|}|G(z/\alpha)|}{S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\} \\ \leq &\alpha \max\left\{\frac{|G^{(S)}(z/\alpha)|}{S!\mathbf{L}^S(z/\alpha)} : \|S\| \leq N(G_\alpha, \mathbf{L}_\alpha)\right\}. \end{aligned} \tag{3.15}$$

Thus, from inequality (3.15) it follows (3.1). □

It is easy to verify that $N(G_\alpha, \mathbf{L}_\alpha) \leq N(G, \mathbf{L})$ for $\alpha \in (0,1)$. Thus, $N(G_\alpha, \mathbf{L}_\alpha)$ in Theorem 3.2 can be replaced by $N(G, \mathbf{L})$.

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References

[1] A. I. Bandura, O. B. Skaskiv and V. L. Tsvigun, *Some characteristic properties of analytic functions in* $\mathbb{D}\times\mathbb{C}$ *of bounded* $\mathbf{L}$*-index in joint variables*, Bukovyn. Mat. Zh. **6** (2018), 21–31.

[2] A. Bandura, N. Petrechko and O. Skaskiv, *Maximum modulus in a bidisc of analytic functions of bounded* $\mathbf{l}$*-index and an analogue of Hayman's theorem*, Math. Bohem. **143** (2018), 339–354.

[3] A. Bandura and O. Skaskiv, *Asymptotic estimates of entire functions of bounded* $\mathbf{L}$*-index in joint variables*, Novi Sad J. Math. **48** (2018), 103–116.

[4] A. Bandura and O. Skaskiv, *Boundedness of the* l*-index in a direction of entire solutions of second order partial differential equation*, Acta Comment. Univ. Tartu. Math. **22** (2018), 223–234.

[5] A. Bandura and O. Skaskiv, *Sufficient conditions of boundedness of* $\mathbf{L}$*-index and analog of Hayman's theorem for analytic functions in a ball*, Stud. Univ. Babeş-Bolyai Math. **63** (2018), 483–501.

[6] A. Bandura and O. Skaskiv, *Analytic functions in the unit ball of bounded* l*-index in joint variables and of bounded* l*-index in direction: a connection between these classes*, Demonstr. Mathem. **52** (2019), 82–87.

[7] A. Bandura, O. Skaskiv and P. Filevych, *Properties of entire solutions of some linear pde's*, J. Appl. Math. Comput. Mech. **16** (2017), 17–28.

[8] A. I. Bandura, *Sum of entire functions of bounded* l*-index in direction*, Mat. Stud. **45** (2016), 149–158.

[9] A. I. Bandura, M. T. Bordulyak and O. B. Skaskiv, *Sufficient conditions of boundedness of* l*-index in joint variables*, Mat. Stud. **45** (2016), 12–26.

[10] A. I. Bandura and N. V. Petrechko, *Sum of entire functions of bounded index in joint variables*, Electr. J. Math. Anal. Appl. **6**(2) (2018), 60–67.

[11] A. I. Bandura, *Product of two entire functions of bounded* L*-index in direction is a function with the same class*, Bukovyn. Mat. Zh. **4**(1–2) (2016), 8–12.

[12] A. I. Bandura and O. B. Skaskiv, *Iyer's metric space, existence theorem and entire functions of bounded* $\mathbf{L}$*-index in joint variables*, Bukovyn. Mat. Zh. **5**(3–4) (2017), 8–14 (in Ukrainian).

[13] M. T. Bordulyak, *A proof of Sheremeta conjecture concerning entire function of bounded* l*-index*, Mat. Stud. **12** (1999), 108–110.

[14] G. J. Krishna and S. M. Shah, *Functions of bounded indices in one and several complex variables*, in: *Mathematical Essays Dedicated to A. J. Macintyre*, Ohio University Press, Athens, Ohio, 1970, 223–235.

[15] A. D. Kuzyk and M. N. Sheremeta, *Entire functions of bounded* l*-distribution of values*, Math. Notes. **39** (1986), 3–8.

[16] B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Proc. Sympos. Pure Math. **11** (1968), 298–307.

[17] F. Nuray and R. F. Patterson, *Entire bivariate functions of exponential type*, Bull. Math. Sci. **5** (2015), 171–177.

[18] F. Nuray and R. F. Patterson, *Multivalence of bivariate functions of bounded index*, Le Matematiche **70** (2015), 225–233.

[19] F. Nuray and R. F. Patterson, *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*, Mat. Stud. **49** (2018), 67–74.

[20] R. F. Patterson and F. Nuray, *A characterization of holomorphic bivariate functions of bounded index*, Math. Slovaca **67** (2017), 731–736.

[21] W. J. Pugh, *Sums of functions of bounded index*, Proc. Amer. Math. Soc. **22** (1969), 319–323.
[22] W. J. Pugh and S. M. Shah, *On the growth of entire functions of bounded index*, Pacific J. Math. **33** (1970), 191–201.
[23] R. Roy and S. M. Shah, *Sums of functions of bounded index and ordinary differential equations*, Complex Var. Elliptic Equ. **12** (1989), 95–100.
[24] M. Salmassi, *Functions of bounded indices in several variables*, Indian J. Math. **31** (1989), 249–257.
[25] M. Sheremeta, *Analytic Functions of Bounded Index*, Mathematical Studies, Monograph Series **6**, VNTL Publishers, Lviv, 1999.

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HANKEL DETERMINANTS FOR A NEW SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

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ABSTRACT. Using the operator $L(a,c)$ defined by Carlson and Shaffer, we defined a new subclass of analytic functions $ML(\lambda,a,c)$. The well known Fekete-Szegö problem, upper bound of Hankel determinant of order two, and coefficient bound of the fourth coefficient is determined. Our investigation generalises some previous results obtained in different articles.

1. INTRODUCTION

We denote by $\mathcal{H}(\mathbb{D})$ the class of functions which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{P}$ be the well-known class of *Carathéodory functions*, that is $P \in \mathcal{H}(\mathbb{D})$ with the power series expansion

$$P(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in \mathbb{D}, \tag{1.2}$$

and $\operatorname{Re} P(z) > 0$ for all $z \in \mathbb{D}$.

For two functions $f, g \in \mathcal{H}(\mathbb{D})$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function $\psi \in \mathcal{H}(\mathbb{D})$, with $|\psi(z)| < 1$,

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$z \in \mathbb{D}$ and $\psi(0) = 0$, such that $f(z) = g(\psi(z))$ for all $z \in \mathbb{D}$. In particular, if g is univalent in $\mathbb{D}$ then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $h_s(z) = \sum_{k=0}^{\infty} a_{k,s}\, z^k$, $s = 1, 2$, which are analytic in $\mathbb{D}$, then the well-known *Hadamard* (*or convolution*) *product* of h_1 and h_2 is given by

$$(h_1 * h_2)(z) := \sum_{k=0}^{\infty} a_{k,1}\, a_{k,2}\, z^k, \quad z \in \mathbb{D}.$$

The *Carlson-Shaffer operator* [2] $L(a,c) : \mathcal{A} \to \mathcal{A}$ is defined by

$$L(a,c)f(z) := \widetilde{\varphi}(a,c;z) * f(z), \quad z \in \mathbb{D}, \tag{1.3}$$

where

$$\widetilde{\varphi}(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad z \in \mathbb{D},\ a \in \mathbb{C},\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mathbb{Z}_0^- := \{\dots, -2, -1, 0\},$$

is the incomplete beta function and $(t)_k$ denotes the *Pochhammer symbol* (or the *shifted factorial*) defined in terms of the *Gamma function* by

$$(t)_k := \frac{\Gamma(t+k)}{\Gamma(t)} = \begin{cases} t(t+1)(t+2)\cdots(t+k-1), & \text{if } k \in \mathbb{N} := \{1, 2, \dots\}, \\ 1, & \text{if } k = 0. \end{cases}$$

For $f \in \mathcal{A}$ is given by (1.1) one can see by using (1.3) that

$$L(a,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1}\, z^{k+1}, \quad z \in \mathbb{D},$$

and

$$zL'(a,c)f(z) = aL(a+1,c)f(z) - (a-1)L(a,c)f(z), \quad z \in \mathbb{D}.$$

Remark 1.1. Next we will emphasize a few special cases of the operator $L(a,c)$, as follows:

(i) $L(a,a)f(z) = f(z)$;

(ii) $L(2,1)f(z) = zf'(z)$;

(iii) $L(3,1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$;

(iv) $L(m+1,1)f(z) =: \mathcal{D}^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z)$, $m \in \mathbb{Z}$, $m > -1$, is the well-known *Ruscheweyh derivative* of f [22];

(v) $L(2, 2-\mu)f(z) =: \Omega_z^{\mu} f(z)$, $0 \le \mu < 1$, is the well-known *Owa-Srivastava fractional differential operator* [18].

For the function $f \in \mathcal{A}$ of the form (1.1) Noonan and Thomas [16] defined q-th Hankel determinant as

$$\mathcal{H}_{q,k}(f) := \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \dots & a_{k+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \dots & a_{k+2q-2} \end{vmatrix}, \quad a_1 = 1,\ q, k \in \mathbb{N}.$$

The above determinant $\mathcal{H}_{q,k}(f)$ has been studied by several authors, for example, Pommerenke [19], Noonan and Thomas [16], Ehrenborg [4] and Noor [17].

These authors studied the Hankel determinant in their own developed way: for instance Noor [17] studied the rate of growth of $\mathcal{H}_{q,k}$ as $k \to \infty$ for functions of the form (1.1) with bounded boundary rotation. Unlike to Noor, Ehrenborg [4] has studied different order Hankel determinants taking a family of exponential polynomials. Layman's article [11] gave some ideas on Hankel transform of an integer sequence, and the article discusses some properties of the transform for integer sequences.

For $k = 1$, $q = 2$, $a_1 = 1$ and $k = q = 2$ the Hankel determinant simplifies to the functionals $|a_3 - a_2^2|$ and $|a_2a_4 - a_3^2|$, called Hankel determinants of order two, denoted by $\wedge_1 := \mathcal{H}_{2,1}(f)$ and $\wedge_2 := \mathcal{H}_{2,2}(f)$, respectively. It is well-known (see Duren [3]) that if f is given by (1.1) and is univalent in $\mathbb{D}$, then $\wedge_1 \leq 1$ occurs, and this result is sharp.

For $\mathcal{T} \subset \mathcal{A}$, to find a sharp (best possible) upper bound of $\widetilde{\wedge}_c := |a_3 - c\, a_2^2|$ for the subclass $\mathcal{T}$ is generally called *Fekete-Szegö problem* for the subclass $\mathcal{T}$, where c is a real or a complex number. There are some subclasses of univalent functions, such that the starlike functions, convex functions and close-to-convex functions, for which the problem of finding sharp upper bounds for the functional $\widetilde{\wedge}_c$ was completely solved (see [5, 8–10]). For the family of analytic functions $\mathcal{R}$, such that for $f \in \mathcal{R}$ we have $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$, Janteng et al. [6, 7] have found the sharp upper bound to the second Hankel determinant $\wedge_2$. For initial work on the class $\mathcal{R}$ one may refer to the article of MacGregor [15].

In our paper we have defined a subclass of $\mathcal{A}$ using the concept of subordination and the linear operator $L(a, c)$.

Definition 1.1. Let $ML(\lambda, a, c)$ denotes the subclass of $\mathcal{A}$, members of which are of the form (1.1) and satisfy the subordination condition

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} \prec \sqrt{1+z}, \tag{1.4}$$

with $\sqrt{1+z}\Big|_{z=0} = 1$ or equivalently

$$\left| \left[\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} \right]^2 - 1 \right| < 1, \quad z \in \mathbb{D},$$

where $a \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $0 \leq \lambda \leq 1$.

Remark 1.2. (i) We will discuss the geometrical significance of the class $ML(\lambda, a, c)$. If we set $h(z) = \sqrt{1+z}$, $z \in \mathbb{D}$, with $h(0) = 1$, and denote

$$\omega := h(e^{i\theta}) = \sqrt{1 + e^{i\theta}}, \quad \theta \in [0, 2\pi] \setminus \{\pi\},$$

this yields $\omega^2 - 1 = e^{i\theta}$ or $|\omega^2 - 1| = 1$. Letting $\omega = u + iv$, $u, v \in \mathbb{R}$, we deduce that

$$\left(u^2 + v^2\right)^2 = 2\left(u^2 - v^2\right).$$

Thus, $h(\mathbb{D})$ is the region bounded by the right-half of the *Bernoulli's lemniscate* given by $\left\{u+iv\in\mathbb{C}:\left(u^2+v^2\right)^2=2\left(u^2-v^2\right)\right\}$, which implies that the functions in $ML(\lambda,a,c)$ have a positive real part.

(ii) Using the point (i) of the Remark 1.1, for $a=c$ we denote $ML(\lambda):=ML(\lambda,a,a)$, and member of this class satisfies the subordination condition

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z}\prec\sqrt{1+z},$$

with $\left.\sqrt{1+z}\right|_{z=0}=1$ or equivalently

$$\left|\left[\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z}\right]^2-1\right|<1,\quad z\in\mathbb{D}.$$

(iii) Remark that the subclass

$$ML(0)=SL^*:=\left\{f\in\mathcal{A}:\left|\left[\frac{zf'(z)}{f(z)}\right]^2-1\right|<1,\ z\in\mathbb{D}\right\}$$

was introduced and studied by Sokól and Stankiewicz [25], and Raza and Mallik [21] determined the upper bound of third Hankel determinant for the class SL^*. Also, the subclass $ML(1):=\left\{f\in\mathcal{A}:\left|[f'(z)]^2-1\right|<1,\ z\in\mathbb{D}\right\}$ was studied by Sahoo and Patel [23].

In our work we have used the techniques of Libera and Zlotkiewicz [12] and Koepf [9], combined with the help of MAPLE™ software to find an upper bound of $\widetilde{\wedge}_\mu$ and $\wedge_2$, and of the coefficient a_4 for the functions belonging to the class $ML(\lambda,a,c)$.

2. Preliminaries

To establish our main results, we shall need the followings lemmas. The first lemma is the well-known *Carathéodory's lemma* (see also [20, Corollary 2.3.]).

Lemma 2.1 ([1]). *If $P\in\mathcal{P}$ and given by* (1.2)*, then $|p_k|\leq 2$ for all $k\geq 1$ and the result is best possible for the function $P_*(z)=\frac{1+\rho z}{1-\rho z}$, $|\rho|=1$.*

The next lemma gives us a majorant for the coefficients of the functions of the class $\mathcal{P}$, and more details may be found in [14, Lemma 1].

Lemma 2.2 ([13]). *Let the function P given by* (1.2) *be a member of the class $\mathcal{P}$. Then*

$$\left|p_2-\nu p_1^2\right|\leq 2\max\left\{1,|2\nu-1|\right\},\quad \textit{where } \nu\in\mathbb{C}. \tag{2.1}$$

The result is sharp for the functions given by

$$P^*(z)=\frac{1+\rho^2z^2}{1-\rho^2z^2}\quad \textit{and}\quad P_*(z)=\frac{1+\rho z}{1-\rho z},\quad |\rho|=1.$$

Lemma 2.3 ([13]). *Let the function P given by* (1.2) *be a member of the class $\mathcal{P}$. Then*

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right] \tag{2.2}$$

and

$$p_3 = \frac{1}{4}\left[p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)(1 - |x|^2)z\right], \tag{2.3}$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$.

Other details regarding the above lemma my be found in [13], relations (3.9) and (3.10).

3. Main Results

In our first result we will determine an upper bound for $\widetilde{\Lambda}_\mu$, and this tends to solve the Fekete-Szegö problem for the subclass $ML(\lambda, a, c)$.

Theorem 3.1. *For $f \in ML(\lambda, a, c)$ and is in the form given by* (1.1) *then, for any $\mu \in \mathbb{C}$ we have*

$$\begin{aligned}\left|a_3 - \mu\, a_2^2\right| \leq &\frac{|(c)_2|}{|(a)_2|} \cdot \frac{1}{2(2+\lambda)}\\ &\times \max\left\{1,\ \frac{|(3\lambda-1)(1+\lambda)a(c+1) + 2\mu(2+\lambda)c(a+1)|}{4(1+\lambda)^2|a(c+1)|}\right\}.\end{aligned} \tag{3.1}$$

Proof. If $f \in ML(\lambda, a, c)$, from (1.4) it follows that there exists a function $\psi \in \mathcal{H}(\mathbb{D})$ satisfying the conditions $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathbb{D}$, such that

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} = \sqrt{1 + \psi(z)}, \quad z \in \mathbb{D}. \tag{3.2}$$

Setting

$$P(z) := \frac{1+\psi(z)}{1-\psi(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in \mathbb{D},$$

then $P \in \mathcal{P}$. From the above relation, we get

$$\psi(z) = \frac{P(z) - 1}{P(z) + 1}, \quad z \in \mathbb{D},$$

and from (3.2) it follows that

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} = \left(\frac{2P(z)}{1+P(z)}\right)^{\frac{1}{2}}, \quad z \in \mathbb{D}. \tag{3.3}$$

It is easy to show that

$$\left(\frac{2P(z)}{1+P(z)}\right)^{\frac{1}{2}} = 1+\frac{1}{4}p_1 z+\left(\frac{1}{4}p_2-\frac{5}{32}p_1^2\right)z^2 + \left(\frac{1}{4}p_3-\frac{5}{16}p_1p_2+\frac{13}{128}p_1^3\right)z^3+\cdots, \quad z\in\mathbb{D},$$

and identifying the coefficients of z, z^2 and z^3 in (3.3) we deduce that

$$a_2=\frac{c}{a}\cdot\frac{p_1}{4(1+\lambda)}, \tag{3.4}$$

$$a_3=\frac{(c)_2}{(a)_2}\cdot\frac{1}{4(2+\lambda)}\left[p_2-\frac{(7\lambda+3)}{8(1+\lambda)}p_1^2\right], \tag{3.5}$$

$$a_4=\frac{(c)_3}{(a)_3}\cdot\frac{1}{4(3+\lambda)}\left[p_3-\frac{7\lambda^2+16\lambda+7}{4(1+\lambda)(2+\lambda)}p_1p_2+\frac{25\lambda^2+40\lambda+13}{32(1+\lambda)(2+\lambda)}p_1^3\right]. \tag{3.6}$$

Thus, from (3.4) and (3.5) we get

$$\left|a_3-\mu\, a_2^2\right|=\frac{1}{4(2+\lambda)}\cdot\frac{|(c)_2|}{|(a)_2|}\left|p_2-\left[\frac{(7\lambda+3)(\lambda+1)a(c+1)+2\mu(2+\lambda)c(a+1)}{8(1+\lambda)^2a(c+1)}\right]p_1^2\right|,$$

which with the aid of the inequality (2.1) of Lemma 2.2 yields the required estimate (3.1). □

For $a=c$ the above theorem reduces to the following special case.

Corollary 3.1. *If $f\in ML(\lambda)$ and is given by* (1.1), *then for any $\mu\in\mathbb{C}$ we have*

$$\left|a_3-\mu\, a_2^2\right|\le\frac{1}{2(2+\lambda)}\max\left\{1,\ \frac{|(3\lambda-1)(1+\lambda)+2\mu(2+\lambda)|}{4(1+\lambda)^2}\right\}.$$

If we take $\mu\in\mathbb{R}$ in Theorem 3.1 we get the next special case.

Corollary 3.2. *If the function $f\in ML(\lambda,a,c)$ and is given by* (1.1), *with $\mu\in\mathbb{R}$ and $a>c\ge0$, then*

$$\left|a_3-\mu\, a_2^2\right|\le\begin{cases}\dfrac{a(c+1)(3\lambda-1)(\lambda+1)+2\mu c(a+1)(2+\lambda)}{8(\lambda+1)^2a(c+1)(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \mu<\delta_1,\\ \dfrac{1}{2(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \delta_1\le\mu\le\delta_2,\\ -\dfrac{a(c+1)(3\lambda-1)(\lambda+1)+2\mu c(a+1)(2+\lambda)}{8(\lambda+1)^2a(c+1)(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \mu>\delta_2,\end{cases}$$

where

$$\delta_1:=-\frac{(7\lambda+3)(\lambda+1)}{2(2+\lambda)}\cdot\frac{a(c+1)}{c(a+1)}\quad\text{and}\quad\delta_2:=\frac{(\lambda+1)(\lambda+5)}{2(2+\lambda)}\cdot\frac{a(c+1)}{c(a+1)}.$$

Remark 3.1. (i) Putting $\lambda=1$ in Corollary 3.1 and Corollary 3.2 we get the recent results due to Sahoo and Patel [23, Theorem 2.1] and [23, Corollary 2.2], respectively.

(ii) For $\lambda = 0$, Corollary 3.1 and Corollary 3.2 reduce to the results of Raza and Malik [21, Theorem 2.1] and [21, Theorem 2.2], respectively.

The next result deals with an upper bound of $\wedge_2$ for the subclass $ML(\lambda, a, c)$.

Theorem 3.2. *For $a \geq c > 0$, if the function f given by* (1.1) *belongs to the class $ML(\lambda, a, c)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{4(2+\lambda)^2}. \tag{3.7}$$

Proof. If $f \in ML(\lambda, a, c)$, using a similar proof like in the proof of Theorem 3.1, from (3.4), (3.5) and (3.6) we get

$$a_2a_4 - a_3^2 = k_1p_1^4 + k_2p_1^2p_2 + k_3p_1p_3 + k_4p_2^2,$$

where

$$\begin{aligned}
k_1 &= \frac{25\lambda^2 + 40\lambda + 13}{512(1+\lambda)^2(2+\lambda)(3+\lambda)} \cdot \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} - \left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{16(2+\lambda)^2}\left(\frac{7\lambda+3}{8(1+\lambda)}\right)^2,\\
k_2 &= \frac{7\lambda+3}{64(2+\lambda)^2(1+\lambda)}\left(\frac{(c)_2}{(a)_2}\right)^2 - \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} \cdot \frac{7\lambda^2+16\lambda+7}{64(1+\lambda)^2(2+\lambda)(3+\lambda)},\\
k_3 &= \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} \cdot \frac{1}{16(1+\lambda)(3+\lambda)},\\
k_4 &= -\left[\left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{16(2+\lambda)^2}\right].
\end{aligned}$$

Using the relations (2.2) and (2.3) of Lemma 2.3, we get

$$\begin{aligned}
&\left|a_2a_4 - a_3^2\right| \qquad (3.8)\\
&= \left|Ap_1^4 + B\left(4-p_1^2\right)xp_1^2 + \left[\frac{k_4}{4}\left(4-p_1^2\right) - \frac{k_3}{4}p_1^2\right]\left(4-p_1^2\right)x^2\right.\\
&\left.\quad + \frac{k_3}{2}p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\right|, \qquad (3.9)
\end{aligned}$$

with $|x| \leq 1$, $|z| \leq 1$ and

$$\begin{aligned}
A :=& \frac{1}{4}\left(4k_1 + 2k_2 + k_3 + k_4\right) = \frac{c(c)_2}{a(a)_2\left[1024(a+1)(a+2)(2+\lambda)^2(1+\lambda)^2(3+\lambda)\right]}\\
&\times \Big[(-4ac - 13c + a - 8)\lambda^3 + (-11ac - 11a - 40c - 22)\lambda^2\\
&+ (19ac + 36c + 21a + 41)\lambda + (3ac + 3c + 5a + 9)\Big],\\
B :=& \frac{1}{2}\left(k_2 + k_3 + k_4\right) = \frac{c(c)_2\Big[3(c-a)\lambda^2 + (ac - 6a + 9c + 2)\lambda - 5ac - 7a\Big]}{a(a)_2\Big[128(1+\lambda)(2+\lambda)^2(3+\lambda)(a+1)(a+2)\Big]}.
\end{aligned}$$

Since $P \in \mathcal{P}$ it follows that $P\left(e^{-i \arg p_1} z\right) \in \mathcal{P}$, hence we may assume without loss of generality that $p := p_1 \geq 0$, and according to Lemma 2.1 it follows that $p \in [0,2]$. Now, using the triangle's inequality in (3.8) and substituting $|x| = t$ we get

$$\begin{aligned}\left|a_2 a_4 - a_3^2\right| \leq & |A|\, p^4 + |B| \left(4-p^2\right) p^2 t + \frac{|k_4|}{4} \left(4-p^2\right)^2 t^2 + \frac{|k_3|}{4} p^2 \left(4-p^2\right) t^2 \\ & + \frac{|k_3|}{2} p \left(4-p^2\right) (1-t^2) =: \mathcal{G}(p,t), \quad 0 \leq p \leq 2,\ 0 \leq t \leq 1.\end{aligned}$$

Next, we will find maximum of $\mathcal{G}(p,t)$ on the closed rectangle $[0,2] \times [0,1]$. Using the MAPLE™ software for the following code, where we denoted $C := k_4$ and $D = E := k_3$,

```
[> G:= abs(A)*p^4+abs(B)*(-p^2+4)*p^2*t+(1/4)*abs(C)*(-p^2+4)^2*t^2
+(1/4)*abs(D)*p^2*(-p^2+4)*t^2+(1/2)*abs(\mathbb{D})*p
*(-p^2+4)*(-t^2+1);
[> maximize(G, p = 0 .. 2, t = 0 .. 1, location);
```

we get

```
max(16 |A|, 4 |C|), {[{p = 2}, 16 |A|], [{p = 0, t = 1}, 4 |C|]}
```

that is

$$\max\left\{\mathcal{G}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = \max\{16|A|, 4|C|\}$$

and

$$16|A| = \mathcal{G}(2,t), \quad 4|C| = \mathcal{G}(0,1).$$

We will prove that under our assumption we have $4|C| \geq 16|A|$ and therefore

$$\max\left\{\mathcal{G}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = 4|C| = 4\,|k_4| = \mathcal{G}(0,1). \tag{3.10}$$

Letting $\alpha := \frac{c}{a} \cdot \frac{(c)_3}{(a)_3}$ and $\beta := \left(\frac{(c)_2}{(a)_2}\right)^2$, since $a \geq c > 0$ it follows that $\alpha \geq \beta > 0$, and first we will show that $A > 0$. A simple computation shows that

$$4A = 4k_1 + 2k_2 + k_3 + k_4 = \alpha\, \frac{5\lambda^2+1}{128(1+\lambda)^2(2+\lambda)(3+\lambda)} - \beta\, \frac{9\lambda^2-6\lambda+1}{256(1+\lambda)^2(2+\lambda)^2},$$

and using the fact that

$$\begin{aligned}&\frac{5\lambda^2+1}{128(1+\lambda)^2(2+\lambda)(3+\lambda)} - \frac{9\lambda^2-6\lambda+1}{256(1+\lambda)^2(2+\lambda)^2} \\ =&\frac{\lambda^3+19\lambda+(1-\lambda^2)}{256(1+\lambda)^2(2+\lambda)^2(3+\lambda)} > 0, \quad 0 \leq \lambda \leq 1,\end{aligned}$$

it follows that $A > 0$. Hence,

$$\begin{aligned}16|A| - 4|C| =&\alpha \left[\frac{5\lambda^2+1}{32(1+\lambda)^2(2+\lambda)(3+\lambda)}\right] - \beta \left[\frac{9\lambda^2-6\lambda+1}{64(1+\lambda)^2(2+\lambda)^2} + \frac{1}{4(2+\lambda)^2}\right] \\ =&\frac{\lambda^3(10\alpha-25\beta)+\lambda^2(20\alpha-101\beta)+\lambda(2\alpha-95\beta)+(4\alpha-51\beta)}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)},\end{aligned}$$

and since $0 \leq \lambda \leq 1$, each term of the numerator is not positive if

$$\frac{\alpha}{\beta} \leq \min\left\{\frac{25}{10}, \frac{101}{20}, \frac{95}{2}, \frac{51}{4}\right\} = \frac{25}{10},$$

which is equivalent to $3ac + a + 8c + 6 \geq 0$. This last inequality holds for all $a > 0$ and $c \geq 0$, and therefore $16|A| \leq 4|C|$. Since (3.10) was proved, the upper bound of $\mathcal{G}(p, t)$ on the closed rectangle $[0, 2] \times [0, 1]$ is attained at $p = 0$ and $t = 1$, which implies the inequality (3.7). $\square$

For $a = c$ Theorem 3.2 reduces to the next special case.

Corollary 3.3. *If the function f given by* (1.1) *belongs to the class $ML(\lambda)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4(2+\lambda)^2}.$$

Remark 3.2. (i) For $\lambda = 1$, Corollary 3.3 reduces to the result due to Sahoo and Patel [23, Theorem 2.2].

(ii) Taking $\lambda = 0$ in Corollary 3.3 we obtain the recent result of Raza and Malik [21, Theorem 2.4].

In our last result we found an upper bound of the fourth coefficient for the functions of $ML(\lambda, a, c)$.

Theorem 3.3. *If $a \geq c > 0$ and the function f given by* (1.1) *belongs to the class $ML(\lambda, a, c)$, then*

$$|a_4| \leq \frac{(c)_3}{(a)_3} \cdot \frac{1}{2(3+\lambda)}.$$

Proof. If $f \in ML(\lambda, a, c)$, using a similar proof like in the proof of Theorem 3.1, from (3.6) we obtain

$$a_4 = \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)}\left[p_3 - \frac{7\lambda^2+16\lambda+7}{4(1+\lambda)(2+\lambda)}p_1p_2 + \frac{25\lambda^2+40\lambda+13}{32(1+\lambda)(2+\lambda)}p_1^3\right]. \tag{3.11}$$

Replacing in (3.11) the values of p_2 and p_3 with those given by the relations (2.2) and (2.3), respectively, and denoting $p := p_1$ we get

$$\begin{aligned} a_4 =& \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)}\left[\frac{5\lambda^2+1}{32(1+\lambda)(2+\lambda)}p^3 - \frac{3\lambda^2+4\lambda-1}{8(1+\lambda)(2+\lambda)}\left(4-p^2\right)px\right. \\ &\left. - \frac{1}{4}\left(4-p^2\right)px^2 + \frac{1}{2}\left(4-p^2\right)(1-|x|^2)z\right], \end{aligned}$$

for some complex numbers x and z, with $|x| < 1$ and $|z| \leq 1$. Using the triangle's inequality and substituting $|x| = y$ we get

$$|a_4| \leq \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)} \times \left[\frac{5\lambda^2+1}{32(1+\lambda)(2+\lambda)} p^3 + \frac{|3\lambda^2+4\lambda-1|}{8(1+\lambda)(2+\lambda)} \left(4-p^2\right) py \right.$$
$$\left. + \frac{1}{4}\left(4-p^2\right) py^2 + \frac{1}{2}\left(4-p^2\right)\left(1-y^2\right) \right] =: \mathcal{T}(p,y), \quad 0 \leq p \leq 2,\ 0 \leq y \leq 1.$$

Now we will find the maximum of the function $\mathcal{T}(p,y)$ on the closed rectangle $[0,2] \times [0,1]$. Denoting

$$\mathcal{H}(p,y) := \frac{5\lambda^2+1}{32(1+\lambda)(2+\lambda)} p^3 + \frac{|3\lambda^2+4\lambda-1|}{8(1+\lambda)(2+\lambda)} \left(4-p^2\right) py$$
$$+ \frac{1}{4}\left(4-p^2\right) py^2 + \frac{1}{2}\left(4-p^2\right)\left(1-y^2\right),$$

and using the MAPLE™ software for the following code

```
[> H := (5*l^2+1)*p^3/((32*(1+l))*(2+l))
+abs(3*l^2+4*l-1)*(-p^2+4)*p*y/((8*(1+l))*(2+l))
+(1/4*(-p^2+4))*p*y^2+(1/2*(-p^2+4))*(-y^2+1);
[> maximize(H, p = 0 .. 2, y = 0 .. 1, location);
```

we get

```
max(2, (1/4)*(5*l^2+1)/((1+l)*(2+l))),
{[{p = 2}, (1/4)*(5*l^2+1)/((1+l)*(2+l))], [{p = 0, y = 0}, 2]}
```

that is

$$\max\left\{\mathcal{H}(p,y) : (p,y) \in [0,2] \times [0,1]\right\} = \max\left\{2, \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)}\right\},$$

and

$$2 = \mathcal{H}(0,0), \quad \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)} = \mathcal{H}(2,y).$$

A simple computation shows that $2 > \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)}$, whenever $\lambda \geq 0$, therefore

$$\max\left\{\mathcal{H}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = 2 = \mathcal{H}(0,0),$$

which implies that

$$\max\left\{\mathcal{T}(p,y) : (p,y) \in [0,2] \times [0,1]\right\} = \frac{(c)_3}{(a)_3} \cdot \frac{1}{2(3+\lambda)} = \mathcal{T}(0,0),$$

and the proof of our theorem is complete. □

Putting $a = c$ in Theorem 3.3 we get the next special case.

Corollary 3.4. *If the function f given by* (1.1) *belongs to the class $ML(\lambda)$, then*

$$|a_4| \leq \frac{1}{2(3+\lambda)}.$$

Remark 3.3. (i) For $\lambda = 1$, Corollary 3.4 reduces to the recent result due to Sahoo and Patel [23, Theorem 2.3].

(ii) Taking $\lambda = 0$ in Corollary 3.4 we get the result due to Sokół [24, Theorem 2].

References

[1] C. Carathéodory, *Über den variabilitätsbereich der Fourier'schen konstanten von positiven harmonischen funktionen*, Rend. Circ. Mat. Palermo **32** (1911), 193–217.

[2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15**(4) (1984), 737–745.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, USA, 1983.

[4] R. Ehrenborg, *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly **107** (2000), 557–560.

[5] M. Fekete and G. Szegö, *Eine Bemerkung über ungerade schlichte Funktionen*, J. Lond. Math. Soc. **8** (1933), 85–89.

[6] A. Janteng, S. A. Halim and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, Journal of Inequalities in Pure and Applied Mathematics **7**(2) (2006), Article ID 50.

[7] A. Janteng, S. A. Halim and M. Darus, *Estimate on the second Hankel functional for functions whose derivative has a positive real part*, Journal of Quality Measurement and Analysis **4**(1) (2008), 189–195.

[8] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[9] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc. **101**(1) (1987), 89–95.

[10] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions-II*, Arch. Math. (Basel) **49** (1987), 420–433.

[11] J. W. Layman, *The Hankel transform and some of its properties*, J. Integer Seq. **4** (2001), 1–11.

[12] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85**(2) (1982), 225–230.

[13] R. J. Libera and E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87**(2) (1983), 251–257.

[14] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, In: Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), *Proceedings of the Conference on Complex Analysis, Tianjin, 1992*, Int. Press, Cambridge, MA, 1994, 157–169.

[15] T. H. MacGregor, *Functions whose derivative have a positive real part*, Trans. Amer. Math. Soc. **104**(3) (1962), 532–537.

[16] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc. **223** (1976), 337–346.

[17] K. I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roumaine Math. Pures Appl. **28**(8) (1983), 731–739.

[18] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39** (1987), 1057–1077.

[19] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc. **41**(1) (1966), 111–122.

[20] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck & Ruprecht, Göttingen, 1975.
[21] M. Raza ans S. Malik, *Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl. **2013**(412) (2013), 1-8.
[22] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49**(1) (1975), 109–115.
[23] A. K. Sahoo and J. Patel, *Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, Int. J. Anal. Appl. **6**(2) (2014), 170–177.
[24] J. Sokół, *Coefficient estimates in a class of strongly starlike functions*, Kyungpook Math. J. **49**(2) (2009), 349–353.
[25] J. Sokół and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Folia scientiarum Universitatis Technicae Resoviensis **19** (1996), 101–105.

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ON FUZZY PRIMARY AND FUZZY QUASI-PRIMARY IDEALS IN LA-SEMIGROUPS

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Abstract. The purpose of this paper is to introduce the notion of a weakly fuzzy quasi-primary ideals in LA-semigroups, we study fuzzy primary, fuzzy quasi-primary, fuzzy completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Some characterizations of weakly fuzzy primary and weakly fuzzy quasi-primary ideals are obtained. Moreover, we investigate relationships between fuzzy completely primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Finally we show that a fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

1. Introduction

The concept of a fuzzy subset of a set was first considered by Zadeh [27] in 1965. In 1988, Zhang [28] studied prime L-fuzzy ideals and primary L-fuzzy ideals in rings where L is a completely distributive lattice. In 2012, Palanivelrajan and Nandakumar [18] introduced the definition and some operations of intuitionistic fuzzy primary and semiprimary ideals.

In 2010, Khan et al. [10] gave the concept of (α, β)-fuzzy interior ideals of AG-groupoids and gave some properties of AG -groupoids in terms of (α, β)-fuzzy interior ideals. In 2012, Khan et al. [14] introduced the concept of $(\in, \in \vee q_k)$-fuzzy bi-ideals, $(\in, \in \vee q_k)$-fuzzy left (right)-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals in AG-groupoids and characterized regular and intera-regular AG -groupoids in terms of the lower parts of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals in AG -groupoids. In 2013, Yaqoob [21] applied the interval valued intuitionistic fuzzy sets

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in regular LA-semigroups and characterized regular LA-semigroups by the properties of interval valued intuitionistic fuzzy left ideals, interval-valued intuitionistic fuzzy right ideal, interval valued intuitionistic fuzzy generalized bi-ideals and interval valued intuitionistic fuzzy bi-ideals. In 2014, Abdullah, Aslam and Amin [1] defined the concept of interval-valued (α, β)-fuzzy ideals, interval-valued and (α, β)-fuzzy generalized bi-ideals in LA-semigroups and characterized the lower part of interval-valued $(\in, \in \vee q)$-fuzzy left ideals, interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideals and interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideals in LA-semigroups. In 2015, Khan, Jun and Yousafzai [11] studied fuzzy left (right, two-sided) ideals, fuzzy (generalized) bi-ideals, fuzzy interior ideals, fuzzy $(1, 2)$-ideals and fuzzy quasi-ideals of right regular LA-semigroups and gave some properties of right regular LA-semigroups in terms of fuzzy left and fuzzy right ideals. In 2016, Yousafzai, Yaqoob and Zeb [26] introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals of ordered AG-groupoids and provided the basic theory for an intra-regular ordered AG-groupoid in terms of generalized fuzzy ideals. In 2018, Rehman et al. [19] studied lower and upper parts of $(\in, \in \vee q)$- fuzzy interior ideals and $(\in, \in \vee q)$-fuzzy bi-ideals in LA-semigroups. In 2019, Nasreen [17] characterized regular (intra-regular, both regular and intra-regular) ordered AG-groupoid in terms of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$. There are many mathematicians who added several results to the theory fuzzy LA-semigroups, see [5, 9, 20, 22, 24, 25].

In this study we followed lines as adopted in [2–4, 6–8, 13, 16, 23] and established the notion of fuzzy subsets of LA-semigroups. Specifically we characterize the fuzzy primary, fuzzy quasi-primary, fuzzy completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Moreover, we investigate relationships between fuzzy completely primary and weakly fuzzy quasi-primary ideals in LA-semigroups.

2. Preliminaries

In this section we refer to [12] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Recall that a function f from S to the unit interval $[0, 1]$ is a fuzzy subset of S.

Definition 2.1 ([12]). A fuzzy subset f of an LA-semigroup S is called a fuzzy LA-subsemigroup of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all x, y in S.

Recall that the LA-semigroup S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$, denoted also by S. Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined $f(x) \leq g(x)$ for all $x \in S$. $f \cap g$ and $f \cup g$ are fuzzy subsets of S defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$, $(f \cup g)(x) = \max\{f(x), g(x)\}$ for all $x \in S$. More generally, if $\{f_\alpha : \alpha \in \beta\}$ is a family of fuzzy subsets of S, then

$\bigcap_{\alpha\in\beta} f_\alpha$ and $\bigcup_{\alpha\in\beta} f_\alpha$ are defined as follows:

$$\left(\bigcap_{\alpha\in\beta} f_\alpha\right)(x) = \bigcap_{\alpha\in\beta} f_\alpha(x) = \inf\{f_\alpha(x) : \alpha\in\beta\},$$
$$\left(\bigcup_{\alpha\in\beta} f_\alpha\right)(x) = \bigcup_{\alpha\in\beta} f_\alpha(x) = \sup\{f_\alpha(x) : \alpha\in\beta\},$$

and will be the intersection and union of the family $\{f_\alpha : \alpha\in\beta\}$ of fuzzy subset of R. The product $f\circ g$ [12] is defined as follows:

$$(f\circ g)(x) = \begin{cases} \bigcup\limits_{x=yz} \min\{f(y), g(z)\}, & \text{exists } y,z\in S, \text{ such that } x=yz, \\ 0, & \text{otherwise.} \end{cases}$$

As is well known [12], this operation "$\circ$" is left invertive.

Lemma 2.1 ([12]). *If f, g and h are fuzzy subsets of an LA-semigroup S, then $(f\circ g)\circ h = (h\circ g)\circ f$.*

Proof. The proof is available in [12]. □

Lemma 2.2 ([12]). *Let f, g, h and k be any fuzzy subsets of an LA-semigroup S with left identity. Then the following properties hold:*

(a) $f\circ(g\circ h) = g\circ(f\circ h)$;
(b) $(f\circ g)\circ(h\circ k) = (k\circ h)\circ(g\circ f)$;
(c) $S\circ S = S$.

Proof. The proof is available in [12]. □

Recall that a fuzzy subset f of an LA-semigroup S is called a fuzzy left (right) ideal of S if $f(xy)\geq f(y)$ $(f(xy)\geq f(x))$ for all $x,y\in S$, if f is both fuzzy left and right ideal of S, then f is called a fuzzy ideal of S.

Remark 2.1. It is easy that f is a fuzzy ideal of an LA-semigroup S if and only if $f(xy)\geq \max\{f(x), f(y)\}$ for all x,y in S and any fuzzy left (right) ideal of S is a fuzzy LA-subsemigroup of S.

Lemma 2.3 ([12]). *Let f be a fuzzy subset of an LA-semigroup S. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S if and only if $f\circ f\subseteq f$.*
(b) *f is a fuzzy left ideal of S if and only if $S\circ f\subseteq f$.*
(c) *f is a fuzzy right ideal of S if and only if $f\circ S\subseteq f$.*
(d) *f is a fuzzy ideal of S if and only if $S\circ f\subseteq f$ and $f\circ S\subseteq f$.*

Proof. The proof is available in [12]. □

Theorem 2.1 ([12]). *Let I be a nonempty subset of an LA-semigroup S and let $f_I : S \to [0,1]$ be a fuzzy subset of S such that*

$$(f_I)(x) = \begin{cases} 1, & x \in I, \\ 0, & \textit{otherwise.} \end{cases}$$

Then the following properties hold.

(a) *I is an LA-subsemigroup of S if and only if f_I is a fuzzy LA-subsemigroup of S.*
(b) *I is a left ideal of S if and only if f_I is a fuzzy left ideal of S.*
(c) *I is a right ideal of S if and only if f_I is a fuzzy right ideal of S.*
(d) *I is an ideal of S if and only if f_I is a fuzzy ideal of S.*

Proof. The proof is available in [12]. □

3. Fuzzy Completely Primary Subsets of LA-Semigroups

In this section, we concentrate our study on the fuzzy completely primary subsets and fuzzy primary ideals of LA-semigroups and investigate their fundamental properties and mutual relationships. Finally, we prove that a fuzzy subset f of an AG-3-band is a fuzzy quasi-primary ideal of S if and only if f is a fuzzy primary ideal in S.

Definition 3.1. Let x and y be any elements of an LA-semigroup S. A fuzzy subset f of S is called fuzzy completely primary if $\max\{f(x), f(y^n)\} \geq f(xy)$ for some positive integer n.

We now present the following example satisfying above definition.

Example 3.1. Let $S = \{0,1,2\}$ be a set under the binary operation defined as in Table 1. Then S is an LA-semigroup. We define a fuzzy subset $f : S \to [0,1]$ by $f(x) = 0$

Table 1. LA-semigroup

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

for all $x \in S$. It is easy to show that f is a fuzzy completely primary subset of S.

In the light of the definition of fuzzy completely primary subsets on an LA-semigroup, we can obtain the following properties.

Theorem 3.1. *If P_i is a fuzzy completely primary subset of an LA-semigroup S, then $\bigcup_{i\in I} P_i$ is a fuzzy completely primary subset of S.*

Proof. Let x and y be any elements of an LA-semigroup S. Since P_i is a fuzzy completely primary subset of S, we have $P_i(xy) \leq \max\{P_i(x), P_i(y^n)\}$ for some positive integer n. Hence, since $\max\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\} \geq P_i(xy)$ for all $i \in I$

$$\max\left\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\right\} \geq \bigcup_{i\in I} P_i(xy).$$

Therefore, $\bigcup_{i\in I} P_i$ is a fuzzy completely primary subset of S. □

Recall that a left ideal P of an LA-semigroup S is called completely quasi-primary if for any two elements x, y of $S, xy \in P$ implies that either $x \in P$ or $y^n \subseteq P$ for some positive integer n.

Theorem 3.2. *Let S be an LA-semigroup S. Then the following properties hold.*

(a) *If P is a completely quasi-primary ideal of S, then f_P is a fuzzy completely primary left ideal of S.*
(b) *If P is a completely quasi-primary ideal of S, then tf_P is a fuzzy completely primary left of S.*

Proof. (a). Let P is a completely quasi-primary ideal of S. It follows from Theorem 2.1 (2) that f_P is a fuzzy left ideal of S. Let x and y be any elements of S. If $xy \notin P$, then $f_P(xy) = 0 \leq \max\{f_P(x), f_P(y^n)\}$ for some positive integer n. Next, if $xy \in P$, then $f_P(xy) = 1$. Since P is a completely quasi-primary ideal of S, we have $x \in P$ or $y^n \in P$ for some positive integer n and so we have $f_P(x) = 1$ or $f_P(y) = 1$. Therefore, $f_P(xy) = 1 = \max\{f_P(x), f_P(y)\}$ and hence f_P is a fuzzy completely primary left ideal of S.

(b) The proof is similar to (a). □

Next, define the notions of fuzzy quasi-primary ideals on an LA-semigroup S.

Definition 3.2. A fuzzy left ideal f of an LA-semigroup S is called fuzzy quasi-primary if for any two fuzzy left ideals g and h of S such that $g \circ h \subseteq f$ implies $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n.

Example 3.2. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be a set under the binary operation defined in Table 2. It is clear that S is an LA-semigroup. We define a fuzzy subset $f : S \to [0, 1]$ by

$$f(x) = \begin{cases} 1, & x \in \{1, 2, 4, 5, 6, 7\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is a fuzzy completely primary subset of S. But f is not a fuzzy quasi-primary ideal of S, since f is not a fuzzy left ideal of S.

Next, define the notions of fuzzy primary ideal on an LA-semigroup S.

Definition 3.3. A fuzzy ideal f of an LA-semigroup S is called fuzzy primary of S if for any two fuzzy ideals g and h of S such that $g \circ h \subseteq f$ implies $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n.

Table 2. LA-semigroup

$\cdot$	0	1	2	3	4	5	6	7	8
0	3	1	6	3	1	6	6	1	3
1	0	3	0	3	8	8	3	0	8
2	8	1	5	3	7	2	6	4	0
3	3	3	3	3	3	3	3	3	3
4	0	6	7	3	5	4	1	2	8
5	8	6	4	3	2	7	1	5	0
6	8	3	8	3	0	0	3	8	0
7	0	1	2	3	4	5	6	7	8
8	3	6	1	3	6	1	1	6	3

Remark 3.1. It is easy to see that every fuzzy primary ideal is fuzzy quasi-primary.

Recall that an LA-semigroup in which $(xx)\,x = x\,(xx) = x$ holds for all x is called an AG-3-band. It is easy to see that every fuzzy left ideal of an AG-3-band S is a fuzzy ideal.

Then we have the following result.

Theorem 3.3. *For an AG-3-band S, the following conditions are equivalent.*

(a) *f is a fuzzy quasi-primary ideal of S.*
(b) *f is a fuzzy primary ideal of S.*

Proof. It is obvious. □

4. Weakly Fuzzy Quasi-Primary Ideals of LA-Semigroups

In this section, we investigate some properties of weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 4.1. *Let A and B be any nonempty subsets of an LA-semigroup S and $t \in (0,1]$. Then the following properties hold:*

(a) $tf_A \circ tf_B = tf_{AB}$;
(b) $tf_A \cap tf_B = tf_{A\cap B}$;
(c) $tf_A \cup tf_B = tf_{A\cup B}$;
(d) $S \circ tf_A = tf_{SA}, tf_A \circ S = tf_{AS}$ *and* $S \circ (tf_A \circ S) = tf_{S(AS)}$.

Proof. It is obvious. □

Definition 4.1 ([12]). Let S be an LA-semigroup, $x \in S$ and $t \in (0,1]$. A fuzzy point x_t of S is defined by the rule that

$$x_t\,(y) = \begin{cases} t, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

It is accepted that x_t is a mapping from S into $[0,1]$, then a fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel.

Lemma 4.2. *Let A be a non-empty subset of an LA-semigroup S. If $t \in (0,1]$, then $tf_A = \bigcup_{a \in A} a_t$.*

Proof. It is obvious. □

Next, defines the notions of weakly fuzzy primary and weakly fuzzy quasi-primary ideals on an LA-semigroup S.

Definition 4.2. A fuzzy ideal f of an LA-semigroup S is called weakly fuzzy primary of S if for any two ideals A and B of S such that $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B^n \subseteq f$ for some positive integer n.

Definition 4.3. A fuzzy left ideal f of an LA-semigroup S is called weakly fuzzy primary of S if for any two left ideals A and B of S such that $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B^n \subseteq f$ for some positive integer n.

Remark 4.1. It is easy to see that every weakly fuzzy primary is weakly fuzzy quasi-primary.

Then we have the following result.

Theorem 4.1. *For an AG-3-band S, the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S.*
(b) *f is a weakly fuzzy primary ideal of S.*

Proof. It is straightforward by Theorem 3.3. □

Theorem 4.2. *Let P be a fuzzy left ideal of an LA-semigroup S with left identity. Then the following statements are equivalent.*

(a) *P is a weakly fuzzy quasi-primary ideal of S.*
(b) *For any $x, y \in S$ and $t \in (0,1]$ if $x_t \circ (S \circ y_t) \subseteq P$, then $x_t \in P$ or $y_t^n \in P$ for some positive integer n.*
(c) *For any $x, y \in S$ and $t \in (0,1]$ if $tf_x \circ tf_y \subseteq P$, then $x_t \in P$ or $y_t^n \in P$ for some positive integer n.*
(d) *If A and B are left ideals of S such that $tf_A \circ tf_B \subseteq P$, then $tf_A \subseteq P$ or $tf_B^n \subseteq P$ for some positive integer n.*

Proof. (a) $\Rightarrow$ (b). First assume that P is a weakly fuzzy quasi-primary ideal of S. Let x and y be any elements of S and $t \in (0,1]$. Since $x_t \circ (S \circ y_t) \subseteq P$, we have

$$\begin{aligned} tf_{(xe)S} \circ tf_{(ye)S} &= (tf_{xe} \circ S) \circ (tf_{ye} \circ S) \\ &= (tf_{xe} \circ tf_{ye}) \circ (S \circ S) \\ &= ((tf_x \circ tf_e) \circ (tf_y \circ tf_e)) \circ (S \circ S) \\ &= ((tf_x \circ tf_y) \circ (tf_e \circ tf_e)) \circ (S \circ S) \\ &= ((tf_e \circ tf_e) \circ (tf_y \circ tf_x)) \circ (S \circ S) \\ &= (tf_{ee} \circ (tf_y \circ tf_x)) \circ (S \circ S) \\ &= (tf_y \circ (tf_e \circ tf_x)) \circ (S \circ S) \\ &= (tf_y \circ tf_{ex}) \circ (S \circ S) \\ &= (S \circ S) \circ (tf_x \circ tf_y) \\ &= S \circ (tf_x \circ tf_y) \\ &= tf_x \circ (S \circ tf_y) \\ &= x_t \circ (S \circ y_t) \\ &\subseteq P. \end{aligned}$$

Then since P is a weakly fuzzy quasi-primary ideal of S, we have

$$x_t = tf_x = tf_{(ee)x} = tf_{(xe)e} \subseteq tf_{(xe)S} \subseteq P,$$

or $y_t^n = tf_{y^n} = tf_{((ee)y)^n} = tf_{((ye)e)^n} \subseteq tf_{((ye)S)^n} = tf^n_{(ye)S} \subseteq P$ for some positive integer n. Thus, $x_t \in P$ or $y_t^n \in P$ and so (a) implies (b).

(b) $\Rightarrow$ (c). Assume that (b) holds. Let x and y be any elements of S and $t \in (0,1]$. Since $tf_x \circ tf_y \subseteq P$, we have

$$\begin{aligned} x_t \circ (S \circ y_t) &\subseteq tf_x \circ (S \circ tf_y) \\ &= S \circ (tf_x \circ tf_y) \\ &\subseteq S \circ P \\ &\subseteq P. \end{aligned}$$

Thus, by hypothesis $x_t \in P$ or $y_t^n \in P$ for some positive integer n. Hence we obtain that (b) implies (c).

(c) $\Rightarrow$ (d). Assume that (c) holds. Let A and B be any left ideals of S. Then it follows from Theorem 2.1 (2) that tf_A and tf_B are fuzzy left ideals of S. Next, let $tf_A \circ tf_B \subseteq P$ such that $tf_B^n \nsubseteq P$ for all positive integer n. Otherwise, there exists $y \in B$ such that $y_t^n \notin P$ for all positive integer n. For any $x \in A$, by Lemma 4.1 and hypothesis

$$tf_x \circ tf_y = tf_{xy} \subseteq tf_{AB} = tf_A \circ tf_B \subseteq P.$$

Since $y_t^n \notin P$, we have $tf_x \subseteq P$ and so $x_t \in P$. By Lemma 4.2, it follows that $tf_A = \bigcup\limits_{x \in A} x_t$. Hence we obtain that (c) implies (d).

(d) $\Rightarrow$ (a). It is obvious. $\square$

As is easily seen, every weakly fuzzy primary ideal of an LA-semigroup S is a weakly fuzzy quasi-primary ideal of S. The following example shows that the converse of this property does not hold in general.

Example 4.1. Let $S = \{0, 1, 2, 3\}$ be a set under the binary operation defined as in Table 3. It is clear that S is an LA-semigroup. We define a fuzzy subset $f : S \to [0, 1]$

TABLE 3. LA-semigroup

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

by

$$f(x) = \begin{cases} 0.9, & x \in \{0\}, \\ 0.5, & x \in \{2\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is a weakly fuzzy quasi-primary ideal of S. But f is not a fuzzy quasi-primary ideal of S, since $g \circ h \subseteq f$, while $g \not\subseteq f$ and $h^n \not\subseteq f$ for all positive integer n, where

$$g(x) = \begin{cases} 0.9, & x \in \{0, 2\}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} 1, & x \in \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Let a and b be any elements of an LA-semigroup S with left identity. If f is a fuzzy quasi-primary subset of S, then* $\inf\{f(a(Sb))\} \le \max\{f(a), f(b^n)\}$ *for some positive integer n.*

Proof. Assume $\inf\{f(a(Sb))\} = m$. Let g and h be any fuzzy subsets of S such that

$$g(x) = \begin{cases} m, & x \in (ae)S, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} m, & x \in (be)S, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Theorem 2.1 (2) that g and h are fuzzy left ideals of S. If

$$(g \circ h)(x) = \bigcup_{x=yz} \min\{g(y), h(z)\} = m,$$

then there exist $u \in (ae)S$ and $v \in (be)S$ such that $uv = x$. Put $u = (ae)t$ and $v = (be)k$ for some $t, k \in S$. Then we have

$$\begin{aligned} f(x) =& f(uv) = f(((ae)t)((be)k)) = f(((ae)(be))(tk)) = f((kt)((be)(ae))) \\ \geq& f((be)(ae)) = f((ba)(ee)) = f((ee)(ab)) = f(e(ab)) = f(a(eb)) \\ \geq& \inf\{f(a(Sb))\} \\ =& m, \end{aligned}$$

so that $g \circ h \subseteq f$. Since f is a fuzzy quasi-primary subset of S, we have $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n. Therefore $g(a) = g((ee)a) = g((ae)e) = mf_{(ae)S}((ae)e) = m$ or

$$\begin{aligned} h^n(b^n) =& g^n((ee)b^n) = g^n((b^n e)e) = mf_{((be)S)}{}^n((b^n e)e) = mf_{(be)^n S^n}((b^n e)e) \\ =& mf_{(b^n e^n)S}((b^n e)e) = mf_{(b^n e)S}((b^n e)e) = m. \end{aligned}$$

But from $m = \max\{f(a), f(b^n)\} < \inf\{f(a(Sb))\} = m$, we have a contradiction. Thus it follows that $\inf\{f(a(Sb))\} \leq \max\{f(a), f(b^n)\}$ for some positive integer n. □

The following corollary can be easily deduced from the theorem.

Corollary 4.1. *Let a and b be any elements of an LA-semigroup S with left identity. If f is a fuzzy quasi-primary left ideal of S, then* $\inf\{f(a(Sb))\} = \max\{f(a), f(b^n)\}$ *for some positive integer n.*

Theorem 4.4. *Let P be a fuzzy left ideal of an LA-semigroup S with left identity and let x and y be any elements of S. If $P(xy) = \max\{P(x), P(y^n)\}$ for some positive integer n, then P is a weakly fuzzy quasi-primary ideal of S.*

Proof. Let x and y be any elements of S and $t \in (0,1]$. Suppose that x_t and y_t are fuzzy points of S such that $x_t \circ (S \circ y_t) \subseteq P$. Then we have $S \circ (xy)_t = S \circ (x_t \circ y_t) = x_t \circ (S \circ y_t) \subseteq P$ and so $P(xy) \geq t$. Since $P(xy) = \max\{P(x), P(y^n)\}$, we have $P(x) \geq t$ or $P(y^n) \geq t$ for some positive integer n. This implies that $x_t \in P$ or $y_t^n \in P$ and hence P is a weakly fuzzy quasi-primary ideal of S. □

Theorem 4.5. *Let P be a fuzzy completely primary subset of an LA-semigroup S with left identity. Then P is weakly fuzzy quasi-primary subset of S.*

Proof. We leave the straightforward proof to the reader. □

Theorem 4.6. *Let S be an LA-semigroup with left identity. Then the following conditions are equivalent.*

(a) *P is a weakly fuzzy quasi-primary subset of S.*
(b) *If $x, y \in S$, then $P(xy) \leq \max\{P(x), P(y^n)\}$ for some positive integer n.*

Proof. First assume that P is a weakly fuzzy quasi-primary subset of an LA-semigroup S with left identity. Let x and y be any elements of S. If $P(xy) > \max\{P(x), P(y^n)\}$, then there exists $t \in (0,1)$ such that $P(xy) > t > \max\{P(x), P(y^n)\}$. Thus, we have

$$x_t \circ (S \circ y_t) = S \circ (x_t \circ y_t) = S \circ (xy)_t \subseteq S \circ P \subseteq P.$$

Since P is a weakly fuzzy quasi-primary subset of S, we have $x_t \in P$ or $y_t^n \in P$ for some positive integer n, but $x_t \notin P$ and $y_t^n \notin P$, which is impossible. Therefore, $P(xy) \leq \max\{P(x), P(y^n)\}$.

Conversely, assume that (b) holds. Let x and y be any elements of S and $t \in (0,1]$. Suppose that x_t and y_t are fuzzy points of S such that $x_t \circ (S \circ y_t) \subseteq P$. Since

$$S \circ (xy)_t = S \circ (x_t \circ y_t) = x_t \circ (S \circ y_t) \subseteq P,$$

we have $P(xy) \geq t$. Then, since $P(xy) \leq \max\{P(x), P(y^n)\}$, we have $P(x) \geq t$ or $P(y^n) \geq t$ for some positive integer n and so $x_t \in P$ or $y_t^n \in P$. Therefore we obtain that, P is a weakly fuzzy quasi-primary subset of S. □

Theorem 4.7. *Let S be an LA-semigroup. Then the following conditions are equivalent.*

(a) *P is a quasi-primary ideal of S.*
(b) *f_P is a weakly fuzzy quasi-primary ideal of S.*

Proof. We leave the straightforward proof to the reader. □

Theorem 4.8. *Let f be a fuzzy subset of an LA-semigroup S. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S.*
(b) *The level subset $U(f,t)$ of f is a quasi-primary ideal of S for every $t \in \mathrm{Im}(f)$.*

Proof. First assume that f is a weakly fuzzy quasi-primary ideal of S. Let $t \in (0,1]$ and let a and b be any elements of S such that $ab \in U(f,t)$. Then we have $f(ab) \geq t$. Since $tf_a \circ tf_b = tf_{ab} \subseteq f$, we have $tf_a \subseteq f$ or $tf_{b^n} \subseteq f$ for some positive integer n and so $f(a) \geq t$ or $f(b^n) \geq t$. Thus, $a \in U(f,t)$ or $b^n \in U(f,t)$ and hence $U(f,t)$ is a quasi-primary ideal of S.

Conversely, assume that $U(f,t)$ is a quasi-prime ideal of S for every $t \in Im(f)$. Let a and b be any elements of S such that $tg_a \circ tg_b \subseteq f$. Since $tg_{ab} = tg_a \circ tg_b$, we have $f(ab) \geq tg_{ab}(ab) = t$ and so $ab \in U(f,t)$. By assumption, $a \in U(f,t)$ or $b^n \in U(f,t)$ for some positive integer n. Suppose $a_t \notin f$ and $b_t^n \notin f$ for all positive integer n. However, $t = a_t(a) > f(a)$ and $t = b_t^n(b^n) > f(b^n)$ and we have a contradiction. Therefore $a_t \in f$ or $b_t^n \in f$ and hence f is a weakly fuzzy quasi-primary ideal of S. □

5. Cartesian Product of Fuzzy Ideals of LA-Semigroups

In this section, we concentrate our study on the cartesian product of fuzzy ideals of an LA-semigroup and investigate their fundamental properties and mutual relationships. Finally we show that a fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

We start with the following theorem that gives a relation between cartesian product of fuzzy ideals in an LA-semigroup. Our starting points are the following definitions:

Let S_1 and S_2 be two LA-semigroups. Then

$$S_1 \times S_2 := \{(x, y) \in S_1 \times S_2 : x \in S_1, y \in S_2\},$$

and for any $(a, b), (c, d) \in S_1 \times S_2$ we define

$$(a, b)(c, d) := (ac, bd),$$

then $S_1 \times S_2$ is an LA-semigroup as well. Let $f : S_1 \to [0, 1]$ and $g : S_2 \to [0, 1]$ be two fuzzy subsets of LA-semigroups S_1 and S_2 respectively. Then the product of fuzzy subsets is denoted by $f \times g$ and defined as $f \times g : S_1 \times S_2 \to [0, 1]$, where $(f \times g)(x, y) = \min\{f(x), g(y)\}$.

In the light of the definition of cartesian product of fuzzy ideals in an LA-semigroup, we can obtain the following properties.

Lemma 5.1. *Let f and g be two fuzzy subsets of LA-simigroups S_1 and S_2, respectively and let $t \in (0, 1]$. Then $(f \times g)_t = f_t \times f_t$.*

Proof. Let f and g be two fuzzy subsets of LA-simigroups S_1 and S_2, respectively and let $t \in (0, 1]$. Next, let (x, y) be any element of $S_1 \times S_2$. Then we have

$$\begin{aligned}(x, y) \in f_t \times g_t &\Leftrightarrow x \in f_t \wedge y \in g_t \\ &\Leftrightarrow f(x) \geq t \wedge g(y) \geq t \\ &\Leftrightarrow \min\{f(x), g(y)\} \geq t \\ &\Leftrightarrow (f \times g)(x, y) \geq t \\ &\Leftrightarrow (x, y) \in (f \times g)_t.\end{aligned}$$

Hence, $(f \times g)_t = f_t \times f_t$. □

By Lemma 5.1, we have the following result.

Corollary 5.1. *Let $f_1, f_2, \ldots, f_n$ be any fuzzy subsets of LA-simigroups $S_1, S_2, \ldots, S_n$, respectively and let $t \in (0, 1]$. Then $\left(\prod_{i=1}^{n} f_i\right)_t = \prod_{i=1}^{n} (f_i)_t$.*

Proof. One can easily show by induction method. □

Theorem 5.1. *Let f_1 and f_2 be two fuzzy subsets of LA-semigroups S_1 and S_2, respectively. Then the following conditions are equivalent.*

(a) *$f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$.*
(b) *The level subset $(f \times g)_t$ of $f \times g$ is a completely primary ideal of $S_1 \times S_2$ for every $t \in \operatorname{Im}(f \times g)$.*

Proof. First assume that $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$. Let (x, y) and (m, n) be any elements of $S_1 \times S_2$ such that $(x, y)(m, n) \in (f \times g)_t$. Then we have $(f \times g)((x, y)(m, n)) \geq t$ and so $(f \times g)(xm, yn) \geq t$. Since $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$, we have

$$(f \times g)\left((x, y)(m, n)\right) \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\},$$

for some positive integer k. If $(f \times g)(x, y) \leq (f \times g)(m, n)^k$, then

$$t \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} = (f \times g)(m, n)^k.$$

Hence, we have $(f \times g)(m, n)^k \geq t$ and so $(m, n)^k \in (f \times g)_t$. Now if $(f \times g)(x, y) > (f \times g)(m, n)^k$, then

$$t \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} = (f \times g)(x, y).$$

Therefore, we obtain that $(f \times g)(x, y) \geq t$ and hence $(x, y) \in (f \times g)_t$. In any case, we have $(f \times g)_t$ is a completely primary ideal of $S_1 \times S_2$.

Conversely, assume that $(f \times g)_t$ is a completely primary ideal of $S_1 \times S_2$ for every $t \in Im(f \times g)$. Let (x, y) and (m, n) be any elements of $S_1 \times S_2$. Otherwise, $(f \times g)((x, y)(m, n)) \geq 0$. Since $(x, y)(m, n) \in (f \times g)_{(f \times g)((x,y)(m,n))}$, by hypothesis, we have $(x, y) \in (f \times g)_{(f \times g)((x,y)(m,n))}$ or $(m, n)^k \in (f \times g)_{(f \times g)((x,y)(m,n))}$ for some positive integer k. Thus, we have $(f \times g)(x, y) \geq (f \times g)((x, y)(m, n))$ or $(f \times g)(m, n)^k \geq (f \times g)((x, y)(m, n))$ and so we have

$$\max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} \geq (f \times g)\left((x, y)(m, n)\right).$$

Thus, it follows from Definition 3.1 that $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$. □

By Theorem 5.1, we have the following result.

Corollary 5.2. *Let $f_1, f_2, \ldots, f_n$ be any fuzzy subsets of LA-semigroups $S_1, S_2, \ldots, S_n$, respectively. Then the following conditions are equivalent.*

(a) *$\prod_{i=1}^n f_i$ is a fuzzy completely primary ideal of $\prod_{i=1}^n S_i$.*
(b) *The level subset $(\prod_{i=1}^n f_i)_t$ is a completely primary ideal of $\prod_{i=1}^n S_i$ for every $t \in \mathrm{Im}(\prod_{i=1}^n f_i)$.*

Proof. One can easily show by induction method. □

Lemma 5.2. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S_1.*
(b) *$f \times S_2$ is a fuzzy LA-subsemigroup of $S_1 \times S_2$.*

Proof. We leave the straightforward proof to the reader. □

By Lemma 5.2, we have the following result.

Corollary 5.3. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S_2.*
(b) *$S_1 \times f$ is a fuzzy LA-subsemigroup of $S_1 \times S_2$.*

Lemma 5.3. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_1.*
(b) *$f \times S_2$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $S_1 \times S_2$.*

Proof. Similar to the proof of Lemma 5.2. □

By Lemma 5.3, we have the following result.

Corollary 5.4. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_2.*
(b) *$S_1 \times f$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $S_1 \times S_2$.*

By Lemmas 5.2, 5.3 and Corollaries 5.3, 5.4, we have the following result.

Corollary 5.5. *Let f_i be a fuzzy subset of an LA-semigroup S_i. Then the following properties hold.*

(a) *f_i is a fuzzy LA-subsemigroup of S_i if and only if $S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a fuzzy LA-subsemigroup of $\prod_{i=1}^{n} S_i$.*
(b) *f_i is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_i if and only if $S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $\prod_{i=1}^{n} S_i$.*

Proof. One can easily show by induction method. □

The following theorem show that the fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

Theorem 5.2. *Let S_1 and S_2 be two LA-semigroups with left identities. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S_1.*
(b) *$f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.*

Proof. First assume that f is a weakly fuzzy quasi-primary ideal of S_1. Let (a, b) and (c, d) be any elements of $S_1 \times S_2$ such that $(ac, bd)_t = (a, b)_t \circ (c, d)_t \in f \times S_2$. Then we have $f(ac) = \min\{f(ac), 1\} = \min\{f(ac), S_2(bd)\} = (f \times S_2)(ac, bd) \geq t$ and so $f(ac) \geq t$. Obviously, $a_t \circ c_t = (ac)_t \in f$. By Theorem 4.2, $a_t \in f$ or $c_t^n \in f$ for some positive integer n, it is clear that $f(a) \geq t$ or $f(c^n) \geq t$ for some positive integer n. Thus, we have

$$(f \times S_2)(a, b) = \min\{f(a), S_2(b)\} \geq \min\{t, 1\} = t$$

or

$$(f \times S_2)(c, d)^n = (f \times S_2)(c^n, d^n) = \min\{f(c^n), S_2(d^n)\} \geq \min\{t, 1\} = t,$$

and so we have $(a,b)_t \in f \times S_2$ or $(c,d)_t^n \in f \times S_2$. Then it follows from Theorem 4.2 that $f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$ and so (a) implies (b).

Conversely, assume that $f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$. Let a and c be any elements of S_1 such that $(ac)_t = a_t \circ c_t \in f$. Then we have $t \leq f(ac) = \min\{f(ac), 1\} = \min\{f(ac), S_2(bd)\} = (f \times S_2)(ac, bd)$ and so $(f \times S_2)(ac, bd) \geq t$ for all $b, d \in S_2$. Obviously, $(a,b)_t \circ (c,d)_t = (ac, bd)_t \in f \times S_2$. By Theorem 4.2, $(a,b)_t \in f \times S_2$ or $(c,d)_t^n \in f \times S_2$ for some positive integer n, it is clear that $(f \times S_2)(a,b) \geq t$ or $(f \times S_2)(c,d)^n \geq t$ for some positive integer n. Thus we have

$$f(a) = \min\{f(a), 1\} = \min\{f(a), S_2(b)\} = (f \times S_2)(a,b) \geq t$$

or

$$f(c^n) = \min\{f(c^n), 1\} = \min\{f(c^n), S_2(d^n)\} = (f \times S_2)(c^n, d^n) = (f \times S_2)(c,d)^n \geq t,$$

and so we have $a_t \in f$ or $c_t^n \in f$. Then it follows from Theorem 4.2 that f is a weakly fuzzy quasi-primary ideal of S_1. □

By Theorem 5.2, we have the following result.

Corollary 5.6. *Let S_1 and S_2 be two LA-semigroups with left identities. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S_2.*
(b) *$S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.*

By Theorem 5.2 and Corollary 5.6, we have the following result.

Theorem 5.3. *Let S_i be an LA-semigroup with left identity. Then the following conditions are equivalent.*

(a) *f_i is a weakly fuzzy quasi-primary ideal of S_i.*
(b) *$S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a weakly fuzzy quasi-primary ideal of $\prod_{i=1}^n S_i$.*

Proof. One can easily show by induction method. □

References

[1] S. Abdullah, S. Aslam and N. U. Amin, *LA-semigroups characterized by the properties of interval valued (α, β)-fuzzy ideals*, J. Appl. Math. Inform. **32**(3-4) (2014), 405–426.

[2] S. Abdullah, M. Aslam, N. Amin and T. Khan, *Direct product of finite fuzzy subsets of LA-semigroups*, Ann. Fuzzy Math. Info. **3**(2) (2012), 281–292.

[3] S. Abdullah, M. Aslam, M. Imran and M. Ibrar, *Direct product of intuitionistic fuzzy sets in LA-semigroups-II*, Ann. Fuzzy Math. Info. **2**(2) (2011), 151–160.

[4] V. Amjad, F. Yousafzai and A. Iampan, *On generalized fuzzy ideals of ordered LA-semigroups*, Boletin de Matematicas **22**(1) (2015), 1–19.

[5] M. Aslam, S. Abdullah and S. Aslam, *Characterization of regular LA-semigroups by interval-valued $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy ideals*, Afr. Mat. **25** (2014), 501–518.

[6] B. Davvaz, M. Khan, S. Anis and S. Haq, *Generalized fuzzy quasi-ideals of an intraregular Abel-Grassmann's groupoid*, J. Appl. Math. **2012** (2012), Article ID 627075, 16 pages.

[7] Faisal, N. Yaqoob and A. Ghareeb, *Left regular AG-groupoids in terms of fuzzy interior ideals*, Afr. Mat. **24** (2013), 577–587.

[8] M. Khan, S. Anis and S. Lodhi, *A study of fuzzy Abel-Grassmann's groupoids*, International Journal of the Physical Sciences **7**(4) (2012), 584–592.

[9] M. Khan, V. L. Fotea and S. Kokab, $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideals of intra-regular Abel Grassmann's-groupoids*, Analele Stiintifice ale Universitatii Ovidius Constanta **22** (2014), 95–113.

[10] M. Khan, Y. B. Jun and T. Mahmood, *Generalized fuzzy interior ideals in Abel Grassmann's groupoids*, Int. J. Math. Math. Sci. **2010** (2010), Article ID 838392, 14 pages.

[11] M. Khan, Y. B. Jun and F. Yousafzai, *Fuzzy ideals in right regular LA-semigroups*, Hacet. J. Math. Stat. **44**(3) (2015), 569–586.

[12] M. Khan and M. N. A. Khan, *Fuzzy Abel-Grassmann's groupoids*, Advances in Fuzzy Mathematics **3** (2010), 349–360.

[13] M. Khan and M. N. Khan, *On fuzzy Abel Grassmann's groupoids*, Advances in Fuzzy Mathematics **5**(3) (2010), 349–360.

[14] A. Khan, N. H. Sarmin, F. M. Khan and B. Davvaz, *Regular AG-groupoids characterized by fuzzy ideals*, Iran. J. Sci. Technol. **36**(2) (2012), 97–113.

[15] A. Khan, M. Shabir, Y. B. Jun, *Generalized fuzzy Abel Grassmann's groupoids*, Int. J. Fuzzy Syst. **12**(4) (2010), 340–349

[16] X. Ma, J. Zhan, M. Khan and T. Aziz, *Some characterizations of intra-regular Abel Grassmann's groupoids*, Italian Journal of Pure and Applied Mathematics **32** (2014), 329–346.

[17] K. Nasreen, *Characterizations of non associative ordered semigroups by the properties of their fuzzy ideals with thresholds* $(\alpha, \beta]$, Prikl. Diskretn. Mat. **43** (2019), 37–59.

[18] M. Palanivelrajan and S. Nandakumar, *Intuitionistic fuzzy primary and semiprimary ideal*, Indian Journal of Applied Research **1**(5) (2012), 159–160.

[19] N. Rehman, N. Shah, M. I. Ali and A. Ali, *Generalised roughness in* $(\in, \in \vee q)$*-fuzzy substructures of LA-semigroups*, Journal of the National Science Foundation of Sri Lanka **46**(3) (2018), 465–473.

[20] T. Shah and N. Kausar, *Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals*, Theoret. Comput. Sci. **529** (2014), 96–110.

[21] N. Yaqoob, *Interval valued intuitionistic fuzzy ideals of regular LA-semigroups*, Thai J. Math. **11**(3) (2013), 683–695.

[22] N. Yaqoob, R. Chinram, A. Ghareeb and M. Aslam, *Left almost semigroups characterized by their interval valued fuzzy ideals*, Afr. Mat. **24** (2013), 231–245.

[23] P. Yiarayong, *On product of fuzzy semiprime ideals in* Γ*-LA-semigroups*, Gazi University Journal of Science **29**(2) (2016), 491–502.

[24] F. Yousafzai, A. Khan, V. Amjad and A. Zeb, *On fuzzy fully regular ordered AG-groupoids*, Journal of Intelligent & Fuzzy Systems **26** (2014), 2973–2982.

[25] F. Yousafzai, W. Khan, A. Khan and K. Hila, *I-V fuzzy set theory applied on po-LA-semigroups*, Afr. Mat. **27** (2016), 23–35.

[26] F. Yousafzai, N. Yaqoob and A. Zeb, *On generalized fuzzy ideals of ordered AG-groupoids*, Int. J. Mach. Learn. & Cyber. **7** (2016), 995–1004.

[27] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1956), 338–353.

[28] Y. Zhang, *Prime L-fuzzy ideals and primary L-fuzzy ideals*, Fuzzy Sets & Syst. **27**(3) (1988), 345–350.

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ITERATIVE CONTINUOUS COLLOCATION METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. This paper is concerned with the numerical solution of nonlinear Volterra integral equations. The main purpose of this work is to provide a new numerical approach based on the use of continuous collocation Lagrange polynomials for the numerical solution of nonlinear Volterra integral equations. It is shown that this method is convergent. The results are compared with the results obtained by other well-known numerical methods to prove the effectiveness of the presented algorithm.

1. Introduction

In this paper, we study a numerical method based on iterative continuous collocation method for the solution of nonlinear Volterra integral equations of the form,

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds, \quad t \in I = [0, T], \tag{1.1}$$

where the functions f, K are sufficiently smooth.

The integral equations are often involved in various fields such as physics and biology (see, for example [5, 14, 15]), and they also occur as reformulations of other mathematical problems, such as ordinary differential equations and partial differential equations (see [14]).

There has been a growing interest in the numerical solution of Equation (1.1) (see, for example, [2, 3, 7, 8, 10, 11, 13, 15–17, 19, 21]) such as, Chebyshev approximation [2], Adomian's method [3, 15], Taylor polynomial approximations [21], homotopy

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perturbation method [10], the series expansion method [11], fixed point method [16], Haar wavelet method [17], rationalized Haar functions method [19]. Moreover, many collocation methods for approximating the solutions for Equation (1.1) have been developed recently (see, [5, 9, 18, 20, 22]) such as Lagrange spline collocation method [5], cubic B-spline collocation method [9], quintic B-spline collocation method [18], Taylor collocation method [20], and sinc-collocation method for Volterra integral equations is used in [22].

The numerical solution of these equations has a high computational cost due to the nonlinearity and most of the collocation methods for nonlinear Volterra integral equations transform (1.1) into a system of nonlinear algebraic equations.

This paper is concerned with the continuous piecewise polynomial collocation method based on the use of Lagrange polynomials. Our goal is to develop an iterative explicit solution to approximate the solution of nonlinear Volterra integral equation (1.1).

The main advantages of the current collocation method are that it is direct and there is no algebraic system to be solved, which makes the proposed algorithm very effective, easy to implement and the calculation cost low.

This paper is organized as follows. In Section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1.1) in each interval by using iterative Lagrange polynomials. Global convergence is established in Section 3. Finally, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in Section 4.

2. Description of the Collocation Method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh$, $n = 0, \dots, N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 \le c_1 < \dots < c_m \le 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1, \dots, m$, $n = 0, \dots, N-1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}]$.

Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree m as follows

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, \dots, N-1\}.$$

It holds for any $y \in C^{m+1}([0, T])$ that

$$y(t_n + sh) = L_0(s)y(t_n) + \sum_{j=1}^{m} L_j(s)y(t_{n,j}) + h^{m+1}\frac{y^{(m+1)}(\zeta_n(s))}{(m+1)!}s\prod_{j=1}^{m}(s - c_j), \tag{2.1}$$

where $s \in [0, 1]$, $L_0(v) = (-1)^m \prod_{l=1}^{m} \frac{v-c_l}{c_l}$ and $L_j(v) = \frac{v}{c_j}\prod_{l\neq j}^{m} \frac{v-c_l}{c_j-c_l}$, $j = 1, \dots, m$, are the Lagrange polynomials associated with the parameters c_j, $j = 1, \dots, m$.

Inserting (2.1) for the function $s \mapsto K(t,s,x(s))ds$ into (1.1), we obtain for each $j = 1, \dots, m$, $n = 0, \dots, N-1$,

$$\begin{aligned} x(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, x(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, x(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j}, t_n, x(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, x(t_{n,v})) + o(h^{m+1}), \end{aligned} \tag{2.2}$$

such that $a_{j,v} = \int_0^{c_j} L_v(\eta)d\eta$ and $b_v = \int_0^1 L_v(\eta)d\eta$, $v = 0, \dots, m$.

It holds for any $u \in S_m^0(I, \Pi_N)$ that

$$u_n(t_n + sh) = L_0(s)u_{n-1}(t_n) + \sum_{j=1}^{m} L_j(s)u_n(t_{n,j}), \quad s \in [0,1]. \tag{2.3}$$

Now, we approximate the exact solution x by $u \in S_m^0(I, \Pi_N)$ such that $u(t_{n,j})$ satisfies the following nonlinear system,

$$\begin{aligned} u_n(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, u_p(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, u_p(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j}, t_n, u_{n-1}(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u_n(t_{n,v})), \end{aligned} \tag{2.4}$$

for $j = 1, \dots, m$, $n = 0, \dots, N-1$, where $u_{-1}(t_0) = x(0) = f(0)$.

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_m^0(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1.1) such that

$$u_n^q(t_n + sh) = L_0(s)u_{n-1}^q(t_n) + \sum_{j=1}^{m} L_j(s)u_n^q(t_{n,j}), \quad s \in [0,1], \tag{2.5}$$

where the coefficients $u_n^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} u_n^q(t_{n,j}) =& f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, u_p^q(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, u_p^q(t_{p,v})) \\ &+ h a_{j,0} K(t_{n,j}, t_n, u_{n-1}^q(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u_n^{q-1}(t_{n,v})), \end{aligned} \tag{2.6}$$

such that $u_{-1}^q(t_0) = f(0)$ for all $q \in \mathbb{N}$ and the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).

The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system. The complexity of the proposed algorithm can be measured in terms of how many times the function K must be evaluated at each collocation point.

From (2.5) it follows that the number of such evaluations is $O(mn)$ for each iteration. Since the optimal number of iterations is $q = m + 1$ (as it will be shown in the next section), we conclude that the total number of evaluations is $O(m^2\, n)$, which makes this method competitive, in comparison with other methods where a nonlinear system of equations is solved by an iterative algorithm.

In the next section, we prove the convergence of the approximate solution u^q to the exact solution x of (1.1) is of order m for all $q \geq m$.

3. Convergence Analysis

In this section, we assume that the function K satisfies the Lipschitz condition with respect to the third variable: there exists $L \geq 0$ such that

$$|K(t,s,y_1) - K(t,s,y_2)| \leq L|y_1 - y_2|.$$

The following lemmas will be used in this section.

Lemma 3.1 ([6]). *Assume that $\{\alpha_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ are given non-negative sequences and the sequence $(\varepsilon_n)_{n\geq 1}$ satisfies $\varepsilon_1 \leq \beta$ and*

$$\varepsilon_n \leq \beta + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \varepsilon_j, \quad n \geq 2,$$

then

$$\varepsilon_n \leq \left(\beta + \sum_{j=1}^{n-1} q_j\right) \exp\left(\sum_{j=1}^{n-1} \alpha_j\right), \quad n \geq 2.$$

Lemma 3.2 ([1]). *If $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ are nonnegative sequences and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0,$$

then

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

Lemma 3.3 ([12]). *Assume that the sequence $\{\varepsilon_n\}_{n\geq 0}$ of nonnegative numbers satisfies*

$$\varepsilon_n \leq A\varepsilon_{n-1} + B\sum_{i=0}^{n-1} \varepsilon_i + K, \quad n \geq 0,$$

where A, B and K are nonnegative constants, then

$$\varepsilon_n \leq \frac{\varepsilon_0}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{K}{R_2 - R_1}[R_2^n - R_1^n],$$

where

$$R_1 = \frac{1 + A + B - \sqrt{(1-A)^2 + B^2 + 2AB + 2B}}{2},$$

$$R_2 = \frac{1 + A + B + \sqrt{(1-A)^2 + B^2 + 2AB + 2B}}{2}. \tag{3.1}$$

Therefore, $0 \leq R_1 \leq 1 \leq R_2$.

The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.4).

Lemma 3.4. *The nonlinear system* (2.4) *has a unique solution* $u \in S_m^0(I, \Pi_N)$ *for sufficiently small* h.

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (2.4) becomes

$$u_0(t_{0,j}) = f(t_{0,j}) + h a_{j,0} K(t_{0,j}, t_0, f(0)) + h \sum_{v=1}^{m} a_{j,v} K(t_{0,j}, t_{0,v}, u_0(t_{0,v})), \quad j = 1, \ldots, m.$$

We consider the operator Ψ defined by

$$\Psi : \mathbb{R}^m \to \mathbb{R}^m,$$
$$x = (x_1, \ldots, x_m) \mapsto \Psi(x) = (\Psi_1(x), \ldots, \Psi_m(x)),$$

such that for $j = 1, \ldots, m$, we have

$$\Psi_j(x) = f(t_{0,j}) + h a_{j,0} K(t_{0,j}, t_0, f(0)) + h \sum_{v=1}^{m} a_{j,v} K(t_{0,j}, t_{0,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq hmaL \|x - y\|,$$

where $a = \max\{|a_{j,v}|, j = 1, \ldots, m, v = 0, \ldots, m\}$.

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (2.4) has a unique solution u_0 on σ_0.

(ii) Suppose that u_i exists and is unique on the intervals σ_i, $i = 0, \ldots, n-1$, for $n \geq 1$. We show that u_n exists and is unique on the interval σ_n.

On the interval σ_n, the nonlinear system (2.4) becomes

$$u_n(t_{n,j}) = F(t_{n,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u_n(t_{n,v})), \tag{3.2}$$

where

$$F(t_{n,j}) = f(t_{n,j}) + h \sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, u_p(t_p))$$
$$+ h \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, u_p(t_{p,v})) + h a_{j,0} K(t_{n,j}, t_n, u_{n-1}(t_n)).$$

We consider the operator Ψ defined by

$$\Psi : \mathbb{R}^m \to \mathbb{R}^m,$$
$$x = (x_1, \ldots, x_m) \mapsto \Psi(x) = (\Psi_1(x), \ldots, \Psi_m(x)),$$

such that for $j = 1, \ldots, m$,

$$\Psi_j(x) = F(t_{n,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|.$$

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (3.2) has a unique solution u_n on σ_n. □

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.1. *Let f, K be $m+1$ times continuously differentiable on their respective domains. If $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, then, for sufficiently small h, the collocation solution u converges to the exact solution x and the resulting error function $e := x - u$ satisfies*

$$\|e\| \leq Ch^{m+1},$$

where C is a finite constant independent of h.

Proof. Define the error e on σ_n by $e(t) = e_n(t) = x(t) - u_n(t)$ for all $n \in \{0, 1, \ldots, N-1\}$.

We have, from (2.4) and (2.2), for all $n = 0, \ldots, N-1$, and $j = 1, \ldots, m$,

$$\begin{aligned} |e_n(t_{n,j})| \leq & hbL \sum_{p=0}^{n-1} |e_p(t_p)| + hbL \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e_p(t_{p,v})| + haL|e_{n-1}(t_n)| \\ & + haL \sum_{v=1}^{m} |e_n(t_{n,v})| + \alpha h^{m+1}, \end{aligned} \tag{3.3}$$

where α is a positive number and $e_{-1}(t_0) = 0$.

We consider the sequence $\varepsilon_n = \sum_{v=1}^{m} |e_n(t_{n,v})|$ for $n = 0, \ldots, N-1$. Then, from (3.3), ε_n satisfies for $n = 0, \ldots, N-1$,

$$\begin{aligned} \varepsilon_n &\leq hbLm \sum_{p=0}^{n-1} |e_p(t_p)| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm|e_{n-1}(t_n)| + haLm\varepsilon_n + \alpha m h^{m+1} \\ &\leq 2hbLm \sum_{p=0}^{n-1} \|e_p\| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm\varepsilon_n + \alpha m h^{m+1}. \end{aligned}$$

Hence, for $\overline{h} < \frac{1}{Lam}$ and $h \in (0, \overline{h}]$, we have

$$\varepsilon_n \leq \underbrace{\frac{2bLm}{1 - Lam\overline{h}}}_{\alpha_1} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\frac{bLm}{1 - Lam\overline{h}}}_{\alpha_2} h \sum_{p=0}^{n-1} \varepsilon_p + \underbrace{\frac{\alpha m}{1 - Lam\overline{h}}}_{\alpha_3} h^{m+1}.$$

Then, by Lemma 3.1, for all $n = 0, \ldots, N-1$,

$$\varepsilon_n \leq \underbrace{\alpha_1 \exp(T\alpha_2)}_{\alpha_4} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\alpha_3 \exp(T\alpha_2)}_{\alpha_5} h^{m+1}.$$

Therefore, by using (2.1) and (2.3), we obtain

$$\begin{aligned} \|e_n\| &\leq |R(\infty)|\|e_{n-1}\| + \rho\varepsilon_n + \beta h^{m+1} \\ &\leq |R(\infty)|\|e_{n-1}\| + \underbrace{\rho\alpha_4}_{\alpha_6} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{(\rho\alpha_5 + \beta)}_{\alpha_7} h^{m+1}, \end{aligned}$$

where $\rho = \max\{|L_j(t)| : t \in [0,1], j = 1, \ldots, m\}$.

Hence, by Lemma 3.3, we obtain for all $n = 0, \ldots, N-1$,

$$\begin{aligned} \|e_n\| &\leq \frac{\|e_0\|}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\leq \frac{\|e_0\|}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \\ &\leq \left(\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right)\alpha_7 h^{m+1}, \end{aligned}$$

where R_1, R_2 are defined by (3.1) such that $A = |R(\infty)|$, $B = \alpha_6 h$, $K = \alpha_7 h^{m+1}$.

Since

$$\begin{aligned} &\lim_{h \longrightarrow 0}\left(\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right) \\ =&\frac{1}{1 - |R(\infty)|}\exp\left(\frac{2T\alpha_6}{1 - |R(\infty)|}\right) < +\infty. \end{aligned}$$

Then there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$

$$\frac{1}{R_2 - R_1}\left[(R_2 - 1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \leq \gamma.$$

Thus, the proof is completed by taking $C = \alpha_7\gamma$. □

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.2. *Consider the iterative collocation solution u^q, $q \geq 1$, defined by (2.5) and (2.6). If $-1 < R(\infty) = (-1)^m \prod_{l=1}^m \frac{1-c_l}{c_l} < 1$, then for any initial condition $u^0(t_{n,j}) \in J$, the iterative collocation solution u^q, $q \geq 1$, converges to the exact solution x for sufficiently small h. Moreover, the following error estimate holds*

$$\|u^q - x\| \leq d\beta^q h^q + Ch^{m+1},$$

where d, β and C are finite constants independent of h.

Proof. We define the errors e^q and ξ^q by $e^q(t) = e^q_n(t) = u^q_n(t) - x(t)$ and $\xi^q = \xi^q_n = u^q_n(t) - u_n(t)$ on σ_n, $n = 0, \dots, N-1$, where u is defined by Lemma 3.4.

We have, from (2.4) and (2.6), for all $n = 0, \dots, N-1$ and $j = 1, \dots, m$,

$$\begin{aligned}|\xi^q_n(t_{n,j})| \leq & hbL \sum_{p=0}^{n-1} |\xi^q_p(t_p)| + hbL \sum_{p=0}^{n-1} \sum_{v=1}^{m} |\xi^q_p(t_{p,v})| + haL|\xi^q_{n-1}(t_n)| \\ & + haL \sum_{v=1}^{m} |\xi^{q-1}_n(t_{n,v})|.\end{aligned}$$

Now, for each fixed $q \geq 1$, we consider the sequence $\eta^q_n = \max\{|\xi^q_n(t_{n,v})| : v = 1, \dots, m\}$ for $n = 0, \dots, N-1$, it follows that

$$\begin{aligned}\eta^q_n &\leq hbL \sum_{p=0}^{n-1} |\xi^q_p(t_p)| + hbLm \sum_{p=0}^{n-1} \eta^q_p + haL|\xi^q_{n-1}(t_n)| + haLm\eta^{q-1}_n \\ &\leq 2hbL \sum_{p=0}^{n-1} \|\xi^q_p\| + hbLm \sum_{p=0}^{n-1} \eta^q_p + haLm\eta^{q-1}_n.\end{aligned}$$

Hence, by Lemma 3.2, for all $n = 0, \dots, N-1$,

$$\begin{aligned}\eta^q_n \leq & 2hbL \sum_{p=0}^{n-1} \|\xi^q_p\| + haLm\eta^{q-1}_n + \exp(TLbm)ab(hLm)^2 \sum_{p=0}^{n-1} \eta^{q-1}_p \\ & + 2\exp(TLbm)Tm(bL)^2 h \sum_{p=0}^{n-1} \|\xi^q_p\|. \end{aligned} \tag{3.4}$$

We consider the sequence $\eta^q = \max\{\eta^q_n, n = 0, \dots, N-1\}$ for $q \geq 1$. Then, from (3.4), η^q satisfies

$$\eta^q_n \leq \underbrace{2(bL + \exp(TLbm)Tm(bL)^2)}_{\alpha_1} h \sum_{p=0}^{n-1} \|\xi^q_p\| + \alpha_2 h\eta^{q-1},$$

where $\alpha_2 = (aLm + \exp(TLbm)abT(Lm)^2)$.

Therefore, by using (2.3) and (2.5), we obtain

$$\|\xi^q_n\| \leq |R(\infty)|\|\xi^q_{n-1}\| + \rho m\eta^q_n \leq |R(\infty)|\|\xi^q_{n-1}\| + \rho m\alpha_1 h \sum_{p=0}^{n-1} \|\xi^q_p\| + \rho m\alpha_2 h\eta^{q-1}.$$

Hence, by Lemma 3.3, we obtain for all $n = 0, \dots, N-1$,

$$\begin{aligned}\|\xi^q_n\| &\leq \frac{\|\xi^q_0\|}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{\rho m\alpha_2 h\eta^{q-1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\leq \frac{\|\xi^q_0\|}{R_2 - R_1}\left[(R_2-1)R_2^{\frac{T}{h}} + 1\right] + \frac{\rho m\alpha_2 h\eta^{q-1}}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right] \\ &\leq \left(\frac{1}{R_2 - R_1}\left[(R_2-1)R_2^{\frac{T}{h}} + 1\right] + \frac{1}{R_2 - R_1}\left[R_2^{\frac{T}{h}}\right]\right)\rho m\alpha_2 h\eta^{q-1}, \end{aligned} \tag{3.5}$$

where R_1 and R_2 are defined by (3.1) such that $A = |R(\infty)|$, $B = \rho m\alpha_1 h$, $K = \rho m\alpha_2 h\eta^{q-1}$. Since

$$\lim_{h\to 0}\left(\frac{1}{R_2-R_1}\left[(R_2-1)R_2^{\frac{T}{h}}+1\right]+\frac{1}{R_2-R_1}\left[R_2^{\frac{T}{h}}\right]\right)=\frac{\exp(\frac{2T\rho m\alpha_1}{1-|R(\infty)|})}{1-|R(\infty)|}<+\infty.$$

Then there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$

$$\frac{1}{R_2-R_1}\left[(R_2-1)R_2^{\frac{T}{h}}+1\right]+\frac{1}{R_2-R_1}\left[R_2^{\frac{T}{h}}\right]\leq\gamma.$$

It follows, from (3.5), that for all $n = 0, \dots, N-1$,

$$\|\xi_n^q\| \leq \gamma\rho m\alpha_2 h\eta^{q-1} \leq \gamma\rho m\alpha_2 h\|\xi^{q-1}\|,$$

which implies, for all $q \geq 1$, that

$$\|\xi^q\| \leq \gamma\rho m\alpha_2 h\|\xi^{q-1}\| \leq \cdots \leq (\gamma\rho m\alpha_2)^q h^q\|\xi^0\|.$$

Since, $u_{-1}^0(t_0) = f(0), u^0(t_{n,j}) \in J$ (bounded interval), then by (2.3) the function u^0 is bounded. Hence, there exists $d > 0$ such that $\|\xi^0\| = \|u^0 - u\| \leq \|u^0 - x\| + \|x - u\| < d$. Which implies that for all $q \geq 1$

$$\|\xi^q\| \leq d(\underbrace{\gamma\rho m\alpha_2}_{\beta})^q h^q.$$

Hence, by Theorem 3.1, we deduce that

$$\|e^q\| \leq \|\xi^q\| + \|u - x\| \leq d\beta^q h^q + Ch^{m+1}.$$

Thus, the proof is completed. □

Remark 3.1. From the error estimate in Theorem 3.2 it follows that the optimal number of iterations is $q = m+1$. Actually, with $m+1$ iterations the total error has the order of $O(h^{m+1})$, which will not be improved if more iterations are performed.

4. Numerical Examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and q. In each example, we calculate the error between x and the iterative collocation solution u^q.

The absolute errors at some particular points are given to compare our solutions with the solutions obtained by [3, 9, 13, 16, 18].

These results of these numerical examples are in agreement with the theory presented in Section 3 and they confirm the advantages of our method in comparison with those described in [3, 9, 13, 16, 18].

Example 4.1 ([9, 13]). Consider the following nonlinear Volterra integral equation

$$x(t) = 1 + (\sin(t))^2 - \int_0^t 3\sin(t-s)(x(s))^2 ds, \quad t \in [0,1],$$

where $u(x) = \cos(x)$ is the exact solution.

The absolute errors for $N = 10, 20$ and $m = q = 3$ at $t = 0, 0.1, \ldots, 1$, are displayed in Table 1. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = -0.02$. The numerical results obtained by the present method are considerably more accurate in comparison with the numerical results obtained in [9,13].

TABLE 1. Comparison of the absolute errors of Example 4.1

t	Method in [9]		Method in [13]		Our method	
	$N = 10$	$N = 20$	$N = 10$	$N = 20$	$N = 10$	$N = 20$
0.1	1.01E−5	1.59E−6	1.24E−5	2.54E−8	3.32E−8	7.92E−9
0.2	2.48E−5	3.26E−6	1.62E−6	3.44E−7	1.84E−9	5.15E−9
0.3	3.65E−5	4.72E−6	2.03E−4	9.19E−7	3.58E−8	3.87E−9
0.4	4.61E−5	5.87E−6	2.07E−5	1.44E−6	5.29E−8	8.00E−9
0.5	5.26E−5	6.63E−6	3.84E−5	1.88E−6	9.91E−8	8.90E−10
0.6	5.59E−5	6.98E−6	5.11E−5	2.18E−6	1.48E−7	5.90E−9
0.7	5.58E−5	6.92E−6	7.22E−5	1.83E−6	1.77E−7	9.71E−9
0.8	5.28E−5	6.47E−6	6.43E−5	6.41E−6	2.00E−7	3.34E−9
0.9	4.65E−5	5.70E−6	1.96E−5	1.00E−4	2.04E−7	2.07E−8
1	3.97E−5	4.71E−6	6.36E−4	9.25E−4	1.95E−7	5.13E−9

Example 4.2 ([3, 18]). Consider the following linear Volterra integral equation with exact solution $x(t) = 1 - \sinh(t)$:

$$x(t) = 1 - t - \frac{t^2}{2} + \int_0^t (t - s)x(s)ds, \quad t \in [0, 1].$$

The absolute errors for $m = q = 3$ and $N = 20$ at $t = 0, 0.1, \ldots, 1$, are displayed in Table 2. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{5}$, $i = 1, \ldots, m$, and $R(\infty) = -0.02$. The numerical results obtained here are compared in Table 2 with the numerical results obtained by using the methods in [3, 18].

It is seen from Table 2 that the results obtained by the present method are much more accurate than those obtained in [3, 18].

The absolute errors for $N = 5$ and $(q, m) \in \{(2, 2), (3, 2), (3, 3), (3, 5), (4, 5)\}$ at $t = 0, 0.1, \ldots, 1$, are presented in Table 3, we note that the absolute error reduces as q or m increases.

We calculate the experimental order of convergence (EOC) at $t = 1$ for $N = 2^l$, $l = 1, 2, 3, 4, 5$, $m = 1, 2, 3$ and $q = m+1$ in Table 4, the result confirms the theoretical result and suggests that the order of convergence with $q = m + 1$ is $m + 1$. As we have remarked (see Remark 3.1) this is the maximal convergence order that can be obtained with the present method.

Moreover, we calculate the run time to solve the approximate solution u for $N = 6, \ldots, 10$, $m = 7, \ldots, 10$, and $q = m + 1$, the numerical results are solved by using Maple version 16.

The computations were performed in a PC with a 2.16 GHz processor, running with 2.00 Go RAM. As it could be expected, the computing time increases with m and N. However, we cannot see a simple relationship between the computing time and the complexity of the algorithm, probably because this time depends on other factors than the number of evaluations of the function K. This table shows that accurate results can be obtained by our method in a small computer with a low computational cost.

TABLE 2. Comparison of the absolute errors of Example 4.2

t	Our method		Method in [3]	Method in [18]
	$N = 10$	$N = 20$		
0.0	0	0	0	1.98E−14
0.1	1.30E−8	1.98E−9	5.38E−6	1.21E−7
0.2	3.35E−8	2.54E−9	2.20E−5	2.35E−7
0.3	3.14E−8	6.55E−9	4.82E−5	3.54E−7
0.4	5.98E−8	5.80E−9	8.33E−5	4.77E−7
0.5	6.94E−8	3.50E−9	1.26E−4	6.05E−7
0.6	8.01E−8	8.51E−10	1.77E−4	7.39E−7
0.7	1.00E−7	5.83E−9	2.34E−4	8.80E−7
0.8	1.15E−7	7.38E−9	2.97E−4	1.03E−6
0.9	1.37E−7	8.90E−9	3.65E−4	1.19E−6
1	1.62E−7	9.38E−9	4.38E−4	1.36E−6

TABLE 3. Absolute errors for Example 4.2

t	$q = 2$, $m = 2$	$q = 3$, $m = 2$	$q = 3$, $m = 3$	$q = 3$, $m = 5$	$q = 4$, $m = 5$
0	0.0	0.0	0.0	0.0	0.0
0.1	8.231E−6	7.282E−6	3.015E−7	7.451E−8	3.701E−8
0.2	8.563E−5	8.373E−5	4.115E−7	1.147E−6	8.474E−7
0.3	1.053E−5	7.583E−6	6.394E−7	5.824E−8	4.007E−8
0.4	1.027E−4	9.863E−5	8.478E−7	8.031E−7	4.328E−7
0.5	1.064E−5	5.410E−6	1.017E−6	2.316E−8	1.897E−8
0.6	1.143E−4	1.070E−4	1.324E−6	1.058E−7	4.785E−8
0.7	1.033E−5	2.283E−6	1.470E−6	1.309E−8	3.040E−8
0.8	1.297E−4	1.175E−4	1.909E−6	1.114E−7	7.258E−8
0.9	9.861E−6	1.815E−6	2.021E−6	8.470E−9	8.137E−10
1	1.514E−4	1.314E−4	2.620E−6	1.156E−7	4.245E−8

Table 4. EOC and the run-time/sec of Example 4.2

N	$m=1$	$m=2$	$m=3$
2			
4	2.04	2.91	4.00
8	2.05	2.96	4.01
16	2.04	2.91	4.00
32	2.04	2.91	4.00

N	$m=7$	$m=8$	$m=9$	$m=10$
6	3.9	5.6	9.9	14.8
7	5.8	9.9	17.9	31.1
8	8.9	16.7	24.3	56.6
9	13.3	32.7	43.9	118.4
10	17.4	35.9	126.3	232.3

Example 4.3 ([16]). We consider the following nonlinear Volterra integral equation

$$x(t) = \frac{t}{e^{t^2}} + \int_0^t 2tse^{-x^2(s)}ds, \quad t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 20$ and $m = 3$, $q = 5$ at $t = 0, 0.2, \dots, 1$, are compared with the absolute error of the method in [16] in Table 5, where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}$, $i = 1, \dots, m$, and $R(\infty) = -0.64$.

Table 5. Comparison of the absolute errors of Example 4.3

t	Method in [16] $N=20$	Our method $N=20$
0	0	0
0.2	1.49E−8	8.9E−9
0.4	7.74E−7	2.69E−8
0.6	9.36E−6	8.90E−9
0.8	4.58E−5	3.39E−8
1	1.29E−4	2.49E−8

Example 4.4 ([16]). We consider the following nonlinear Volterra integral equation

$$x(t) = t\cos(t) + \int_0^t t\sin(x(s))ds, \quad t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 25$ and $m = q = 4$ at $t = 0.001, 0.2, 0.4, 0.6, 0.8, 1$ are compared with the absolute error of the method in [16] in Table 6.

Where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}$, $i = 1, \dots, m$, and $R(\infty) = 0.35$.

It is seen from Table 6 that the results obtained by the present method is very superior to that obtained by the method in [16].

5. Conclusion

In this paper, we have used an iterative collocation method based on the Lagrange polynomials for the numerical solution of nonlinear Volterra integral equations (1.1)

Table 6. Comparison of the absolute errors of Example 4.4

t	Method in [16] $N = 25$	Our method $N = 25$
0.001	1.25E−12	1.75E−11
0.2	3.53E−6	6.30E−8
0.4	5.81E−6	5.40E−8
0.6	7.74E−7	9.60E−8
0.8	1.20E−5	6.00E−9
1	3.98E−5	7.20E−8

in the spline space $S_m^{(0)}(\Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples showing that the method is convergent with a good accuracy and the comparison of the results obtained by the present method with the other methods reveals that the method is very effective and convenient.

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References

[1] R. P. Agrwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Second edition, Marcel Dekker, New York, 2000.
[2] E. Babolian, F. Fattahzadeh and E. Golpar Raboky, *A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type*, Appl. Math. Comput. **189** (2007), 641–646.
[3] E. Babolian and A. Davary, *Numerical implementation of Adomian decomposition method for linear Volterra integral equations for the second kind*, Appl. Math. Comput. **165** (2005), 223–227.
[4] R. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J. **10**(4) (1943), 643–647.
[5] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004.
[6] H. Brunner and P. J. van der Houwen, *The numerical solution of Volterra equations*, Elsevier Science Pub, Amsterdam, 1986.
[7] D. Costarelli and R. Spigler, *Solving Volterra integral equations of the second kind by sigmoidal functions approximation*, J. Integral Equations Appl. **25**(2) (2013), 193–222.
[8] D. Costarelli, *Approximate solutions of Volterra integral equations by an interpolation method based on ramp functions*, Comput. Appl. Math. **38**(4) (2019), DOI 10.1007/s40314-019-0946-x.
[9] N. Ebrahimi and J. Rashidinia, *Collocation method for linear and nonlinear Fredholm and Volterra integral equations*, Appl. Math. Comput. **270** (2015), 156–164.
[10] M. Ghasemi, M. T. Kajani and E. Babolian, *Numerical solution of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method*, Appl. Math. Comput. **188** (2007), 446–449.

[11] H. Guoqiang, *Asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations*, Appl. Numer. Math. **13** (1993), 357–369.
[12] E. Hairer, C. Lubich and S. P. Nørsett, *Order of convergence of one-step methods for Volterra integral equations of the second kind*, SIAM J. Numer. Anal. **20**(3) (1983), 569–579.
[13] SH. Javadi, A. Davari and E. Babolian, *Numerical implementation of the Adomian decomposition method for nonlinear Volterra integral equations of the second kind*, Int. J. Comput. Math. **84**(1) (2007), 75–79.
[14] A.J. Jerri, *Introduction to Integral Equations with Application*, Wiley, New York, 1999.
[15] N. M. Madbouly, D. F. McGhee and G. F. Roach, *Adomian's method for Hammerstein integral equations arising from chemical reactor theory*, Appl. Math. Comput. **117** (2001), 241–249.
[16] K. Maleknejad and P. Torabi, *Application of fixed point method for solving nonlinear Volterra-Hammerstein integral equation*, UPB Scientific Bulletin, Series A **74**(1) (2012), 45–56.
[17] K. Maleknejad and F. Mirzaee, *Using rationalized Haar wavelet for solving linear integral equations*, Appl. Math. Comput. **160** (2005), 579–587.
[18] J. Rashidinia and Z. Mahmoodi, *Collocation method for Fredholm and Volterra integral equations*, Kybernetes **42**(3) (2013), 400–412.
[19] M. H. Reihani and Z. Abadi, *Rationalized Haar functions method for solving Fredholm and Volterra integral equations*, J. Comput. Appl. Math. **200** (2007), 12–20.
[20] M. Sezer and M. Gülsu, *Polynomial solution of the most general linear Fredholm-Volterra integrodifferential-difference equations by means of Taylor collocation method*, Appl. Math. Comput. **185** (2007), 646–657.
[21] S. Yalçinbaş, *Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations*, Appl. Math. Comput. **127** (2002), 195–206.
[22] M. Zarebnia and J. Rashidinia, *Convergence of the sinc method applied to Volterra integral equations*, Appl. Appl. Math. **5**(1) (2010), 198–216.

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TWO-DIMENSIONAL WAVELET WITH MATRIX DILATION $M = 2I$ AND ITS APPLICATION IN SOLVING INTEGRAL EQUATIONS

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Abstract. In this study, using a one-dimensionl MRA we constructed a two-dimensional wavelet as well as four masks which are not related to the MRA. Finally, we provide some examples to prove the applicability of our construction in case of finding numerical solution of two-dimensional first kind Fredholm integral equations.

1. Introduction

Let $\{V_j\}$ be a one-dimensional multiresolution analysis (MRA) with scaling function ϕ and mother wavelet ψ, then $\Phi(x,y) = \phi(x)\phi(y)$ is a scaling function for two-dimensional MRA and in this case, we have 3 mother wavelets

$$\Psi^a(x,y) = \phi(x)\psi(y), \quad \Psi^b(x,y) = \psi(x)\phi(y), \quad \Psi^d(x,y) = \psi(x)\psi(y). \tag{1.1}$$

It means that $\{\Psi^r_{j,(s,t)} : j, s, t \in \mathbb{Z}, r = a, b, d\}$ consists of an orthonormal basis for $L^2(\mathbb{R})$. For more details see [1, 3, 9].

In applications, finding a way to construct a wavelet with a smaller frequency domain and correspondingly increase in time domain is of great importance. The higher the number of mother wavelets, the more accurate the answer would be. Finding a way to minimizing frequency domain and so maximizing accuracy is so important. For more details see [3].

In Section 2, first we refer to the meaning of a two-dimensional wavelet by matrix dilation and then we present a way to construct a two-dimensional wavelet with small frequency domain and high accuracy by using a two-dimensional MRA and four masks

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which are not related to the MRA. In Section 3, we will find numerical solution for two first kind Fredholm integral equations. This kind of equations provide an ill-posed system, i.e., there might be no solutions or no unique solution and even no stable solution. Solving this type of integral equation is not easy.

2. Two-Dimensional Wavelet with Matrix Dilation $M = 2I$

In the subject of wavelet with matrix dilation M, we shall assume that M is a fixed quadratic integer matrix such that all its eigenvalues are greater than one in modulus, $m = |\det M|$. In this paper we consider $M = 2I$, especially.

Definition 2.1 ([8])**.** A collection of closed subspaces $V_j \subset L^2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, is called a multiresolution analysis (MRA) in $L^2(\mathbb{R}^2)$ with matrix dilation M if the following conditions hold:

MRA1: $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
MRA2: $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^2)$;
MRA3: $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
MRA4: $f \in V_j$ if and only if $f(M^{-j}\cdot) \in V_0$ for all $j \in \mathbb{Z}$;
MRA5: there exists a function $\phi \in V_0$ such that the sequence $\{\phi(\cdot + m, \cdot + n)\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis in V_0 (ϕ is called scaling function).

Let ϕ be a scaling function for an MRA. Using properties MRA1, MRA5 and notation

$$f_{j,(s,t)} := m^{j/2} f(M^j \cdot + (s,t)), \quad j, s, t \in \mathbb{Z},$$

we get the refinement equation

$$\phi = \sum_{s,t\in\mathbb{Z}} h_{s,t}\phi_{1,(s,t)}, \quad \sum_{s,t\in\mathbb{Z}} |h_{s,t}|^2 < \infty. \tag{2.1}$$

Applying the Fourier transform,

$$\hat{\phi}(\xi_1, \xi_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x,y) e^{-2\pi i(x\xi_1 + y\xi_2)} dxdy,$$

to both sides of above equality, we get

$$\widehat{\phi}(\xi_1, \xi_2) = m_0(M^{*-1}(\xi_1, \xi_2))\widehat{\phi}(M^{*-1}(\xi_1, \xi_2)),$$

where

$$m_0(\eta_1, \eta_2) = m^{-\frac{1}{2}} \sum_{s,t\in\mathbb{Z}} h_{s,t} e^{2\pi i(s\eta_1 + t\eta_2)}.$$

As in the one-dimensional case, the function m_0 is called a mask. For more details see [8].

Example 2.1. Let $M = 2I$ and $\phi(x,y) = \chi_{[0,1)\times[0,1)}(x,y)$. From (2.1), we have

$$\phi(x,y) = 2 \sum_{s,t\in\mathbb{Z}} h_{s,t}\phi(2x + s, 2y + t),$$

and we conclude that

$$\phi(x,y) = 2[h_{-1,-1}\phi(2x-1,2y-1)+h_{-1,0}\phi(2x-1,y)+h_{0,-1}\phi(2x,2y-1)+h_{0,0}\phi(x,y)],$$

where $h_{-1,-1}, h_{-1,0}, h_{0,-1}, h_{0,0} = \frac{1}{2}$.

Hence,

$$m_0(\xi_1,\xi_2) = \frac{1}{2}\ \left(\frac{1}{2}e^{2\pi i(-\xi_1-\xi_2)} + \frac{1}{2}e^{2\pi i(-\xi_1)} + \frac{1}{2}e^{2\pi i(-\xi_2)} + \frac{1}{2}\right).$$

Definition 2.2 ([8])**.** If A is a nonsingular integer 2×2 matrix, we say the vectors $(k,l),(s,t)\in\mathbb{Z}^2$ are congruent modulo A and write $(k,l)\equiv(s,t) \pmod A$ if $(k,l)-(s,t) = A(p,q)$ for some $(p,q)\in\mathbb{Z}^2$. The integer lattice $\mathbb{Z}^2$ is partitioned into cosets with respect to the congruence introduced above. Any set containing only one representative of each coset is called a set of digits of the matrix A. When it does not matter which set of digits is chosen, we shall assume that it is chosen arbitrarily and denote it by $D(A)$.

Example 2.2. For $M = 2I$, we consider

$$D(M) = \{s_0 = (0,0),\ s_1 = (0,-1),\ s_2 = (-1,0),\ s_3 = (-1,-1)\}.$$

Theorem 2.1 ([8])**.** *Suppose an MRA $\{V_j\}_{j\in\mathbb{Z}}$ is generated by a scaling function ϕ with mask m_0 and the system $\{\phi(\cdot+(s,t))\}_{s,t\in\mathbb{Z}}$ is orthonormal. Let $D(M^*) = \{s_0,\dots,s_{m-1}\}$. Let there exist functions $m_\nu \in L^2([0,1]\times[0,1])$, $\nu = 0,\dots,m-1$, such that the matrix*

$$\mathcal{M} = \{m_\nu((\xi_1,\xi_2) + M^{*-1}s_k)\}_{\nu,k=0}^{m-1} \tag{2.2}$$

is unitary, that is, $\mathcal{M}\mathcal{M}^* = \mathcal{M}^*\mathcal{M} = I$. Take the functions $\psi^{(\nu)}$, $\nu = 1,\dots,m-1$, defined by the equalities

$$\widehat{\psi}^{(\nu)}(\xi_1,\xi_2) = m_\nu(M^{*-1}(\xi_1,\xi_2))\widehat{\phi}(M^{*-1}(\xi_1,\xi_2)).$$

Then the system $\{\psi^{(\nu)}_{j,(k,l)}\}$ is an orthonormal basis in the space $L^2(\mathbb{R}^2)$.

The following lemma is a portrait of some remark in [8, page 93].

Lemma 2.1. *Let ϕ be a scaling function with mask*

$$m_0(\xi_1,\xi_2) = m^{-\frac{1}{2}}\sum_{k=0}^{m-1} h^0_{0,k}e^{2\pi i\langle s_k,(\xi_1,\xi_2)\rangle},$$

such that $h_{0,k}$'s are real numbers, $k = 0,\dots,m-1$, and $\sum_{k=0}^{m-1}|h_{0,k}|^2 = 1$. Define

$$m_\nu(\xi_1,\xi_2) = m^{-\frac{1}{2}}\sum_{k=0}^{m-1} h^\nu_{\nu,k}e^{2\pi i\langle s_k,(\xi_1,\xi_2)\rangle},$$

where $h^\nu_{\nu,0} = h^0_{0,\nu}$, $h^\nu_{\nu,k} = \delta_{\nu,k} - \frac{h^0_{0,k}h^0_{0,\nu}}{1-h^0_{0,0}}$, $\nu = 1,\dots,m-1$. Then the matrix

$$\mathcal{M} = \{m_\nu((\xi_1,\xi_2) + M^{*-1}s_k)\}_{\nu,k=0}^{m-1}$$

is unitary.

Proof. Since $\sum_{k=0}^{m-1} |h^{\nu}_{\nu,k}|^2 = 1$ for $\nu = 1, \dots, m-1$, and $\sum_{k=0}^{m-1} h^{\nu}_{\nu,k} h^{\mu}_{\mu,k} = 0$ for $\nu \neq \mu$, $\mathcal{M}$ is unitary. $\square$

Example 2.3. Consider mask m_0 in example (2.1),

$$m_0(\xi_1, \xi_2) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} e^{-2\pi i \xi_2} + \frac{1}{2} e^{-2\pi i \xi_1} + \frac{1}{2} e^{-2\pi i (\xi_1 + \xi_2)} \right).$$

Take

$$m_n(\xi_1, \xi_2) = \frac{1}{2} \left(\frac{1}{2} + (-1)^{[\frac{n}{2}]} \frac{1}{2} e^{-2\pi i \xi_2} + (-1)^n \frac{1}{2} e^{-2\pi i \xi_1} + (-1)^{[\frac{n+1}{2}]} \frac{1}{2} e^{-2\pi i (\xi_1 + \xi_2)} \right),$$

for $n = 1, 2, 3$, where $[\cdot]$ denotes integer part. Hence,

$$\mathcal{M} = \begin{pmatrix} m_0(\xi_1,\xi_2) & m_0\Big(\xi_1, \xi_2 - \frac{1}{2}\Big) & m_0\Big(\xi_1 - \frac{1}{2}, \xi_2\Big) & m_0\Big(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\Big) \\ m_1(\xi_1,\xi_2) & m_1\Big(\xi_1, \xi_2 - \frac{1}{2}\Big) & m_1\Big(\xi_1 - \frac{1}{2}, \xi_2\Big) & m_1\Big(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\Big) \\ m_2(\xi_1,\xi_2) & m_2\Big(\xi_1, \xi_2 - \frac{1}{2}\Big) & m_2\Big(\xi_1 - \frac{1}{2}, \xi_2\Big) & m_2\Big(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\Big) \\ m_3(\xi_1,\xi_2) & m_3\Big(\xi_1, \xi_2 - \frac{1}{2}\Big) & m_3\Big(\xi_1 - \frac{1}{2}, \xi_2\Big) & m_3\Big(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\Big) \end{pmatrix} \tag{2.3}$$

is unitary.

Since the matrix (2.2) is unitary we have some useful formulas for m_ν, $\nu = 0, \dots, m-1$, as

$$\sum_{k=0}^{m-1} |m_\nu((\xi_1, \xi_2) + M^{*-1} s_k)|^2 = 1. \tag{2.4}$$

For all $\nu, \mu = 0, \dots, m-1$,

$$\sum_{k=0}^{m-1} m_\nu((\xi_1, \xi_2) + M^{*-1} s_k) \overline{m_\mu}((\xi_1, \xi_2) + M^{*-1} s_k) = 0, \quad \text{for } \nu \neq \mu, \tag{2.5}$$

and

$$\sum_{\nu=0}^{m-1} |m_\nu((\xi_1, \xi_2) + M^{*-1} s_k)|^2 = 1, \quad \text{for } k = 0, \dots, m-1,$$

and for all $k = 1, \dots, m-1$,

$$\sum_{\nu=0}^{m-1} m_\nu((\xi_1, \xi_2) + M^{*-1} s_k) \overline{m_\nu(\xi_1, \xi_2)} = 0. \tag{2.6}$$

Theorem 2.2. *Let $f(x,y)$ be a function such that $\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$ is an orthonormal system and let $m_0(\xi_1, \xi_2) = \frac{1}{2} \sum_{s,t \in \mathbb{Z}} h^0_{s,t} e^{2\pi i (s\xi_1 + t\xi_2)}$ and $m_\nu(\xi_1, \xi_2) = \frac{1}{2} \sum_{s,t \in \mathbb{Z}} h^\nu_{s,t} e^{2\pi i (s\xi_1 + t\xi_2)}$, $\nu = 1, 2, 3$, are masks with matrix dilation $M = 2I$ such that (2.3) is unitary. Define*

$$F_n(x, y) = \sum_{s,t} h^n_{s,t} f(x - s, y - t), \quad n = 0, 1, 2, 3.$$

Then

$$\{F_n(\cdot - 2k, \cdot - 2l) : n = 0, \dots, 3, \ k, l \in \mathbb{Z}\} \tag{2.7}$$

is an orthonormal basis for $\overline{\text{span}}\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$.

Proof. First of all, we calculate some useful formulas.

By definition of F_ν, for $\nu = 1, 2, 3$,

$$\widehat{F_\nu}(\xi_1, \xi_2) = 2\overline{m_\nu(\xi_1, \xi_2)}\hat{f}(\xi_1, \xi_2). \tag{2.8}$$

Since $\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$ is an orthonormal set so

$$\langle f, f(\cdot - s, \cdot - t)\rangle = \begin{cases} 1, & \text{if } s = t = 0, \\ 0, & \text{o.w}, \end{cases}$$

and since

$$\begin{aligned}
\langle f, f(\cdot - s, \cdot - t)\rangle &= \left\langle \hat{f}(\cdot, \cdot), e^{-2\pi i(s\cdot + t\cdot)}\hat{f}(\cdot, \cdot)\right\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{f}(\xi_1, \xi_2)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} \\
&= \sum_{k,l}\int_l^{l+1}\int_k^{k+1} |\hat{f}(\xi_1, \xi_2)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2 \\
&= \int_0^1\int_0^1 \sum_{k,l} |\hat{f}(\xi_1 + k, \xi_2 + l)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2,
\end{aligned}$$

noticing Fourier cofficients, we conclude that

$$\sum_{k,l} |\hat{f}(\xi_1 + k, \xi_2 + l)|^2 = 1 \quad \text{a.e.} \tag{2.9}$$

Also, by (2.4), (2.5), (2.8) and (2.9), we have

$$\begin{aligned}
&\sum_{k,l} \overline{\widehat{F_\nu}\left(\xi_1 - \frac{k}{2}, \xi_2 - \frac{l}{2}\right)}\widehat{F_\mu}\left(\xi_1 - \frac{k}{2}, \xi_2 - \frac{l}{2}\right) \\
=&\sum_{k,l} \overline{\widehat{F_\nu}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right)}\widehat{F_\mu}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right) \\
&+ \sum_{k,l} \overline{\widehat{F_\nu}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right)}\widehat{F_\mu}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right) \\
&+ \sum_{k,l} \overline{\widehat{F_\nu}\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right)}\widehat{F_\mu}\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right) \\
&+ \sum_{k,l} \overline{\widehat{F_\nu}\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right)}\widehat{F_\mu}\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right) \\
=&4\sum_{k,l} m_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right)\overline{m_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right)}\left|\hat{f}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right)\right|^2 \\
&+ 4\sum_{k,l} m_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right)\overline{m_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\times\left|\hat{f}\left(\xi_1-\frac{2k}{2},\xi_2-\frac{2l+1}{2}\right)\right|^2\\
&+4\sum_{k,l} m_\nu\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l}{2}\right)\overline{m_\mu\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l}{2}\right)}\\
&\times\left|\hat{f}\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l}{2}\right)\right|^2\\
&+4\sum_{k,l} m_\nu\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l+1}{2}\right)\overline{m_\mu\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l+1}{2}\right)}\\
&\times\left|\hat{f}\left(\xi_1-\frac{2k+1}{2},\xi_2-\frac{2l+1}{2}\right)\right|^2\\
=&4\left[m_\nu(\xi_1,\xi_2)\overline{m_\mu(\xi_1,\xi_2)}+m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)\overline{m_\mu\left(\xi_1,\xi_2-\frac{1}{2}\right)}\right.\\
&\left.+m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)\overline{m_\mu\left(\xi_1-\frac{1}{2},\xi_2\right)}+m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\overline{m_\mu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)}\right]
\end{aligned}
$$

(2.10)

$$
=\begin{cases} 4, & \text{if } \mu=\nu,\\ 0, & \text{if } \mu\neq\nu.\end{cases}
$$

Now we are ready to show that $\{F_\nu(\cdot-2k,\cdot-2l):\nu=0,\dots,3\}_{k,l\in\mathbb{Z}}$ is an orthonormal set. By (2.10),

$$
\begin{aligned}
&\langle F_\nu, F_\mu(\cdot-2s,\cdot-2t)\rangle\\
=&\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{F_\nu}(\xi_1,\xi_2)\overline{\widehat{F_\mu}(\xi_1,\xi_2)}e^{4\pi i(s\xi_1+t\xi_2)}d\xi_1 d\xi_2\\
=&\sum_{k,l}\int_{\frac{l}{2}}^{(l+1)/2}\int_{\frac{k}{2}}^{(k+1)/2}\widehat{F_\nu}(\xi_1,\xi_2)\overline{\widehat{F_\mu}(\xi_1,\xi_2)}e^{4\pi i(s\xi_1+t\xi_2)}d\xi_1 d\xi_2\\
=&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\sum_{k,l}\widehat{F_\nu}\left(\xi_1+\frac{k}{2},\xi_2+\frac{l}{2}\right)\overline{\widehat{F_\mu}\left(\xi_1+\frac{k}{2},\xi_2+\frac{l}{2}\right)}e^{4\pi i(s\xi_1+t\xi_2)}d\xi_1 d\xi_2\\
=&\begin{cases} 1, & \text{if } \nu=\mu \text{ and } s=t=1,\\ 0, & \text{o.w.}\end{cases}
\end{aligned}
$$

Now we will show that the set (2.7) will generate the set $\{f(\cdot-s,\cdot-t):s,t\in\mathbb{Z}\}$. In this order we will use the bellow equalities:

$$
\begin{aligned}
&m_\nu(\xi_1,\xi_2)+m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)+m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)+m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\\
=&2\sum_{s,t}h^\nu_{2s,2t}e^{2\pi i(2s\xi_1+2t\xi_2)},
\end{aligned}
$$

$$
\begin{aligned}
&m_\nu(\xi_1,\xi_2) - m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right) + m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right) - m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\\
&=2\sum_{s,t} h^\nu_{2s,2t+1} e^{2\pi i(2s\xi_1+(2t+1)\xi_2)},\\
&m_\nu(\xi_1,\xi_2) + m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right) - m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right) - m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\\
&=2\sum_{s,t} h^\nu_{2s+1,2t} e^{2\pi i((2s+1)\xi_1+2t\xi_2)},\\
&m_\nu(\xi_1,\xi_2) - m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right) - m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right) + m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\\
&=2\sum_{s,t} h^\nu_{2s+1,2t+1} e^{2\pi i((2s+1)\xi_1+(2t+1)\xi_2)}. \qquad (2.11)
\end{aligned}
$$

Hence, by (2.4), (2.6) and (2.11), we have

$$
\begin{aligned}
&\Bigg[\sum_{k,l} h^0_{2k,2l}F_0(\cdot+2k,\cdot+2l) + h^1_{2k,2l}F_1(\cdot+2k,\cdot+2l) + h^2_{2k,2l}F_2(\cdot+2k,\cdot+2l)\\
&\quad + h^3_{2k,2l}F_3(\cdot+2k,\cdot+2l)\Bigg]^{\widehat{}}(\xi_1,\xi_2)\\
&=2\sum_{\nu=0}^{3}\sum_{k,l} h^\nu_{2k,2l} e^{2\pi i(2k\xi_1+2l\xi_2)}\overline{m_\nu(\xi_1,\xi_2)}\ \hat{f}(\xi_1,\xi_2)\\
&=\Bigg[\ m_0(\xi_1,\xi_2) + m_0\left(\xi_1,\xi_2-\frac{1}{2}\right) + m_0\left(\xi_1-\frac{1}{2},\xi_2\right)\\
&\quad + m_0\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\Bigg]\overline{m_0(\xi_1,\xi_2)}\hat{f}(\xi_1,\xi_2) + \Bigg[m_1(\xi_1,\xi_2) + m_1\left(\xi_1,\xi_2-\frac{1}{2}\right)\\
&\quad + m_1\left(\xi_1-\frac{1}{2},\xi_2\right) + m_1\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\Bigg]\overline{m_1(\xi_1,\xi_2)}\ \hat{f}(\xi_1,\xi_2)\\
&\quad + \Bigg[m_2(\xi_1,\xi_2) + m_2\left(\xi_1,\xi_2-\frac{1}{2}\right) + m_2\left(\xi_1-\frac{1}{2},\xi_2\right)\\
&\quad + m_2\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\Bigg]\overline{m_2(\xi_1,\xi_2)}\hat{f}(\xi_1,\xi_2) + \Bigg[m_3(\xi_1,\xi_2) + m_3\left(\xi_1,\xi_2-\frac{1}{2}\right)\\
&\quad +m_3\left(\xi_1-\frac{1}{2},\xi_2\right) + m_3\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\Bigg]\overline{m_3(\xi_1,\xi_2)}\ \hat{f}(\xi_1,\xi_2)\\
&=\hat{f}(\xi_1,\xi_2)\ \sum_{\nu=0}^{3}|m_\nu(\xi_1,\xi_1)|^2 + \hat{f}(\xi_1,\xi_2)\ \sum_{\nu=0}^{3} m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)\overline{m_\nu(\xi_1,\xi_2)}\\
&\quad + \hat{f}(\xi_1,\xi_2)\ \sum_{\nu=0}^{3} m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)\overline{m_\nu(\xi_1,\xi_2)}\\
&\quad + \hat{f}(\xi_1,\xi_2)\ \sum_{\nu=0}^{3} m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\overline{m_\nu(\xi_1,\xi_2)}
\end{aligned}
$$

$$=\hat{f}(\xi_1,\xi_2).$$

Similarly, we have

$$\Bigg[\sum_{k,l} h^0_{2k,2l+1}F_0\left(\cdot+2k,\cdot+2l\right)+\ h^1_{2k,2l+1}F_1\left(\cdot+2k,\cdot+2l\right)$$

$$+\,h^2_{2k,2l+1}F_2\left(\cdot+2k,\cdot+2l\right)+\ h^3_{2k,2l+1}F_3\left(\cdot+2k,\cdot+2l\right)\Bigg]^{\widehat{}}(\xi_1,\xi_2)$$

$$=2\sum_{\nu=0}^{3}\sum_{k,l} h^\nu_{2k,2l+1}e^{2\pi i(2k\xi_1+2l\xi_2)}\overline{m_\nu(\xi_1,\xi_2)}\hat{f}(\xi_1,\xi_2)$$

$$=e^{-2\pi i\xi_2}\hat{f}(\xi_1,\xi_2)\sum_{\nu=0}^{3}\left[m_\nu(\xi_1,\xi_2)-m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)+m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)\right.$$

$$\left.-\,m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\right]\overline{m_\nu(\xi_1,\xi_2)}$$

$$=[f(\cdot,\cdot-1)]^{\widehat{}}(\xi_1,\xi_2)$$

and

$$\Bigg[\sum_{k,l} h^0_{2k+1,2l}F_0(\cdot+2k,\cdot+2l)+\ h^1_{2k+1,2l}F_1(\cdot+2k,\cdot+2l)+h^2_{2k+1,2l}F_2(\cdot+2k,\cdot+2l)$$

$$+\,h^3_{2k+1,2l}F_3(\cdot+2k,\cdot+2l)\Bigg]^{\widehat{}}(\xi_1,\xi_2)$$

$$=2\sum_{\nu=0}^{3}\sum_{k,l} h^\nu_{2k+1,2l}e^{2\pi i(2k\xi_1+2l\xi_2)}\overline{m_\nu(\xi_1,\xi_2)}\hat{f}(\xi_1,\xi_2)$$

$$=e^{-2\pi i\xi_1}\hat{f}(\xi_1,\xi_2)\sum_{\nu=0}^{3}\left[m_\nu(\xi_1,\xi_2)+m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)-m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)\right.$$

$$\left.-\,m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\right]\overline{m_\nu(\xi_1,\xi_2)}=[f(\cdot-1,\cdot)]^{\widehat{}}(\xi_1,\xi_2)$$

and

$$\Bigg[\sum_{k,l} h^0_{2k+1,2l+1}F_0(\cdot+2k,\cdot+2l)+h^1_{2k+1,2l+1}F_1(\cdot+2k,\cdot+2l)$$

$$+\,h^2_{2k+1,2l+1}F_2(\cdot+2k,\cdot+2l)+h^3_{2k+1,2l+1}F_3(\cdot+2k,\cdot+2l)\Bigg]^{\widehat{}}(\xi_1,\xi_2)$$

$$=2\sum_{\nu=0}^{3}\sum_{k,l} h^\nu_{2k,2l}e^{2\pi i(2k\xi_1+2l\xi_2)}\overline{m_\nu(\xi_1,\xi_2)}$$

$$
\begin{aligned}
=&e^{-2\pi i(\xi_1+\xi_2)}\hat{f}(\xi_1,\xi_2)\sum_{\nu=0}^{3}\left[m_\nu(\xi_1,\xi_2)-m_\nu\left(\xi_1,\xi_2-\frac{1}{2}\right)-m_\nu\left(\xi_1-\frac{1}{2},\xi_2\right)\right.\\
&\left.+m_\nu\left(\xi_1-\frac{1}{2},\xi_2-\frac{1}{2}\right)\right]\overline{m_\nu(\xi_1,\xi_2)}\\
=&[f(\cdot-1,\cdot-1)]\widehat{\ }(\xi_1,\xi_2). \qquad \square
\end{aligned}
$$

The above theorem shows that wavelet filters can be used to split any space spanned by two-dimensional orthonormal functions $f(\cdot-s,\cdot-t)$ into four parts. We can apply this method to the space W_0 spanned by the $\psi(\cdot-s,\cdot-t)$ in a two-dimensional multiresolution analysis with matrix dilation $2I$. In particular, if we choose arbitrary functions m_ν in Theorem 2.2 we have the following.

Corollary 2.1. *Let $\psi\in L^2(\mathbb{R}^2)$ be a two-dimensional wavelet which is generated by an MRA $\{V_j\}_{j\in\mathbb{Z}}$ and let m_ν, $\nu=0,\dots,3$, are masks not necessary related to ψ. Define*

$$\psi^\nu(x,y)=\sum_{s,t}h^\nu_{s,t}\psi(x-s,y-t),\quad \nu=0,\dots,3.$$

Then $\{\psi^\nu_{j,(2k,2l)}:\nu=0,\dots,3\}_{j,k,l\in\mathbb{Z}}$ consists of an orthonormal basis for $\overline{\mathrm{span}}\{\psi^\nu_{j,(2k,2l)}:\nu=0,\dots,3\}_{j,k,l\in\mathbb{Z}}$.

Proof. Let W_j be the orthonormal complement in V_{j+1} of V_j. Since $W_0=\overline{\mathrm{span}}\{\psi(\cdot-k,\cdot-l):k,l\in\mathbb{Z}\}$, by preceding theorem,

$$
\begin{aligned}
W_0=&\overline{\mathrm{span}}\{\psi^0(\cdot-2k,\cdot-2l):l,k\in\mathbb{Z}\}\bigoplus\overline{\mathrm{span}}\{\psi^1(\cdot-2k,\cdot-2l):l,k\in\mathbb{Z}\}\\
&\bigoplus\overline{\mathrm{span}}\{\psi^2(\cdot-2k,\cdot-2l):l,k\in\mathbb{Z}\}\bigoplus\overline{\mathrm{span}}\{\psi^3(\cdot-2k,\cdot-2l):l,k\in\mathbb{Z}\}\\
=&W_0^0\bigoplus W_0^1\bigoplus W_0^2\bigoplus W_0^3.
\end{aligned}
$$

Since each W_0^j, $j=0,\dots,3$, is generated by translations of $\psi^j(\cdot-2k,\cdot-2l)$, by dilation we can construct corresponding orthonormal bases for each W_m and their union is again a basis for $\overline{\mathrm{span}}\{\psi^\nu_{j,(2k,2l)}:\nu=0,\dots,3\}_{j,k,l\in\mathbb{Z}}$. $\square$

Corollary 2.2. *Let ψ be a one dimensional wavelet with scaling function ϕ. Consider masks m_ν, $\nu=0,\dots,3$, which is asserted in Theorem 2.2. Define*

$$
\begin{aligned}
\Psi^{a,i}=&\sum_{s,t}h^i_{s,t}\Psi^a(\cdot-s,\cdot-t),\quad i=1,\dots,3,\\
\Psi^{b,i}=&\sum_{s,t}h^i_{s,t}\Psi^b(\cdot-s,\cdot-t),\quad i=1,\dots,3,\\
\Psi^{d,i}=&\sum_{s,t}h^i_{s,t}\Psi^d(\cdot-s,\cdot-t),\quad i=1,\dots,3.
\end{aligned}
$$

Then, by Corollary 2.1,

$$\left\{\Psi^{r,i}_{j,(2s,2t)}:r=a,b,d,\ i=0,\dots,3\right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^2)$, where Ψ^r, $r=a,b,d$ are defined as (1.1).

3. Constructing an Example and Application of Example

3.1. **Example.** Consider one-dimensional Haar wavelet with scaling function $\phi = \chi_{[0,1)}$

$$\psi(x) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right), \\ 0, & \text{o.w.,} \end{cases}$$

then

$$\Psi^a(x,y) = \begin{cases} 1, & \text{if } x \in [0,1), y \in \left[0, \frac{1}{2}\right), \\ -1, & \text{if } x \in [0,1), y \in \left[\frac{1}{2}, 1\right), \\ 0, & \text{o.w.,} \end{cases}$$

$$\Psi^b(x,y) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{2}\right), y \in [0,1), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right), y \in [0,1), \\ 0, & \text{o.w.,} \end{cases}$$

$$\Psi^d(x,y) = \begin{cases} 1, & \text{if } x, y \in \left[0, \frac{1}{2}\right) \text{ or } x, y \in \left[\frac{1}{2}, 1\right), \\ -1, & \text{if } x \in \left[0, \frac{1}{2}\right), y \in \left[\frac{1}{2}, 1\right) \text{ or } x \in \left[\frac{1}{2}, 1\right), y \in \left[0, \frac{1}{2}\right), \\ 0, & \text{o.w.} \end{cases}$$

Now consider the masks in the Example (2.3), we have

$$\begin{aligned} h^0_{0,0} =& h^0_{0,-1} = h^0_{-1,0} = h^0_{-1,-1} = \frac{1}{2}, \\ h^1_{0,0} =& h^1_{0,-1} = \frac{1}{2},\ h^1_{-1,0} = h^1_{-1,-1} = -\frac{1}{2}, \\ h^2_{0,0} =& h^2_{-1,0} = \frac{1}{2},\ h^2_{0,-1} = h^2_{-1,-1} = -\frac{1}{2}, \\ h^3_{0,0} =& h^3_{-1,-1} = \frac{1}{2},\ h^3_{0,-1} = h^3_{-1,0} = -\frac{1}{2}. \end{aligned}$$

So, we have

$$\begin{aligned} \Psi^{a,0}(x,y) =& \sum_{s,t=-1}^{0} h^0_{s,t} \Psi^a(x-s, y-t) \\ =& \frac{1}{2}[\Psi^a(x,y) + \Psi^a(x,y+1) + \Psi^a(x+1,y) + \Psi^a(x+1,y+1)] \end{aligned}$$

$$= \begin{cases} \frac{1}{2}, & \text{if } (x,y) \in [0,1) \times \left[0, \frac{1}{2}\right) \text{ or } (x,y) \in [-1,1) \times \left[0, \frac{1}{2}\right) \\ & \text{or } (x,y) \in [-1,1) \times \left[-1, -\frac{1}{2}\right), \\ -\frac{1}{2}, & \text{if } (x,y) \in [-1,1) \times \left[\frac{1}{2}, 1\right) \text{ or } (x,y) \in [-1,1) \times \left[-\frac{1}{2}, 0\right), \\ 0, & \text{o.w.} \end{cases}$$

Similary,

$$\Psi^{r,0}(x,y) = \frac{1}{2}[\Psi^r(x,y) + \Psi^r(x,y+1) + \Psi^r(x+1,y) + \Psi^r(x+1,y+1)], \quad r = b, d,$$

and

$$\Psi^{r,1}(x,y) = \frac{1}{2}[\Psi^r(x,y) + \Psi^r(x,y+1) - \Psi^r(x+1,y) - \Psi^r(x+1,y+1)], \quad r = a, b, d,$$

$$\Psi^{r,2}(x,y) = \frac{1}{2}[\Psi^r(x,y) - \Psi^r(x,y+1) + \Psi^r(x+1,y) - \Psi^r(x+1,y+1)], \quad r = a, b, d,$$

$$\Psi^{r,3}(x,y) = \frac{1}{2}[\Psi^r(x,y) - \Psi^r(x,y+1) - \Psi^r(x+1,y) + \Psi^r(x+1,y+1)], \quad r = a, b, d.$$

Hence, $\{2^j\Psi^{r,i}(2^j \cdot -2k, 2^j \cdot -2l) : i = 0, \ldots, 3,\ r = a, b, d,\ j, k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

The diagram of the 12 mother wavelets is shown in Figures 1–12.

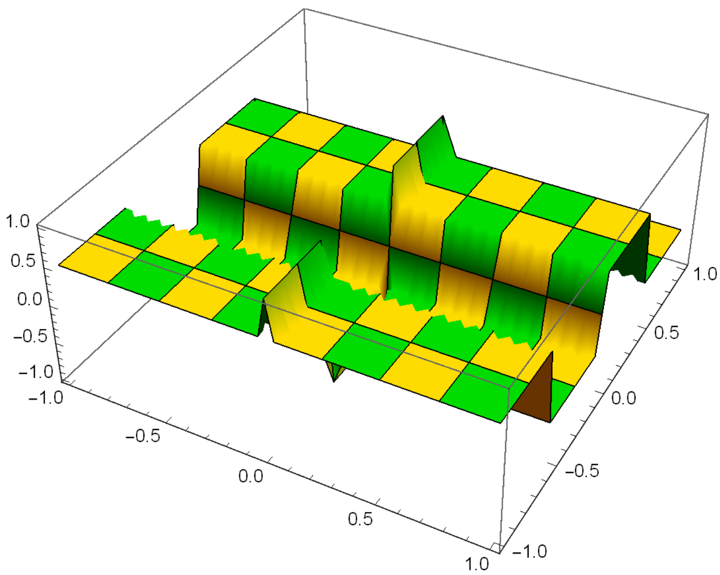

FIGURE 1. $\Psi^{a,0}(x,y)$

3.2. **Application of example in solving two dimensional first kind Fredholm integral equation.** Now we are going to show that our example consists a useful

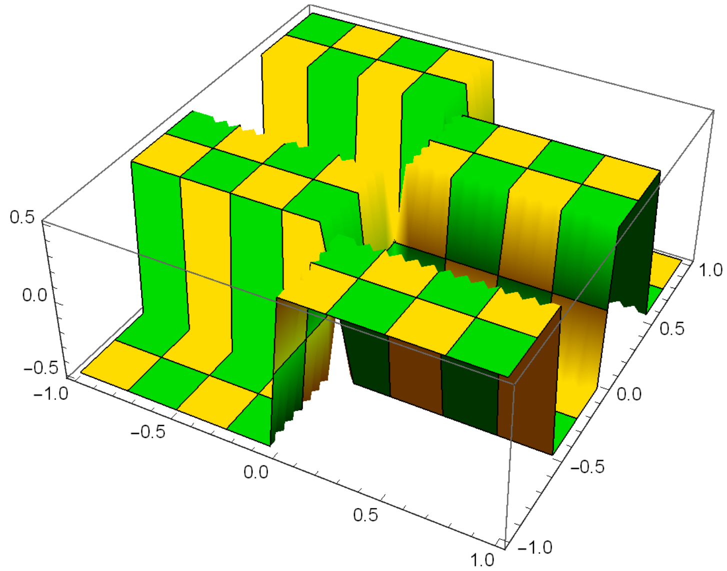

FIGURE 2. $\Psi^{a,1}(x, y)$

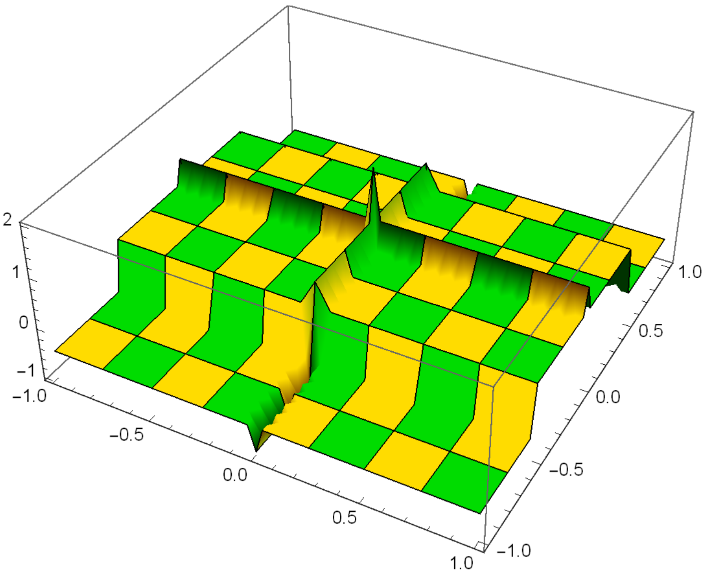

FIGURE 3. $\Psi^{a,2}(x, y)$

basis wavelet to find numerical solution for the first kind Fredholm integral equations. A two-dimensional first kind Fredholm integral equation has the following form

$$f(x, y) = \int_c^d \int_a^b k(x, y, s, t)\mathcal{G}(u(s, t))dsdt, \tag{3.1}$$

where $k(x, y, s, t)$ and $f(x, y)$ are known functions and $u(x, y)$ is an unknown function to be determined. To solve (3.1), if $\{\Psi_{jk} : j, k \in \mathbb{Z}\}$ is a wavelet basis let β be a finite

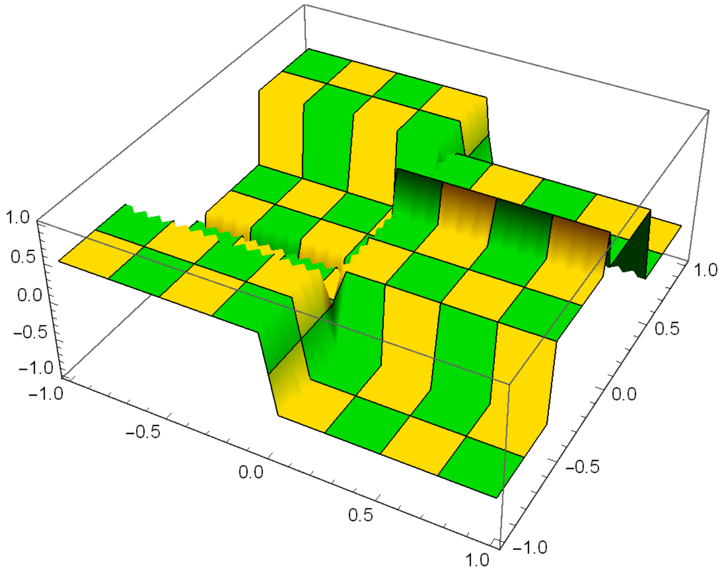

FIGURE 4. $\Psi^{a,3}(x, y)$

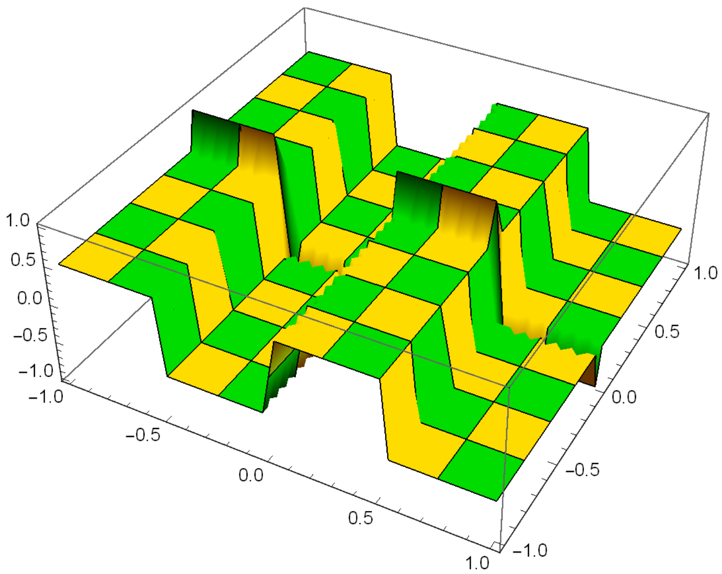

FIGURE 5. $\Psi^{b,0}(x, y)$

subset of it. If $\tilde{u}$ is an approximated solution for (3.1) which is compute by β take

$$\tilde{r} = \int_c^d \int_a^b k(x, y, s, t)\mathcal{G}(\tilde{u}(s, t))dsdt - f(x, y).$$

To find $\tilde{u}$, we have to solve the system $\langle \tilde{r}, y \rangle = 0$ for all $y \in \beta$. Since first kind Fredholm integral equations generate ill-conditioned systems, to solve the mentioned system we use Tikhonov regularization. Note that if $\tilde{u}$ is a solution for (3.1), then

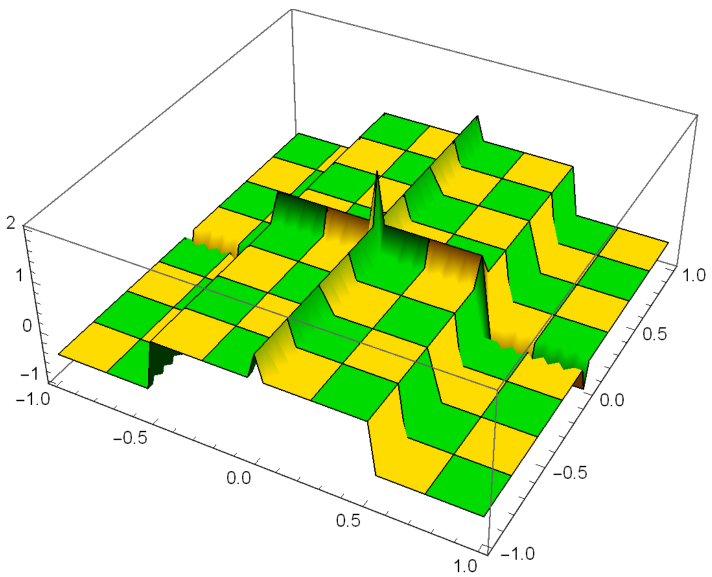

FIGURE 6. $\Psi^{b,1}(x,y)$

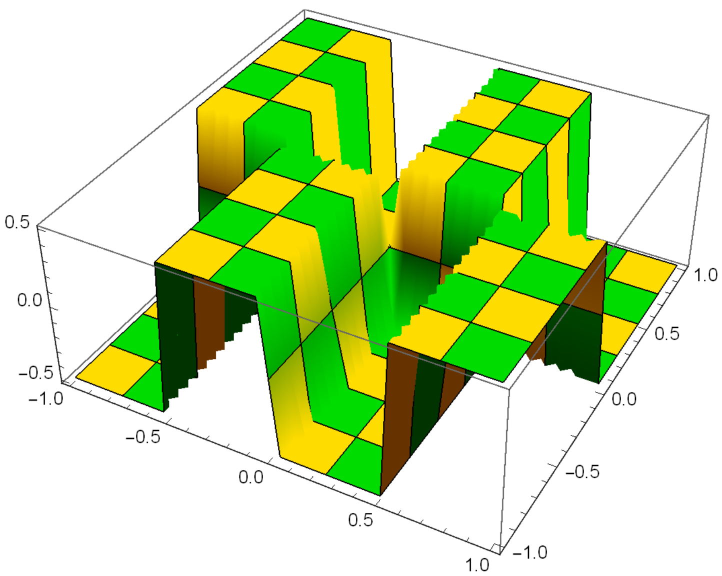

FIGURE 7. $\Psi^{b,2}(x,y)$

$\|\tilde{r}\|_2 = 0$. Then we are going to find $\tilde{u}$ such that $\|\tilde{r}\|_2$ be the smallest. We named the value of $\|\tilde{r}\|_2$ as L^2-norm of error. For more details see [2, 4, 7, 9].

Now by using wavelet basis which was presented in the former subsection, we are going to solve two following examples.

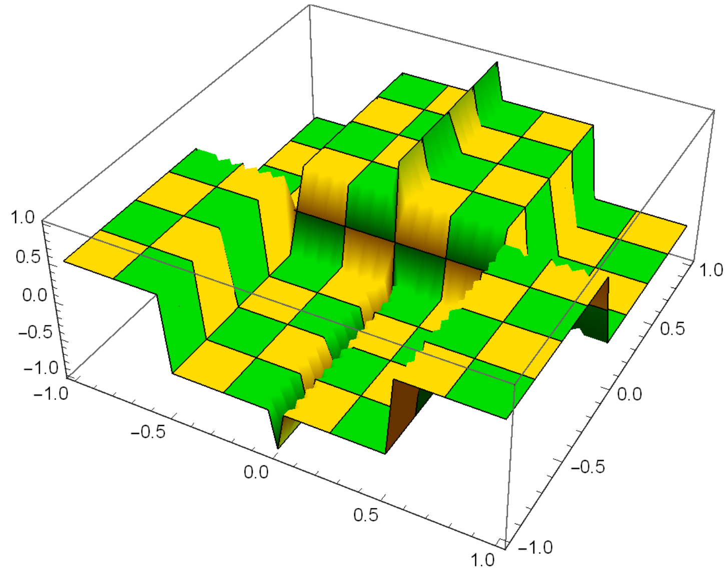

FIGURE 8. $\Psi^{b,3}(x,y)$

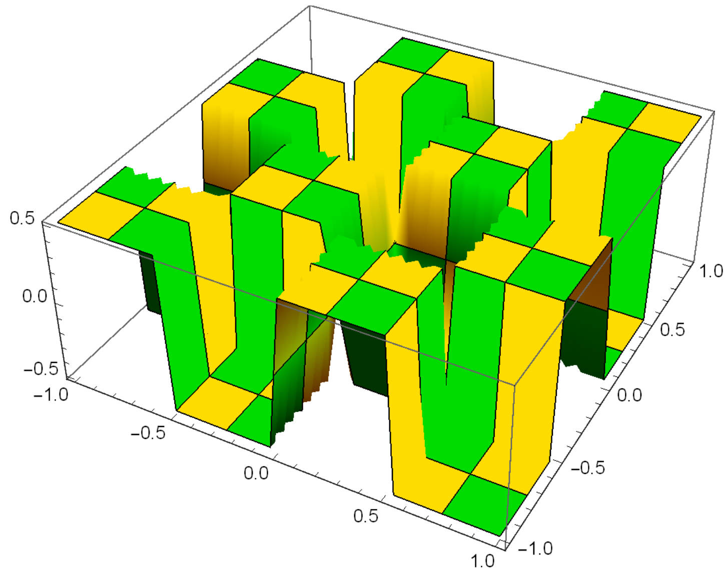

FIGURE 9. $\Psi^{d,0}(x,y)$

Example 3.1. Consider the integral equation

$$\int_0^1 \int_0^1 (x^2 s + yt^2) u(s,t) ds dt = \frac{15(9944x^2 + 16549y)}{131072}.$$

We have

$$\begin{aligned} \tilde{u}(x,y) = & -3.78777\Psi^{a,0}(x,y) - 3.78777\Psi^{a,1}(x,y) - 3.78777\Psi^{a,2}(x,y) \\ & -3.78777\Psi^{a,3}(x,y) - 2.276\Psi^{b,0}(x,y) - 2.276\Psi^{b,1}(x,y) \end{aligned}$$

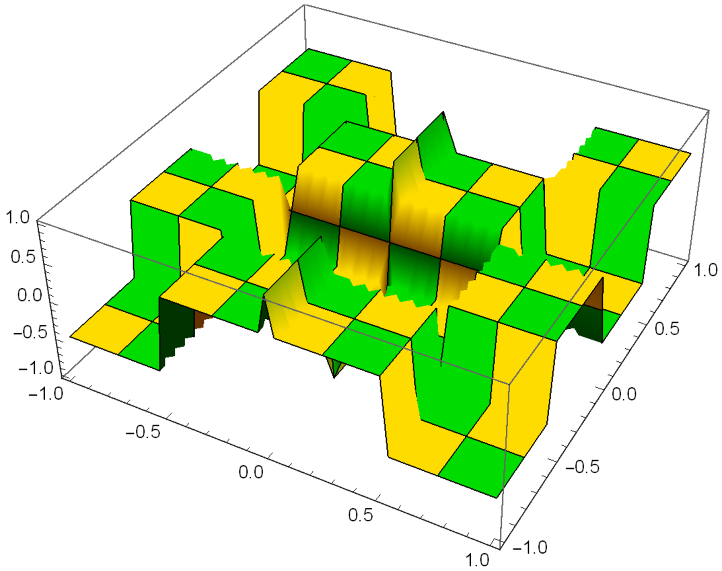

FIGURE 10. $\Psi^{d,1}(x,y)$

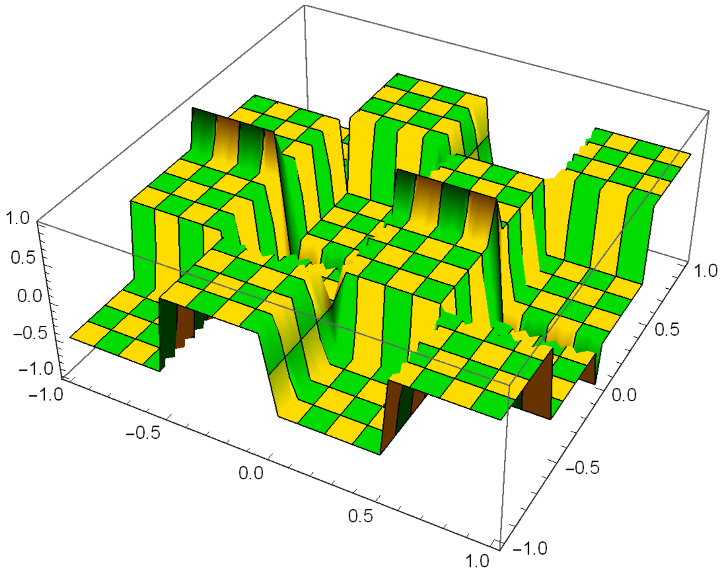

FIGURE 11. $\Psi^{d,2}(x,y)$

$$- 2.276\Psi^{b,2}(x,y)2.276\Psi^{b,3}(x,y) + \Psi^{d,0}(x,y) + \Psi^{d,1}(x,y) + \Psi^{d,2}(x,y) + \Psi^{d,3}(x,y).$$

The L^2-norm of error equals 1.42968×10^{-11}.

If we use two-dimensional Haar wavelets which are made by (1.1), our numerical solution is $-7.57553\Psi^a(x,y) - 4.552\Psi^b(x,y) + \Psi^d(x,y)$ and the L^2-norm of error equals 1.03387×10^{-8}.

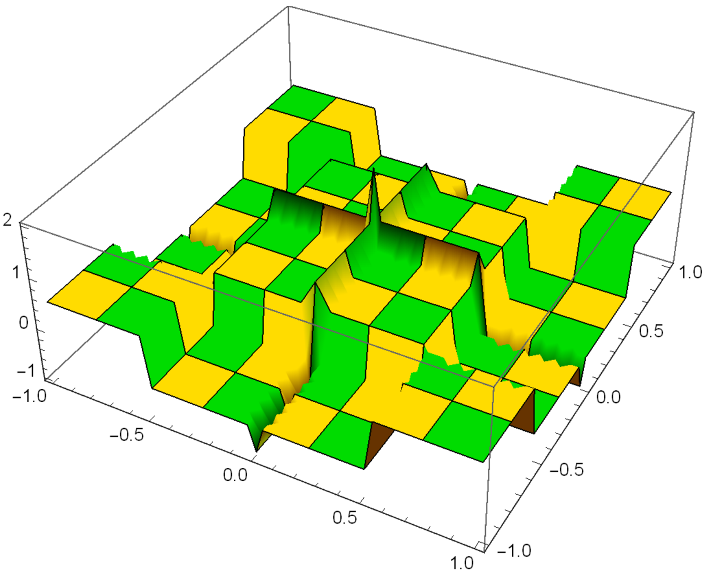

FIGURE 12. $\Psi^{d,3}(x,y)$

Example 3.2. Consider integral equation

$$\int_0^1\int_0^1 (x^2+yt)(u^2(s,t)-u(s,t))dsdt = \frac{(-12+5\ \pi)(2x^2+y)}{12\ \pi}.$$

We have

$$\begin{aligned}\tilde{u}(x,y) =&0.0841832\Psi^{a,0}(x,y)+0.0841832\Psi^{a,1}(x,y)+0.0841832\Psi^{a,2}(x,y)\\&+0.0841832\Psi^{a,3}(x,y)+0.145072\Psi^{b,0}(x,y)+0.145072\Psi^{b,1}(x,y)\\&+0.145072\Psi^{b,2}(x,y)+0.145072\Psi^{b,3}(x,y)+0.145072\Psi^{d,0}(x,y)\\&+0.145072\Psi^{d,1}(x,y)+0.145072\Psi^{d,2}(x,y)+0.145072\Psi^{d,3}(x,y).\end{aligned}$$

The L^2-norm of error equals 1.13239×10^{-9}.

REFERENCES

[1] J. J.-P. Antoine, R. Murenzi, P. Vandergheynst and S. T. Ali, *Two-Dimensional Wavelets and their Relatives*, Cambridge University Press, Cambridge, 2004.
[2] C. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.
[3] I. Daubechies, *Ten Lecture on Wavelets*, SIAM, Philaderphia, PA, 1992.
[4] G. H. Golub, P. C. Hansen and D. P. O'Leary, *Tikhonov regularization and total least squares*, SIAM J. Matrix Anal. Appl. **21**(1) (1999), 185–194.
[5] C. W. Groetsch, *Integral equations of the first kind, inverse problems and regularization: a crash course*, J. Phys. Conf. Ser. **73**(1) (2007), Paper ID 012001.
[6] A. Krisch, *An Introduction to the Mathematical Theory of Inverse Problems*, Second Edition, Springer-Verlag, New York, 2013.
[7] D. Lesnic, S. A. Yousefi and M. Ivanchov, *Determination of a time-dependent diffusivity from nonlocal conditions*, Int. J. Appl. Math. Comput. Sci. **41** (2013), 301–320.

[8] I. Ya. Novikov, V. Yu. Protasova and M. A. Skopina, *Wavelet Theory*, American Mathematical Soc., Providence, Rhode Island, 2011.
[9] M. Tahami, A. Askari Hemmat and S. A. Yousefi, *Numerical solution of two-dimensional first kind Fredholm integral equations by using linear Legendre wavelet*, Int. J. Wavelets Multiresolut. Inf. Process. **14**(1) (2016), DOI 10.1142/S0219691316500041.

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