Kragujevac Journal of Mathematics
Volume 46(4) (2022), Pages 575–594.

$\mathcal{N}$-CUBIC SETS APPLIED TO LINEAR SPACES

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Abstract. The concept of $\mathcal{N}$-fuzzy sets is a good mathematical tool to deal with uncertainties that use the co-domain $[-1, 0]$ for the membership function. The notion of $\mathcal{N}$-cubic sets is defined by combining interval-valued $\mathcal{N}$-fuzzy sets and $\mathcal{N}$-fuzzy sets. Using this $\mathcal{N}$-cubic sets, we initiate a new theory called $\mathcal{N}$-cubic linear spaces. Motivated by the notion of cubic linear spaces we define P-union (resp. R-union), P-intersection (resp. R-intersection) of $\mathcal{N}$-cubic linear spaces. The notion of internal and external $\mathcal{N}$-cubic linear spaces and their properties are investigated.

1. Introduction

The classical set theory failed to handle uncertain, vague and clearly not defined objects because of its limitation to a bivalent condition which is precise in character - an element either belongs or does not belong to the set. As it is well known that Zadeh [19] pioneered the study of fuzzy sets in 1965, which can handle various types of uncertainties successfully in different fields. In contrast to classical set theory fuzzy set theory permits gradual assessment of membership of elements in a set. Fuzzy set theory has rich potential for application in several directions such as topology, analysis, logic, group theory and, semigroup theory. After a decade in 1975, Zadeh [20] introduced interval-valued fuzzy sets as a generalization of a fuzzy set whose members are mapped to the collection of closed subintervals of $[0, 1]$. Attansov [1, 2], further extended the idea of fuzzy sets to intuitionistic fuzzy sets where one can handle membership as well as non-membership of an element. This approach gradually replaced fuzzy sets in dealing with uncertanity and vagueness.

Key words and phrases. $\mathcal{N}$-Interval number, interval-valued $\mathcal{N}$-fuzzy linear space, $\mathcal{N}$-cubic linear space, internal and external $\mathcal{N}$-cubic linear spaces, P-intersection and P-union, R-intersection and R-union.

2010 *Mathematics Subject Classification.* Primary: 08A72. Secondary: 03E72.

DOI 10.46793/KgJMat2204.575K

Received: November 18, 2019.

Accepted: February 21, 2020.

Another extension of fuzzy set theory is cubic set theory introduced by Jun et al. [5] in 2010 and examined many properties of cubic sets like internal cubic sets, external cubic sets, P-union, P-intersection, R-union and R-intersection of internal and external cubic sets. Since cubic sets undertake positive part of many physical problems and took no notice of negative aspects wholly. Jun et al [4] brought up a negative valued function and formulated $\mathcal{N}$-structures. Moreover, they applied $\mathcal{N}$ structure theory to subtraction algebra and BCK/BCI algebra [6]. This paved way to the idea of $\mathcal{N}$-cubic sets introduced by Jun [9] combining $\mathcal{N}$-fuzzy sets and interval-valued $\mathcal{N}$-fuzzy sets to cover the negative part of cubic sets along the codomain $[-1, 0]$.

An abundant measure of efforts was executed by researchers in extending fuzzy sets to groups, rings, vector spaces and other branches of mathematics. G. Lubczonok and V. Murali [10] introduced the theory of flags and fuzzy subspaces of vector spaces. Kastras and Liu [7] applied the concept of fuzzy sets to the elementary theory of vector spaces and topological vector spaces. Nanda [11] introduced the concept of fuzzy linear space. Later Gu Wexiang and Lu [18] redefined the concept of fuzzy field and fuzzy linear space and gave some fundamental properties. Vijaybalaji et al. further advanced the theory to cubic linear space combining interval-valued fuzzy linear space and fuzzy linear space and their properties are presented in [17].

In this paper we present the notion of $\mathcal{N}$-cubic linear spaces. After providing essential background on cubic sets, $\mathcal{N}$-cubic linear spaces and their intersection and union properties we confine section 3 to define the concept of $\mathcal{N}$-cubic linear spaces. We introduce the P-union (resp. P-intersection) and R-union (resp. R-intersection) in $\mathcal{N}$-Cubic linear spaces. We show that $\mathcal{N}$-cubic linear space is closed with respect to R-intersection. By giving examples we disprove that R-union, P-union and P-intersection of two $\mathcal{N}$-cubic linear spaces is again a $\mathcal{N}$-cubic linear space. In section 4, we introduce the concept of internal $\mathcal{N}$-cubic linear space and external $\mathcal{N}$-cubic linear space. We also show that internal $\mathcal{N}$-cubic linear space is not closed with respect to P-union, P-intersection and R-union (resp. external) by providing counter examples.

2. Preliminaries

Definition 2.1 ([20])**.** An $\mathcal{N}$-interval number is a closed subinterval of $[-1, 0]$ and the collection of all closed subintervals of $[-1, 0]$ is denoted by $D[-1, 0]$. It is of the form $D[-1, 0] = \{\hat{i} = [i^-, i^+] : i^- \leq i^+, i^-, i^+ \in [0, 1]\}$. Notably the operations "$\geq$", "$\leq$", "$=$", "max", "min" are defined as follows:

(i) $\hat{i_1} \geq \hat{i_2}$ if and only if ${i_1}^- \geq {i_2}^-$ and ${i_1}^+ \geq {i_2}^+$;

(ii) $\hat{i_1} \leq \hat{i_2}$ if and only if ${i_1}^- \leq {i_2}^-$ and ${i_1}^+ \leq {i_2}^+$;

(iii) $\hat{i_1} = \hat{i_2}$ if and only if ${i_1}^- = {i_2}^-$ and ${i_1}^+ = {i_2}^+$;

(iv) $\min\{\hat{i_1}, \hat{i_2}\} = [\min\{{i_1}^-, {i_2}^-\}, \min\{{i_1}^+, {i_2}^+\}]$;

(v) $\max\{\hat{i_1}, \hat{i_2}\} = [\max\{{i_1}^-, {i_2}^-\}, \max\{{i_1}^+, {i_2}^+\}]$.

Definition 2.2 ([20])**.** For an $\mathcal{N}$-interval number $\hat{i_t} \in D[-1,0]$, where $t \in \Lambda$. We define

$$\inf \hat{i_t} = \left[\inf_{t\in\Lambda} \hat{i_t}^-, \inf_{t\in\Lambda} \hat{i_t}^+\right] \quad \text{and} \quad \sup \hat{i_t} = \left[\sup_{t\in\Lambda} \hat{i_t}^-, \sup_{t\in\Lambda} \hat{i_t}^+\right].$$

Definition 2.3 ([20])**.** An interval valued $\mathcal{N}$-fuzzy set denoted by $\mathcal{J}^{\mathcal{N}}$ on Y is of the form $\mathcal{J}^{\mathcal{N}} = \{< y, \mathcal{J}^{\mathcal{N}}(y) : y \in \mathrm{Y} >\}$, where $\mathcal{J}^{\mathcal{N}} : \mathrm{Y} \to D[0,1]$ and $\mathcal{J}^{\mathcal{N}}(y) = [\vartheta^-_{\mathcal{J}^{\mathcal{N}}}(y), \vartheta^+_{\mathcal{J}^{\mathcal{N}}}(y)]$ for all $y \in \mathrm{Y}$. Here $\vartheta^-_{\mathcal{J}^{\mathcal{N}}}(y) : \mathrm{Y} \to [0,1]$ and $\vartheta^+_{\mathcal{J}^{\mathcal{N}}}(y)] : \mathrm{Y} \to [0,1]$ are fuzzy sets in Y such that $\vartheta^-_{\mathcal{J}^{\mathcal{N}}}(y) \leq \vartheta^+_{\mathcal{J}^{\mathcal{N}}}(y)$.

Definition 2.4 ([5])**.** Let Y be a non-empty set. A cubic set $\mathbf{C}$ of Y is a structure $\mathbf{C} = \{y, \hat{\vartheta}_{\mathbf{C}}(y), \lambda_{\mathbf{C}}(y) | y \in \mathrm{Y}\}$ in which $\hat{\vartheta}_{\mathbf{C}} : \mathrm{Y} \to D[0,1]$ and $\lambda_{\mathbf{C}} : \mathrm{Y} \to [0,1]$.

Definition 2.5 ([5])**.** A cubic set $\mathbf{C} = (\hat{\vartheta}_{\mathbf{C}}, \lambda_{\mathbf{C}})$ in a non-empty set Y is said to be an internal cubic set (in brief, ICS) if $\vartheta^-_{\mathbf{C}}(y) \leq \lambda_{\mathbf{C}}(y) \leq \vartheta^+_{\mathbf{C}}(y)$ for all $y \in \mathrm{Y}$. For an external cubic set (in brief, ECS) it is $\lambda_{\mathbf{C}}(y) \notin (\vartheta^-_{\mathbf{C}}(y), \vartheta^+_{\mathbf{C}}(y)$ for all $y \in \mathrm{Y}$.

Definition 2.6 ([17])**.** Let W be a linear space over field F, $(\mathrm{W}, \hat{\vartheta})$ be an interval valued fuzzy linear space, (W, λ) be a fuzzy linear space. A cubic set $\mathbf{C} = (\hat{\vartheta}_{\mathbf{C}}, \lambda_{\mathbf{C}})$ is called a cubic linear space of W if for all $\sigma, \tau \in F$
(i) $\hat{\vartheta}(\sigma a * \tau b) \geq \min\{\hat{\vartheta}(a), \hat{\vartheta}(b)\}$;
(ii) $\lambda(\sigma a * \tau b) \leq \max\{\lambda(a), \lambda(b)\}$.

Definition 2.7 ([9])**.** Let Y be a fixed set. A $\mathcal{N}$-fuzzy set in Y is defined as $\mathbf{N}^F = \{y, \lambda_{\mathbf{N}^F}(y)\} : y \in \mathrm{Y}$ and $\lambda_{\mathbf{N}^F} : \mathrm{Y} \to [-1,0]$ a membership function for all $y \in \mathrm{Y}$.

Definition 2.8 ([9])**.** Let Y be a non-empty set. A $\mathcal{N}$-cubic set in Y is a structure $\mathbf{N}^{\mathbf{C}} = \Big\{\langle y, \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}(y), \lambda_{\mathbf{N}^{\mathbf{C}}}(y)\rangle | y \in \mathrm{Y}\Big\}$ is briefly denoted by $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}}\rangle$ in which $\hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}} = [\vartheta^-_{\mathbf{N}^{\mathbf{C}}}, \vartheta^+_{\mathbf{N}^{\mathbf{C}}}]$ an interval valued fuzzy set and $\lambda_{\mathbf{N}^{\mathbf{C}}} : \mathrm{Y} \to [-1,0]$ is a fuzzy set in Y.

Definition 2.9 ([9])**.** Let Y be a non-empty set. An $\mathcal{N}$-cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}}\rangle$ in Y is said to be an internal $\mathcal{N}$-cubic set (INCS) if $\vartheta^-_{\mathbf{N}^{\mathbf{C}}}(y) \leq \lambda_{\mathbf{N}^{\mathbf{C}}} \leq \vartheta^+_{\mathbf{N}^{\mathbf{C}}}(y)$ for all $y \in \mathrm{Y}$. Similarly, external $\mathcal{N}$-cubic set (ENCS) if $\lambda_{\mathbf{N}^{\mathbf{C}}}(y) \notin (\vartheta^-_{\mathbf{N}^{\mathbf{C}}}(y), \vartheta^+_{\mathbf{N}^{\mathbf{C}}}(y))$.

Definition 2.10 ([9])**.** For any $\mathbf{N}^{\mathbf{C}}_i = \Big\{\langle y, \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i}(y), \lambda_{\mathbf{N}^{\mathbf{C}}_i}(y)\rangle : y \in \mathrm{Y}\Big\}$, where $i \in \Lambda$, we define
(a) $\bigcup_{\mathrm{R}\, i\in\Lambda} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (R-union);
(b) $\bigcup_{\mathrm{P}\, i\in\Lambda} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (P-union);
(c) $\bigcap_{\mathrm{R}\, i\in\Lambda} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (P-intersection);
(d) $\bigcap_{\mathrm{R}\, i\in\Lambda} \mathbf{N}^{\mathbf{C}}_i = \Big\{\Big\langle y, (\bigcup_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_i})(y), (\bigcup_{i\in\Lambda} \lambda_{\mathbf{N}^{\mathbf{C}}_i})(y) : y \in \mathrm{Y}\Big\rangle\Big\}$ (R-intersection).

3. Results

In this section, we come across the notion of $\mathcal{N}$-cubic linear space. We also discuss some results in connection with the $\mathcal{N}$-cubic linear space.

3.1. $\mathcal{N}$-cubic linear spaces.

Definition 3.1. For a linear space W over a field F a $\mathcal{N}$-fuzzy set $\mathbf{N}^F$=(W, $\lambda_{\mathbf{N}^F}$) in W is said to be a $\mathcal{N}$-fuzzy linear space $\mathbf{W^F}$=$\{(w, \lambda_{\mathbf{N}^F}(w)) : w \in \mathrm{W}, \lambda_{\mathbf{N}^F}(w) \in [-1, 0]\}$ if it satisfies

$$\lambda_{\mathbf{W^F}}(\sigma a * \tau b) \leq \lambda_{\mathbf{W^F}}(a) \cup \lambda_{\mathbf{W^F}}(a),$$

for any $\sigma, \tau \in F$ and $a, b \in \mathrm{W}$.

Definition 3.2. An interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}} : \mathrm{W} \to D[-1, 0]$ is said to be an interval-valued $\mathcal{N}$-fuzzy linear space where W over field F if the latter conditions are satisfied

$$\hat{\vartheta}_{\mathbf{N}}(\sigma a * \tau b) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}}(a), \hat{\vartheta}_{\mathbf{N}}(b)\right\},$$

for any for any $\sigma, \tau \in F$ and $a, b \in \mathrm{W}$.

Definition 3.3. Let W be a linear space over field F, (W, $\hat{\vartheta}_{\mathbf{IF}}$) an interval-valued $\mathcal{N}$-fuzzy linear space, (W, $\lambda_{\mathbf{W^F}}$) a $\mathcal{N}$-fuzzy linear space. A **N**-cubic set $\mathbf{N^C} = \langle \hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}} \rangle$ in Y is said to be a **N**-cubic linear space of W if

(i) $\hat{\vartheta}_{\mathbf{N^C}}(\sigma a * \tau b) \leq \max\{\hat{\vartheta}_{\mathbf{N^C}}(a), \hat{\vartheta}_{\mathbf{N^C}}(b)\}$;

(ii) $\lambda_{\mathbf{N^C}}(\sigma a * \tau b) \geq \min\{\lambda_{\mathbf{N^C}}(a), \lambda_{\mathbf{N^C}}(b)\}$,

for all $a, b \in \mathrm{W}$ and $\sigma, \tau \in F$.

Example 3.1. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as follows

$$\mathrm{W} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

such that $w_{11} + w_{12} = w_{21}$. Then W is a vector space over the field $GF(2)$.

Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\begin{aligned} \hat{\vartheta}_{\mathbf{N}}(a) =& [-0.9, -0.8], \\ \hat{\vartheta}_{\mathbf{N}}(b) =& [-0.6, -0.3], \\ \hat{\vartheta}_{\mathbf{N}}(c) =& [-0.4, -0.1], \\ \hat{\vartheta}_{\mathbf{N}}(d) =& [-0.8, -0.7]. \end{aligned}$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space.

Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\begin{aligned}\lambda_{\mathbf{N}}(a) &= -\,0.4,\\ \lambda_{\mathbf{N}}(b) &= -\,0.6,\\ \lambda_{\mathbf{N}}(c) &= -\,0.25,\\ \lambda_{\mathbf{N}}(d) &= -\,0.9.\end{aligned}$$

Here λ is a $\mathcal{N}$-fuzzy linear space of W.

Consequently, the above example satisfied the conditions required for a cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ to be a $\mathcal{N}$-cubic linear space.

Remark 3.1. For any family of real numbers $\{b_j : j \in \Lambda\}$ we define

(i)

$$\bigcup\{b_j : j \in W\} = \begin{cases} \max\{b_j : j \in \Lambda\}, & \text{if } \Lambda \text{ is finite,}\\ \sup\{b_j : j \in \Lambda\}, & \text{otherwise.}\end{cases}$$

(ii)

$$\bigcap\{b_j : j \in W\} = \begin{cases} \min\{b_j : j \in \Lambda\}, & \text{if } \Lambda \text{ is finite,}\\ \inf\{b_j : j \in \Lambda\}, & \text{otherwise.}\end{cases}$$

In the following proposition, we prove that the R-union of a family of $\mathbf{N}$-cubic linear spaces is again a $\mathbf{N}$-cubic linear space.

Definition 3.4. Let $(\mathrm{W}, \hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}})$ and $(\mathrm{W}, \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}})$ be two interval-valued $\mathbf{N}$-fuzzy linear spaces. Then the union and intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces can be defined as

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N_1}} \cap \hat{\vartheta}_{\mathbf{N_2}}(w) &= \min\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W},\\ \hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) &= \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W}.\end{aligned}$$

Definition 3.5. Let $(\mathrm{W}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ and $(\mathrm{W}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ be two interval-valued $\mathcal{N}$-fuzzy linear spaces. Then the union and intersection of $\mathcal{N}$-fuzzy linear spaces can be defined as

$$\begin{aligned}\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w) &= \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W},\\ \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w) &= \max\{\lambda_{\mathbf{N_1}}(w).\lambda_{\mathbf{N_2}}(w)\}, \quad w \in \mathrm{W}.\end{aligned}$$

Proposition 3.1. *Let* $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their R-intersection* $(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *is again an* $\mathcal{N}$*-cubic linear space.*

Proof. Since $\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) = \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$. We have

$$\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) = \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\right\},$$

for $w_1, w_1 \in \mathrm{W}$ and $\sigma, \tau \in F$.

From Definition 3.3 we have

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) =& \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \hat{\vartheta}_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\right\},\\ \leq& \max\left\{\max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_1), \hat{\vartheta}_{\mathbf{N_1}}(w_2)\right\}, \max\left\{\hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}\right\},\\ =& \max\left\{\max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_1), \hat{\vartheta}_{\mathbf{N_2}}(w_1)\right\}, \max\left\{\hat{\vartheta}_{\mathbf{N_1}}(w_2), \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}\right\},\\ =& \max\left\{\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\},\end{aligned}$$

which imply

$$(\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}})(\sigma w_1 * \tau w_2) \leq \max\left\{\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_1), \hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w_2)\right\}. \tag{3.1}$$

Hence, $\bigcup\limits_{i\in\Lambda} \hat{\vartheta}_{\mathbf{N_i}}$ is an interval-valued **N**-fuzzy linear space. Since $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N_2}}(w) = \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$. We have

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) = \min\{\lambda_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\},$$

for $w_1, w_1 \in \mathrm{W}$ and $\sigma, \tau \in F$.

From Definition 3.3 we have

$$\begin{aligned}\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2) =& \min\{\lambda_{\mathbf{N_1}}(\sigma w_1 * \tau w_2), \lambda_{\mathbf{N_2}}(\sigma w_1 * \tau w_2)\},\\ \geq& \min\{\min\{\lambda_{\mathbf{N_1}}(w_1), \lambda_{\mathbf{N_1}}(w_2)\}, \max\{\lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_2}}(w_2)\}\},\\ =& \min\{\min\{\lambda_{\mathbf{N_1}}(w_1), \lambda_{\mathbf{N_2}}(w_1)\}, \min\{\lambda_{\mathbf{N_1}}(w_2), \lambda_{\mathbf{N_2}}(w_2)\}\},\\ =& \min\{\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_2)\},\end{aligned}$$

which imply

$$(\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}})(\sigma w_1 * \tau w_2) \geq \min\{\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_1), \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_2)\}. \tag{3.2}$$

Hence, $\bigcap\limits_{i\in\Lambda} \lambda_{\mathbf{N_i}}$ is an interval-valued **N**-fuzzy linear space.

Thus from (3.1) and (3.2) the conditions required for R- intersection to be a $\mathcal{N}$-cubic linear space are satisfied. ☐

Remark 3.2. By taking an example we prove that the intersection of two interval-valued **N**-fuzzy linear spaces do not satisfy the first condition of **N**-cubic linear space as in Definition 3.3.

Example 3.2. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Consider two interval-valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ in W as given in the Table 1. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W.

From the Definition 3.4

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) =& [-0.7, -0.6], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.5, -0.4],\\ \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) =& [-0.4, -0.2], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.6, -0.4].\end{aligned}$$

TABLE 1. Values of interval-valued N-fuzzy sets $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})=[-0.7,-0.5]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})=[-0.7,-0.6]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})=[-0.4,-0.1]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})=[-0.5,-0.4]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})=[-0.4,-0.2]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})=[-0.4,-0.1]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})=[-0.3,-0.1]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})=[-0.6,-0.4]$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued N-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}+w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.7,-0.6],[-0.5,-0.4]\} = [-0.5,-0.4],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.4,-0.2] \leq [-0.5,-0.4]$, which is non-sensical.

From the above example, it is clear that the intersection of two interval-valued **N**-fuzzy linear spaces need not be an interval-valued **N**-fuzzy linear space.

Remark 3.3. Similarly, by taking an example, we prove that the union of two **N**-fuzzy linear spaces does not satisfy the second condition of **N**-cubic linear space as in Definition 3.3.

Example 3.3. Let W = $M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Consider a N-fuzzy set $\lambda_{\mathbf{N}}$ in W as given in the Table 2. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are N-fuzzy linear spaces in W. From Definition 3.5 we have

TABLE 2. Values of N-fuzzy sets $\lambda_{\mathbf{N}}$

$\lambda_{\mathbf{N}_1}(w_{11})=-0.5$	$\lambda_{\mathbf{N}_2}(w_{11})=-0.2$
$\lambda_{\mathbf{N}_1}(w_{12})=-0.3$	$\lambda_{\mathbf{N}_2}(w_{12})=-0.85$
$\lambda_{\mathbf{N}_1}(w_{21})=-0.4$	$\lambda_{\mathbf{N}_2}(w_{21})=-0.7$
$\lambda_{\mathbf{N}_1}(w_{22})=-0.2$	$\lambda_{\mathbf{N}_2}(w_{22})=-0.6$

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.3,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.4, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.2.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are N-fuzzy sets in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}+w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.2,-0.3\} = -0.3,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.3$, which is non-sensical.

From the above example, it is clear that the intersection of two $\mathcal{N}$-fuzzy linear spaces need not be a $\mathcal{N}$-fuzzy linear space.

Lemma 3.1. *From the above theorem and examples following statements can be proved.*

(i) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their R-union* $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

(ii) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their P-union* $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

(iii) Let $\mathcal{N}_1^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^{\mathbf{C}} = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two* $\mathcal{N}$*-cubic linear spaces. Then their P-intersection* $(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be a* $\mathcal{N}$*-cubic linear space.*

Proof. (i) From Example 3.2 we can observe that intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ do not satisfy the first condition of $\mathcal{N}$-cubic linear space as in Definition 3.3 and from Example 3.3 union of two $\mathcal{N}$-fuzzy linear spaces $(\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2})$ do not satisfy the second condition of $\mathcal{N}$-cubic linear space as in Definition 3.3. Therefore, the R-union $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ is not a $\mathcal{N}$-cubic linear space.

(ii) Consider $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ as in Example 3.3. Now by Definition 3.5 $\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w) = \min\{\lambda_{\mathbf{N_1}}(w), \lambda_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$. Therefore,

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}) = -0.5, \quad \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12}) = -0.85,$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.7, \quad \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{22}) = -0.6.$$

We note that $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}$ is an $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\begin{aligned} \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) &\geq \min\left\{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\right\}, \\ \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) &\geq \min\left\{-0.5, -0.85\right\} = -0.85, \end{aligned}$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.7 \geq -0.85$. Certainly, $(\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2})$ satisfies the second condition of $\mathcal{N}$-cubic linear spaces. But from Example 3.2 $(\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2})$ is not an interval-valued **N**-fuzzy linear space. Therefore, P-union $(\mathcal{N}_1^{\mathbf{C}} \cup \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N_1^C}} \cap \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ is not a $\mathcal{N}$-cubic linear space.

(iii) Consider $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ as in Example 3.2. Now by Definition 3.4 $\hat{\vartheta}_{\mathbf{N_1}} \cup \hat{\vartheta}_{\mathbf{N_2}}(w) = \max\{\hat{\vartheta}_{\mathbf{N_1}}(w), \hat{\vartheta}_{\mathbf{N_2}}(w)\}$, $w \in \mathrm{W}$, we have

$$\begin{aligned} \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) &= [-0.7, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.4, -0.1], \\ \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) &= [-0.4, -0.1], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.3, -0.1]. \end{aligned}$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.7, -0.5], [-0.4, -0.1]\} = [-0.4, -0.1],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.4, -0.1] \leq [-0.4, -0.1]$. Certainly, $(\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2})$ satisfies the second condition of $\mathcal{N}$-cubic linear spaces. But from Example 3.3 $(\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2})$ is not a $\mathcal{N}$-fuzzy linear space. Therefore, P-intersection

$$(\mathcal{N}_1^{\mathbf{C}} \cap \mathcal{N}_2^{\mathbf{C}})_P = (\hat{\vartheta}_{\mathbf{N}_1^{\mathbf{C}}} \cup \hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N_1 C}} \cup \lambda_{\mathbf{N_2 C}})$$

need not be a $\mathcal{N}$-cubic linear space. □

4. Internal and External $\mathcal{N}$-Cubic Linear Spaces

In this section, we come out with the notion of internal and external $\mathcal{N}$-cubic linear spaces and confer some of their properties.

Definition 4.1. Suppose W be a linear space over a field F. A $\mathcal{N}$-cubic set $\mathbf{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ is said to be an internal $\mathcal{N}$-cubic linear space (shortly, INCLS) if

$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}}(\sigma w_1 * \tau w_2),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$.

Example 4.1. Let W $= M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\hat{\vartheta}_{\mathbf{N}}(w_{11}) = [-0.5, -0.3], \quad \hat{\vartheta}_{\mathbf{N}}(w_{12}) = [-0.4, -0.1],$$
$$\hat{\vartheta}_{\mathbf{N}}(w_{21}) = [-0.8, -0.7], \quad \hat{\vartheta}_{\mathbf{N}}(w_{22}) = [-0.6, -0.4].$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space.

Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\lambda_{\mathbf{N}}(w_{11}) = -0.7, \quad \lambda_{\mathbf{N}}(w_{12}) = -0.6,$$
$$\lambda_{\mathbf{N}}(w_{21}) = -0.85, \quad \lambda_{\mathbf{N}}(w_{22}) = -0.4.$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}}(w_{21}),$$

which imply $-0.85 \in [-0.8, -0.7]$. So, $\mathcal{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}}, \lambda_{\mathbf{N}^{\mathbf{C}}} \rangle$ is an INCLS.

Definition 4.2. Suppose W be a linear space over a field F. A $\mathcal{N}$-cubic set $\mathbf{N^C} = \langle \hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}} \rangle$ is said to be an external $\mathcal{N}$-cubic linear space (shortly, ENCLS) if

$$\lambda_{\mathbf{N^C}}(\sigma w_1 * \tau w_2) \notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(\sigma w_1 * \tau w_2), \hat{\vartheta}^+_{\mathbf{N^C}}(\sigma w_1 * \tau w_2)\right),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$.

Example 4.2. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Consider an interval-valued $\mathcal{N}$-fuzzy set $\hat{\vartheta}_{\mathbf{N}}$ in W as

$$\begin{aligned}\hat{\vartheta}_{\mathbf{N}}(w_{11}) =& [-0.5, -0.1], \quad \hat{\vartheta}_{\mathbf{N}}(w_{12}) = [-0.7, -0.4],\\ \hat{\vartheta}_{\mathbf{N}}(w_{21}) =& [-0.8, -0.6], \quad \hat{\vartheta}_{\mathbf{N}}(w_{22}) = [-0.6, -0.3].\end{aligned}$$

Here $\hat{\vartheta}_{\mathbf{N}}$ is an interval-valued $\mathcal{N}$-fuzzy linear space. Consider a $\mathcal{N}$-fuzzy set λ in W as

$$\lambda_{\mathbf{N}}(w) = \begin{cases} -0.9, & \text{when } w = w_{11},\\ -0.95, & \text{otherwise.}\end{cases}$$

For $\sigma = \tau = 1$ in Definition 4.2 we have

$$\begin{aligned}\lambda_{\mathbf{N^C}}(w_{11} + w_{12}) &\notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(w_{11} + w_{12}), \hat{\vartheta}^+_{\mathbf{N^C}}(w_{11} + w_{12})\right),\\ \lambda_{\mathbf{N^C}}(w_{21}) &\notin \left(\hat{\vartheta}^-_{\mathbf{N^C}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N^C}}(w_{21})\right),\end{aligned}$$

which imply $-0.95 \notin [-0.8, -0.6]$. So, $\mathcal{N}^{\mathbf{C}} = \langle \hat{\vartheta}_{\mathbf{N^C}}, \lambda_{\mathbf{N^C}} \rangle$ is an ENCLS.

Remark 4.1. In the following proposition, we present that the R-intersection of a family of INCLS's is again an INCLS (resp. ENCLS).

Proposition 4.1. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1^C}}, \lambda_{\mathbf{N_1^C}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_2^C}})$ *be two INCLS. Then their R-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)^I_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1^C}} \cap \lambda_{\mathbf{N_2^C}})$ *is an INCLS.*

Proof. Considering the fact that $\mathcal{N}_1^I$ and $\mathcal{N}_2^I$ are INCLS in W, we have

$$\begin{aligned}\hat{\vartheta}^-_{\mathbf{N_1^C}}(\sigma w_1 * \tau w_2) \le \lambda_{\mathbf{N_1^C}}(\sigma w_1 * \tau w_2) \le \hat{\vartheta}^+_{\mathbf{N_1^C}}(\sigma w_1 * \tau w_2),\\ \hat{\vartheta}^-_{\mathbf{N_2^C}}(\sigma w_1 * \tau w_2) \le \lambda_{\mathbf{N_2^C}}(\sigma w_1 * \tau w_2) \le \hat{\vartheta}^+_{\mathbf{N_2^C}}(\sigma w_1 * \tau w_2),\end{aligned}$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$. Now since the union of interval-valued fuzzy linear spaces is again an interval-valued $\mathcal{N}$-fuzzy linear space and intersection of $\mathcal{N}$-fuzzy linear space is again a fuzzy linear space. We have

$$\begin{aligned}(\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}})^-(\sigma w_1 * \tau w_2) \le& (\lambda_{\mathbf{N_1^C}} \cap \lambda_{\mathbf{N_2^C}})(\sigma w_1 * \tau w_2)\\ \le& (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}})^+(\sigma w_1 * \tau w_2).\end{aligned}$$

Therefore, $(\mathcal{N}_1 \cap \mathcal{N}_2)^I_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1^C}} \cap \lambda_{\mathbf{N_2^C}})$ is an INCLS. □

Proposition 4.2. *Let* $\mathcal{N}_1^E = (\hat{\vartheta}_{\mathbf{N_1^C}}, \lambda_{\mathbf{N_1^C}})$ *and* $\mathcal{N}_2^E = (\hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_2^C}})$ *be two ENCLS. Then their R-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)^E_R = (\hat{\vartheta}_{\mathbf{N_1^C}} \cup \hat{\vartheta}_{\mathbf{N_2^C}}, \lambda_{\mathbf{N_1^C}} \cap \lambda_{\mathbf{N_2^C}})$ *is again an ENCLS.*

Proof. Considering the fact that $\mathcal{N}_1^I$ and $\mathcal{N}_2^I$ are ENCLS in W, we have

$$\lambda_{\mathbf{N_1}\mathbf{c}}(\sigma w_1 * \tau w_2) \notin \left((\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}})^+(\sigma w_1 * \tau w_2)\right),$$
$$\lambda_{\mathbf{N_2}\mathbf{c}}(\sigma w_1 * \tau w_2) \notin \left((\hat{\vartheta}_{\mathbf{N_2}\mathbf{c}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_2}\mathbf{c}})^+(\sigma w_1 * \tau w_2)\right),$$

for all $w_1, w_2 \in$ W and $\sigma, \tau \in F$. Now since the union of interval-valued fuzzy linear spaces is again an interval-valued $\mathcal{N}$-fuzzy linear space and intersection of $\mathcal{N}$-fuzzy linear space is again a fuzzy linear space. We have

$$\begin{aligned}&(\lambda_{\mathbf{N_1}\mathbf{c}} \cap \lambda_{\mathbf{N_2}\mathbf{c}})(\sigma w_1 * \tau w_2)\\ &\notin \left((\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}} \cup \hat{\vartheta}_{\mathbf{N_2}\mathbf{c}})^-(\sigma w_1 * \tau w_2), (\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}} \cup \hat{\vartheta}_{\mathbf{N_2}\mathbf{c}})^+(\sigma w_1 * \tau w_2)\right).\end{aligned}$$

Therefore, $(\mathcal{N}_1 \cap \mathcal{N}_2)_R^I = (\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}} \cup \hat{\vartheta}_{\mathbf{N_2}\mathbf{c}}, \lambda_{\mathbf{N_1}\mathbf{c}} \cap \lambda_{\mathbf{N_2}\mathbf{c}})$ is an ENCLS. □

Remark 4.2. By taking an example, we disprove the statement that the P-intersection of two interior $\mathcal{N}$-cubic linear spaces is again an interior $\mathcal{N}$-cubic linear space.

Proposition 4.3. *Let $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}}, \lambda_{\mathbf{N_1}\mathbf{c}})$ and $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}\mathbf{c}}, \lambda_{\mathbf{N_2}\mathbf{c}})$ be two INCLS. Then their P-intersection $(\mathcal{N}_1 \cap \mathcal{N}_2)_P^I = (\hat{\vartheta}_{\mathbf{N_1}\mathbf{c}} \cup \hat{\vartheta}_{\mathbf{N_2}\mathbf{c}}, \lambda_{\mathbf{N_1}\mathbf{c}} \cup \lambda_{\mathbf{N_2}\mathbf{c}})$ need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.3. Let W $= M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 3. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can

Table 3. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=$[-0.8, -0.5]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=$[-0.9, -0.8]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=$[-1, -0.9]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=$[-0.6, -0.3]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=$[-0.85, -0.7]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=$[-1, -0.93]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=$[-0.9, -0.78]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=$[-0.7, -0.4]$

check by simple calculation using Definition 3.2. From the Definition 3.4

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.8, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.6, -0.3],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.85, -0.7], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.7, -0.4].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.8, -0.5], [-0.6, -0.3]\} = [-0.6, -0.3],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.85, -0.7] \leq [-0.6, -0.3]$.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 4. We note that

Table 4. Value of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=$-0.42$	$\lambda_{\mathbf{N}_2}(w_{11})$=$-0.9$
$\lambda_{\mathbf{N}_1}(w_{12})$=$-0.3$	$\lambda_{\mathbf{N}_2}(w_{12})$=$-0.2$
$\lambda_{\mathbf{N}_1}(w_{21})$=$-0.8$	$\lambda_{\mathbf{N}_2}(w_{21})$=$-0.98$
$\lambda_{\mathbf{N}_1}(w_{22})$=$-0.6$	$\lambda_{\mathbf{N}_2}(w_{22})$=$-0.1$

$\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W.

From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.42, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.2,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.8, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.1.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W.

Since ${\mathcal{N}_1}^I$ and ${\mathcal{N}_2}^I$ are INCLS the example that we have taken will satisfy the condition mentioned in Definition 4.1. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_1^\mathbf{C}}(w_{21}) \leq \lambda_{\mathbf{N}_1^\mathbf{C}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^\mathbf{C}}(w_{21}),$$

which imply $0.8 \in [-0.85, -0.7]$.

Similarly, for ${\mathcal{N}_2}^I = (\hat{\vartheta}_{\mathbf{N}_2^\mathbf{C}}, \lambda_{\mathbf{N}_2^\mathbf{C}})$, when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_2^\mathbf{C}}(w_{21}) \leq \lambda_{\mathbf{N}_2^\mathbf{C}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^\mathbf{C}}(w_{21}),$$

which imply $0.98 \in [-1, -0.93]$.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\left\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\right\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.42, -0.2\} = -0.42,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.8 \geq -0.42$, which is non-sensical.

Therefore, the P-intersection of two INCLS need not be an INCLS. □

Remark 4.3. By taking an example, we disprove the statement that the P-intersection of two exterior $\mathcal{N}$-cubic linear spaces is again an exterior $\mathcal{N}$-cubic linear space.

Proposition 4.4. *Let* $\mathcal{N}_1^E = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}})$ *and* $\mathcal{N}_2^E = (\hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_2}^\mathbf{C}})$ *be two ENCLS. Then their* P*-intersection* $(\mathcal{N}_1 \cap \mathcal{N}_2)_P^E = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cup \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cap \lambda_{\mathbf{N_2}^\mathbf{C}})$ *need not be an ENCLS.*

Proof. The proof of the above statement follows by the example.

Example 4.4. The proof of the above statement follows by the example. Let W = $M_{2\times 2}$(R) over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued N-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 5. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued N-fuzzy linear spaces in W and that we can

TABLE 5. Values of two interval valued N-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=[−0.55, −0.3]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=[−0.75, −0.5]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=[−0.9, −0.8]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=[−0.4, −0.05]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=[−0.6, −0.4]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=[−0.5, −0.2]
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=[−0.7, −0.5]	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=[−0.38, −0.08]

check by simple calculation using Definition 3.2. From Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) =[-0.55, -0.3], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.4, -0.05],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) =[-0.5, -0.2], \quad \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.38, -0.08].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued N-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.55, -0.3], [-0.4, -0.05]\} = [-0.4, -0.05],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cup \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.5, -0.2] \leq [-0.4, -0.05]$.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in Table 6.

TABLE 6. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=−0.2	$\lambda_{\mathbf{N}_2}(w_{11})$=−0.84
$\lambda_{\mathbf{N}_1}(w_{12})$=−0.32	$\lambda_{\mathbf{N}_2}(w_{12})$=−0.4
$\lambda_{\mathbf{N}_1}(w_{21})$=−0.7	$\lambda_{\mathbf{N}_2}(w_{21})$=−0.9
$\lambda_{\mathbf{N}_1}(w_{22})$=−0.25	$\lambda_{\mathbf{N}_2}(w_{22})$=−0.1

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are N-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.32,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.7, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.1.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. Since ${\mathcal{N}_1}^I$ and ${\mathcal{N}_2}^I$ are ENCLS the example that we have taken will satisfy the condition mentioned in Definition 4.2. For $\sigma=\tau=1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12}), \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12})\right),$$
$$\lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{21})\right),$$

which imply $-0.7 \notin [-0.6, -0.4]$.

Similarly, for ${\mathcal{N}_2}^I = (\hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N}_2^{\mathbf{C}}})$, when $\sigma=\tau=1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12}), \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12})\right),$$
$$\lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{21})\right),$$

which imply $-0.95 \notin [-0.5, -0.2]$. For $\sigma=\tau=1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}+w_{12}) \geq \min\left\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\right\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\left\{-0.2, -0.32\right\} = -0.32,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.7 \geq -0.42$, which is non-sensical.

Therefore, the P-intersection of two ENCLS need not be an ENCLS. □

Remark 4.4. By taking an example, we disprove the statement that the P-union of two interior $\mathcal{N}$-cubic linear spaces is again an interior $\mathcal{N}$-cubic linear space.

Proposition 4.5. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two INCLS. Then their P-union* $(\mathcal{N}_1 \cup \mathcal{N}_2)^I_P = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cap \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.5. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 7. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces

TABLE 7. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=$[-0.3, -0.1]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=$[-0.95, -0.85]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=$[-0.5, -0.45]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=$[-1, -0.93]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=$[-0.4, -0.2]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=$[-0.63, -0.5]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=$[-1, -0.9]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=$[-0.8, -0.7]$

in W and that we can check by simple calculation using Definition 3.2. From the

Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.3, -0.1], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-1, -0.93],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.63, -0.5], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-1, -0.9].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.95, -0.85], [-1, -0.93]\} = [-0.95, -0.85],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.63, -0.5] \leq [-0.95, -0.85]$, which is non-sensical. Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 8. We note that

TABLE 8. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11}) = -0.65$	$\lambda_{\mathbf{N}_2}(w_{11}) = -0.8$
$\lambda_{\mathbf{N}_1}(w_{12}) = -0.12$	$\lambda_{\mathbf{N}_2}(w_{12}) = -0.25$
$\lambda_{\mathbf{N}_1}(w_{21}) = -0.42$	$\lambda_{\mathbf{N}_2}(w_{21}) = -0.68$
$\lambda_{\mathbf{N}_1}(w_{22}) = -0.3$	$\lambda_{\mathbf{N}_2}(w_{22}) = -0.9$

$\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{11}) = -0.8, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{12}) = -0.25,$$
$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{21}) = -0.68, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{22}) = -0.9.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.1

$$\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}),$$

which imply $0.42 \in [-0.4, -0.2]$.

Similarly, for $\mathcal{N}_2{}^I = (\hat{\vartheta}_{\mathbf{N}_2^{\mathbf{C}}}, \lambda_{\mathbf{N}_2^{\mathbf{C}}})$ when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}),$$

which imply $0.68 \in [-0.63, -0.5]$.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.8, -0.25\} = -0.8,$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.68 \geq -0.8$. Even though the intersection of two $\mathcal{N}$-fuzzy linear spaces satisfies the first condition of Definition 3.3 the intersection of interval-valued $\mathcal{N}$-fuzzy linear spaces failed to satisfy the second condition of Definition 3.3.

Therefore, the P-union of two INCLS need not be an INCLS. □

Remark 4.5. In the latter example, we show that the P-union of two exterior $\mathcal{N}$-cubic linear spaces need not be an exterior $\mathcal{N}$-cubic linear space.

Proposition 4.6. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two ENCLS. Then their* P*-union* $(\mathcal{N}_1 \cup \mathcal{N}_2)_P^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cap \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cap \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be an ENCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.6. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1.

Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 9.

TABLE 9. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})=[-1, -0.7]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})=[-0.9, -0.5]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})=[-0.75, -0.2]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})=[-1, -0.6]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})=[-0.95, -0.55]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})=[-0.85, -0.4]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})=[-0.6, -0.1]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})=[-0.8, -0.4]$

Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can check by simple calculation using Definition 3.2. From the Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-1, -0.7], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-1, -0.6],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.95, -0.55], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-0.8, -0.4].$$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W.

For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.1, -0.7], [-1, -0.6]\} = [-1, -0.6],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.95, -0.55] \leq [-1, -0.6]$, which is non-sensical.

Now define two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 10. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{11}) = -0.8, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{12}) = -0.75,$$
$$\lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{21}) = -0.4, \quad \lambda_{\mathbf{N_1}} \cap \lambda_{\mathbf{N_2}}(w_{22}) = -0.7.$$

TABLE 10. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=$-0.8$	$\lambda_{\mathbf{N}_2}(w_{11})$=$-0.25$
$\lambda_{\mathbf{N}_1}(w_{12})$=$-0.6$	$\lambda_{\mathbf{N}_2}(w_{12})$=$-0.75$
$\lambda_{\mathbf{N}_1}(w_{21})$=$-0.4$	$\lambda_{\mathbf{N}_2}(w_{21})$=$-0.35$
$\lambda_{\mathbf{N}_1}(w_{22})$=$-0.3$	$\lambda_{\mathbf{N}_2}(w_{22})$=$-0.7$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.2 we have

$$\lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12}), \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{11}+w_{12})\right),$$
$$\lambda_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_1^{\mathbf{C}}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_1^{\mathbf{C}}}(w_{21})\right),$$

which imply $-0.4 \notin [-0.95, -0.55]$. Also,

$$\lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12}), \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{11}+w_{12})\right),$$
$$\lambda_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}) \notin \left(\hat{\vartheta}^-_{\mathbf{N}_2^{\mathbf{C}}}(w_{21}), \hat{\vartheta}^+_{\mathbf{N}_2^{\mathbf{C}}}(w_{21})\right),$$

which imply $-0.35 \notin [-0.85, -0.4]$. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}+w_{12}) \geq \min\left\{\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{12})\right\},$$
$$\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.8, -0.75\} = -0.8,$$

which imply $\lambda_{\mathbf{N}_1} \cap \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.8$.

Therefore, the P-union of two ENCLS need not be an ENCLS. □

Remark 4.6. In the latter example, we show that the R-union of two interior $\mathcal{N}$-cubic linear spaces need not be an interior $\mathcal{N}$-cubic linear space.

Proposition 4.7. *Let* $\mathcal{N}_1^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}})$ *and* $\mathcal{N}_2^I = (\hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *be two INCLS. Then their R-union* $(\mathcal{N}_1 \cap \mathcal{N}_2)_R^I = (\hat{\vartheta}_{\mathbf{N_1}^{\mathbf{C}}} \cap \hat{\vartheta}_{\mathbf{N_2}^{\mathbf{C}}}, \lambda_{\mathbf{N_1}^{\mathbf{C}}} \cup \lambda_{\mathbf{N_2}^{\mathbf{C}}})$ *need not be an INCLS.*

Proof. The statement can be proved by giving an example below.

Example 4.7. Let $\mathrm{W} = M_{2\times 2}(\mathrm{R})$ over the field $GF(2)$ with the binary operation "+" as in the Example 3.1. Define two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$ in W as given in the Table 11. Here $\hat{\vartheta}_{\mathbf{N}_1}$ and $\hat{\vartheta}_{\mathbf{N}_2}$ are interval-valued $\mathcal{N}$-fuzzy linear spaces in W and that we can check by simple calculation using Definition 3.2. From the Definition 3.4 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}) = [-0.65, -0.45], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12}) = [-0.9, -0.8],$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.6, -0.4], \quad \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{22}) = [-1, -0.95].$$

TABLE 11. Values of two interval valued $\mathcal{N}$-fuzzy sets $\hat{\vartheta}_{\mathbf{N_1}}$ and $\hat{\vartheta}_{\mathbf{N_2}}$

$\hat{\vartheta}_{\mathbf{N}_1}(w_{11})$=$[-0.65, -0.5]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{11})$=$[-0.55, -0.25]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{12})$=$[-0.8, -0.7]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{12})$=$[-0.9, -0.8]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{21})$=$[-0.5, -0.3]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{21})$=$[-0.6, -0.4]$
$\hat{\vartheta}_{\mathbf{N}_1}(w_{22})$=$[-1, -0.95]$	$\hat{\vartheta}_{\mathbf{N}_2}(w_{22})$=$[-1, -0.85]$

TABLE 12. Values of two fuzzy sets $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$

$\lambda_{\mathbf{N}_1}(w_{11})$=$-0.2$	$\lambda_{\mathbf{N}_2}(w_{11})$=$-0.35$
$\lambda_{\mathbf{N}_1}(w_{12})$=$-0.7$	$\lambda_{\mathbf{N}_2}(w_{12})$=$-0.3$
$\lambda_{\mathbf{N}_1}(w_{21})$=$-0.4$	$\lambda_{\mathbf{N}_2}(w_{21})$=$-0.5$
$\lambda_{\mathbf{N}_1}(w_{22})$=$-0.8$	$\lambda_{\mathbf{N}_2}(w_{22})$=$-0.45$

We note that $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}$ is an interval-valued $\mathcal{N}$-fuzzy set in W. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11} + w_{12}) \leq \max\left\{\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{11}), \hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{12})\right\},$$
$$\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) \leq \max\{[-0.65, -0.5], [-0.9, -0.8]\} = [-0.65, -0.5],$$

which imply $\hat{\vartheta}_{\mathbf{N}_1} \cap \hat{\vartheta}_{\mathbf{N}_2}(w_{21}) = [-0.6, -0.4] \leq [-0.65, -0.5]$, which is non-sensical.

Now define two fuzzy sets $\lambda_{\mathbf{N_1}}$ and $\lambda_{\mathbf{N}_2}$ in W as given in the Table 12. We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy linear spaces in W. From Definition 3.5 we have

$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{11}) = -0.2, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{12}) = -0.3,$$
$$\lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{21}) = -0.4, \quad \lambda_{\mathbf{N_1}} \cup \lambda_{\mathbf{N_2}}(w_{22}) = -0.45.$$

We note that $\lambda_{\mathbf{N}_1}$ and $\lambda_{\mathbf{N}_2}$ are $\mathcal{N}$-fuzzy sets in W. For $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}_1}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}_1}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}_1}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}_1}(w_{21}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}_1}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}_1}(w_{21}),$$

which imply $-0.4 \in [-0.5, -0.3]$.

Similarly, for $\mathcal{N}_2{}^I$= $(\hat{\vartheta}_{\mathbf{N}^{\mathbf{C}}_2}, \lambda_{\mathbf{N}^{\mathbf{C}}_2})$, when $\sigma = \tau = 1$ in Definition 4.1 we have

$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}_2}(w_{11} + w_{12}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}_2}(w_{11} + w_{12}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}_2}(w_{11} + w_{12}),$$
$$\hat{\vartheta}^-_{\mathbf{N}^{\mathbf{C}}_2}(w_{21}) \leq \lambda_{\mathbf{N}^{\mathbf{C}}_2}(w_{21}) \leq \hat{\vartheta}^+_{\mathbf{N}^{\mathbf{C}}_2}(w_{21}),$$

which imply $0.5 \in [-0.6, -0.4]$. For $\sigma = \tau = 1$ in Definition 3.3 we have

$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11} + w_{12}) \geq \min\{\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{11}), \lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{12})\},$$
$$\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) \geq \min\{-0.2, -0.3\} = -0.3,$$

which imply $\lambda_{\mathbf{N}_1} \cup \lambda_{\mathbf{N}_2}(w_{21}) = -0.4 \geq -0.3$, which is non-sensical.

Therefore, the R-union of two INCLS need not be an INCLS. □

Remark 4.7. Finally, we show that the R-union of two exterior $\mathcal{N}$-cubic linear spaces need not be an exterior $\mathcal{N}$-cubic linear space. From Example 4.5 we can observe that intersection of two interval-valued $\mathcal{N}$-fuzzy linear spaces do not satisfy the first condition of $\mathcal{N}$-cubic linear spaces and in Example 4.7 we can observe that union of two $\mathcal{N}$-fuzzy linear spaces do not satisfy the second condition of $\mathcal{N}$-cubic linear spaces. Hence,

$$(\mathcal{N}_1 \cap \mathcal{N}_2)^I_R = (\hat{\vartheta}_{\mathbf{N_1}^\mathbf{C}} \cap \hat{\vartheta}_{\mathbf{N_2}^\mathbf{C}}, \lambda_{\mathbf{N_1}^\mathbf{C}} \cup \lambda_{\mathbf{N_2}^\mathbf{C}})$$

need not be an ENCLS.

5. Conclusion

We find huge literature for dealing the uncertain problems like fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, cubic sets, $\mathcal{N}$-fuzzy sets, and $\mathcal{N}$-cubic sets. But in all most all cases we see these sets are properly used for applications in algebra and topology. In order, to extend this idea to linear spaces, we in this paper have introduced the notion of $\mathcal{N}$-cubic linear spaces which also handles the negative features of certain things like side effects of certain medicine. The main rationale of this paper is to extend the idea of $\mathcal{N}$-cubic sets to $\mathcal{N}$-cubic linear spaces and discuss in detail two types of $\mathcal{N}$-cubic linear spaces called ENCLS and INCLS with examples. We also discuss the basic operations like P-union (resp. intersection) and R-union (resp. intersection) intersection of $\mathcal{N}$-cubic linear spaces, ENCLS and INCLS. Sooner, different aggregation operators can be dealt with $\mathcal{N}$-cubic linear spaces. We will define Pythagorean fuzzy linear spaces by using the idea presented in this paper and [16].

Acknowledgements. This work is financially supported by Council of Scientific and Industrial Research (CSIR)

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