EXTENSIONS OF MEIR-KEELER CONTRACTION VIA
$w$-DISTANCES WITH AN APPLICATION

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Abstract. In this article, we conceive the notion of a generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mapping and then we investigate a fixed point theorem involving such kind of contractions in the setting of a complete metric space via a $w$-distance. Our obtained result extends and generalizes some of the previously derived fixed point theorems in the literature via $w$-distances. In addition, to validate the novelty of our findings, we illustrate a couple of constructive numerical examples. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

1. Introduction and Preliminaries

In this paper, we introduce the notion of a generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mapping and investigate fixed points for such operators in the context of complete metric spaces via a $w$-distance. For this purpose we first recall the outstanding result of Meir-Keeler [14] (see also [10]).

Theorem 1.1 ([14]). Let $f$ be a self-map defined on a complete metric space $(M, d)$. Also assume that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\varepsilon \leq d(\rho, \varrho) < \varepsilon + \delta \quad \text{implies} \quad d(f\rho, f\varrho) < \varepsilon,$$

for all $\rho, \varrho \in M$. Then $f$ has a unique fixed point.

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This result is also known as a uniform contraction and it has been studied and extended by a number of researchers in many directions (see [16,20]). Now we recall the notion of $w$-distance introduced by Kada et al. [12].

**Definition 1.1** ([12]). Let $(M,d)$ be a metric space. A mapping $q : M \times M \to [0,\infty)$ is said to be a $w$-distance on $M$ if

(i) $q(\rho,\sigma) \leq q(\rho,\varrho) + q(\varrho,\sigma)$ for any $\rho,\varrho,\sigma \in M$;

(ii) $q$ is a lower semi-continuous map in the second variable, that is, when $\rho \in M$ and $\sigma_n \to \sigma$ in $M$, then we have $q(\rho,\sigma) \leq \lim\inf_n q(\rho,\sigma_n)$;

(iii) for every $\epsilon > 0$, there is a $\delta > 0$ which $q(\sigma,\rho) \leq \delta$ and $q(\sigma,\varrho) \leq \delta$ imply that $d(\rho,\varrho) \leq \epsilon$.

Let $T : M \to M$ and $\alpha : M \times M \to [0,\infty)$. We say that $T$ is $\alpha$-orbital admissible (see [17]) if

$$\alpha(p,Tp) \geq 1 \quad \text{implies} \quad \alpha(Tp,T^2p) \geq 1,$$

for all $p \in M$. By using this auxiliary function, it is possible to combine several existing results in the literature, see, e.g. [9,15,18,19] and the related references therein. In particular, Lakzian et al. [13] introduced the concept of $(\alpha,\psi,q)$-contractive mappings in metric spaces via $w$-distances and proved fixed point results via this notion.

On the other hand, inspired by the notion of Meir-Keeler contractions, Chen [11] introduced the concept of a weaker Meir-Keeler function as follows.

**Definition 1.2** ([11]). A mapping $\psi : [0,\infty) \to [0,\infty)$ is said to be a weaker Meir-Keeler function if, for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $\tau \in [0,\infty)$ with $\epsilon \leq \tau < \epsilon + \delta$, we have an $n_0 \in \mathbb{N}$ satisfying $\psi^{n_0}(\tau) < \epsilon$.

Regarding [11], we also consider the family $\Psi$ of weaker Meir-Keeler functions $\psi : [0,\infty) \to [0,\infty)$ fulfilling the subsequent properties:

$(\psi_1)$ $\psi(\tau) > 0$ whenever $\tau > 0$ and $\psi(0) = 0$;

$(\psi_2)$ $\sum_{n=1}^{\infty} \psi^n(\tau) < \infty$, $\tau \in (0,\infty)$;

$(\psi_3)$ for each $y_n \in [0,\infty)$, the following hold:

(i) when $\lim_{n \to \infty} y_n = \ell > 0$, then $\lim_{n \to \infty} \psi(y_n) < \ell$;

(ii) whenever $\lim_{n \to \infty} y_n = 0$, we have $\lim_{n \to \infty} \psi(y_n) = 0$.

Along with the aforementioned terminologies, the following lemma is also playing a crucial role in our subsequent studies.

**Lemma 1.1** ([12]). Suppose that $(M,d)$ is a metric space with a $w$-distance $q$.

(i) For any sequence $\{\rho_n\}$ in $M$ with $\lim_n q(\rho_n,\rho) = \lim_n q(\rho_n,\varrho) = 0$, we have $\rho = \varrho$. Additionally, $q(\sigma,\rho) = q(\sigma,\varrho) = 0$ implies $\rho = \varrho$.

(ii) For two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,\infty)$ converging to 0, whenever $q(\rho_n,\alpha_n) \leq \alpha_n$, $q(\rho_n,\varrho) \leq \beta_n$ hold for each $n \in \mathbb{N}$, then the sequence $\{q_n\}$ converges to $\varrho$. 
(iii) Suppose that $\{\rho_n\}$ is a sequence in $M$ such that for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ with $m > n > N_\varepsilon$ implies that $q(\rho_n, \rho_m) < \varepsilon$ (or $\lim_{m,n} q(\rho_n, \rho_m) = 0$). Then $\{\rho_n\}$ is a Cauchy sequence.

In this paper, we define the concept of generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mappings and by using this new concept, we give some fixed point results. Furthermore, some significant non-trivial numerical examples are investigated to authenticate our findings. Moreover, as an application, the existence of the solution for a non-linear Fredholm integral equation is investigated.

2. $(\alpha, \psi, q)$-Meir-Keeler Contractions

This section brings the idea of generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mappings with the help of a weaker Meir-Keeler function. Also, we conceive a fixed point result concerning such kinds of mappings. Now we consider the following expressions:

$$
M_q(\rho, \varrho) = \max \left\{ q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho), \frac{q(\rho, f\varrho) + q(f\rho, \varrho)}{2} \right\}
$$

and

$$
m(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, f\rho), d(\varrho, f\varrho), \frac{d(\rho, f\varrho) + d(f\rho, \varrho)}{2} \right\}.
$$

Here, we propose the idea of generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mappings.

**Definition 2.1.** Suppose that $(M, d)$ is a metric space with a $w$-distance $q$ and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0, \infty)$ and an $\alpha$-orbital admissible map $f$. Then $f$ is called a generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mapping if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In addition, for $q = d$ and $M_q(\rho, \varrho) = m(\rho, \varrho)$, the mapping $f$ is said to be a generalized $(\alpha, \psi)$-Meir-Keeler-contractive. Furthermore, $f$ is a $(\alpha, \psi, q)$-Meir-Keeler contractive map, when $M_q(\rho, \varrho) = q(\rho, \varrho)$ for each $\rho, \varrho \in M$.

The succeeding theorem deals with an interesting fixed point result involving the previously discussed type of maps.

**Theorem 2.1.** Suppose that $(M, d)$ is a complete metric space with a $w$-distance $q$. Also assume that $f$ is a generalized $(\alpha, \psi, q)$-Meir-Keeler contractive map such that there is $\rho_0 \in M$ with $q(f^n\rho_0, f^n\rho_0) = 0$ for all non-negative integers $n$ and $\alpha(\rho_0, f\rho_0) \geq 1$. Suppose that one of the following conditions holds.

(i) For each $w \in M$ satisfying $w \neq fw$, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.

(ii) $f$ is continuous.

(iii) If for some sequence $\{\rho_n\}$, $\lim_{n\to\infty} q(\rho_n, \rho) = \lim_{n\to\infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.

Then $f$ owns a fixed point $u \in M$, with $q(u, u) = 0$. 
Proof. We construct a sequence \( \{\rho_n\} \) in \( M \) such that \( 
olinebreak[4]\rho_{n+1} = f\rho_n = f^{n+1}\rho_0 \) for each \( n \in \mathbb{N} \). When \( \rho_{n_0} = \rho_{n_0+1} \) for some positive integer \( n_0 \), then \( u = \rho_{n_0} \) is a fixed point of \( f \). Hence, without loss of generality consider that,

\[ \rho_n \neq \rho_{n+1}, \quad \text{for all } n \in \mathbb{N}. \]

As \( f \) is \( \alpha \)-orbital admissible, we have

\[ \alpha(\rho_0, \rho_1) = \alpha(\rho_0, f\rho_0) \geq 1 \quad \text{implies} \quad \alpha(f\rho_0, f\rho_1) = \alpha(\rho_1, \rho_2) \geq 1. \]

Using mathematical induction, it follows that \( \alpha(\rho_n, \rho_{n+1}) \geq 1 \) for each \( n \in \mathbb{N} \). Now, we divide the entire proof into four steps and discuss one by one.

**Step 1.** We first prove that for each \( n \in \mathbb{N} \)

\[ q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n). \]

Note that for every natural number \( n \), we have \( q(\rho_n, \rho_{n+1}) > 0 \). Since, otherwise by the combination of \( q(\rho_n, \rho_{n+1}) = 0 \) and the assumption \( q(\rho_n, \rho_n) = 0 \) and applying Lemma 1.1 we get \( \rho_n = \rho_{n+1} \), which is a contradiction. Therefore, we find that

\[ M_q(\rho_{n-1}, \rho_n) = \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_n, \rho_{n+1}) + q(\rho_{n+1}, \rho_n)}{2} \right\} > 0. \]

Hence, we obtain \( \psi(M_q(\rho_{n-1}, \rho_n)) > 0 \). Now, from the hypothesis and Definition 2.1 for \( \eta = \psi(M_q(\rho_{n-1}, \rho_n)) \), there exists a \( \delta > 0 \) such that for \( \rho, g \in M \), when \( \eta \leq \psi(M_q(\rho, g)) < \eta + \delta \), we have \( \alpha(\rho, g)f(\rho, g) < \eta \).

In particular, since for each \( \tau > 0 \), \( \psi(\tau) < \tau \), we have

\[ q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \eta = \psi(M_q(\rho_{n-1}, \rho_n)) < M_q(\rho_{n-1}, \rho_n). \]

Since

\[ \frac{q(\rho_{n-1}, \rho_n) + q(\rho_{n+1}, \rho_n)}{2} \leq \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}, \]

we have

\[ M_q(\rho_{n-1}, \rho_n) = \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \]

\[ = \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1})}{2} \right\} \]

\[ = \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}. \]

So, \( q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n) = \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\} \) and this implies that

\[ M_q(\rho_{n-1}, \rho_n) = q(\rho_{n-1}, \rho_n) \quad \text{and} \quad q(\rho_n, \rho_{n+1}) < q(\rho_{n-1}, \rho_n). \]

Now, since \( \{q(\rho_{n-1}, \rho_n)\} \) is decreasing and bounded below, it is convergent to \( t \geq 0 \) such that \( q(\rho_n, \rho_{n+1}) \geq t \) for each \( n \). Assume that \( t \neq 0 \) and \( \xi = \lim_{n} \psi(q(\rho_n, \rho_{n+1})) \). Then by (\( \psi_3 \)), \( 0 < \xi < t \) and by Definition 2.1, we can find \( \delta > 0 \) satisfying

\[ \quad \text{when} \quad \xi \leq \psi(M_q(\rho, g)) < \xi + \delta, \quad \text{we have} \quad \alpha(\rho, g)f(\rho, g) < \xi, \]


for $\rho, g \in M$. Consider $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$ and $\frac{1}{k_0} < \xi$. Then for each $k \geq k_0$ there is $\delta_k \leq \frac{1}{k}$ such that

\begin{equation}
\xi - \frac{1}{k} \leq \psi(M_q(0)) < \xi - \frac{1}{k} + \delta_k \quad \text{implies} \quad \alpha(\rho, g)p(f \rho, f g) < \xi - \frac{1}{k} < \xi.
\end{equation}

Also there is $k_2 \in \mathbb{N}$ such that for each $n \geq k_2$ one obtains

\[
\xi - \frac{1}{k_2} < \psi(q(\rho_{n-1}, \rho_n)) = \psi(M_q(\rho_{n-1}, \rho_n)) < \xi + \frac{1}{k_2} < \xi + \delta.
\]

Now, when $\xi \leq \psi(M_q(\rho_{n-1}, \rho_n)) \leq \xi + \frac{1}{k_0}$, by (2.2), we have

\[
q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t,
\]

and when $\xi - \frac{1}{k_0} \leq \psi(M_q(\rho_{n-1}, \rho_n)) < \xi$ by (2.3) and since

\[
[\xi - \frac{1}{k_0}, \xi] \subseteq \bigcup_{k \geq k_0} [\xi - \frac{1}{k}, \xi - \frac{1}{k} + \delta_k),
\]

we have $q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t$, which is a contradiction. Therefore, $t = 0$ and so

\begin{equation}
\lim_n q(\rho_n, \rho_{n+1}) = 0.
\end{equation}

**Step 2.** We prove that $\{\rho_n\}$ is a Cauchy sequence. Alternatively, from the inequality (2.1), we arrive at

\begin{equation}
q(\rho_n, \rho_{n+1}) \leq \psi(q(\rho_{n-1}, \rho_n)), \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Indeed, if there exists some $n^*$ such that

\[
q(\rho_{n^*}, \rho_{n^*+1}) \leq \psi(q(\rho_{n^*}, \rho_{n^*+1})) < q(\rho_{n^*}, \rho_{n^*+1}),
\]

we get a contradiction. Hence, (2.5) holds. Inductively, we derive, from (2.5), that

\[
q(\rho_n, \rho_{n+1}) \leq \psi^n(q(\rho_0, \rho_1)), \quad \text{for all } n \in \mathbb{N}.
\]

Fix $\varepsilon$ and let $n_\varepsilon \in \mathbb{N}$ such that $\sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)) < \varepsilon$. Furthermore, for $m > n > n_\varepsilon$, we can find that

\[
q(\rho_n, \rho_m) \leq q(\rho_n, \rho_{n+1}) + \cdots + q(\rho_{m-1}, \rho_m)
\]

\[
\leq \sum_{k=n}^{m-1} \psi^k(q(\rho_1, \rho_0))
\]

\[
\leq \sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)).
\]

Hence, we conclude that the sequence $\{\rho_n\}$ is Cauchy. Now, since $(M, d)$ is complete, we can get $u \in M$ with $\rho_n \to u$ in $M$.

**Step 3.** $u$ is a fixed point of $f$.

Case (i). For each $g \in M$ satisfying $g \neq f g$, we have $\inf\{q(\rho, g) + q(\rho, f \rho) : \rho \in M\} > 0$. It implies that for every $\varepsilon > 0$, there is a natural number $N$ such that for
n > N_ε, we have q(ρ_{N_ε}, ρ_n) < ε. Since, ρ_n → u and q(ρ, ·) is a lower semi-continuous map, we have
\[ q(ρ_{N_ε}, u) \leq \liminf_{n→∞} q(ρ_{N_ε}, ρ_n) \leq ε. \]
Putting ε = \frac{1}{k} and N_ε = n_k, we have
\[ (2.6) \quad \lim_{k→∞} q(ρ_{n_k}, u) = 0. \]

Assume that u ≠ f u. Then
\[ 0 < \inf\{q(σ, u) + q(σ, f σ) : σ ∈ M\} \leq \inf\{q(ρ_{n_k}, u) + q(ρ_{n_k}, ρ_{n_k+1}) : k ∈ N\}. \]
From (2.4) and (2.6), we derive
\[ \inf\{q(σ, u) + q(σ, f σ) : σ ∈ M\} = 0, \]
which contradicts the given hypothesis. Therefore, f u = u.

Case (ii). Let f be continuous.
Using the triangular inequality, we have
\[ q(ρ_n, f^2ρ_n) \leq q(ρ_n, fρ_n) + q(fρ_n, f^2ρ_n). \]
Accordingly, letting n → ∞, we obtain q(ρ_n, f^2ρ_n) → 0. Further, Lemma 1.1 confirms that \{f^2ρ_n\} → u as n → ∞. As f is continuous, we have
\[ f u = f(\lim_{n→∞} fρ_n) = \lim_{n→∞} f^2ρ_n = u. \]
Hence, u is a fixed point of f.

Case (iii). Here, \lim_{n→∞} q(fρ_n, u) = \lim_{n→∞} q(ρ_{n+1}, u) = \lim_{n→∞} q(ρ_n, u). Hence, f u = u.

**Step 4.** u is a fixed point with q(u, u) = 0.

Conversely, suppose that q(u, u) > 0. Then from (2.1), we get
\[ 0 < q(u, u) = q(f u, f u) \leq ψ(Mq(u, u)) < Mq(u, u) = q(u, u), \]
and this is impossible. Hence, our claim is verified.

The fixed point obtained in the previous theorem may be not unique. The following examples validate our claim.

**Example 2.1.** Suppose that G is a locally compact group, M = L^1(G) and
\[ q(f, g) = \|g\|_1, \quad f, g ∈ L^1(G). \]
Then q is a w-distance. Define ψ(t) = \begin{cases} \frac{t}{2}, & t ∈ [0, 1], \\ \frac{1}{2}, & t ∈ (1, ∞), \end{cases}
and
\[ α(f, g) = \begin{cases} \frac{2}{2\|g\|_1}, & g = 0 \ (a.e.), \\ ψ(Mq(f, g)), & \text{otherwise}, \end{cases} \]
and for an arbitrary x ∈ G
\[ T_x : L^1(G) → L^1(G), \quad f → \frac{1}{8} L_x f, \]
where $L_x f(y) = f(x^{-1} y)$. Then for each $f \in L^1(G)$ and $x \in G$, since $\|L_x f\|_1 = \|f\|_1$, we conclude that $M_q(f, g) = \max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \}$ and so

$$\alpha(f, g) = \frac{\psi(M_q(f, g))}{2 \| g \|_1} = \frac{\psi(\max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \})}{2 \| g \|_1} \geq 1.$$ 

In each of the cases $0 \leq \max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \} \leq 1$, $1 \leq \max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \} \leq 8$ and $8 \leq \max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \}$ we conclude that

$$\alpha(T_x f, T_x g) = \frac{\psi(\max \{ \frac{1}{8} \| f \|_1, \| g \|_1 \})}{\| g \|_1} > 1.$$ 

So, $T_x$ is $\alpha$-orbital admissible. Now for each $\eta > 0$ and $\delta = \eta$, if $\eta \leq \psi(M_q(f, g)) < 2\eta$, then for $g \neq 0$ we have

$$\alpha(f, g) q \left( \frac{1}{8} L_x f, \frac{1}{8} L_x g \right) = \frac{\psi(M_q(f, g))}{2 \| g \|_1} \left( \frac{1}{8} \right) \| g \|_1 \leq \frac{1}{8} \eta < \eta,$$

and for $g = 0$, since $q(T_x f, T_x g) = 0$, we are done. So, $T_x$ is a generalized $(\alpha, \psi, q)$-Meir-keeler contractive map. Moreover, $\alpha(0, T_x 0) = \alpha(0, 0) = 2 > 1$,

$$q(T_x^n 0, T_x^n 0) = q(0, 0) = \| 0 \|_1 = 0,$$

and $T_x$ is continuous. Therefore, all the hypotheses of Theorem 2.1 hold and so, $T_x$ has a fixed point (which is $f = 0$, satisfying $q(0, 0) = 0$). Note that for each $f \in L^1(G)$, we have

$$\lim \| T_x^n f \|_1 = \lim \frac{1}{8^n} \| f \|_1 = 0.$$ 

Therefore, $T_x^n f$ converges to 0 and so 0 is the only fixed point of $T_x$.

**Example 2.2.** Suppose that $M = \{ \frac{1}{2^n} : n \in \mathbb{N} \} \cup \{ 0 \}$ is equipped with the usual metric on $\mathbb{R}$. Consider

$$q(\rho, \varrho) = \begin{cases} \frac{1}{n} + \frac{1}{m}, & \varrho = \frac{1}{2^m}, \rho = \frac{1}{2^n}, \\ 0, & \rho = 0 \text{ or } \varrho = 0, \end{cases}$$

$$\alpha(\rho, \varrho) = \begin{cases} \frac{n}{m}, & \varrho = \frac{1}{2^m}, \rho = \frac{1}{2^n} \text{ and } 2m \geq m \geq n, \\ 1, & \rho = 0 \text{ or } \varrho = 0, \\ \frac{1}{n}, & \text{otherwise}, \end{cases}$$

and $f \rho = \rho^8$. Then $\alpha(0, f 0) = 1$, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and $f$ is continuous and also $\alpha$-orbital admissible. Since if $\alpha(\rho, f \rho) \geq 1$, then $\rho = 0$, since if $\rho = \frac{1}{2^n}$ for some $n$, then $n \leq 8n \leq 2n$ is impossible. Therefore, $\alpha(f \rho, f^2 \rho) \geq 1$. Also if

$$\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$$
Remark 2.1

2.3

Again, that is for each \( n \). Therefore, we have

\[
\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq 2 \left( \frac{1}{8n} + \frac{1}{8m} \right) = \frac{1}{4} q(\rho, \varrho) \leq \frac{1}{4} M_\rho(\rho, \varrho) = \frac{1}{2} \psi(M_\rho(\rho, \varrho)) < \eta.
\]

Also, if one of \( \rho \) or \( \varrho \) is zero, then \( \alpha(\rho, \varrho)q(f\rho, f\varrho) = 0 \leq \eta \). So, \( f \) is a generalized \((\alpha, \psi, q)\)-Meir-Keeler contractive map. Therefore, all the conditions of Theorem 2.1 hold. Hence, \( \rho = 0 \) is the unique fixed point of \( f \).

Example 2.3. Let \( M = [0, 1] \) be equipped with the usual metric. Also let us consider the \( w \)-distance as \( q(\rho, \varrho) = |\rho - \varrho| \) for each \( \rho, \varrho \in M \). Further, we define

\[
f\rho = \begin{cases} \frac{\rho}{20}, & \rho \in [0, 1), \\ 1, & \rho = 1, \end{cases}
\]

\[
\alpha(\rho, \varrho) = \begin{cases} 1, & \rho, \varrho \in [0, 1), \\ 0, & \rho = 1, \end{cases}
\]

\[
\psi(\rho) = \begin{cases} 0, & \rho = 0, \\ \frac{1}{3}, & \rho \in \left(0, \frac{1}{2}\right), \\ \frac{\rho}{2}, & \rho \in \left[\frac{1}{2}, 1\right], \\ \frac{1}{2}, & \rho \in (1, \infty). \end{cases}
\]

Hence, for every \( w \in M \) with \( f\rho \neq w \), we obtain \( w \neq 0, 1 \) and so

\[
\lim_{\rho \to w} (|w - \rho| + |\rho - f\rho|) \geq \frac{19}{20} w > 0.
\]

Again,

\[
\lim_{\rho \to \varrho} (|w - \rho| + |\rho - f\rho|) \geq |w - \varrho| > 0, \quad \varrho \neq w.
\]

Therefore, we have \( \inf \{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0 \) for each \( w \in M \) satisfying \( w \neq f\varrho \). Besides, for every \( \rho \in M \), we obtain \( |f^n\rho - f^n\varrho| = 0 \). Now for each \( \eta > 0 \), put \( \delta = \eta \). Then, \( \rho = \varrho \) implies \( M_\varrho(\rho, \varrho) = 0 \) and when \( \rho \neq \varrho \), \( M_\varrho(\rho, \varrho) \neq 0 \) and further, \( \psi(M_\varrho(\rho, \varrho)) \geq \frac{1}{4} \). Therefore, for \( \eta > \frac{1}{8} \), there is no \( \rho, \varrho \in M \) satisfying

\[
\frac{1}{8} \leq \psi(M_\varrho(\rho, \varrho)) < \frac{1}{4}.
\]

On the other hand, for \( \eta \leq \frac{1}{8} \), if \( \eta \leq \psi(M_\varrho(\rho, \varrho)) < \eta + \eta = 2\eta \), we have

\[
\alpha(\rho, \varrho)|f\rho - f\varrho| \leq |f\rho - f\varrho| = \left| \frac{\rho}{20} - \frac{\varrho}{20} \right| \leq \frac{2}{20} < \frac{1}{8} \leq \eta.
\]

That is for each \( \rho, \varrho \), if \( \eta \leq \psi(M_\varrho(\rho, \varrho)) < \eta + \eta = 2\eta \), then \( \alpha(\rho, \varrho)|f\rho - f\varrho| \leq \eta \). Note that 0, 1 are the fixed points of \( f \).

Remark 2.1. In the case where \( q(\rho, \varrho) = \varrho \) for each \( \rho, \varrho \in M \), the assumption \( q(f^n\rho, f^n\varrho) = 0 \), for some \( \rho \in M \) and for each \( n \in \mathbb{N} \), imply that \( f^n\rho = 0 \) for each \( n \). Therefore, in this case without any another condition, since \( \rho_0 = 0 = \rho_{n+1} \),
the first part of the Theorem 2.1 implies that \( f \) possesses a fixed point. For example, let \( M = \{ \frac{1}{2^k} : n \in \mathbb{N} \} \cup \{ 0 \}, \)

\[
\alpha(\rho, \varrho) = \begin{cases} 0, & \varrho \in \left\{ \frac{1}{2^k} : k \in \mathbb{N} \right\}, \\ 1, & \text{otherwise}, \end{cases}
\]
and \( f \rho = \begin{cases} \rho, & \rho \in \left\{ \frac{1}{2^k} : k \in \mathbb{N} \right\}, \\ 1, & \text{otherwise}. \end{cases} \)

Then \( f \) is continuous, \( q(f^n 0, f^n 0) = 0 \) for each \( n \in \mathbb{N} \) and \( \rho, \varrho \in M \) and \( \eta, \delta > 0 \), if \( \eta \leq \psi(M q(\rho, \varrho)) < \eta + \delta \), then we have

\[
0 = \alpha(\rho, \varrho) q(f \rho, f \varrho) \leq \eta.
\]

Note that 0 is a fixed point of \( f \), since here we require only \( q(f^n 0, f^n 0) = 0 \).

Now we put down the following additional hypothesis. To attest the uniqueness of the fixed point of \( f \), this condition along with those of Theorem 2.1 is required.

**Property U.** Let \( \alpha(u, v) < 1 \), implies that at least one of \( u \) or \( v \) is not a fixed point of \( f \).

For example if \( \alpha(u, v) \geq 1 \) for each \( u, v \in M \), then the property \( U \) is valid.

**Theorem 2.2.** Suppose that \((M, d)\) is a metric space with a \( w \)-distance \( q \). Also assume that \( f \) is a generalized \((\alpha, \psi, p)\)-Meir-Keeler contractive mapping and satisfies all the hypotheses of Theorem 2.1 along with the additional property \( U \). Then we can claim the uniqueness of the fixed point of \( f \) obtained in Theorem 2.1.

**Proof.** We suppose that \( u, v \in M \) are two distinct fixed points of \( f \). Then \( \alpha(u, v) \geq 1 \), \( fu = u, fv = v, q(u, u) = 0 \) and \( q(v, v) = 0 \). Using the aforementioned criteria and (2.1), we obtain

\[
q(u, v) = q(fu, fv) \leq \alpha(u, v) q(fu, fv) \leq \psi(M q(u, v)) = \psi(q(u, v)) < q(u, v),
\]

and this is impossible. Hence, \( f \) possesses a unique fixed point. \(\square\)

3. Consequences

This section deals with a few immediate corollaries of our obtained Theorem 2.1.

**Corollary 3.1.** Suppose that \((M, d)\) is a complete metric space with a \( w \)-distance \( q \). Also let \( f \) be an \((\alpha, \psi, q)\)-Meir-Keeler contractive mapping with the fact that there is some \( \rho_0 \in M \), with \( q(f^n \rho_0, f^n \rho_0) = 0 \) for all non-negative integers \( n \) and \( \alpha(\rho_0, f \rho_0) \geq 1 \). Suppose that one of the following holds.

(i) For each \( w \in M \) satisfying \( w \neq f w \), we have \( \inf \{ q(\rho, w) + q(\rho, f \rho) : \rho \in M \} > 0 \).

(ii) \( f \) is continuous.

(iii) If for some sequence \( \{ \rho_n \} \), \( \lim_{n \to \infty} q(\rho_n, \rho) = \lim_{n \to \infty} q(f \rho_n, \rho) \), then \( f \rho = \rho \).

Then \( f \) possesses a fixed point \( u \in M \), with \( q(u, u) = 0 \).
Putting $\alpha \equiv 1$ in Theorem 2.1, we obtain the trailing important corollary.

**Corollary 3.2.** Suppose that $(M, d)$ is a complete metric space with a $w$-distance $q$. Also let $f$ be a $(\psi, q)$-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n \rho_0, f^n \rho_0) = 0$ for all non-negative integers $n$. Suppose that one of the following conditions holds.

(i) For each $w \in M$ satisfying $w \neq f w$, we have $\inf\{q(\rho, w) + q(\rho, f \rho) : \rho \in M\} > 0$.

(ii) $f$ is continuous.

(iii) If for some sequence $\{\rho_n\}$, $\lim_{n \to \infty} q(\rho_n, \rho) = \lim_{n \to \infty} q(f \rho_n, \rho)$, then $f \rho = \rho$.

Then $f$ possesses a fixed point $u \in M$.

Considering $q = d$ in Theorem 2.1, we deduce the subsequent corollary.

**Corollary 3.3.** Suppose that $(M, d)$ is a complete metric space and $f$ be an $(\alpha, \psi)$-Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$ with $\alpha(\rho_0, f \rho_0) \geq 1$ or $\alpha(f \rho_0, \rho_0) \geq 1$. Suppose that one of the following conditions holds.

(i) For each $w \in M$ satisfying $w \neq f w$, we have $\inf\{d(\rho, w) + d(\rho, f \rho) : \rho \in M\} > 0$.

(ii) $f$ is continuous.

(iii) For some sequence $\{\rho_n\}$ in $M$ with $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all natural numbers $n$ and $\rho_n \to \rho \in M$ as $n \to \infty$, then $\alpha(\rho_n, \rho) \geq 1$ for every $n \in \mathbb{N}$.

Then $f$ possesses a fixed point $u \in M$.

Taking $\alpha \equiv 1$ in Corollary 3.3, we get the succeeding consequence.

**Corollary 3.4.** Suppose that $(M, d)$ is a complete metric space and $f$ be a $\psi$-Meir-Keeler contractive mapping. Suppose that either $f$ is continuous or $\inf\{d(\rho, w) + d(\rho, f \rho) : \rho \in M\} > 0$ for each $w \in M$ with $w \neq f w$. Then $f$ possesses a fixed point $u \in M$.

**Definition 3.1.** Suppose that $(M, d)$ is a metric space with a $w$-distance $q$ and consider the functions $\psi \in \Psi$, $\alpha : M \times M \to [0, \infty)$ and a self-map $f$. Then $f$ is said to be a generalized $(\alpha, \psi, q)$-Meir-Keeler contractive mapping of

(a) Banach type if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\eta \leq \psi(q(\rho, \varrho)) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta;$$

(b) Kannan type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\eta \leq \psi\left(\frac{q(\rho, f \rho) + q(\varrho, f \varrho)}{2}\right) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta;$$

(c) Kannan type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$

$$\eta \leq \psi(\max\{q(\rho, f \rho), q(\varrho, f \varrho)\}) < \eta + \delta, \quad \text{we have} \quad \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta;$$
(d) Chatterjea type I if for every \( \eta > 0 \), there exists a \( \delta > 0 \) such that for \( \rho, \varrho \in M \) when \( \eta \leq \psi \left( \frac{q(\rho, f \varrho) + q(\varrho, f \rho)}{2} \right) < \eta + \delta \), we have \( \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta \);

(e) Chatterjea type II if for every \( \eta > 0 \), there exists a \( \delta > 0 \) such that for \( \rho, \varrho \in M \) when \( \eta \leq \psi(\max\{q(\rho, f \varrho), q(\varrho, f \rho)\}) < \eta + \delta \), we have \( \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta \);

(f) Reich type I if for every \( \eta > 0 \), there exists a \( \delta > 0 \) such that for \( \rho, \varrho \in M \) when \( \eta \leq \psi \left( \frac{q(\rho, \varrho) + q(\rho, f \rho) + q(\varrho, f \varrho)}{3} \right) < \eta + \delta \), we have \( \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta \);

(g) Reich type II if for every \( \eta > 0 \), there exists a \( \delta > 0 \) such that for \( \rho, \varrho \in M \) when \( \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f \rho), q(\varrho, f \varrho)\}) < \eta + \delta \), we have \( \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta \);

(h) Reich type III if for every \( \eta > 0 \), there exists a \( \delta > 0 \) such that for \( \rho, \varrho \in M \) when \( \eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f \rho), q(\varrho, f \varrho)\}) < \eta + \delta \), we have \( \alpha(\rho, \varrho)p(f \rho, f \varrho) < \eta \).

In addition, for taking \( q = d \) in the inequalities above, we can get several other kind of contractions in the context of metric spaces.

If in Theorem 2.1, we change the contraction condition ‘generalized \((\alpha, \psi, q)\)-Meir-Keeler contractive mapping’ with one of the new contractions defined in Definition 3.1, then we may obtain a similar result as Theorem 2.1. Furthermore, as in Corollary 3.3 and Corollary 3.4, we may get some more results by letting \( q = d \). Also, notice that by choosing the auxiliary function \( \alpha \) in a proper way in Theorem 2.1, we can deduce more consequences related to cyclic contractions and results in metric spaces endowed with a partially ordered set, see for example [1–8].

4. An Application

In this section, we discuss an application of our obtained fixed point result to a certain kind of non-linear Fredholm integral equations. First of all, we prove a proposition which is going to play a crucial role here.

**Proposition 4.1.** Suppose that \((M, d)\) is a metric space with a \(w\)-distance \(q\). Also, assume that \(f\) is a self-mapping on \(M\) satisfying

\[
\alpha(\rho, \varrho)q(f \rho, f \varrho) \leq k\psi(M_q(\rho, \varrho)),
\]

for all \(\rho, \varrho \in M\) and for some \(k \in (0, 1)\). Then \(f\) is a generalized \((\alpha, \psi, q)\)-Meir-Keeler contractive mapping.

**Proof.** Consider \(\delta = (\frac{1}{k} - 1)\eta\) in Definition 2.1. Accordingly, we derive

\[
\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta < \eta + \left(\frac{1}{k} - 1\right) \eta = \frac{\eta}{k},
\]

\(1\).
and so, for every $\rho, \varrho \in M$, we obtain $\kappa \eta \leq k \psi(M_\eta(\rho, \varrho)) < \eta$. Using (4.1), we get

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k \psi(M_\eta(\rho, \varrho)) < \eta.$$  

Hence, $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$ and therefore, $f$ is an $(\alpha, \psi, q)$-Meir-Keeler contractive mapping. \qed

Now, we try to obtain a criterion to ensure the existence of a solution for a type of non-linear Fredholm integral equation.

**Theorem 4.1.** Let us consider the non-linear Fredholm integral equation

$$\tag{4.2} (fx)(t) = g(t) + \int_a^b H(t, s, x(s))ds,$$

for some $a, b \in \mathbb{R}$, with $a < b$, $g : [a, b] \to \mathbb{R}$ and $H : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$ be two continuous maps. Also, assume that the subsequent properties hold:

(i) $f : C[a, b] \to C[a, b]$ is a continuous mapping;

(ii) there exists a weaker Meir-Keeler function $\psi$ and $k \in [0, 1)$ satisfying

$$k \left[ \psi \left( \max \left\{ \left| x(t) \right| + \left| y(t) \right|, \left| x(t) \right| + \left| (fx)(t) \right|, \left| y(t) \right| + \left| (fy)(t) \right| \right) \right] \leq \frac{b - a}{2 \left( \left| x(t) \right| + \left| (fx)(t) \right| + \left| (fy)(t) \right| \right) + \left( \left| x(t) \right| + \left| y(t) \right| \right)} - 2 \left| g(t) \right|,$$

for all $t, s \in [a, b]$. Then the non-linear Fredholm integral equation (4.2) owns a unique solution in $C[a, b]$.

**Proof.** Suppose $M = C[a, b]$. Obviously, $M$ is complete with respect to the metric $d : M \times M \to \mathbb{R}^+$ defined as

$$d(x, y) = \sup_{t \in [a, b]} \left| x(t) - y(t) \right|,$$

where $x, y \in M$. Now, we consider the map $q : M \times M \to \mathbb{R}^+$ given by

$$q(x, y) = \sup_{t \in [a, b]} \left| x(t) \right| + \sup_{t \in [a, b]} \left| y(t) \right|,$$

where $x, y \in M$. One can easily check that, $q$ is a $w$-distance on $M$. Here we have

$$|(fx)(t)| + |(fy)(t)|$$

$$= \left| g(t) + \int_a^b H(t, s, x(s))ds \right| + \left| g(t) + \int_a^b H(t, s, y(s))ds \right|$$

$$\leq \left| g(t) \right| + \left| \int_a^b H(t, s, x(s))ds \right| + \left| g(t) \right| + \left| \int_a^b H(t, s, y(s))ds \right|$$

$$\leq 2 \left| g(t) \right| + \left| \int_a^b H(t, s, x(s))ds \right| + \left| \int_a^b H(t, s, y(s))ds \right|$$
This implies that equation (4.2) has a solution. □

contractive mapping. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (6) (2016), 1095–1108.

Thus, \[
\sup_{t \in [a,b]} |(f(x))(t)| + \sup_{t \in [a,b]} |(Ty)(t)| \leq k \left[ \psi (M_q(x,y)) \right],
\]
and therefore for each \( x, y \in M \)
\[
q(fx, fy) \leq k \left[ \psi (M_q(x,y)) \right].
\]

This implies that \( f \) satisfies Proposition 4.1 and hence it is an \((\alpha, \psi, q)\)-Meir-Keeler contractive mapping. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (4.2) has a solution. □

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