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ON DISTANCE IRREGULAR LABELING OF DISCONNECTED GRAPHS

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Abstract. A distance irregular k-labeling of a graph G is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that the weights of all vertices are distinct. The weight of a vertex v, denoted by $wt(v)$, is the sum of labels of all vertices adjacent to v (distance 1 from v), that is, $wt(v) = \sum_{u \in N(v)} f(u)$. If the graph G admits a distance irregular labeling then G is called a distance irregular graph. The distance irregularity strength of G is the minimum k for which G has a distance irregular k-labeling and is denoted by $\text{dis}(G)$. In this paper, we derive a new lower bound of distance irregularity strength for graphs with t pendant vertices. We also determine the distance irregularity strength of some families of disconnected graphs namely disjoint union of paths, suns, helms and friendships.

1. Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the set of neighbors of v is denoted by $N(v)$. We write $\deg(v)$ to represent the degree of v. The vertex v is called an *isolated vertex* if $\deg(v) = 0$. Meanwhile, if $\deg(v) = 1$, we then call such a vertex as a *pendant*. Other basic definitions and terminologies about graph theory not mentioned here, we refer the reader to a book [4]. By notation $[a, b]$ with integers a, b we mean the set of all integers x such that $a \leqslant x \leqslant b$.

A graph labeling is a mapping that carries some sets of graph elements to a set of positive integers, called *labels*, such that satisfies certain conditions. If the domain is vertex-set or edge-set, the labelings are called *vertex labelings* or *edge labelings*,

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respectively. If the domain is $V(G) \cup E(G)$, then it is called a *total labeling*. More details about recent results of graph labelings can be found in a great survey by Gallian [5].

One of interesting topics in graph labelings is a *distance irregular labeling*. This labeling is motivated by three concepts in graph labelings, namely a *distance magic labeling* [6], an (a, d)-*distance antimagic labeling* [1] and an *irregular labeling* [3]. For a graph G, a vertex labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is said to be a *distance irregular k-labeling* of G if the weights of all vertices are distinct. The weight of a vertex v, denoted by $wt(v)$, is the sum of labels of all vertices adjacent to v (distance 1 from v), that is, $wt(v) = \sum_{u \in N(v)} f(u)$. If the graph G admits a distance irregular labeling then G is called a *distance irregular graph*. The *distance irregularity strength* of G is the minimum k for which G has a distance irregular k-labeling and is denoted by $\mathrm{dis}(G)$.

The notion of distance irregular labeling was firstly introduced by Slamin in 2017 [8]. In his paper, he showed some particular graphs that admit a distance irregular labeling, such as paths with $\mathrm{dis}(P_n) = \lceil n/2 \rceil$ for $n \geqslant 4$, complete graphs with $\mathrm{dis}(K_n) = n$ for $n \geqslant 3$, cycles with $\mathrm{dis}(C_n) = \lceil (n+1)/2 \rceil$ for $n \equiv 0, 1, 2, 5 \pmod 8$, and wheels with $\mathrm{dis}(W_n) = \lceil (n+1)/2 \rceil$ for $n \equiv 0, 1, 2, 5 \pmod 8$. He also proved that for any two different vertices u and v of a graph G, if u and v have the same neighbors, then G has no distance irregular labeling. As a consequence of this property, he showed that some classes of graphs such as complete bipartite graphs, complete multipartite graphs, stars and trees containing vertex with at least two leaves, have no distance irregular labeling. Novindasari, Marjono and Abusini in [7] determined the distance irregularity strength of ladder graph and triangular ladder graph. Recently, in [2], Bong et al. completed the results for the distance irregularity strength of C_n and W_n, for $n \equiv 3, 4, 6, 7 \pmod 8$. In the same paper, they also determined the distance irregularity strength of m-book graphs B_m and $G + K_1$ for any connected graph G admitting a distance irregular labeling.

So far, all papers concerning distance irregular labeling have presented the results only for connected graphs. Meanwhile, determining the distance irregularity strength for disconnected graphs has still never been studied. Motivated by this, in this paper, we study the distance irregular labeling for disconnected graphs. We derive a new lower bound of distance irregularity strength for graphs with t pendant vertices. Also, the distance irregularity strength for some classes of disconnected graphs especially disjoint union of paths, suns, helms and friendships will be determined through this paper.

The following lemma gives the general lower bound for distance irregularity strength of graphs found by Slamin [8].

Lemma 1.1 ([8]). *Let G be a graph on p vertices with minimum degree δ and maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. Then*

$$\operatorname{dis}(G) \geqslant \left\lceil \frac{\delta + p - 1}{\Delta} \right\rceil .$$

2. Main Results

Our first result gives a lower bound of distance irregularity strength for a graph having t pendant vertices. We note that the graph is not necessarily connected.

Lemma 2.1. *Let G be a graph on p vertices with maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. If G has t pendant vertices, then*

$$\operatorname{dis}(G) \geqslant \max\left\{ t, \left\lceil \frac{p}{\Delta} \right\rceil \right\} .$$

Proof. Let G be a graph on p vertices with maximum degree Δ containing no isolated vertex and no vertices with identical neighbors. For a positive integer t, let $x_1, x_2, \ldots, x_t$ be the pendant vertices of G. Since the weight of every vertex of G must be distinct, then the labels of neighbor of all x_is must be distinct, that is, $f(N(x_1)) \neq f(N(x_2)) \neq \ldots \neq f(N(x_t))$. So, $\operatorname{dis}(G) \geqslant t$. Combining with the lower bound for $\delta = 1$ (since the minimum degree of G is 1) in Lemma 1.1, we have $\operatorname{dis}(G) \geqslant \max\{t, \lceil p/\Delta \rceil\}$. □

The lower bound in Lemma 2.1 is tight as can be seen from Theorem 2.1, 2.2 and 2.3, which present the exact value of distance irregularity strength for disconnected paths, suns and helms, respectively.

2.1. Disjoint union of paths.

In this subsection, we deal with a distance irregular labeling of disconnected paths. Let mP_n be a disjoint union of m identical copies of paths with vertex set $V(mP_n) = \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mP_n) = \{v_i^j v_{i+1}^j : i \in [1, n-1], j \in [1, m]\}$. For $m \geqslant 2$ and $n = 3$, there exist vertices having the same neighbors. Consequently, the graph mP_3 has no distance irregular labeling. However, for $m \geqslant 2$ and $n \geqslant 4$, the graph mP_n admits a distance irregular labeling and its distance irregularity strength will be determined by the following theorem.

Theorem 2.1. *For each $m \geqslant 2$ and $n \geqslant 4$, $\operatorname{dis}(mP_n) = \lceil mn/2 \rceil$.*

Proof. As $n \geqslant 4$, it follows from Lemma 2.1 that $\operatorname{dis}(mP_n) \geqslant \lceil mn/2 \rceil$. To prove the reverse inequality, define a vertex labeling $f : V(mP_n) \to \{1, 2, \ldots, \lceil mn/2 \rceil\}$ as follows.

Case 1. Let $n \equiv 0 \pmod 4$.

For $j \in [1, m]$, label each vertex in the following way:

$$
\begin{aligned}
f(v_i^j) &= \frac{1}{2}(n+1-i)m, && \text{for } i = 1, 5, \dots, n-3,\\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6, \dots, n-2,\\
f(v_i^j) &= \frac{1}{2}(n+1-i)m + j, && \text{for } i = 3, 7, \dots, n-1,\\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8, \dots, n.
\end{aligned}
$$

Hence, for $j \in [1, m]$, the labeling gives the vertex weights as follows:

$$
\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, \dots, n-1,\\
wt(v_i^j) &= (n+1-i)m + j, && \text{for } i = 2, 4, \dots, n.
\end{aligned}
$$

Case 2. Let $n \equiv 1 \pmod 4$.

For $n = 5$, first, label all vertices except v_1^j, $j \in [1, m]$, in the following way:

$$
\begin{aligned}
f(v_2^j) &= \frac{5j}{2} - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
f(v_2^j) &= \left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
f(v_3^j) &= \left\lfloor \frac{5(m+j)}{2} \right\rfloor - \left\lceil \frac{5m}{2} \right\rceil, && \text{for } j \in [1, m],\\
f(v_4^j) &= \left\lfloor \frac{5j}{2} \right\rfloor, && \text{for } j \in [1, m],\\
f(v_5^j) &= \left\lceil \frac{5m}{2} \right\rceil, && \text{for } j \in [1, m].
\end{aligned}
$$

Then, we obtain all vertex weights except $wt(v_2^j)$, $j \in [1, m]$:

$$
\begin{aligned}
wt(v_1^j) &= \frac{5j}{2} - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
wt(v_1^j) &= \left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
wt(v_3^j) &= 5j - 2, && \text{for } j \equiv 2^t \pmod{2^{t+1}},\ t \text{ is even},\ t \geqslant 2,\\
wt(v_3^j) &= 2\left\lfloor \frac{5j}{2} \right\rfloor - 1, && \text{for other } j,\\
wt(v_4^j) &= \left\lfloor \frac{5(m+j)}{2} \right\rfloor, && \text{for } j \in [1, m],\\
wt(v_5^j) &= \left\lfloor \frac{5j}{2} \right\rfloor, && \text{for } j \in [1, m].
\end{aligned}
$$

Next, for $j \in [1, m]$, the label of v_1^j and the weight of v_2^j will be determined by using the following algorithm.

1. Let $W = \left\{ wt(v_3^j) : j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m \right] \right\}$.

2. For j from 1 up to m, do
   a. $p = f(v_3^j) = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - \left\lceil \frac{5m}{2} \right\rceil$;
   b. $q = wt(v_4^j) = \left\lfloor \frac{5(m+j)}{2} \right\rfloor$.
   c. If $(q-1)$ is contained in W, then
      1) $f(v_1^j) = q - p - 2 = \left\lceil \frac{5m}{2} \right\rceil - 2$;
      2) $wt(v_2^j) = q - 2 = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - 2$;
      3) $W = W \backslash \{q-1\}$.
   d. Else
      1) $f(v_1^j) = q - p - 1 = \left\lceil \frac{5m}{2} \right\rceil - 1$;
      2) $wt(v_2^j) = q - 1 = \left\lfloor \frac{5(m+j)}{2} \right\rfloor - 1$.

For $n \geqslant 9$ and $j \in [1, m]$, label each vertex in the following way:

$$
\begin{aligned}
f(v_i^j) =& \frac{1}{4}(3n-4+i)m - \left\lfloor \frac{mn}{2} \right\rfloor + j - 1, \\
& \text{for } i = 1, 5, \dots, \frac{n-7}{2} \quad (\text{if } n \equiv 1 \pmod 8), \\
f(v_i^j) =& \frac{1}{4}(3n+i)m - \left\lfloor \frac{mn}{2} \right\rfloor - j + 1, \\
& \text{for } i = 1, 5, \dots, \frac{n-11}{2} \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_i^j) =& \frac{1}{4}(i-2)m + j, \quad \text{for } i = 2, 6, \dots, n-3, \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n+2-i)m + 1, \\
& \text{for } i = 3, 7, \dots, \frac{n-11}{2} \quad (\text{if } n \equiv 1 \pmod 8), \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n+6-i)m + 2j - 1, \\
& \text{for } i = 3, 7, \dots, \frac{n-7}{2} \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_i^j) =& \frac{mi}{4}, \quad \text{for } i = 4, 8, \dots, n-5, \\
f(v_i^j) =& \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(n+2-i)m + j, \\
& \text{for } i = \frac{n-3}{2}, \frac{n+5}{2}, \dots, n-2 \quad (\text{if } n \equiv 1 \pmod 8) \text{ or} \\
& \text{for } i = \frac{n+1}{2}, \frac{n+9}{2}, \dots, n-2 \quad (\text{if } n \equiv 5 \pmod 8),
\end{aligned}
$$

$$
\begin{aligned}
f(v_i^j) =& \frac{1}{4}(3n+i)m - \left\lfloor \frac{mn}{2} \right\rfloor, \\
& \text{for } i = \frac{n+1}{2}, \frac{n+9}{2}, \ldots, n-4 \quad (\text{if } n \equiv 1 \pmod 8) \text{ or} \\
& \text{for } i = \frac{n-3}{2}, \frac{n+5}{2}, \ldots, n-4 \quad (\text{if } n \equiv 5 \pmod 8), \\
f(v_{n-1}^j) =& \frac{1}{2}(n-3)m + j, \\
f(v_n^j) =& \left\lceil \frac{mn}{2} \right\rceil.
\end{aligned}
$$

Thus, for $j \in [1, m]$, the labeling provides the following vertex weights:

$$
\begin{aligned}
wt(v_i^j) =& \frac{1}{2}(i-1)m + j, && \text{for } i = 1, 3, \ldots, n-4, \\
wt(v_i^j) =& \frac{1}{2}(n-3+i)m + j, && \text{for } i = 2, 4, \ldots, \frac{n-9}{2}, \\
wt\left(v_{\frac{n-5}{2}}^j\right) =& && \frac{1}{4}(3n-11)m + 2j - 1, \\
wt(v_i^j) =& \frac{1}{2}(n-1+i)m + j, && \text{for } i = \frac{n-1}{2}, \frac{n+3}{2}, \ldots, n-1, \\
wt(v_{n-2}^j) =& \frac{1}{4}(3n-11)m + 2j, \\
wt(v_n^j) =& \frac{1}{2}(n-3)m + j.
\end{aligned}
$$

Case 3. Let $n \equiv 2 \pmod 4$.
For $j \in [1, m]$, label each vertex in the following way:

$$
\begin{aligned}
f(v_1^j) &= \frac{1}{2}(n-2)m + j, \\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6, \ldots, n-4, \\
f(v_3^j) &= \frac{mn}{2}, \\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8, \ldots, n-2, \\
f(v_i^j) &= \frac{1}{2}(n+1-i)m + j, && \text{for } i = 5, 9, \ldots, n-1, \\
f(v_i^j) &= \frac{1}{2}(n+1-i)m, && \text{for } i = 7, 11, \ldots, n-3, \\
f(v_n^j) &= \frac{1}{2}(n-4)m + j.
\end{aligned}
$$

Hence, for $j \in [1, m]$, the labeling provides the following vertex weights:

$$\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, \ldots, n-3,\\
wt(v_i^j) &= \frac{1}{2}(2n-i)m + j, && \text{for } i = 2, 4,\\
wt(v_i^j) &= (n+1-i)m + j, && \text{for } i = 6, 8, \ldots, n,\\
wt(v_{n-1}^j) &= (n-3)m + j.
\end{aligned}$$

Case 4. Let $n \equiv 3 \pmod 4$.

For $n = 7$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_i^j) &= \frac{1}{4}(i+23)m - \left\lceil \frac{7m}{2} \right\rceil, && \text{for } i = 1, 5,\\
f(v_i^j) &= \frac{1}{4}(i-2)m + j, && \text{for } i = 2, 6,\\
f(v_i^j) &= \frac{1}{4}(i-11)m + \left\lceil \frac{7m}{2} \right\rceil + j, && \text{for } i = 3, 7,\\
f(v_4^j) &= 2m.
\end{aligned}$$

Then, for $j \in [1, m]$, the labeling yields the following vertex weights:

$$\begin{aligned}
wt(v_1^j) &= j,\\
wt(v_i^j) &= \frac{1}{2}(i+6)m + j, && \text{for } i = 2, 4, 6,\\
wt(v_i^j) &= \frac{1}{2}(i+1)m + j, && \text{for } i = 3, 5,\\
wt(v_7^j) &= m + j.
\end{aligned}$$

For $n = 11$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(v_1^j) &= \left\lfloor \frac{11m}{2} \right\rfloor - 5m,\\
f(v_i^j) &= \frac{1}{2}(i-2)m + j, && \text{for } i = 2, 6,\\
f(v_i^j) &= \frac{1}{4}(i-15)m + \left\lceil \frac{11m}{2} \right\rceil + j, && \text{for } i = 3, 7, 11,\\
f(v_i^j) &= \frac{mi}{2}, && \text{for } i = 4, 8,\\
f(v_i^j) &= \frac{1}{4}(i-9)m + \left\lfloor \frac{11m}{2} \right\rfloor, && \text{for } i = 5, 9,\\
f(v_{10}^j) &= m + j.
\end{aligned}$$

So, for $j \in [1, m]$, the labeling gives the following vertex weights:

$$
\begin{aligned}
wt(v_i^j) &= (i-1)m + j, && \text{for } i = 1, 3, 5, 7,\\
wt(v_2^j) &= 3m + j, &&\\
wt(v_i^j) &= \frac{1}{2}(i+10)m + j, && \text{for } i = 4, 6, 8, 10,\\
wt(v_i^j) &= (23-2i)m + j, && \text{for } i = 9, 11.
\end{aligned}
$$

For $n \geqslant 15$ and $j \in [1, m]$, label each vertex in the following way:

$$
\begin{aligned}
f(v_i^j) &= \left\lceil \frac{mn}{2} \right\rceil - \frac{1}{4}(i-1)m,\\
&\quad \text{for } i = 1, 5, \dots, n-2 \quad (\text{if } n \equiv 3 \pmod 8) \text{ or}\\
&\quad \text{for } i = 1, 5, \dots, \frac{n+11}{2} \quad (\text{if } n \equiv 7 \pmod 8)),\\
f(v_i^j) &= \frac{1}{4}(i-2)m + j, \quad \text{for } i = 2, 6, \dots, n-5,\\
f(v_i^j) &= \left\lfloor \frac{mn}{2} \right\rfloor - \frac{1}{4}(i+1)m + j,\\
&\quad \text{for } i = 3, 7, \dots, \frac{n+11}{2} \quad (\text{if } n \equiv 3 \pmod 8)) \text{ or}\\
&\quad \text{for } i = 3, 7, \dots, n \quad (\text{if } n \equiv 7 \pmod 8)),\\
f(v_i^j) &= \frac{1}{4}(i+4)m, \quad \text{for } i = 4, 8, \dots, n-7,\\
f(v_i^j) &= \frac{1}{4}(4n-5-i)m - \left\lceil \frac{mn}{2} \right\rceil + j,\\
&\quad \text{for } i = \frac{n+19}{2}, \frac{n+27}{2}, \dots, n \quad (\text{if } n \equiv 3 \pmod 8)),\\
f(v_i^j) &= \frac{1}{4}(4n-3-i)m - \left\lfloor \frac{mn}{2} \right\rfloor,\\
&\quad \text{for } i = \frac{n+19}{2}, \frac{n+27}{2}, \dots, n-2 \quad (\text{if } n \equiv 7 \pmod 8)),\\
f(v_{n-3}^j) &= \frac{1}{2}(n-5)m,\\
f(v_{n-1}^j) &= m + j.
\end{aligned}
$$

Thus, for $j \in [1, m]$, the labeling yields the following vertex weights:

$$
\begin{aligned}
wt(v_1^j) &= j, && \\
wt(v_i^j) &= \frac{1}{2}(2n-i)m + j, && \text{for } i = 2, 4, \ldots, \frac{n+13}{2}, \\
wt(v_i^j) &= \frac{1}{2}(i+1)m + j, && \text{for } i = 3, 5, \ldots, n-6, \\
wt(v_i^j) &= \frac{1}{2}(2n-2-i)m + j, && \text{for } i = \frac{n+17}{2}, \frac{n+21}{2}, \ldots, n-1, \\
wt(v_{n-4}^j) &= \frac{1}{4}(3n-17)m + j, && \\
wt(v_{n-2}^j) &= \frac{1}{2}(n-3)m + j, && \\
wt(v_n^j) &= m + j. &&
\end{aligned}
$$

From all cases, it can be checked that the vertex weights form the set $\{1, 2, \ldots, mn\}$ and the labels used in the labelings are at most $\lceil mn/2 \rceil$. Thus, $\text{dis}(mP_n) \leqslant \lceil mn/2 \rceil$. As $\lceil mn/2 \rceil \leqslant \text{dis}(mP_n) \leqslant \lceil mn/2 \rceil$, we can conclude that $\text{dis}(mP_n) = \lceil mn/2 \rceil$. $\square$

As an illustration, a distance irregular labeling of $6P_5$ is given in Figure 1, where red numbers show the vertex weights and black numbers represent the label of the vertices.

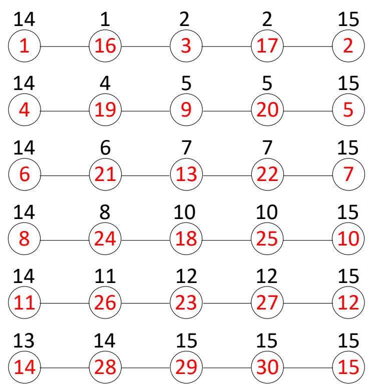

FIGURE 1. A distance irregular 15-labeling of $6P_5$.

2.2. **Disjoint union of suns.** A *sun*, denoted by S_n, is a graph with $2n$ vertices obtained from a cycle by attaching a *pendant vertex* to each cycle's vertex. We then call all vertices adjacent to such pendant vertices as the *rim vertices* of S_n. Now, let us denote by mS_n a disjoint union of m identical copies of sun graphs with vertex set $V(mS_n) = \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mS_n) = \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j u_{i+1}^j : i \in [1, n], j \in [1, m]\}$ where the index i is taken modulo n. Next, we will determine the distance irregularity strength of mS_n in the following theorem.

Theorem 2.2. *For each $m \geqslant 2$ and $n \geqslant 3$,* $\operatorname{dis}(mS_n) = mn$.

Proof. Consider the graph mS_n, with $2mn$ vertices. Since mS_n has mn pendant vertices, according to Lemma 2.1, we have $\operatorname{dis}(mS_n) \geqslant mn$. To prove that mn is the upper bound of $\operatorname{dis}(mS_n)$, it is sufficient to show the existence of a distance irregular mn-labeling of mS_n. To do that, let us define $f : V(mS_n) \to \{1, 2, \dots, mn\}$ as follows.

For $n = 3$ and $j \in [1, m]$, label every vertex in the following way:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,3], \\ f(v_i^j) &= 2m - j, && \text{for } i \in [1,3]. \end{aligned}$$

Hence, for $j \in [1, m]$, we obtain the following vertex weights:

$$\begin{aligned} wt(u_i^j) &= j - (i-6)m, && \text{for } i \in [1,3], \\ wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1,3]. \end{aligned}$$

For $n = 4$ and $j \in [1, m]$, label every vertex in the following way:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,4], \\ f(v_i^j) &= 3m - j, && \text{for } i = 1, 4, \\ f(v_i^j) &= 2m - j, && \text{for } i = 2, 3. \end{aligned}$$

So, for $j \in [1, m]$, we can get the weight of each vertex as follows:

$$\begin{aligned} wt(u_i^j) &= \frac{1}{2}(15-i)m + j, && \text{for } i = 1, 3, \\ wt(u_i^j) &= \frac{1}{2}(i+6)m + j, && \text{for } i = 2, 4, \\ wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1,4]. \end{aligned}$$

For $n \geqslant 5$ and $j \in [1, m]$, label each vertex as follows:

$$\begin{aligned} f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1,n], \\ f(v_i^j) &= m \left\lfloor \frac{n+1}{2} \right\rfloor - j, && \text{for } i = 1, n, \\ f(v_i^j) &= (n-i)m - j, && \text{for } i \in \left[2, \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right], \\ f\left(v^j_{\lfloor \frac{n+1}{2} \rfloor}\right) &= \left(n + 1 - \left\lfloor \frac{n+1}{2} \right\rfloor\right) m - j, \\ f(v_i^j) &= (n+2-i)m - j, && \text{for } i \in \left[\left\lfloor \frac{n+1}{2} \right\rfloor + 1, n-1\right]. \end{aligned}$$

Then, for $j \in [1, m]$, we obtain the weight of each vertex as follows:

$$
\begin{aligned}
wt(u_i^j) &= \left(n - \frac{2(i-1)}{n-1} + \left\lfloor \frac{n+1}{2} \right\rfloor\right) m + j, && \text{for } i = 1, n,\\
wt(u_i^j) &= (n - 2 + i)m + j, && \text{for } i \in \left[2, \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right],\\
wt\left(u^j_{\lfloor \frac{n+1}{2} \rfloor}\right) &= \left(n - 1 + \left\lfloor \frac{n+1}{2} \right\rfloor\right) m + j, &&\\
wt(u_i^j) &= (n + i)m + j, && \text{for } i \in \left[\left\lfloor \frac{n+1}{2} \right\rfloor + 1, n-1\right],\\
wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1, n].
\end{aligned}
$$

Clearly, the largest label appearing on the vertices is mn for each $n \geqslant 3$. Moreover, it can be checked that vertex weights of the pendant vertices and the rim vertices of mS_n constitute the set $\{1, 2, \dots, mn\}$ and the set $\{mn+1, mn+2, \dots, 2mn\}$, respectively. It means that f is a distance irregular mn-labeling of mS_n. The proof is complete. $\square$

In Figure 2, as an illustralion, a distance irregular labeling of $3S_5$ is shown.

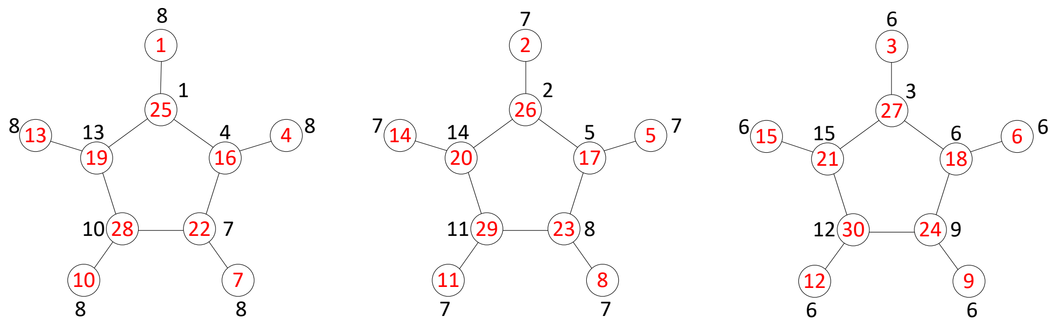

FIGURE 2. A distance irregular 15-labeling of $3S_5$.

2.3. **Disjoint union of helms.** A *helm*, denoted by H_n, is a graph constructed from a sun S_n by joining a new vertex, called *center vertex*, to all the rim vertices of S_n. Next, we focus on a disjoint union of m identical copies of helm graphs mH_n with vertex set $V(mH_n) = \{c^j : j \in [1, m]\} \cup \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mH_n) = \{c^j u_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j u_{i+1}^j : i \in [1, n], j \in [1, m]\}$ where the index i is taken modulo n.

Let us recall the labeling formula of the rim vertices of mS_n defined in the previous theorem, that is, for $m \geqslant 2$, $n \geqslant 3$, $i \in [1, n]$ and $j \in [1, m]$,

$$f(u_i) = j + (i-1)m.$$

The sum of labels of such rim vertices is

$$\sum_{i=1}^{n} f(u_i) = \sum_{i=1}^{n} (j + (i-1)m) = \frac{n}{2}(2j + (n-1)m). \tag{2.1}$$

Next, consider the set of vertex weights of mS_n obtained from Theorem 2.2, namely $\{1, 2, \ldots, 2mn\}$. We want to find all possible n such that the Equation (2.1) is different from all such vertex weights for every $m \geqslant 2$ and $j \in [1, m]$. Therefore,

$$\frac{n}{2}(2j + (n-1)m) > 2mn. \tag{2.2}$$

It is not difficult to show that (2.2) happens if and only if $n \geqslant 5$. Thus, we can use this characteristic to construct a distance irregular labeling of mH_n from the described distance irregular labeling of mS_n for case $n \geqslant 5$.

Next, we will present the distance irregularity strength of mH_n in the following theorem.

Theorem 2.3. *For each $m \geqslant 2$ and $n \geqslant 3$, $\operatorname{dis}(mH_n) = mn$.*

Proof. Consider the graph mH_n on $(2n+1)m$ vertices. Since mH_n has mn pendant vertices, by Lemma 2.1, we get $\operatorname{dis}(mH_n) \geqslant mn$. To prove that mn is the upper bound of $\operatorname{dis}(mH_n)$, it is sufficient to show the existence of an optimal distance irregular mn-labeling of mH_n. Let $f : V(mH_n) \to \{1, 2, \ldots, mn\}$ be a vertex labeling defined as follows.

For $n = 3$ and $j \in [1, m]$, label each vertex in the following way:

$$\begin{aligned}
f(c^j) &= 1, \\
f(u_i^j) &= j + (i-1)m, && \text{for } i \in [1, 3], \\
f(v_1^j) &= 3m - j - 1, \\
f(v_2^j) &= \frac{1}{2}(5m - 3) - \left\lceil \frac{j}{2} \right\rceil, && \text{if } m \text{ is odd}, \\
f(v_2^j) &= \frac{1}{2}(5m - 4) - \left\lfloor \frac{j}{2} \right\rfloor, && \text{if } m \text{ is even}, \\
f(v_3^j) &= 2m - 2 - \left\lceil \frac{j-1}{2} \right\rceil.
\end{aligned}$$

Therefore, for $j \in [1, m]$, we obtain the following vertex weights:

$$\begin{aligned}
wt(c^j) &= 3(m + j), \\
wt(u_1^j) &= 6m + j, \\
wt(u_2^j) &= \frac{1}{2}(9m - 1) + \left\lfloor \frac{3j}{2} \right\rfloor, && \text{if } m \text{ is odd}, \\
wt(u_2^j) &= \frac{1}{2}(9m - 2) + \left\lceil \frac{3j}{2} \right\rceil, && \text{if } m \text{ is even}, \\
wt(u_3^j) &= 3m - 1 + \left\lfloor \frac{3j+1}{2} \right\rfloor, \\
wt(v_i^j) &= j + (i-1)m, && \text{for } i \in [1, 3].
\end{aligned}$$

For $n = 4$ and $j \in [1, m]$, label every vertex in the following way:

$$
\begin{aligned}
&f(c^j) = 1, \\
&f(u_i^j) = j + (i-1)m, && \text{for } i \in [1,4], \\
&f(v_1^j) = \frac{1}{3}(10m-6) - \left\lceil \frac{2j-2}{3} \right\rceil, && \text{if } m \equiv 0 \pmod 3, \\
&f(v_1^j) = \frac{1}{3}(10m-4) - \left\lceil \frac{2j}{3} \right\rceil, && \text{if } m \equiv 1 \pmod 3, \\
&f(v_1^j) = \frac{1}{3}(10m-5) - \left\lceil \frac{2j-1}{3} \right\rceil, && \text{if } m \equiv 2 \pmod 3, \\
&f(v_2^j) = 2m - j - 1, \\
&f(v_3^j) = 2m - \left\lceil \frac{2j+4}{3} \right\rceil, \\
&f(v_4^j) = 3m - j - 1.
\end{aligned}
$$

So, for $j \in [1, m]$, we get the vertex weights as follows:

$$
\begin{aligned}
&wt(c^j) = 6m + 4j, \\
&wt(u_1^j) = \frac{1}{3}(22m-3) + \left\lfloor \frac{4j+2}{3} \right\rfloor, && \text{if } m \equiv 0 \pmod 3, \\
&wt(u_1^j) = \frac{1}{3}(22m-1) + \left\lfloor \frac{4j}{3} \right\rfloor, && \text{if } m \equiv 1 \pmod 3, \\
&wt(u_1^j) = \frac{1}{3}(22m-2) + \left\lfloor \frac{4j+1}{3} \right\rfloor, && \text{if } m \equiv 2 \pmod 3, \\
&wt(u_2^j) = 4m + j, \\
&wt(u_3^j) = 6m + \left\lfloor \frac{4j-1}{3} \right\rfloor, \\
&wt(u_4^j) = 5m + j, \\
&wt(v_i^j) = j + (i-1)m, && \text{for } i \in [1,4].
\end{aligned}
$$

Now, let $n \geqslant 5$. For the proof purpose only, first, let us denote the described vertex labelings and vertex weights formula of mS_n, $n \geqslant 5$, by f^* and by wt^*, respectively. Next, for $j \in [1, m]$, label every vertex of mH_n such that

$$
\begin{aligned}
&f(c^j) = 1, \\
&f(u_i^j) = f^*(u_i^j), \\
&f(v_i^j) = f^*(v_i^j) - 1.
\end{aligned}
$$

Then, for $j \in [1, m]$, we obtain the vertex weights as follows:

$$\begin{aligned} wt(c^j) &= \frac{n}{2}((n-1)m + 2j), \\ wt(u_i^j) &= wt^*(u_i^j), \\ wt(v_i^j) &= wt^*(v_i^j). \end{aligned}$$

It can be verified that all the vertex weights are distinct for all pairs of distinct vertices and the largest label is mn, which lead to $\text{dis}(mH_n) \leqslant mn$. Combining with the lower bound, we have $\text{dis}(mH_n) = mn$. □

We show in Figure 3 a distance irregular labeling of $3H_5$ as an illustration.

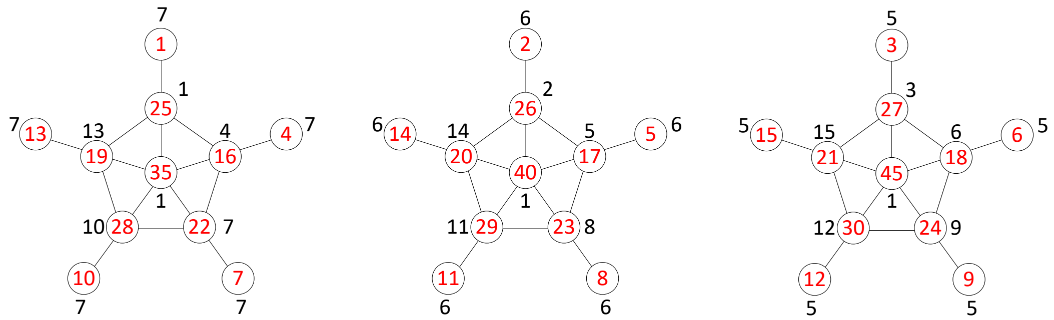

FIGURE 3. A distance irregular 15-labeling of $3H_5$.

2.4. **Disjoint union of friendships.** A *friendship* f_n is a graph obtained by identifying a vertex from n copies of *triangles* K_3. The vertex of degree $2n$ is called the *center vertex* and the remaining vertices are called the *rim vertices*. Now, we focus on a disjoint union of m identical copies of friendships mf_n with vertex set $V(mf_n) = \{c^j : j \in [1, m]\} \cup \{u_i^j : i \in [1, n], j \in [1, m]\} \cup \{v_i^j : i \in [1, n], j \in [1, m]\}$ and edge set $E(mf_n) = \{c^j u_i^j, c^j v_i^j : i \in [1, n], j \in [1, m]\} \cup \{u_i^j v_i^j : i \in [1, n], j \in [1, m]\}$.

First, let us consider a single copy of friendship f_n. In the following lemma, we give a necessary condition for f_n to be a distance irregular graph.

Lemma 2.2. *If f_n is a distance irregular graph, then the labels of all rim vertices of f_n must be distinct.*

Proof. Let f be a distance irregular labeling of f_n. Let x, y be any two rim vertices of f_n. We show that $f(x) \neq f(y)$. Let c be the center vertex and let x', y' be rim vertices adjacent to x and y, respectively. We know that $wt(x) = f(c) + f(x')$ and $wt(y) = f(c) + f(y')$. Since $wt(x)$ and $wt(y)$ must be distinct, we get $f(x') \neq f(y')$. Since x, y are arbitrarily two rim vertices in the graph f_n and x', y' are also the rim vertices of f_n, it naturally implies that $f(x) \neq f(y)$. □

It is coherent to say that the property in Lemma 2.2 holds also for disconnected version of friendships. Thus, in any distance irregular labeling of mf_n, the labels of all rim vertices in the j^{th}-copy of f_n are distinct for $j \in [1, m]$. Next, we will determine the distance irregularity strength of mf_n in the following theorem.

Theorem 2.4. *For each $n \geqslant 2$ and $m \in [2, n]$, $\mathrm{dis}(mf_n) = mn + 1$.*

Proof. Firstly, we determine the lower bound of $\mathrm{dis}(mf_n)$. Let k be the largest label of the graph mf_n. The optimal weights of the vertices of mf_n are $2, 3, \ldots, 2mn + 1, wt(c^1), wt(c^2), \ldots, wt(c^m)$. Next, for some $i \in [1, n]$ and some $s \in [1, m]$, let $wt(c^s)$ and $wt(v_i^s)$, be the largest weight of the center vertices of mf_n and the largest weight of the rim vertices of mf_n, respectively. Furthermore, it follows from Lemma 2.2 that the labels of every rim vertex in the j^{th}-copy of f_n, $j \in [1, m]$, must be distinct. Since the center vertex c^s is adjacent to all rim vertices in the s^{th}-copy of f_n, then the largest label used in the computation of $wt(c^s)$ is at most k. On the other hand, we have $wt(v_i^s) \geqslant 2mn + 1$. Since $\deg(v_i^s) = 2$, we obtain $\mathrm{dis}(mf_n) = k \geqslant \lceil (2mn + 1)/2 \rceil = mn + 1$. Next, for the upper bound of $\mathrm{dis}(mf_n)$, construct a vertex labeling $f : V(mf_n) \to \{1, 2, , \ldots, mn + 1\}$ as follows:

$$\begin{aligned}
f(c^j) &= (2n-1)(j-1)+1, && \text{for } j \in [1,2],\\
f(c^j) &= nj, && \text{for } j \in [3,m],\\
f(u_i^j) &= 2i+j-1, && \text{for } i \in [1,n] \text{ and } j \in [1,2],\\
f(u_i^j) &= 2i+1+(j-2)n, && \text{for } i \in [1,n] \text{ and } j \in [3,m],\\
f(v_i^j) &= 2i+j-2, && \text{for } i \in [1,n] \text{ and } j \in [1,2],\\
f(v_i^j) &= 2i+(j-2)n, && \text{for } i \in [1,n] \text{ and } j \in [3,m].
\end{aligned}$$

Therefore, we get the vertex weights as follows:

$$\begin{aligned}
wt(c^j) &= 2n^2+(2j-1)n, && \text{for } j \in [1,2],\\
wt(c^j) &= 2n^2(j-1)+3n, && \text{for } j \in [3,m],\\
wt(u_i^j) &= 2n(j-1)+2i, && \text{for } i \in [1,n] \text{ and } j \in [1,m],\\
wt(v_i^j) &= 2n(j-1)+2i+1, && \text{for } i \in [1,n] \text{ and } j \in [1,m].
\end{aligned}$$

It can be verified that f is a distance irregular $(mn+1)$-labeling of mf_n as the vertex weights are unique and the labels appearing on the vertices are at most $mn+1$. Thus $\mathrm{dis}(mf_n) \leqslant mn + 1$. This concludes the proof. $\square$

An example of distance irregular labeling of mf_n is described in Figure 4.

3. Conclusion

In this paper we initiated to study the distance irregular labeling of disconnected graphs. A new lower bound of the distance irregularity strength for a graph G having

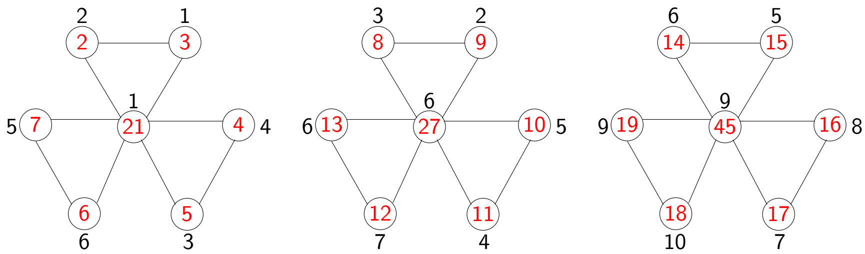

FIGURE 4. A distance irregular 10-labeling of $3f_3$.

t pendant vertices was introduced and we proved that $\mathrm{dis}(G) \geqslant \max\{t, \lceil p/\Delta \rceil\}$. We also showed that this lower bound is sharp for disconnected paths, suns and helms.

Because of the limitation of results we found related to this parameter for disconnected graphs, we propose the open problem below.

Open Problem 1. Determine the distance irregularity strength of other classes of disconnected graphs.

In relation with our lower bound in Lemma 2.1 which works for graphs containing t pendant vertices ($\delta = 1$), the following open problems are also interesting to be studied.

Open Problem 2. Characterize all graphs containing t pendant vertices having distance irregularity strength t. Particularly, characterize all trees with t leaves having distance irregularity strength t.

Open Problem 3. Characterize all graphs containing t pendant vertices having distance irregularity strength $\lceil p/\Delta \rceil$. Specifically, characterize all trees with t leaves having distance irregularity strength $\lceil p/\Delta \rceil$.

In Theorem 2.4, we determined the distance irregularity strength of disconnected friendships mf_n only for $m \leqslant n$. Meanwhile, this parameter is still unsolved for the remaining case of mf_n. Therefore, we also give the following open problem.

Open Problem 4. Determine the distance irregularity strength of mf_n for $m > n$.

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