Kragujevac Journal of Mathematics
Volume 46(4) (2022), Pages 617–633.

ON FUZZY PRIMARY AND FUZZY QUASI-PRIMARY IDEALS IN LA-SEMIGROUPS

PAIROTE YIARAYONG[1]

Abstract. The purpose of this paper is to introduce the notion of a weakly fuzzy quasi-primary ideals in LA-semigroups, we study fuzzy primary, fuzzy quasi-primary, fuzzy completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Some characterizations of weakly fuzzy primary and weakly fuzzy quasi-primary ideals are obtained. Moreover, we investigate relationships between fuzzy completely primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Finally we show that a fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

1. Introduction

The concept of a fuzzy subset of a set was first considered by Zadeh [27] in 1965. In 1988, Zhang [28] studied prime L-fuzzy ideals and primary L-fuzzy ideals in rings where L is a completely distributive lattice. In 2012, Palanivelrajan and Nandakumar [18] introduced the definition and some operations of intuitionistic fuzzy primary and semiprimary ideals.

In 2010, Khan et al. [10] gave the concept of (α, β)-fuzzy interior ideals of AG-groupoids and gave some properties of AG -groupoids in terms of (α, β)-fuzzy interior ideals. In 2012, Khan et al. [14] introduced the concept of $(\in, \in \vee q_k)$-fuzzy bi-ideals, $(\in, \in \vee q_k)$-fuzzy left (right)-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals in AG-groupoids and characterized regular and intera-regular AG -groupoids in terms of the lower parts of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals in AG -groupoids. In 2013, Yaqoob [21] applied the interval valued intuitionistic fuzzy sets

Key words and phrases. LA-semigroup, fuzzy primary ideal, weakly fuzzy quasi-primary ideal, fuzzy completely primary ideal, fuzzy quasi-primary ideal.

2010 *Mathematics Subject Classification.* Primary: 03E72. Secondary: 94D05, 18B40.

DOI 10.46793/KgJMat2204.617Y

Received: February 29, 2020.

Accepted: February 08, 2020.

in regular LA-semigroups and characterized regular LA-semigroups by the properties of interval valued intuitionistic fuzzy left ideals, interval-valued intuitionistic fuzzy right ideal, interval valued intuitionistic fuzzy generalized bi-ideals and interval valued intuitionistic fuzzy bi-ideals. In 2014, Abdullah, Aslam and Amin [1] defined the concept of interval-valued (α, β)-fuzzy ideals, interval-valued and (α, β)-fuzzy generalized bi-ideals in LA-semigroups and characterized the lower part of interval-valued $(\in, \in \vee q)$-fuzzy left ideals, interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideals and interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideals in LA-semigroups. In 2015, Khan, Jun and Yousafzai [11] studied fuzzy left (right, two-sided) ideals, fuzzy (generalized) bi-ideals, fuzzy interior ideals, fuzzy $(1, 2)$-ideals and fuzzy quasi-ideals of right regular LA-semigroups and gave some properties of right regular LA-semigroups in terms of fuzzy left and fuzzy right ideals. In 2016, Yousafzai, Yaqoob and Zeb [26] introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, bi-) ideals of ordered AG-groupoids and provided the basic theory for an intra-regular ordered AG-groupoid in terms of generalized fuzzy ideals. In 2018, Rehman et al. [19] studied lower and upper parts of $(\in, \in \vee q)$- fuzzy interior ideals and $(\in, \in \vee q)$-fuzzy bi-ideals in LA-semigroups. In 2019, Nasreen [17] characterized regular (intra-regular, both regular and intra-regular) ordered AG-groupoid in terms of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$. There are many mathematicians who added several results to the theory fuzzy LA-semigroups, see [5, 9, 20, 22, 24, 25].

In this study we followed lines as adopted in [2–4, 6–8, 13, 16, 23] and established the notion of fuzzy subsets of LA-semigroups. Specifically we characterize the fuzzy primary, fuzzy quasi-primary, fuzzy completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups. Moreover, we investigate relationships between fuzzy completely primary and weakly fuzzy quasi-primary ideals in LA-semigroups.

2. Preliminaries

In this section we refer to [12] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Recall that a function f from S to the unit interval $[0, 1]$ is a fuzzy subset of S.

Definition 2.1 ([12]). A fuzzy subset f of an LA-semigroup S is called a fuzzy LA-subsemigroup of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all x, y in S.

Recall that the LA-semigroup S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$, denoted also by S. Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined $f(x) \leq g(x)$ for all $x \in S$. $f \cap g$ and $f \cup g$ are fuzzy subsets of S defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$, $(f \cup g)(x) = \max\{f(x), g(x)\}$ for all $x \in S$. More generally, if $\{f_\alpha : \alpha \in \beta\}$ is a family of fuzzy subsets of S, then

$\bigcap_{\alpha\in\beta} f_\alpha$ and $\bigcup_{\alpha\in\beta} f_\alpha$ are defined as follows:

$$\left(\bigcap_{\alpha\in\beta} f_\alpha\right)(x) = \bigcap_{\alpha\in\beta} f_\alpha(x) = \inf\left\{f_\alpha(x) : \alpha\in\beta\right\},$$
$$\left(\bigcup_{\alpha\in\beta} f_\alpha\right)(x) = \bigcup_{\alpha\in\beta} f_\alpha(x) = \sup\left\{f_\alpha(x) : \alpha\in\beta\right\},$$

and will be the intersection and union of the family $\{f_\alpha : \alpha\in\beta\}$ of fuzzy subset of R. The product $f\circ g$ [12] is defined as follows:

$$(f\circ g)(x) = \begin{cases} \bigcup\limits_{x=yz} \min\left\{f(y), g(z)\right\}, & \text{exists } y,z\in S, \text{ such that } x = yz, \\ 0, & \text{otherwise.} \end{cases}$$

As is well known [12], this operation "$\circ$" is left invertive.

Lemma 2.1 ([12])**.** *If f, g and h are fuzzy subsets of an LA-semigroup S, then $(f\circ g)\circ h = (h\circ g)\circ f$.*

Proof. The proof is available in [12]. □

Lemma 2.2 ([12])**.** *Let f, g, h and k be any fuzzy subsets of an LA-semigroup S with left identity. Then the following properties hold:*

(a) $f\circ(g\circ h) = g\circ(f\circ h)$;
(b) $(f\circ g)\circ(h\circ k) = (k\circ h)\circ(g\circ f)$;
(c) $S\circ S = S$.

Proof. The proof is available in [12]. □

Recall that a fuzzy subset f of an LA-semigroup S is called a fuzzy left (right) ideal of S if $f(xy)\geq f(y)$ $(f(xy)\geq f(x))$ for all $x,y\in S$, if f is both fuzzy left and right ideal of S, then f is called a fuzzy ideal of S.

Remark 2.1. It is easy that f is a fuzzy ideal of an LA-semigroup S if and only if $f(xy)\geq \max\{f(x), f(y)\}$ for all x,y in S and any fuzzy left (right) ideal of S is a fuzzy LA-subsemigroup of S.

Lemma 2.3 ([12])**.** *Let f be a fuzzy subset of an LA-semigroup S. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S if and only if $f\circ f\subseteq f$.*
(b) *f is a fuzzy left ideal of S if and only if $S\circ f\subseteq f$.*
(c) *f is a fuzzy right ideal of S if and only if $f\circ S\subseteq f$.*
(d) *f is a fuzzy ideal of S if and only if $S\circ f\subseteq f$ and $f\circ S\subseteq f$.*

Proof. The proof is available in [12]. □

Theorem 2.1 ([12]). *Let I be a nonempty subset of an LA-semigroup S and let $f_I : S \to [0,1]$ be a fuzzy subset of S such that*

$$(f_I)(x) = \begin{cases} 1, & x \in I, \\ 0, & \textit{otherwise.} \end{cases}$$

Then the following properties hold.

(a) *I is an LA-subsemigroup of S if and only if f_I is a fuzzy LA-subsemigroup of S.*
(b) *I is a left ideal of S if and only if f_I is a fuzzy left ideal of S.*
(c) *I is a right ideal of S if and only if f_I is a fuzzy right ideal of S.*
(d) *I is an ideal of S if and only if f_I is a fuzzy ideal of S.*

Proof. The proof is available in [12]. □

3. Fuzzy Completely Primary Subsets of LA-Semigroups

In this section, we concentrate our study on the fuzzy completely primary subsets and fuzzy primary ideals of LA-semigroups and investigate their fundamental properties and mutual relationships. Finally, we prove that a fuzzy subset f of an AG-3-band is a fuzzy quasi-primary ideal of S if and only if f is a fuzzy primary ideal in S.

Definition 3.1. Let x and y be any elements of an LA-semigroup S. A fuzzy subset f of S is called fuzzy completely primary if $\max\{f(x), f(y^n)\} \geq f(xy)$ for some positive integer n.

We now present the following example satisfying above definition.

Example 3.1. Let $S = \{0, 1, 2\}$ be a set under the binary operation defined as in Table 1. Then S is an LA-semigroup. We define a fuzzy subset $f : S \to [0,1]$ by $f(x) = 0$

Table 1. LA-semigroup

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

for all $x \in S$. It is easy to show that f is a fuzzy completely primary subset of S.

In the light of the definition of fuzzy completely primary subsets on an LA-semigroup, we can obtain the following properties.

Theorem 3.1. *If P_i is a fuzzy completely primary subset of an LA-semigroup S, then $\bigcup_{i \in I} P_i$ is a fuzzy completely primary subset of S.*

Proof. Let x and y be any elements of an LA-semigroup S. Since P_i is a fuzzy completely primary subset of S, we have $P_i(xy) \leq \max\{P_i(x), P_i(y^n)\}$ for some positive integer n. Hence, since $\max\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\} \geq P_i(xy)$ for all $i \in I$

$$\max\left\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\right\} \geq \bigcup_{i\in I} P_i(xy).$$

Therefore, $\bigcup_{i\in I} P_i$ is a fuzzy completely primary subset of S. □

Recall that a left ideal P of an LA-semigroup S is called completely quasi-primary if for any two elements x, y of $S, xy \in P$ implies that either $x \in P$ or $y^n \subseteq P$ for some positive integer n.

Theorem 3.2. *Let S be an LA-semigroup S. Then the following properties hold.*

(a) *If P is a completely quasi-primary ideal of S, then f_P is a fuzzy completely primary left ideal of S.*

(b) *If P is a completely quasi-primary ideal of S, then tf_P is a fuzzy completely primary left of S.*

Proof. (a). Let P is a completely quasi-primary ideal of S. It follows from Theorem 2.1 (2) that f_P is a fuzzy left ideal of S. Let x and y be any elements of S. If $xy \notin P$, then $f_P(xy) = 0 \leq \max\{f_P(x), f_P(y^n)\}$ for some positive integer n. Next, if $xy \in P$, then $f_P(xy) = 1$. Since P is a completely quasi-primary ideal of S, we have $x \in P$ or $y^n \in P$ for some positive integer n and so we have $f_P(x) = 1$ or $f_P(y) = 1$. Therefore, $f_P(xy) = 1 = \max\{f_P(x), f_P(y)\}$ and hence f_P is a fuzzy completely primary left ideal of S.

(b) The proof is similar to (a). □

Next, define the notions of fuzzy quasi-primary ideals on an LA-semigroup S.

Definition 3.2. A fuzzy left ideal f of an LA-semigroup S is called fuzzy quasi-primary if for any two fuzzy left ideals g and h of S such that $g \circ h \subseteq f$ implies $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n.

Example 3.2. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be a set under the binary operation defined in Table 2. It is clear that S is an LA-semigroup. We define a fuzzy subset $f : S \to [0, 1]$ by

$$f(x) = \begin{cases} 1, & x \in \{1, 2, 4, 5, 6, 7\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is a fuzzy completely primary subset of S. But f is not a fuzzy quasi-primary ideal of S, since f is not a fuzzy left ideal of S.

Next, define the notions of fuzzy primary ideal on an LA-semigroup S.

Definition 3.3. A fuzzy ideal f of an LA-semigroup S is called fuzzy primary of S if for any two fuzzy ideals g and h of S such that $g \circ h \subseteq f$ implies $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n.

Table 2. LA-semigroup

·	0	1	2	3	4	5	6	7	8
0	3	1	6	3	1	6	6	1	3
1	0	3	0	3	8	8	3	0	8
2	8	1	5	3	7	2	6	4	0
3	3	3	3	3	3	3	3	3	3
4	0	6	7	3	5	4	1	2	8
5	8	6	4	3	2	7	1	5	0
6	8	3	8	3	0	0	3	8	0
7	0	1	2	3	4	5	6	7	8
8	3	6	1	3	6	1	1	6	3

Remark 3.1. It is easy to see that every fuzzy primary ideal is fuzzy quasi-primary.

Recall that an LA-semigroup in which $(xx)\,x = x\,(xx) = x$ holds for all x is called an AG-3-band. It is easy to see that every fuzzy left ideal of an AG-3-band S is a fuzzy ideal.

Then we have the following result.

Theorem 3.3. *For an AG-3-band S, the following conditions are equivalent.*

(a) *f is a fuzzy quasi-primary ideal of S.*
(b) *f is a fuzzy primary ideal of S.*

Proof. It is obvious. □

4. Weakly Fuzzy Quasi-Primary Ideals of LA-Semigroups

In this section, we investigate some properties of weakly fuzzy primary and weakly fuzzy quasi-primary ideals in LA-semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 4.1. *Let A and B be any nonempty subsets of an LA-semigroup S and $t \in (0,1]$. Then the following properties hold:*

(a) $tf_A \circ tf_B = tf_{AB}$;
(b) $tf_A \cap tf_B = tf_{A\cap B}$;
(c) $tf_A \cup tf_B = tf_{A\cup B}$;
(d) $S \circ tf_A = tf_{SA}, tf_A \circ S = tf_{AS}$ *and* $S \circ (tf_A \circ S) = tf_{S(AS)}$.

Proof. It is obvious. □

Definition 4.1 ([12])**.** Let S be an LA-semigroup, $x \in S$ and $t \in (0,1]$. A fuzzy point x_t of S is defined by the rule that

$$x_t\,(y) = \begin{cases} t, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

It is accepted that x_t is a mapping from S into $[0,1]$, then a fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel.

Lemma 4.2. *Let A be a non-empty subset of an LA-semigroup S. If $t \in (0,1]$, then $tf_A = \bigcup_{a\in A} a_t$.*

Proof. It is obvious. □

Next, defines the notions of weakly fuzzy primary and weakly fuzzy quasi-primary ideals on an LA-semigroup S.

Definition 4.2. A fuzzy ideal f of an LA-semigroup S is called weakly fuzzy primary of S if for any two ideals A and B of S such that $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B^n \subseteq f$ for some positive integer n.

Definition 4.3. A fuzzy left ideal f of an LA-semigroup S is called weakly fuzzy primary of S if for any two left ideals A and B of S such that $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B^n \subseteq f$ for some positive integer n.

Remark 4.1. It is easy to see that every weakly fuzzy primary is weakly fuzzy quasi-primary.

Then we have the following result.

Theorem 4.1. *For an AG-3-band S, the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S.*
(b) *f is a weakly fuzzy primary ideal of S.*

Proof. It is straightforward by Theorem 3.3. □

Theorem 4.2. *Let P be a fuzzy left ideal of an LA-semigroup S with left identity. Then the following statements are equivalent.*

(a) *P is a weakly fuzzy quasi-primary ideal of S.*
(b) *For any $x, y \in S$ and $t \in (0,1]$ if $x_t \circ (S \circ y_t) \subseteq P$, then $x_t \in P$ or $y_t^n \in P$ for some positive integer n.*
(c) *For any $x, y \in S$ and $t \in (0,1]$ if $tf_x \circ tf_y \subseteq P$, then $x_t \in P$ or $y_t^n \in P$ for some positive integer n.*
(d) *If A and B are left ideals of S such that $tf_A \circ tf_B \subseteq P$, then $tf_A \subseteq P$ or $tf_B^n \subseteq P$ for some positive integer n.*

Proof. (a) $\Rightarrow$ (b). First assume that P is a weakly fuzzy quasi-primary ideal of S. Let x and y be any elements of S and $t \in (0,1]$. Since $x_t \circ (S \circ y_t) \subseteq P$, we have

$$\begin{aligned} tf_{(xe)S} \circ tf_{(ye)S} &= (tf_{xe} \circ S) \circ (tf_{ye} \circ S) \\ &= (tf_{xe} \circ tf_{ye}) \circ (S \circ S) \\ &= ((tf_x \circ tf_e) \circ (tf_y \circ tf_e)) \circ (S \circ S) \\ &= ((tf_x \circ tf_y) \circ (tf_e \circ tf_e)) \circ (S \circ S) \\ &= ((tf_e \circ tf_e) \circ (tf_y \circ tf_x)) \circ (S \circ S) \\ &= (tf_{ee} \circ (tf_y \circ tf_x)) \circ (S \circ S) \\ &= (tf_y \circ (tf_e \circ tf_x)) \circ (S \circ S) \\ &= (tf_y \circ tf_{ex}) \circ (S \circ S) \\ &= (S \circ S) \circ (tf_x \circ tf_y) \\ &= S \circ (tf_x \circ tf_y) \\ &= tf_x \circ (S \circ tf_y) \\ &= x_t \circ (S \circ y_t) \\ &\subseteq P. \end{aligned}$$

Then since P is a weakly fuzzy quasi-primary ideal of S, we have

$$x_t = tf_x = tf_{(ee)x} = tf_{(xe)e} \subseteq tf_{(xe)S} \subseteq P,$$

or $y_t^n = tf_{y^n} = tf_{((ee)y)^n} = tf_{((ye)e)^n} \subseteq tf_{((ye)S)^n} = tf^n_{(ye)S} \subseteq P$ for some positive integer n. Thus, $x_t \in P$ or $y_t^n \in P$ and so (a) implies (b).

(b) $\Rightarrow$ (c). Assume that (b) holds. Let x and y be any elements of S and $t \in (0,1]$. Since $tf_x \circ tf_y \subseteq P$, we have

$$\begin{aligned} x_t \circ (S \circ y_t) &\subseteq tf_x \circ (S \circ tf_y) \\ &= S \circ (tf_x \circ tf_y) \\ &\subseteq S \circ P \\ &\subseteq P. \end{aligned}$$

Thus, by hypothesis $x_t \in P$ or $y_t^n \in P$ for some positive integer n. Hence we obtain that (b) implies (c).

(c) $\Rightarrow$ (d). Assume that (c) holds. Let A and B be any left ideals of S. Then it follows from Theorem 2.1 (2) that tf_A and tf_B are fuzzy left ideals of S. Next, let $tf_A \circ tf_B \subseteq P$ such that $tf_B^n \nsubseteq P$ for all positive integer n. Otherwise, there exists $y \in B$ such that $y_t^n \notin P$ for all positive integer n. For any $x \in A$, by Lemma 4.1 and hypothesis

$$tf_x \circ tf_y = tf_{xy} \subseteq tf_{AB} = tf_A \circ tf_B \subseteq P.$$

Since $y_t^n \notin P$, we have $tf_x \subseteq P$ and so $x_t \in P$. By Lemma 4.2, it follows that $tf_A = \bigcup_{x \in A} x_t$. Hence we obtain that (c) implies (d).

(d) $\Rightarrow$ (a). It is obvious. □

As is easily seen, every weakly fuzzy primary ideal of an LA-semigroup S is a weakly fuzzy quasi-primary ideal of S. The following example shows that the converse of this property does not hold in general.

Example 4.1. Let $S = \{0, 1, 2, 3\}$ be a set under the binary operation defined as in Table 3. It is clear that S is an LA-semigroup. We define a fuzzy subset $f : S \to [0, 1]$

TABLE 3. LA-semigroup

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

by

$$f(x) = \begin{cases} 0.9, & x \in \{0\}, \\ 0.5, & x \in \{2\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is a weakly fuzzy quasi-primary ideal of S. But f is not a fuzzy quasi-primary ideal of S, since $g \circ h \subseteq f$, while $g \not\subseteq f$ and $h^n \not\subseteq f$ for all positive integer n, where

$$g(x) = \begin{cases} 0.9, & x \in \{0, 2\}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} 1, & x \in \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Let a and b be any elements of an LA-semigroup S with left identity. If f is a fuzzy quasi-primary subset of S, then* $\inf\{f(a(Sb))\} \leq \max\{f(a), f(b^n)\}$ *for some positive integer n.*

Proof. Assume $\inf\{f(a(Sb))\} = m$. Let g and h be any fuzzy subsets of S such that

$$g(x) = \begin{cases} m, & x \in (ae)S, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} m, & x \in (be)S, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Theorem 2.1 (2) that g and h are fuzzy left ideals of S. If

$$(g \circ h)(x) = \bigcup_{x=yz} \min\{g(y), h(z)\} = m,$$

then there exist $u \in (ae)S$ and $v \in (be)S$ such that $uv = x$. Put $u = (ae)t$ and $v = (be)k$ for some $t, k \in S$. Then we have

$$\begin{aligned} f(x) =& f(uv) = f(((ae)t)((be)k)) = f(((ae)(be))(tk)) = f((kt)((be)(ae))) \\ \geq& f((be)(ae)) = f((ba)(ee)) = f((ee)(ab)) = f(e(ab)) = f(a(eb)) \\ \geq& \inf\{f(a(Sb))\} \\ =& m, \end{aligned}$$

so that $g \circ h \subseteq f$. Since f is a fuzzy quasi-primary subset of S, we have $g \subseteq f$ or $h^n \subseteq f$ for some positive integer n. Therefore $g(a) = g((ee)a) = g((ae)e) = mf_{(ae)S}((ae)e) = m$ or

$$\begin{aligned} h^n(b^n) =& g^n((ee)b^n) = g^n((b^n e)e) = mf_{((be)S)}{}^n((b^n e)e) = mf_{(be)^n S^n}((b^n e)e) \\ =& mf_{(b^n e^n)S}((b^n e)e) = mf_{(b^n e)S}((b^n e)e) = m. \end{aligned}$$

But from $m = \max\{f(a), f(b^n)\} < \inf\{f(a(Sb))\} = m$, we have a contradiction. Thus it follows that $\inf\{f(a(Sb))\} \leq \max\{f(a), f(b^n)\}$ for some positive integer n. □

The following corollary can be easily deduced from the theorem.

Corollary 4.1. *Let a and b be any elements of an LA-semigroup S with left identity. If f is a fuzzy quasi-primary left ideal of S, then* $\inf\{f(a(Sb))\} = \max\{f(a), f(b^n)\}$ *for some positive integer n.*

Theorem 4.4. *Let P be a fuzzy left ideal of an LA-semigroup S with left identity and let x and y be any elements of S. If* $P(xy) = \max\{P(x), P(y^n)\}$ *for some positive integer n, then P is a weakly fuzzy quasi-primary ideal of S.*

Proof. Let x and y be any elements of S and $t \in (0,1]$. Suppose that x_t and y_t are fuzzy points of S such that $x_t \circ (S \circ y_t) \subseteq P$. Then we have $S \circ (xy)_t = S \circ (x_t \circ y_t) = x_t \circ (S \circ y_t) \subseteq P$ and so $P(xy) \geq t$. Since $P(xy) = \max\{P(x), P(y^n)\}$, we have $P(x) \geq t$ or $P(y^n) \geq t$ for some positive integer n. This implies that $x_t \in P$ or $y_t^n \in P$ and hence P is a weakly fuzzy quasi-primary ideal of S. □

Theorem 4.5. *Let P be a fuzzy completely primary subset of an LA-semigroup S with left identity. Then P is weakly fuzzy quasi-primary subset of S.*

Proof. We leave the straightforward proof to the reader. □

Theorem 4.6. *Let S be an LA-semigroup with left identity. Then the following conditions are equivalent.*

(a) *P is a weakly fuzzy quasi-primary subset of S.*
(b) *If $x, y \in S$, then* $P(xy) \leq \max\{P(x), P(y^n)\}$ *for some positive integer n.*

Proof. First assume that P is a weakly fuzzy quasi-primary subset of an LA-semigroup S with left identity. Let x and y be any elements of S. If $P(xy) > \max\{P(x), P(y^n)\}$, then there exists $t \in (0,1)$ such that $P(xy) > t > \max\{P(x), P(y^n)\}$. Thus, we have

$$x_t \circ (S \circ y_t) = S \circ (x_t \circ y_t) = S \circ (xy)_t \subseteq S \circ P \subseteq P.$$

Since P is a weakly fuzzy quasi-primary subset of S, we have $x_t \in P$ or $y_t^n \in P$ for some positive integer n, but $x_t \notin P$ and $y_t^n \notin P$, which is impossible. Therefore, $P(xy) \leq \max\{P(x), P(y^n)\}$.

Conversely, assume that (b) holds. Let x and y be any elements of S and $t \in (0,1]$. Suppose that x_t and y_t are fuzzy points of S such that $x_t \circ (S \circ y_t) \subseteq P$. Since

$$S \circ (xy)_t = S \circ (x_t \circ y_t) = x_t \circ (S \circ y_t) \subseteq P,$$

we have $P(xy) \geq t$. Then, since $P(xy) \leq \max\{P(x), P(y^n)\}$, we have $P(x) \geq t$ or $P(y^n) \geq t$ for some positive integer n and so $x_t \in P$ or $y_t^n \in P$. Therefore we obtain that, P is a weakly fuzzy quasi-primary subset of S. □

Theorem 4.7. *Let S be an LA-semigroup. Then the following conditions are equivalent.*

(a) *P is a quasi-primary ideal of S.*
(b) *f_P is a weakly fuzzy quasi-primary ideal of S.*

Proof. We leave the straightforward proof to the reader. □

Theorem 4.8. *Let f be a fuzzy subset of an LA-semigroup S. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S.*
(b) *The level subset $U(f,t)$ of f is a quasi-primary ideal of S for every $t \in \mathrm{Im}(f)$.*

Proof. First assume that f is a weakly fuzzy quasi-primary ideal of S. Let $t \in (0,1]$ and let a and b be any elements of S such that $ab \in U(f,t)$. Then we have $f(ab) \geq t$. Since $tf_a \circ tf_b = tf_{ab} \subseteq f$, we have $tf_a \subseteq f$ or $tf_{b^n} \subseteq f$ for some positive integer n and so $f(a) \geq t$ or $f(b^n) \geq t$. Thus, $a \in U(f,t)$ or $b^n \in U(f,t)$ and hence $U(f,t)$ is a quasi-primary ideal of S.

Conversely, assume that $U(f,t)$ is a quasi-prime ideal of S for every $t \in Im(f)$. Let a and b be any elements of S such that $tg_a \circ tg_b \subseteq f$. Since $tg_{ab} = tg_a \circ tg_b$, we have $f(ab) \geq tg_{ab}(ab) = t$ and so $ab \in U(f,t)$. By assumption, $a \in U(f,t)$ or $b^n \in U(f,t)$ for some positive integer n. Suppose $a_t \notin f$ and $b_t^n \notin f$ for all positive integer n. However, $t = a_t(a) > f(a)$ and $t = b_t^n(b^n) > f(b^n)$ and we have a contradiction. Therefore $a_t \in f$ or $b_t^n \in f$ and hence f is a weakly fuzzy quasi-primary ideal of S. □

5. Cartesian Product of Fuzzy Ideals of LA-Semigroups

In this section, we concentrate our study on the cartesian product of fuzzy ideals of an LA-semigroup and investigate their fundamental properties and mutual relationships. Finally we show that a fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

We start with the following theorem that gives a relation between cartesian product of fuzzy ideals in an LA-semigroup. Our starting points are the following definitions:

Let S_1 and S_2 be two LA-semigroups. Then

$$S_1 \times S_2 := \{(x, y) \in S_1 \times S_2 : x \in S_1, y \in S_2\},$$

and for any $(a, b), (c, d) \in S_1 \times S_2$ we define

$$(a, b)(c, d) := (ac, bd),$$

then $S_1 \times S_2$ is an LA-semigroup as well. Let $f : S_1 \to [0, 1]$ and $g : S_2 \to [0, 1]$ be two fuzzy subsets of LA-semigroups S_1 and S_2 respectively. Then the product of fuzzy subsets is denoted by $f \times g$ and defined as $f \times g : S_1 \times S_2 \to [0, 1]$, where $(f \times g)(x, y) = \min\{f(x), g(y)\}$.

In the light of the definition of cartesian product of fuzzy ideals in an LA-semigroup, we can obtain the following properties.

Lemma 5.1. *Let f and g be two fuzzy subsets of LA-simigroups S_1 and S_2, respectively and let $t \in (0, 1]$. Then $(f \times g)_t = f_t \times f_t$.*

Proof. Let f and g be two fuzzy subsets of LA-simigroups S_1 and S_2, respectively and let $t \in (0, 1]$. Next, let (x, y) be any element of $S_1 \times S_2$. Then we have

$$\begin{aligned} (x, y) \in f_t \times g_t &\Leftrightarrow x \in f_t \wedge y \in g_t \\ &\Leftrightarrow f(x) \geq t \wedge g(y) \geq t \\ &\Leftrightarrow \min\{f(x), g(y)\} \geq t \\ &\Leftrightarrow (f \times g)(x, y) \geq t \\ &\Leftrightarrow (x, y) \in (f \times g)_t . \end{aligned}$$

Hence, $(f \times g)_t = f_t \times f_t$. □

By Lemma 5.1, we have the following result.

Corollary 5.1. *Let $f_1, f_2, \ldots, f_n$ be any fuzzy subsets of LA-simigroups $S_1, S_2, \ldots, S_n$, respectively and let $t \in (0, 1]$. Then $\left(\prod_{i=1}^{n} f_i\right)_t = \prod_{i=1}^{n} (f_i)_t$.*

Proof. One can easily show by induction method. □

Theorem 5.1. *Let f_1 and f_2 be two fuzzy subsets of LA-semigroups S_1 and S_2, respectively. Then the following conditions are equivalent.*

(a) *$f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$.*
(b) *The level subset $(f \times g)_t$ of $f \times g$ is a completely primary ideal of $S_1 \times S_2$ for every $t \in \mathrm{Im}(f \times g)$.*

Proof. First assume that $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$. Let (x, y) and (m, n) be any elements of $S_1 \times S_2$ such that $(x, y)(m, n) \in (f \times g)_t$. Then we have $(f \times g)((x, y)(m, n)) \geq t$ and so $(f \times g)(xm, yn) \geq t$. Since $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$, we have

$$(f \times g)((x, y)(m, n)) \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\},$$

for some positive integer k. If $(f \times g)(x, y) \leq (f \times g)(m, n)^k$, then

$$t \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} = (f \times g)(m, n)^k.$$

Hence, we have $(f \times g)(m, n)^k \geq t$ and so $(m, n)^k \in (f \times g)_t$. Now if $(f \times g)(x, y) > (f \times g)(m, n)^k$, then

$$t \leq \max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} = (f \times g)(x, y).$$

Therefore, we obtain that $(f \times g)(x, y) \geq t$ and hence $(x, y) \in (f \times g)_t$. In any case, we have $(f \times g)_t$ is a completely primary ideal of $S_1 \times S_2$.

Conversely, assume that $(f \times g)_t$ is a completely primary ideal of $S_1 \times S_2$ for every $t \in Im(f \times g)$. Let (x, y) and (m, n) be any elements of $S_1 \times S_2$. Otherwise, $(f \times g)((x, y)(m, n)) \geq 0$. Since $(x, y)(m, n) \in (f \times g)_{(f \times g)((x,y)(m,n))}$, by hypothesis, we have $(x, y) \in (f \times g)_{(f \times g)((x,y)(m,n))}$ or $(m, n)^k \in (f \times g)_{(f \times g)((x,y)(m,n))}$ for some positive integer k. Thus, we have $(f \times g)(x, y) \geq (f \times g)((x, y)(m, n))$ or $(f \times g)(m, n)^k \geq (f \times g)((x, y)(m, n))$ and so we have

$$\max\left\{(f \times g)(x, y), (f \times g)(m, n)^k\right\} \geq (f \times g)((x, y)(m, n)).$$

Thus, it follows from Definition 3.1 that $f \times g$ is a fuzzy completely primary ideal of $S_1 \times S_2$. □

By Theorem 5.1, we have the following result.

Corollary 5.2. *Let $f_1, f_2, \ldots, f_n$ be any fuzzy subsets of LA-semigroups $S_1, S_2, \ldots, S_n$, respectively. Then the following conditions are equivalent.*

(a) *$\prod_{i=1}^n f_i$ is a fuzzy completely primary ideal of $\prod_{i=1}^n S_i$.*

(b) *The level subset $(\prod_{i=1}^n f_i)_t$ is a completely primary ideal of $\prod_{i=1}^n S_i$ for every $t \in \operatorname{Im}(\prod_{i=1}^n f_i)$.*

Proof. One can easily show by induction method. □

Lemma 5.2. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S_1.*

(b) *$f \times S_2$ is a fuzzy LA-subsemigroup of $S_1 \times S_2$.*

Proof. We leave the straightforward proof to the reader. □

By Lemma 5.2, we have the following result.

Corollary 5.3. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy LA-subsemigroup of S_2.*

(b) *$S_1 \times f$ is a fuzzy LA-subsemigroup of $S_1 \times S_2$.*

Lemma 5.3. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_1.*
(b) *$f \times S_2$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $S_1 \times S_2$.*

Proof. Similar to the proof of Lemma 5.2. □

By Lemma 5.3, we have the following result.

Corollary 5.4. *Let S_1 and S_2 be two LA-semigroups. Then the following properties hold.*

(a) *f is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_2.*
(b) *$S_1 \times f$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $S_1 \times S_2$.*

By Lemmas 5.2, 5.3 and Corollaries 5.3, 5.4, we have the following result.

Corollary 5.5. *Let f_i be a fuzzy subset of an LA-semigroup S_i. Then the following properties hold.*

(a) *f_i is a fuzzy LA-subsemigroup of S_i if and only if $S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a fuzzy LA-subsemigroup of $\prod_{i=1}^{n} S_i$.*
(b) *f_i is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of S_i if and only if $S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $\prod_{i=1}^{n} S_i$.*

Proof. One can easily show by induction method. □

The following theorem show that the fuzzy left ideal f is a weakly fuzzy quasi-primary ideal of S_2 if and only if $S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.

Theorem 5.2. *Let S_1 and S_2 be two LA-semigroups with left identities. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S_1.*
(b) *$f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.*

Proof. First assume that f is a weakly fuzzy quasi-primary ideal of S_1. Let (a, b) and (c, d) be any elements of $S_1 \times S_2$ such that $(ac, bd)_t = (a, b)_t \circ (c, d)_t \in f \times S_2$. Then we have $f(ac) = \min\{f(ac), 1\} = \min\{f(ac), S_2(bd)\} = (f \times S_2)(ac, bd) \geq t$ and so $f(ac) \geq t$. Obviously, $a_t \circ c_t = (ac)_t \in f$. By Theorem 4.2, $a_t \in f$ or $c_t^n \in f$ for some positive integer n, it is clear that $f(a) \geq t$ or $f(c^n) \geq t$ for some positive integer n. Thus, we have

$$(f \times S_2)(a, b) = \min\{f(a), S_2(b)\} \geq \min\{t, 1\} = t$$

or

$$(f \times S_2)(c, d)^n = (f \times S_2)(c^n, d^n) = \min\{f(c^n), S_2(d^n)\} \geq \min\{t, 1\} = t,$$

and so we have $(a,b)_t \in f \times S_2$ or $(c,d)_t^n \in f \times S_2$. Then it follows from Theorem 4.2 that $f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$ and so (a) implies (b).

Conversely, assume that $f \times S_2$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$. Let a and c be any elements of S_1 such that $(ac)_t = a_t \circ c_t \in f$. Then we have $t \leq f(ac) = \min\{f(ac), 1\} = \min\{f(ac), S_2(bd)\} = (f \times S_2)(ac, bd)$ and so $(f \times S_2)(ac, bd) \geq t$ for all $b, d \in S_2$. Obviously, $(a,b)_t \circ (c,d)_t = (ac, bd)_t \in f \times S_2$. By Theorem 4.2, $(a,b)_t \in f \times S_2$ or $(c,d)_t^n \in f \times S_2$ for some positive integer n, it is clear that $(f \times S_2)(a,b) \geq t$ or $(f \times S_2)(c,d)^n \geq t$ for some positive integer n. Thus we have

$$f(a) = \min\{f(a), 1\} = \min\{f(a), S_2(b)\} = (f \times S_2)(a,b) \geq t$$

or

$$f(c^n) = \min\{f(c^n), 1\} = \min\{f(c^n), S_2(d^n)\} = (f \times S_2)(c^n, d^n) = (f \times S_2)(c,d)^n \geq t,$$

and so we have $a_t \in f$ or $c_t^n \in f$. Then it follows from Theorem 4.2 that f is a weakly fuzzy quasi-primary ideal of S_1. □

By Theorem 5.2, we have the following result.

Corollary 5.6. *Let S_1 and S_2 be two LA-semigroups with left identities. Then the following conditions are equivalent.*

(a) *f is a weakly fuzzy quasi-primary ideal of S_2.*
(b) *$S_1 \times f$ is a weakly fuzzy quasi-primary ideal of $S_1 \times S_2$.*

By Theorem 5.2 and Corollary 5.6, we have the following result.

Theorem 5.3. *Let S_i be an LA-semigroup with left identity. Then the following conditions are equivalent.*

(a) *f_i is a weakly fuzzy quasi-primary ideal of S_i.*
(b) *$S_1 \times \cdots \times S_{i-1} \times f_i \times S_{i+1} \times \cdots \times S_n$ is a weakly fuzzy quasi-primary ideal of $\prod_{i=1}^n S_i$.*

Proof. One can easily show by induction method. □

References

[1] S. Abdullah, S. Aslam and N. U. Amin, *LA-semigroups characterized by the properties of interval valued (α, β)-fuzzy ideals*, J. Appl. Math. Inform. **32**(3-4) (2014), 405–426.

[2] S. Abdullah, M. Aslam, N. Amin and T. Khan, *Direct product of finite fuzzy subsets of LA-semigroups*, Ann. Fuzzy Math. Info. **3**(2) (2012), 281–292.

[3] S. Abdullah, M. Aslam, M. Imran and M. Ibrar, *Direct product of intuitionistic fuzzy sets in LA-semigroups-II*, Ann. Fuzzy Math. Info. **2**(2) (2011), 151–160.

[4] V. Amjad, F. Yousafzai and A. Iampan, *On generalized fuzzy ideals of ordered LA-semigroups*, Boletin de Matematicas **22**(1) (2015), 1–19.

[5] M. Aslam, S. Abdullah and S. Aslam, *Characterization of regular LA-semigroups by interval-valued $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy ideals*, Afr. Mat. **25** (2014), 501–518.

[6] B. Davvaz, M. Khan, S. Anis and S. Haq, *Generalized fuzzy quasi-ideals of an intraregular Abel-Grassmann's groupoid*, J. Appl. Math. **2012** (2012), Article ID 627075, 16 pages.
[7] Faisal, N. Yaqoob and A. Ghareeb, *Left regular AG-groupoids in terms of fuzzy interior ideals*, Afr. Mat. **24** (2013), 577–587.
[8] M. Khan, S. Anis and S. Lodhi, *A study of fuzzy Abel-Grassmann's groupoids*, International Journal of the Physical Sciences **7**(4) (2012), 584–592.
[9] M. Khan, V. L. Fotea and S. Kokab, $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideals of intra-regular Abel Grassmann's-groupoids*, Analele Stiintifice ale Universitatii Ovidius Constanta **22** (2014), 95–113.
[10] M. Khan, Y. B. Jun and T. Mahmood, *Generalized fuzzy interior ideals in Abel Grassmann's groupoids*, Int. J. Math. Math. Sci. **2010** (2010), Article ID 838392, 14 pages.
[11] M. Khan, Y. B. Jun and F. Yousafzai, *Fuzzy ideals in right regular LA-semigroups*, Hacet. J. Math. Stat. **44**(3) (2015), 569–586.
[12] M. Khan and M. N. A. Khan, *Fuzzy Abel-Grassmann's groupoids*, Advances in Fuzzy Mathematics **3** (2010), 349–360.
[13] M. Khan and M. N. Khan, *On fuzzy Abel Grassmann's groupoids*, Advances in Fuzzy Mathematics **5**(3) (2010), 349–360.
[14] A. Khan, N. H. Sarmin, F. M. Khan and B. Davvaz, *Regular AG-groupoids characterized by fuzzy ideals*, Iran. J. Sci. Technol. **36**(2) (2012), 97–113.
[15] A. Khan, M. Shabir, Y. B. Jun, *Generalized fuzzy Abel Grassmann's groupoids*, Int. J. Fuzzy Syst. **12**(4) (2010), 340–349
[16] X. Ma, J. Zhan, M. Khan and T. Aziz, *Some characterizations of intra-regular Abel Grassmann's groupoids*, Italian Journal of Pure and Applied Mathematics **32** (2014), 329–346.
[17] K. Nasreen, *Characterizations of non associative ordered semigroups by the properties of their fuzzy ideals with thresholds* $(\alpha, \beta]$, Prikl. Diskretn. Mat. **43** (2019), 37–59.
[18] M. Palanivelrajan and S. Nandakumar, *Intuitionistic fuzzy primary and semiprimary ideal*, Indian Journal of Applied Research **1**(5) (2012), 159–160.
[19] N. Rehman, N. Shah, M. I. Ali and A. Ali, *Generalised roughness in* $(\in, \in \vee q)$*-fuzzy substructures of LA-semigroups*, Journal of the National Science Foundation of Sri Lanka **46**(3) (2018), 465–473.
[20] T. Shah and N. Kausar, *Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals*, Theoret. Comput. Sci. **529** (2014), 96–110.
[21] N. Yaqoob, *Interval valued intuitionistic fuzzy ideals of regular LA-semigroups*, Thai J. Math. **11**(3) (2013), 683–695.
[22] N. Yaqoob, R. Chinram, A. Ghareeb and M. Aslam, *Left almost semigroups characterized by their interval valued fuzzy ideals*, Afr. Mat. **24** (2013), 231–245.
[23] P. Yiarayong, *On product of fuzzy semiprime ideals in* Γ*-LA-semigroups*, Gazi University Journal of Science **29**(2) (2016), 491–502.
[24] F. Yousafzai, A. Khan, V. Amjad and A. Zeb, *On fuzzy fully regular ordered AG-groupoids*, Journal of Intelligent & Fuzzy Systems **26** (2014), 2973–2982.
[25] F. Yousafzai, W. Khan, A. Khan and K. Hila, *I-V fuzzy set theory applied on po-LA-semigroups*, Afr. Mat. **27** (2016), 23–35.
[26] F. Yousafzai, N. Yaqoob and A. Zeb, *On generalized fuzzy ideals of ordered AG-groupoids*, Int. J. Mach. Learn. & Cyber. **7** (2016), 995–1004.
[27] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1956), 338–353.
[28] Y. Zhang, *Prime L-fuzzy ideals and primary L-fuzzy ideals*, Fuzzy Sets & Syst. **27**(3) (1988), 345–350.

[1]Department of Mathematics, Faculty of Science and Technology
Pibulsongkram Rajabhat University,
Phitsanulok 65000, Thailand
Email address: pairote0027@hotmail.com