

**TWO-DIMENSIONAL WAVELET WITH MATRIX DILATION
 $M = 2I$ AND ITS APPLICATION IN SOLVING INTEGRAL
EQUATIONS**

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ABSTRACT. In this study, using a one-dimensional MRA we constructed a two-dimensional wavelet as well as four masks which are not related to the MRA. Finally, we provide some examples to prove the applicability of our construction in case of finding numerical solution of two-dimensional first kind Fredholm integral equations.

1. INTRODUCTION

Let $\{V_j\}$ be a one-dimensional multiresolution analysis (MRA) with scaling function ϕ and mother wavelet ψ , then $\Phi(x, y) = \phi(x)\phi(y)$ is a scaling function for two-dimensional MRA and in this case, we have 3 mother wavelets

$$(1.1) \quad \Psi^a(x, y) = \phi(x)\psi(y), \quad \Psi^b(x, y) = \psi(x)\phi(y), \quad \Psi^d(x, y) = \psi(x)\psi(y).$$

It means that $\{\Psi_{j,(s,t)}^r : j, s, t \in \mathbb{Z}, r = a, b, d\}$ consists of an orthonormal basis for $L^2(\mathbb{R})$. For more details see [1, 3, 9].

In applications, finding a way to construct a wavelet with a smaller frequency domain and correspondingly increase in time domain is of great importance. The higher the number of mother wavelets, the more accurate the answer would be. Finding a way to minimizing frequency domain and so maximizing accuracy is so important. For more details see [3].

In Section 2, first we refer to the meaning of a two-dimensional wavelet by matrix dilation and then we present a way to construct a two-dimensional wavelet with small frequency domain and high accuracy by using a two-dimensional MRA and four masks

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which are not related to the MRA. In Section 3, we will find numerical solution for two first kind Fredholm integral equations. This kind of equations provide an ill-posed system, i.e., there might be no solutions or no unique solution and even no stable solution. Solving this type of integral equation is not easy.

2. TWO-DIMENSIONAL WAVELET WITH MATRIX DILATION $M = 2I$

In the subject of wavelet with matrix dilation M , we shall assume that M is a fixed quadratic integer matrix such that all its eigenvalues are greater than one in modulus, $m = |\det M|$. In this paper we consider $M = 2I$, especially.

Definition 2.1 ([8]). A collection of closed subspaces $V_j \subset L^2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, is called a multiresolution analysis (MRA) in $L^2(\mathbb{R}^2)$ with matrix dilation M if the following conditions hold:

- MRA1: $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- MRA2: $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^2)$;
- MRA3: $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- MRA4: $f \in V_j$ if and only if $f(M^{-j}\cdot) \in V_0$ for all $j \in \mathbb{Z}$;
- MRA5: there exists a function $\phi \in V_0$ such that the sequence $\{\phi(\cdot + m, \cdot + n)\}_{m,n \in \mathbb{Z}}$ forms an orthonormal basis in V_0 (ϕ is called scaling function).

Let ϕ be a scaling function for an MRA. Using properties MRA1, MRA5 and notation

$$f_{j,(s,t)} := m^{j/2} f(M^j \cdot + (s, t)), \quad j, s, t \in \mathbb{Z},$$

we get the refinement equation

$$(2.1) \quad \phi = \sum_{s,t \in \mathbb{Z}} h_{s,t} \phi_{1,(s,t)}, \quad \sum_{s,t \in \mathbb{Z}} |h_{s,t}|^2 < \infty.$$

Applying the Fourier transform,

$$\hat{\phi}(\xi_1, \xi_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x, y) e^{-2\pi i(x\xi_1 + y\xi_2)} dx dy,$$

to both sides of above equality, we get

$$\hat{\phi}(\xi_1, \xi_2) = m_0(M^{*-1}(\xi_1, \xi_2)) \hat{\phi}(M^{*-1}(\xi_1, \xi_2)),$$

where

$$m_0(\eta_1, \eta_2) = m^{-\frac{1}{2}} \sum_{s,t \in \mathbb{Z}} h_{s,t} e^{2\pi i(s\eta_1 + t\eta_2)}.$$

As in the one-dimensional case, the function m_0 is called a mask. For more details see [8].

Example 2.1. Let $M = 2I$ and $\phi(x, y) = \chi_{[0,1) \times [0,1)}(x, y)$. From (2.1), we have

$$\phi(x, y) = 2 \sum_{s,t \in \mathbb{Z}} h_{s,t} \phi(2x + s, 2y + t),$$

and we conclude that

$$\phi(x, y) = 2[h_{-1,-1}\phi(2x - 1, 2y - 1) + h_{-1,0}\phi(2x - 1, y) + h_{0,-1}\phi(2x, 2y - 1) + h_{0,0}\phi(x, y)],$$

where $h_{-1,-1}, h_{-1,0}, h_{0,-1}, h_{0,0} = \frac{1}{2}$.

Hence,

$$m_0(\xi_1, \xi_2) = \frac{1}{2} \left(\frac{1}{2}e^{2\pi i(-\xi_1-\xi_2)} + \frac{1}{2}e^{2\pi i(-\xi_1)} + \frac{1}{2}e^{2\pi i(-\xi_2)} + \frac{1}{2} \right).$$

Definition 2.2 ([8]). If A is a nonsingular integer 2×2 matrix, we say the vectors $(k, l), (s, t) \in \mathbb{Z}^2$ are congruent modulo A and write $(k, l) \equiv (s, t) \pmod{A}$ if $(k, l) - (s, t) = A(p, q)$ for some $(p, q) \in \mathbb{Z}^2$. The integer lattice \mathbb{Z}^2 is partitioned into cosets with respect to the congruence introduced above. Any set containing only one representative of each coset is called a set of digits of the matrix A . When it does not matter which set of digits is chosen, we shall assume that it is chosen arbitrarily and denote it by $D(A)$.

Example 2.2. For $M = 2I$, we consider

$$D(M) = \{s_0 = (0, 0), s_1 = (0, -1), s_2 = (-1, 0), s_3 = (-1, -1)\}.$$

Theorem 2.1 ([8]). Suppose an MRA $\{V_j\}_{j \in \mathbb{Z}}$ is generated by a scaling function ϕ with mask m_0 and the system $\{\phi(\cdot + (s, t))\}_{s, t \in \mathbb{Z}}$ is orthonormal. Let $D(M^*) = \{s_0, \dots, s_{m-1}\}$. Let there exist functions $m_\nu \in L^2([0, 1] \times [0, 1])$, $\nu = 0, \dots, m - 1$, such that the matrix

$$(2.2) \quad \mathcal{M} = \{m_\nu((\xi_1, \xi_2) + M^{*-1}s_k)\}_{\nu, k=0}^{m-1}$$

is unitary, that is, $\mathcal{M}\mathcal{M}^* = \mathcal{M}^*\mathcal{M} = I$. Take the functions $\psi^{(\nu)}$, $\nu = 1, \dots, m - 1$, defined by the equalities

$$\widehat{\psi}^{(\nu)}(\xi_1, \xi_2) = m_\nu(M^{*-1}(\xi_1, \xi_2))\widehat{\phi}(M^{*-1}(\xi_1, \xi_2)).$$

Then the system $\{\psi_{j, (k, l)}^{(\nu)}\}$ is an orthonormal basis in the space $L^2(\mathbb{R}^2)$.

The following lemma is a portrait of some remark in [8, page 93].

Lemma 2.1. Let ϕ be a scaling function with mask

$$m_0(\xi_1, \xi_2) = m^{-\frac{1}{2}} \sum_{k=0}^{m-1} h_{0,k}^0 e^{2\pi i \langle s_k, (\xi_1, \xi_2) \rangle},$$

such that $h_{0,k}$'s are real numbers, $k = 0, \dots, m - 1$, and $\sum_{k=0}^{m-1} |h_{0,k}|^2 = 1$. Define

$$m_\nu(\xi_1, \xi_2) = m^{-\frac{1}{2}} \sum_{k=0}^{m-1} h_{\nu,k}^\nu e^{2\pi i \langle s_k, (\xi_1, \xi_2) \rangle},$$

where $h_{\nu,0}^\nu = h_{0,\nu}^0$, $h_{\nu,k}^\nu = \delta_{\nu,k} - \frac{h_{0,k}^0 h_{0,\nu}^0}{1 - h_{0,0}^0}$, $\nu = 1, \dots, m - 1$. Then the matrix

$$\mathcal{M} = \{m_\nu((\xi_1, \xi_2) + M^{*-1}s_k)\}_{\nu, k=0}^{m-1}$$

is unitary.

Proof. Since $\sum_{k=0}^{m-1} |h_{\nu,k}^\nu|^2 = 1$ for $\nu = 1, \dots, m - 1$, and $\sum_{k=0}^{m-1} h_{\nu,k}^\nu h_{\mu,k}^\mu = 0$ for $\nu \neq \mu$, \mathcal{M} is unitary. \square

Example 2.3. Consider mask m_0 in example (2.1),

$$m_0(\xi_1, \xi_2) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} e^{-2\pi i \xi_2} + \frac{1}{2} e^{-2\pi i \xi_1} + \frac{1}{2} e^{-2\pi i (\xi_1 + \xi_2)} \right).$$

Take

$$m_n(\xi_1, \xi_2) = \frac{1}{2} \left(\frac{1}{2} + (-1)^{[\frac{n}{2}]} \frac{1}{2} e^{-2\pi i \xi_2} + (-1)^n \frac{1}{2} e^{-2\pi i \xi_1} + (-1)^{[\frac{n+1}{2}]} \frac{1}{2} e^{-2\pi i (\xi_1 + \xi_2)} \right),$$

for $n = 1, 2, 3$, where $[\cdot]$ denotes integer part. Hence,

$$(2.3) \quad \mathcal{M} = \begin{pmatrix} m_0(\xi_1, \xi_2) & m_0(\xi_1, \xi_2 - \frac{1}{2}) & m_0(\xi_1 - \frac{1}{2}, \xi_2) & m_0(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}) \\ m_1(\xi_1, \xi_2) & m_1(\xi_1, \xi_2 - \frac{1}{2}) & m_1(\xi_1 - \frac{1}{2}, \xi_2) & m_1(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}) \\ m_2(\xi_1, \xi_2) & m_2(\xi_1, \xi_2 - \frac{1}{2}) & m_2(\xi_1 - \frac{1}{2}, \xi_2) & m_2(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}) \\ m_3(\xi_1, \xi_2) & m_3(\xi_1, \xi_2 - \frac{1}{2}) & m_3(\xi_1 - \frac{1}{2}, \xi_2) & m_3(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}) \end{pmatrix}$$

is unitary.

Since the matrix (2.2) is unitary we have some useful formulas for m_ν , $\nu = 0, \dots, m - 1$, as

$$(2.4) \quad \sum_{k=0}^{m-1} |m_\nu((\xi_1, \xi_2) + M^{*-1} s_k)|^2 = 1.$$

For all $\nu, \mu = 0, \dots, m - 1$,

$$(2.5) \quad \sum_{k=0}^{m-1} m_\nu((\xi_1, \xi_2) + M^{*-1} s_k) \overline{m_\mu((\xi_1, \xi_2) + M^{*-1} s_k)} = 0, \quad \text{for } \nu \neq \mu,$$

and

$$\sum_{\nu=0}^{m-1} |m_\nu((\xi_1, \xi_2) + M^{*-1} s_k)|^2 = 1, \quad \text{for } k = 0, \dots, m - 1,$$

and for all $k = 1, \dots, m - 1$,

$$(2.6) \quad \sum_{\nu=0}^{m-1} m_\nu((\xi_1, \xi_2) + M^{*-1} s_k) \overline{m_\nu(\xi_1, \xi_2)} = 0.$$

Theorem 2.2. Let $f(x, y)$ be a function such that $\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$ is an orthonormal system and let $m_0(\xi_1, \xi_2) = \frac{1}{2} \sum_{s,t \in \mathbb{Z}} h_{s,t}^0 e^{2\pi i (s\xi_1 + t\xi_2)}$ and $m_\nu(\xi_1, \xi_2) = \frac{1}{2} \sum_{s,t \in \mathbb{Z}} h_{s,t}^\nu e^{2\pi i (s\xi_1 + t\xi_2)}$, $\nu = 1, 2, 3$, are masks with matrix dilation $M = 2I$ such that (2.3) is unitary. Define

$$F_n(x, y) = \sum_{s,t} h_{s,t}^n f(x - s, y - t), \quad n = 0, 1, 2, 3.$$

Then

$$(2.7) \quad \{F_n(\cdot - 2k, \cdot - 2l) : n = 0, \dots, 3, k, l \in \mathbb{Z}\}$$

is an orthonormal basis for $\overline{\text{span}}\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$.

Proof. First of all, we calculate some useful formulas.

By definition of F_ν , for $\nu = 1, 2, 3$,

$$(2.8) \quad \widehat{F}_\nu(\xi_1, \xi_2) = 2\overline{m_\nu(\xi_1, \xi_2)}\widehat{f}(\xi_1, \xi_2).$$

Since $\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$ is an orthonormal set so

$$\langle f, f(\cdot - s, \cdot - t) \rangle = \begin{cases} 1, & \text{if } s = t = 0, \\ 0, & \text{o.w,} \end{cases}$$

and since

$$\begin{aligned} \langle f, f(\cdot - s, \cdot - t) \rangle &= \langle \widehat{f}(\cdot, \cdot), e^{-2\pi i(s\cdot + t\cdot)}\widehat{f}(\cdot, \cdot) \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{f}(\xi_1, \xi_2)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} \\ &= \sum_{k,l} \int_l^{l+1} \int_k^{k+1} |\widehat{f}(\xi_1, \xi_2)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2 \\ &= \int_0^1 \int_0^1 \sum_{k,l} |\widehat{f}(\xi_1 + k, \xi_2 + l)|^2 e^{2\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2, \end{aligned}$$

noticing Fourier coefficients, we conclude that

$$(2.9) \quad \sum_{k,l} |\widehat{f}(\xi_1 + k, \xi_2 + l)|^2 = 1 \quad \text{a.e.}$$

Also, by (2.4), (2.5), (2.8) and (2.9), we have

$$\begin{aligned} &\sum_{k,l} \overline{\widehat{F}_\nu\left(\xi_1 - \frac{k}{2}, \xi_2 - \frac{l}{2}\right)} \widehat{F}_\mu\left(\xi_1 - \frac{k}{2}, \xi_2 - \frac{l}{2}\right) \\ &= \sum_{k,l} \overline{\widehat{F}_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right)} \widehat{F}_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right) \\ &\quad + \sum_{k,l} \overline{\widehat{F}_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right)} \widehat{F}_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right) \\ &\quad + \sum_{k,l} \overline{\widehat{F}_\nu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right)} \widehat{F}_\mu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right) \\ &\quad + \sum_{k,l} \overline{\widehat{F}_\nu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right)} \widehat{F}_\mu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right) \\ &= 4 \sum_{k,l} m_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right) m_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right) \left| \widehat{f}\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l}{2}\right) \right|^2 \\ &\quad + 4 \sum_{k,l} m_\nu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right) \overline{m_\mu\left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2}\right)} \\ &\quad + 4 \sum_{k,l} m_\nu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right) \overline{m_\mu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2}\right)} \\ &\quad + 4 \sum_{k,l} m_\nu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right) \overline{m_\mu\left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2}\right)} \end{aligned}$$

$$\begin{aligned}
 & \times \left| \hat{f} \left(\xi_1 - \frac{2k}{2}, \xi_2 - \frac{2l+1}{2} \right) \right|^2 \\
 & + 4 \sum_{k,l} m_\nu \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2} \right) \overline{m_\mu \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2} \right)} \\
 & \times \left| \hat{f} \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l}{2} \right) \right|^2 \\
 & + 4 \sum_{k,l} m_\nu \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2} \right) \overline{m_\mu \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2} \right)} \\
 & \times \left| \hat{f} \left(\xi_1 - \frac{2k+1}{2}, \xi_2 - \frac{2l+1}{2} \right) \right|^2 \\
 & = 4 \left[m_\nu(\xi_1, \xi_2) \overline{m_\mu(\xi_1, \xi_2)} + m_\nu \left(\xi_1, \xi_2 - \frac{1}{2} \right) \overline{m_\mu \left(\xi_1, \xi_2 - \frac{1}{2} \right)} \right. \\
 & \quad \left. + m_\nu \left(\xi_1 - \frac{1}{2}, \xi_2 \right) \overline{m_\mu \left(\xi_1 - \frac{1}{2}, \xi_2 \right)} + m_\nu \left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2} \right) \overline{m_\mu \left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2} \right)} \right] \\
 & (2.10) \\
 & = \begin{cases} 4, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu. \end{cases}
 \end{aligned}$$

Now we are ready to show that $\{F_\nu(\cdot - 2k, \cdot - 2l) : \nu = 0, \dots, 3\}_{k,l \in \mathbb{Z}}$ is an orthonormal set. By (2.10),

$$\begin{aligned}
 & \langle F_\nu, F_\mu(\cdot - 2s, \cdot - 2t) \rangle \\
 & = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{F}_\nu(\xi_1, \xi_2) \overline{\widehat{F}_\mu(\xi_1, \xi_2)} e^{4\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2 \\
 & = \sum_{k,l} \int_{\frac{l}{2}}^{(l+1)/2} \int_{\frac{k}{2}}^{(k+1)/2} \widehat{F}_\nu(\xi_1, \xi_2) \overline{\widehat{F}_\mu(\xi_1, \xi_2)} e^{4\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2 \\
 & = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \sum_{k,l} \widehat{F}_\nu \left(\xi_1 + \frac{k}{2}, \xi_2 + \frac{l}{2} \right) \overline{\widehat{F}_\mu \left(\xi_1 + \frac{k}{2}, \xi_2 + \frac{l}{2} \right)} e^{4\pi i(s\xi_1 + t\xi_2)} d\xi_1 d\xi_2 \\
 & = \begin{cases} 1, & \text{if } \nu = \mu \text{ and } s = t = 1, \\ 0, & \text{o.w.} \end{cases}
 \end{aligned}$$

Now we will show that the set (2.7) will generate the set $\{f(\cdot - s, \cdot - t) : s, t \in \mathbb{Z}\}$. In this order we will use the bellow equalities:

$$\begin{aligned}
 & m_\nu(\xi_1, \xi_2) + m_\nu \left(\xi_1, \xi_2 - \frac{1}{2} \right) + m_\nu \left(\xi_1 - \frac{1}{2}, \xi_2 \right) + m_\nu \left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2} \right) \\
 & = 2 \sum_{s,t} h_{2s,2t}^\nu e^{2\pi i(2s\xi_1 + 2t\xi_2)},
 \end{aligned}$$

$$\begin{aligned}
 & m_\nu(\xi_1, \xi_2) - m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) + m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \\
 &= 2 \sum_{s,t} h_{2s,2t+1}^\nu e^{2\pi i(2s\xi_1+(2t+1)\xi_2)}, \\
 & m_\nu(\xi_1, \xi_2) + m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \\
 &= 2 \sum_{s,t} h_{2s+1,2t}^\nu e^{2\pi i((2s+1)\xi_1+2t\xi_2)}, \\
 & m_\nu(\xi_1, \xi_2) - m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) + m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \\
 (2.11) \quad &= 2 \sum_{s,t} h_{2s+1,2t+1}^\nu e^{2\pi i((2s+1)\xi_1+(2t+1)\xi_2)}.
 \end{aligned}$$

Hence, by (2.4), (2.6) and (2.11), we have

$$\begin{aligned}
 & \left[\sum_{k,l} h_{2k,2l}^0 F_0(\cdot + 2k, \cdot + 2l) + h_{2k,2l}^1 F_1(\cdot + 2k, \cdot + 2l) + h_{2k,2l}^2 F_2(\cdot + 2k, \cdot + 2l) \right. \\
 & \left. + h_{2k,2l}^3 F_3(\cdot + 2k, \cdot + 2l) \right] \widehat{f}(\xi_1, \xi_2) \\
 &= 2 \sum_{\nu=0}^3 \sum_{k,l} h_{2k,2l}^\nu e^{2\pi i(2k\xi_1+2l\xi_2)} \overline{m_\nu(\xi_1, \xi_2)} \widehat{f}(\xi_1, \xi_2) \\
 &= \left[m_0(\xi_1, \xi_2) + m_0\left(\xi_1, \xi_2 - \frac{1}{2}\right) + m_0\left(\xi_1 - \frac{1}{2}, \xi_2\right) \right. \\
 & \quad \left. + m_0\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_0(\xi_1, \xi_2)} \widehat{f}(\xi_1, \xi_2) + \left[m_1(\xi_1, \xi_2) + m_1\left(\xi_1, \xi_2 - \frac{1}{2}\right) \right. \\
 & \quad \left. + m_1\left(\xi_1 - \frac{1}{2}, \xi_2\right) + m_1\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_1(\xi_1, \xi_2)} \widehat{f}(\xi_1, \xi_2) \\
 & \quad + \left[m_2(\xi_1, \xi_2) + m_2\left(\xi_1, \xi_2 - \frac{1}{2}\right) + m_2\left(\xi_1 - \frac{1}{2}, \xi_2\right) \right. \\
 & \quad \left. + m_2\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_2(\xi_1, \xi_2)} \widehat{f}(\xi_1, \xi_2) + \left[m_3(\xi_1, \xi_2) + m_3\left(\xi_1, \xi_2 - \frac{1}{2}\right) \right. \\
 & \quad \left. + m_3\left(\xi_1 - \frac{1}{2}, \xi_2\right) + m_3\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_3(\xi_1, \xi_2)} \widehat{f}(\xi_1, \xi_2) \\
 &= \widehat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 |m_\nu(\xi_1, \xi_1)|^2 + \widehat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) \overline{m_\nu(\xi_1, \xi_2)} \\
 & \quad + \widehat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) \overline{m_\nu(\xi_1, \xi_2)} \\
 & \quad + \widehat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \overline{m_\nu(\xi_1, \xi_2)}
 \end{aligned}$$

$$= \hat{f}(\xi_1, \xi_2).$$

Similarly, we have

$$\begin{aligned} & \left[\sum_{k,l} h_{2k,2l+1}^0 F_0(\cdot + 2k, \cdot + 2l) + h_{2k,2l+1}^1 F_1(\cdot + 2k, \cdot + 2l) \right. \\ & \quad \left. + h_{2k,2l+1}^2 F_2(\cdot + 2k, \cdot + 2l) + h_{2k,2l+1}^3 F_3(\cdot + 2k, \cdot + 2l) \right] (\xi_1, \xi_2) \\ &= 2 \sum_{\nu=0}^3 \sum_{k,l} h_{2k,2l+1}^\nu e^{2\pi i(2k\xi_1+2l\xi_2)} \overline{m_\nu(\xi_1, \xi_2)} \hat{f}(\xi_1, \xi_2) \\ &= e^{-2\pi i\xi_2} \hat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 \left[m_\nu(\xi_1, \xi_2) - m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) + m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) \right. \\ & \quad \left. - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_\nu(\xi_1, \xi_2)} \\ &= [f(\cdot, \cdot - 1)](\xi_1, \xi_2) \end{aligned}$$

and

$$\begin{aligned} & \left[\sum_{k,l} h_{2k+1,2l}^0 F_0(\cdot + 2k, \cdot + 2l) + h_{2k+1,2l}^1 F_1(\cdot + 2k, \cdot + 2l) + h_{2k+1,2l}^2 F_2(\cdot + 2k, \cdot + 2l) \right. \\ & \quad \left. + h_{2k+1,2l}^3 F_3(\cdot + 2k, \cdot + 2l) \right] (\xi_1, \xi_2) \\ &= 2 \sum_{\nu=0}^3 \sum_{k,l} h_{2k+1,2l}^\nu e^{2\pi i(2k\xi_1+2l\xi_2)} \overline{m_\nu(\xi_1, \xi_2)} \hat{f}(\xi_1, \xi_2) \\ &= e^{-2\pi i\xi_1} \hat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 \left[m_\nu(\xi_1, \xi_2) + m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) \right. \\ & \quad \left. - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_\nu(\xi_1, \xi_2)} = [f(\cdot - 1, \cdot)](\xi_1, \xi_2) \end{aligned}$$

and

$$\begin{aligned} & \left[\sum_{k,l} h_{2k+1,2l+1}^0 F_0(\cdot + 2k, \cdot + 2l) + h_{2k+1,2l+1}^1 F_1(\cdot + 2k, \cdot + 2l) \right. \\ & \quad \left. + h_{2k+1,2l+1}^2 F_2(\cdot + 2k, \cdot + 2l) + h_{2k+1,2l+1}^3 F_3(\cdot + 2k, \cdot + 2l) \right] (\xi_1, \xi_2) \\ &= 2 \sum_{\nu=0}^3 \sum_{k,l} h_{2k,2l}^\nu e^{2\pi i(2k\xi_1+2l\xi_2)} \overline{m_\nu(\xi_1, \xi_2)} \end{aligned}$$

$$\begin{aligned}
 &= e^{-2\pi i(\xi_1 + \xi_2)} \hat{f}(\xi_1, \xi_2) \sum_{\nu=0}^3 \left[m_\nu(\xi_1, \xi_2) - m_\nu\left(\xi_1, \xi_2 - \frac{1}{2}\right) - m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2\right) \right. \\
 &\quad \left. + m_\nu\left(\xi_1 - \frac{1}{2}, \xi_2 - \frac{1}{2}\right) \right] \overline{m_\nu(\xi_1, \xi_2)} \\
 &= [f(\cdot - 1, \cdot - 1)] \hat{f}(\xi_1, \xi_2). \quad \square
 \end{aligned}$$

The above theorem shows that wavelet filters can be used to split any space spanned by two-dimensional orthonormal functions $f(\cdot - s, \cdot - t)$ into four parts. We can apply this method to the space W_0 spanned by the $\psi(\cdot - s, \cdot - t)$ in a two-dimensional multiresolution analysis with matrix dilation $2I$. In particular, if we choose arbitrary functions m_ν in Theorem 2.2 we have the following.

Corollary 2.1. *Let $\psi \in L^2(\mathbb{R}^2)$ be a two-dimensional wavelet which is generated by an MRA $\{V_j\}_{j \in \mathbb{Z}}$ and let $m_\nu, \nu = 0, \dots, 3$, are masks not necessary related to ψ . Define*

$$\psi^\nu(x, y) = \sum_{s,t} h_{s,t}^\nu \psi(x - s, y - t), \quad \nu = 0, \dots, 3.$$

Then $\{\psi_{j,(2k,2l)}^\nu : \nu = 0, \dots, 3\}_{j,k,l \in \mathbb{Z}}$ consists of an orthonormal basis for $\overline{\text{span}}\{\psi_{j,(2k,2l)}^\nu : \nu = 0, \dots, 3\}_{j,k,l \in \mathbb{Z}}$.

Proof. Let W_j be the orthonormal complement in V_{j+1} of V_j . Since $W_0 = \overline{\text{span}}\{\psi(\cdot - k, \cdot - l) : k, l \in \mathbb{Z}\}$, by preceding theorem,

$$\begin{aligned}
 W_0 &= \overline{\text{span}}\{\psi^0(\cdot - 2k, \cdot - 2l) : l, k \in \mathbb{Z}\} \oplus \overline{\text{span}}\{\psi^1(\cdot - 2k, \cdot - 2l) : l, k \in \mathbb{Z}\} \\
 &\quad \oplus \overline{\text{span}}\{\psi^2(\cdot - 2k, \cdot - 2l) : l, k \in \mathbb{Z}\} \oplus \overline{\text{span}}\{\psi^3(\cdot - 2k, \cdot - 2l) : l, k \in \mathbb{Z}\} \\
 &= W_0^0 \oplus W_0^1 \oplus W_0^2 \oplus W_0^3.
 \end{aligned}$$

Since each $W_0^j, j = 0, \dots, 3$, is generated by translations of $\psi^j(\cdot - 2k, \cdot - 2l)$, by dilation we can construct corresponding orthonormal bases for each W_m and their union is again a basis for $\overline{\text{span}}\{\psi_{j,(2k,2l)}^\nu : \nu = 0, \dots, 3\}_{j,k,l \in \mathbb{Z}}$. \square

Corollary 2.2. *Let ψ be a one dimensional wavelet with scaling function ϕ . Consider masks $m_\nu, \nu = 0, \dots, 3$, which is asserted in Theorem 2.2. Define*

$$\begin{aligned}
 \Psi^{a,i} &= \sum_{s,t} h_{s,t}^i \Psi^a(\cdot - s, \cdot - t), \quad i = 1, \dots, 3, \\
 \Psi^{b,i} &= \sum_{s,t} h_{s,t}^i \Psi^b(\cdot - s, \cdot - t), \quad i = 1, \dots, 3, \\
 \Psi^{d,i} &= \sum_{s,t} h_{s,t}^i \Psi^d(\cdot - s, \cdot - t), \quad i = 1, \dots, 3.
 \end{aligned}$$

Then, by Corollary 2.1,

$$\left\{ \Psi_{j,(2s,2t)}^{r,i} : r = a, b, d, i = 0, \dots, 3 \right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^2)$, where $\Psi^r, r = a, b, d$ are defined as (1.1).

3. CONSTRUCTING AN EXAMPLE AND APPLICATION OF EXAMPLE

3.1. **Example.** Consider one-dimensional Haar wavelet with scaling function $\phi = \chi_{[0,1)}$

$$\psi(x) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right), \\ 0, & \text{o.w.,} \end{cases}$$

then

$$\Psi^a(x, y) = \begin{cases} 1, & \text{if } x \in [0, 1), y \in \left[0, \frac{1}{2}\right), \\ -1, & \text{if } x \in [0, 1), y \in \left[\frac{1}{2}, 1\right), \\ 0, & \text{o.w.,} \end{cases}$$

$$\Psi^b(x, y) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{2}\right), y \in [0, 1), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right), y \in [0, 1), \\ 0, & \text{o.w.,} \end{cases}$$

$$\Psi^d(x, y) = \begin{cases} 1, & \text{if } x, y \in \left[0, \frac{1}{2}\right) \text{ or } x, y \in \left[\frac{1}{2}, 1\right), \\ -1, & \text{if } x \in \left[0, \frac{1}{2}\right), y \in \left[\frac{1}{2}, 1\right) \text{ or } x \in \left[\frac{1}{2}, 1\right), y \in \left[0, \frac{1}{2}\right), \\ 0, & \text{o.w.} \end{cases}$$

Now consider the masks in the Example (2.3), we have

$$\begin{aligned} h_{0,0}^0 &= h_{0,-1}^0 = h_{-1,0}^0 = h_{-1,-1}^0 = \frac{1}{2}, \\ h_{0,0}^1 &= h_{0,-1}^1 = \frac{1}{2}, \quad h_{-1,0}^1 = h_{-1,-1}^1 = -\frac{1}{2}, \\ h_{0,0}^2 &= h_{-1,0}^2 = \frac{1}{2}, \quad h_{0,-1}^2 = h_{-1,-1}^2 = -\frac{1}{2}, \\ h_{0,0}^3 &= h_{-1,-1}^3 = \frac{1}{2}, \quad h_{0,-1}^3 = h_{-1,0}^3 = -\frac{1}{2}. \end{aligned}$$

So, we have

$$\begin{aligned} \Psi^{a,0}(x, y) &= \sum_{s,t=-1}^0 h_{s,t}^0 \Psi^a(x-s, y-t) \\ &= \frac{1}{2} [\Psi^a(x, y) + \Psi^a(x, y+1) + \Psi^a(x+1, y) + \Psi^a(x+1, y+1)] \end{aligned}$$

$$= \begin{cases} \frac{1}{2}, & \text{if } (x, y) \in [0, 1) \times \left[0, \frac{1}{2}\right) \text{ or } (x, y) \in [-1, 1) \times \left[0, \frac{1}{2}\right) \\ & \text{or } (x, y) \in [-1, 1) \times \left[-1, -\frac{1}{2}\right), \\ -\frac{1}{2}, & \text{if } (x, y) \in [-1, 1) \times \left[\frac{1}{2}, 1\right) \text{ or } (x, y) \in [-1, 1) \times \left[-\frac{1}{2}, 0\right), \\ 0, & \text{o.w.} \end{cases}$$

Similarly,

$$\Psi^{r,0}(x, y) = \frac{1}{2}[\Psi^r(x, y) + \Psi^r(x, y + 1) + \Psi^r(x + 1, y) + \Psi^r(x + 1, y + 1)], \quad r = b, d,$$

and

$$\Psi^{r,1}(x, y) = \frac{1}{2}[\Psi^r(x, y) + \Psi^r(x, y + 1) - \Psi^r(x + 1, y) - \Psi^r(x + 1, y + 1)], \quad r = a, b, d,$$

$$\Psi^{r,2}(x, y) = \frac{1}{2}[\Psi^r(x, y) - \Psi^r(x, y + 1) + \Psi^r(x + 1, y) - \Psi^r(x + 1, y + 1)], \quad r = a, b, d,$$

$$\Psi^{r,3}(x, y) = \frac{1}{2}[\Psi^r(x, y) - \Psi^r(x, y + 1) - \Psi^r(x + 1, y) + \Psi^r(x + 1, y + 1)], \quad r = a, b, d.$$

Hence, $\{2^j \Psi^{r,i}(2^j \cdot - 2k, 2^j \cdot - 2l) : i = 0, \dots, 3, r = a, b, d, j, k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

The diagram of the 12 mother wavelets is shown in Figures 1–12.

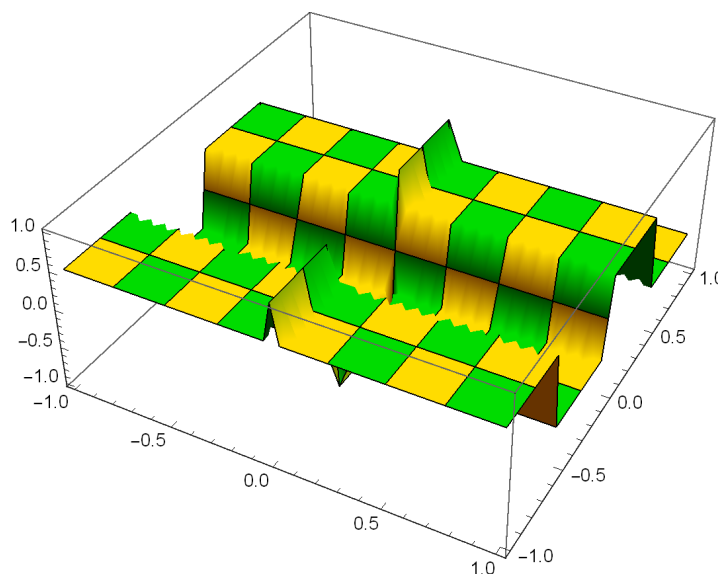
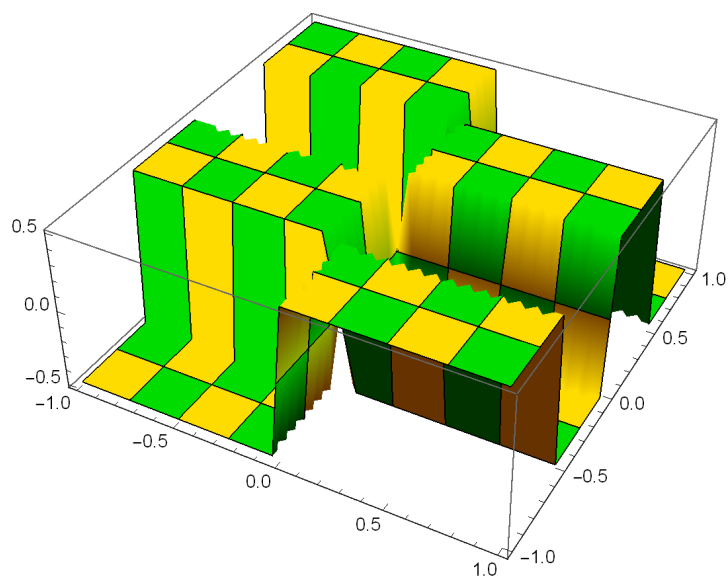
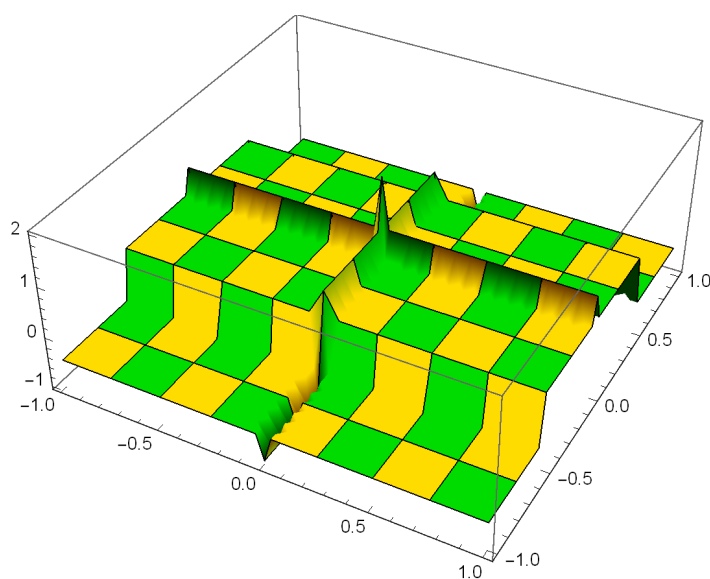


FIGURE 1. $\Psi^{a,0}(x, y)$

3.2. Application of example in solving two dimensional first kind Fredholm integral equation. Now we are going to show that our example consists a useful

FIGURE 2. $\Psi^{a,1}(x, y)$ FIGURE 3. $\Psi^{a,2}(x, y)$

basis wavelet to find numerical solution for the first kind Fredholm integral equations. A two-dimensional first kind Fredholm integral equation has the following form

$$(3.1) \quad f(x, y) = \int_c^d \int_a^b k(x, y, s, t) \mathcal{G}(u(s, t)) ds dt,$$

where $k(x, y, s, t)$ and $f(x, y)$ are known functions and $u(x, y)$ is an unknown function to be determined. To solve (3.1), if $\{\Psi_{jk} : j, k \in \mathbb{Z}\}$ is a wavelet basis let β be a finite

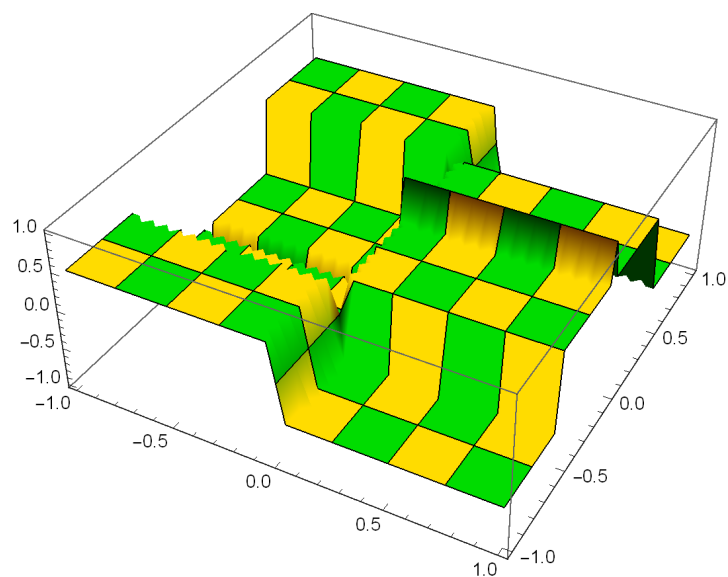


FIGURE 4. $\Psi^{a,3}(x, y)$

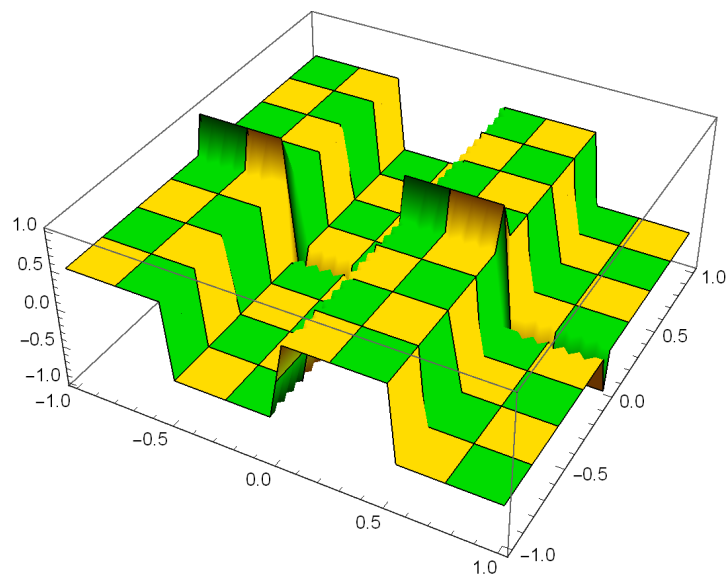
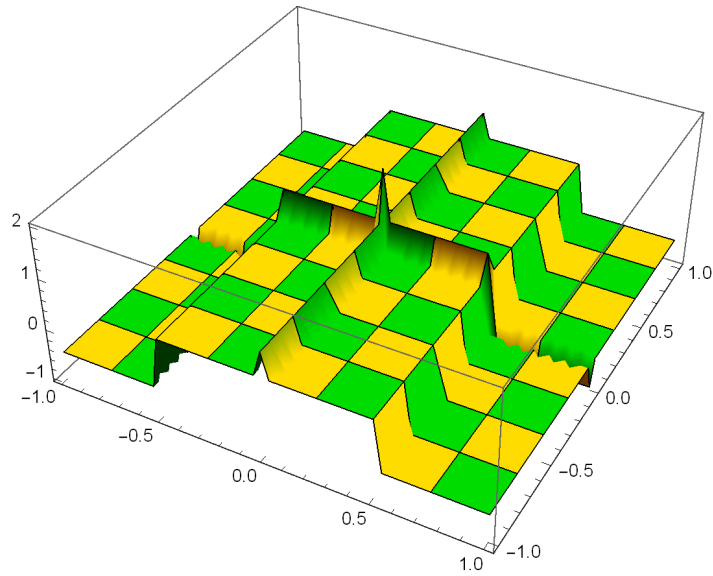
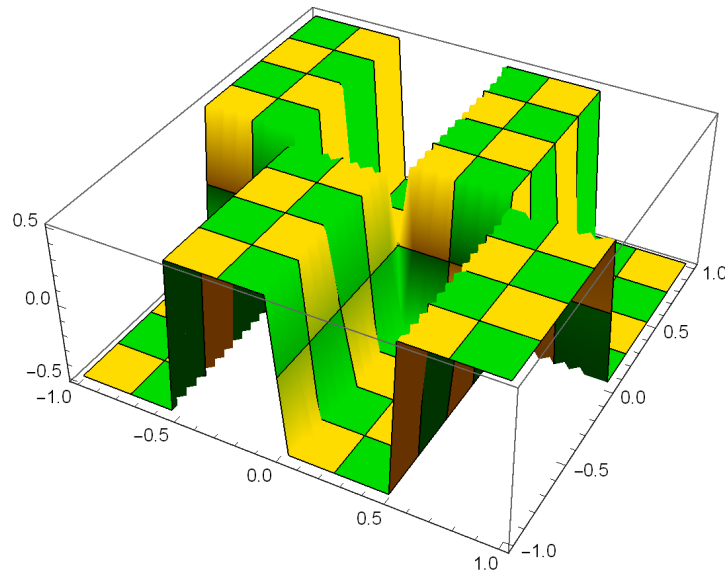


FIGURE 5. $\Psi^{b,0}(x, y)$

subset of it. If \tilde{u} is an approximated solution for (3.1) which is compute by β take

$$\tilde{r} = \int_c^d \int_a^b k(x, y, s, t) \mathcal{G}(\tilde{u}(s, t)) ds dt - f(x, y).$$

To find \tilde{u} , we have to solve the system $\langle \tilde{r}, y \rangle = 0$ for all $y \in \beta$. Since first kind Fredholm integral equations generate ill-conditioned systems, to solve the mentioned system we use Tikhonov regularization. Note that if \tilde{u} is a solution for (3.1), then

FIGURE 6. $\Psi^{b,1}(x, y)$ FIGURE 7. $\Psi^{b,2}(x, y)$

$\|\tilde{r}\|_2 = 0$. Then we are going to find \tilde{u} such that $\|\tilde{r}\|_2$ be the smallest. We named the value of $\|\tilde{r}\|_2$ as L^2 -norm of error. For more details see [2, 4, 7, 9].

Now by using wavelet basis which was presented in the former subsection, we are going to solve two following examples.

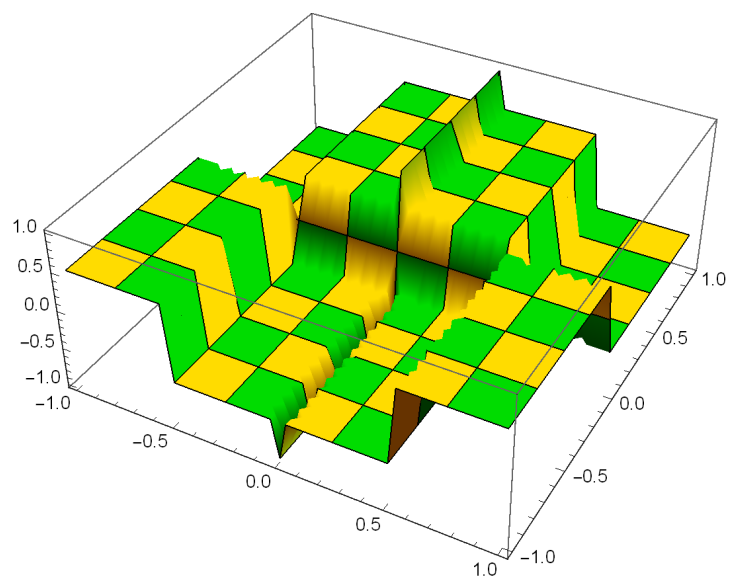


FIGURE 8. $\Psi^{b,3}(x, y)$

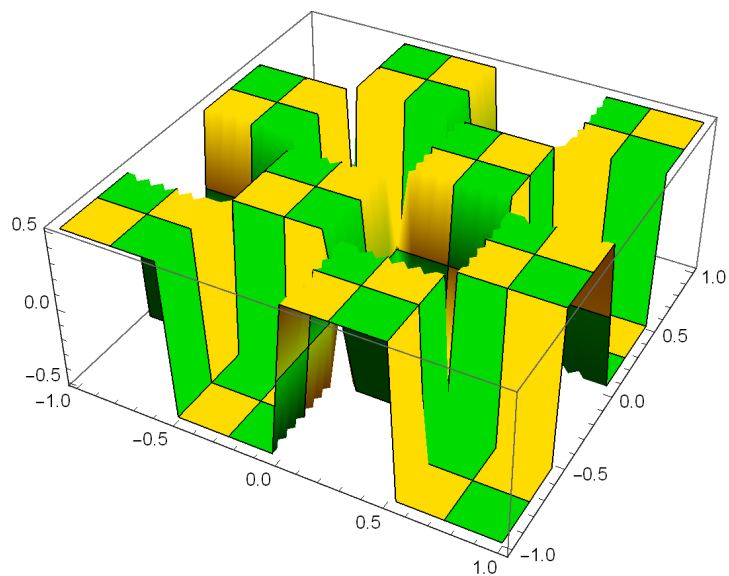


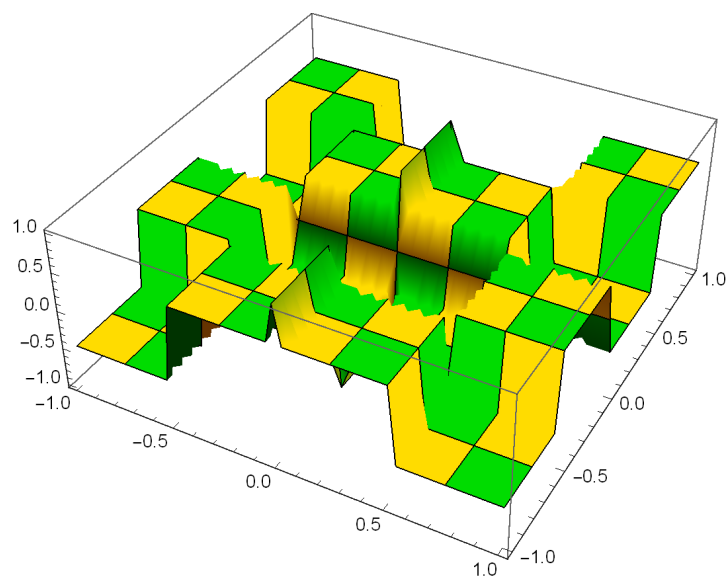
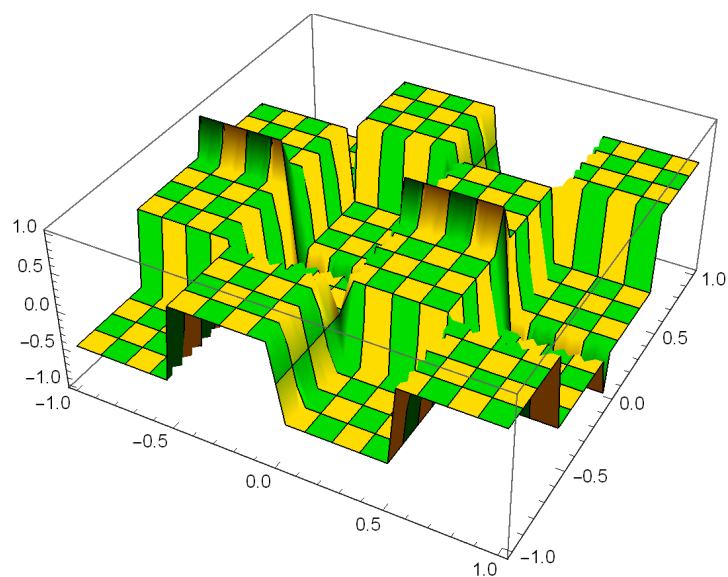
FIGURE 9. $\Psi^{d,0}(x, y)$

Example 3.1. Consider the integral equation

$$\int_0^1 \int_0^1 (x^2s + yt^2)u(s, t)dsdt = \frac{15(9944x^2 + 16549y)}{131072}.$$

We have

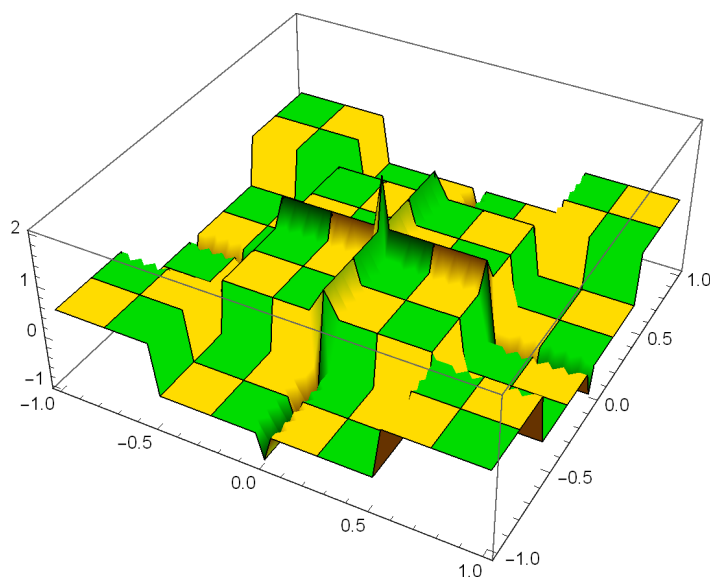
$$\begin{aligned} \tilde{u}(x, y) = & -3.78777\Psi^{a,0}(x, y) - 3.78777\Psi^{a,1}(x, y) - 3.78777\Psi^{a,2}(x, y) \\ & - 3.78777\Psi^{a,3}(x, y) - 2.276\Psi^{b,0}(x, y) - 2.276\Psi^{b,1}(x, y) \end{aligned}$$

FIGURE 10. $\Psi^{d,1}(x, y)$ FIGURE 11. $\Psi^{d,2}(x, y)$

$$- 2.276\Psi^{b,2}(x, y)2.276\Psi^{b,3}(x, y) + \Psi^{d,0}(x, y) + \Psi^{d,1}(x, y) + \Psi^{d,2}(x, y) + \Psi^{d,3}(x, y).$$

The L^2 -norm of error equals 1.42968×10^{-11} .

If we use two-dimensional Haar wavelets which are made by (1.1), our numerical solution is $-7.57553\Psi^a(x, y) - 4.552\Psi^b(x, y) + \Psi^d(x, y)$ and the L^2 -norm of error equals 1.03387×10^{-8} .

FIGURE 12. $\Psi^{d,3}(x, y)$

Example 3.2. Consider integral equation

$$\int_0^1 \int_0^1 (x^2 + yt)(u^2(s, t) - u(s, t)) ds dt = \frac{(-12 + 5\pi)(2x^2 + y)}{12\pi}.$$

We have

$$\begin{aligned} \tilde{u}(x, y) = & 0.0841832\Psi^{a,0}(x, y) + 0.0841832\Psi^{a,1}(x, y) + 0.0841832\Psi^{a,2}(x, y) \\ & + 0.0841832\Psi^{a,3}(x, y) + 0.145072\Psi^{b,0}(x, y) + 0.145072\Psi^{b,1}(x, y) \\ & + 0.145072\Psi^{b,2}(x, y) + 0.145072\Psi^{b,3}(x, y) + 0.145072\Psi^{d,0}(x, y) \\ & + 0.145072\Psi^{d,1}(x, y) + 0.145072\Psi^{d,2}(x, y) + 0.145072\Psi^{d,3}(x, y). \end{aligned}$$

The L^2 -norm of error equals 1.13239×10^{-9} .

REFERENCES

- [1] J. J.-P. Antoine, R. Murenzi, P. Vandergheynst and S. T. Ali, *Two-Dimensional Wavelets and their Relatives*, Cambridge University Press, Cambridge, 2004.
- [2] C. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.
- [3] I. Daubechies, *Ten Lecture on Wavelets*, SIAM, Philaderphia, PA, 1992.
- [4] G. H. Golub, P. C. Hansen and D. P. O'Leary, *Tikhonov regularization and total least squares*, SIAM J. Matrix Anal. Appl. **21**(1) (1999), 185–194.
- [5] C. W. Groetsch, *Integral equations of the first kind, inverse problems and regularization: a crash course*, J. Phys. Conf. Ser. **73**(1) (2007), Paper ID 012001.
- [6] A. Krisch, *An Introduction to the Mathematical Theory of Inverse Problems*, Second Edition, Springer-Verlag, New York, 2013.
- [7] D. Lesnic, S. A. Yousefi and M. Ivanhov, *Determination of a time-dependent diffusivity from nonlocal conditions*, Int. J. Appl. Math. Comput. Sci. **41** (2013), 301–320.

- [8] I. Ya. Novikov, V. Yu. Protasova and M. A. Skopina, *Wavelet Theory*, American Mathematical Soc., Providence, Rhode Island, 2011.
- [9] M. Tahami, A. Askari Hemmat and S. A. Yousefi, *Numerical solution of two-dimensional first kind Fredholm integral equations by using linear Legendre wavelet*, Int. J. Wavelets Multiresolut. Inf. Process. **14**(1) (2016), DOI 10.1142/S0219691316500041.

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