ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 46, Number 5, 2022

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić
. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Belgrade : Donat Graf). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2205

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
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Website: http://kjm.pmf.kg.ac.rs

Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: Donat Graf, Belgrade, Serbia
From 2021 the journal appears in one volume and six issues per annum.

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KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 46(5) (2022), PAGES 675–683.

ENTIRE FUNCTION SHARING ENTIRE FUNCTION WITH ITS FIRST DERIVATIVE

SUJOY MAJUMDER[1] AND JEET SARKAR[2]

ABSTRACT. In this paper, we use the idea of normal family to investigate the problem of entire function that share entire function with its first derivative.

1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $|z|<r$, the poles are counted with their multiplicities. We call the quantity

$$N(r,\infty;f)=\int\limits_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

as the integrated counting function or simply the counting function of poles of f and

$$m(r,\infty;f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(re^{i\theta})|d\theta$$

as the proximity function of poles of f, where $\log^+ x=\log x$, if $x\geq 1$ and $\log^+ x=0$, if $0\leq x<1$.

We use the notation $T(r,f)$ for the sum $m(r,\infty;f)+N(r,\infty;f)$ and it is called the Nevanlinna characteristic function of f. We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $\frac{S(r,f)}{T(r,f)}\to 0$ as $r\to\infty$ except possibly a set of finite linear measure.

Key words and phrases. Meromorphic functions, derivative, nevanlinna theory, normal family.
2010 *Mathematics Subject Classification.* Primary: 30D35. Secondary: 30D30.
DOI 10.46793/KgJMat2205.675M
Received: December 25, 2018.
Accepted: March 07, 2020.

For $a \in \mathbb{C}$, we write $N(r,a;f) = N(r,\infty;\frac{1}{f-a})$ and $m(r,a;f) = m(r,\infty;\frac{1}{f-a})$.

Again we denote by $\overline{n}(r,a;f)$ the number of distinct a points of f lying in $|z| < r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r,a;f) = \int_0^r \frac{\overline{n}(t,a;f) - \overline{n}(0,a;f)}{t} dt + \overline{n}(0,a;f) \log r$$

denotes the reduced counting function of a points of f (see, e.g., [6, 15]).

A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$, i.e., if $T(r,a) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ and Q be a polynomial or a finite complex number. If $g(z) - Q(z) = 0$ whenever $f(z) - Q(z) = 0$, we write $f = Q \Rightarrow g = Q$.

Let f and g be two non-constant meromorphic functions. Let a be a small function with respect to both f and g. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities then we say that f and g share a with CM (counting multiplicities) and if we do not consider the multiplicities then we say that f and g share a with IM (ignoring multiplicities).

We recall that the order $\rho(f)$ of meromorphic function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [13]).

Rubel and Yang [12] were the first authors to study the entire functions that share values with their derivatives. In 1977, they proved the following important result.

Theorem A ([12]). *Let a and b be complex numbers such that $b \neq a$ and let f be a non-constant entire function. If f and f' share the values a and b CM, then $f \equiv f'$.*

In 1979, Mues and Steinmetz [11] generalized Theorem A from sharing values CM to IM and obtained the following result.

Theorem B ([11]). *Let a and b be complex numbers such that $b \neq a$ and f a non-constant entire function. If f and f' share the values a and b IM, then $f \equiv f'$.*

In 1983, Gundersen [4] improved Theorem A from entire function to meromorphic function and obtained the following result.

Theorem C ([4]). *Let f be a non-constant meromorphic function, a and b two distinct finite values. If f and f' share the values a and b CM, then $f \equiv f'$.*

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. Let f be a non-constant entire function. Suppose

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a} = c, \tag{1.1}$$

for some non-zero constant c.

By the solutions of the differential equations

$$\frac{f'-a}{f-a} = e^{z^n} \quad \text{and} \quad \frac{f'-a}{f-a} = e^{e^z}, \tag{1.2}$$

we see that when $\rho_1(f)$ is a positive integer or infinite, the conjecture does not hold.

Conjecture A for the case $a = 0$ had been proved by Brück [1]. In the same paper Brück [1] proved that the growth restriction on f is not necessary when $N(r, 0; f') = S(r, f)$.

Gundersen and Yang [5] proved that Conjecture A is true when f is of finite order. Further Chen and Shon [3] proved that Conjecture A is also true when f is of infinite order with $\rho_1(f) < \frac{1}{2}$. Recently Cao [2] proved that Brück conjecture is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open.

Since then, shared value problems, especially the case of f and $f^{(k)}$, where $k \in \mathbb{N}$ sharing one value or small function have undergone various extensions and improvements (see [15]).

Now it is interesting to know what happens if f is replaced by f^n in Conjecture A. From (1.2), we see that Conjecture A does not hold when $n = 1$. Thus, we have to discuss the problem only when $n \geq 2$.

Yang and Zhang [14] proved that Conjecture A holds for the function f^n without imposing the order restriction on f if n is relatively large. Actually they proved the following result.

Theorem D ([14]). *Let f be a non-constant entire function, $n \in \mathbb{N} \setminus \{1, 2, \ldots, 6\}$ and $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2009, Lü, Xu and Chen [8] improved Theorem D in the following manner.

Theorem E ([8]). *Let $a(\not\equiv 0)$ be a polynomial and $n \in \mathbb{N} \setminus \{1\}$, f a transcendental entire function and $F = f^n$. If F and F' share a CM, then conclusion of Theorem D holds.*

In 2011, Lü [9] further improved Theorem E as follows.

Theorem F ([9]). Let f be a transcendental meromorphic function with finitely many poles, $n \in \mathbb{N} \setminus \{1\}$ and $\alpha = Pe^Q(\not\equiv \alpha')$ an entire function such that the order of α is less than that of f, where P, Q are two polynomials. If f^n and $(f^n)'$ share α CM, then conclusion of Theorem D holds.

Remark 1.1. If Q is a constant, then Theorem F still holds without the assumption that $\rho(\alpha) < \rho(f)$.

In 2014, Zhang, Kang and Liao [17] improved Theorem F in a different direction as follows.

Theorem G ([17]). *Let f be a transcendental entire function, $a = a(z)(\not\equiv 0, \infty)$ a small function of f such that order of a is less than that of f and $n \in \mathbb{N} \setminus \{1\}$. If f^n and $(f^n)'$ share a CM, then conclusion of Theorem D holds.*

Naturally, one can ask whether the conclusion of Theorem E still holds if F and F' share a CM is replaced by share a IM. In 2015, Lü and Yi [10] gave an affirmative answer and obtained the following result.

Theorem H ([10]). *Let $a(\not\equiv 0)$ be a polynomial and $n \in \mathbb{N} \setminus \{1\}$. Let f be a transcendental entire function and $F = f^n$. If F and F' share a IM, then conclusion of Theorem D holds.*

We now emerge the following question as an open problem.
Question 1. What happens if F and F' share a CM is replaced by share Pe^Q IM, where $P(\not\equiv 0)$ and Q are polynomials in Theorem E?
In the paper we prove the following result that answer the above question.

Theorem 1.1. *Let f be a transcendental entire function and $n \in \mathbb{N} \setminus \{1\}$. Let $\alpha = Pe^Q(\not\equiv \alpha')$, where $P(\not\equiv 0)$ and Q are polynomials such that $2\rho(\alpha) < \rho(f)$. If f^n and $(f^n)'$ share α IM, then conclusion of Theorem D holds.*

Remark 1.2. If Q is a constant, then Theorem 1.1 still holds without the assumption that $2\rho(\alpha) < \rho(f)$. Also from Theorem 1.1, it is clear that Theorem 1.1 is the generalization of Theorem H.

2. Lemmas

In this section we present the lemmas which will be needed in the sequel.

Lemma 2.1 ([8]). *Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc Δ. If $a_n \to a$, $|a| < 1$ and $f_n^{\#}(a_n) \to \infty$, then there exist*

(a) *a subsequence of* f_n (*which we still write as* $\{f_n\}$);
(b) *points* $z_n \to z_0$, $|z_0| < 1$;
(c) *positive numbers* $\rho_n \to 0$,

such that $f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ *locally uniformly, where* g *is a non-constant meromorphic* (*entire*) *function on* $\mathbb{C}$ *such that*

$$\rho_n \leq \frac{M}{f_n^{\#}(a_n)},$$

where M *is a constant which is independent of* n.

Lemma 2.2 ([16])**.** *Let* f *be a meromorphic function in the complex plane and* $\rho(f) > 2$. *Then for each* $0 < \mu < \frac{\rho(f)-2}{2}$, *there exist points* $a_n \to \infty$, $n \to \infty$, *such that*

$$\lim_{n\to\infty} \frac{f^{\#}(a_n)}{|a_n|^{\mu}} = +\infty.$$

Lemma 2.3 ([7])**.** *Let* f *be a meromorphic function of infinite order on* $\mathbb{C}$. *Then there exist points* $z_n \to \infty$ *such that for every* $N > 0$, $f^{\#}(z_n) > |z_n|^N$, *if* n *is sufficiently large.*

3. Proof of the Theorem 1.1

Proof. Let $F = \frac{f^n}{\alpha}$ and $G = \frac{(f^n)'}{\alpha}$. Now we consider following two cases.
Case 1. Suppose $\rho(f) < +\infty$. Clearly $\rho(\alpha) = \deg(Q)$ and $\rho(f) = \rho(f^n)$. Since $\rho(\alpha) < \rho(f)$, we have $\rho(\alpha) < \rho(f^n)$. Note that $\rho\left(\frac{f^n}{\alpha}\right) \leq \max\{\rho(f^n), \rho(\alpha)\} = \rho(f^n)$. Since $\rho(\alpha) < \rho(f^n)$, it follow that $\rho(f^n) = \rho\left(\frac{f^n}{\alpha}\alpha\right) \leq \max\left\{\rho\left(\frac{f^n}{\alpha}\right), \rho(\alpha)\right\} = \rho\left(\frac{f^n}{\alpha}\right)$. Consequently, $\rho(f^n) = \rho\left(\frac{f^n}{\alpha}\right) = \rho(F)$. Therefore,

$$\rho(f) = \rho\Big(f^n\Big) = \rho\Big(\frac{f^n}{\alpha}\Big) = \rho(F) < +\infty.$$

Since $\rho((f^n)') = \rho(f^n) < +\infty$, we have $\rho(G) \leq \max\{\rho((f^n)'), \rho(\alpha)\} < +\infty$. Following two sub-cases are immediately.
Sub-case 1.1. Suppose Q is a constant. In that case α reduces to a polynomial. Then by Theorem H, we have $F \equiv G$, i.e., $f^n \equiv (f^n)'$ and so $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.
Sub-case 1.2. Suppose Q is non-constant. Let $\mu_1 = 2\deg(Q) \geq 2$ and $\mu_2 = \frac{\mu_1 - 2}{2}$. Since $\mu_1 < \rho(f)$, we have $0 \leq \mu_2 < \frac{\rho(f)-2}{2}$. Let $0 < \varepsilon < \frac{\rho(f)-\mu_1}{2}$. Then $0 \leq \mu_2 < \mu_2 + \varepsilon < \frac{\rho(f)-2}{2}$. Let $\mu = \mu_2 + \varepsilon$. Now by Lemma 2.2, for $0 < \mu < \frac{\rho(f)-2}{2}$, there exists a sequence $\{w_n\}_n$ such that $w_n \to \infty$, $n \to \infty$, and

$$\lim_{n\to\infty} \frac{F^{\#}(w_n)}{|w_n|^{\mu}} = +\infty. \tag{3.1}$$

Since P is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \geq r_1$, we have

$$0 \leftarrow \left|\frac{P'(z)}{P(z)}\right| \leq \frac{M_1}{|z|} < 1, \quad P(z) \neq 0.$$

Let $r > r_1$ and $D = \{z : |z| \geq r\}$. Then F is analytic in D. Since $w_n \to \infty$ as $n \to \infty$, without loss of generality we may assume that $|w_n| \geq r + 1$ for all n. Let $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f^n(w_n + z)}{\alpha(w_n + z)}.$$

Since $|w_n + z| \geq |w_n| - |z|$, it follows that $w_n + z \in D$ for all $z \in D_1$. Also, since $F(z)$ is analytic in D, it follows that $F_n(z)$ is analytic in D_1 for all n. Thus, we have structured a family $(F_n)_n$ of holomorphic functions. Note that $F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now it follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$. Let $a_n = 0$ for all n and $a = 0$. Then $a_n \to a$ and $|a| < 1$. Also, $F_n^{\#}(a_n) = F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now we apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$, if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ such that $|z_n| < r < 1$, $z_n \to 0$, $\rho_n \to 0$ and that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{f^n(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{3.2}$$

converges locally and uniformly in $\mathbb{C}$, where $g(\zeta)$ is a non-constant entire function. By Hurwitz's theorem, we conclude that zeros of g are of multiplicities at least n. Also,

$$\rho_n \leq \frac{M}{F_n^{\#}(a_n)} = \frac{M}{F^{\#}(w_n)}, \tag{3.3}$$

for a positive number M. Now from (3.1) and (3.3), we deduce that

$$\rho_n \leq \frac{M}{F^{\#}(w_n)} \leq M_1|w_n|^{-\mu}, \tag{3.4}$$

for sufficiently large values of n, where M_1 is a positive constant.

Also from (3.2), we see that

$$\begin{aligned} \rho_n\frac{(f^n)'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} =& g_n'(\zeta) + \rho_n\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha^2(w_n + z_n + \rho_n\zeta)}f^n(w_n + z_n + \rho_n\zeta) \\ =& g_n'(\zeta) + \rho_n\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)}g_n(\zeta). \end{aligned} \tag{3.5}$$

Note that

$$\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} = \frac{P'(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} + Q'(w_n + z_n + \rho_n\zeta). \tag{3.6}$$

Observe that $\frac{P'(w_n+z_n+\rho_n\zeta)}{P(w_n+z_n+\rho_n\zeta)} \to 0$ as $n \to \infty$. Let $s = \deg(Q')$. Since $2\deg(Q) \leq \mu_1$, it follows that $0 \leq s \leq \mu_2 < \mu$. Therefore, from (3.4), we have

$$\lim_{n\to\infty} \rho_n|w_n|^s \leq \lim_{n\to\infty} M_1|w_n|^{s-\mu} = 0. \tag{3.7}$$

Note that $|Q'(w_n+z_n+\rho_n\zeta)| = O(|w_n|^s)$ and so from (3.7), we have

$$\rho_n|Q'(w_n+z_n+\rho_n\zeta)| = O(\rho_n|w_n|^s) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.8}$$

From (3.6) and (3.8), we have

$$\rho_n\frac{\alpha'(w_n+z_n+\rho_n\zeta)}{\alpha(w_n+z_n+\rho_n\zeta)} \to 0 \quad \text{as} \quad n\to\infty. \tag{3.9}$$

Now from (3.2), (3.5) and (3.9), we observe that

$$\rho_n\frac{(f^n)'(w_n+z_n+\rho_n\zeta)}{\alpha(w_n+z_n+\rho_n\zeta)} \to g'(\zeta). \tag{3.10}$$

Clearly $g'(z)\not\equiv 0$, for otherwise $g(z)$ would be a polynomial of degree at most 1 and so $g(z)$ could not have zero of multiplicity at least $n(\geq 2)$.

Firstly we claim that $g=1 \Rightarrow g'=0$. Suppose that $g(\eta_0)=1$. Then by Hurwitz's theorem there exists a sequence $(\eta_n)_n$, $\eta_n\to\eta_0$ such that (for sufficiently large n)

$$g_n(\eta_n) = \frac{f^n(w_n+z_n+\rho_n\eta_n)}{\alpha(w_n+z_n+\rho_n\eta_n)} = 1,$$

i.e., $f^n(w_n+z_n+\rho_n\eta_n) = \alpha(w_n+z_n+\rho_n\eta_n)$. By the given condition, we have

$$(f^n)'(w_n+z_n+\rho_n\eta_n) = \alpha(w_n+z_n+\rho_n\eta_n). \tag{3.11}$$

Now from (3.10) and (3.11), we see that

$$g'(\eta_0) = \lim_{n\to\infty} g'(\eta_n) = \lim_{n\to\infty}\rho_n\frac{(f^n)'(w_n+z_n+\rho_n\eta_n)}{\alpha(w_n+z_n+\rho_n\eta_n)} = \lim_{n\to\infty}\rho_n = 0.$$

Thus, $g=1\Rightarrow g'=0$. Finally we want to prove that $g'=0\Rightarrow g=1$. Now from (3.10), we see that

$$\rho_n\frac{(f^n)'(w_n+z_n+\rho_n\zeta)-\alpha(w_n+z_n+\rho_n\zeta)}{\alpha(w_n+z_n+\rho_n\zeta)} \to g'(\zeta). \tag{3.12}$$

Suppose that $g'(\xi_0)=0$. Then by (3.12) and Hurwitz's theorem, there exists a sequence $(\xi_n)_n$, $\xi_n\to\xi_0$ such that (for sufficiently large n) $(f^n)'(w_n+z_n+\rho_n\xi_n) = \alpha(w_n+z_n+\rho_n\xi_n)$. By the given condition, we have

$$f^n(w_n+z_n+\rho_n\xi_n) = \alpha(w_n+z_n+\rho_n\xi_n).$$

Therefore, from (3.2), we have

$$g(\xi_0) = \lim_{n\to\infty}\frac{f^n(w_n+z_n+\rho_n\xi_n)}{\alpha(w_n+z_n+\rho_n\xi_n)} = 1.$$

Thus $g'=0\Rightarrow g=1$. As a result we have (1) $g=0\Rightarrow g'=0$ and (2) $g=1\Leftrightarrow g'=0$. From (1) and (2), one can easily deduce that $g\neq 0$. Also from (2), we see that zeros of $g-1$ are of multiplicities at least 2. Now by the second fundamental theorem, we have

$$T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;g) + S(r,g) \leq \frac{1}{2}\, N(r,1;g) + S(r,g)$$
$$\leq \frac{1}{2}\, T(r,g) + S(r,g),$$

which is a contradiction.

Case 2. Suppose $\rho(f) = +\infty$. Then $\rho(f^n) = +\infty$. Since $\rho(\alpha) < +\infty$, it follows that $\rho(F) = +\infty$. Now by Lemma 2.3, there exist $\{w_n\}_n$ satisfying $w_n \to \infty$, $n \to \infty$, such that for every $N > 0$,

$$F^{\#}(w_n) > |w_n|^N, \tag{3.13}$$

if n is sufficiently large. Then from (3.3) and (3.13), we deduce for every $N > 0$ that

$$\rho_n < M|w_n|^{-N}, \tag{3.14}$$

if n is sufficiently large. If we take $N > s$, then from (3.14) we deduce that $\lim_{n\to\infty} \rho_n |w_n|^s = 0$ and so (3.9) holds. We omit the proof since the proof of Case 2 can be carried out in the line of proof of Sub-case 1.2. □

Acknowledgements. The authors are grateful to the referee for his/her valuable comments and suggestions to-wards the improvement of the paper.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 685–699.

EXISTENCE AND STABILITY RESULTS OF A NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper deals with the stability results for solution of a fractional integro-differential problem with integral conditions. Using the Krasnoselskii's, Banach fixed point theorems, we proof the existence and uniqueness results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system. The results are illustrated by an example.

1. introduction and Formulation of the problem

So far, similar to the simplest case-solution of a system of linear ordinary differential equations, the fractional derivative is not explicitly presented, and therefore it makes sense to consider for $t \in [0,1]$, $0 < \alpha, \beta < 1$, the problem for the system

$$\begin{cases} {}^C D_{0^+}^{\alpha+\beta} u(t) = h(t,u(t)) + I_{0^+}^{\alpha} f(t,u(t)) + \int_0^t K(t,s,u(s))ds, \\ u(0) = b \int\limits_0^\eta u(s)ds, \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where b is a real constant, $0 < \alpha + \beta \leq 1$, ${}^C D_{0^+}^{\alpha+\beta}$ is the Caputo fractional derivative of order $\alpha + \beta$, $I_{0^+}^{\alpha}$ denotes the left sided Riemann-Liouville fractional integral of order α and f, h, K defined as

$$\begin{aligned} f &: [0,1] \times X \to X, \\ h &: [0,1] \times X \to X, \\ K &: [0,1] \times [0,1] \times X \to X, \end{aligned} \tag{1.2}$$

Key words and phrases. Fractional integro-differential equation, existence, stability, nonlocal conditions, fixed point theory, single valued maps.

2010 *Mathematics Subject Classification.* Primary: 34A08, 26A33. Secondary: 34B15, 34K20.

DOI 10.46793/KgJMat2205.685A

Received: February 01, 2020.

Accepted: March 12, 2020.

are an appropriate functions satisfying some conditions which will be stated later. X here is a Banach space. It is also interesting to study solution to fractional integro-differential problem with integral conditions, which will allow a generalized stability. The fractional differential equation

$$D_*^\alpha x(t) = f(t, x(t)), \quad \alpha \in \mathbb{R},\ 0 < \alpha < 1, \tag{1.3}$$

was considered in [4, 5, 8] and results related to the existence and uniqueness for solution, with some analytical properties and useful inequalities, were obtained. Next, it is shown in [9] that, in a real n-dimensional Euclidean space, the local and global solutions exist for the following Cauchy problem

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = f(t, u(t)) + \int_{t_0}^t K(t, s, u(s))ds, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, $f \in C\Big([0,1] \times \mathbb{R}^n, \mathbb{R}^n\Big)$, $K \in C\Big([0,1] \times [0,1] \times \mathbb{R}^n, \mathbb{R}^n\Big)$ and ${}^C D_{0+}^\alpha$ is the Caputo fractional operator.

A class of abstract delayed fractional neutral integro-differential equations was introduced in [11]

$$\begin{cases} D_t^\alpha N(x_t) = AN(x_t) + \int_0^t B(t-s)N(x_s)ds + f(t, x_{\rho(t,x_t)}), \\ x_0 = \phi, \quad x'(0) = 0, \quad \alpha \in (1, 2), \end{cases} \tag{1.5}$$

Using the Leray-Schauder alternative fixed point theorem, the existence results were obtained (for more details, please see [10]). Recently, much attention has been paid to the study of differential equations with fractional derivatives [2, 3], mainly to the questions of the existence and stability for a fractional order differential equation with non-conjugate Riemann-Stieltjes Integro-multipoint boundary conditions.

Note that in [3], the authors introduced and studied a related problem. Precisely the authors studied the existence for the following problem

$$\begin{cases} {}^C D_{0+}^p \{ {}^C D_{0+}^q x(t) + f(t, x(t)) \} = g(t, x(t)), \quad t \in [0, 1], \\ x(0) = \sum_{j=1}^{j=m} \beta_j x(\sigma_j), \\ bx(1) = a \int_0^1 x(s)dH(s) + \sum_{i=1}^{i=n} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \end{cases} \tag{1.6}$$

where

$$0 < \sigma_j < \xi_i < \eta_i < 1, \quad 0 < p, q < 1, \quad \beta_j, \alpha_i \in \mathbb{R}, \quad i = 1, 2, \dots n,\ j = 1, 2, \dots, m.$$

${}^C D_{0+}^p$ is the Caputo fractional derivative of order p, f, g, are given continuous functions. By using a classical tools of fixed point theory, the existence and uniqueness results were obtained. On an arbitrary domain, in [2], the authors study the existence and stability results for a fractional order differential equation with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions by using a new tools on function analysis.

Here we focused our study on the question of existence and uniqueness in section 3. Section 4 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. The paper is ended by an example illustrating our results.

2. Notations and Notions Preliminaries

In the present section, we present some notations, definitions and auxiliary lemmas concerning fractional calculus and fixed point theorems. Let $J=[0,1]$, X is Banach space equipped with the norm $\|\cdot\|$ and $C(J,X)$, $C^n(J,X)$ denotes respectively the Banach spaces of all continuous bounded functions and n times continuously differentiable functions on J. In addition, we define the norm $\|g\|=\max\{|g(t)|:t\in J\}$ for any continuous function $g:J\to X$.

Definition 2.1 ([1,6])**.** Let $\alpha>0$ and $g:J\to X$. The left sided Riemann-Liouville fractional integral of order α of a function g is defined by

$$I_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,\quad t\in J.$$

Definition 2.2 ([1,7])**.** Let $n-1<\alpha<n$, $n\in\mathbb{N}^\star$, and $g\in C^n(J,X)$. The left sided Caputo fractional derivative of order α of a function g is given by

$${}^{C}D_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}g^{(n)}(s)ds=I_{0^+}^{n-\alpha}\frac{d^n}{dt^n}g(t),\quad t\in J,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 ([1,7])**.** *For real numbers $\alpha,\beta>0$ and appropriate function g, we have for all $t\in J$:*

1) $I_{0^+}^{\alpha}I_{0^+}^{\beta}g(t)=I_{0^+}^{\beta}I_{0^+}^{\alpha}g(t)=I_{0^+}^{\alpha+\beta}g(t)$;
2) $I_{0^+}^{\alpha}\,{}^{C}D_{0^+}^{\alpha}g(t)=g(t)-g(0)$, $0<\alpha<1$;
3) ${}^{C}D_{0^+}^{\alpha}I_{0^+}^{\alpha}g(t)=g(t)$.

Lemma 2.2 (Banach fixed point theorem, [12])**.** *Let U be a non-empty complete metric space and $T:U\to U$ is contraction mapping. Then, there exists a unique point $u\in U$ such that $T(u)=u$.*

Lemma 2.3 (Krasnoselskii fixed point theorem, [12])**.** *Let E be bounded, closed and convex subset in a Banach space X. If $T_1,T_2:E\to E$ are two applications satisfying the following conditions:*

1) *$T_1x+T_2y\in E$ for every $x,y\in E$;*
2) *T_1 is a contraction;*
3) *T_2 is compact and continuous.*

Then there exists $z\in E$ such that $T_1z+T_2z=z$.

Before presenting our main results, we need the following auxiliary lemma.

Lemma 2.4. *Let $0<\alpha+\beta<1$ and $b\neq\frac{1}{\eta}$. Assume that h,f and K are three continuous functions. If $u\in C(J,X)$, then u is solution of* (1.1) *if and only if u*

satisfies the integral equation

$$
\begin{aligned}
u(t) =&\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds \\
&+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\bigg] d\tau.
\end{aligned}
\tag{2.1}
$$

Proof. Let $u \in C(J,X)$ be a solution of (1.1). Firstly, we show that u is solution of integral equation (2.1). By Lemma 2.1, we obtain

$$
I_{0^+}^{\alpha+\beta C} D_{0^+}^{\alpha+\beta} u(t) = u(t) - u(0).
\tag{2.2}
$$

In addition, from equation in (1.1) and Definition 2.1, and use the assumption 1) of Lemma 2.1 we have

$$
\begin{aligned}
I_{0^+}^{\alpha+\beta C} D_{0^+}^{\alpha+\beta} u(t) =&I_{0^+}^{\alpha+\beta}\left(h(t,u(t)) + \int_0^t K(t,s,u(s))ds + I_{0^+}^{\alpha} f(t,u(t))\right)ds \\
=&\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds.
\end{aligned}
\tag{2.3}
$$

By substituting (2.3) in (2.2) with nonlocal condition in problem (2.1), we get the following integral equation:

$$
\begin{aligned}
u(t) =& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds + u(0).
\end{aligned}
\tag{2.4}
$$

From integral boundary condition of our problem with using Fubini's thorem and after some computations, we get:

$$
\begin{aligned}
u(0) =&b\int_0^\eta u(s)ds \\
=&b\int_0^\eta \bigg[\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg(h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\bigg) d\tau\bigg] ds + b\eta u(0) \\
=&b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(\tau,u(\tau))d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&+ b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau ds \\
&+ b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma d\tau ds + b\eta u(0) \\
=&b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds h(\tau,u(\tau))d\tau \\
&+ b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau \\
&+ b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma d\tau + b\eta u(0),
\end{aligned}
$$

that is

$$
\begin{aligned}
u(0) =& \frac{b}{1-b\eta} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma \Bigg] d\tau.
\end{aligned} \tag{2.5}
$$

Finally, by substituting (2.5) in (2.4) we find (2.1).

Conversely, from Lemma 2.1 and by applying the operator ${}^C D_{0^+}^{\alpha+\beta}$ on both sides of (2.1), we find

$$
{}^C D_{0^+}^{\alpha+\beta} u(t) = {}^C D_{0^+}^{\alpha+\beta} I_{0^+}^{\alpha+\beta} \left[h(t,u(t)) + \int_0^t K(t,s,u(s))ds + I_{0^+}^{\alpha} f(t,u(t)) \right] + {}^C D_{0^+}^{\alpha+\beta} u(0).
$$

$$
= h(t,u(t)) + I_{0^+}^{\alpha} f(t,u(t)) + \int_0^t K(t,s,u(s))ds, \tag{2.6}
$$

this means that u satisfies the equation in problem (1.1). Furthermore, by substituting t by 0 in integral equation (2.1), we have clearly that the integral boundary condition in (1.1) holds. Therefore, u is solution of problem (1.1), which completes the proof. ☐

3. Existence Results

In order to prove the existence and uniqueness of solution for the problem (1.1) in $C([0,1],X)$, we use two fixed point theorem.

Firstly, we transform the system (1.1) into fixed point problem as $u = Tu$, where $T : C(J,X) \to C(J,X)$ is an operator defined by following

$$
\begin{aligned}
Tu(t) =& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau \Bigg] ds \\
&+ \frac{b}{1-b\eta} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma
\end{aligned}
$$

$$+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Bigg]d\tau. \tag{3.1}$$

In order to simplify the computations, we offer the following notations

$$\begin{aligned}\Delta =&\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\\ &+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+2)}\\ &+\frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\end{aligned} \tag{3.2}$$

and

$$\Delta_1=\frac{|b|}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}+\frac{\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]. \tag{3.3}$$

3.1. **Existence result by Krasnoselskii's fixed point.**

Theorem 3.1. *Let $h,f:[0,1]\times X\to X$ and $K:[0,1]\times[0,1]\times X\longrightarrow X$ be continuous functions satisfying the following.*

$(H1)$ *The inequalities*

$$\begin{aligned}\|h(t,u(t))-h(t,v(t))\|\leq& L_1\|u(t)-v(t)\|, \quad t\in[0,1],\ u,v\in X,\\ \|f(t,u(t))-f(t,v(t))\|\leq& L_2\|u(t)-v(t)\|, \quad t\in[0,1],\ u,v\in X,\\ \|K(t,s,u(s))-K(t,s,v(s))\|\leq& L_3\|u(s)-v(s)\|, \quad (t,s)\in G,\ u,v\in X,\end{aligned}$$

hold, where $L_1,L_2,L_3\geq 0$, with $L=\max\{L_1,L_2,L_3\}$ and $G=\{(t,s):0\leq s\leq t\leq 1\}$.

$(H2)$ *There exist three functions $\mu_1,\mu_2,\mu_3\in L^\infty([0,1],\mathbb{R}^+)$ such that*

$$\begin{aligned}\|h(t,u(t))\|\leq&\mu_1(t)\|u(t)\|, \quad t\in[0,1],\ u\in X,\\ \|f(t,u(t))\|\leq&\mu_2(t)\|u(t)\|, \quad t\in[0,1],\ u\in X,\\ \|K(t,s,u(s))\|\leq&\mu_3(t)\|u(s)\|, \quad (t,s)\in G,\ u\in X.\end{aligned}$$

If $\Delta\leq 1$ and $L\Delta_1\leq 1$, then the problem (1.1) *has at least one solution on* $[0,1]$.

Proof. For any function $u\in C(J,X)$ we define the norm

$$\|u\|_1=\max\{e^{-t}\|u(t)\|:t\in[0,1]\},$$

and consider the closed ball

$$B_r=\{u\in C(J,X):\|u\|_1\leq r\}.$$

Next, let us define the operators T_1,T_2 on B_r as follows

$$T_1u(t)=\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[h(s,u(s))+\int_0^s K(s,\tau,u(\tau))d\tau$$

$$+\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\Bigg]ds \tag{3.4}$$

and

$$\begin{aligned} T_2u(t) =&\frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[h(\tau,u(\tau))+\int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\ &+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Bigg]d\tau. \end{aligned} \tag{3.5}$$

For $u, v \in B_r$, $t \in [0,1]$ and by the assumption $(H2)$, we find

$$\begin{aligned} \|T_1u(t)+T_2v(t)\| \le &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\|h(s,u(s))\|+\int_0^s \|K(s,\tau,u(\tau))\|d\tau \\ &+\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau))\|d\tau\Bigg]ds \\ &+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|h(\tau,v(\tau))\|+\int_0^\tau \|K(\tau,\sigma,v(\sigma))\|d\sigma \\ &+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,v(\sigma))\|d\sigma\Bigg]d\tau \\ \le &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\mu_1(s)\|u(s)\|+\int_0^s \mu_3(s)\|u(\tau)\|d\tau \\ &+\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mu_2(\tau)\|u(\tau)\|d\tau\Bigg]ds \\ &+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\mu_1(\tau)\|v(\tau)\|+\int_0^\tau \mu_3(\tau)\|v(\sigma)\|d\sigma \\ &+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\mu_2(\sigma)\|v(\sigma)\|d\sigma\Bigg]d\tau \\ \le &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\|\mu_1\|_{L^\infty}\|u\|_1 e^s+\|\mu_3\|_{L^\infty}\|u\|_1(e^s-1) \\ &+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}e^\tau d\tau\Bigg]ds \\ &+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|\mu_1\|_{L^\infty}\|v\|_1 e^\tau+\|\mu_3\|_{L^\infty}\|v\|_1(e^\tau-1) \\ &+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}e^\sigma d\sigma\Bigg]d\tau. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\|T_1u+T_2v\|_1 \leq &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\right.\\
&\left.+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t}d\tau\right]ds\\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[\|\mu_1\|_{L^\infty}\|v\|_1\frac{e^\tau}{e^t}+\|\mu_3\|_{L^\infty}\|v\|_1\frac{(e^\tau-1)}{e^t}\right.\\
&\left.+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau\frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\sigma}{e^t}d\sigma\right]d\tau.\\
\leq &r\left[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta+1}s^\alpha ds\right.\\
&+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+1)}\\
&\left.+\frac{|b|\|\mu_2\|_{L^\infty}}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\int_0^\eta(\eta-\tau)^{\alpha+\beta}\tau^\alpha d\tau\right]\\
= &r\left[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\right.\\
&+\frac{|b|}{|1-b\eta|}\left(\frac{\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\right.\\
&\left.\left.+\frac{\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right)\right]\\
= &r\Delta\leq r. \qquad (3.6)
\end{aligned}
$$

This implies that $(T_1u+T_2v)\in B_r$. Here we used the computations

$$
\begin{aligned}
\int_0^1(1-s)^{\alpha+\beta}s^\alpha ds =&\beta(\alpha+1,\alpha+\beta),\\
\int_0^\eta(\eta-\tau)^{\alpha+\beta}\tau^\alpha ds =&\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1),
\end{aligned}
$$

and the estimations: $\frac{e^s}{e^t}\leq 1$, $\frac{e^\tau}{e^t}\leq 1$, $\frac{e^\sigma}{e^t}\leq 1$. Now, we establish that T_2 is a contraction mapping. For $u,v\in X$ and $t\in[0,1]$, we have

$$
\begin{aligned}
\|T_2u(t)-T_2v(t)\|\leq &\frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left[\|h(\tau,u(\tau))-h(\tau,v(\tau))\|\right.\\
&+\int_0^\tau\|K(\tau,\sigma,u(\sigma))-K(\tau,\sigma,v(\sigma))\|d\sigma\\
&\left.+\int_0^\tau\frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,u(\sigma))-f(\sigma,v(\sigma))\|d\sigma\right]d\tau
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L_1\|u-v\|_1 e^\tau + \int_0^\tau L_3\|u-v\|_1 e^\sigma d\sigma \\
& + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} L_2\|u-v\|_1 e^\sigma d\sigma\Big] d\tau \\
\leq & \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L\|u-v\|_1 e^\tau + L\|u-v\|_1 (e^\tau-1) \\
& + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} L\|u-v\|_1 e^\sigma d\sigma\Big] d\tau.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|T_2u - T_2v\|_1 \leq & \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L\|u-v\|_1 \frac{e^\tau}{e^t} + L\|u-v\|_1 \frac{(e^\tau-1)}{e^t} \\
& + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} L\|u-v\|_1 \frac{e^\sigma}{e^t} d\sigma\Big] d\tau \\
\leq & \frac{|b|L}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]\|u-v\|_1.
\end{aligned}
$$

Then since $L\Delta_1 \leq 1$, T_2 is a contraction mapping. The continuity of the functions h, f and K implies that the operator T_1 is continuous. Also, $T_1B_r \subset B_r$, for each $u \in B_r$,, i.e., T_1 is uniformly bounded on B_r as

$$
\begin{aligned}
\|(T_1u)(t)\| \leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|h(s,u(s))\| + \int_0^s \|K(s,\tau,u(\tau))\| d\tau \\
& + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau))\| d\tau\Big] ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|T_1u\|_1 \leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1 \frac{e^s}{e^t} + \|\mu_3\|_{L^\infty}\|u\|_1 \frac{(e^s-1)}{e^t} \\
& + \|\mu_2\|_{L^\infty}\|u\|_1 \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t} d\tau\Big] ds \\
\leq & r\left[\frac{\|\mu_1\|_{L^\infty} + \|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)} + \frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\right] \\
\leq & r\Delta. \\
\leq & r. \qquad (3.7)
\end{aligned}
$$

Finally, we will show that $(\overline{T_1}B_r)$ is equi-continuous. For this end, we define

$$
\begin{aligned}
\overline{h} &= \sup_{(s,u)\in[0,1]\times B_r}\|h(s,u)\|, \\
\overline{f} &= \sup_{(s,u)\in[0,1]\times B_r}\|f(s,u)\|,
\end{aligned}
$$

$$\overline{K} = \sup_{(s,\tau,u)\in G\times B_r} \int_0^s \|K(t,s,u)\| d\tau.$$

Let for any $u \in B_r$ and for each $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$, we have:

$$\begin{aligned}
&\|(T_1u)(t_2) - (T_1u)(t_1)\| \\
\leq &\frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} \Bigg[\|h(s,u(s))\| + \int_0^s \|K(s,\tau,u(\tau))\| d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|f(\tau,u(\tau))\| d\tau \Bigg] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{\alpha+\beta-1} - (t_2-s)^{\alpha+\beta-1} \right] \Bigg[\|h(s,u(s))\| \\
&+ \int_0^s \|K(s,\tau,u(\tau))\| d\tau + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|f(\tau,u(\tau))\| d\tau \Bigg] ds \\
\leq &\frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} \left[\overline{h} + \overline{K} + \frac{\overline{f}}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} d\tau \right] ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left[(t_1-s)^{\alpha+\beta-1} - (t_2-s)^{\alpha+\beta-1} \right] \left[\overline{h} + \overline{K} + \frac{\overline{f}}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} d\tau \right] ds \\
\leq &\frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} \left[\overline{h} + \overline{K} + \frac{\overline{f}}{\Gamma(\alpha+1)} \right] ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left[(t_1-s)^{\alpha+\beta-1} - (t_2-s)^{\alpha+\beta-1} \right] \left[\overline{h} + \overline{K} + \frac{\overline{f}}{\Gamma(\alpha+1)} \right] ds \\
= &\frac{1}{\Gamma(\alpha+\beta+1)} \left[\overline{h} + \overline{K} + \frac{\overline{f}}{\Gamma(\alpha+1)} \right] \left[2(t_2-t_1)^{\alpha+\beta} + t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right].
\end{aligned}$$

The RHS of the last inequality is independent of u and tends to zero when $|t_2 - t_1| \to 0$, this means that $|T_1u(t_2) - T_1u(t_1)| \to 0$, which implies that $(\overline{T_1B_r})$ is equi-continuous, then T_1 is relatively compact on B_r. Hence by Arzela-Ascoli theorem, T_1 is compact on B_r. Now, all hypothesis of Theorem 3.2 hold, therefore the operator $T_1 + T_2$ has a fixed point on B_r. So the problem (1.1) has at least one solution on $[0,1]$. This proves the theorem. □

3.2. **Existence and uniqueness result.**

Theorem 3.2. *Assume that* $(H1)$ *holds. If* $L\Delta < 1$, *then the BVP* (1.1) *has a unique solution on* $[0,1]$.

Proof. Define $M = \max\{M_1, M_2, M_3\}$, where M_1, M_2, M_3 are positive numbers such that:

$$M_1 = \sup_{t\in[0,1]} \|h(t,0)\|, \quad M_2 = \sup_{t\in[0,1]} \|f(t,0)\|, \quad M_3 = \sup_{(t,s)\in G} \|K(t,s,0)\|.$$

We fix $r \geq \frac{M\Delta}{1-L\Delta}$ and we consider

$$D_r = \{x \in C([0,1], X) : \|u\| \leq r\}.$$

Then, in view of the assumption $(H1)$, we have

$$\begin{aligned}\|h(t,u(t))\| =& \|h(t,u(t)) - h(t,0) + h(t,0)\| \leq \|h(t,u(t)) - h(t,0)\| + \|h(t,0)\| \\ \leq& L_1\|u\| + M_1, \\ \|f(t,u(t))\| \leq& L_2\|u\| + M_2,\end{aligned}$$

and

$$\|K(t,s,u(s))\| \leq L_3\|u\| + M_3.$$

First step. We show that $TD_r \subset D_r$. For each $t \in [0,1]$ and for any $u \in D_r$

$$\begin{aligned}\|(Tu)(t)\| \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\|h(s,u(s))\| + \int_0^s \|K(s,\tau,u(\tau))\|d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau))\|d\tau\Bigg]ds \\ &+ \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|h(\tau,v(\tau))\| + \int_0^\tau \|K(\tau,\sigma,v(\sigma))\|d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,v(\sigma))\|d\sigma\Bigg]d\tau \\ \leq& (Lr+M)\Delta \\ \leq& r.\end{aligned}$$

Hence, $TD_r \subset D_r$.

Second step. We shall show that $T : D_r \to D_r$ is a contraction. From the assumption $(H1)$ we have for any $u, v \in D_r$ and for each $t \in [0,1]$

$$\begin{aligned}&\|(Tu)(t) - (Tv)(t)\| \\ &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\|h(s,u(s)) - h(s,v(s))\| \qquad (3.8)\\ &+ \int_0^s \|K(s,\tau,u(\tau)) - K(s,\tau,v(\tau))\|d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau)) - f(\tau,v(\tau))\|d\tau\Bigg]ds \\ &+ \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|h(\tau,u(\tau)) - h(\tau,v(\tau))\| \qquad (3.9)\\ &+ \int_0^\tau \|K(\tau,\sigma,u(\sigma)) - K(\tau,\sigma,v(\sigma))\|d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))\|d\sigma\Bigg]d\tau\end{aligned}$$

$$\leq L\Delta\|u-v\|.$$

Since $L\Delta < 1$, it follows that T is a contraction. All assumptions of Lemma 2.2 are satisfied, then there exists $u \in C(J, X)$ such that $Tu = u$, which is the unique solution of the problem (1.1) in $C(J, X)$. □

4. Generalized Ulam Stabilities

The aim is to discus the Ulam stability for (1.1), by using the integration

$$\begin{aligned} v(t) = & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Bigg[h(s, v(s)) + \int_0^s K(s, \tau, v(\tau)) d\tau \\ & + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, v(\tau)) d\tau \Bigg] ds \\ & + \frac{b}{1-b\eta} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Bigg[h(\tau, v(\tau)) + \int_0^\tau K(\tau, \sigma, v(\sigma)) d\sigma \\ & + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma, v(\sigma)) d\sigma \Bigg] d\tau. \end{aligned} \tag{4.1}$$

Here $v \in C([0,1], X)$ possess a fractional derivative of order $\alpha+\beta$, where $0 < \alpha+\beta < 1$ and

$$f, h : [0,1] \times X \to X,$$

and

$$K : [0,1] \times [0,1] \times X \to X,$$

are continuous functions. Then we define the nonlinear continuous operator

$$P : C([0,1], X) \to C([0,1], X),$$

as follows

$$Pv(t) = {}^C D^{\alpha+\beta} v(t) - I_{0+}^\alpha f(t, v(t)) - h(t, v(t)) - \int_0^t K(t, s, v(s)) ds.$$

Definition 4.1. For each $\epsilon > 0$ and for each solution v of (1.1), such that

$$\|Pv\| \leq \epsilon, \tag{4.2}$$

the problem (1.1), is said to be Ulam-Hyers stable if we can find a positive real number ν and a solution $u \in C([0,1], X)$ of (1.1), satisfying the inequality

$$\|u - v\| \leq \nu \epsilon^*, \tag{4.3}$$

where ϵ^* is a positive real number depending on ϵ.

Definition 4.2. Let $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each solution v of (1.1), we can find a solution $u \in C([0,1], X)$ of (1.1) such that

$$\|u(t) - v(t)\| \leq m(\epsilon), \quad t \in [0,1]. \tag{4.4}$$

Then the problem (1.1), is said to be generalized Ulam-Hyers stable.

Definition 4.3. For each $\epsilon > 0$ and for each solution v of (1.1), the problem (1.1) is called Ulam-Hyers-Rassias stable with respect to $\theta \in C([0,1], \mathbb{R}^+)$ if

$$\|Pv(t)\| \leq \epsilon\theta(t), \quad t \in [0,1], \tag{4.5}$$

and there exist a real number $\nu > 0$ and a solution $v \in C([0,1], X)$ of (1.1) such that

$$\|u(t) - v(t\| \leq \nu\epsilon_*\theta(t), \quad t \in [0,1], \tag{4.6}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Under assumption* $(H1)$ *in Theorem* 3.1*, with* $L\Delta < 1$*. The problem* (1.1) *is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C([0,1], X)$ be a solution of (1.1), satisfying (2.1) in the sense of Theorem 3.2. Let v be any solution satisfying (4.2). Lemma 2.4 implies the equivalence between the operators P and $T - Id$ (where Id is the identity operator) for every solution $v \in C([0,1], X)$ of (1.1) satisfying $L\Delta < 1$. Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned}\|v(t) - u(t\| &= \|v(t) - Tv(t) + Tv(t) - u(t)\| = \|v(t) - Tv(t) + Tv(t) - Tu(t)\| \\ &\leq \|Tv(t) - Tu(t)\| + |Tv(t) - v(t)\| \leq L\Delta\|u - v\| + \epsilon,\end{aligned}$$

because $L\Delta < 1$ and $\epsilon > 0$, we find

$$\|u - v\| \leq \frac{\epsilon}{1 - L\Delta}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-L\Delta}$ and $\nu = 1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m(\epsilon) = \frac{\epsilon}{1-L\Delta}$. □

Theorem 4.2. *Assume that* $(H1)$ *holds with* $L < \Delta^{-1}$*, and there exists a function* $\theta \in C([0,1], \mathbb{R}^+)$ *satisfying the condition* (4.5)*. Then the problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* θ*.*

Proof. We have from the proof of Theorem 4.1,

$$\|u(t) - v(t\| \leq \epsilon_*\theta(t), \quad t \in [0,1],$$

where $\epsilon_* = \frac{\epsilon}{1-L\Delta}$. This completes the proof. □

Example 4.1. Consider the following fractional integro-differential problem

$$\begin{cases} {}^C D_{0^+}^{\frac{2}{5}} u(t) = h(t, u(t)) + I_{0^+}^{\alpha} f(t, u(t)) + \int_0^t K(t, s, u(s))ds, & t \in [0,1], \\ u(0) = 3\int_0^{\frac{1}{5}} u(s)ds, & 0 < \eta < 1, \end{cases} \tag{4.7}$$

where $\alpha = \beta = \frac{1}{5}$, $b = 3$, $\eta = \frac{1}{5}$. By the above, we find that $\Delta = 0.4602$, $\Delta_1 = 4.3755$. To illustrate our results in Theorem 3.1 and Theorem 4.1, we take for $u, v \in X = \mathbb{R}^+$ and $t \in [0,1]$ the following continuous functions:

$$h(t, u(t)) = \frac{(2-t)u(t)}{60}, \quad f(t, u(t)) = \frac{3-t^2}{72}u(t), \quad K(t, s, u(s)) = \frac{e^{-(s+t)}}{64}u(s).$$

Note that we can find

$$L_1 = \frac{1}{20}, \quad L_2 = \frac{1}{18}, \quad L_3 = \frac{1}{64},$$

Moreover,

$$\mu_1(t) = \frac{2-t}{60}, \quad \mu_2(t) = \frac{3-t^2}{72}, \quad \mu_3(t) = \frac{e^{-t}}{64}.$$

Obviously,

$$\|\mu_1\|_{L_\infty} = \frac{1}{30}, \quad \|\mu_2\|_{L_\infty} = \frac{1}{24}, \quad \|\mu_3\|_{L_\infty} = \frac{1}{64},$$

and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{18}.$$

Then, we get

$$L\Delta_1 = 0.2431 < 1, \quad \Delta = 0.3229 < 1.$$

All assumptions of Theorem 3.1 are satisfied. Hence, there exists at least one solution for the problem (4.7) on $[0, 1]$.

By take the same functions, we result the assumption

$$L\Delta = 0.0179 < 1,$$

then there exists a unique solution of (4.7) on $[0, 1]$.

In order to illustrate our stability result, we consider the same above example:

$$L = \frac{1}{18}, \quad L\Delta_1 = 0.2431$$

This implies that the system (4.7) is Ulam-Hyers stable, then it is generalized Ulam-Hyers stable. It is Ulam-Hyers-Rassias stable if there exists a continuous and positive function.

Acknowledgements. The authors would like to express their gratitude to the anonymous referees for helpful and very careful reading.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 701–709.

LOWER BOUNDS FOR ENERGY OF MATRICES AND ENERGY OF REGULAR GRAPHS

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Abstract. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. The energy of A, denoted by $\mathcal{E}(A)$, is defined as $|\lambda_1| + \cdots + |\lambda_n|$. We prove that if A is non-zero and $|\lambda_1| \geq \cdots \geq |\lambda_n|$, then

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1\leq i,j\leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}. \tag{0.1}$$

In particular, we show that $\Psi(A)\mathcal{E}(A) \geq \sum_{1\leq i,j\leq n} a_{ij}^2$, where $\Psi(A)$ is the maximum value of the sequence $\sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \dots, \sum_{j=1}^n |a_{nj}|$. The energy of a simple graph G, denoted by $\mathcal{E}(G)$, is defined as the energy of its adjacency matrix. As an application of inequality (0.1) we show that if G is a t- regular graph ($t \neq 0$) of order n with no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \geq \frac{2tn}{t+1}$ and the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K_{t+1}^\star$.

1. Introduction

Throughout this paper the matrices are complex and the graphs are simple (that is graphs are finite and undirected, without loops and multiple edges). The *conjugate transpose* of a complex matrix A is denoted by A^*. We recall that a *Hermitian* matrix (or *self-adjoint* matrix) is a complex square matrix that is equal to its own *conjugate transpose*. If A is a real matrix, then A is Hermitian if and only if A is symmetric. It is well known that the eigenvalues of Hermitian matrices (in particular, the eigenvalues of real symmetric matrices) are real. A complex square matrix A is called *normal* if it commutes with its conjugate transpose, that is $AA^* = A^*A$. For example, every

Key words and phrases. Energy of matrices, energy of graphs, energy of regular graphs.
2010 *Mathematics Subject Classification.* Primary: 05C31. Secondary: 05C50, 15A18.
DOI 10.46793/KgJMat2205.701O
Received: December 02, 2019.
Accepted: March 13, 2020.

real symmetric matrix is normal. For every complex square matrix A, the *trace* of A, denoted by $tr(A)$, is defined to be the sum of the entries on the main diagonal of A. The *energy* of a square complex matrix A, denoted by $\mathcal{E}(A)$, is defined as the sum of the absolute values of its eigenvalues. In other words, if A is an $n \times n$ complex matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\mathcal{E}(A) = |\lambda_1| + \cdots + |\lambda_n|. \tag{1.1}$$

Nikiforov [9] defined the energy of any complex matrix A by considering the *singular values*. This definition of energy of matrices coincides with the previous definition of energy of matrices if and only if the matrix is normal [1].

Let $G = (V(G), E(G))$ be a simple graph. The *order* of G denotes the number of vertices of G. For two vertices u and v by $e = uv$ we mean the edge e between u and v. For a vertex v of G, the *degree* of v is the number of edges incident with v. A k–regular graph is a graph such that every vertex of that has degree k. Let $B \subseteq V(G)$ ($B \subseteq E(G)$). By $G \setminus B$ we mean the graph that obtained from G by removing the vertices of B (the edges of B). The *complement* of G, denoted by $\overline{G}$, is the simple graph with vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. For two disjoint graphs G_1 and G_2, the *disjoint union* of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. A *matching* in G is a set of edges of G without common vertices. A *perfect matching* of G is a matching in which every vertex of G is incident to exactly one edge of the matching. The *edgeless graph* (*empty graph*), the *complete graph* and the *cycle* of order n, are denoted by $\overline{K_n}$, K_n and C_n, respectively. The *complete bipartite graph* with part sizes m and n is denoted by $K_{m,n}$. Let $t \geq 0$ be an integer and M be a perfect matching of $K_{t+1,t+1}$. By $K_{t+1}^{\star}$ we mean the t–regular graph $K_{t+1,t+1} \setminus M$. The graph $K_{t+1}^{\star}$ is called the *crown* graph of order $2t+2$. For example $K_1^{\star} = 2K_1$, $K_2^{\star} = 2K_2$ and $K_3^{\star} = C_6$.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix such that the (i, j)-entry is 1 if v_i and v_j are adjacent, and otherwise is 0. Since $A(G)$ is symmetric, all of the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. By *Spec*(G) we mean the multiset of all eigenvalues of G. The *energy* of G, denoted by $\mathcal{E}(G)$, is defined as the energy of the adjacency matrix of G. In other words, the energy of G is the sum of the absolute values of all eigenvalues of G. More precisely, $\mathcal{E}(G) = |\lambda_1| + \cdots + |\lambda_n|$, where $\mathrm{Spec}(G) = \{\lambda_1, \ldots, \lambda_n\}$. The energy of graphs was defined by Ivan Gutman in 1978. For example, since the eigenvalues of the complete graph K_n are $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$), so $\mathcal{E}(K_n) = 2n-2$. See [4,5] for more details. Many papers are devoted to studying the properties of the spectra of adjacency matrix, in particular studying the energy of graphs. For instance see [1–20] and the references therein. There are many other matrices associated to graphs such as *Laplacian matrix*, *signless Laplacian matrix* [20] and *distance matrix* [18]. For instance the Laplacian matrix of a graph G, denoted by $L(G)$, is defined as $D(G) - A(G)$, where $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of G.

We note that the definition (1.1) for energy of matrices is significant for square real symmetric matrices whose trace are equal to zero. In other words, this definition maybe not notable for matrices with non-zero trace. For instance consider the Laplacian matrix of a graph G of order n. It is well known that the eigenvalues of Laplacian matrix of graphs are real and non-negative. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of the Laplacian matrix of G (in fact $\mu_n = 0$). By definition (1.1), the energy of $L(G)$ is $\mathcal{E}(L(G)) = \mu_1 + \cdots + \mu_n = tr(L(G)) = 2m$, where m is the number of edges of G. We remark that in [7] the authors define the Laplacian energy of graphs in another way.

In this paper first we obtain a new lower bound for energy of real symmetric matrices. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. If $A = 0$, then clearly $\mathcal{E}(A) = 0$. We show that if $A \neq 0$, then

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1\leq i,j\leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}, \tag{1.2}$$

where $|\lambda_1| \geq \cdots \geq |\lambda_n|$. By studying the lower bound (1.2) we obtain a simple lower bound for energy of matrices. Let $\Psi(A)$ be the maximum value of the sequence of real numbers $\sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \ldots, \sum_{j=1}^n |a_{nj}|$. In other words, $\Psi(A)$ is the maximum value of the sum of the absolute values of the entries of rows of A. We prove that if $A \neq 0$, then

$$\mathcal{E}(A) \geq \frac{\sum_{1\leq i,j\leq n} a_{ij}^2}{\Psi(A)}.$$

Finally we study the energy of regular graphs. Let G be a t–regular graph of order n and $t \neq 0$. In [6] (see also [4]) it was shown that $\mathcal{E}(G) \geq n$. By applying the lower bound (1.2) we improve this result and prove that if G has no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \geq \frac{2tn}{t+1}$. In addition we show that the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K_{t+1}^\star$.

2. Energy of Matrices

In this section we obtain some lower bounds for the energy of matrices. At first similar to Lemma 1 of [19] we prove the inequality (1.2).

Theorem 2.1. *Let $n \geq 2$ be an integer and $A = [a_{ij}] \neq 0$ be an $n \times n$ real symmetric matrix. Assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then*

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1\leq i,j\leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}.$$

Moreover, the equality holds if and only if for some $r \in \{1, \ldots, n\}$, $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$.

Proof. We note that for every $j \in \{1,\dots,n\}$, $|\lambda_1| \geq |\lambda_j| \geq |\lambda_n|$. Thus, for $j = 1,\dots,n$, $(|\lambda_1| - |\lambda_j|)(|\lambda_j| - |\lambda_n|) \geq 0$. In addition, the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. On the other hand

$$|\lambda_j|(|\lambda_1| + |\lambda_n|) - (|\lambda_j|^2 + |\lambda_1||\lambda_n|) = (|\lambda_1| - |\lambda_j|)(|\lambda_j| - |\lambda_n|).$$

Hence, $|\lambda_j|(|\lambda_1| + |\lambda_n|) - (|\lambda_j|^2 + |\lambda_1||\lambda_n|) \geq 0$ and the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. So, for every $j \in \{1,\dots,n\}$

$$|\lambda_j|(|\lambda_1| + |\lambda_n|) \geq |\lambda_j|^2 + |\lambda_1||\lambda_n|, \tag{2.1}$$

and the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. Now by summing the sides of (2.1) for $j = 1,\dots,n$, we find that

$$(|\lambda_1| + |\lambda_n|)\Big(|\lambda_1| + \cdots + |\lambda_n|\Big) \geq |\lambda_1|^2 + \cdots + |\lambda_n|^2 + n|\lambda_1||\lambda_n|, \tag{2.2}$$

and the equality holds if and only if for some $r \in \{1,\dots,n\}$, $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$. On the other hand $\mathcal{E}(A) = |\lambda_1| + \cdots + |\lambda_n|$ and

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 = \lambda_1^2 + \cdots + \lambda_n^2 = tr(A^2) = \sum_{1\leq i,j\leq n} a_{ij}^2. \tag{2.3}$$

We note that because of A is symmetric, we have $tr(A^2) = \sum_{1\leq i,j\leq n} a_{ij}^2$. Since $A \neq 0$, clearly $\sum_{1\leq i,j\leq n} a_{ij}^2 \neq 0$. Thus, by (2.3), A has at least one non-zero eigenvalue. Thus, $|\lambda_1| > 0$. So, $|\lambda_1| + |\lambda_n| \neq 0$. Now by dividing the sides of (2.2) by $|\lambda_1| + |\lambda_n|$ and using (2.3) the result follows. □

Remark 2.1. We note that in Theorem 2.1 the equality holds for some family of matrices. For example for diagonal matrices such as $\mathrm{diag}(a,\dots,a,b,\dots,b)$, where a and b are real. Since the eigenvalues of the complete bipartite graph $K_{p,q}$ are $-\sqrt{pq}$ (with multiplicity 1), 0 (with multiplicity $p+q-2$) and $\sqrt{pq}$ (with multiplicity 1), the adjacency matrix of $K_{p,q}$ also satisfying in the equality of Theorem 2.1.

We are interested in to obtain a suitable estimation for the lower bound of Theorem 2.1 in terms of the entries of the matrix. First we prove the following lemma.

Lemma 2.1. *Let a and b be some positive real numbers. Let α, β, x and y be some non-negative real numbers such that $\beta \geq y \geq \sqrt{\frac{a}{b}} \geq x \geq \alpha$. Then*

$$\frac{a + bxy}{x + y} \geq \frac{a + b\alpha\beta}{\alpha + \beta},$$

and the equality holds if and only if $x = \alpha = \sqrt{\frac{a}{b}}$ or $x = \beta = \sqrt{\frac{a}{b}}$ or $y = \beta = \sqrt{\frac{a}{b}}$ or $x = \alpha$ and $y = \beta$.

Proof. Let d be a positive real number and $f_d(t) = \frac{a+bdt}{d+t}$ be the one-variable function on t, where $t \geq 0$. So the derivative of $f_d(t)$ with respect to t is

$$f_d'(t) = \frac{bd^2 - a}{(d+t)^2}.$$

This shows that if $d > \sqrt{\frac{a}{b}}$, then $f_d(t)$ is strictly increasing on the interval $[0, \infty)$ and if $d < \sqrt{\frac{a}{b}}$, then $f_d(t)$ is strictly decreasing on the interval $[0, \infty)$. We note that if $d = \sqrt{\frac{a}{b}}$, then for every $t \geq 0$, $f_d(t) = \frac{a}{d} = \sqrt{ab}$.

Since $y \leq \beta$ and $f_x(t)$ is strictly decreasing on the interval $[0, \infty)$, if $x < \sqrt{\frac{a}{b}}$,

$$f_x(y) \geq f_x(\beta) \quad (\text{if} \beta > y \text{ and } x \neq \sqrt{\frac{a}{b}}, \text{ then } f_x(y) > f_x(\beta)). \tag{2.4}$$

On the other hand, since $x \geq \alpha$ and $f_\beta(t)$ is strictly increasing on the interval $[0, \infty)$, if $\beta > \sqrt{\frac{a}{b}}$,

$$f_\beta(x) \geq f_\beta(\alpha) \quad (\text{if } x > \alpha \text{ and } \beta \neq \sqrt{\frac{a}{b}}, \text{ then } f_\beta(x) > f_\beta(\alpha)). \tag{2.5}$$

Since $f_x(\beta) = f_\beta(x)$, (2.4) and (2.5) show that $f_x(y) \geq f_\beta(\alpha)$. In other words, we obtain that $\frac{a+bxy}{x+y} \geq \frac{a+b\alpha\beta}{\alpha+\beta}$.

Now we consider the equality. Assume that $\frac{a+bxy}{x+y} = \frac{a+b\alpha\beta}{\alpha+\beta}$. So, $f_x(y) = f_\beta(\alpha)$. Hence, by (2.4) and (2.5) we find that $f_x(y) = f_x(\beta)$ and $f_\beta(x) = f_\beta(\alpha)$. Using (2.4) and (2.5) one can easily obtain the result. □

Let $A = [a_{ij}]$ be a complex $n \times n$ matrix, where $n \geq 1$ be an integer. As we mentioned before, $\Psi(A)$ denotes the maximum value of the sequence of real numbers $\sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \ldots, \sum_{j=1}^{n} |a_{nj}|$. We need the following result.

Theorem 2.2 ([8]). *Let A be a complex square matrix and λ be an eigenvalue of A. Then $|\lambda| \leq \Psi(A)$.*

Now we obtain a lower bound for the energy of matrices in terms of their entries.

Theorem 2.3. *Let $n \geq 2$ be an integer and $A = [a_{ij}] \neq 0$ be an $n \times n$ real symmetric matrix. Then*

$$\mathcal{E}(A) \geq \frac{\sum_{1 \leq i,j \leq n} a_{ij}^2}{\Psi(A)}.$$

Proof. Assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. We note that the eigenvalues of A are real. Since A is symmetric, $tr(A^2) = \sum_{1 \leq i,j \leq n} a_{ij}^2$. On the other hand $\lambda_1^2 + \cdots + \lambda_n^2 = tr(A^2)$ and

$$n|\lambda_n|^2 \leq |\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq n|\lambda_1|^2. \tag{2.6}$$

Therefore,

$$n|\lambda_n|^2 \leq \sum_{1 \leq i,j \leq n} a_{ij}^2 \leq n|\lambda_1|^2. \tag{2.7}$$

Hence,

$$|\lambda_n| \leq \sqrt{\frac{\sum_{1 \leq i,j \leq n} a_{ij}^2}{n}} \leq |\lambda_1|. \tag{2.8}$$

Let $\alpha = 0$, $\beta = \Psi(A)$, $a = \sum_{1\le i,j\le n} a_{ij}^2$, $b = n$, $x = |\lambda_n|$ and $y = |\lambda_1|$. We note that since $A \neq 0$, $a > 0$. Using Theorem 2.2 and (2.8) we deduce that

$$\beta \ge y \ge \sqrt{\frac{a}{b}} \ge x \ge \alpha.$$

Thus by applying Theorem 2.1 and Lemma 2.1 we obtain that

$$\mathcal{E}(A) \ge \frac{a+bxy}{x+y} \ge \frac{a+b\alpha\beta}{\alpha+\beta} = \frac{a}{\beta} = \frac{\sum_{1\le i,j\le n} a_{ij}^2}{\Psi(A)}.$$

This completes the proof. □

3. Energy of Regular Graphs

In [6] it was proved that if G is a t-regular graph of order n where $t \neq 0$, then $\mathcal{E}(G) \ge n$. In this section by applying Theorem 2.1 we improve this result and show that if G has no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \ge \frac{2tn}{t+1}$. Two examples of this kind of regular graphs are the cycle C_6 (with spectrum $\{2, 1, 1, -1, -1, -2\}$) and the *Petersen* graph (with spectrum $\{3, 1, 1, 1, 1, 1, -2, -2, -2, -2\}$). First we recall some results.

Theorem 3.1 ([2])**.** *Let G be a graph and $\rho(G)$ be the largest eigenvalue (the spectral radius) of G. Then the following hold:*

(*i*) *if G is connected, then the multiplicity of $\rho(G)$ is one;*
(*ii*) *for every eigenvalue λ of G, $|\lambda| \le \rho(G)$.*

Theorem 3.2. [2] *Let G be a graph. Then the following hold:*

(*i*) *G is bipartite if and only if for every eigenvalue λ of G, also $-\lambda$ is an eigenvalue of G, with the same multiplicity.*
(*ii*) *If G is connected with largest eigenvalue θ, then G is bipartite if and only if $-\theta$ is an eigenvalue of G.*

Lemma 3.1 ([19])**.** *Let H be a connected t-regular graph where $t \ge 2$. Assume that*

$$\mathrm{Spec}(H) = \{t, \underbrace{1, \dots, 1}_{b}, \underbrace{-1, \dots, -1}_{c}\},$$

where b and c are some non-negative integers. Then H is the complete graph K_{t+1}.

Lemma 3.2 ([19])**.** *Let H be a connected bipartite t-regular graph where $t \ge 2$. Assume that*

$$\mathrm{Spec}(H) = \{t, \underbrace{1, \dots, 1}_{b}, \underbrace{-1, \dots, -1}_{c}, -t\},$$

where b and c are some non-negative integers. Then H is the crown graph $K_{t+1}^{\star}$.

Lemma 3.3 ([19])**.** *Let $t \ge 0$ be an integer. Then*

$$\mathrm{Spec}(K_{t+1}^{\star}) = \{t, \underbrace{1, \dots, 1}_{t}, \underbrace{-1, \dots, -1}_{t}, -t\}.$$

Now we prove the main result of this section.

Theorem 3.3. *Let G be a t–regular graph of order n where $t \neq 0$. Suppose that G has no eigenvalue in the interval* $(-1, 1)$. *Then*

$$\mathcal{E}(G) \geq \frac{2tn}{t+1}. \tag{3.1}$$

In particular, if $t \geq 2$, then $\mathcal{E}(G) \geq \frac{4n}{3}$. Moreover in (3.1) *the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K^\star_{t+1}$.*

Proof. Note that if H is a 0-regular graph, then $\mathcal{E}(H) = 0$. Let $A = A(G) = [a_{ij}]$ be the adjacency matrix of G. Assume that $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G (the eigenvalues of A) such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Thus, $|\lambda_1| \geq |\lambda_i|$, for $i = 1, \ldots, n$. Since $Aj = tj$ (j is the vector of size n such that all of its entries are equal to 1), t is one of the eigenvalues of G. Hence, $|\lambda_1| \geq t$. On the other hand $\Psi(A) = t$. So, by Theorem 2.2, $|\lambda_1| \leq t$. Thus, $|\lambda_1| = t$. In fact t is the largest eigenvalue of G. Since G has no eigenvalue in the interval $(-1, 1)$, $|\lambda_n| \geq 1$. As we see in the proof of Theorem 2.3

$$|\lambda_n| \leq \sqrt{\frac{\sum_{1\leq i,j\leq n} a_{ij}^2}{n}} \leq |\lambda_1|. \tag{3.2}$$

Let $\alpha = 1$, $\beta = t$, $a = \sum_{1\leq i,j\leq n} a_{ij}^2$, $b = n$, $x = |\lambda_n|$ and $y = |\lambda_1|$. Since $t \neq 0$, $A \neq 0$. Therefore, $a > 0$. In fact $a = nt$. By (3.2) we find that

$$\beta \geq y \geq \sqrt{\frac{a}{b}} \geq x \geq \alpha.$$

Now, by applying Theorem 2.1 and Lemma 2.1 ,we find that

$$\mathcal{E}(G) = \mathcal{E}(A) \geq \frac{a + bxy}{x + y} \geq \frac{a + b\alpha\beta}{\alpha + \beta} = \frac{2tn}{t+1}. \tag{3.3}$$

Hence,

$$\mathcal{E}(G) \geq \frac{2tn}{t+1}. \tag{3.4}$$

If $t \geq 2$, then $\frac{2t}{t+1} \geq \frac{4}{3}$ and so (3.4) implies that $\mathcal{E}(G) \geq \frac{4n}{3}$.

Now we investigate the equality of (3.4). We note that for every disjoint graphs G_1 and G_2, $\mathcal{E}(G_1 \cup G_2) = \mathcal{E}(G_1) + \mathcal{E}(G_2)$. Since $\mathcal{E}(K_{t+1}) = 2t$ and $\mathcal{E}(K^\star_{t+1}) = 4t$ (by Lemma 3.3), it is easy to check that if $G = pK_{t+1} \cup qK^\star_{t+1}$, where p and q are some non-negative integers, then the equality holds. Hence, it remains to consider the converse. Thus, assume that $t \neq 0$ and G is a t-regular graph of order n such that G has no eigenvalue in the interval $(-1, 1)$ and $\mathcal{E}(G) = \frac{2tn}{t+1}$. Using (3.3) we obtain that

$$\mathcal{E}(A) = \frac{a + bxy}{x + y} \quad \text{and} \quad \frac{a + bxy}{x + y} = \frac{a + b\alpha\beta}{\alpha + \beta}. \tag{3.5}$$

Thus, the equality hold in Theorem 2.1 and in Lemma 2.1. By Theorem 2.1, there exists $r \in \{1, \ldots, n\}$ such that $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$ (we note that $|\lambda_1| = t$). On the other hand, by Lemma 2.1, we deduce that $|\lambda_1| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = 1$. If $|\lambda_1| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = \sqrt{\frac{a}{b}}$, then by (2.6) we obtain that $|\lambda_1| = \cdots = |\lambda_n|$. By combining these conditions, we find that there are two following cases.

(I) $|\lambda_1| = \cdots = |\lambda_n| = t$. Hence, every eigenvalue of G is t or $-t$. By the fact that $\lambda_1^2 + \cdots + \lambda_n^2 = 2m$, where m is the number of edges of G, we conclude that $nt^2 = nt$. Thus, $t = 1$. Since G is t–regular, this shows that every connected component of G is K_2.

(II) $|\lambda_1| = \cdots = |\lambda_r| = t$ and $|\lambda_{r+1}| = \cdots = |\lambda_n| = 1$. Thus, every eigenvalue of G is t or $-t$ or 1 or -1. If $t = 1$, then every connected component of G is K_2. Thus assume that $t \geq 2$. Let H be a connected component of G. Since H is t-regular, t is the largest eigenvalue of H (we note that since H is connected, by the first part of Theorem 3.1, the multiplicity of t as an eigenvalue of H is one). First suppose that H is bipartite. Thus, by the first part of Theorem 3.2, $-t$ is also one of the eigenvalues of H with multiplicity one. Thus $\mathrm{Spec}(H)$ is consist of one t, one $-t$ and the other elements are 1 or -1. Thus, by Lemma 3.2, H is $K_{t+1}^{\star}$. Now assume that H is not bipartite. Since t is the largest eigenvalue of H, by the second part of Theorem 3.2, $-t$ is not an eigenvalue of H. Therefore, $\mathrm{Spec}(H)$ is consist of one t and the other elements are 1 or -1. Hence, by Lemma 3.1, H is K_{t+1}. The proof is complete. □

Acknowledgements. The author is grateful to the referee for helpful comments.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 711–719.

COMPUTING THE TOTAL VERTEX IRREGULARITY STRENGTH ASSOCIATED WITH ZERO DIVISOR GRAPH OF COMMUTATIVE RING

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Abstract. Let R be a commutative ring and $Z(R)$ be the set of all zero divisors of R. $\Gamma(R)$ is said to be a zero divisor graph if $x, y \in V(\Gamma(R)) = Z(R)$ and $(x, y) \in E(\Gamma(R))$ if and only if $x.y = 0$. In this paper, we determine the total vertex irregularity strength of zero divisor graphs associated with the commutative rings $\mathbb{Z}_{p^2} \times Z_q$ for p, q prime numbers.

1. Introduction

Let $G = (V, E)$ be a simple graph, the weight of a vertex $x \in V(G)$ for an edge k-labeling $\varphi : E(G) \to \{1, 2, \ldots, k\}$ is $w_\varphi(x) = \sum_{xy \in E(G)} \varphi(xy)$. For all distinct vertices $x, y \in V(G)$, with $w_\varphi(x) \neq w_\varphi(y)$, an edge k-labeling φ is called a vertex irregular k-labeling of G. The minimum value of k for which G has an edge k-labeling φ with labels at most k is known as irregularity strength, $s(G)$, of a graph G. This labeling is also called irregular assignments and introduced by Chartrand et al. in [12]. For further results on irregularity strength, one can see [11, 13, 15] and a detailed survey [14].

Motivated by these papers, Bača et al. in [9] introduced an edge irregular total k-labeling and a vertex irregular total k-labeling. For a graph $G = (V, E)$, a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is defined to be a vertex irregular total k-labeling, if for every two distinct vertices $x, y \in V(G)$ is $wt_\varphi(x) \neq wt_\varphi(y)$, where the weight of a vertex $x \in E(G)$ is $wt_\varphi(x) = \varphi(x) + \sum_{z \in N(x)} \varphi(xz)$ and $N(x)$ is the

Key words and phrases. Total vertex irregularity strength, zero divisor graph, commutative ring.
2010 *Mathematics Subject Classification.* Primary: 05C78. Secondary: 05C25, 05C12.
DOI 10.46793/KgJMat2205.711A
Received: May 12, 2019.
Accepted: March 14, 2020.

set of neighbors of x. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, denoted by $\mathrm{tvs}(G)$.

In [9] several exact values and bounds of tvs(G) were determined for different types of graphs. Among others, the authors proved the following theorem.

Theorem 1.1 ([9])**.** *Let G be a $(|V(G)|, |E(G)|)$-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{|V(G)| + \delta}{\Delta + 1} \right\rceil \leq \mathrm{tvs}(G) \leq |V(G)| + \Delta - 2\delta + 1.$$

Przybylo [17] improved the results for sparse graphs and for graphs with large minimum degree. In the latter case the bounds $\mathrm{tvs}(G) < \frac{32|V(G)|}{\delta} + 8$ in general and $\mathrm{tvs}(G) < \frac{8|V(G)|}{r} + 1$ for r-regular $(|V(G)|, |E(G)|)$-graphs were proved to hold. Anholcer et al. [8] determined a new upper bound of the form

$$\mathrm{tvs}(G) \leq \frac{3|V(G)|}{\delta} + 1. \tag{1.1}$$

Some results on total vertex irregularity strength can be found in [1–3, 16]. The main aim of this paper is to find an exact value of the total vertex irregularity strength of certain classes of zero divisor graph of commutative rings which is much closer to the lower bound in Theorem 1.1 than to the upper bound in (1.1).

2. Results and Discussion

Let R be a commutative ring and $Z(R)$ be the set of all zero divisors of R. $\Gamma(R)$ is said to be a zero divisor graph if $x, y \in V(\Gamma(R)) = Z(R)$ and $(x, y) \in E(\Gamma(R))$ if and only if $x.y = 0$. Beck [10] introduced the notion of zero divisor graph. Anderson and Livingston [6] proved that $G(R)$ is always connected if R is commutative. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [4, 5, 7].

Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ denotes the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and is defined as following. For $x \in \mathbb{Z}_{p^2}$ and $y \in \mathbb{Z}_q$, $(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$ if and only if $x \neq p, 2p, 3p, \ldots, (p-1)p$ and $y \neq 0$. Let $I = \{(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) : x \neq p, 2p, 3p, \ldots, (p-1)p \text{ and } y \neq 0\}$, then $|I| = (p^2 - p)(q - 1)$. The vertices of the set I are the non zero divisors of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Also $(0, 0) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ is a non zero divisor. Therefore, the total number of non zero divisors are: $|I| + 1 = (p^2 - p)(q - 1) + 1 = p^2q - p^2 - pq + p + 1$. There are p^2q total vertices of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Hence, there are $p^2q - (p^2q - p^2 - pq + p + 1) = p^2 + pq - p - 1$ total number of zero divisors. This implies that the order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $p^2 + pq - p - 1$, i.e., $|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = p^2 + pq - p - 1$.

In the following theorem, we determine the lower bound of total vertex irregularity strength for zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$.

Lemma 2.1. *Let p, q be two prime numbers and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$, $p > q$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times Z_q$ with maximum degree $\Delta = \Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$ and minimum degree $\delta = \delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$. Then*

$$\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \geq \max\left\{\left\lceil\frac{p^2-p+q-1}{q}\right\rceil, p+q-2, \left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil, \left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil\right\}.$$

Proof. The order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $p^2+pq-p-1$, i.e.,

$$|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = p^2+pq-p-1.$$

The degree of each vertex $(u, v) \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$, is discussed as follows.

If $x = 0$ and $y \in \{1, 2, 3, \ldots, q-1\}$, then each such vertex $(0, y)$ is only adjacent to the vertices $(x', 0)$ for every $x' \in \{1, 2, 3, \ldots, p^2-1\}$. Hence, the degree of each vertex $(0, y)$ is p^2-1 and the number of vertices of type $(0, y)$, with $y \neq 0$ are $q-1$. Similarly, the degree of each vertex of type $(x, 0)$, $x \neq 0, p, 2p, \ldots, (p-1)p$ is $q-1$ and the number of vertices of type $(x, 0)$, with $x \neq 0, p, 2p, \ldots, (p-1)p$ are p^2-p.

If $x = kp$, $1 \leq k \leq p-1$ and $y \in \{1, 2, 3, \ldots, q-1\}$, then each such vertex (x, y) is only adjacent to the vertices $(x', 0)$ for every $x' = kp$, $1 \leq k \leq p-1$. Hence, the degree of each vertex (x, y) is $p-1$ and the number of vertices of this type is $(p-1)(q-1)$.

If $x = kp$, $1 \leq k \leq p-1$ and $y = 0$, then each such vertex $(x, 0)$ is adjacent to the vertices $(0, y'), (x', 0)$, with $x \neq x'$ and (x', y') for every $y' \in \{1, 2, 3, \ldots, q-1\}$ and $x' = kp$, $1 \leq k \leq p-1$. Hence, the degree of each vertex $(x, 0)$ is $(q-1)+(p-2)+(pq-p-q+1) = pq-2$.

Let V_a denotes the vertex partition of zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ of degree a and n_a denotes the number of vertices in the partition V_a. Therefore, $n_{q-1} = p^2-p$, $n_{pq-2} = p-1$, $n_{p^2-1} = q-1$ and $n_{p-1} = (p-1)(q-1)$. As $p > q$, this implies that $\Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = p^2-1$ and $\delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = q-1$.

To prove the lower bound consider the weights of the vertices. The smallest weight among all vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is at least $q-1$, so the largest weight of vertex of degree $q-1$ is at least $p^2-p+q-1$. Since the weight of any vertex of degree $q-1$ is the sum of q positive integers, so at least one label is at least $\left\lceil\frac{p^2-p+q-1}{q}\right\rceil$.

The largest value among the weights of vertices of degree $q-1$ and $p-1$ is at least $p^2+pq-2p$ and this weight is the sum of at most p integers. Hence, the largest label contributing to this weight must be at least $\left\lceil\frac{p^2+pq-2p}{p}\right\rceil = p+q-2$.

The largest value among the weights of vertices of degree $q-1$, $p-1$ and $pq-2$ is at least $p^2+pq-p-1$ and this weight is the sum of at most $pq-1$ integers. Hence, the largest label contributing to this weight must be at least $\left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil$.

If we consider all vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ then the lower bound

$$\left\lceil\frac{|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| + \delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))}{\Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))+1}\right\rceil = \left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil$$

follows from Theorem 1.1. This gives

$$\begin{aligned}\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \geq \max\Bigg\{&\left\lceil\frac{p^2-p+q-1}{q}\right\rceil, p+q-2, \left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil,\\ &\left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil\Bigg\}\end{aligned}$$

and we are done. □

In the following theorems, we determine the total vertex irregularity strength of zero divisor graph $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ associative with commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_q$, for p,q prime numbers.

Theorem 2.1. *Let p be a prime number and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$, $p\geq 3$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_2$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) = \left\lceil\frac{p^2-p+1}{2}\right\rceil$.

Proof. Let $(x,y)\in\mathbb{Z}_{p^2}\times\mathbb{Z}_2$, with $(x,y)\neq(0,0)$, such that $x\in\mathbb{Z}_{p^2}$ and $y\in\mathbb{Z}_2=\{0,1\}$. For our convenient, we partitioned the vertices of type $(x,0)$ into two distinct partitions as: If $x\in\mathbb{Z}_{p^2}\setminus\{0,p,2p,3p,\dots,(p-1)p\}$, then we denotes such vertices as x_i and the number of these vertices is p^2-p, the degree of each vertex is 1. If $x\in\{p,2p,3p,\dots,(p-1)p\}$, then we denote such vertices as z_j, $1\leq j\leq p-1$. This implies that $(x,0)=(x_i,0)\cup(z_j,0)$ for $1\leq i\leq p^2-p$, $1\leq j\leq p-1$. The degree of each vertex of type $(z_j,0)$ and $(z_j,1)$ is $2p-2$ and $p-1$, respectively. The vertex $(0,1)$ is the only one vertex with degree p^2-1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$ are defined as:

$$\begin{aligned}V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) =&\{(x_i,0): 1\leq i\leq p^2-p\}\\ &\cup\{(z_j,0),(z_j,1): 1\leq j\leq p-1\}\cup\{(0,1)\},\\ E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) =&\{(x_i,0)(0,1): 1\leq i\leq p^2-p\}\\ &\cup\{(z_j,0)(z_t,1),(z_j,0)(0,1): 1\leq j,t\leq p-1\}\\ &\cup\{(z_j,0)(z_t,0): 1\leq j,t\leq p-1, j\neq t\}.\end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\geq\lceil\frac{p^2-p+1}{2}\rceil$. Put $k=\lceil\frac{p^2-p+1}{2}\rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi: V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\cup E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\to\{1,2,\dots,k\}$ as:

$$\varphi((0,1))=k,\quad \varphi((x_i,0))=\max\{1,i+1-k\},$$

for $1\leq i\leq p^2-p$ and $\varphi((z_j,0))=\varphi((z_j,1))=i$ for $1\leq j\leq p-1$, $\varphi((x_i,0)(0,1))=\min\{i,k\}$ for $1\leq i\leq p^2-p$, $\varphi((z_j,0)(0,1))=\varphi((z_j,0)(z_t,1))=k$, for $1\leq j,t\leq p-1$ and $\varphi((z_j,0)(z_t,0))=k$ for $1\leq j,t\leq p-1$, $j\neq t$.

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$ are as follows:

$$\begin{aligned}wt_\varphi((x_i,0)) =&i+1,\quad \text{for } 1\leq i\leq p^2-p,\\ wt_\varphi((z_j,1)) =&(p-1)k+i,\quad \text{for } 1\leq j\leq p-1,\\ wt_\varphi((z_j,0)) =&2k(p-1)+i,\quad \text{for } 1\leq j\leq p-1,\end{aligned}$$

$$wt_{\varphi}((0,1)) = \frac{k(k+1)}{2} + (p^2-k)k.$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k = \lceil \frac{p^2-p+1}{2} \rceil$. Thus, the labeling φ is the desired vertex irregular total $\lceil \frac{p^2-p+1}{2} \rceil$-labeling. This completes the proof. $\square$

Theorem 2.2. *Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$, $p > 3$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_3$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) = \left\lceil \frac{p^2-p+2}{3} \right\rceil$.

Proof. Let us consider the vertex partition of type $(x, 0)$ as defined in the proof of Theorem 2.1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$ are defined as:

$$\begin{aligned}
V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) =& \{(x_i,0) : 1 \le i \le p^2-p\} \cup \{(z_j,t) : 1 \le j \le p-1, 0 \le t \le 2\} \\
& \cup \{(0,1),(0,2)\}, \\
E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) =& \{(x_i,0)(0,1),(x_i,0)(0,2) : 1 \le i \le p^2-p\} \\
& \cup \{(z_j,0)(z_t,1),(z_j,0)(z_t,2),(z_j,0)(0,1),(z_j,0)(0,2) : \\
& 1 \le j,t \le p-1\} \cup \{(z_j,0)(z_t,0) : 1 \le j,t \le p-1, j \ne t\}.
\end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \ge \lceil \frac{p^2-p+2}{3} \rceil$. Put $k = \lceil \frac{p^2-p+2}{3} \rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi : V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \cup E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \to \{1,2,\dots,k\}$ as $\varphi((x_i,0)) = \max\{1, i+2-2k\}$, for $1 \le i \le p^2-p$, $\varphi((0,1)) = k$, $\varphi((0,2)) = k-1$, $\varphi((z_j,0)) = \varphi((z_j,1)) = j$ for $1 \le j \le p-1$ and $\varphi((z_j,2)) = p+j-1$, $1 \le j \le p-1$.

For $1 \le i \le p^2-p$, $\varphi((0,1)(x_i,0)) = \min\{i,k\}$, $\varphi((0,2)(x_i,0)) = \min\{\max\{1, i+1-k\}, k\}$ and $\varphi((0,1)(z_j,0)) = \varphi((0,2)(z_j,0)) = \varphi((z_j,1)(z_j,0)) = \varphi((z_j,2)(z_j,0)) = \varphi((z_j,0)(z_s,0)) = k$, for $1 \le j \le p-1$, $j \ne s$.

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$ are as follows:

$$\begin{aligned}
wt_{\varphi}((x_i,0)) =& i+2, \quad \text{for } 1 \le i \le p^2-p, \\
wt_{\varphi}((z_j,t)) =& (p-1)(k+t-1)+j, \quad \text{for } 1 \le j \le p-1, 1 \le t \le 2, \\
wt_{\varphi}((z_j,0)) =& k(pq-q+1)+j, \quad \text{for } 1 \le j \le p-1, \\
wt_{\varphi}((0,1)) =& k\left(\frac{2p^2+1-k}{2}\right), \\
wt_{\varphi}((0,2)) =& k\left(\frac{2p^2+5-3k}{2}\right) - 2.
\end{aligned}$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k = \lceil \frac{p^2-p+2}{3} \rceil$. Thus, the labeling φ is the desired vertex irregular total $\lceil \frac{p^2-p+2}{3} \rceil$-labeling. This completes the proof. $\square$

Theorem 2.3. *Let $p > q > 3$, $(p-q)(p-1) \ge (q-1)^2$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = \left\lceil \frac{p^2-p+q-1}{q} \right\rceil$.

Proof. Let us consider the vertex partition of type $(x,0)$ as defined in the proof of Theorem 2.1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ are defined as:

$$\begin{aligned}V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) =& \{(x_i,0): 1\le i\le p^2-p\}\cup\{(z_j,t): 1\le j\le p-1, 0\le t\le q-1\}\\ &\cup\{(0,t): 1\le t\le q-1\},\\ E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) =& \{(x_i,0)(0,t): 1\le i\le p^2-p, 1\le t\le q-1\}\\ &\cup\{(z_j,0)(z_s,t),(z_j,0)(0,t): 1\le j,s\le p-1, 1\le t\le q-1\}\\ &\cup\{(z_j,0)(z_s,0): 1\le j,s\le p-1, j\ne s\}.\end{aligned}$$

According to Lemma 2.1, we have $\mathrm{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q))\ge\lceil\frac{p^2-p+q-1}{q}\rceil$ for $p>q>3$, $(p-q)(p-1)\ge(q-1)^2$. Put $k=\lceil\frac{p^2-p+q-1}{q}\rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi: V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q))\cup E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q))\to\{1,2,\dots,k\}$ as: $\varphi((0,t))=k+1-t$, $1\le t\le q-1$, $\varphi((z_j,0))=j$ for $1\le j\le p-1$, $\varphi((z_j,t))=j+(t-1)(p-1)$ for $1\le j\le p-1$, $1\le t\le q-1$ and

$$\varphi((x_i,0))=\begin{cases}1, & \text{for } 1\le i\le (q-1)k-q+2,\\ i+(q-1)(1-k), & \text{for } (q-1)k-q+3\le i\le p^2-p,\end{cases}$$

$\varphi((z_j,0)(0,t))=k$ for $1\le j\le p-1$, $1\le t\le q-1$, $\varphi((z_j,0)(z_s,0))=k$ for $1\le j,s\le p-1$, $j\ne s$, $\varphi((z_j,0)(z_s,t))=k$ for $1\le j,s\le p-1$, $1\le t\le q-1$, $\varphi((0,1)(x_i,0))=\min\{i,k\}$ for $1\le i\le p^2-p$ and for $2\le t\le q-1$

$$\varphi((0,t)(x_i,0))=\begin{cases}1, & \text{for } 1\le i\le (t-1)(k-1)+1,\\ i+(t-1)(1-k), & \text{for } (t-1)(k-1)+2\le i\le t(k-1)+1,\\ k, & \text{for } t(k-1)+2\le i\le p^2-p.\end{cases}$$

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ are as follows:

$$\begin{aligned}wt_\varphi((x_i,0)) =& i+q-1, \quad \text{for } 1\le i\le p^2-p,\\ wt_\varphi((z_j,t)) =& j+(p-1)(t-1+k), \quad \text{for } 1\le t\le q-1,\ 1\le j\le p-1,\\ wt_\varphi((z_j,0)) =& j+(pq-2)k, \quad \text{for } 1\le j\le p-1,\\ wt_\varphi((0,1)) =& \frac{k(2p^2-k+1)}{2},\\ wt_\varphi((0,t)) =& \sum_{i=1}^{(t-1)(k-1)+1} 1+\sum_{i=(t-1)(k-1)+2}^{t(k-1)+1}(i+(t-1)(1-k)),\\ &+(p^2-tk+t-1)k-1-t, \quad \text{for } 2\le t\le q-1.\end{aligned}$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k=\lceil\frac{p^2-p+q-1}{q}\rceil$. Thus the labeling φ is the desired vertex irregular total $\lceil\frac{p^2-p+q-1}{q}\rceil$-labeling. This completes the proof. □

Theorem 2.4. *Let $p \geq q > 3$, $(p+q)(q+1) \geq p^2+4q-1$ and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_q$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) = p+q-2$.

Proof. For our convenient, we partitioned the vertices of the graph $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ as:

$$\begin{aligned}
A =& \{(x,0) : x \in \mathbb{Z}_{p^2} \setminus \{0, p, 2p, \dots, (p-1)p\}\} = \left\{a_i : 1 \leq i \leq p^2 - p\right\},\\
B =& \{(x,y) : x = p, 2p, \dots, (p-1)p \text{ and } y \in \mathbb{Z}_q \setminus \{0\}\}\\
=& \{b_j : 1 \leq j \leq (p-1)(q-1)\},\\
C =& \{(0,y) : y \in \mathbb{Z}_q \setminus \{0\}\} = \{c_t : 1 \leq t \leq q-1\},\\
D =& \{(x,0) : x = p, 2p, \dots, (p-1)p\} = \{d_s : 1 \leq s \leq p-1\}.
\end{aligned}$$

This implies that $V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) = A \cup B \cup C \cup D$ and the edge set is

$$\begin{aligned}
E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) =& \{a_i c_t : 1 \leq i \leq p^2 - p, 1 \leq t \leq q-1\}\\
&\cup \{c_t d_s : 1 \leq s \leq p-1, 1 \leq t \leq q-1\}\\
&\cup \{d_s b_j : 1 \leq s \leq p-1, 1 \leq j \leq (p-1)(q-1)\}\\
&\cup \{d_s d_{s'} : 1 \leq s,\, s' \leq p-1, s \neq s'\}.
\end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \geq p+q-2$, for $p \geq q > 3, (p+q)(q+1) \geq p^2+4q-1$. It is enough to describe a suitable vertex irregular total $(p+q-2)$-labeling. We define a labeling $\varphi : V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \cup E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \to \{1, 2, \dots, p+q-2\}$ as $\varphi(a_i) = 1$ for $1 \leq i \leq p^2-p$, $\varphi(c_t) = t$ for $1 \leq t \leq q-1$, $\varphi(d_s) = s$ for $1 \leq s \leq p-1$

$$\varphi(b_j) = \begin{cases} q, & \text{for } 1 \leq j \leq pq-2p-q+3,\\ j - pq + 2p + 2q - 3, & \text{for } pq-2p-q+4 \leq j \leq (p-1)(q-1), \end{cases}$$

$\varphi(c_t d_s) = \varphi(d_s d_{s'}) = p+q-2$ for $1 \leq t \leq q-1$, $1 \leq s, s' \leq p-1$, $s \neq s'$, $\varphi(c_1 a_i) = \min\{i, p+q-2\}$ for $1 \leq i \leq p^2-p$, $\varphi(d_1 b_j) = \min\{p+j, p+q-2\}$ for $1 \leq j \leq (p-1)(q-1)$. For $2 \leq t \leq q-1$

$$\varphi(c_t a_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq (t-1)(p+q-3)+1,\\ i-(t-1)(p+q-3), & \text{for } (t-1)(p+q-3)+2 \leq i \leq t(p+q-3)+1,\\ p+q-2, & \text{for } t(p+q-3)+2 \leq i \leq p^2-p. \end{cases}$$

For $2 \leq s \leq p-1$

$$\varphi(d_s b_j) = \begin{cases} p, & \text{for } 1 \leq j \leq (s-1)(q-3)+1,\\ p+j-(s-1)(q-3), & \text{for } (s-1)(q-3)+2 \leq j \leq s(q-3)+1,\\ p+q-2, & \text{for } s(q-3)+2 \leq j \leq (p-1)(q-1), \end{cases}$$

$$\begin{aligned}
wt_\varphi(a_i) =& i+q-1, \quad \text{for } 1 \leq i \leq p^2-p,\\
wt_\varphi(b_j) =& p^2-p+q-1+j, \quad \text{for } 1 \leq j \leq (p-1)(q-1),\\
wt_\varphi(c_1) =& \frac{p+q-2}{2}(2p^2-p-q+1)+1,
\end{aligned}$$

$$\begin{aligned}
wt_\varphi(c_t) =& t + \sum_{i=1}^{(t-1)(p+q-3)+1} (1) + \sum_{i=(t-1)(p+q-3)+2}^{t(p+q-3)+1} (i-(t-1)(p+q-3)) \\
&+ \sum_{i=t(p+q-3)+2}^{p^2-p} (p+q-2), \quad \text{for } 2 \le t \le q-1, \\
wt_\varphi(d_1) =& \frac{p+q-2}{2}(3p+3q-7) + 1 - \frac{p(p+1)}{2}, \\
wt_\varphi(d_s) =& s + (p+q-2)(p+q-3) + \sum_{j=1}^{(s-1)(q-3)+1} p \\
&+ \sum_{j=(s-1)(q-3)+2}^{s(q-3)+1} (p+j-(s-1)(q-3)) \\
&+ \sum_{j=s(q-3)+2}^{(p-1)(q-1)} (p+q-2), \quad \text{for } 2 \le s \le p-1.
\end{aligned}$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $p+q-2$. Thus the labeling φ is the desired vertex irregular total $(p+q-2)$-labeling. This completes the proof. $\square$

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 721–732.

STABILITY OF NONLINEAR NEUTRAL MIXED TYPE LIVEN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for a nonlinear neutral mixed type Levin-Nohel integro-differential equation. An asymptotic stability theorem with a necessary and sufficient condition is proved. An example is also given to illustrate our main results.

1. Introduction

The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory, see [1–28] and the references therein. The fixed point theory does not only solve the problems on stability but have other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise, see [8] and the references therein.

Key words and phrases. Asymptotic stability, contraction mapping theorem, neutral integro-differential equations, mixed type.
2010 *Mathematics Subject Classification.* Primary: 34K20. Secondary: 45J05, 45D05.
DOI 10.46793/KgJMat2205.721B
Received: February 08, 2020.
Accepted: March 17, 2020.

In paper, we consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation

$$\frac{d}{dt}x(t) = -\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,x(s)\,ds - \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,x(s)\,ds$$

$$+\frac{d}{dt}g\left(t, x\left(t-\tau_1(t)\right), \ldots, x\left(t-\tau_m(t)\right)\right), \tag{1.1}$$

with an assumed initial condition

$$x(t) = \phi(t), \quad t \in [m(t_0), t_0],$$

where $\phi \in C([m(t_0), t_0], \mathbb{R})$ and

$$m_j(t_0) = \inf\{t - \tau_j(t)\}, \quad m(t_0) = \min\{m_j(t_0) : 1 \leq j \leq m\}.$$

Throughout this chapter, we assume that $a_j \in C([t_0, +\infty) \times [m(t_0), +\infty), \mathbb{R})$, $b_j \in C([t_0, +\infty) \times [t_0, +\infty), \mathbb{R})$ and $\tau_j, \sigma_j \in C([t_0, +\infty), \mathbb{R}^+)$, with $t - \tau_j(t) \to +\infty$ as $t \to +\infty$ and $t + \sigma_j(t) \to +\infty$ as $t \to +\infty$, $(1 \leq j \leq m)$. The functions g is globally Lipschitz continuous in x. That is, there are positive constants E_j, $1 \leq j \leq m$, such that

$$|g(t, x_1, x_2, \ldots, x_m) - g(t, y_1, y_2, \ldots, y_m)| \leq \sum_{j=1}^{m} E_j |x_j - y_j|, \quad g(t, 0, \ldots, 0) = 0. \tag{1.2}$$

In this paper, our purpose is to use the contraction mapping theorem [26] to show the asymptotic stability of the zero solution for (1.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In the special case $b_j(t,s) = 0$, $1 \leq j \leq m$ and $g(t, x_1, x_2, \ldots, x_m) = \sum_{j=1}^{m} g_j(t, x_j)$, Bessioud, Ardjouni and Djoudi [5] proved the zero solution of (1.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem. Then, the results presented in this paper extend the main results in [5]. An example is also given to illustrate our main results.

2. Main Results

For each t_0, we denote $C(t_0)$ the space of continuous functions on $[m(t_0), t_0]$ with the supremum norm $\|\cdot\|_{t_0}$. For each $(t_0, \phi) \in [0, +\infty) \times C(t_0)$, denote by $x(t) = x(t, t_0, \phi)$ the unique solution of (1.1).

Definition 2.1. The zero solution of (1.1) is called

(i) stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|x(t, t_0, \phi)| < \epsilon$ for all $t \geq t_0$ if $\|\phi\|_{t_0} < \delta$,

(ii) asymptotically stable if it is stable and $\lim\limits_{t\to+\infty} |x(t, t_0, \phi)| = 0$.

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. For this, we use the variation of parameter formula and the integration by parts.

Lemma 2.1. *x is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) = {} & (\phi(t_0) - G_\phi(t_0)) e^{-\int_{t_0}^t A(z)dz} + G_x(t) \\ & - \int_{t_0}^t [L_x(s) + A(s) G_x(s)] e^{-\int_s^t A(z)dz} ds - \int_{t_0}^t N_x(s) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{2.1}$$

where

$$G_x(t) = g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))), \tag{2.2}$$

$$\begin{aligned} L_x(t) = {} & \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(\int_s^t \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^m \int_u^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} N_x(t) = {} & \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(\int_s^t \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^m \int_u^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds, \end{aligned} \tag{2.4}$$

and

$$A(t) = \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) ds. \tag{2.5}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du.$$

Inserting this relation into (1.1), we get

$$\begin{aligned} & \frac{d}{dt} x(t) + \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds \\ & + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds - \frac{d}{dt} G_x(t) = 0, \end{aligned}$$

where G_x is given by (2.2). Or equivalently

$$\begin{aligned} & \frac{d}{dt} x(t) + x(t) \left(\sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) ds \right) \\ & - \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u) du \right) ds \\ & - \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u) du \right) ds - \frac{d}{dt} G_x(t) = 0. \end{aligned}$$

Substituting $\frac{\partial x}{\partial u}$ from (1.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}x(t) + x(t)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,ds + \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,ds\right)\\
&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u} a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\
&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)} b_k(u,\nu)\,x(\nu)\,d\nu + \frac{\partial}{\partial u}G_x(u)\right)du\right]ds\\
&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u} a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\
&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)} b_k(u,\nu)\,x(\nu)\,d\nu + \frac{\partial}{\partial u}G_x(u)\right)du\right]ds - \frac{d}{dt}G_x(t) \;=\; 0.
\end{aligned} \tag{2.6}$$

By performing the integration, we have

$$\int_s^t \frac{\partial}{\partial u}G_x(u)\,du = G_x(t) - G_x(s)\,. \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\frac{d}{dt}x(t) + A(t)\,x(t) + L_x(t) + N_x(t) - \frac{d}{dt}G_x(t) = 0, \quad t \geq t_0,$$

where A and L_x and N_x are given by (2.5) and (2.3) and (2.4) respectively. By the variation of constants formula, we get

$$\begin{aligned}
x(t) =&\phi(t_0)\,e^{-\int_{t_0}^{t} A(z)dz} - \int_{t_0}^{t}\left[L_x(s) + N_x(s)\right]e^{-\int_s^t A(z)dz}ds\\
&+\int_{t_0}^{t}\left(\frac{\partial}{\partial s}G_x(s)\right)e^{-\int_s^t A(z)dz}ds.
\end{aligned} \tag{2.8}$$

By using the integration by parts, we obtain

$$\begin{aligned}
&\int_{t_0}^{t}\left(\frac{\partial}{\partial s}G_x(s)\right)e^{-\int_s^t A(z)dz}ds\\
=&G_x(t) - G_x(t_0)\,e^{-\int_{t_0}^{t} A(z)dz} - \int_{t_0}^{t} A(s)\,G_x(s)\,e^{-\int_s^t A(z)dz}ds.
\end{aligned} \tag{2.9}$$

Finally, we obtain (2.1) by substituting (2.9) in (2.8). Since each step is reversible, the converse follows easily. This completes the proof. □

Theorem 2.1. *Let* (1.2) *holds and suppose that the following two conditions hold*

$$\lim_{t\to+\infty}\inf\int_0^t A(z)\,dz > -\infty, \tag{2.10}$$

$$\sup_{t\geq 0}\left(\sum_{j=1}^{m}E_j + \int_0^t \omega(s)\,e^{-\int_s^t A(z)dz}ds\right) = \alpha < 1, \tag{2.11}$$

where

$$\begin{aligned}\omega(s) = & \sum_{j=1}^{m}\int_{s-\tau_j(s)}^{s}|a_j(s,w)|\left(\int_w^s\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}|a_k(u,\nu)|\,d\nu\right.\right.\\ & \left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}|b_k(u,\nu)|\,d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw\\ & +\sum_{j=1}^{m}\int_{s}^{s+\sigma_j(s)}|b_j(s,w)|\left(\int_w^s\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}|a_k(u,\nu)|\,d\nu\right.\right.\\ & \left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}|b_k(u,\nu)|\,d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw+|A(s)|\sum_{j=1}^{m}E_j.\end{aligned}$$

Then the zero solution of (1.1) *is asymptotically stable if and only if*

$$\int_0^t A(z)\,dz \to +\infty \quad \text{as} \quad t\to+\infty. \tag{2.12}$$

Proof. **Sufficient condition.** Suppose that (2.12) holds. Denoted by C the space of continuous bounded functions $x:[m(t_0),+\infty)\to\mathbb{R}$ such that $x(t)=\phi(t)$, $t\in[m(t_0),t_0]$. It is known that C is a complete metric space endowed with a metric $\|x\|=\sup_{t\geq m(t_0)}|x(t)|$. Define the operator P on C by $(Px)(t)=\phi(t)$, $t\in[m(t_0),t_0]$ and

$$\begin{aligned}(Px)(t) = & (\phi(t_0)-G_\phi(t_0))\,e^{-\int_{t_0}^{t}A(z)dz}+G_x(t)\\ & -\int_{t_0}^{t}[L_x(s)+A(s)G_x(s)]\,e^{-\int_s^t A(z)dz}ds-\int_{t_0}^{t}N_x(s)\,e^{-\int_s^t A(z)dz}ds.\end{aligned}$$

Obviously, Px is continuous for each $x\in C$. Moreover, it is a contraction operator. Indeed, let $x,y\in C$

$$\begin{aligned}& |(Px)(t)-(Py)(t)|\\ \leq & |G_x(t)-G_y(t)|+\int_{t_0}^{t}[|L_x(s)-L_y(s)|+|N_x(s)-N_y(s)|\\ & +|A(s)|\,|G_x(s)-G_y(s)|]\,e^{-\int_s^t A(z)dz}ds.\end{aligned}$$

Since $x(t)=y(t)=\phi(t)$ for all $t\in[m(t_0),t_0]$, this implies that

$$\begin{aligned}& |L_x(s)-L_y(s)|\\ \leq & \sum_{j=1}^{m}\int_{s-\tau_j(s)}^{s}|a_j(s,w)|\left(\int_w^s\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}|a_k(u,\nu)|\,d\nu\right.\right.\\ & \left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}|b_k(u,\nu)|\,d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw\,\|x-y\|\end{aligned}$$

and

$$\begin{aligned}
&\left|N_x(s)-N_y(s)\right| \\
\leq&\sum_{j=1}^{m}\int_{s}^{s+\sigma_j(s)}\left|b_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right.\\
&\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw\left\|x-y\right\|.
\end{aligned}$$

Consequently, it holds for all $t\geq t_0$ that

$$\begin{aligned}
&\left|(Px)(t)-(Py)(t)\right| \\
\leq&\sum_{j=1}^{m}E_j\left\|x-y\right\|+\int_{t_0}^{t}\left(\sum_{j=1}^{m}\int_{s-\tau_j(s)}^{s}\left|a_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right.\right.\\
&\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw\\
&+\sum_{j=1}^{m}\int_{s}^{s+\sigma_j(s)}\left|b_j(s,w)\right|\left(\int_{w}^{s}\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k(u,\nu)\right|d\nu\right.\right.\\
&\left.\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k(u,\nu)\right|d\nu\right)du+2\sum_{k=1}^{m}E_k\right)dw+\left|A(s)\right|\sum_{j=1}^{m}E_j\right)\\
&\times e^{-\int_s^t A(z)dz}ds\left\|x-y\right\|.
\end{aligned}$$

Hence, it follows from (2.11) that

$$\left|(Px)(t)-(Py)(t)\right|\leq\alpha\left\|x-y\right\|,\quad t\geq t_0.$$

Thus P is a contraction operator on C. We now consider a closed subspace S of C that is defined by

$$S=\left\{x\in C:\left|x(t)\right|\to 0\text{ as }t\to+\infty\right\}.$$

We will show that $P(S)\subset S$. To do this, we need to point out that for each $x\in S$, $\left|(Px)(t)\right|\to 0$ as $t\to+\infty$. Let $x\in S$, by the definition of P we have

$$\begin{aligned}
(Px)(t)=&\left(\phi(t_0)-G_\phi(t_0)\right)e^{-\int_{t_0}^{t}A(z)dz}+G_x(t)\\
&-\int_{t_0}^{t}\left[L_x(s)+A(s)G_x(s)\right]e^{-\int_s^t A(z)dz}ds-\int_{t_0}^{t}N_x(s)e^{-\int_s^t A(z)dz}ds\\
=&I_1+I_2+I_3+I_4.
\end{aligned}$$

The first term I_1 tends to 0 by (2.12) and I_2 tends to 0 by (1.2) and $t-\tau_j(t)\to+\infty$ as $t\to+\infty$, and $t+\sigma_j(t)\to+\infty$ as $t\to+\infty$. For any $T\in(t_0,t)$, we have the following estimate for the third term

$$
\begin{aligned}
|I_3| \leq & \left| \int_{t_0}^{T} \left[L_x(s) + A(s) G_x(s) \right] e^{-\int_s^t A(z)dz} ds \right| \\
& + \left| \int_{T}^{t} \left[L_x(s) + A(s) G_x(s) \right] e^{-\int_s^t A(z)dz} ds \right| \\
\leq & \left[\int_{t_0}^{T} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right. \right. \right. \right. \\
& \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} b_k(u,\nu) \, d\nu \right) du \\
& \left. \left. + 2 \sum_{k=1}^{m} E_k \right) dw + |A(s)| \sum_{j=1}^{m} E_j \right) e^{-\int_s^t A(z)dz} ds \Bigg] \left(\|x\| + \|\phi\|_{t_0} \right) \\
& + \int_{T}^{t} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, |x(\nu)| \, d\nu \right. \right. \right. \\
& \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} |b_k(u,\nu)| \, |x(v)| \, d\nu \right) du \\
& \left. + \sum_{k=1}^{m} E_k \left| x(s-\tau_k(s)) \right| + \sum_{k=1}^{m} E_k \left| x(w-\tau_k(w)) \right| \right) dw \\
& \left. + |A(s)| \sum_{j=1}^{m} E_j \left| x(s-\tau_j(s)) \right| \right) e^{-\int_s^t A(z)dz} ds \\
= & I_{31} + I_{32}.
\end{aligned}
$$

Since $t - \tau_j(t) \to +\infty$ as $t \to +\infty$, and $t + \sigma_j(t) \to +\infty$ as $t \to +\infty$, this implies that $u - \tau_k(u) \to +\infty$, and $u + \sigma_k(u) \to +\infty$ as $T \to +\infty$. Thus, from the fact $|x(\nu)| \to 0$, $\nu \to +\infty$, we can infer that for any $\epsilon > 0$ there exists $T_1 = T > t_0$ such that

$$
\begin{aligned}
I_{32} < & \frac{\epsilon}{2\alpha} \int_{T_1}^{t} \left(\sum_{j=1}^{m} \int_{s-\tau_j(s)}^{s} |a_j(s,w)| \left(\int_{w}^{s} \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)| \, d\nu \right. \right. \right. \\
& \left. \left. \left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} |b_k(u,\nu)| \, d\nu \right) du + 2 \sum_{k=1}^{m} E_k \right) dw + |A(s)| \sum_{j=1}^{m} E_j \right) e^{-\int_s^t A(z)dz} ds,
\end{aligned}
$$

and hence, $I_{32} < \frac{\epsilon}{2}$ for all $t \geq T_1$. On the other hand, $\|x\| < +\infty$ because $x \in S$. This combined with (2.12) yields $I_{31} \to 0$ as $t \to +\infty$. As a consequence, there exists $T_2 \geq T_1$ such that $I_{31} < \frac{\epsilon}{2}$ for all $t \geq T_2$. Thus, $I_3 < \epsilon$ for all $t \geq T_2$, that is, $I_3 \to 0$ as $t \to +\infty$. Similarly, $I_4 \to 0$ as $t \to +\infty$. So, $P(S) \subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (1.1) with $x(t) = \phi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \phi) \to 0$ as

$t \to +\infty$. To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. By condition (2.10), we can define

$$K = \sup_{t \geq 0} e^{-\int_0^t A(z)dz} < +\infty. \tag{2.13}$$

Using the formula (2.1) and condition (2.11), we can obtain

$$|x(t)| \leq K\left(1+\sum_{j=1}^{m} E_j\right) \|\phi\|_{t_0} e^{\int_0^{t_0} A(z)dz} + \alpha\left(\|x\| + \|\phi\|_{t_0}\right), \quad t \geq t_0,$$

which leads us to

$$\|x\| \leq \frac{K\left(1+\sum\limits_{j=1}^{m} E_j\right) e^{\int_0^{t_0} A(z)dz} + \alpha}{1-\alpha} \|\phi\|_{t_0}. \tag{2.14}$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \epsilon$. This shows that the zero solution of (1.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (1.1) is asymptotically stable and that the condition (2.12) fails. It follows from (2.10) that there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \int_0^{t_n} A(z)\, dz$ exists and is finite. Hence, we can choose a positive constant L satisfying

$$-L < \lim_{n\to+\infty} \int_0^{t_n} A(z)\, dz < L, \quad \text{for all } n \geq 1. \tag{2.15}$$

Then condition (2.11) gives us

$$c_n = \int_0^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds \leq \alpha e^{\int_0^{t_n} A(z)dz} \leq e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds < \frac{\delta_0}{2K}, \quad \text{for all } n \geq n_0, \tag{2.16}$$

where K is as in (2.13). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{K\left(1+\sum_{j=1}^m E_j\right)e^L+1}$, and consider the solution $x(t) = x(t, t_{n_0}, \phi)$ of (1.1), with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \leq \delta_0$, $s < t_{n_0}$. It follows from (2.14) that

$$|x(t)| \leq 1-\delta_0, \quad \text{for all } t \geq t_{n_0}. \tag{2.17}$$

Applying the fundamental inequality $|a-b| \geq |a| - |b|$ and then using (2.17), (2.16) and (2.15), we get

$$\begin{aligned}
&|x(t_n) - G_x(t_n)| \\
\geq &\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} - \int_{t_{n_0}}^{t_n} \omega(s)\, e^{-\int_s^{t_n} A(z)dz} ds \\
\geq &e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - e^{-\int_0^{t_{n_0}} A(z)dz} \int_{t_{n_0}}^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds \right) \\
\geq &e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s)\, e^{\int_0^s A(z)dz} ds \right) \\
\geq &\frac{1}{2}\delta_0 e^{-\int_{t_{n_0}}^{t_n} A(z)dz} \geq \frac{1}{2}\delta_0 e^{-2L} > 0,
\end{aligned}$$

which is a contradiction because, then $(x(t_n) - G_x(t_n)) \to 0$ as $t_n \to +\infty$. The proof is complete. □

Letting $G_x(t_n) = 0$, we get the following result.

Corollary 2.1. *Suppose that the following two conditions hold:*

$$\liminf_{t \to +\infty} \int_0^t A_0(z)\, dz > -\infty \tag{2.18}$$

and

$$\begin{aligned}
&\sup_{t \geq 0} \int_0^t \Bigg[\sum_{j=1}^m \int_{s-\tau_j(s)}^s |a_j(s,w)| \left(\int_w^s \left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u |a_k(u,\nu)|\, d\nu \right.\right.\\
&+ \left.\left. \sum_{k=1}^m \int_u^{u+\sigma_k(u)} |b_k(u,\nu)|\, d\nu \right) du \right) dw \\
&+ \sum_{j=1}^m \int_s^{s+\sigma_j(s)} |b_j(s,w)| \int_w^s \left(\left(\sum_{k=1}^m \int_{u-\tau_k(u)}^u |a_k(u,\nu)|\, d\nu \right.\right.\\
&+ \left.\left. \sum_{k=1}^m \int_u^{u+\sigma_k(u)} |b_k(u,\nu)|\, d\nu \right) du \right) dw \Bigg] e^{-\int_s^t A_0(z)dz} ds \\
&= \alpha < 1,
\end{aligned} \tag{2.19}$$

where

$$A_0(z) = \sum_{j=1}^m \int_{z-\tau_j(z)}^z a_j(z,s)\, ds + \sum_{j=1}^m \int_z^{z+\sigma_j(z)} b_j(z,s)\, ds.$$

Then the zero solution of

$$\frac{d}{dt} x(t) + \sum_{j=1}^m \int_{t-\tau_j(t)}^t a_j(t,s)\, x(s)\, ds + \sum_{j=1}^m \int_t^{t+\sigma_j(t)} b_j(t,s)\, x(s)\, ds = 0$$

is asymptotically stable if and only if

$$\int_0^t A_0(z)\,dz \to +\infty \quad as \quad t \to +\infty. \tag{2.20}$$

Example 2.1. Consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation

$$\frac{d}{dt}x(t) = \int_{t-\tau(t)}^{t} a(t,s)\,ds + \int_{t}^{t+\sigma(t)} b(t,s)\,x(s)\,ds + \frac{d}{dt}g(t, x(t-\tau(t))), \tag{2.21}$$

where $a(t,s) = \frac{2}{t^2+1}$, $\tau(t) = \frac{t}{2}$, $b(t,s) = \frac{1}{t^2+1}$, $\sigma(t) = t$, $g(t,x) = 0.08(x\cos t + 3) + 0.09x\sin t^2$. Then the zero solution of (2.21) is asymptotically stable.

Proof. We have

$$A(t) = \int_{t-\tau(t)}^{t} a(t,s)\,ds + \int_{t}^{t+\sigma(t)} b(t,s)\,ds = 2\frac{t}{t^2+1}, \quad \int_0^t A(z)\,dz = \ln\left(t^2+1\right),$$

$$\begin{aligned}
\omega(s) = & \int_{\frac{s}{2}}^{s} \frac{2}{s^2+1}\left(\int_w^s\left(\int_{\frac{u}{2}}^{u}\frac{2}{u^2+1}d\nu + \int_u^{2u}\frac{1}{u^2+1}d\nu\right)du + 0.34\right)dw \\
& + \int_s^{2s} \frac{1}{s^2+1}\left(\int_w^s\left(\int_{\frac{u}{2}}^{u}\frac{2}{u^2+1}d\nu + \int_u^{2u}\frac{1}{u^2+1}d\nu\right)du + 0.34\right)dw + \frac{0.34s}{s^2+1} \\
= & \frac{1}{s^2+1}\Bigg[s\ln\left(s^2+1\right) + 4\arctan\frac{s}{2} - 2\arctan s \\
& + s\ln\left(\frac{s^2}{4}+1\right) - 2\arctan 2s - 2s\ln\left(4s^2+1\right) + 5.02s\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \omega(s)\,e^{-\int_s^t A(z)dz}dz \\
= & \frac{1}{t^2+1}\int_0^t\Bigg[s\ln\left(s^2+1\right) + 4\arctan\frac{s}{2} - 2\arctan s \\
& + s\ln\left(\frac{s^2}{4}+1\right) - 2\arctan 2s - 2s\ln\left(4s^2+1\right) + 5.02s\Bigg]ds \\
= & \frac{1}{t^2+1}\Bigg[\frac{1}{2}\left(t^2+3\right)\ln\left(t^2+1\right) + 2\left(\frac{t^2}{4}-1\right)\ln\left(\frac{t^2}{4}+1\right) - \frac{1}{4}\left(4t^2-1\right)\ln\left(4t^2+1\right) \\
& - 2t\arctan t + 4t\arctan\frac{t}{2} - 2t\arctan 2t + 2.51t^2\Bigg] \\
\leq & 0.43056.
\end{aligned}$$

Then

$$\sup_{t\geq 0}\left(E + M + \int_0^t \omega(s)\,e^{-\int_s^t A(z)dz}dz\right) \leq 0.60.$$

It is easy to see that all conditions of Theorem 2.1 hold for $\alpha = 0.60 < 1$. Thus, Theorem 2.1 implies that the zero solution of (2.21) is asymptotically stable. □

Acknowledgements. The authors gratefully acknowledge the reviewers for their helpful comments.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 733–742.

ROUGH STATISTICAL CONVERGENCE FOR DIFFERENCE SEQUENCES

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Abstract. As known, difference sequences have their own characteristics. In this paper, we study the concept of rough statistical convergence for difference sequences in a finite dimensional normed space. At the same time, we examine some properties of the set $st-\lim^r_{\Delta x_i} = \left\{x_* \in X : \Delta x_i \xrightarrow{r} x_*\right\}$, which is called as r-statistical limit set of the difference sequence (Δx_i).

1. Introduction and Background

In this study, since the concept of rough statistical convergence will be studied for difference sequences, it is important to give some literature knowledge about difference sequences. Kizmaz [19] defined the concept of difference sequence such that $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$, where $x = (x_i)$ is a real sequence for all $i \in \mathbb{N}$ (the set of all natural numbers). In this paper, he also defined $c_0(\Delta) = \{x = (x_i) : \Delta x \in c_0\}$, $c(\Delta) = \{x = (x_i) : \Delta x \in c\}$ and $l_\infty(\Delta) = \{x = (x_i) : \Delta x \in l_\infty\}$ spaces, where l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. Furthermore, he investigated relations between these spaces and obtained $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$.

After this study, which can be considered as a base about difference sequences, Et [11], Et and Çolak [12], Başarır [5], Et and Nuray [15], Gümüş and Nuray [18], Aydın and Başar [1], Bektaş et al. [6], Et and Esi [14], Savaş [23] and many others researched various properties of this concept. Et and Çolak [12] generalized Kızmaz's results for generalized difference sequences.

Key words and phrases. Statistical convergence, difference sequences, rough convergence, statistical limit set.
2010 *Mathematics Subject Classification.* Primary: 40G15. Secondary: 40A35.
DOI 10.46793/KgJMat2205.733D
Received: November 20, 2019.
Accepted: March 17, 2020.

One of the other basic concepts of this study is the concept of statistical convergence. Statistical convergence was defined by Fast [16] and Steinhaus' [25], independently. Schoenberg's work [24] for this kind of convergence can be shown as one of the important studies in summability theory. Since the concept of statistical convergence has been applied to many fields by many researchers, a wide area of use has emerged. Some of these areas are number theory [10], measure theory [20], trigonometric series [30] and summability theory [17].

Statistical convergence has recently been studied by Ulusu and Nuray [27, 29] and Ulusu and Dündar [28] for set sequences.

The concept of density is quite wide and is defined in many different ways such as natural density (asymptotic density), uniform density, density of rational and real numbers, density of ratio sets. Natural density will also form the basis of statistical convergence. Let $K \subseteq \mathbb{N}$ be a subset of $\mathbb{N}$. $d(K) := \lim_n \frac{1}{n}\sum_{j=1}^{n}\chi_K(j)$ is said to be natural density of K whenever the limit exists, where χ_K is the characteristic function of K. According to the definition of statistical convergence, sets with natural density zero will be important for us. In more detail we can say that, if K is a finite set, then it is clear that $d(K) = 0$. Another notation that we will use during our studies will be the notation that a P feature is provided for almost all $i \in \mathbb{N}$. If a sequence $x = (x_i)$ provides any P property for all other elements except the elements with zero natural density then the sequence is called "provides the P property for almost all i" and is abbreviated by writing $(a.a.i.)$. Now, it is possible to give the definition of statistical convergence as follows.

Definition 1.1 ([16])**.** Let $x = (x_i)$ be a real or complex sequence. x is statistically convergent to L if

$$\lim_n \frac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| = 0,$$

for each $\varepsilon > 0$ or equivalently

$$|x_i - L| < \varepsilon \quad (a.a.i).$$

This is indicated by $st\text{-}\lim x = L$. So, it is easy to say that each sequence that convergent is also statistical convergent.

Basarir [5] defined the concept of Δ-statistical convergence as follows.

Definition 1.2 ([5])**.** Let $x = (x_i)$ be a real sequence and $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$. For each $\varepsilon > 0$ if

$$\lim_n \frac{1}{n} |\{i \leq n : |\Delta x_i - L| \geq \varepsilon\}| = 0,$$

or equivalently

$$|\Delta x_i - L| < \varepsilon \quad (a.a.i),$$

then x is Δ-statistically convergent to L. The set of all Δ-statistically convergent sequences is denoted by $S(\Delta)$.

The concept of rough convergence is based on the idea of defining a new convergence type by extending the radius of convergence of a non-convergent but bounded sequence. Rough convergence is defined by Phu [21] in finite dimensional normed spaces. This concept was later extended by Phu [22] to infinite dimensional normed spaces. The definition of rough convergence in a finite dimensional normed space can be given as follows.

Definition 1.3 ([21])**.** Let $(X, \|\cdot\|)$ be a normed linear space and r be a non-negative real number. Then the sequence $x = (x_i)$ in X is said to be rough convergent (or r-convergent) to x_*, if for any $\varepsilon > 0$, there exists an $i_\varepsilon \in \mathbb{N}$ such that

$$\|x_i - x_*\| < r + \varepsilon,$$

for all $i \geq i_\varepsilon$. This expression means that

$$\limsup \|x_i - x_*\| < r,$$

and r is called as roughness degree. In this definition, we say that x_* is an r-limit point of the sequence (x_i) and it is denoted by $x_i \xrightarrow{r} x_*$.

Let (x_i) be a rough convergent sequence in a finite dimensional normed space $(X, \|\cdot\|)$ and r be a non-negative real number. For each $r > 0$ we obtain a different x_* point. So, this point which is called as the r-limit point of the sequence may not be unique. Therefore, a set of these points can be mentioned. This set is called as the set of r-limit points and is indicated by $\lim_{x_i}^r$. As seen, the topological and analytical features of this set are very important. The r-limit points set of the sequence (x_i) is defined by

$$\lim_{x_i}^r = \left\{ x_* \in X : x_i \xrightarrow{r} x_* \right\}.$$

Following Phu's definition [21], Aytar [2] described rough statistically convergent sequences as follows.

Definition 1.4 ([2])**.** Let $(X, \|\cdot\|)$ be a normed linear space and r be a non-negative real number. The sequence $x = (x_i)$ in X is said to be rough statistically convergent (or r-statistically convergent) to x_*, if the set

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. This expression means that

$$st - \limsup \|x_i - x_*\| \leq r,$$

and it is denoted by $x_i \xrightarrow{rst} x_*$.

Aytar [3, 4] also studied with rough limit set and rough cluster points. After these studies, Demir [7] and Demir and Gümüş [8] studied the concept of rough convergence for difference sequences and proved some basic theorems. On the other hand, Dündar and Çakan [9] define rough $\mathcal{I}$-convergence.

2. Our Aim

The idea of rough statistical convergence has developed a new perspective for non-convergent sequences. Applying this new perspective to difference sequences, which are known with their own properties, will produce very interesting results.

3. Main Results

In this part we investigate the concept of rough statistical convergence for difference sequences in $(\mathbb{R}^n, \|\cdot\|)$ space, where $\mathbb{R}^n$ is real n-dimensional normed space and we prove some important theorems.

Definition 3.1. Let $(\mathbb{R}^n, \|\cdot\|)$ be the real n-dimensional normed space and r be a non-negative real number. A difference sequence $\Delta x = (\Delta x_i)$ in $\mathbb{R}^n$ is said to be rough statistically convergent (or r-statistically convergent) to x_*, provided that the set

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$ or equivalently

$$st - \limsup \|\Delta x_i - x_*\| \leq r.$$

In this case we write $\Delta x_i \overset{r-st}{\to} x_*$.

The set of all r-st-limit points of a difference sequence Δx is indicated by

$$st - \lim{}^r_{\Delta x_i} = \left\{ x_* \in \mathbb{R}^n : \Delta x_i \overset{r-st}{\to} x_* \right\}.$$

The notation r denotes the degree of roughness and it is easy to see that if $r = 0$, then statistical convergence is obtained.

The following example gives us an example of a difference sequence which is not statistically convergent but r-statistically convergent.

Example 3.1. Let the difference sequence $\Delta y = (\Delta y_i)$ be a statistically convergent to y_* and cannot be measured exactly. Additionally, let $\Delta x = (\Delta x_i)$ be a sequence that provides the property $\|\Delta x_i - \Delta y_i\| \leq r$ $(a.a.i.)$. Then the sets

$$\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\}$$

and

$$\{i \in \mathbb{N} : \|\Delta x_i - \Delta y_i\| \geq \varepsilon\},$$

have natural density zero for any $\varepsilon > 0$. According to these informations we can not say that Δx is statistically convergent. But we know that

$$\{i \in \mathbb{N} : \|\Delta x_i - y_*\| \geq r + \varepsilon\} \subseteq \{i \in \mathbb{N} : \|\Delta y_i - y_*\| \geq \varepsilon\}$$

and this relation gives us that the natural density of the set on the left will be zero. So, the sequence Δx is r-statistically convergent.

For the set of all $r-st$-limit points of Δx, if $st-\lim^r_{\Delta x_i}\neq\emptyset$, then $st-\lim^r_{\Delta x_i}=[st-\limsup\Delta x-r, st-\liminf\Delta x+r]$. On the other hand, we know that if Δx is unbounded, then the set of r-limit points is empty, i.e., $\lim^r_{\Delta x_i}=\emptyset$. Whereas this sequence might be rough statistically convergent. The following example explains this situation.

Example 3.2. Let

$$\Delta x_i=\begin{cases}(-1)^i, & \text{if } i=k^2,\\ i, & \text{otherwise,}\end{cases}$$

i.e.,

$$(\Delta x_i)=(-1,2,3,1,5,6,7,8,-1,\dots).$$

Then

$$\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}=\{1,4,9,16,\dots\}$$

and this set has natural density zero. So, we obtain

$$st-\lim^r_{\Delta x_i}=\begin{cases}\emptyset, & \text{if } r<1,\\ [1-r,r-1], & \text{otherwise.}\end{cases}$$

Corollary 3.1. *$st-\lim^r_{\Delta x_i}\neq\emptyset$ does not imply $\lim^r_{\Delta x_i}\neq\emptyset$, but $\lim^r_{\Delta x_i}\neq\emptyset$ implies $st-\lim^r_{\Delta x_i}\neq\emptyset$. Therefore,*

$$\lim^r_{\Delta x_i}\subseteq st-\lim^r_{\Delta x_i}$$

and

$$\operatorname{diam}(\lim^r_{\Delta x_i})\subseteq\operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right).$$

Theorem 3.1. *For any difference sequence $\Delta x=(\Delta x_i)$, diameter of $st-\lim^r_{\Delta x_i}$ is not greater than $2r$. Generally, there is no smaller bound.*

Proof. Suppose that $\operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right)>2r$. Then there exist $y,z\in st-\lim^r_{\Delta x_i}$ such that

$$d:=\|y-z\|>2r.$$

Take an arbitrary $\varepsilon\in\left(0,\frac{d}{2}-r\right)$. Define A_1 and A_2 sets such that

$$A_1:=\{i\in\mathbb{N}:\|\Delta x_i-y\|\geq r+\varepsilon\}$$

and

$$A_2:=\{i\in\mathbb{N}:\|\Delta x_i-z\|\geq r+\varepsilon\}.$$

Because $y,z\in st-\lim^r_{\Delta x_i}$, we have $d(A_1)=0$, $d(A_2)=0$ and from the properties of natural density, $d(A_1^c\cap A_2^c)=1$. So,

$$\|y-z\|\leq\|\Delta x_i-y\|+\|\Delta x_i-z\|<2(r+\varepsilon)<2r+2\left(\frac{d}{2}-r\right)=d=\|y-z\|,$$

for all $i\in A_1^c\cap A_2^c$. This is a contradiction. Therefore, $\operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right)\leq 2r$.

Now, let's show that there is generally no smaller bound. For this, we show that $st-\lim^r_{\Delta x_i}=\bar{B}_r(x_*)$. We know that diam $\left(\bar{B}_r(x_*)\right)=2r$ for

$$\bar{B}_r(x_*):=\{y\in X:\|x_*-y\|\leq r\}.$$

Choose a difference sequence (Δx_i), with $st-\lim\Delta x=x_*$. For each $\varepsilon>0$ we have

$$d(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq\varepsilon\})=0.$$

Then

$$\|\Delta x_i-y\|\leq\|\Delta x_i-x_*\|+\|x_*-y\|\leq\|\Delta x_i-x_*\|+r,$$

for each $y\in\bar{B}_r(x_*)$. In this case,

$$\|\Delta x_i-y\|<r+\varepsilon,$$

for each $i\in\{i\in\mathbb{N}:\|\Delta x_i-x_*\|<\varepsilon\}$. At the same time, we know that

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|<\varepsilon\}\right)=1$$

and so, $y\in st-\lim^r_{\Delta x_i}$. Then we have $st-\lim^r_{\Delta x_i}=\bar{B}_r(x_*)$. □

Theorem 3.2. *For a bounded sequence* (Δx_i), *there is a non-negative real number* r *such that* $st-\lim^r_{\Delta x_i}\neq 0$.

The question of whether the converse of the above theorem is also valid is a question that can immediately come to mind. The answer is no. But if the sequence is statistically bounded, the converse is valid. The theorem that gives this case is below.

Theorem 3.3. (Δx_i) *is statistically bounded if and only if there exists a non-negative real number* r *such that* $st-\lim^r_{\Delta x_i}\neq 0$.

Proof. First, let's show that $st-\lim^r_{\Delta x_i}\neq 0$, when Δx is statistically bounded. From the definition of statistically boundedness, there exists a positive real number M such that

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i\|\geq M\}\right)=0.$$

Let's define $r':=\sup\{\|\Delta x_i\|:i\in K^c\}$, where $K=\{i\in\mathbb{N}:\|\Delta x_i\|\geq M\}$. Then $st-\lim^{r'}_{\Delta x_i}$ contains the origin of $\mathbb{R}^n$ and $st-\lim^{r'}_{\Delta x_i}\neq\emptyset$.

Now, assume that $st-\lim^{r'}_{\Delta x_i}\neq\emptyset$ for some $r\geq 0$. Then we have an x_* such that $x_*\in st-\lim^{r'}_{\Delta x_i}$. In that case

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}\right)=0,$$

for each $\varepsilon>0$. So, we can say that almost all Δx_i are contained in some ball with any radius greater than r and Δx_i is statistically bounded. □

In rough convergence, we know that when $\left(\Delta x_{i_j}\right)$ is a subset of (Δx_i)

$$\lim^r_{\Delta x_i}\subseteq\lim^r_{\Delta x_{i_j}}.$$

In the case of rough statistical convergence, the subsequence must be non-thin to satisfy this condition.

Example 3.3. Let $\Delta x_i := \begin{cases} i, & \text{if } i = k^2, \\ 0, & \text{otherwise,} \end{cases}$ is a difference sequence of real numbers. Then $\left(\Delta x_{i_j}\right) := (1, 4, 9, 16, \dots)$ is a subsequence of (Δx_i). We have $st - \lim^r_{\Delta x_i} = [-r, r]$ and $st - \lim^r_{\Delta x_{i_j}} = \emptyset$.

Definition 3.2. $\left(\Delta x_{i_j}\right)$ is a non-thin subsequence of (Δx_i) provided that the set B does not have natural density zero where $B = \{i_j : j \in \mathbb{N}\}$.

Theorem 3.4. *If* $\left(\Delta x_{i_j}\right)$ *is a non-thin subsequence of* (Δx_i), *then* $st - \lim^r_{\Delta x_i} \subseteq st - \lim^r_{\Delta x_{i_j}}$.

Theorem 3.5. $st - \lim^r_{\Delta x_i}$ *is closed.*

Proof. For this proof, we use one of the well-known theorems of functional analysis. According to this theorem, "For a convergent sequence $\Delta y_i \to y_*$, when $\Delta y \in st - \lim^r_{\Delta x_i}$ (at the same time $y_* \in st - \lim^r_{\Delta x_i}$), then $st - \lim^r_{\Delta x_i}$ is closed". If $st - \lim^r_{\Delta x_i} = \emptyset$, then the proof is trivial. Suppose that $st - \lim^r_{\Delta x_i} \neq \emptyset$. Then we have a sequence $(\Delta y_i) \subseteq st - \lim^r_{\Delta x_i}$ such that $\Delta y_i \to y_*$. From the definition of convergence, for each $\varepsilon > 0$ there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $\|\Delta y_i - y_*\| < \frac{\varepsilon}{2}$ for all $i > i_{\frac{\varepsilon}{2}}$. Choose an $i_0 \in \mathbb{N}$ such that $i_0 > i_{\frac{\varepsilon}{2}}$. Then $\|\Delta y_{i_0} - y_*\| < \frac{\varepsilon}{2}$.

On the other hand, since $\Delta y_i \subseteq st - \lim^r_{\Delta x_i}$, we have $y_{i_0} \in st - \lim^r_{\Delta x_i}$, i.e.,

$$d\left(\left\{i \in N : \|\Delta x_i - y_{i_0}\| \geq r + \frac{\varepsilon}{2}\right\}\right) = 0.$$

Now, we need to show following inclusion

$$\{i \in \mathbb{N} : \|\Delta x_i - y_*\| < r + \varepsilon\} \supseteq \left\{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\right\}.$$

Let $k \in \left\{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\right\}$. Then $\|\Delta x_k - y_{i_0}\| < r + \frac{\varepsilon}{2}$ and hence

$$\|\Delta x_k - y_*\| \leq \|\Delta x_k - y_{i_0}\| + \|y_{i_0} - y_*\| < r + \varepsilon.$$

It means $k \in \{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \varepsilon\}$ and we have the proof. $\square$

Theorem 3.6. $st - \lim^r_{\Delta x_i}$ *is convex.*

Proof. Suppose that $y_0, y_1 \in st - \lim^r_{\Delta x_i}$ and let $\varepsilon > 0$ be given. Define the sets

$$K_1 := \{i \in \mathbb{N} : \|\Delta x_i - y_0\| \geq r + \varepsilon\}$$

and

$$K_2 := \{i \in \mathbb{N} : \|\Delta x_i - y_1\| \geq r + \varepsilon\}.$$

We know that $d(K_1) = d(K_2) = 0$ and $d(K_1^c \cap K_2^c) = 1$ from the assumption. Then we have

$$\|\Delta x_i - [(1-\lambda) y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta x_i - y_0) + \lambda(\Delta x_i - y_1)\| < r + \varepsilon,$$

for each $i \in K_1^c \cap K_2^c$ and each $\lambda \in [0, 1]$. We get

$$d(\{i \in \mathbb{N} : \|\Delta x_i - [(1-\lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon\}) = 0,$$

this means $[(1-\lambda)\,y_0+\lambda y_1]\in st-\lim^r_{\Delta x_i}$ and so $st-\lim^r_{\Delta x_i}$ is convex. $\square$

Theorem 3.7. *The sequence (Δx_i) is r-statistically convergent to x_* if only if there exists a difference sequence $\Delta y=(\Delta y_i)$ such that $st-\lim \Delta y=x_*$ and $\|\Delta x_i-\Delta y_i\|\leq r$ for each $i\in\mathbb{N}$.*

Proof. For the necessity part, suppose that (Δx_i) is r-statistically convergent to x_*. From the definition

$$st-\limsup\|\Delta x_i-x_*\|\leq r.$$

Let's define the sequence (Δy_i) as follows:

$$\Delta y_i:=\begin{cases} x_*, & \text{if } \|\Delta x_i-x_*\|\leq r,\\ \Delta x_i+r\dfrac{x_*-\Delta x_i}{\|x_*-\Delta x_i\|}, & \text{otherwise.}\end{cases}$$

Then it is easy to see that

$$\|\Delta y_i-x_*\|=\begin{cases} 0, & \text{if } \|\Delta x_i-x_*\|\leq r,\\ \|\Delta x_i-x_*\|-r, & \text{otherwise,}\end{cases}$$

and $\|\Delta x_i-\Delta y_i\|\leq r$ for each $i\in\mathbb{N}$.

For the sufficiency, suppose that $st-\lim\Delta y=x_*$ and $\|\Delta x_i-\Delta y_i\|\leq r$ for each $i\in\mathbb{N}$. From the definiton of statistical convergence, for each $\varepsilon>0$ we get

$$d\left(\{i\in\mathbb{N}:\|\Delta y_i-x_*\|\geq\varepsilon\}\right)=0.$$

We know that

$$\{i\in\mathbb{N}:\|\Delta y_i-x_*\|\geq\varepsilon\}\supseteq\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\},$$

and we have

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}\right)=0.$$ $\square$

In order to prove the next theorem, we will need the following lemma, which is related to statistical cluster points.

Lemma 3.1. *Let $\Gamma_{\Delta x}$ be the set of all statistical cluster points of Δx and c be an arbitrary element of this set. For all $x_*\in st-\lim^r_{\Delta x_i}$ we have $\|x_*-c\|\leq r$.*

Proof. Let's accept the contrary of the lemma and find the contradiction. Assume that there exist a point $c\in\Gamma_{\Delta x}$ and $x_*\in st-\lim^r_{\Delta x_i}$ such that $\|x_*-c\|>r$. Define $\varepsilon=\frac{\|x_*-c\|-r}{3}$. In that case,

$$\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}\supseteq\{i\in\mathbb{N}:\|\Delta x_i-c\|<\varepsilon\}.$$

From the fact that $c\in\Gamma_{\Delta x}$, we know that the natural density of the set

$$\{i\in\mathbb{N}:\|\Delta x_i-c\|<\varepsilon\}$$

is not zero. So, by using the inclusion above, we obtain

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}\right)\neq 0,$$

and this completes the proof. $\square$

Theorem 3.8. *For a difference sequence* $\Delta x = (\Delta x_i)$, $\Delta x_i \overset{r-st}{\to} x_*$ *if and only if* $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$.

Proof. In Theorem 3.1, we proved the necessity part. So, we need to prove if $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$, then $\Delta x_i \overset{r-st}{\to} x_*$. We know that if the statistical cluster point of a statistically bounded sequence is unique, then the sequence is statistically convergent to this point.

In that case, if $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*) \neq \emptyset$, then (Δx_i) is statistically bounded. Let (Δx_i) sequence has two different statistical cluster points, such as x_* and x'_*. Then the point

$$\bar{x}_* := x_* + \frac{r}{\|x_* - x'_*\|}\left(x_* - x'_*\right),$$

satisfies

$$\|\bar{x}_* - x'_*\| = \left(\frac{r}{\|x_* - x'_*\|} + 1\right)\|x_* - x'_*\| = r + \|x_* - x'_*\| > r.$$

From the previous lemma, $\bar{x}_* \notin st - \lim^r_{\Delta x_i}$ but this contradicts the fact that $\|\bar{x}_* - x_*\| = r$ and $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$. This means that x_* is the unique statistical cluster point of Δx. So, Δx is statistically convergent to x_*. □

4. Conclusions and Future Developments

In our paper, we obtain some different results by defining the concept of rough statistical convergence for difference sequences. Later on, we investigate some properties of r-statistical limit point set of a difference sequence. In addition, it may be of interest to investigate similar results for generalized difference sequences.

Acknowledgements. The authors are grateful to the referees and the editor for their corrections and suggestions, which have greatly improved the readability of the paper.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 743–757.

QUOTIENT HOOPS INDUCED BY QUASI-VALUATION MAPS

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Abstract. In this paper, our aim was making a metric space on hoop algebras, because of that, we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduced a quasi-metric space. The continuity of operations of a hoop is studied with topology induced by a quasi-valuation. Also, we studied hoop homomorphism and investigated that under which condition this homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. So, we defined a congruence relation by F-quasi-valuation map and proved that the quotient is a hoop.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. In order to research the many-valued logical system whose propositional value is given in a lattice, Bosbach in [14, 15], proposed the concept of hoops, and discussed their some properties. Hoops are naturally ordered commutative residuated integral monoids. In the last years, hoops theory and related structues was enriched with deep structure theorems [1, 3–10, 12, 13, 16–18, 22, 24, 27]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced

Key words and phrases. Hoop, quasi-valuation map, $S_{\odot}$-quasi-valuation map, $S_{\rightarrow}$-quasi-valuation map, F-quasi-valuation map, (pseudo) metric space.

2010 *Mathematics Subject Classification.* Primary: 06B30. Secondary: 03G25, 06D20.

DOI 10.46793/KgJMat2205.743B

Received: July 27, 2019.

Accepted: March 17, 2020.

by Hájek in [21]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0,1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures. During these years, many researchers study on hoops in different way, and got some results on hoops [11, 20, 23, 26]. Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields and topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to topological algebra. Song, Roh and Jun, in [25] introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. In a BCI-algebra, they gave a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra, and found conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, they constructed (pseudo) metric spaces, and shew that the binary operation $\star$ in BCK-algebras is uniformly continuous. In [2], Aaly and Rezaei, introduced the notion of quasi-valuation maps such as ($S_{\odot}$, $S_{\rightarrow}$) S-quasi-valuation maps and F-quasi-valuation map based on subalgebras and filters and related properties of them are investigated. Also, they studied the relation between them and proved that every F-quasi-valuation map is an S-quasi-valuation map. Finally, by using the notion F-quasi-valuation map, they introduced a metric space and proved that if λ is an F-quasi-valuation map of hoop H then all operation of H are continuous.

In this paper, our aim was making a metric space on hoop algebras, because of that, we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduced a quasi-metric space. The continuity of operations of a hoop is studied with topology induced by a quasi-valuation. Also, we studied hoop homomorphism and investigated that under which condition these homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. So, we defined a congruence relation by F-quasi-valuation map and proved that the quotient is a hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H, \odot, \rightarrow, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and for all $x, y, z \in H$ the following assertions are valid:

(H1) $x \rightarrow x = 1$;
(H2) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$;
(H3) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \Leftrightarrow x \rightarrow y = 1).$$

It is easy to see that $(H, \leq)$ is a poset. A hoop H is *bounded* if there is an element $0 \in H$ such that, for all $x \in H$, $0 \leq x$. Let $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. If H is a bounded hoop, then we define a negation " $'$ " on H such that, for all $x \in H$, $x' = x \rightarrow 0$. A nonempty subset S of H is called a *subhoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\ x \rightarrow y \in S).$$

Note that every subhoop contains the element 1.

Proposition 2.1 ([19]). *Let $(H, \odot, \rightarrow, 1)$ be a hoop. For any $x, y, z \in H$, the following conditions hold:*

($a1$) *$(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \rightarrow y)$;*
($a2$) *$x \odot y \leq z$ if and anly if $x \leq y \rightarrow z$;*
($a3$) *$x \odot y \leq x, y$ and $x^n \leq x$ for any $n \in \mathbb{N}$;*
($a4$) *$x \leq y \rightarrow x$;*
($a5$) *$1 \rightarrow x = x$ and $x \rightarrow 1 = 1$;*
($a6$) *$x \odot (x \rightarrow y) \leq y$ and $x \odot y \leq x \wedge y \leq x \rightarrow y$;*
($a7$) *$x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;*
($a8$) *$x \leq y$ implies $x \odot z \leq y \odot z$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;*
($a9$) *$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$.*

A nonempty subset F of a hoop H is called a *filter* of H (see [19]) if the following assertions are valid:

$$(\forall x, y \in H)(x, y \in F \Rightarrow x \odot y \in F), \tag{2.1}$$

$$(\forall x, y \in H)(x \in F,\ x \leq y \Rightarrow y \in F). \tag{2.2}$$

Note that the conditions (2.1) and (2.2) mean that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note that a subset F of a hoop H is a filter of H if and only if the following assertions are valid (see [19]):

$$1 \in F,$$
$$(\forall x, y \in H)\,(x \to y \in F,\ x \in F \Rightarrow y \in F)\,.$$

Definition 2.1 ([2])**.** A real valued function λ of H is called
- an $S_{\odot}$-*quasi-valuation map* of H if

$$(\forall x, y \in H)(\lambda(x \odot y) \geq \lambda(x) + \lambda(y));$$

- an $S_{\to}$-*quasi-valuation map* of H if

$$(\forall x, y \in H)(\lambda(x \to y) \geq \lambda(x) + \lambda(y));$$

- an S-*quasi-valuation map* of H if it is an $S_{\odot}$-quasi-valuation map and an $S_{\to}$-quasi-valuation map of H.

Definition 2.2 ([2])**.** A real valued function λ of H is called an F-*quasi-valuation map of* H if

$$\lambda(1) = 0,$$
$$(\forall x, y \in H)(\lambda(y) \geq \lambda(x) + \lambda(x \to y)).$$

Proposition 2.2 ([2])**.** *Let λ be an F-quasi-valuation map on H. Then the following statements hold:*
(i) *λ is an S-quasi-valuation map on H;*
(ii) *λ is an order preserving map;*
(iii) *for any $x \in H$, $\lambda(x) \leq 0$.*

Theorem 2.1 ([2])**.** *If an F-quasi-valuation map λ of H satisfies the following condition*

$$(\forall x \in H)(\lambda(x) = 0 \Rightarrow x = 1),$$

then (H, d_{λ}) is a metric space.

Note. In what follows, let H denote a hoop unless otherwise specified.

3. Quasi-Valuation Maps on Hoops

In this section, we introduce the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduce a quasi-metric space. The continuity of operations of a hoop was studied with topology induced by a quasi-valuation.

If a F-quasi-valuation map λ of H satisfies:

$$(\forall x \in H)\,(x \neq 1 \Rightarrow \lambda(x) \neq 0)\,,$$

then we say that λ is an F-*valuation map* of H.

TABLE 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

TABLE 2. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Example 3.1. Let $H = \{0, a, b, 1\}$ be a set with Cayley tables (Table 1 and 2). Then $(H, \odot, \rightarrow, 1)$ is a bounded hoop. Define a map λ on H as follows:

$$\lambda : H \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} -30, & \text{if } x = 0, \\ -25, & \text{if } x = a, \\ -20, & \text{if } x = b, \\ 0, & \text{if } x = 1. \end{cases}$$

It is routine to verify that λ is an F-valuation map of H.

For any non-empty subset F of H and a negative real number k, define a real valued function λ_F on H as follows:

$$\lambda_F : H \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0, & \text{if } x \in F, \\ k, & \text{otherwise.} \end{cases} \tag{3.1}$$

Lemma 3.1 ([2]). *If F is a filter of H, then the function λ_F in* (3.1) *is an F-quasi-valuation map of H and $F_{\lambda_F} = F$.*

Theorem 3.1. *The function λ_F in* (3.1) *is an F-valuation map of H if and only if F is the trivial filter of H, that is $F = \{1\}$.*

Proof. Straightforward. □

In the following, we introduce quasi-metric space by using the notion of valuation maps from F-quasi-valuation map based on hoops. The continuity of operations of a hoop will study with topology induced by a quasi-valuation.

Definition 3.1. A function $d : H \times H \to \mathbb{R}$ is called a *quasi-metric* on H if it satisfies:

$$(\forall x, y \in H)\,(d(x, y) \leq 0,\ d(x, x) = 0)\,,$$
$$(\forall x, y \in H)\,(d(x, y) = d(y, x))\,,$$
$$(\forall x, y, z \in H)\,(d(x, z) \geq d(x, y) + d(y, z))\,.$$

We say that the pair (H, d) is a *quasi-metric space.*

Theorem 3.2. *If λ is an F-quasi-valuation map of H, then (H, d_λ) is a quasi-metric space which is called the quasi-metric space induced by λ, where*

$$d_\lambda : H \times H \to \mathbb{R}, \quad (x, y) \mapsto \lambda(x \to y) + \lambda(y \to x).$$

Proof. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and for any $x \in H$, $\lambda(x) \leq 0$. Thus, according to definition of d_λ, it is clear that, for any $x, y \in H$, $d_\lambda(x, y) \leq 0$. Let $x \in H$. Then $d_\lambda(x, x) = \lambda(x \to x) = \lambda(1) = 0$. Also, for any $x, y \in H$,

$$d_\lambda(x, y) = \lambda(x \to y) + \lambda(y \to x) = \lambda(y \to x) + \lambda(x \to y) = d_\lambda(y, x).$$

Moreover, by Proposition 2.1 ($a7$), for any $x, y, z \in H$, $(x \to y) \odot (y \to z) \leq x \to z$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then

$$\lambda(x \to y) + \lambda(y \to z) \leq \lambda((x \to y) \odot (y \to z)) \leq \lambda(x \to z).$$

By the similar way, $\lambda(z \to y) + \lambda(y \to x) \leq \lambda((z \to y) \odot (y \to x)) \leq \lambda(z \to x)$. Hence,

$$\begin{aligned} d_\lambda(x, y) + d_\lambda(y, z) &= \lambda(x \to y) + \lambda(y \to x) + \lambda(y \to z) + \lambda(z \to y) \\ &\leq \lambda(x \to z) + \lambda(z \to x) = d_\lambda(x, z). \end{aligned}$$

Therefore, (H, d_λ) is a quasi-metric space which is called the quasi-metric space induced by λ. □

Proposition 3.1. *Every quasi-metric space (H, d_λ) induced by an F-quasi-valuation map λ of H satisfies:*

$$\begin{aligned} d_\lambda(x, y) &\leq \min\{d_\lambda(x \to a, y \to a), d_\lambda(a \to x, a \to y)\}, \\ d_\lambda(x \to y, a \to b) &\geq d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b), \\ d_\lambda(x \odot y, a \odot b) &\geq d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b), \end{aligned}$$

for all $a, b, x, y \in H$.

Proof. Let (H, d_λ) be a quasi-metric space. By Proposition 2.1 ($a7$) for any $x, y, z \in H$, $x \to y \leq (y \to a) \to (x \to a)$ and $y \to x \leq (x \to a) \to (y \to a)$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then

$$\begin{aligned} d_\lambda(x, y) &\leq \lambda(x \to y) + \lambda(y \to x) \leq \lambda((y \to a) \to (x \to a)) + \lambda((x \to a) \to (y \to a)) \\ &= d_\lambda(x \to a, y \to a). \end{aligned}$$

By the similar way, $d_\lambda(x,y) \leq d_\lambda(a \to x, a \to y)$. Hence,

$$d_\lambda(x,y) \leq \min\{d_\lambda(x \to a, y \to a), d_\lambda(a \to x, a \to y)\}.$$

Now, let $x, y, a \in H$. Then by Proposition 2.1 ($a7$), we have

$$((x \to y) \to (y \to a)) \odot ((y \to a) \to (a \to b)) \leq (x \to y) \to (a \to b).$$

By the similar way,

$$((a \to b) \to (a \to y)) \odot ((a \to y) \to (x \to y)) \leq (a \to b) \to (x \to y).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then it is clear that

$$d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b) \leq d_\lambda(x \to y, a \to b).$$

Also, since $y \odot a \leq y \odot a$, by Proposition 2.1 ($a2$) and ($a8$), we have $y \leq a \to (y \odot a)$, and so $x \to y \leq (x \odot a) \to (y \odot a)$. Then, by Proposition 2.2, λ is order preserving, thus, $\lambda(x \to y) \leq \lambda((x \odot a) \to (y \odot a))$. By the similar way, $\lambda(y \to x) \leq \lambda((y \odot a) \to (x \odot a))$. Hence,

$$\begin{aligned} d_\lambda(x,y) =& \lambda(x \to y) + \lambda(y \to x) \leq \lambda((x \odot a) \to (y \odot a)) + \lambda((y \odot a) \to (x \odot a)) \\ =& d_\lambda(x \odot a, y \odot a). \end{aligned}$$

Then, for any $x, y, a, b \in H$, since (H, d_λ) is a quasi-metric space, we have,

$$d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b) \leq d_\lambda(x \odot y, a \odot b). \qquad \square$$

Theorem 3.3. *If λ is an F-valuation map of H, then the quasi-metric space induced by λ satisfies the following assertion,*

$$(\forall x, y \in H)\,(d_\lambda(x,y) = 0 \Rightarrow x = y)\,. \tag{3.2}$$

Proof. Let λ be an F-valuation map of H. Then λ is an F-quasi-valuation map of H. Thus, by Theorem 3.2, $d_\lambda(x,y)$ is quasi-metric. Now, for any $x, y \in H$, if $d_\lambda(x,y) = 0$, then $\lambda(x \to y) + \lambda(y \to x) = 0$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. So, by routine calculations, it is clear that $\lambda(x \to y) = \lambda(y \to x) = 0$, and so by Theorem 2.1, $x \to y = 1$ and $y \to x = 1$. Therefore, $x = y$. $\square$

We consider conditions for an F-quasi-valuation map to be an F-valuation map.

Theorem 3.4. *If the quasi-metric space (H, d_λ) induced by an F-quasi-valuation map λ of H satisfies the condition* (3.2)*, then λ is an F-valuation map of H.*

Proof. Let λ be an F-quasi-valuation map of H and there exists $1 \neq x \in H$ such that $\lambda(x) = 0$. Since λ is an F-quasi-valuation map of H that satisfying the condition (3.2), we have

$$d_\lambda(1,x) = \lambda(1 \to x) + \lambda(x \to 1) = \lambda(x) + \lambda(1) = 0.$$

Then $d_\lambda(1, x) = 0$. Since (H, d_λ) is a quasi metric, we have $x = 1$, which is a contradiction. Hence, for any $1 \neq x \in H$, $\lambda(x) \neq 0$. Therefore, λ is an F-valuation map of H. □

Note. If (H, d_λ) is a quasi-metric space, then for any $x \in H$ and $\varepsilon < 0$ the set $B_\varepsilon(x) = \{y \in H \mid d_\lambda(x, y) > \varepsilon\}$ is called a *ball of radius* $|\varepsilon|$ *with center at* x. The set $U \subseteq H$ is open in (H, d_λ) if, for any $x \in U$, there is an $\varepsilon < 0$ such that $B_\varepsilon(x) \subseteq U$. The topology $\mathcal{T}_{d_\lambda}$ induced by d_λ is the collection of all open sets in (H, d_λ).

Theorem 3.5. *If $\mathcal{T}_\lambda$ is an induced topology by d_λ, then $(H, \odot, \rightarrow, \mathcal{T}_\lambda)$ is a topological hoop.*

Proof. By Theorem 3.3, (H, d_λ) is a quasi-metric space. Let $x, y \in H$ such that $x \rightarrow y \in B_\varepsilon(x \rightarrow y)$ for any $\varepsilon < 0$. We claim that $B_\varepsilon(x) \rightarrow B_\varepsilon(y) \subseteq B_\varepsilon(x \rightarrow y)$. For this, suppose $z \in B_\varepsilon(x) \rightarrow B_\varepsilon(y)$. Then there exist $p \in B_\varepsilon(x)$ and $q \in B_\varepsilon(y)$ such that $z = p \rightarrow q$. Thus, $d_\lambda(x, p) \geq \frac{\varepsilon}{2}$ and $d_\lambda(y, q) \geq \frac{\varepsilon}{2}$. By Proposition 3.1, it is clear that $d_\lambda(x \rightarrow y, p \rightarrow y) \geq d_\lambda(x, p)$ and $d_\lambda(p \rightarrow y, p \rightarrow q) \geq d_\lambda(y, q)$. Thus,

$$d_\lambda(x \rightarrow y, p \rightarrow q) \geq d_\lambda(x \rightarrow y, p \rightarrow y) + d_\lambda(p \rightarrow y, p \rightarrow q) \geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $d_\lambda(x \rightarrow y, p \rightarrow q) \geq \varepsilon$. Hence, $z \in B_\varepsilon(x \rightarrow y)$ and so, $(H, \rightarrow, \mathcal{T}_\lambda)$ is a topological hoop. By the similar way, we can prove that $(H, \odot, \mathcal{T}_\lambda)$ is a topological hoop. Therefore, $(H, \odot, \rightarrow, \mathcal{T}_\lambda)$ is a topological hoop. □

Theorem 3.6. *For any F-quasi-valuation map λ of H, if we define a relation R_λ on H as follows:*

$$(\forall x, y)\left((x, y) \in R_\lambda \Leftrightarrow d_\lambda(x, y) = 0\right),$$

then R_λ is a congruence relation on H.

We say that R_λ is a congruence relation on H induced by λ.

Proof. Let $x, y, z \in H$. For proving that R_λ is a congruence relation on H, first of all we have to prove that R_λ is an equivalence relation on H. It is clear that R_λ is reflexive and symetric relation on H. Suppose $(x, y) \in R_\lambda$ and $(y, z) \in R_\lambda$. Then $d_\lambda(x, y) = 0$ and $d_\lambda(y, z) = 0$. By Proposition 2.1 $(a7)$, for any $x, y, z \in H$, we have $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H, then $\lambda(x \rightarrow y) + \lambda(y \rightarrow z) \leq \lambda(x \rightarrow z)$. By the similar way, $\lambda(z \rightarrow y) + \lambda(y \rightarrow x) \leq \lambda(z \rightarrow x)$. Hence,

$$0 = \lambda(x \rightarrow y) + \lambda(y \rightarrow x) + \lambda(y \rightarrow z) + \lambda(z \rightarrow y) \leq \lambda(x \rightarrow z) + \lambda(z \rightarrow x) = d_\lambda(x, z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. Then $d_\lambda(x, z) = 0$ and so $(x, z) \in R_\lambda$. Therefore, R_λ is a transitive relation on H. Now, we prove that R_λ is a congruence relation on H. For any $x, y, z \in H$ such that $(x, y) \in R_\lambda$. Since $y \leq z \rightarrow (y \odot z)$, by Proposition 2.1 $(a8)$,

$$x \rightarrow y \leq x \rightarrow (z \rightarrow (y \odot z)) = (x \odot z) \rightarrow (y \odot z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving, then $\lambda(x \to y) \leq \lambda((x \odot z) \to (y \odot z))$. By the similar way, it is clear that

$$\lambda(y \to x) \leq \lambda((y \odot z) \to (x \odot z)).$$

Hence,

$$0 = d_\lambda(x, y) \leq \lambda((x \odot z) \to (y \odot z)) + \lambda((y \odot z) \to (x \odot z)) = d_\lambda(x \odot z, y \odot z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. Then $d_\lambda(x \odot z, y \odot z) = 0$ and so $(x \odot z, y \odot z) \in R_\lambda$. Moreover, if $(x, y) \in R_\lambda$, then by Proposition 3.1, it is clear that $(x \to z, y \to z) \in R_\lambda$ and $(z \to x, z \to y) \in R_\lambda$. Therefore, R_λ is a congruence relation on H induced by λ. □

For any congruence relation R_λ induced by F-quasi-valuation map λ of H, let H_λ denote the set of all equivalence classes, that is,

$$H_\lambda := \{x_\lambda \mid x \in H\},$$

where $x_\lambda := \{y \in H \mid (x, y) \in R_\lambda\}$.

Theorem 3.7. *If λ is an F-quasi-valuation map of H, then $(H_\lambda, \boxdot, \twoheadrightarrow, 1_\lambda)$ is a hoop, where*

$$(\forall x_\lambda, y_\lambda \in H_\lambda)\left(x_\lambda \boxdot y_\lambda = (x \odot y)_\lambda,\ x_\lambda \twoheadrightarrow y_\lambda = (x \to y)_\lambda\right).$$

Proof. Let $x \in H$. Then it is clear that $x_\lambda \twoheadrightarrow x_\lambda = (x \to x) = 1_\lambda$ and $(H_\lambda, \boxdot, 1_\lambda)$ is a commutative monoid. Suppose $x_\lambda, y_\lambda, z_\lambda \in H_\lambda$. Then

$$\begin{aligned}(x_\lambda \boxdot y_\lambda) \twoheadrightarrow z_\lambda =& (x \odot y)_\lambda \twoheadrightarrow z_\lambda = ((x \odot y) \to z)_\lambda = (x \to (y \to z))_\lambda \\ =& x_\lambda \twoheadrightarrow (y_\lambda \twoheadrightarrow z_\lambda).\end{aligned}$$

Moreover, by routine calculations, we have

$$x_\lambda \boxdot (x_\lambda \twoheadrightarrow y_\lambda) = (x \odot (x \to y))_\lambda = (y \odot (y \to x))_\lambda = y_\lambda \boxdot (y_\lambda \twoheadrightarrow x_\lambda).$$

Therefore, $(H_\lambda, \boxdot, \twoheadrightarrow, 1_\lambda)$ is a hoop. □

Theorem 3.7 is illustrated by the following example.

Example 3.2. According to Example 3.1, $H_\lambda = \{1_\lambda, x_\lambda, y_\lambda, 0_\lambda\}$.

Lemma 3.2 ([2])**.** *If $\lambda : H \to \mathbb{R}$ is an F-quasi-valuation map of H, then the set*

$$F_\lambda := \{x \in H \mid \lambda(x) = 0\}$$

is a filter of H.

Proposition 3.2. *If λ is an F-quasi-valuation map of H, then $F_\lambda = 1_\lambda$.*

Proof. Let λ be an F-quasi-valuation map of H. Then, by Lemma 3.2, we have

$$\begin{aligned} F_\lambda = \{x \in H \mid \lambda(x) = 0\} &= \{x \in H \mid \lambda(1 \to x) + \lambda(x \to 1) = 0\} \\ &= \{x \in H \mid d_\lambda(1, x) = 0\} \\ &= \{x \in H \mid (x, 1) \in R_\lambda\} \\ &= 1_\lambda. \end{aligned}$$

□

For any filter F of H, let η_F be a relation on H defined by

$$(\forall x, y \in H)\, ((x, y) \in \eta_F \Leftrightarrow x \to y \in F,\ y \to x \in F)\,.$$

Then η_F is a congruence relation on H (induced by F). Denote by H/F the set of all equivalence classes, that is,

$$H/F := \{[x] \mid x \in H\},$$

where $[x] = \{y \in H \mid (x, y) \in \eta_F\}$.

Theorem 3.8. *If λ is an F-quasi-valuation map of H, then $\eta_{F_\lambda} = R_\lambda$.*

Proof. Let $x, y \in H$. Then

$$\begin{aligned} (x, y) \in \eta_{F_\lambda} &\Leftrightarrow x \to y \in F_\lambda \text{ and } y \to x \in F_\lambda \\ &\Leftrightarrow \lambda(x \to y) = \lambda(y \to x) = 0 \\ &\Leftrightarrow \lambda(x \to y) + \lambda(y \to x) = 0 \\ &\Leftrightarrow d_\lambda(x, y) = 0 \\ &\Leftrightarrow (x, y) \in R_\lambda. \end{aligned}$$

□

Theorem 3.9. *Let λ and g be F-quasi-valuation maps of H with $\lambda \neq g$. If $1_\lambda = 1_g$, then R_λ and R_g coincide and so $H_\lambda = H_g$.*

Proof. By routine calculations, we can see that $1_\lambda = \{x \in H \mid \lambda(x) = 0\}$. Suppose $x, y \in H$ such that $(x, y) \in R_\lambda$. Then $d_\lambda(x, y) = 0$ and so $\lambda(x \to y) + \lambda(y \to x) = 0$. Thus, $\lambda(x \to y) \geq -\lambda(y \to x)$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, we get that $\lambda(x \to y) = \lambda(y \to x) = 0$. Thus, $x \to y, y \to x \in 1_\lambda$. By assumption, $1_\lambda = 1_g$ we get that $x \to y$, $y \to x \in 1_g$, and so $g(x \to y) = g(y \to x) = 0$. Hence, $g(x \to y) + g(y \to x) = 0$, and so $d_g(x, y) = 0$. So $(x, y) \in R_g$. The proof of converse is similar. Therefore, R_λ and R_g coincide and so $H_\lambda = H_g$. □

Theorem 3.10. *For any filter F and any F-quasi-valuation map λ of H such that $1_\lambda \subseteq F$ consider the set*

$$\overline{F_\lambda} := \{x_\lambda \mid x \in F\}.$$

Then the following assertions are valid:

(1) $(\forall x \in H) \left(x \in F \Leftrightarrow x_\lambda \in \overline{F_\lambda}\right)$;

(2) F_λ *is a filter of* H_λ.

Proof. (1) It is clear that if $x \in F$, then $x_\lambda \in \overline{F_\lambda}$. Suppose $x_\lambda \in \overline{F_\lambda}$. Then there exists $y \in F$ such that $x_\lambda = y_\lambda$. Thus, $(x, y) \in R_\lambda$. Since R_λ is a congruence relation on H, we have $(y \to x, 1) = (y \to x, y \to y) \in R_\lambda$. Hence, $y \to x \in 1_\lambda$. Since $1_\lambda \subseteq F$, we have $y \to x \in F$. Moreover, $y \in F$ and F is a filter of H, then $x \in F$.

(2) Since F is a filter of H, $1 \in F$, and so $1_\lambda \in \overline{F_\lambda}$. Suppose $x_\lambda, x_\lambda \twoheadrightarrow y_\lambda \in \overline{F_\lambda}$. Then by (1), $x \in F$ and $x \to y \in F$. Since F is a filter of H, $y \in F$. Thus, by (1), $y_\lambda \in \overline{F_\lambda}$. Therefore, $\overline{F_\lambda}$ is a filter of H_λ. ☐

Proposition 3.3. *For any F-quasi-valuation map λ of H, let F^* be a filter of F_λ. Then the set*

$$F := \{x \in H \mid x_\lambda \in F^*\}$$

is a filter of H and $1_\lambda \subseteq F$.

Proof. Since F^* is a filter of $\overline{F_\lambda}$, $1_\lambda \in F^*$ and so $1 \in F$. Now, suppose $x, x \to y \in F$. Then $x_\lambda, (x \to y)_\lambda \in F^*$. Since F^* is a filter of $\overline{F_\lambda}$, we have $y_\lambda \in F^*$ and so $y \in F$. Hence, F is a filter of H. ☐

Let $\mathcal{F}(H_\lambda)$ denote the set of all filters of F_λ and let $\mathcal{F}(H, \lambda)$ denote the set of all filters of H containing 1_λ. Then there exists a bijection between $\mathcal{F}(H_\lambda)$ and $\mathcal{F}(H, \lambda)$, that is,

$$f : \mathcal{F}(H_\lambda) \to \mathcal{F}(H, \lambda), \quad F \mapsto F_\lambda$$

is a bijection.

Theorem 3.11. *Let $g : H \to G$ be a homomorphism of hoops. Then the following hold.*

(1) *If λ is an F-quasi-valuation map of G, then the composition $\lambda \circ g$ of λ and g is an F-quasi-valuation map of H.*
(2) *If g is an isomorphism and if λ is an F-quasi-valuation map of G, then $H_{\lambda \circ g}$ and G_λ are isomorphic.*

Proof. (1) Since g is a homomorphism of hoops, we have $(\lambda \circ g)(1) = \lambda(g(1)) = \lambda(1)$. Since λ is an F-quasi-valuation map of H, we have $\lambda(1) = 0$ and so $(\lambda \circ g)(1) = 0$. Now, suppose $x, y \in H$. Since λ is an F-quasi-valuation map of H, we have

$$\begin{aligned}(\lambda \circ g)(x \to y) + (\lambda \circ g)(x) &= \lambda(g(x \to y)) + \lambda(g(x)) \\ &= \lambda(g(x) \to g(y)) + \lambda(g(x)) \\ &\leq \lambda(g(y)) = (\lambda \circ g)(y).\end{aligned}$$

(2) Let define the map $\phi : H_{\lambda \circ g} \to G_\lambda$ such that, for any $x_{\lambda \circ g} \in H_{\lambda \circ g}$, $\phi(x_{\lambda \circ g}) = (g(x))_\lambda$. Now, we prove that ϕ is an isomorphism. For this, let $x_{\lambda \circ g}, y_{\lambda \circ g} \in H_{\lambda \circ g}$. Then

$$\begin{aligned}\phi(x_{\lambda \circ g} \boxdot y_{\lambda \circ g}) &= \phi((x \odot y)_{\lambda \circ g}) \\ &= (g(x \odot y))_\lambda = (g(x) \odot g(y))_\lambda \\ &= (g(x))_\lambda \boxdot (g(y))_\lambda = \phi(x_{\lambda \circ g}) \boxdot \phi(y_{\lambda \circ g})\end{aligned}$$

and

$$\begin{aligned}\phi(x_{\lambda\circ g} \twoheadrightarrow y_{\lambda\circ g}) &= \phi((x \to y)_{\lambda\circ g})\\ &= (g(x \to y))_\lambda = (g(x) \to g(y))_\lambda\\ &= (g(x))_\lambda \twoheadrightarrow (g(y))_\lambda = \phi(x_{\lambda\circ g}) \twoheadrightarrow \phi(y_{\lambda\circ g}),\end{aligned}$$

Hence, ϕ is a homomorphism of hoop.

Let $x, y \in H$ such that $\phi(x) = \phi(y)$. Then $\lambda \circ g(x) = \lambda \circ g(y)$. Thus, $g(x)_\lambda = g(y)_\lambda$, and so $(g(x), g(y)) \in R_\lambda$. Hence, $d_\lambda(g(x), g(y)) = 0$. Since d_λ is a quasi-metric and g is an isomorphism, we have $g(x) = g(y)$ and so $x = y$. Hence, ϕ is a one to one homomorphism.

Let $x_\lambda \in G_\lambda$. Since g is unto, there exists $y \in G$, such that $g(y)_\lambda = x_\lambda$. Then $(\lambda \circ g)(y) = x_\lambda$, thus, $\phi(y) = x_\lambda$. Hence, ϕ is an isomorphism and so $H_{\lambda\circ g}$ and G_λ are isomorphic. □

Theorem 3.12. *For any F-quasi-valuation map λ of H, we have the following assertions.*

(1) *The map $\pi : H \to H_\lambda$, $x \mapsto x_\lambda$, is an onto homomorphism.*
(2) *For any F-quasi-valuation map ϕ^* of H_λ, there exists an F-quasi-valuation map ϕ of H such that $\phi = \phi^* \circ \pi$.*
(3) *The map $\lambda^* : H_\lambda \to \mathbb{R}$, $x_\lambda \mapsto \lambda(x)$, is an F-quasi-valuation map of H_λ.*

Proof. (1) By definition of H_λ, the proof is clear.

(2) Let define $\phi = \phi^*(x_\lambda)$. We show that ϕ is an F-quasi-valuation map of H. For this, since $1 \in H$, we have $\phi(1) = \phi^*(1_\lambda)$. Moreover, ϕ^* is an F-quasi-valuation map of H, $\phi(1) = 0$. Suppose $x, y \in H$ such that

$$\phi(x) + \phi(x \to y) = \phi^*(x_\lambda) + \phi^*(x_\lambda \twoheadrightarrow y_\lambda) \leq \phi^*(y_\lambda).$$

Since ϕ^* is an F-quasi-valuation map of H, we have $\phi(x) + \phi(x \to y) \leq \phi(y)$.

(3) Let $x_\lambda \in H_\lambda$. Since λ is an F-quasi-valuation map of H, we have

$$\lambda^*(x_\lambda) + \lambda^*(x_\lambda \twoheadrightarrow y_\lambda) = \lambda(x) + \lambda(x \to y) \leq \lambda(y) = \lambda^*(y_\lambda).$$ □

Proposition 3.4. *Let H and G be two hoops and $\lambda : H \to \mathbb{R}$ and $\gamma : G \to \mathbb{R}$ be quasi-valuations. If $f : H \to G$ is a homomorphism, then the following statements are equivalent:*

(i) *f is a quasi-valuation preserving;*
(ii) *f is an isometry.*

Proof. $(i) \Rightarrow (ii)$ Let f be a quasi-valuation preserving. Then, for any $x \in H$, define $\gamma(f(x)) = \lambda(x)$. For any $x, y \in H$, we have

$$\begin{aligned} d_\gamma(f(x), f(y)) &= \gamma(f(x) \to f(y)) + \gamma(f(y) \to f(x)) \\ &= \gamma(f(x \to y)) + \gamma(f(y \to x)) \\ &= \gamma \circ f(x \to y) + \gamma \circ f(y \to x) \\ &= \lambda(x \to y) + \lambda(y \to x) \\ &= d_\lambda(x, y). \end{aligned}$$

Hence, f is an isometry.

$(ii) \Rightarrow (i)$ Let f be an isometry. Then, for any $x \in H$,

$$\begin{aligned} \lambda(x) =& d_\lambda(x, 1) = d_\gamma(f(x), f(1)) = \gamma(f(x) \to f(1)) + \gamma(f(1) \to f(x)) = \gamma(f(x)) \\ =& \gamma \circ f(x). \end{aligned}$$

Hence, f is a quasi-valuation preserving. □

Proposition 3.5. *Let $f : H \to G$ be a hoop isomorphism. If λ is a quasi-valuation on H, then $\gamma : G \to \mathbb{R}$ that, for any $y \in G$, is defined by $\gamma(y) = \lambda \circ f^{-1}(y)$ is a quasi-valuation. Moreover, if λ is an F-quasi-valuation on H, then γ is an F-quasi-valuation on G.*

Proof. Let $y_1, y_2 \in G$. Since f is an isomorphism, there exist $x_1, x_2 \in H$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned} \gamma(y_1 \to y_2) &= \lambda \circ f^{-1}(y_1 \to y_2) = \lambda(f^{-1}(y_1 \to y_2)) \\ &= \lambda(f^{-1}(y_1) \to f^{-1}(y_2)) = \lambda(x_1 \to x_2) \\ &\geq \lambda(x_1) + \lambda(x_2) = \lambda(f^{-1}(y_1)) + \lambda(f^{-1}(y_2)) \\ &= \gamma(y_1) + \gamma(y_2). \end{aligned}$$

By the similar way, we can prove that $\gamma(y_1 \odot y_2) \geq \gamma(y_1) + \gamma(y_2)$. Hence, γ is a quasi-valuation.

Since f is a hoop isomorphism, it is clear that $f(1_H) = 1_G$. Since λ is an F-quasi-valuation on H, we have $\gamma(1_G) = \lambda \circ f^{-1}(1_G) = \lambda(f^{-1}(1_G)) = \lambda(1_H) = 0$, and so $\gamma(1_G) = 0$. Let $x, y \in H$ and λ be an F-quasi-valuation on H. Since f is an isomorphism, there exist $a, b \in H$ such that $f(a) = x$ and $f(b) = y$. Then

$$\begin{aligned} \gamma(y) &= \lambda \circ f^{-1}(y) = \lambda(f^{-1}(y)) = \lambda(b) \\ &\geq \lambda(a \to b) + \lambda(a) = \lambda(f^{-1}(x)) + \lambda(f^{-1}(x) \to f^{-1}(y)) \\ &= \lambda(f^{-1}(x)) + \lambda(f^{-1}(x \to y)) = \lambda \circ f^{-1}(x) + \lambda \circ f^{-1}(x \to y) \\ &= \gamma(x) + \gamma(x \to y). \end{aligned}$$

□

4. Conclusions and Future Works

In this paper, our aim was making a metric space on hoop algebras, because of that we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduce a quasi-metric space. The continuity of operations of a hoop was studied with topology induced by a quasi-valuation. Also, we study hoop homomorphism and investigate that under which condition these homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. Because of that, we define a congruence relation by F-quasi-valuation map and prove that the quotient is a hoop. In our future work, we want to study about the product of finite number of this quasi-metric space and investigate that the quotient space of hoop has a quasi-metric or not. Finally we study the completion of this pace.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 759–772.

OBTAINING VOIGT FUNCTIONS VIA QUADRATURE FORMULA FOR THE FRACTIONAL IN TIME DIFFUSION AND WAVE PROBLEM

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Abstract. In many given physical problems and in the course of dispersion curve through a spectral line under the influence of the Doppler-effect and in collision damping, the Voigt functions have been widely utilized. By taking advantage of the fractional calculus in spectral theory and the Sturm-Liouville problems, in this paper, we obtain the Voigt functions via the quadrature formulae of one dimensional fractional in time evolution diffusion and wave problems consisting of different initial and inhomogeneous boundary conditions.

1. Introduction

The Voigt functions $V_{\gamma,\nu}(x, y, z)$ in generalized form have been studied by many authors (e.g., [10, 19, 25] and [26]) for getting various connections with a class of special functions and the numbers. In astrophysics the fundamental equations of stellar statistics are of this type. Other remarkable examples are the Voigt functions which occurs and utilized frequently in the course of the dispersion curve through a spectral line under the influence of the Doppler-effect and collision damping. The Voigt functions $V_{\gamma,\nu}(x, y, z)$ which play an essential role in spectroscopy, neutron physics and in several diverse field of physics and harmonic analysis are generally investigated from the viewpoint of integral operators.

Key words and phrases. Caputo fractional derivative, Sturm-Liouville diffusion and wave problem, non-zero zeros of Bessel function, Voigt functions.

2010 *Mathematics Subject Classification.* Primary: 26A33, 35M33, 35M12, 44A30.

DOI 10.46793/KgJMat2205.759K

Received: September 27, 2019.

Accepted: March 20, 2020.

In 1991, Klusch [10] defined the generalized Voigt function of the second kind by the Hankel integral transform

$$V_{\gamma,\nu}(x,y,z)=\sqrt{\frac{x}{2}}\int_0^\infty t^\gamma e^{-yt-zt^2}J_\nu(xt)dt,\quad x,y,z\in\mathbb{R}^+,\ \Re(\gamma+\nu)>-1, \tag{1.1}$$

where $J_\nu(\cdot)$ is the classical Bessel function (see [1,22] and [24]) defined by

$$J_v(z)=\sum_{n=0}^\infty\frac{(-1)^n(z/2)^{v+2n}}{\Gamma(n+1)\Gamma(v+n+1)},\quad |z|<\infty.$$

Again, we note that $J_v(z)$ is the defining oscillatory kernel of Hankel's integral transform

$$(H_vf)(x)=\int_0^\infty f(t)J_v(xt)dt.$$

Furthermore, the relation of the Bessel functions with the trigonometrical functions is given by

$$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}z^{-\frac{1}{2}}\sin z\quad\text{ and }\quad J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}z^{\frac{-1}{2}}\cos z.$$

To explore new ideas for representing the relation of the Voigt functions (1.1) with the quadrature formula of the solution of fractional in time diffusion and wave problem, in our current investigation, we present following fractional in time Sturm-Liouville type diffusion and wave equation in the form:

$${}_t^CD_{0+}^\alpha Y(x,t)=\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right]Y(x,t)-q(x)Y(x,t)+f(x,t),\quad 0<\alpha\le 2, \tag{1.2}$$

for all $(x,t)\in(0,l)\times(0,\infty)$, for the function defined by $f:[0,l]\times[0,\infty)\to\mathbb{R}$, $[0,l]\subset\mathbb{R}$.

Throughout this paper l is taken greater than zero, and also subjected to the initial and inhomogeneous boundary values

$$\begin{aligned}
Y(x,0)=&g(x)+\left(\frac{x}{l}-1\right)\varphi_1(0)-\frac{x}{l}\varphi_2(0), \\
\frac{\partial}{\partial t}Y(x,t)|_{t=0}=&\left(\frac{x}{l}-1\right)\varphi_1'(0)-\frac{x}{l}\varphi_2'(0),\quad (x,t)\in[0,l]\times\{0\},\\
Y(0,t)+\varphi_1(t)=&0,\quad \frac{\partial}{\partial x}Y(x,t)|_{x=0}=1+\frac{1}{l}(\varphi_1(t)-\varphi_2(t)),\quad (x,t)\in\{0\}\times[0,\infty),\\
Y(l,t)+\varphi_2(t)=&0,\quad \text{for all } (x,t)\in\{l\}\times[0,\infty).
\end{aligned} \tag{1.3}$$

Here in (1.2), the Caputo fractional derivative ${}_t^CD_{0+}^\alpha$, $m-1<\alpha\le m$, of function $Y(t)$ is given by

$$({}_t^CD_{0+}^\alpha Y)(t)=(I^{m-\alpha}Y^{(m)})(t),\quad \text{for all } m\in\mathbb{N}, \tag{1.4}$$

where $Y^{(m)}(t) = \frac{d^m Y}{dt^m}(t)$, $I^{m-\alpha}$ being the Riemann-Liouville fractional integral (see, Diethelm [2, p. 49])

$$(I^{m-\alpha}Y)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} Y(\tau) d\tau, & t > 0,\, m-1 < \alpha \leq m, \\ Y(t), \alpha = m, & \text{for all } m \in \mathbb{N}. \end{cases}$$

In this work, we also use the Laplace transformation of Caputo derivative (1.4), for $L[Y(t)] = \overline{Y}(s)$, $s > 0$, (see, Kilbas, Srivastava and Trujillo [8, p. 312]), given by

$$\begin{aligned} L[({}_t^C D_{0+}^{\alpha} Y)(t)] =& s^{\alpha}\overline{Y}(s) - s^{\alpha-1}Y(0) - s^{\alpha-2}Y^{(1)}(0) - \cdots - s^{\alpha-m}Y^{(m-1)}(0), \\ & m-1 < \alpha \leq m. \end{aligned} \tag{1.5}$$

It may be observed that for $\alpha = 1$, the equation (1.2) converts into a linear second order parabolic partial differential equation and a diffusion problem with initial and boundary conditions given in (1.3). For $\alpha = 2$, equation (1.2) reduces to a linear second order elliptic partial differential equation of wave problem with given initial and inhomogeneous boundary conditions (see Evans [3]). On the other hand, when $0 < \alpha \leq 1$, the above problem becomes identical to the initial-boundary value problem for the one dimensional time fractional diffusion equation because of the availability of the vast literature due to the researchers and authors (e.g., [4, 9, 14, 15]) with some additional boundary conditions. The analytic solutions of the space-time fractional differential equations with initial and boundary value problems are computed by the authors ([11, 14]). The computation of anomalous diffusion problems in the form of integral equations can be found in ([5, 12] and [13]). For the theory and analysis of the fractional differential equations, we refer the work of the researchers including authors (e.g., [2, 6–8, 18, 21] and [23]).

We will focus on the relations of the Voigt functions with the quadrature formula of the solution of fractional in time diffusion and wave problem. We first convert this fractional in time problem into the Sturm-Liouville problem and then find out its solution on using Green function in the form of Mercer formula [20]. The theory and applications of Sturm-Liouville problems are studied and computed by various authors (e.g., [5, 17, 28]).

2. Solution of the Problem (1.2)–(1.3)

We solve our problem (1.2)–(1.3) by setting $Y(x,t) = y(x,t) + \frac{x}{l}(\varphi_1(t) - \varphi_2(t)) - \varphi_1(t)$ and to get

$$ {}_t^C D_{0+}^{\alpha} y(x,t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] y(x,t) - q(x)y(x,t) + f_1(x,t), \quad 0 < \alpha \leq 2, \tag{2.1}$$

for all $(x,t) \in (0,l) \times (0,\infty)$, where

$$f_1(x,t) = \left(1 - \frac{x}{l}\right) {}_t^C D_{0+}^{\alpha}\varphi_1(t) + \frac{x}{l}\, {}_t^C D_{0+}^{\alpha}\varphi_2(t) + \left[\left(1 - \frac{x}{l}\right) q(x) + \frac{p'(x)}{l}\right]\varphi_1(t)$$

$$+\left[\frac{x}{l}q(x)-\frac{p^{'}(x)}{l}\right]\varphi_2(t)+f(x,t),$$

along with initial and homogeneous boundary conditions, given by

$$\begin{aligned}(2.2)\qquad & y(x,0)=g(x),\quad \frac{\partial}{\partial t}y(x,t)|_{t=0}=0,\quad \text{for all } (x,t)\in[0,l]\times\{0\},\\ & y(0,t)=0,\quad \frac{\partial}{\partial x}y(x,t)|_{x=0}=1,\quad \text{for all } (x,t)\in\{0\}\times[0,\infty),\\ & y(l,t)=0,\quad \text{for all } (x,t)\in\{l\}\times[0,\infty).\end{aligned}$$

Then consider $L\{y(x,t)\}=\bar{y}(x,s)$ for $s>0$. Now using the result (1.5), and then taking Laplace transformation of (2.1) and (2.2), we find that in the form of Sturm-Liouville problem [1]

$$(2.3)\qquad \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right]\bar{y}(x,s)-\{q(x)+s^{\alpha}\}\bar{y}(x,s)=\bar{f}_1(x,s),$$

where,

$$\begin{aligned}\bar{f}_1(x,s)=&-s^{\alpha-1}\left\{g(x)-\left(1-\frac{x}{l}\right)\varphi_1(0)-\frac{x}{l}\varphi_2(0)\right\}\\ &+s^{\alpha-2}\left\{\left(1-\frac{x}{l}\right)\varphi_1^{'}(0)+\frac{x}{l}\varphi_2^{'}(0)\right\}-s^{\alpha}\left\{\left(1-\frac{x}{l}\right)\bar{\varphi}_1(s)+\frac{x}{l}\bar{\varphi}_2(s)\right\}\\ &-\left\{\left(1-\frac{x}{l}\right)q(x)+\frac{p^{'}(x)}{l}\right\}\bar{\varphi}_1(s)-\left\{\frac{x}{l}q(x)-\frac{p^{'}(x)}{l}\right\}\bar{\varphi}_2(s)-\bar{f}(x,s),\end{aligned}$$

$0<\alpha\leq 2$ for all $x\in(0,l)$ and $s>0$, along with homogeneous boundary conditions

$$\begin{aligned}(2.4)\qquad & \bar{y}(0,s)=0,\quad \text{for all } (x,s)\in\{0\}\times(0,\infty),\ s>0,\\ & \bar{y}(l,s)=0,\quad \text{for all } (x,s)\in\{l\}\times(0,\infty),\ s>0.\end{aligned}$$

Again, letting $\mathfrak{L}\bar{y}(x,s)=\{\frac{\partial}{\partial x}[p(x)\frac{\partial}{\partial x}]-q(x)\}\bar{y}(x,s)$, we may write the problem (2.3)–(2.4) in the form

$$(2.5)\qquad \mathfrak{L}\bar{y}(x,s)-s^{\alpha}\bar{y}(x,s)=\bar{f}_1(x,s),\quad 0<\alpha\leq 2,$$

for all $x\in(0,l)$, $(0,l)\subset\mathbb{R}$, and $s>0$, along with the boundary conditions given in (2.4).

Now, to solve the differential equation (2.5), with boundary conditions (2.4), first we construct a Green function and consider the normalized eigenfunctions (see, Churchill [1, p. 291]) $\Psi_n(x)$, for all $n=1,2,3,\ldots$, where $\Psi_n(x)=\frac{\bar{y}_n(x,s_n)}{\|\bar{y}_n(x,s_n)\|}$ for $s\geq s_n$, $s_n>0$ for all $n=1,2,3,\ldots$, and the orthonormalized property, given by

$$\int_0^l \Psi_n(t)\Psi_m(t)dt=\begin{cases}0, & m\neq n,\\ 1, & m=n.\end{cases}$$

Thus, by differential equation (2.5) with boundary conditions (2.4), we have the following homogeneous differential equation

$$\mathfrak{L}\Psi_n(x) - s_n^\alpha \Psi_n(x) = 0, \Psi_n(0) = 0, \Psi_n(l) = 0, \quad \text{for all } x \in [0,l],\ n = 1,2,3,\ldots \tag{2.6}$$

Again then, in (2.5) and (2.6), we introduce two series

$$\bar{f}_1(x,s) = \sum_{n=1}^{\infty} A_n \Psi_n(x), \bar{y}(x,s) = \sum_{n=1}^{\infty} C_n \Psi_n(x), \tag{2.7}$$

for all $s \geq s_n,\ s_n > 0,\ A_n \neq 0,\ n = 1,2,3,\ldots$

Then on using the relations from (2.5) to (2.7), we find following equations

$$\sum_{n=1}^{\infty} \mathfrak{L} C_n \Psi_n(x) - s^\alpha \sum_{n=1}^{\infty} C_n \Psi_n(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} \mathfrak{L} C_n \Psi_n(x) - \sum_{n=1}^{\infty} s_n^\alpha C_n \Psi_n(x) = 0. \tag{2.9}$$

Therefore, on use of (2.8) and (2.9), we find that

$$-\sum_{n=1}^{\infty} (s^\alpha - s_n^\alpha) C_n \int_0^l \Psi_n(x)\Psi_m(x)dx = \sum_{n=1}^{\infty} A_n \int_0^l \Psi_n(x)\Psi_m(x)dx. \tag{2.10}$$

Now, for obtaining the solution of the problem (1.2)-(1.3), for $s \geq s_n,\ s_n > 0$, we use the orthogonal property given in (2.5) and consider that $A_n[[\alpha]]_{s,s_n} = \left(\frac{B_{n_o}}{s-s_n}\right)$, $B_{n_0} \neq 0$, when $s \to s_n$ for all $n \geq n_0$ and

$$C_n = -\frac{A_n}{H(\alpha; s, s_n)}, \quad \alpha > 0, s\ \geq s_n,\ s_n > 0 \text{ for all } n = 1,2,3,\ldots,$$

where

$$H(\alpha; s, s_n) = \begin{cases} (s - s_n)[[\alpha]]_{s,s_n}, & s > s_n > 0, \\ (s - s_n)^{-2}, & s \to s_n,\ s_n > 0, \end{cases}$$

for all $n = 1,2,3,\ldots,\ \alpha > 0$. Here $[[\alpha]]_{s,s_n} = (s^{m-1} + s^{m-2}s_n + \ldots + ss_n^{m-2} + s_n^{m-1})$, $[\alpha] = m$, m is the smallest integer greater than or equal to α, then

$$C_n = \begin{cases} \dfrac{-A_n}{s^\alpha - s_n^\alpha}, & s > s_n, s_n > 0, \\ 0, & s \to s_n,\ \text{for all } n = 1,2,3,\ldots \end{cases}$$

Again then, for $s \geq s_n,\ s_n > 0$ for all $n = 1,2,3,\ldots$, by (2.7) and (2.10), and the orthogonal property (2.5), we may write

$$\bar{y}(x,s) = -\sum_{n=1}^{\infty} \frac{A_n}{(s^\alpha - s_n^\alpha)} \Psi_n(x), \tag{2.11}$$

and further for all $s \geq s_n$, $s_n > 0$ for all $n = 1, 2, 3, \ldots$, and by relation (2.7), we get an equality as

$$\sum_{m=1}^{\infty} \int_0^l \frac{\bar{f}_1(\xi, s)}{(s^\alpha - s_n^\alpha)} \Psi_m(x)\Psi_m(\xi) d\xi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_n}{(s^\alpha - s_n^\alpha)} \Psi_m(x) \int_0^l \Psi_n(\xi)\Psi_m(\xi) d\xi.$$

Therefore, for all $m = n$, by using the orthogonal property (2.5), and the relations given in (2.11), we obtain an identity

$$\bar{y}(x, s) = \int_0^l G(x, \xi, s)\bar{f}_1(\xi, s) d\xi, \tag{2.12}$$

where the following Green function in form of Mercer formula [20] is obtained as

$$G(x, \xi, s) = -\sum_{n=1}^{\infty} \frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha - s_n^\alpha)}, \quad s \geq s_n,\ s_n > 0 \text{ for all } n = 1, 2, 3, \ldots \tag{2.13}$$

Here in (2.12), the value of $\bar{f}_1(x, s)$ is given in (2.3) and the functions $\Psi_n(x)$ for all $n = 1, 2, 3, \ldots$, are found by the problem (2.6). Thus, by (2.3), (2.12) and (2.13), the solution of the problem (2.5) with the conditions (2.4) may be computed in the form

$$\begin{aligned}
&\bar{y}(x, s) \\
=&\sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)s^{\alpha-1}}{\|\bar{y}_n(x, s_n)\|^2(s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n)(g(\xi) - \left(1 - \frac{\xi}{l}\right)\varphi_1(0) - \frac{\xi}{l}\varphi_2(0))d\xi \\
&-\sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)s^{\alpha-1}}{\|\bar{y}_n(x, s_n)\|^2 s(s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n)\left(\left(1 - \frac{\xi}{l}\right)\varphi_1'(0) + \frac{\xi}{l}\varphi_2'(0)\right)d\xi \\
&+\sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)}{\|\bar{y}_n(x, s_n)\|^2} \int_0^l \bar{y}_n(\xi, s_n)\left(\left(1 - \frac{\xi}{l}\right)\bar{\varphi}_1(s) + \frac{\xi}{l}\bar{\varphi}_2(s)\right)d\xi \\
&+\sum_{n=1}^{\infty} \frac{(s_n)^\alpha \bar{y}_n(x, s_n)}{\|\bar{y}_n(x, s_n)\|^2(s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n)\left(\left(1 - \frac{\xi}{l}\right)\bar{\varphi}_1(s) + \frac{\xi}{l}\bar{\varphi}_2(s)\right)d\xi \\
&+\sum_{n=1}^{\infty} \frac{\bar{y}_n(x, s_n)(s_n)^\alpha}{(s_n)^\alpha\|\bar{y}_n(x, s_n)\|^2(s^\alpha - (s_n)^\alpha)} \int_0^l \bar{y}_n(\xi, s_n)\left(\left(\left(1 - \frac{\xi}{l}\right)q(\xi) + \frac{p'(\xi)}{l}\right)\bar{\varphi}_1(s)\right. \\
&+\left.\left(\frac{\xi}{l}q(\xi) - \frac{p'(\xi)}{l}\right)\bar{\varphi}_2(s) + \bar{f}(x, s)\right)d\xi.
\end{aligned} \tag{2.14}$$

Now, to take the inverse Laplace transformation on both of the sides of result (2.14), we have the following formulae. For $0 < \alpha \leq 2$, $|\theta| < |s^\alpha|$ and $s \geq s_n$, $s_n > 0$ for all $n = 1, 2, 3, \ldots$, the inverse Laplace transformation formula of Mittag-Leffler function $E_\alpha(z)$, where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ (see Mathai and Haubold [16, p. 80], and Kilbas, Srivastava and Trujillo [8, p. 313]), is given by

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha - \theta}\right\} = E_\alpha(\theta t^\alpha), \quad 0 < |\theta| < |s^\alpha|. \tag{2.15}$$

Again, the Laplace transformation formula of the derivative of the Mittag-Leffler function (see in (2.15)), with the aid of formula (1.5), is found in the form

$$L\left\{\frac{d}{dt}E_\alpha(\theta t^\alpha)\right\} = \frac{s^\alpha}{s^\alpha-\theta} - 1 = \frac{\theta}{s^\alpha-\theta}, \tag{2.16}$$

so that the relation (2.16) gives us

$$L^{-1}\left\{\frac{1}{s^\alpha-\theta}\right\} = \frac{1}{\theta}\frac{d}{dt}E_\alpha(\theta t^\alpha), \quad 0<|\theta|<|s^\alpha|. \tag{2.17}$$

Finally, on making an application of the results (2.15)–(2.17), we obtain

$$L^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha-\theta}\right\} = L^{-1}\left\{\frac{1}{s}\frac{s^{\alpha-1}}{s^\alpha-\theta}\right\} = \int_0^t E_\alpha(\theta\tau^\alpha)d\tau. \tag{2.18}$$

Thus, on using above results of (2.15)–(2.18) into the result (2.14), we obtain the solution of the problem (2.3)–(2.4) for all $x\in(0,l)$, and $t>0$, $s\geq s_n$, $s_n>0$ for all $n=1,2,3,\ldots$, in the form

$$\begin{aligned}
(2.19)\quad & y(x,t)\\
&=\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}E_\alpha((s_n)^\alpha t^\alpha)\int_0^l y_n(\xi,s_n)(g(\xi)-\left(1-\frac{\xi}{l}\right)\varphi_1(0)-\frac{\xi}{l}\varphi_2(0))d\xi\\
&-\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^t E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\varphi_1'(0)+\frac{\xi}{l}\varphi_2'(0)\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\varphi_1(t)+\frac{\xi}{l}\varphi_2(t)\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\bigg\{\left(1-\frac{\xi}{l}\right)\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\\
&+\frac{\xi}{l}\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\bigg\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{(s_n)^\alpha\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\bigg\{\left(\left(1-\frac{\xi}{l}\right)q(\xi)+\frac{p'(\xi)}{l}\right)\\
&\times\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau+\left(\frac{\xi}{l}q(\xi)-\frac{p'(\xi)}{l}\right)\\
&\times\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau+\int_0^t f(x,t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\bigg\}d\xi.
\end{aligned}$$

Finally, putting

$$y(x,t) = Y(x,t) - \frac{x}{l}(\varphi_1(t)-\varphi_2(t)) + \varphi_1(t)$$

in solution (2.19), we obtain the solution of the problem (1.2)–(1.3) for all $x\in(0,l)$ and $t>0$, $s_n>0$, for all $n=1,2,3,\ldots$, in the form

$$(2.20)\quad Y(x,t)$$

$$
\begin{aligned}
&=\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}E_\alpha((s_n)^\alpha t^\alpha)\int_0^l y_n(\xi,s_n)\left(g(\xi)-\left(1-\frac{\xi}{l}\right)\varphi_1(0)-\frac{\xi}{l}\varphi_2(0)\right)d\xi\\
&-\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^t E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\int_0^l y_n(\xi,s_n)\left(\left(1-\frac{\xi}{l}\right)\varphi_1^{'}(0)+\frac{\xi}{l}\varphi_2^{'}(0)\right)d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\varphi_1(t)+\frac{\xi}{l}\varphi_2(t)\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\left\{\left(1-\frac{\xi}{l}\right)\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\right.\\
&\left.+\frac{\xi}{l}\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\right\}d\xi\\
&+\sum_{n=1}^{\infty}\frac{y_n(x,s_n)}{(s_n)^\alpha\|y_n(x,s_n)\|^2}\int_0^l y_n(\xi,s_n)\{\left(\left(1-\frac{\xi}{l}\right)q(\xi)+\frac{p^{'}(\xi)}{l}\right)\\
&\times\int_0^t\varphi_1(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\\
&+\left(\frac{\xi}{l}q(\xi)-\frac{p^{'}(\xi)}{l}\right)\int_0^t\varphi_2(t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\\
&+\int_0^t f(x,t-\tau)\frac{d}{d\tau}E_\alpha((s_n)^\alpha\tau^\alpha)d\tau\}d\xi+\frac{x}{l}(\varphi_1(t)-\varphi_2(t))-\varphi_1(t).
\end{aligned}
$$

Hence, from the study of the results (1.2) to (2.20), we find the following.

Theorem 2.1. *If $l>0$ and f is a function defined by $f:[0,l]\times[0,\infty)\to\mathbb{R}$, $[0,l]\subset\mathbb{R}$, $q(x)$, $p(x)$, $p'(x)$, $p''(x)$ are continuous real valued functions of x on $0\le x\le l$. Then there exists the normalized eigenfunctions*

$$\Psi_n(x)=\frac{y_n(x,s_n)}{\|y_n(x,s_n)\|},\quad s_n>0 \text{ for all } n\in\mathbb{N},$$

by the solution of boundary value problem (2.5) and with boundary conditions (2.4) simultaneously give the Green's function

$$G(x,\xi,s)=-\sum_{n=1}^{\infty}\frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha-s_n^\alpha)},$$

provided that $s\ge s_n$, $s_n>0$ for all $n\in\mathbb{N}$ and $0<\alpha\le 2$, for all $x,\xi\in(0,l)$, $(0,l)\subset\mathbb{R}$ and $s>0$, which gives the solution of the problem (2.3) with boundary values (2.4), in the form

$$\bar{y}(x,s)=-\sum_{n=1}^{\infty}\int_0^l\bar{f}_1(\xi,s)\frac{\Psi_n(x)\Psi_n(\xi)}{(s^\alpha-s_n^\alpha)}d\xi,\quad s\ge s_n,\ s_n>0, \text{ for all } n=1,2,3\ldots,$$

the function $\bar{f}_1(x,s)$ is given in the (2.3).

Finally, its inverse Laplace transformation gives the solution (2.20) of the fractional in time Sturm-Liouville type diffusion and wave problem (1.2)–(1.3) for all x, $0<x<l$, $t>0$, $s\ge s_n$, $s_n>0$ for all $n\in\mathbb{N}$.

3. The Voigt Functions via Solution of the Problem (1.2)–(1.3) in Various Conditions

In any given physical problem, a numerical, computational or analytical evaluation of the Voigt functions (or of their variants) is required. We begin our study of Voigt functions and their relations with quadrature formula of the solution of the fractional in time Sturm-Liouville type diffusion and wave problem (1.2)–(1.3) in different particular cases and conditions.

3.1. **The Voigt functions via non-homogeneous Bessel type diffusion and wave problem, when $0 < \alpha \le 2$.**

Theorem 3.1. *If we put $f = 0$, $q(x) = 0$, $p(x) = x$ for all x, $0 \le x \le l \subset \mathbb{R}$, $\varphi_1(t) = t$, $\varphi_2(t) = t^2$, $0 < \alpha \le 2$, $t \ge 0$, in the problem (1.2)–(1.3), then our problem becomes Bessel type fractional in time diffusion-wave problem of the form*

$$
{}^{C}_{t}D^{\alpha}_{0^+}Y(x,t) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\right]Y(x,t), \quad 0 < \alpha \le 2, \tag{3.1}
$$

for all $(x,t) \in (0,l) \times (0,\infty)$, $(0,l) \subset \mathbb{R}$, subjected to the initial and inhomogeneous boundary values

$$
\begin{aligned}
(3.2)\quad Y(x,0) =& g(x), \quad \frac{\partial}{\partial t}Y(x,t)|_{t=0} = \left(\frac{x}{l} - 1\right), \quad \textit{for all } (x,t) \in [0,l] \times \{0\},\ [0,l] \subset \mathbb{R},\\
Y(0,t) + t =& 0, \quad \frac{\partial}{\partial x}Y(x,t)|_{x=0} = 1 + \frac{1}{l}(t - t^2), \quad \textit{for all } (x,t) \in \{0\} \times [0,\infty),\\
Y(l,t) + t^2 =& 0, \quad \textit{for all } (x,t) \in \{l\} \times [0,\infty).
\end{aligned}
$$

Then, solution of problem (3.1)–(3.2) has the form

$$
\begin{aligned}
(3.3)\qquad & Y(x,t)\\
=& \frac{1}{l}\sum_{n=1}^{\infty}\frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{[J_1(-\mu_n)]^2}E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right)\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)g(\xi)d\xi\\
&+\frac{2}{l}\sum_{n=1}^{\infty}\frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{[J_1(-\mu_n)]^2}\int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)\xi d\xi\\
&+8l\sum_{n=1}^{\infty}\frac{J_0(-\mu_n\sqrt{\frac{x}{l}})}{(\mu_n)^3J_1(-\mu_n)}\\
&\times\left[\int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau - \int_0^t (t-\tau)^2\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right]\\
&+\left(\frac{x}{l} - 1\right)t - \frac{x}{l}t^2, \quad \textit{for all } \mu_n \in \mathbb{R}^-,\ n = 1,2,3,\ldots
\end{aligned}
$$

Proof. Here, put $Y(x,t) = y(x,t) + \frac{x}{l}(t - t^2) - t$, in differential equation (3.1) and boundary values (3.2), and then make an appeal to the techniques applied for finding out the solution (2.20) of the problem (1.2)–(1.3) and with the aid of Theorem 2.1, we obtain the solution (3.3) of the problem (3.1)–(3.2). □

Corollary 3.1. *If $J_0(-\mu_n) = 0$ for all $n = 1, 2, 3\ldots$, and for all $x, y, z \in \mathbb{R}^+$, $\Re(\gamma+\nu) > -1$, then for all $\mu_n \in \mathbb{R}^-$ under the conditions given in the differential equation* (3.1) *and boundary values* (3.2)*, the quadrature formula of the solution* (3.3) *exists and is given by the relation*

(3.4)

$$\begin{aligned}
&\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right) Y(xu^2,t)du \\
&= \sum_{n=1}^\infty \{H_1(t;l,\mu_n) + H_2(t;l,\mu_n) + H_3(t;l,\mu_n)\} \\
&\times \left[\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^\nu \sum_{m=0}^\infty \frac{\Gamma(1+\nu+2m)}{(\Gamma(1+\nu+m)(1)_m)^2}\left(-(\mu_n)^2\frac{x}{4l}\right)^m I_m^{(1)}(\gamma,\nu,y,z)\right] + x\left(\frac{t}{l} - \frac{t^2}{l}\right) \\
&\times \left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma+2,\nu}\left(-\mu_n\sqrt{\frac{x}{l}}, y, z\right) - t\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma,\nu}\left(-\mu_n\sqrt{\frac{x}{l}}, y, z\right).
\end{aligned}$$

Here in (3.4)*, we have*

$$\begin{aligned}
(3.5)\quad H_1(t;l,\mu_n) =& \frac{1}{l[J_1(-\mu_n)]^2} E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right) \int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right) g(\xi)d\xi, \\
H_2(t;l,\mu_n) =& \frac{2}{l[J_1(-\mu_n)]^2} \int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau \int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right)\xi d\xi, \\
H_3(t;l,\mu_n) =& \frac{8l}{(\mu_n)^3 J_1(-\mu_n)}\left[\int_0^t (t-\tau)\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right. \\
&\left.\int_0^t (t-\tau)^2\frac{d}{d\tau}E_\alpha\left(-\frac{(\mu_n)^2}{4l}\tau^\alpha\right)d\tau\right], \quad \mu_n \in \mathbb{R}^-, n = 1, 2, 3, \ldots,
\end{aligned}$$

and

$$I_m^{(1)}(\gamma,\nu,\theta,\phi) = \int_0^\infty u^{\gamma+\nu+2m} e^{-\theta u - \phi u^2} du, \quad \text{for all } m \in \mathbb{N},\ \theta, \phi \in \mathbb{R}^+,$$

(*see* [10]).

Proof. In both sides of (3.3), replace x by xu^2 and then multiply by $u^\gamma e^{-yu-zu^2} J_\nu(-\mu_{n'} u\sqrt{\frac{x}{l}})$ and then integrate both the sides with respect to u from 0 to ∞, and use (1.1) and (3.5), to get the relation

$$\begin{aligned}
(3.6)\quad &\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_{n'} u\sqrt{\frac{x}{l}}\right) Y(xu^2,t)du \\
&= \sum_{n=1}^\infty H_1(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_{n'} u\sqrt{\frac{x}{l}}\right) J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du \\
&+ \sum_{n=1}^\infty H_2(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_{n'} u\sqrt{\frac{x}{l}}\right) J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du \\
&+ \sum_{n=1}^\infty H_3(t;l,\mu_n)\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_{n'} u\sqrt{\frac{x}{l}}\right) J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right)du
\end{aligned}$$

$$+ x\left(\frac{t}{l}-\frac{t^2}{l}\right)\left(-\frac{\mu_{n'}}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma+2,\nu}\left(-\mu_{n'}\sqrt{\frac{x}{l}}, y, z\right)$$
$$- t\left(-\frac{\mu_{n'}}{2}\sqrt{\frac{x}{l}}\right)^{-1/2} V_{\gamma,\nu}\left(-\mu_{n'}\sqrt{\frac{x}{l}}, y, z\right), \quad \mu_n, \mu_{n'} \in \mathbb{R}^-,\ n, n' = 1, 2, 3, \ldots.$$

Now in both sides of equation (3.6), replacing n' by n and then using the following result given in Rainville [22, p. 121]

$$J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right) J_0\left(-\mu_n u\sqrt{\frac{x}{l}}\right) = \frac{\left(-u(\frac{\mu_n}{2})\sqrt{\frac{x}{l}}\right)^\nu}{\Gamma(\nu+1)} {}_2F_3\left[\begin{matrix}\frac{1}{2}(1+\nu), \frac{1}{2}(2+\nu); \\ 1, 1+\nu, 1+\nu;\end{matrix} - u^2(\mu_n)^2\frac{x}{l}\right],$$

and the sequence of functions of mathematical physics due to [14], given by

$$I_m^{(1)}(\gamma,\nu,\theta,\phi) = \int_0^\infty u^{\gamma+\nu+2m} e^{-\theta u-\phi u^2} du, \quad \text{for all } m \in \mathbb{N},\ \theta, \phi \in \mathbb{R}^+,$$

to obtain the result (3.4). □

3.2. **The Voigt functions via homogeneous Bessel type diffusion problem, when $0 < \alpha \leq 1$.** In a similar manner of the Theorem 3.1, we present and prove the following.

Theorem 3.2. *If we put $f = 0$, $q(x) = 0$, $p(x) = x$ for all x, $0 \leq x \leq l \subset \mathbb{R}$, $\varphi_1(t) = 0$, $\varphi_2(t) = 0$, $0 < \alpha \leq 1$, $t > 0$, in the equations* (1.2)–(1.3)*, then we have Bessel type one dimensional time fractional diffusion problem*

$$(3.7) \qquad {}_t^C D_{0+}^\alpha Y(x,t) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\right] Y(x,t), \quad 0 < \alpha \leq 1,$$

for all $(x,t) \in (0,l)\times(0,\infty)$, $(0,l) \subset \mathbb{R}$, *subjected to the initial and inhomogeneous boundary values*

$$(3.8) \qquad Y(x,0) = g(x), \quad \frac{\partial}{\partial t}Y(x,t)|_{t=0} = 0, \quad \textit{for all } (x,t) \in [0,l]\times\{0\},\ [0,l] \subset \mathbb{R},$$
$$Y(0,t) = 0, \quad \frac{\partial}{\partial x}Y(x,t)|_{x=0} = 1, \quad \textit{for all } (x,t) \in \{0\}\times[0,\infty),$$
$$Y(l,t) = 0, \quad \textit{for all } (x,t) \in \{l\}\times[0,\infty).$$

Then there exists

$$(3.9) \qquad Y(x,t) = \frac{1}{l}\sum_{n=1}^\infty \frac{J_0\left(-\mu_n\sqrt{\frac{x}{l}}\right)}{[J_1(-\mu_n)]^2} E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right)\int_0^l J_0\left(-\mu_n\sqrt{\frac{\xi}{l}}\right) g(\xi)d\xi,$$

$0 < x < l$, $t > 0$.

Proof. With the aid of Theorem 2.1 and Subsection 3.1, the solution of the problems (3.7)–(3.8) is found by result (3.9). □

Corollary 3.2. *If $J_0(-\mu_n) = 0$ for all $n = 1, 2, 3 \ldots$, and for all $x, y, z \in \mathbb{R}^+$, $\Re(\gamma+\nu) > -1$, then for all $\mu_n \in \mathbb{R}^-$ under the conditions given in* (3.7) *and* (3.8)*, the quadrature formula of the solution* (3.9) *exists and is given by the relation*

(3.10)

$$\int_0^\infty u^\gamma e^{-yu-zu^2} J_\nu\left(-\mu_n u\sqrt{\frac{x}{l}}\right) Y(xu^2,t)du = \sum_{n=1}^\infty H_1(t;l,\mu_n)\left(-\frac{\mu_n}{2}\sqrt{\frac{x}{l}}\right)^\nu$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(1+\nu+2m)}{(\Gamma(1+\nu+m)(1)_m)^2} \left(-(\mu_n)^2 \frac{x}{4l}\right)^m \times I_m^{(1)}(\gamma,\nu,y,z),$$

where $H_1(t;l,\mu_n)$ *and* $I_m^{(1)}(\gamma,\nu,\theta,\phi)$ *are given in* (3.5).

4. Numerical Example

In this section, we consider more briefly a computational formula starting from $Y(x,t)$, $0<x<l$, $t\in\mathbb{R}^+$ and using the Theorem 3.2.

If we set $g(x)=\frac{1}{2}$ for all x, $0<x<l$, in (3.9), we find a numerical formula

$$Y(x,t)=\sum_{n=1}^{\infty} \frac{J_0\left(-\mu_n\sqrt{\frac{x}{l}}\right)}{(-\mu_n)J_1(-\mu_n)} E_\alpha\left(-\frac{(\mu_n)^2}{4l}t^\alpha\right). \tag{4.1}$$

A fairly immediate consequence of this result is its use for obtaining the approximate various real values of $Y(x,t)$. According to our formalism we now in (4.1), introduce the approximate value of $E_\alpha(-x)$, given by (see [27])

$$E_\alpha(-x)=\frac{1+\frac{1}{\Gamma(1-\alpha)q_0^*}x}{1+\frac{q_1^*}{q_0^*}x+\frac{1}{q_0^*}x^2},$$

where

$$q_0^*=\frac{\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}-\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}}{\Gamma(1+\alpha)\Gamma(1-\alpha)-1}$$

and

$$q_1^*=\frac{\Gamma(1+\alpha)-\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}}{\Gamma(1+\alpha)\Gamma(1-\alpha)-1}.$$

Again from the formula (1.1), it follows that

$$J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}, \quad J_1(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)(m!)^2}\left(\frac{x}{2}\right)^{2m+1}.$$

Now putting the zeros of $J_0(x)$ as μ_i, $i=1,2,\ldots,n$, together with the values of α and l such that $0<\alpha<1$ and $l>0$, we can provide several examples with selected values of n to compute and approximate various real values of $Y(x,t)$, for all $x,t\in\mathbb{R}^+$. We omit them due to lack of space and left them for further researchers in the field of computer science and technology.

Conclusion

Explicit expressions for the generalized Voigt functions [10, 19, 25] and [26] of the second kind defined by the Hankel integral transform (1.1) are given in terms of relatively more familiar special functions of one and more variables, indeed, each of these representations will naturally lead to various other needed properties of the Voigt functions. Here, in our work, we have obtained the relations of the Voigt functions with the quadrature formula of the solution of fractional in time diffusion and wave problem by first converting it into the Sturm-Liouville problems and then looked out for its solutions. This concept may provide the basis of investigations and further extensions for a high voltage technology to compute the fractional differential equations, anomalous diffusion problems and fractional in time and space diffusion and wave problems with the help of Voigt functions.

To explore new ideas for representing the relation of the Voigt functions (1.1) with the quadrature formula of the solution of fractional in time diffusion and wave problem, in our current investigation, we have presented fractional in time Sturm-Liouville type diffusion and wave equation. In the paper of Luchko [14] (see also [15]), some initial-boundary-value problems with the Dirichlet boundary conditions for the time-fractional diffusion equation were considered. Of course, the same method can be applied for the initial boundary value problems with the Neumann, Robin, or mixed boundary conditions.

Besides establishing some interesting integral and series representations of special functions, the results given in [13] and [14] may provide a new way of solution of a space-time fractional anomalous diffusion problem using the series of bilateral eigenfunctions and series solution for initial value problems of time fractional generalized anomalous diffusion equations as on the lines of [11, 12] and [13].

Acknowledgements. The authors are highly grateful to the editor and reviewer for their valuable comments and suggestions to improve the presentation of the paper.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 773–787.

(F, G)-DERIVATIONS ON A LATTICE

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Abstract. In the present paper, we introduce the notion of (F, G)-derivation on a lattice as a generalization of the notion of $(\wedge, \vee)$-derivation. This newly notion is based on two arbitrary binary operations F and G instead of the meet $(\wedge)$ and the join $(\vee)$ operations. Also, we investigate properties of (F, G)-derivation on a lattice in details. Furthermore, we define and study the notion of principal (F, G)-derivations as a particular class of (F, G)-derivations. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations.

1. Introduction

Binary operations are among the oldest fundamental concepts in algebraic structures. Since their introduction, they have become the key notion in the consepts of groups, monoids, semigroups, rings, and in more algebraic structures studied in abstract algebra [6, 15]. Binary operations have become essential tools in lattice theory and its applications [8]. Several notions and properties, and the notion of the lattice itself can be interpreted in terms of binary operations on it [5, 21]. Furthermore, it is not surprising that binary operations with specific properties appear in various theoretical and application fields. For instance, aggregation functions (as binary operations with specific properties) on bounded lattices and their wide use in various fields of applied sciences, including, computer and information sciences, economics, and social sciences (see, e.g., [9, 11] and [12, 13, 16, 18]). Also, they play an important role (as generalization of the basic connectives between fuzzy sets) in theories of fuzzy sets and logic [3].

Key words and phrases. Lattice, (F, G)-derivation, principal (F, G)-derivation, lattice representation.

2010 *Mathematics Subject Classification.* Primary: 06B05, 03G10. Secondary: 06B99.

DOI 10.46793/KgJMat2205.773A

Received: August 03, 2019.

Accepted: March 25, 2020.

The notion of derivation appeared first on the ring structures and it has many applications (see, e.g., [2]). Szász [23] has extended this notion to the lattice structures based on the meet and the join operations ($(\wedge, \vee)$-derivation, for short), i.e., a $(\wedge, \vee)$-derivation on given lattice L is a function d of L into itself satisfying the following two conditions: $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ and $d(x \vee y) = d(x) \vee d(y)$ for any $x, y \in L$. Ferrari [7] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin et al. [27] have ameliorated the notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on a distributive lattice. In the same paper, they have characterized the distributive and modular lattices in terms of their isotone derivations. Later on, Xin [26] has focused his attention to the structure of the set of fixed points of a derivation on a lattice and has shown some relationships between lattice ideals and this set of fixed points.

The notion of $(\wedge, \vee)$-derivation on a lattice is witnessing increased attention. It studies, among others, in partially ordered sets [1, 31], in distributive lattices [30], in semilattices [29], in bounded hyperlattices [24], in quantales and residuated lattices [10, 25] and in several kinds of algebras [14, 17, 19]. Furthermore, it used in the definition of congruences and ideals in a distributive lattice [20].

In this paper, we generalize the notion of $(\wedge, \vee)$-derivation on a lattice to the (F, G)-derivation, where F and G are arbitrary binary operations on that lattice. More precisely, we introduce the notion of derivation on a lattice L with respect to two arbitrary binary operations F and G on L instead of the meet $(\wedge)$ and the join $(\vee)$ operations of L. Also, we investigate their properties in details. Furthermore, we define the principal (F, G)-derivations as a particular class of (F, G)-derivations on a lattice, and we study their various properties. Specific attention is paid to the lattice structure of the poset of principal (F, G)-derivations on a lattice. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. These representations are draw upon some properties of binary operations on a lattice we investigated in [28].

The remainder of the paper is structured as follows. In Section 2, we recall the necessary basic concepts and properties of lattices and binary operations on lattices. In Section 3, we introduce the notion of (F, G)-derivation on a lattice and investigate their properties. In Section 4, we define the principal (F, G)-derivations on a lattice and study their various properties. In Section 5, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. Finally, we present some concluding remarks in Section 6.

2. Basic concepts

In this section, we recall the necessary basic concepts and properties of lattices and binary operations on lattices.

2.1. **Lattice.** An *order relation* $\leqslant$ on a set X is a binary relation on X that is *reflexive* (i.e., $x \leqslant x$, for any $x \in X$), *antisymmetric* (i.e., $x \leqslant y$ and $y \leqslant x$ imply $x = y$, for any $x, y \in X$) and *transitive* (i.e., $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$, for any $x, y, z \in X$). A set X equipped with an order relation $\leqslant$ is called a *partially ordered set* (a *poset*, for short), denoted $(X, \leqslant)$. Let $(X, \leqslant)$ be a poset and A be a subset of X. An element $x_0 \in X$ is called a *lower bound* of A if $x_0 \leqslant x$, for any $x \in A$. x_0 is called the *greatest lower bound* (or the *infimum*) of A if x_0 is a lower bound and $m \leqslant x_0$, for any lower bound m of A. *Upper bound* and *least upper bound* (or *supremum*) are defined dually.

A poset $(L, \leqslant)$ is called a $\wedge$-semi-lattice if any two elements x and y have a greatest lower bound, denoted by $x \wedge y$ and called the meet (infimum) of x and y. Analogously, it is called a $\vee$-semi-lattice if any two elements x and y have a smallest upper bound, denoted by $x \vee y$ and called the join (supremum) of x and y. A poset $(L, \leqslant)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice. Usually, the notation $(L, \leqslant, \wedge, \vee)$ is used. A poset $(L, \leqslant)$ is called bounded if it has a smallest and a greatest element, respectively denoted by 0 and 1. Often, the notation $(L, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice. A lattice $(L, \leqslant, \wedge, \vee)$ is called distributive if one of the following two equivalent conditions hold:

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$;
(a^δ) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$.

Let $(X, \leqslant)$ and $(Y, \preceq)$ be two posets. A mapping φ from X into Y is called an *order isomorphism* if it is surjective and satisfies

$$x \leqslant y \quad \text{if and only if} \quad \varphi(x) \preceq \varphi(y), \quad \text{for any } x, y \in X.$$

Let $(L, \leqslant, \wedge, \vee)$ and $(M, \preceq, \frown, \smile)$ be two lattices. A mapping φ from L into M is called a *lattice homomorphism*, if it satisfies $\varphi(x \wedge y) = \varphi(x) \frown \varphi(y)$ and $\varphi(x \vee y) = \varphi(x) \smile \varphi(y)$, for any $x, y \in L$. A *lattice isomorphism* is a bijective lattice homomorphism.

The following proposition shows that an order isomorphism between two lattices is a lattice isomorphism.

Proposition 2.1 ([5])**.** *Let L, M be two lattices and φ be a mapping from L into M. The following statements are equivalent:*

(i) *φ is an order isomorphism;*
(ii) *φ is a lattice isomorphism.*

For more details on lattices, we refer to [5, 15, 21, 22].

2.2. **Binary operations on a lattice.** In this subsection, we recall properties of binary operations on a lattice.

Let F be a binary operation on a non-empty set X. An element $e \in X$ is called:

(i) a right- (resp. left-) neutral element of F, if $F(x, e) = x$ (resp. $F(e, x) = x$) for any $x \in X$;

(ii) neutral element of F, if it is right- and left-neutral element, i.e., $F(e,x) = F(x,e) = x$ for any $x \in X$.

An element $k \in X$ is called:

(i) a right- (resp. left-) absorbing element of F, if $F(x,k) = k$ (resp. $F(k,x) = k$) for any $x \in X$;

(ii) an absorbing element of F, if it is right- and left-absorbing element, i.e.,

$$F(x,k) = F(k,x) = k, \quad \text{for any } x \in X.$$

The following properties of binary operations on a lattice are interest in this paper.

Definition 2.1 ([28])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. F is called:

(i) *idempotent*, if $F(x,x) = x$ for any $x \in L$;

(ii) *conjunctive*, if $F(x,y) \leqslant x \wedge y$ for any $x, y \in L$;

(iii) *increasing with respect to the first variable*, if $x \leqslant y$, implies $F(x,z) \leqslant F(y,z)$ for any $x, y, z \in L$;

(iv) *increasing*, if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ implies $F(x_1, y_1) \leqslant F(x_2, y_2)$ for any $x_1, y_1, x_2, y_2 \in L$.

In what follows, the statement $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, can be written shortly by using the coordinate-wise order $(x_1, y_1) \leqslant_{L \times L} (x_2, y_2)$.

Definition 2.2 ([4])**.** A *triangular norm* (*t-norm*, for short) T on a bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is a commutative, associative and increasing binary operation on L, and it has the neutral element $1 \in L$. Dually, a *triangular conorm* (*t-conorm*, for short) S on L, is a commutative, associative and increasing binary operation on L, and it has the neutral element $0 \in L$.

3. (F, G)-Derivations on a Lattice

In this section, we introduce the notion of (F, G)-derivation on a lattice and investigate their properties. This newly notion is a natural generalization of the notion of derivation on a lattice given by Xin et al. [27] with respect to the meet and join operations.

3.1. **Definitions and examples.** The notion of derivation on a lattice was first introduced by Szász [23].

Definition 3.1 ([23])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a *derivation* on L if it satisfies the following two conditions:

(D_1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ for any $x, y \in L$;

(D_2) $d(x \vee y) = d(x) \vee d(y)$ for any $x, y \in L$.

Later on, Xin et al. [27] have reduced the above conditions by considering only the condition (D_1). Moreover, they have shown that the condition (D_2) obviously holds for the isotone derivations on a distributive lattice.

Definition 3.2 ([27])**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a *derivation* on L if it satisfies the (D_1) condition, i.e.,

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \quad \text{for any } x, y \in L.$$

Inspired by the above Definition 3.2, we introduce the core definition of this paper. It is based on two arbitrary binary operations on a lattice.

Definition 3.3. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L. A function $d : L \to L$ is called an (F, G)-*derivation* on L if it satisfies the following condition:

$$d(F(x, y)) = G(F(d(x), y), F(x, d(y))), \quad \text{for any } x, y \in L.$$

In the rest of the paper, we shortly write dx instead of $d(x)$ and $dF(x, y)$ instead of $d(F(x, y))$.

In the following, we present some illustrative examples of (F, G)-derivations on a lattice.

Example 3.1. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that G is idempotent. The identity function of L is an (F, G)-derivation on L. Indeed, suppose that d is the identity function of L. The fact that G is idempotent implies that $dF(x, y) = F(x, y) = G(F(x, y), F(x, y)) = G(F(dx, y), F(x, dy))$ for any $x, y \in L$. Hence, d is an (F, G)-derivation on L.

Example 3.2. Let $(\mathbb{N}^*, \leqslant, \min, \max)$ be the lattice of positive integers and $\alpha \in \mathbb{N}^*$. Then the following hold.

- (i) The null function of $\mathbb{N}$ is a $(\cdot, +)$-derivation on $\mathbb{N}$, but it is not a $(+, \cdot)$-derivation on $\mathbb{N}$.
- (ii) The translation function d_1 of $\mathbb{N}^*$ defined by $d_1(x) = x + \alpha$ for any $x \in \mathbb{N}^*$, it is both $(+, \mathrm{gcd})$-derivation and $(+, \mathrm{lcm})$-derivation on $\mathbb{N}^*$.
- (iii) The homothety function d_2 of $\mathbb{N}^*$ defined by $d_2(x) = \alpha \cdot x$ for any $x \in \mathbb{N}^*$, it is both $(\cdot, \mathrm{gcd})$-derivation and $(\cdot, \mathrm{lcm})$-derivation on $\mathbb{N}^*$.

Remark 3.1. We note that any derivation on a lattice L is a $(\wedge, \vee)$-derivation on L.

Definition 3.4. Let $(L, \leqslant, \wedge, \vee)$ be a lattice. An (F, G)-derivation d on L is called *isotone* if it satisfies the following condition:

$$x \leqslant y \quad \text{implies} \quad dx \leqslant dy, \quad \text{for any } x, y \in L.$$

Example 3.3. The translation (resp. homothety) function given in Example 3.2 is both isotone $(+, \mathrm{gcd})$-derivation and isotone $(+, \mathrm{lcm})$-derivation (resp. isotone $(\cdot, \mathrm{gcd})$-derivation and isotone $(\cdot, \mathrm{lcm})$-derivation) on $\mathbb{N}^*$.

3.2. **Properties of (F, G)-derivations on a lattice.** In this subsection, we investigate some properties of (F, G)-derivations on a lattice.

Proposition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. The following implications hold.*

(i) *If* G *is conjunctive, then* $dF(x,y) \leqslant F(dx,y) \wedge F(x,dy)$ *for any* $x,y \in L$.
(ii) *If* G *is disjunctive, then* $F(dx,y) \vee F(x,dy) \leqslant dF(x,y)$ *for any* $x,y \in L$.

Proof. (i) The fact that d is an (F,G)-derivation on L and G is conjunctive imply that $dF(x,y) = G(F(dx,y),F(x,dy)) \leqslant F(dx,y) \wedge F(x,dy)$, for any $x,y \in L$. Thus, $dF(x,y) \leqslant F(dx,y) \wedge F(x,dy)$ for any $x,y \in L$.

(ii) The proof is similar to that of (i). □

The above proposition leads to the following corollary.

Corollary 3.1. *Let* $(L,\leqslant,\wedge,\vee)$ *be a lattice and* d *be an* (F,G)*-derivation on* L*. The following implications hold.*

(i) *If* F *and* G *are conjunctive, then* $dF(x,y) \leqslant dx \wedge dy \wedge x \wedge y$ *for any* $x,y \in L$.
(ii) *If* F *and* G *are disjunctive, then* $dx \vee dy \vee x \vee y \leqslant dF(x,y)$ *for any* $x,y \in L$.

Proposition 3.2. *Let* $(L,\leqslant,\wedge,\vee)$ *be a lattice and* d *be an* (F,G)*-derivation on* L*. The following implications hold.*

(i) *If* F *is conjunctive and* G *is increasing, then* $dF(x,y) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy)$ *for any* $x,y \in L$.
(ii) *If* F *is disjunctive and* G *is increasing, then* $G(dx,dy) \vee G(y,x) \vee G(dx,x) \vee G(y,dy) \leqslant dF(x,y)$ *for any* $x,y \in L$.

Proof. (i) Let $x,y \in L$, the conjunctivity of F guarantees that $F(dx,y) \leqslant dx \wedge y$ and $F(x,dy) \leqslant x \wedge dy$. Since G is increasing, it holds that

$$G(F(dx,y),F(x,dy)) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy).$$

The fact that d is an (F,G)-derivation on L implies that

$$dF(x,y) \leqslant G(dx,dy) \wedge G(y,x) \wedge G(dx,x) \wedge G(y,dy).$$

(ii) The proof is similar to that of (i). □

Theorem 3.1. *Let* $(L,\leqslant,\wedge,\vee)$ *be a lattice and* d *be an* (F,G)*-derivation on* L*. If* F *and* G *are increasing and conjunctive, then the following statements hold for any* $x,y \in L$*:*

(i) $G(F(dx,dF(y,y)),F(dF(x,x),dy)) \leqslant dF(x,y) \wedge F(dx,dy)$;
(ii) $G(F(dF(x,x),y),F(x,dF(y,y))) \leqslant dF(x,y) \wedge F(x,y)$;
(iii)

$$\begin{aligned} &G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \\ \leqslant &dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy). \end{aligned}$$

Proof. To prove (i), suppose that d is an (F,G)-derivation on L and let $x,y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then

$$\begin{cases} ((dx,dF(y,y)),(dF(x,x),dy)) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)), \\ ((dx,dF(y,y)),(dF(x,x),dy)) \leqslant_{L^2\times L^2} ((dx,dy),(dx,dy)). \end{cases}$$

The fact that F is increasing implies that

$$\begin{cases} (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx,y), F(x,dy)), \\ (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx,dy), F(dx,dy)). \end{cases}$$

Since G is increasing and conjunctive, and d is an (F, G)-derivation on L, then it follows that

$$\begin{cases} G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx,y), F(x,dy)) = dF(x,y), \\ G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx,dy), F(dx,dy)) \leqslant F(dx,dy). \end{cases}$$

Thus, $G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant dF(x,y) \wedge F(dx,dy)$.

To demonstrate (ii), let $x, y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then

$$\begin{cases} ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)), \\ ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((x,y),(x,y)). \end{cases}$$

Since F is increasing, it holds that

$$\begin{cases} (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(dx,y), F(x,dy)), \\ (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(x,y), F(x,y)). \end{cases}$$

The fact that G is increasing and conjunctive, and d is an (F, G)-derivation on L, it implies that

$$\begin{cases} G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(dx,y), F(x,dy)) = dF(x,y), \\ G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(x,y), F(x,y)) \leqslant F(x,y). \end{cases}$$

Therefore, $G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant dF(x,y) \wedge F(x,y)$.

For proving (iii), let $x, y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then

$$\begin{cases} ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)), \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((x,y),(x,y)), \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((x,dy),(x,dy)), \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(dx,y)), \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,dy),(dx,dy)). \end{cases}$$

The fact that F is increasing implies that

$$\begin{cases} (F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant_{L\times L} (F(dx,y), F(x,dy)), \\ (F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant_{L\times L} (F(x,y), F(x,y)), \\ (F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant_{L\times L} (F(x,dy), F(x,dy)), \\ (F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant_{L\times L} (F(dx,y), F(dx,y)), \\ (F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant_{L\times L} (F(dx,dy), F(dx,dy)). \end{cases}$$

Since G is increasing, it holds that

$$\begin{cases} G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(x,dy)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,y),F(x,y)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,dy),F(x,dy)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(dx,y)), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,dy),F(dx,dy)). \end{cases}$$

The conjunctivity of G and the fact that d is an (F,G)-derivation on L assure that

$$\begin{cases} G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,dy), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,y), \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,dy). \end{cases}$$

Thus, $G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy)$. □

Analogously, we obtain the following result for increasing and disjunctive binary operations.

Theorem 3.2. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and d be an (F,G)-derivation on L. If F and G are increasing and disjunctive, then the following statements hold for any $x,y \in L$:*

- (i) $dF(x,y) \vee F(dx,dy) \leqslant G(F(dx,dF(y,y)),F(dF(x,x),dy))$;
- (ii) $dF(x,y) \vee F(x,y) \leqslant G(F(dF(x,x),y),F(x,dF(y,y)))$;
- (iii) $dF(x,y) \vee F(x,y) \vee F(x,dy) \vee F(dx,y) \vee F(dx,dy) \leqslant G(F(dF(x,x),dF(y,y)), F(dF(x,x),dF(y,y)))$.

Proposition 3.3. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and d be an (F,G)-derivation on L. The following implications hold:*

- (i) *If F is conjunctive, G is increasing and idempotent, then $dF(x,x) \leqslant x \wedge dx$ for any $x \in L$;*
- (ii) *If F is disjunctive, G is increasing and idempotent, then $x \vee dx \leqslant dF(x,x)$ for any $x \in L$.*

Proof. (i) Let $x \in L$, since F is conjunctive, it holds that $(F(dx,x),F(x,dx)) \leqslant_{L\times L} (x \wedge dx, x \wedge dx)$. The fact that G is increasing implies that

$$G(F(dx,x),F(x,dx)) \leqslant G(x \wedge dx, x \wedge dx).$$

Since d is an (F,G)-derivation on L and G is idempotent, it follows that $dF(x,x) \leqslant x \wedge dx$.

(ii) The proof is similar to that of (i). □

The following theorems give alternative conditions to that of Theorems 3.1 and 3.2. Their proofs can be done in a similar way.

Theorem 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F is increasing and conjunctive, G is increasing and idempotent, then the following statements hold for any $x, y \in L$:*

(i) $G(F(dx, dF(y, y)), F(dF(x, x), dy)) \leqslant dF(x, y) \wedge F(dx, dy)$;

(ii) $G(F(dF(x, x), y), F(x, dF(y, y))) \leqslant dF(x, y) \wedge F(x, y)$;

(iii) $F(dF(x, x), dF(y, y)) \leqslant dF(x, y) \wedge F(x, y) \wedge F(x, dy) \wedge F(dx, y) \wedge F(dx, dy)$.

Theorem 3.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F is increasing and disjunctive, G is increasing and idempotent, then the following statements hold for any $x, y \in L$:*

(i) $dF(x, y) \vee F(dx, dy) \leqslant G(F(dx, dF(y, y)), F(dF(x, x), dy))$;

(ii) $dF(x, y) \vee F(x, y) \leqslant G(F(dF(x, x), y), F(x, dF(y, y)))$;

(iii) $dF(x, y) \vee F(x, y) \wedge F(x, dy) \vee F(dx, y) \vee F(dx, dy) \leqslant F(dF(x, x), dF(y, y))$.

Proposition 3.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F is commutative and G is idempotent, then*

$$dF(x, x) = F(dx, x) = F(x, dx), \quad \text{for any } x \in L.$$

Proof. Since d is an (F, G)-derivation on L, F is commutative and G is idempotent, it holds that $dF(x, x) = G(F(dx, x), F(x, dx)) = G(F(dx, x), F(dx, x)) = F(dx, x)$ for any $x \in L$. □

Proposition 3.4 leads to the following corollary.

Corollary 3.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. The following implications hold:*

(i) *If F is commutative and conjunctive and G is idempotent, then*

$$dF(x, x) \leqslant x \wedge dx, \quad \text{for any } x \in L;$$

(ii) *If F is commutative and disjunctive and G is idempotent, then*

$$x \vee dx \leqslant dF(x, x), \quad \text{for any } x \in L.$$

Proposition 3.5. *Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with the least element $0 \in L$ and d be an (F, G)-derivation on L. The following implications hold.*

(i) *If F and G are conjunctive, then $d0 = 0$ (i.e., 0 is a fixed point of d).*

(ii) *If F is conjunctive, G is increasing and idempotent, then $d0 = 0$.*

(iii) *If F is commutative and conjunctive, and G is idempotent, then $d0 = 0$.*

Proof. (i) The conjunctivity of F implies that $F(0, 0) = 0$. Then $d0 = dF(0, 0)$. From Corollary 3.1 (i), it holds that $dF(0, 0) \leqslant 0$. Thus, $d0 = 0$.

(ii) The proof is similar to that of (i) by using Proposition 3.3 (i).

(iii) The proof is similar to that of (i) by using Corollary 3.2 (i). □

In the same line, we obtain the following result.

Proposition 3.6. *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and d be an (F, G)-derivation on L. The following implications hold.*

(i) *If F and G are disjunctive, then $d1 = 1$ (i.e., 1 is a fixed point of d).*
(ii) *If F is disjunctive, G is increasing and idempotent, then $d1 = 1$.*
(iii) *If F is commutative and disjunctive and G is idempotent, then $d1 = 1$.*

Proposition 3.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F has a right- (resp. a left-) neutral element $e \in L$, then $dx = G(dx, F(x, de))$ (resp. $dx = G(F(de, x), dx))$ for any $x \in L$.*

Proof. The fact that e is a right- (resp. a left-) neutral element of F and d is an (F, G)-derivation on L imply that $dx = dF(x, e) = G(F(dx, e), F(x, de)) = G(dx, F(x, de))$ (resp. $dx = dF(e, x) = G(F(de, x), F(e, dx)) = G(F(de, x), dx)$) for any $x \in L$. ☐

Proposition 3.8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F has a right- (resp. a left-) absorbing element $k \in L$, then $dk = G(k, F(x, dk))$ (resp. $dk = G(F(dk, x), k))$ for any $x \in L$.*

Proof. Since k is a right- (resp. a left-) absorbing element of F and d is an (F, G)-derivation on L, then $dk = dF(x, k) = G(F(dx, k), F(x, dk)) = G(k, F(x, dk))$ (resp. $dk = dF(k, x) = G(F(dk, x), F(k, dx)) = G(F(de, x), k)$) for any $x \in L$. ☐

The above Proposition 3.8 leads to the following corollary.

Corollary 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an (F, G)-derivation on L. If F and G have a right- or a left-absorbing element $k \in L$, then $dk = k$ (i.e., k is a fixed point of d).*

4. Principal (F, G)-Derivations on a Lattice

In this section, we introduce the notion of principal (F, G)-derivation on a lattice and investigate their various properties.

4.1. Definitions and auxiliary results.

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. For any element $\alpha \in L$, there exists an F-function $f_\alpha : L \to L$ defined as:

$$f_\alpha(x) = F(\alpha, x), \quad \text{for any } x \in L.$$

Let $\mathcal{A}_F(L)$ be the set of the f_α functions on L, i.e., $\mathcal{A}_F(L) = \{f_\alpha \mid \alpha \in L\}$. One can easily verify that $\mathcal{A}_F(L)$ equipped with the usual order of functions (i.e., $f_\alpha \preceq f_\beta$ if and only if $f_\alpha(x) \leqslant f_\beta(x)$ for any $x \in L$) is a poset.

The following propositions show some cases that the poset $(\mathcal{A}_F(L), \preceq)$ has a lattice structure.

Proposition 4.1 ([28])**.** *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and F be a binary operation on L. If F is increasing with respect to the first variable and having a right-neutral element $e\in L$, then the poset $(\mathcal{A}_F(L),\preceq)$ is a lattice. Where, the meet $\frown$ and the join $\smile$ operations of $\mathcal{A}_F(L)$ are defined as $f_\alpha\frown f_\beta=f_{\alpha\wedge\beta}$ and $f_\alpha\smile f_\beta=f_{\alpha\vee\beta}$ for any $f_\alpha,f_\beta\in\mathcal{A}_F(L)$.*

Proposition 4.2 ([28])**.** *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and F be a binary operation on L. If F is the meet (resp. the join) operation of L, then $(\mathcal{A}_F(L),\preceq,\frown,\smile)$ is a lattice.*

Proposition 4.3 ([28])**.** *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and F be a binary operation on L. Then the following hold.*

(*i*) *If F is increasing with respect to the first variable and having a right-neutral element $e\in L$, then for any $\alpha,\beta\in L$, we obtain that*
$$\alpha\leqslant\beta \quad \text{if and only if} \quad f_\alpha\preceq f_\beta.$$
(*ii*) *If F is idempotent, conjunctive or disjunctive and increasing with respect to the first variable, then for any $\alpha,\beta\in L$, we obtain that*
$$\alpha\leqslant\beta \quad \text{if and only if} \quad f_\alpha\preceq f_\beta.$$

The following proposition provides some conditions that the elements of $\mathcal{A}_F(L)$ are (F,G)-derivations on L.

Proposition 4.4. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and F,G be two binary operations on L such that F is commutative and associative, and G is idempotent. Then any elements of $\mathcal{A}_F(L)$ is an (F,G)-derivation on L.*

Proof. Let $f_\alpha\in\mathcal{A}_F(L)$ and $x,y\in L$. We will show that
$$f_\alpha(F(x,y))=G(F(f_\alpha(x),y),F(x,f_\alpha(y)))\,.$$
Since F is commutative and associative, and G is idempotent, it follows that
$$\begin{aligned}
f_\alpha(F(x,y))&=F(\alpha,F(x,y))\\
&=G(F(\alpha,F(x,y)),F(\alpha,F(x,y)))\\
&=G(F(\alpha,F(x,y)),F(F(\alpha,x),y))\\
&=G(F(\alpha,F(x,y)),F(F(x,\alpha),y))\\
&=G(F(F(\alpha,x),y),F(x,F(\alpha,y)))\\
&=G(F(f_\alpha(x),y),F(x,f_\alpha(y)))\,.
\end{aligned}$$
Thus, f_α is an (F,G)-derivation on L. □

In view of Proposition 4.4, we introduce the notion of principal (F,G)-derivation on a given lattice.

Definition 4.1. Let $(L,\leqslant,\wedge,\vee)$ be a lattice and F,G be two binary operations on L such that F is commutative and associative, and G is idempotent. Any f_α function is called *principal (F,G)-derivation* on L. In this case, $\mathcal{A}_F(L)$ denotes the set of the principal (F,G)-derivations on L.

Example 4.1. Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, $\alpha \in L$, T be a t-norm on L and S a t-conorm on L. The function t_α (resp. s_α) defined for any $x \in L$ by $t_\alpha(x) = T(\alpha, x)$ (resp. $s_\alpha(x) = S(\alpha, x)$) is a principal (T, G)-derivation (resp. principal (S, G)-derivation) on L, for any idempotent binary operation G on L.

4.2. **Properties of principal (F, G)-derivations on a lattice.** In this subsection, we investigate some properties of principal (F, G)-derivations on a given lattice.

Proposition 4.5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. Then it holds that*

$$f_\alpha \circ f_\alpha(F(x, y)) = F(f_\alpha(x), f_\alpha(y)), \quad \text{for any } x, y \in L\,.$$

Proof. Let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L and $x, y \in L$. The facts that F is commutative and associative imply that

$$\begin{aligned} f_\alpha \circ f_\alpha(F(x, y)) &= f_\alpha(f_\alpha(F(x, y))) = f_\alpha F(\alpha, F(x, y)) = F(\alpha, F(\alpha, F(x, y))) \\ &= F(\alpha, F(F(\alpha, x), y)) = F(F(\alpha, F(\alpha, x)), y) = F(F(F(\alpha, x), \alpha), y) \\ &= F(F(\alpha, x), F(\alpha, y)) = F(f_\alpha(x), f_\alpha(y)). \end{aligned}$$

□

Proposition 4.6. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. If F is increasing, then f_α is isotone, i.e., if $x \leqslant y$, then $f_\alpha(x) \leqslant f_\alpha(y)$, for any $x, y \in L$.*

Proof. Let $x, y \in L$ such that $x \leqslant y$. Since F is increasing, it holds that $F(\alpha, x) \leqslant F(\alpha, y)$, i.e., $f_\alpha(x) \leqslant f_\alpha(y)$. Thus, the principal (F, G)-derivation f_α is isotone. □

One can easily verify that if the principal (F, G)-derivations on L are isotone, then F is increasing.

Proposition 4.6 leads to the following corollary.

Corollary 4.1. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, T be a t-norm and S be a t-conorm on L. Then the principal (T, G)-derivations (resp. principal (S, G)-derivations) on L are isotone for any idempotent binary operation G on L.*

Proposition 4.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. If F is increasing, then f_α satisfies that if $x \leqslant y$, then $f_\alpha(F(x, x)) \leqslant f_\alpha(F(y, y))$ for any $x, y \in L$.*

Proof. Let $x, y \in L$ such that $x \leqslant y$. The fact that F is associative implies that $f_\alpha(F(x, x)) = F(\alpha, F(x, x)) = F(F(\alpha, x), x)$ and $f_\alpha(F(y, y)) = F(F(\alpha, y), y)$. Since $x \leqslant y$ and F is increasing, it follows that $F(F(\alpha, x), x) \leqslant F(F(\alpha, y), y)$. Thus, $f_\alpha(F(x, x)) \leqslant f_\alpha(F(y, y))$. □

5. Representations of a Lattice in Terms of its Principal (F, G)-Derivations

In this section, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. We start by the first one.

Theorem 5.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. If F is increasing and having a neutral element $e \in L$, then the lattice $(L, \leqslant, \wedge, \vee)$ is isomorphic to the lattice $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F,G)-derivations on L.*

Proof. Proposition 4.1 guarantees that the poset $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F,G)-derivations on L is a lattice. Next, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha) = f_\alpha$, for any $\alpha \in L$. One can easily verify that ψ is surjective. Furthermore, from Proposition 4.3 (i), it holds that

$$\alpha \leqslant \beta \quad \text{if and only if} \quad \psi(\alpha) \preceq \psi(\beta), \quad \text{for any } \alpha, \beta \in L.$$

Now, ψ is an order isomorphism between L and $\mathcal{A}_F(L)$. Thus, Proposition 2.1 guarantees that ψ is a lattice isomorphism. Therefore, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ are isomorphic. □

In the following, we present an illustrative example of Theorem 5.1.

Example 5.1. Let $L = \mathbb{R}_+^*$ be the lattice of the positive real numbers ordered by the usual order, and let F, G be two binary operations on $\mathbb{R}_+^*$ defined for any $x, y \in \mathbb{R}_+^*$ as $F(x,y) = x \cdot y$ and

$$G(x,y) = \begin{cases} x, & \text{if } x = y, \\ x + y, & \text{otherwise.} \end{cases}$$

One can easily verify that F is commutative and associative, and G is idempotent. Furthermore, F is increasing and having the neutral element $1 \in \mathbb{R}_+^*$. Theorem 5.1 guarantees that the lattice $(\mathbb{R}_+^*, \leqslant, \min, \max)$ is isomorphic to the lattice $(\mathcal{A}_F(\mathbb{R}_+^*), \preceq, \frown, \smile)$ of principal (F,G)-derivations on $\mathbb{R}_+^*$.

Theorem 5.1 leads to the following corollary.

Corollary 5.1. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, G be an idempotent binary operation, T be a t-norm and S be a t-conorm on L. Then the following hold.*

(i) The bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is isomorphic to the bounded lattice $(\mathcal{A}_T(L), \preceq, \frown, \smile, t_0, t_1)$ of principal (T,G)-derivations on L, where $t_0(x) = T(0,x) = 0$ and $t_1(x) = T(1,x) = x$ for any $x \in L$.

(ii) The bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is isomorphic to the bounded lattice $(\mathcal{A}_S(L), \preceq, \frown, \smile, s_0, s_1)$ of principal (S,G)-derivations on L, where $s_0(x) = S(0,x) = x$ and $s_1(x) = S(1,x) = 1$ for any $x \in L$.

The following theorem provides the second representation of a lattice in terms of its principal (F,G)-derivations.

Theorem 5.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F, G be two binary operations on L such that G is idempotent. If F is the meet (resp. the join) operation of L, then the lattice $(L, \leqslant, \wedge, \vee)$ is isomorphic to the lattice $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F,G)-derivations on L.*

Proof. Since F is the meet (resp. the join) operation of L, it follows from Proposition 4.2 that the poset $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L is a lattice. Now, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha) = f_\alpha$, for any $\alpha \in L$. Using the same steps as Theorem 5.2, we obtain that ψ is a lattice isomorphism. Thus, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ are isomorphic. □

Remark 5.1. We note the following.

(i) If $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice, F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee, 0, 1)$ can be represented by using both Theorems 5.1 and 5.2.

(ii) If $(L, \leqslant, \wedge, \vee)$ is a latticea and it does not have the least element 0 (resp. greatest element 1), F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee)$ can be represented only by Theorem 5.2.

(iii) If $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and F, G are respectively the meet and the join operations of L, then Theorem 5.2 coincides with representation theorem given by Xin et al. (Theorem 3.29 in [27]). Thus, Theorem 5.2 is a generalization of Theorem 3.29 to an arbitrary lattice.

6. Conclusion

In this work, based on two arbitrary binary operations F and G on a given lattice, we have introduced the notion of (F, G)-derivation on a lattice as a generalization to the notion of $(\wedge, \vee)$-derivation. Also, we have investigated their various properties. We have defined and studied the principal (F, G)-derivations as a particular class of (F, G)-derivations on a lattice. As applications, we have provided two representations of a given lattice in terms of its principal (F, G)-derivations.

Acknowledgements. The authors thank the editors and the referees for their valuable comments and suggestions for improving the article.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 789–795.

INFINITELY MANY SOLUTIONS TO A FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATION WITH TWO CONTROL PARAMETERS

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Abstract. In this article, we give some new criteria to guarantee the infinitely many solutions for a fourth-order impulsive boundary value problem. Our main tool to ensure the existence of infinitely many solutions is the classical Ricceri's Variational Principle.

1. Introduction.

In this paper, we consider the following boundary value problem for a fourth-order impulsive differential equation:

$$\begin{cases} u^{(4)}(t) + Au''(t) + Bu(t) = \lambda f(t, u(t)) + \mu g(t, u(t)), \quad t \neq t_j, t \in [0, 1], \\ \triangle u''(t_j) = I_{1j}(u'(t_j)), \quad -\triangle u'''(t_j) = I_{2j}(u(t_j)), \quad j = 1, 2, \dots, m, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where A, B are two real constants, $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory functions, $I_{1j}, I_{2j} \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, the operator Δ is defined as $\Delta U(t_j) = U(t_j^+) - U(t_j^-)$, where $U(t_j^+)$ $\left(U(t_j^-)\right)$ denotes the right-hand (left-hand) limit of U at t_j and $\lambda > 0$ and $\mu \geq 0$ are referred to as control parameters.

In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs for short).

Some classical tools have been used to study such problems in the literatures. These techniques include the coincidence degree theory of Mawhin, the method of upper and

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Key words and phrases. Infinitely many solutions, impulsive differential equations, critical points, variational methods.

2010 *Mathematics Subject Classification.* Primary: 34B15, 34B37, 58E30.

DOI 10.46793/KgJMat2205.789H

Received: January 07, 2020.

Accepted: April 03, 2020.

lower solutions with monotone iterative technique, and some fixed point theorems in cones.

On the other hand, in the last few years, many researchers have used variational methods to study the existence of solutions for IBVPs. We refer the interested readers to [1–4, 6–8].

Motivated by the paper [1], in the present paper, by employing the classical Ricceri's Variational Principle, we obtain a sequence of solutions to problem (1.1) which is unbounded. Note that when $\mu = 0$ system (1.1) reduces to the one studied in [8]. Our results extend those ones in [8].

The remaining part of this paper is organized as follows. Some fundamental facts will be given in Section 2 and the main result of this paper will be presented in Section 3.

2. Preliminaries

Our main tool to ensure the existence of infinitely many solutions for the problem (1.1) is the classical Ricceri's Variational Principle ([5, Theorem 2.5]) that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space. Let $\phi, \psi : X \to \mathbb{R}$ be two Gateaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous and coercive and ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \phi$, let us put*

$$\varphi(r) = \inf_{u \in \phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \phi^{-1}(]-\infty, r])} \psi(v) - \psi(u)}{r - \phi(u)}.$$

If $\gamma := \liminf_{r \to +\infty} \varphi(r) < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, only one of the following statements holds to the functional $I_\lambda := \phi - \lambda\psi$:

(A1) *I_λ possesses a global minimum;*
(A2) *there is a sequence (u_n) of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \phi(u_n) = +\infty.$$

Here and in the sequel, we suppose that A and B satisfy the following condition:

$$A \leq 0 \leq B. \tag{2.1}$$

Define

$$H_0^1([0,1]) := \{u \in L^2([0,1]) : u' \in L^2([0,1]), u(0) = u(1) = 0\},$$
$$H^2([0,1]) := \{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\}.$$

Take $X := H^2([0,1]) \bigcap H_0^1([0,1])$ and define

$$\|u\|_X = \left(\int_0^1 |u''(t)|^2 - A|u'(t)|^2 + B|u(t)|^2 dt \right)^{\frac{1}{2}}. \tag{2.2}$$

Since A, B satisfy (2.1), it is straightforward to verify that (2.2) defines a norm for the Sobolev space X and this norm is equivalent to the usual norm defined as follows:

$$\|u\| := \|u''\|_{L^2([0,1])}.$$

It follows from (2.1) that $\|u\| \le \|u\|_X$. For the norm in $C^1([0,1])$

$$\|u\|_\infty = \max\left\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\right\},$$

we have the following relation.

Lemma 2.1 ([8]). *Let $M_1 := 1 + \frac{1}{\pi}$. Then $\|u\|_\infty \le M_1\|u\|_X$ for all $u \in X$.*

Definition 2.1. By a weak solution of the problem (1.1), we mean any $u \in X$ such that

$$\int_0^1 [u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)]dt + \sum_{j=1}^m I_{2j}(u(t_j))v(t_j) + \sum_{j=1}^m I_{1j}(u'(t_j))v'(t_j)$$
$$= \lambda \int_0^1 f(t, u(t))v(t)dt + \mu \int_0^1 g(t, u(t))v(t)dt$$

holds for every $v \in X$.

Put

$$F(t, x) := \int_0^x f(t, \xi)d\xi, \quad G(t, x) := \int_0^x g(t, \xi)d\xi,$$

for all $(t, x) \in [0, 1] \times \mathbb{R}$.

3. Main Results

In this section, we present our main results. To this end, we need the following assumptions.

$(H1)$ Assume that there exist two positive constants k_1 and k_2 such that for each $u \in X$

$$0 \le \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(s)ds \le k_1 \max_{j\in\{1,2,\ldots,m\}} |u'(t_j)|^2$$

and

$$0 \le \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(s)ds \le k_2 \max_{j\in\{1,2,\ldots,m\}} |u(t_j)|^2.$$

Also put $k_3 := 2.048(\frac{3}{8} - \frac{9}{10.4^4}A + \frac{79}{14.4^8}B)$ and $k_4 := k_2 + k_3$. These constants will be used in the hypotheses of Theorem 3.1.

$(H2)$ Assume that $\{t_1, t_2, \ldots, t_m\} \subseteq [\frac{1}{4}, \frac{3}{4}]$.

$(H3)$ Assume that $F(t, u) \ge 0$ for $(t, u) \in \left([0, \frac{1}{4}] \bigcup [\frac{3}{4}, 1]\right) \times \mathbb{R}$.

Theorem 3.1. *Suppose that* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied. Also* $(H4)$

$$M_1^2 \liminf_{\xi \to +\infty} \frac{\int_0^1 \sup_{|x| \le \xi} F(t,x)dt}{\xi^2} < \frac{1}{k_4} \limsup_{\xi \to +\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2},$$

and $\lambda \in]\lambda_1, \lambda_2[$, *where*

$$\lambda_1 := \frac{k_4}{\limsup\limits_{\xi \to +\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2}}, \quad \lambda_2 := \frac{1}{M_1^2 \liminf\limits_{\xi \to +\infty} \frac{\int_0^1 \sup_{|x| \le \xi} F(t,x)dt}{\xi^2}}.$$

If G *is a nonnegative function satisfying the condition*

$$g_\infty := \lim_{\xi \to +\infty} \frac{\int_0^1 \sup_{|x| \le \xi} G(t,x)dt}{\left(\frac{\xi^2}{M_1^2}\right)} < +\infty, \tag{3.1}$$

then for every $\mu \in [0, \mu_{g,\lambda}[$, *where*

$$\mu_{g,\lambda} := \frac{1}{g_\infty}\left(1 - \lambda M_1^2 \liminf_{\xi \to +\infty} \frac{\int_0^1 \sup_{|x| \le \xi} F(t,x)dt}{\xi^2}\right),$$

the problem (1.1) *has an unbounded sequence of weak solutions in* X.

Proof. Fix $\lambda \in]\lambda_1, \lambda_2[$ and let g be a function satisfying the condition (3.1). Since $\lambda < \lambda_2$, one has $\mu_{g,\lambda} > 0$. Fix $\mu \in [0, \mu_{g,\lambda}[$ and put

$$v_1 := \lambda_1, \quad v_2 := \frac{\lambda_2}{1 + (\frac{\mu}{\lambda})\lambda_2 g_\infty}.$$

If $g_\infty = 0$, clearly, $v_1 = \lambda_1$, $v_2 = \lambda_2$ and $\lambda \in]v_1, v_2[$. If $g_\infty \neq 0$, since $\mu < \mu_{g,\lambda}$, we obtain $\frac{\lambda}{\lambda_2} + \mu g_\infty < 1$, and so $v_2 > \lambda$. Hence, since $\lambda > \lambda_1 = v_1$, one has $\lambda \in]v_1, v_2[$. Take $X = H^2\big([0,1]\big) \cap H_0^1\big([0,1]\big)$ and define in X the functional I_λ for each $u \in X$, as follows

$$I_\lambda(u) := \phi(u) - \lambda \psi(u),$$

where

$$\phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(s)ds + \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(s)ds,$$

$$\psi(u) = \int_0^1 F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^1 G(t,u(t))dt.$$

It is not hard to show that every critical point of I_λ is a weak solution of system (1.1). So, our goal is to apply Theorem 2.1 to ϕ and ψ. In the first step, it is well known that $\phi, \psi : X \to \mathbb{R}$ are two Gateaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous and coercive.

Moreover, ψ is sequentially weakly upper semicontinuous. Now, we wish to prove that $\gamma = \liminf_{r\to+\infty} \varphi(r) < \infty$, where

$$\varphi(r) = \inf_{\phi(u)<r} \frac{\sup\limits_{\phi(v)\le r} \psi(v) - \psi(u)}{r - \phi(u)}.$$

Let

$$Q(t,x) := F(t,x) + \frac{\mu}{\lambda} G(t,x), \quad (t,x) \in [0,1] \times \mathbb{R}.$$

Let (ξ_n) be a real sequence such that $\xi_n > 0$ for all $n \in \mathbb{N}$ and $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \sup\limits_{|x|\le\xi_n} Q(t,x)dt}{\xi_n^2} = \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup\limits_{|x|\le\xi} Q(t,x)dt}{\xi^2}.$$

Put $r_n = \dfrac{\xi_n^2}{2M_1^2}$ for all $n \in \mathbb{N}$. Then for every $u \in X$, with $\phi(u) < r_n$, we have

$$\|u\|_\infty^2 \le M_1^2 \|u\|_X^2 \le 2M_1^2 \phi(u) < 2M_1^2 r_n = \xi_n^2,$$

thus

$$\phi^{-1}(]-\infty, r_n[) \subseteq \{u \in X : \|u\|_\infty \le \xi_n\}.$$

Hence, taking into account that $\phi(0) = \psi(0) = 0$ for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{\phi(u)<r_n} \frac{\sup\limits_{\phi(v)\le r_n} \psi(v) - \psi(u)}{r_n - \phi(u)} \le \frac{\sup\limits_{\phi(v)\le r_n} \psi(v) - \psi(0)}{r_n - \phi(0)} \\
&= \frac{1}{r_n} \sup_{\phi(v)\le r_n} \psi(v) = \frac{2M_1^2}{\xi_n^2} \sup_{\phi(v)\le r_n} \psi(v) \\
&\le \frac{2M_1^2}{\xi_n^2} \int_0^1 \sup_{|x|\le\xi_n} Q(t,x)dt \\
&\le \frac{\int_0^1 \sup\limits_{|x|\le\xi_n} F(t,x)dt}{\frac{\xi_n^2}{2M_1^2}} + \frac{\mu}{\lambda} \cdot \frac{\int_0^1 \sup\limits_{|x|\le\xi_n} G(t,x)dt}{\frac{\xi_n^2}{2M_1^2}}.
\end{aligned}$$

Therefore, it follows from $(H4)$ and condition (3.1) that

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \lim_{n\to+\infty} \frac{\int_0^1 \sup\limits_{|x|\le\xi_n} F(t,x) + (\frac{\mu}{\lambda})G(t,x)dt}{\frac{\xi_n^2}{2M_1^2}} < +\infty. \tag{3.2}$$

Here, we can observe that $]\lambda_1, \lambda_2[\subseteq]0, \frac{1}{\gamma}[$. Hence, for the fixed $\lambda \in]\lambda_1, \lambda_2[$, the inequality (3.2) assures that Theorem 2.1 can be used and either I_λ has a global minimum or there exists a sequence (u_n) of weak solutions of the problem (1.1) such that $\lim\limits_{n\to\infty} \|u_n\|_X = +\infty$.

The other step is to verity that the functional I_λ has no global minimum. Since

$$\frac{k_4}{\lambda} < \limsup_{\xi \to +\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)dt}{\xi^2},$$

we can consider a real sequence (γ_n) and a positive constant τ such that $\gamma_n \to +\infty$ as $n \to \infty$ and

$$\frac{k_4}{\lambda} < \tau < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt}{\gamma_n^2}, \tag{3.3}$$

for each $n \in \mathbb{N}$ large enough. Thus, if we consider a sequence (w_n) in X defined by setting

$$w_n(t) = \begin{cases} 64\gamma_n \left(t^3 - \frac{3}{4}t^2 + \frac{3}{16}t\right), & t \in \left[0, \frac{1}{4}\right[, \\ \gamma_n, & t \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 64\gamma_n \left(-t^3 + \frac{9}{4}t^2 - \frac{27}{16}t + \frac{7}{16}\right), & t \in \left]\frac{3}{4}, 1\right], \end{cases}$$

then, taking $(H1)$ and $(H2)$ into account, we conclude

$$\phi(w_n) = 2.048\left(\frac{3}{8} - \frac{9}{10.4^4}A + \frac{79}{14.4^8}B\right)\gamma_n^2 + \sum_{j=1}^{m}\int_0^{\gamma_n} I_{2j}(s)ds \le k_3\gamma_n^2 + k_2\gamma_n^2 = k_4\gamma_n^2.$$

On the other hand, since G is nonnegative, we observe

$$\psi(w_n) \ge \int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt.$$

So, from (3.3), we conclude

$$I_\lambda(w_n) = \phi(w_n) - \lambda\psi(w_n) \le k_4\gamma_n^2 - \lambda\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\gamma_n)dt \le \gamma_n^2(k_4 - \lambda\tau),$$

for every $n \in \mathbb{N}$ large enough. Note that $k_4 - \lambda\tau < 0$. Hence, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum and we have the conclusion. □

We have the following corollary as a special case of Theorem 3.1.

Corollary 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function and let $F(x) = \int_0^x f(\xi)d\xi$ for all $x \in \mathbb{R}$. Also,*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then for every continuous function $g : \mathbb{R} \to \mathbb{R}$ whose $G(x) = \int_0^x g(\xi)d\xi$ for every $x \in \mathbb{R}$, is a nonnegative function satisfying the condition

$$g_* := \lim_{\xi \to +\infty} \frac{\sup_{|x| \le \xi} G(x)}{\frac{\xi^2}{M_1^2}} < +\infty,$$

and for every $\mu \in [0, \mu_*[$, where $\mu_* := \frac{1}{g_*}\left(1 - M_1^2 \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}\right)$, the problem

$$\begin{cases} u^{(4)}(t) + Au''(t) + Bu(t) = f(u(t)) + \mu g(u(t)), & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

has an unbounded sequence of weak solutions.

Acknowledgements. The authors would like to thank the two referees for a very careful reading of this manuscript and for making good suggestions for the improvement of this article.

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 797–813.

ON SOME STATISTICAL APPROXIMATION PROPERTIES OF GENERALIZED LUPAŞ-STANCU OPERATORS

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Abstract. The purpose of this paper is to introduce Stancu variant of generalized Lupaş operators whose construction depends on a continuously differentiable, increasing and unbounded function ρ. Depending on the selection of γ and δ, these operators are more flexible than the generalized Lupaş operators while retaining their approximation properties. For these operators we give weighted approximation, Voronovskaya type theorem, quantitative estimates for the local approximation. Finally, we investigate the statistical approximation property of the new operators with the aid of a Korovkin type statistical approximation theorem.

1. Introduction

Approximation theory rudimentary deals with approximation of functions by simpler functions or more facilely calculated functions. Broadly it is divided into theoretical and constructive approximation. Inspired by the binomial probability distribution, in 1912 S. N. Bernstein [3] was the first to construct sequence of positive linear operators to provide a constructive proof of prominent Weierstrass approximation theorem [33] using probabilistic approach. One can find a detailed monograph about the Bernstein polynomials in [19, 21].

In order to obtain more flexibility, Stancu [32] applied another technique for choosing nodes. He observed that the distance between two successive nodes and between 0 and first node and similarly between last and 1 goes to zero when $m \to \infty$. After

Key words and phrases. Generalized Lupaş-Stancu operators, Korovkin's type theorem, convergence theorems, Voronovskaya type theorem, statistical approximation.
2010 *Mathematics Subject Classification.* Primary: 41A10. Secondary: 41A25, 41A36.
DOI 10.46793/KgJMat2205.797Q
Received: December 28, 2019.
Accepted: April 07, 2020.

these observation Stancu introduced the following positive linear operators

$$\left(P_m^{(\gamma,\delta)}f\right)(u)=\sum_{k=0}^{m}\binom{m}{k}u^k(1-u)^{m-k}f\left(\frac{k+\gamma}{m+\delta}\right) \tag{1.1}$$

converge to continuous function $f(u)$ uniformly in $[0,1]$ for each real γ,δ such that $0\leq\gamma\leq\delta$. For more recent literatures on Stancu type operators on can see [1,4,7,15–17,23–31].

In another development in approximation theory Cárdenas et al. [5], in 2011 defined the Bernstein type operators by $B_m(f\circ\tau^{-1})\circ\tau$ and showed that its Korovkin set is $\{e_0,\tau,\tau^2\}$ instead of $\{e_0,e_1,e_2\}$. Recently, Aral et al. [18] in 2014 defined a similar modification of Szász-Mirakyan type operators obtaining approximation properties of these operators on the interval $[0,\infty)$.

Very recently motivated by the above work İlarslan et al. [14] introduced a new modification of Lupaş operators [22] using a suitable function ρ, which satisfies following properties:

(ρ_1) ρ be a continuously differentiable function on $[0,\infty)$;
(ρ_2) $\rho(0)=0$ and $\inf_{u\in[0,\infty)}\rho^{'}(u)\geq 1$.

The generalized Lupaş operators are defined as

$$\mathcal{L}_m^{\rho}(f;u)=2^{-m\rho(u)}\sum_{\ell=0}^{\infty}\frac{(m\rho(u))_\ell}{2^\ell\ell!}\left(f\circ\rho^{-1}\right)\left(\frac{\ell}{m}\right), \tag{1.2}$$

for $m\geq1$, $u\geq0$, and suitable functions f defined on $[0,\infty)$. If $\rho(u)=u$, then (1.2) reduces to the Lupaş operators defined in [22].

The purpose of this paper is to define the Stancu type variant of operators (1.2) which depend on ρ. The present work is organized as follows. In the Section 2, we give the definition of a new family of the generalized Lupaş-Stancu operators and calculate its moments and central moments. In the Section 3, we study convergence properties of new constructed operators in the light of weighted space. In Section 4, we obtain the order of approximation of generalized Lupaş-Stancu operators associated with the weighted modulus of continuity. In Section 5, a Voronovskaya type result is obtained. In Section 6, we obtain some local approximation results related to $\mathcal{K}$-functional also we define a Lipschitz-type functions, as well as related results. Finally, in last section, we investigate the statistical approximation property of the new operators with the aid of a Korovkin type statistical approximation theorem

2. Construction of the Generalized Lupaş-Stancu Operators

Persuaded by the above mentioned work, we introduce Stancu variant of operators (1.2), which depend on a suitable function ρ as follows.

Definition 2.1. Let $0 \le \gamma \le \delta$ and $m \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$, we define generalized Lupaş-Stancu operators as

$$\mathcal{T}_{m,\rho}^{\gamma,\delta}(f; u) = 2^{-m\rho(u)} \sum_{\ell=0}^{\infty} \frac{m\rho(u))_\ell}{2^\ell \ell!} \left(f \circ \rho^{-1}\right) \left(\frac{\ell + \gamma}{m + \delta}\right), \tag{2.1}$$

where $(m\rho(u))_l$ is the rising factorial defined as:

$$\begin{aligned}(m\rho(u))_0 =& 1,\\ (m\rho(u))_l =& (m\rho(u))(m\rho(u) + 1)(m\rho(u) + 2) \cdots (m\rho(u) + l - 1), \quad l \ge 0.\end{aligned}$$

The operators (2.1) are linear and positive. For $\gamma = \delta = 0$, the operators (2.1) turn out to be generalized Lupaş operators defined in (1.2). Next, we prove some auxiliary results for (2.1).

Lemma 2.1. *Let $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ be given by (2.1). Then for each $u \ge 0$ and $m \in \mathbb{N}$ we have*

(i) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(1; u) = 1$;

(ii) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho; u) = \frac{m}{m+\delta}\rho(u) + \frac{\gamma}{m+\delta}$;

(iii) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) = \frac{m^2}{(m+\delta)^2}\rho^2(u) + \frac{2\gamma m+2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2}$;

(iv) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^3; u) = \frac{m^3}{(m+\delta)^3}\rho^3(u) + \frac{6m^2+3\gamma m^2}{(m+\delta)^3}\rho^2(u) + \frac{6m+6\gamma m+3\gamma^2 m}{(m+\delta)^3}\rho(u) + \frac{\gamma^3}{(m+\delta)^3}$;

(v)

$$\begin{aligned}\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^4; u) =& \frac{m^4}{(m+\delta)^4}\rho^4(u) + \frac{12m^3 + 4\gamma m^3}{(m+\delta)^4}\rho^3(u) + \frac{36m^2 + 6\gamma m^2 m^2 + 24\gamma m^2}{(m+\delta)^4}\rho^2(u)\\ &+ \frac{12\gamma^2 m + 24\gamma m + 26m}{(m+\delta)^4}\rho(u) + \frac{\gamma^4}{(m+\delta)^4}.\end{aligned}$$

Corollary 2.1. *For $n = 1, 2, 3, 4$ the n^{th} order central moments of $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ defined as $\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w) - \rho(u))^n; u)$, we have*

(i) $\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho(w) - \rho(u); u) = \left(\frac{m}{m+\delta} - 1\right)\rho(u) + \frac{\gamma}{m+\delta}$;

(ii)

$$\begin{aligned}&\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w) - \rho(u))^2; u)\\ =& \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2} = \sigma_m(u);\end{aligned}$$

(iii)

$$\begin{aligned}&\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w) - \rho(u))^3; u)\\ =& \left(\frac{m^3}{(m+\delta)^3} - \frac{3m^2}{(m+\delta)^2} + \frac{3m}{m+\delta} - 1\right)\rho^3(u) + \left(\frac{6m^2 + 3\gamma m^2}{(m+\delta)^3} - \frac{6\gamma m + 6m}{(m+\delta)^2}\right.\\ &\left.+ \frac{3\gamma}{(m+\delta)}\right)\rho^2(u) + \left(\frac{6m + 6\gamma m + 3\gamma^2 m}{(m+\delta)^3}\right)\rho(u) + \frac{\gamma^3}{(m+\delta)^3};\end{aligned}$$

(iv)

$$
\begin{aligned}
&\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w)-\rho(u))^4;u)\\
=&\left(\frac{m^4}{(m+\delta)^4}-\frac{4m^3}{(m+\delta)^3}+\frac{6m^2}{(m+\delta)^2}-\frac{4m}{(m+\delta)}+1\right)\rho^4(u)\\
&+\left(\frac{12m^3+4m^3\gamma}{(m+\delta)^4}+\frac{12m^2\gamma+24m^2}{(m+\delta)^3}+\frac{12m\gamma+12m}{(m+\delta)^2}-\frac{4\gamma}{(m+\delta)}\right)\rho^3(u)\\
&+\left(\frac{36m^2+6m^2\gamma^2+24m^2\gamma}{(m+\delta)^4}-\frac{24m\gamma m+24m+12m\gamma^2}{(m+\delta)^3}+\frac{6\gamma^2}{(m+\delta)^2}\right)\rho^2(u)\\
&+\left(\frac{26m+12m\gamma^2+24m\gamma}{(m+\delta)^4}\right)\rho(u)+\frac{\gamma^4}{(m+\delta)^4}-\frac{4\gamma^3}{(m+\delta)^3}.
\end{aligned}
$$

Remark 2.1. It is observed from Lemma 2.1 and Corollary 2.1 that for $\gamma=\delta=0$, we get the moments and central moments of generalized Lupaş operators [14].

3. Weighted Approximation

We start by noting that ρ not only defines a Korovkin-type set $\{1,\rho,\rho^2\}$ but also characterizes growth of the functions which are approximated.

Let $\phi(u)=1+\rho^2(u)$ be a weight function satisfying the conditions (ρ_1) and (ρ_2) given above let $\mathcal{B}_\phi[0,\infty)$ be the weighted space defined by

$$\mathcal{B}_\phi[0,\infty)=\{f:[0,\infty)\to\mathbb{R}\mid |f(u)|\leq \mathcal{K}_f\phi(u), u\geq 0\},$$

where $\mathcal{K}_f$ is a constant which depends only on f. $\mathcal{B}_\phi[0,\infty)$ is a normed linear space equipped with the norm

$$\|f\|_\phi=\sup_{u\in[0,\infty)}\frac{|f(u)|}{\phi(u)}.$$

Also, we define the following subspaces of $\mathcal{B}_\phi[0,\infty)$ as

$$
\begin{aligned}
\mathcal{C}_\phi[0,\infty)=&\{f\in\mathcal{B}_\phi[0,\infty): f \text{ is continuous on } [0,\infty)\},\\
\mathcal{C}^*_\phi[0,\infty)=&\left\{f\in\mathcal{C}_\phi[0,\infty):\lim_{u\to\infty}\frac{f(u)}{\phi(u)}=\mathcal{K}_f\right\},
\end{aligned}
$$

where $\mathcal{K}_f$ is a constant depending on f and

$$U_\phi[0,\infty)=\left\{f\in\mathcal{C}_\phi[0,\infty):\frac{f(u)}{\phi(u)} \text{ is uniformly continuous on } [0,\infty)\right\}.$$

Obviously,

$$\mathcal{C}^*_\phi[0,\infty)\subset U_\phi[0,\infty)\subset\mathcal{C}_\phi[0,\infty)\subset\mathcal{B}_\phi[0,\infty).$$

For the weighted uniform approximation by linear positive operators acting from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$, we state the following results due to Gadjiev in [12] and [9].

Lemma 3.1 ([12]). *Let $(\mathcal{A}_m)_{m\geq 1}$ be a sequence of positive linear operators which acts from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$ if and only if the inequality*

$$|\mathcal{A}_m(\phi;u)| \leq \mathcal{K}_m\phi(u), \quad u \geq 0,$$

holds, where $\mathcal{K}_m > 0$ is a constant depending on m.

Theorem 3.1 ([9]). *Let $(\mathcal{A}_m)_{m\geq 1}$ be a sequence of positive linear operators, acting from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$ and satisfying*

$$\lim_{m\to\infty} \|\mathcal{A}_m\rho^i - \rho^i\|_\phi = 0, \quad i = 0,1,2.$$

Then we have

$$\lim_{m\to\infty} \|\mathcal{A}_m(f) - f\|_\phi = 0, \quad \textit{for any } f \in C^*_\phi[0,\infty).$$

Remark 3.1. It is clear from Lemma 2.1 and Lemma 3.1 that the operators $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ act from $\mathcal{C}_\phi[0,\infty)$ to $\mathcal{B}_\phi[0,\infty)$.

Theorem 3.2. *Let $0 \leq \gamma \leq \delta$ and for each function $f \in C^*_\phi[0,\infty)$ we have*

$$\lim_{m\to\infty} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f) - f\|_\phi = 0.$$

Proof. By Lemma 2.1 (i) and (ii), it is clear that

$$\begin{aligned}
&\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(1;u) - 1\|_\phi = 0,\\
&\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho;u) - \rho\|_\phi = \left(\frac{m}{m+\delta} - 1\right) \sup_{u\in[0,\infty)} \frac{\rho(u)}{1+\rho^2(u)} + \frac{\gamma}{m+\delta} \leq \frac{\gamma-\delta}{m+\delta}.
\end{aligned}$$

Again by Lemma 2.1 (iii), we have

$$\begin{aligned}
\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^2;u) - \rho^2\|_\phi &= \left(\frac{m^2}{(m+\delta)^2} - 1\right) \sup_{u\in[0,\infty)} \frac{\rho^2(u)}{1+\rho^2(u)}\\
&\quad + \frac{2\gamma m + 2m}{(m+\delta)^2} \sup_{u\in[0,\infty)} \frac{\rho(u)}{1+\rho^2(u)} + \frac{\gamma^2}{(m+\delta)^2}\\
&\leq \frac{\gamma^2 - \delta^2 - 2m\delta + 2m\gamma + 2m}{(m+\delta)^2}.
\end{aligned} \tag{3.1}$$

Then from Lemma 2.1 and (3.1) we get $\lim_{m\to\infty} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho^i) - \rho^i\|_\phi = 0$, $i = 0,1,2$. Hence, the proof is completed. □

4. Rate of Convergence

In this section, we determine the rate of convergence for $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ by weighted modulus of continuity $\omega_\rho(f;\sigma)$ which was recently considered by Holhoş [13] as follows:

$$\omega_\rho(f;\sigma) = \sup_{u,\zeta\in[0,\infty), |\rho(\zeta)-\rho(u)|\leq\sigma} \frac{|f(\zeta) - f(u)|}{\phi(\zeta) + \phi(u)}, \quad \sigma > 0, \tag{4.1}$$

where $f \in \mathcal{C}_\phi[0,\infty)$, with the following properties:

(i) $\omega_\rho(f;0)=0$;
(ii) $\omega_\rho(f;\sigma)\geq 0$, $\sigma\geq 0$ for $f\in \mathcal{C}_\phi[0,\infty)$;
(ii) $\lim_{\sigma\to 0}\omega_\rho(f;\sigma)=0$ for each $f\in U_\phi[0,\infty)$.

Theorem 4.1 ([13]). *Let $\mathcal{A}_m:\mathcal{C}_\phi[0,\infty)\to\mathcal{B}_\phi[0,\infty)$ be a sequence of positive linear operators with*

$$\|\mathcal{A}_m(\rho^0)-\rho^0\|_{\phi^0}=a_m, \tag{4.2}$$
$$\|\mathcal{A}_m(\rho)-\rho\|_{\phi^{\frac{1}{2}}}=b_m, \tag{4.3}$$
$$\|\mathcal{A}_m(\rho^2)-\rho^2\|_{\phi}=c_m, \tag{4.4}$$
$$\|\mathcal{A}_m(\rho^3)-\rho^3\|_{\phi^{\frac{3}{2}}}=d_m, \tag{4.5}$$

where the sequences (a_m), (b_m), (c_m) and (d_m) converge to zero as $m\to\infty$. Then

$$\|\mathcal{A}_m(f)-f\|_{\phi^{\frac{3}{2}}}\leq (7+4a_m+2c_m)\omega_\rho(f;\sigma_m)+\|f\|_\phi a_m, \tag{4.6}$$

for all $f\in\mathcal{C}_\phi[0,\infty)$, where

$$\sigma_m=2\sqrt{(a_m+2b_m+c_m)(1+a_m)}+a_m+3b_m+3c_m+d_m.$$

Theorem 4.2. *Let for each $f\in\mathcal{C}_\phi[0,\infty)$, with $0\leq\gamma\leq\delta$. Then we have*

$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f)-f\|_{\phi^{\frac{3}{2}}}\leq\left(7+\frac{2\gamma^2-2\delta^2-4m\delta+4m\gamma+4m}{(m+\delta)^2}\right)\omega_\rho(f;\sigma_m),$$

where ω_ρ is the weighted modulus of continuity defined in (4.1) *and*

$$\begin{aligned}\sigma_m=&2\sqrt{\frac{2\gamma-2\delta}{m+\delta}+\frac{\gamma^2-\delta^2-2m\delta+2m\gamma+2m}{(m+\delta)^2}}\\&+\frac{3\gamma-3\delta}{m+\delta}+\frac{3\gamma^2-3\delta^2-6m\delta+6m\gamma+6m}{(m+\delta)^2}\\&+\frac{6m^2+3\gamma m^2+6m+6\gamma m+3\gamma^2 m+\gamma^3-\delta^3-3m^2\delta-3m\delta^2}{(m+\delta)^3}.\end{aligned}$$

Proof. If we calculate the sequences (a_m), (b_m), (c_m) and (d_m), then by using Lemma 2.1, clearly we have

$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^0)-\rho^0\|_{\phi^0}=0=a_m,$$
$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho)-\rho\|_{\phi^{\frac{1}{2}}}\leq\frac{\gamma-\delta}{m+\delta}=b_{m,q},$$

and

$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2)-\rho^2\|_{\phi}\leq\frac{\gamma^2-\delta^2-2m\delta+2m\gamma+2m}{(m+\delta)^2}=c_m.$$

Finally,

$$\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^3)-\rho^3\|_{\phi^{\frac{3}{2}}} \tag{4.7}$$

$$\leq \frac{6m^2+3\gamma m^2+6m+6\gamma m+3\gamma^2 m+\gamma^3-\delta^3-3m^2\delta-3m\delta^2}{(m+\delta)^3}=d_{m,q}.$$

Thus the conditions (4.1)–(4.2) are satisfied. Now, by Theorem 4.1, we obtain the desired result. ☐

Remark 4.1. For $\lim_{\delta\to 0}\omega_\rho(f;\delta)=0$ in Theorem 4.2, we get

$$\lim_{m\to\infty}\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f)-f\|_{\phi^{\frac{3}{2}}}=0,\quad \text{for } f\in U_\phi[0,\infty).$$

5. Voronovskaya Type Theorem

In this section, by using a technique which is developed in [5] by Cardenas-Morales, Garrancho and Raşa, we prove pointwise convergence of $\mathcal{T}^{\gamma,\delta}_{m,\rho}$ by obtaining Voronovskaya-type theorems.

Theorem 5.1. *Let* $f\in\mathcal{C}_\phi[0,\infty)$, $u\in[0,\infty)$, *with* $0\leq\gamma\leq\delta$ *and suppose that* $(f\circ\rho^{-1})'$ *and* $(f\circ\rho^{-1})''$ *exist at* $\rho(u)$. *If* $(f\circ\rho^{-1})''$ *is bounded on* $[0,\infty)$, *then we have*

$$\lim_{m\to\infty} m\left[\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u)-f(u)\right]=\rho(u)\left(f\circ\rho^{-1}\right)'\gamma+\rho(u)\left(f\circ\rho^{-1}\right)''\rho(u).$$

Proof. By using Taylor expansion of $f\circ\rho^{-1}$ at $\rho(u)\in[0,\infty)$, we have

(5.1)

$$\begin{aligned}f(w)=\left(f\circ\rho^{-1}\right)(\rho(w))&=\left(f\circ\rho^{-1}\right)(\rho(u))+\left(f\circ\rho^{-1}\right)'(\rho(u))\,(\rho(w)-\rho(u))\\&+\frac{\left(f\circ\rho^{-1}\right)''(\rho(u))\,(\rho(w)-\rho(u))^2}{2}+\lambda_u(w)\,(\rho(w)-\rho(u))^2,\end{aligned}$$

where

$$\lambda_u(w)=\frac{\left(f\circ\rho^{-1}\right)''(\rho(w))-\left(f\circ\rho^{-1}\right)''(\rho(u))}{2}. \tag{5.2}$$

Therefore, by (5.2) together with the assumption on f ensures that

$$|\lambda_u(w)|\leq\mathcal{K},\quad \text{for all } w\in[0,\infty),$$

and is convergent to zero as $w\to u$. Now, applying the operators (2.1) to the equality (5.1), we obtain

$$\begin{aligned}\left[\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u)-f(u)\right]&=\left(f\circ\rho^{-1}\right)'(\rho(u))\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(u));u\right)\\&+\frac{\left(f\circ\rho^{-1}\right)''(\rho(u))\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(y))^2;u\right)}{2}\\&+\mathcal{T}^{\gamma,\delta}_{m,\rho}\left(\lambda^u(w)\left((\rho(w)-\rho(u))^2;u\right)\right).\end{aligned} \tag{5.3}$$

By Lemma 2.1 and Corollary 2.1, we get

$$\lim_{m\to\infty} m\mathcal{T}^{\gamma,\delta}_{m,\rho}\left((\rho(w)-\rho(u));u\right)=\gamma \tag{5.4}$$

and

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^2;u\right)=2\rho(u). \tag{5.5}$$

By estimating the last term on the right hand side of equality (5.3), we will get the proof.

Since from (5.2) for every $\epsilon>0$, $\lim_{w\to u}\lambda_u(w)=0$. Let $\sigma>0$ such that $|\lambda_u(w)|<\epsilon$ for every $w\geq 0$. By Cauchy-Schwartz inequality, we get

$$\begin{aligned}\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\lambda_u(w)|\left(\rho(w)-\rho(u)\right)^2;u\right) \leq& \epsilon\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^2;u\right)\\ &+\frac{\mathcal{K}}{\sigma^2}\lim_{m\to\infty}\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^4;u\right).\end{aligned}$$

Since

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left((\rho(w)-\rho(u))^4;u\right)=0, \tag{5.6}$$

we obtain

$$\lim_{m\to\infty} m\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\lambda_u(w)|\left(\rho(w)-\rho(y)\right)^2;y\right)=0. \tag{5.7}$$

Thus, by taking into account the equations (5.4), (5.5) and (5.7) to (5.3) the proof is completed. ☐

6. Local Approximation

In this section, we present local approximation theorems for the operators $\mathcal{T}_{m,\rho}^{\gamma,\delta}$. By $\mathcal{C}_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $\mathcal{C}_B[0,\infty)$ is given by

$$\|f\|=\sup_{0\leq u<\infty}|f(x)|.$$

Further let us consider the following $\mathcal{K}$-functional:

$$\mathcal{K}_2(f,\sigma)=\inf_{s\in W^2}\{\|f-s\|+\sigma\|g''\|\},$$

where $\sigma>0$ and $W^2=\{s\in\mathcal{C}_B[0,\infty): s', s''\in\mathcal{C}_B[0,\infty)\}$. By Devore and Lorentz [6, Theorem 2.4, p. 177], there exists an absolute constant $\mathcal{C}>0$ such that

$$\mathcal{K}(f,\sigma)\leq\mathcal{C}\omega_2(f,\sqrt{\sigma}). \tag{6.1}$$

Second order modulus of smoothness is as follows

$$\omega_2(f,\sqrt{\sigma})=\sup_{0<h\leq\sqrt{\sigma}}\sup_{u\in[0,\infty)}|f(u+2h)-2f(u+h)+f(u)|,$$

where $f\in C_B[0,\infty)$. The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega(f,\sigma)=\sup_{0<h\leq\sigma}\sup_{u\in[0,\infty)}|f(u+h)-f(u)|.$$

Theorem 6.1. *Let $f \in \mathcal{C}_B[0,\infty)$, with $0 \leq \gamma \leq \delta$. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2) and $||\rho''||$ is finite. Then, there exists an absolute constant $\mathcal{C} > 0$ such that*

$$\left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) - f(u)\right| \leq \mathcal{C}\mathcal{K}\left(f, \sigma_m(u)\right),$$

where

$$\sigma_m(u) = \left\{\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. Let $s \in W^2$ and $u, w \in [0,\infty)$. By using Taylor's formula we have

$$s(w) = s(u) + \left(s \circ \rho^{-1}\right)'(\rho(u))(\rho(w) - \rho(u)) + \int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\left(s \circ \rho^{-1}\right)''(v)dv. \tag{6.2}$$

Now, put $v = \rho(y)$ in the last term of (6.2) and by using the equality

$$\left(s \circ \rho^{-1}\right)''(\rho(u)) = \frac{s''(u)}{(\rho'(u))^2} - s''(u)\frac{\rho''(u)}{(\rho'(u))^3}. \tag{6.3}$$

we get

$$\begin{aligned}
\int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\left(s \circ \rho^{-1}\right)''(v)dv &= \int_u^w (\rho(w) - \rho(y))\left[\frac{s''(y)\rho'(y) - s'(y)\rho''(v)}{(\rho'(y))^2}\right]dy \\
&= \int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv \\
&\quad - \int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv.
\end{aligned} \tag{6.4}$$

By using Lemma 2.1 and (6.4) and applying the operator (2.1) to the both sides of equality (6.2), we deduce

$$\begin{aligned}
\mathcal{T}^{\gamma,\delta}_{m,\rho}(s;u) =& s(u) + \mathcal{T}^{\gamma,\delta}_{m,\rho}\left(\int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv; u\right) \\
&- \mathcal{T}^{\gamma,\delta}_{m,\rho}\left(\int_{\rho(u)}^{\rho(w)} (\rho(w) - v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv; u\right).
\end{aligned}$$

As we know ρ is strictly increasing on $[0,\infty)$ and with condition (ρ_2), we get

$$\left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(s;u) - s(u)\right| \leq \mathcal{M}^{\rho}_{m,2}(u)\Big(\|s''\| + \|s'\|\|\rho''\|\Big),$$

where

$$\mathcal{M}^{\rho}_{m,2}(u) = \mathcal{T}^{\gamma,\delta}_{m,\rho}((\rho(t) - \rho(u))^2; u).$$

For $f \in \mathcal{C}_B[0,\infty)$, we have

$$\left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(s;u)\right| \leq \|f \circ \rho^{-1}\| 2^{-m\rho(u)} \sum_{\ell=0}^{\infty} \frac{(m\rho(u))_\ell}{2^\ell \ell!} \leq \|f\|\mathcal{T}^{\gamma,\delta}_{m,\rho}(1;u) = \|f\|. \tag{6.5}$$

Hence, we have

$$\left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) - f(u)\right| \leq \left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f - s;u)\right| + \left|\mathcal{T}^{\gamma,\delta}_{m,\rho}(s;u) - s(u)\right| + \left|s(u) - f(u)\right|$$

$$\leq 2\|f-g\| + \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) \right.$$
$$\left. + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\} \Big(\|s''\| + \|s'\| \|\rho''\| \Big).$$

If we choose $\mathcal{C} = \max\{2, \|\rho''\|\}$, then

$$\left| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \right| \leq \mathcal{C} \left(2\|f-g\| + \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) \right. \right.$$
$$\left. \left. + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\} \|s''\|_{W^2} \right).$$

Taking infimum over all $s \in W^2$ we obtain

$$\left| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \right| \leq \mathcal{C} \mathcal{K} \left(f, \sigma_m(u) \right). \qquad \square$$

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given in [10]. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2), $0 < \alpha \leq 1$, and $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $\mathcal{M} \geq 0$, is the set of functions f satisfying the inequality

$$\left| f(w) - f(u) \right| \leq \mathcal{M} \left| \rho(w) - \rho(u) \right|^{\alpha}, \quad u, w \geq 0.$$

Moreover, for a bounded subset $\mathcal{E} \subset [0,\infty)$, we say that the function $f \in \mathcal{C}_B[0,\infty)$ belongs to $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$, on $\mathcal{E}$ if

$$\left| f(w) - f(u) \right| \leq \mathcal{M}_{\alpha,f} \left| \rho(w) - \rho(u) \right|^{\alpha}, \quad u \in \mathcal{E} \text{ and } w \geq 0,$$

where $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f.

Theorem 6.2. *Let ρ be a function satisfying the conditions (ρ_1), (ρ_2). Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$, with $0 \leq \gamma \leq \delta$ and for every $u \in (0,\infty)$, $m \in \mathbb{N}$, we have*

$$\left| \mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u) \right| \leq \mathcal{M} \left(\sigma_m(u) \right)^{\frac{\alpha}{2}}, \tag{6.6}$$

where

$$\sigma_m(u) = \left\{ \left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1 \right) \rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta} \right) \rho(u) + \frac{\gamma^2}{(m+\delta)^2} \right\}.$$

Proof. Assume that $\alpha = 1$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned} |\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| &\leq \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(w) - f(u)|;u) \\ &\leq \mathcal{M} \mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w) - f(u)|;u). \end{aligned}$$

By applying Cauchy-Schwartz inequality, we get

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{M} \Big[\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(t) - \rho(u))^2;u) \Big]^{\frac{1}{2}} \leq \mathcal{M} \sqrt{\sigma_m(u)}.$$

Let us assume that $\alpha \in (0,1)$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(w) - f(u)|;u) \leq \mathcal{M}\mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w) - f(u)|^{\alpha};u).$$

From Hölder's inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, for $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{M}\Big[\mathcal{T}_{m,\rho}^{\gamma,\delta}(|(\rho(t) - \rho(u)|;u)\Big]^{\alpha}.$$

Finally by Cauchy-Schwartz inequality, we get

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\Big| \leq \mathcal{M}\left(\sigma_m(u)\right)^{\frac{\alpha}{2}}. \qquad \square$$

A relationship between local smoothness of functions and the local approximation was given by Agratini in [2]. Here we will prove the similar result for operators $\mathcal{T}_{m,\rho}^{\gamma,\delta}$, $m \in \mathbb{N}$, for functions from $Lip_{\mathcal{M}}(\rho(u))$ on a bounded subset.

Theorem 6.3. *Let $\mathcal{E}$ be a bounded subset of $[0,\infty)$ and ρ be a function satisfying the conditions (ρ_1), (ρ_2). Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$ on $\mathcal{E}$ $\alpha \in (0,1]$, we have*

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\Big| \leq \mathcal{M}_{\alpha,f}\left\{(\sigma_m(u))^{\frac{\alpha}{2}} + 2[\rho'(u)]^{\alpha} d^{\alpha}(u,\mathcal{E})\right\}, \quad u \in [0,\infty), m \in \mathbb{N},$$

where $d(u,\mathcal{E}) = \inf\{\|u - y\| : y \in \mathcal{E}\}$ and $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f,

$$\sigma_m(u) = \left\{\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. Let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in $[0,\infty)$. Then there exists a point $u_0 \in \overline{\mathcal{E}}$ such that $d(u,\mathcal{E}) = |u - u_0|$.

Using the monotonicity of $\mathcal{T}_{m,\rho}^{\gamma,\delta}$ and the hypothesis of f, we obtain

$$\begin{aligned}
|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| &\leq \mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|f(w) - f(u_0)|;u\right) + \mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|f(u) - f(u_0)|;u\right)\\
&\leq \mathcal{M}_{\alpha,f}\left\{\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\rho(w) - \rho(u_0)|^{\alpha};u\right) + |\rho(u) - \rho(u_0)|^{\alpha}\right\}\\
&\leq \mathcal{M}_{\alpha,f}\left\{\mathcal{T}_{m,\rho}^{\gamma,\delta}\left(|\rho(w) - \rho(u)|^{\alpha};u\right) + 2|\rho(u) - \rho(u_0)|^{\alpha}\right\},
\end{aligned}$$

by choosing $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, as well as the fact $|\rho(u) - \rho(u_0)| = \rho'(u)|\rho(u) - \rho(u_0)|$ in the last inequality. Then by using Hölder's inequality we easily conclude

$$\Big|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\Big| \leq \mathcal{M}_{\alpha,f}\left\{\Big[\mathcal{T}_{m,\rho}^{\gamma,\delta}((\rho(w) - \rho(u))^2;u)\Big]^{\frac{1}{2}} + 2[\rho'(u)|\rho(u) - \rho(u_0)|]^{\alpha}\right\}.$$

Hence, by Corollary 2.1 we get the proof. $\square$

Now, for $f \in \mathcal{C}_B[0,\infty)$, we recall local approximation in terms of α order generalized Lipschitz-type maximal function given by Lenze [20] as

$$(6.7) \qquad \widetilde{\omega}_{\alpha}^{\rho}(f;u) = \sup_{w \neq u, w \in (0,\infty)} \frac{|f(w) - f(u)|}{|w - u|^{\alpha}}, \quad u \in [0,\infty) \text{ and } \alpha \in (0,1].$$

Then we get the following result.

Theorem 6.4. *Let $f \in \mathcal{C}_B[0,\infty)$ and $\alpha \in (0,1]$, with $0 \le \gamma \le \delta$. Then for all $u \in [0,\infty)$ we have*

$$\left|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)\right| \le \widetilde{\omega}_{\alpha}^{\rho}(f;u)\,(\sigma_m(u))^{\frac{\alpha}{2}},$$

where

$$\sigma_m(u) = \left\{\left(\frac{m^2}{(m+\delta)^2} - \frac{2m}{m+\delta} + 1\right)\rho^2(u) + \left(\frac{2\gamma m + 2m}{(m+\delta)^2} - \frac{2\gamma}{m+\delta}\right)\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right\}.$$

Proof. We know that

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \le \mathcal{T}_{m,\rho}^{\gamma,\delta}(|f(t) - f(u)|;u).$$

From (6.7), we have

$$|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| \le \widetilde{\omega}_{\alpha}^{\rho}(f;u)\mathcal{T}_{m,\rho}^{\gamma,\delta}(|\rho(w) - \rho(u)|^{\alpha};u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f;u) - f(u)| &\le \widetilde{\omega}_{\alpha}^{\rho}(f;u)\left[\mathcal{T}_{m,q}^{\rho}((\rho(t) - \rho(u))^2;u)\right]^{\frac{\alpha}{2}}\\ &\le \widetilde{\omega}_{\alpha}^{\rho}(f;u)\,(\sigma_m(u))^{\frac{\alpha}{2}},\end{aligned}$$

which proves the desired result □

7. Statistical Approximation

In this section we obtain the Korovkin type weighted statistical approximation by the operators defined in (2.1). Let us recall the concept of statistical convergence which was given by Fast [8] and further studied by many authors.

Let $\mathcal{K} \subseteq N$ and $\mathcal{K}_m = \{i \le m : i \in \mathcal{K}\}$. Then the natural density or we can say asymptotic density of $\mathcal{K}$ is defined by $\sigma(\mathcal{K}) = \lim_m \frac{1}{m}|\mathcal{K}_m|$ whenever the limit exists, where $|\mathcal{K}_m|$ denotes the cardinality of the set $\mathcal{K}_m$.

A sequence $u = (u_i)$ of real numbers is said to be statistically convergent to $\mathcal{L}$ if for every $\epsilon > 0$ the set $\{i \in N : |u_i - \mathcal{L}| \ge \epsilon\}$ has natural density zero; that is, for each $\epsilon > 0$,

$$\lim_m \frac{1}{m}|\{i \le m : |u_i - \mathcal{L}| \ge \epsilon\}| = 0.$$

In this case, we write $st - \lim_m u_m = \mathcal{L}$. Note that convergent sequences are statistically convergent since all finite subset of of natural no have density zero. However, its converse is not true. This is demonstrated by the following example.

Example 7.1. Let us consider the sequences,

$$u = (u_m) := \begin{cases} \dfrac{1}{2m} + 1, & \text{otherwise,} \\ 0, & m = i^2 \text{ for some } i, \end{cases}$$

and

$$v = (v_m) := \begin{cases} 1, & m = i^2 \text{ for some } i, \\ 0, & m = i^2 + 1 \text{ for some } i, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that the sequence u and v are not convergent in the ordinary sense, but $st - \lim_m u_m = 1$ and $st - \lim_m v_m = 2$. All properties of convergent sequences are not true for statistical convergence. For instance, it is known that a subsequence of a convergent sequence is convergent. However, for statistical convergence this is not true. Indeed, the sequence $l = \{i : i = 1, 2, 3, \dots\}$ is a subsequence of the statistically convergent sequence u from Example 7.1. At the same time, l is statistically divergent.

Gadjiev and Orhan [11] introduced the concept of statistical convergence in approximation theory and prove the following Bohman-Korovkintype approximation theorem for statistical convergence.

Theorem 7.1 ([11])**.** *If the sequence of positive linear operators $\mathcal{A}_n : \mathcal{C}_{\mathcal{M}}[a,b] \to \mathcal{C}[a,b]$ satisfies the conditions $st - \lim_{n\to\infty} ||\mathcal{A}_n(e_v;\cdot) - e_v||_{\mathcal{C}[a,b]} = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$, then for any function $f \in \mathcal{C}_{\mathcal{M}}[a,b]$, we have*

$$st - \lim_{n\to\infty} ||\mathcal{A}_n(f;\cdot) - f||_{\mathcal{C}[a,b]} = 0,$$

where $\mathcal{C}_{\mathcal{M}}[a,b]$ denotes the space of all functions f which are continuous in $[a,b]$ and bounded on the all positive axis.

Now our first result is as follows.

Theorem 7.2. *Let $\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u)$ be the sequence of operators* (2.1)*, then for any function $f \in \mathcal{C}_B[0,\infty)$ we have*

$$st - \lim_m \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) - f\|_\phi = 0. \tag{7.1}$$

Proof. Clearly for $\nu = 0$, $\mathcal{T}^{\gamma,\delta}_{m,\rho}(f;u) = 1$, which implies

$$st - \lim_m \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(1;u) \ - 1 \ \|_\phi = 0.$$

For $\nu = 1$

$$\begin{aligned} \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho;u) - \rho\|_\phi &\le \left|\frac{m}{m+\delta}\rho(u) + \frac{\gamma}{m+\delta} - \rho(u)\right| \\ &= \left|\left(\frac{m}{m+\delta} - 1\right)\rho(u) - \frac{\gamma}{m+\delta}\right| \\ &\le \left|\frac{m}{m+\delta} - 1\right| + \left|\frac{\gamma}{m+\delta}\right|. \end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets:

$$\mathcal{W} = \{m : \|\mathcal{T}^{\gamma,\delta}_{m,\rho}(\rho;u) - \rho\|_\phi \ge \epsilon\},$$

$$\mathcal{W}' = \left\{m : 1 - \frac{m}{m+\delta} \geq \epsilon\right\},$$
$$\mathcal{W}'' = \left\{m : \frac{\gamma}{m+\delta} \geq \epsilon\right\}.$$

It is obvious that $\mathcal{W} \subseteq \mathcal{W}'' \cup \mathcal{W}'$. Then it can be written as:

$$\begin{aligned}\sigma\{i \leq m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho; u) - \rho\|_\phi \geq \epsilon\} \leq & \sigma\left\{i \leq m : 1 - \frac{m}{m+\delta} \geq \epsilon\right\} \\ & + \sigma\left\{i \leq m : \frac{\gamma}{m+\delta}\| \geq \epsilon\right\}.\end{aligned}$$

Then we have

$$st - \lim_m \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(f; u) - f\|_\phi = 0.$$

Lastly for $\nu = 2$, we have

$$\begin{aligned}\|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi \leq & \left|\frac{m^2}{(m+\delta)^2}\rho^2(u) + \frac{2\gamma m + 2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2} - \rho^2(u)\right| \\ \leq & \left|\left(\frac{m^2}{(m+\delta)^2} - 1\right)\rho^2(u) + \frac{2\gamma m + 2m}{(m+\delta)^2}\rho(u) + \frac{\gamma^2}{(m+\delta)^2}\right| \\ \leq & \left|\frac{m^2}{(m+\delta)^2} - 1\right| + \left|\frac{2\gamma m + 2m}{(m+\delta)^2}\right| + \left|\frac{\gamma^2}{(m+\delta)^2}\right|.\end{aligned}$$

If we choose

$$\alpha_m = \frac{m^2}{(m+\delta)^2} - 1, \quad \beta_m = \frac{2\gamma m + 2m}{(m+\delta)^2}, \quad \gamma_m = \frac{\gamma^2}{(m+\delta)^2},$$

then

$$st - \lim_m \alpha_m = st - \lim_m \beta_m = st - \lim_m \gamma_m = 0. \tag{7.2}$$

Now given $\epsilon > 0$, we define the following four sets:

$$\mathcal{W} = \{m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi \geq \epsilon\},$$
$$\mathcal{W}_1 = \left\{m : \alpha_m \geq \frac{\epsilon}{3}\right\},$$
$$\mathcal{W}_2 = \left\{m : \beta_m \geq \frac{\epsilon}{3}\right\},$$
$$\mathcal{W}_3 = \left\{m : \gamma_m \geq \frac{\epsilon}{3}\right\}.$$

It is obvious that $\mathcal{W} \subseteq \mathcal{W}_1 \bigcup \mathcal{W}_2 \bigcup \mathcal{W}_3$. Thus, we obtain

$$\begin{aligned}\delta\{i \leq m : \|\mathcal{T}_{m,\rho}^{\gamma,\delta}(\rho^2; u) - \rho^2\|_\phi \geq \epsilon\} \leq & \delta\left\{i \leq m : \alpha_m \geq \frac{\epsilon}{3}\right\} \\ & + \delta\left\{i \leq m : \beta_m \geq \frac{\epsilon}{3}\right\} + \delta\left\{i \leq m : \gamma_m \geq \frac{\epsilon}{3}\right\}.\end{aligned}$$

Using (7.2), we get

$$st - \lim_m \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(\rho^2; u) - \rho^2\|_\phi = 0$$

and thus the proof is completed.

Since

$$\|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(f; u) - f\|_\phi \leq \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(\rho^2; u) - \rho^2\|_\phi + \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(\rho; u) - \rho\|_\phi + \|S^{(\gamma,\delta)}_{m,q_m}(1; u) - 1\|_\phi,$$

we get

$$st - \lim_m \| \ \mathfrak{T}^{\gamma,\delta}_{m,\rho}(f; u) - f\|_\phi$$

$$\leq st - \lim_m \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(\rho^2; u) - \rho^2\|_\phi + st - \lim_m \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(\rho; u) - \rho\|_\phi + st - \lim_m \|\mathfrak{T}^{\gamma,\delta}_{m,\rho}(1; u) - 1\|_\phi,$$

which implies that

$$st - \lim_m \| \ \mathfrak{T}^{\gamma,\delta}_{m,\rho}(f; u) - f\|_\phi = 0.$$

This completes the proof. ☐

Acknowledgements. The first author is grateful to Council of Scientific and Industrial Research (CSIR), India, for providing the Senior Research Fellowship under the grant no (09/1172(0001)/2017-EMR-I).

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Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 815–832.

RIGHT AND LEFT MAPPINGS IN EQUALITY ALGEBRAS

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Abstract. The notion of (right) left mapping on equality algebras is introduced, and related properties are investigated. In order for the kernel of (right) left mapping to be filter, we investigate what conditions are required. Relations between left mapping and $\rightarrow$-endomorphism are investigated. Using left mapping and $\rightarrow$-endomorphism, a characterization of positive implicative equality algebra is established. By using the notion of left mapping, we define $\rightarrow$-endomorphism and prove that the set of all $\rightarrow$-endomorphisms on equality algebra is a commutative semigroup with zero element. Also, we show that the set of all right mappings on positive implicative equality algebra makes a dual BCK-algebra.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. A crucial question for every many-valued logic is, what should be structure of its truth values. It is generally accepted that in fuzzy logic, it should be a residuated lattice, possibly fulfilling some additional properties. On the basis of that, we may now distinguish various kinds of formal fuzzy logics. Most important among them seem to be BL-logics, MTL-logics and IMTL-logics. The answer to the above question is positive and the fuzzy type theory (FTT) has indeed been introduced in [12]. However, the basic connective in FTT is a fuzzy equality since it is developed as a generalization of the elegant classical

Key words and phrases. Equality algebra, &-equality algebra, positive implicative equality algebra, filter, left mapping, right mapping.
2010 *Mathematics Subject Classification.* Primary: 03G25. Secondary: 06B10, 06B99.
DOI 10.46793/KgJMat2205.815K
Received: January 31, 2020.
Accepted: April 11, 2020.

formal system originated by Henkin (see [5]). So Novák in [13] introduced a special algebra so called EQ-algebra and that reflects directly the syntax of FTT. Viewing the axioms of EQ-algebras with a purely algebraic eye it appears that unlike in the case of residuated lattices where the adjointness condition ties product with implication, the product in EQ-algebras is quite loosely related to the other connectives. For instance, a moment's reflection shows that one can replace the product of an EQ-algebra by any other binary operation which is smaller or equal than the original product (viewed as a two-place function) and still obtains an EQ-algebra. However, the huge freedom in choosing the product might prohibit to find deep related algebraic results, hence our aim was to find something similar to EQ-algebras but without a product: an axiomatic treatment of equality/equivalence. Because of that Jenei in [9] introduced a new structure, called equality algebras. It has two connectives, a meet operation and an equivalence, and a constant 1.

Left and right mappings are very important concepts and mathematicians have used them in various mathematical fields. For example, Kondo [11] introduced the notion of left mapping on BCK-algebras and investigated some properties of it. He showed that in a positive implicative BCK-algebra, if a left map is surjective, then it is also an injective one. Borzooei and Aaly [2], introduced left and right stabilizers by using a fixed point sets of right and left mappings. They investigated that under which conditions these sets can be equal. Also, by using the (right) left stabilizers, produced a basis for a topology on hoops and showed that the generated topology by this basis is Baire, connected, locally connected and separable. Moreover, Hail, Abu baker and Mohd [4], by using the notion of (right) left mapping defined different kinds of derivation on BCK/BCI-algebras. The notion of derivation and extended of that are introduced on different kinds of logical algebras such as UP-algebras, MV-algebras and etc. In UP-algebras, Iampan in [6] proved that the fixed point set and the kernel of left derivation are UP-subalgebras and investigated under which condition they can be an ideal or filter. Kamali in [10], extended the notion of derivation on MV-algebras by using left and right mappings and investigate some properties of them.

Now, in this paper, we introduce the concept of (right) left mapping on equality algebras and investigate several properties. Then by using of left and right mapping on equality algebras, we construct a commutative monoid, a commutative semigroup with zero element and a dual BCK-algebra.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1 ([8])**.** By an *equality algebra*, we mean an algebra $(X, \wedge, \sim, 1)$ satisfying the following conditions:

(E1) $(X, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., meet semilattice with the top element 1);

(E2) The operation "$\sim$" is commutative;
(E3) $(\forall a \in X)(a \sim a = 1)$;
(E4) $(\forall a \in X)(a \sim 1 = a)$;
(E5) $(\forall a, b, c \in X)(a \leq b \leq c \Rightarrow a \sim c \leq b \sim c, a \sim c \leq a \sim b)$;
(E6) $(\forall a, b, c \in X)(a \sim b \leq (a \wedge c) \sim (b \wedge c))$;
(E7) $(\forall a, b, c \in X)(a \sim b \leq (a \sim c) \sim (b \sim c))$,

where $a \leq b$ if and only if $a \wedge b = a$. The equality algebra $(X, \wedge, \sim, 1)$ is simply denoted by X only.

In an equality algebra $(X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$a \rightarrow b := a \sim (a \wedge b),$$
$$a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a).$$

Proposition 2.1 ([8]). *Let $(X, \wedge, \sim, 1)$ be an equality algebra. Then for all $a, b, c \in X$, the following assertions are valid:*

$$a \rightarrow b = 1 \Leftrightarrow a \leq b,$$
$$a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c), \tag{2.1}$$
$$1 \rightarrow a = a, \quad a \rightarrow 1 = 1, \quad a \rightarrow a = 1, \tag{2.2}$$
$$a \leq b \rightarrow c \Leftrightarrow b \leq a \rightarrow c,$$
$$a \leq b \rightarrow a, \tag{2.3}$$
$$a \leq (a \rightarrow b) \rightarrow b, \tag{2.4}$$
$$a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c), \tag{2.5}$$
$$b \leq a \Rightarrow a \leftrightarrow b = a \rightarrow b = a \sim b,$$
$$a \sim b \leq a \leftrightarrow b \leq a \rightarrow b,$$
$$a \leq b \;\Rightarrow \begin{cases} b \rightarrow c \leq a \rightarrow c, \\ c \rightarrow a \leq c \rightarrow b, \end{cases} \tag{2.6}$$
$$((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b. \tag{2.7}$$

An equality algebra X is said to be *bounded* if there exists an element $0 \in X$ such that $0 \leq a$ for all $a \in X$. In a bounded equality algebra X, we define the negation "$\neg$" on X by $\neg a = a \rightarrow 0 = a \sim 0$ for all $a \in X$.

A subset A of X is called a *deductive system* (or *filter*) of X (see [9]) if it satisfies

$$1 \in A, \tag{2.8}$$
$$(\forall a, b \in X)(a \in A, a \leq b \;\Rightarrow\; b \in A),$$
$$(\forall a, b \in X)(a \in A, a \sim b \in A \;\Rightarrow\; b \in A).$$

Denote by $\mathcal{DS}(X)$ the set of all deductive systems of X.

Lemma 2.1 ([7]). *Let X be an equality algebra. A subset A of X is a deductive system of X if and only if it satisfies* (2.8) *and*

$$(\forall a, b \in X)(a \in A, a \to b \in A \Rightarrow b \in A).$$

Definition 2.2 ([14]). An equality algebra X is said to be *commutative* if it satisfies:

$$(\forall x, y \in X)((x \to y) \to y = (y \to x) \to x).$$

Definition 2.3 ([1]). Given an equality algebra $(X, \wedge, \sim, 1)$ and $a, b \in X$, we define

$$X(a, b) := \{x \in X \mid a \leq b \to x\}.$$

It is clear that 1, a and b are contained in $X(a, b)$.

Definition 2.4 ([1]). An equality algebra $(X, \wedge, \sim, 1)$ is called an *&-equality algebra* if for all $a, b \in X$, the set $X(a, b)$ has the least element which is denoted by $a \odot b$.

Proposition 2.2 ([1]). *If $X = (X, \wedge, \sim, 1)$ is an &-equality algebra, then*

$$(\forall a, b \in X)(a \odot b = b \odot a),$$
$$(\forall a, b, c \in X)((a \odot b) \odot c = a \odot (b \odot c)),$$
$$(\forall a, b, c \in X)(a \leq b \Rightarrow a \odot c \leq b \odot c).$$

Lemma 2.2 ([1]). *Let $X = (X, \wedge, \sim, 1)$ be an equality algebra in which there exists a binary operation "$\odot$" such that*

$$(\forall a, b, c \in X)(a \to (b \to c) = (a \odot b) \to c).$$

Then $\mathfrak{X} = (X, \wedge, \sim, 1)$ is an &-equality algebra.

3. Left Mappings

In this section, we define the notion of left mapping on equality algebra and investigate some properties of it. Moreover, we define the notions of $\to$-homomorphism, positive implicative and &-equality algebras and study the relation among them.

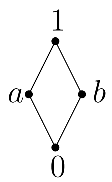
Definition 3.1. Given a fixed element a in an equality algebra X, we define a self-mapping f_a of X by

$$f_a : X \to X, \quad x \mapsto a \to x,$$

and we say that f_a is a *left mapping* on X.

Let $\mathcal{L}(X)$ denote the set of all left mappings on an equality algebra X.

Example 3.1. Let $X = \{0, a, b, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 1. Then $(X, \wedge, \sim, 1)$ is an equality algebra, and the

TABLE 1. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

implication "$\to$" is given by Table 2.

TABLE 2. Cayley table for the implication "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Let f_a and f_b be self mappings of X defined by

$$f_a(0) = f_a(b) = b, \quad f_a(a) = f_a(1) = 1$$

and

$$f_b(0) = f_b(a) = a, \quad f_b(b) = f_b(1) = 1,$$

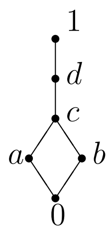
respectively. It is routine to verify that f_a and f_b are left mappings on X.

Remark 3.1. It is clear that $f_0(x) = 1$ and $f_1(x) = x$ for all x in a bounded equality algebra X.

Question 1. If f_a is a left mapping on X, then is f_a^2 a left mapping on X?

The following example shows that the answer to the above question is false.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 3. Then $\mathcal{E} = (X, \wedge, \sim, 1)$ is an equality algebra, and

TABLE 3. Cayley table for the implication "$\rightsquigarrow$"

$\rightsquigarrow$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

TABLE 4. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

the implication "$\to$" is given by Table 4. Define a mapping $f_a : X \to X$ by $f_a(0) = f_a(b) = d$ and $f_a(a) = f_a(c) = f_a(d) = f_a(1) = 1$. Then f_a is a left mapping on X, but f_a^2 is not a left mapping on X since

$$d = a \to 0 = f_a(0) \neq f_a^2(0) = a \to (a \to 0) = a \to d = 1.$$

Definition 3.2. An equality algebra X is said to be *positive implicative* if it satisfies

$$(\forall x, y, z \in X)(x \to (y \to z) = (x \to y) \to (x \to z)). \tag{3.1}$$

Theorem 3.1. *In a positive implicative equality algebra X, if f_a is a left mapping on X, then so is f_a^2.*

Proof. For any $x \in X$, we have

$$f_a^2(x) = f_a(f_a(x)) = a \to (a \to x) = (a \to a) \to (a \to x) = 1 \to (a \to x) = a \to x.$$

Therefore, f_a^2 is a left mapping on X. □

Corollary 3.1. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then so is f_a^n for every $n \in \mathbb{N}$.*

Proof. It is by mathematical induction. □

Proposition 3.1. *Let X be an equality algebra and f_a be a left mapping on X. Then the following statements hold:*

(1) $f_a(x) \to f_a(y) \le f_a(x \to y)$ *and the equality is true when* X *is positive implicative;*

(2) *the left mapping* f_a *on* X *is isotone, that is, if* $x \le y$*, then* $f_a(x) \le f_a(y)$*;*

(3) $x \le f_a(x) \le f_a^2(x) \le \cdots$ *and the equality is true when* $a = 1$*;*

(4) $x \to y \le f_a^n(x) \to f_a^n(y)$ *for any* $n \in \mathbb{N}$ *and the equality is true when* $a = 1$*;*

(5) $\mathrm{Im}(f_a^n) \subseteq \cdots \subseteq \mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a)$*;*

(6) $\mathrm{Fix}(f_a) \subseteq \mathrm{Fix}(f_a^2) \subseteq \cdots$*, where* $\mathrm{Fix}(f_a) := \{x \in X \mid f_a(x) = x\}$*;*

(7) $\ker(f_a) \subseteq \ker(f_a^2) \subseteq \cdots$ *and the equality is true when* $a = 1$*, where* $\ker(f_a) := \{x \in X \mid f_a(x) = 1\}$*;*

(8) $\mathrm{Fix}(f_a^n) \subseteq \mathrm{Im}(f_a^n)$ *for any* $n \in \mathbb{N}$ *and the equality is true when* X *is positive implicative;*

(9) $\mathrm{Fix}(f_a^n) \cap \ker(f_a^n) = \{1\}$ *for any* $n \in \mathbb{N}$*, for all* $a, x, y \in X$*;*

(10) *if* X *is an &-equality algebra, then* $f_a^2 = f_a$ *for any* $a \in X$*, with* $a \odot a = a$*.*

Proof. Let $a, x, y \in X$. Using (2.1) and (2.6), we have

$$f_a(x) \to f_a(y) = (a \to x) \to (a \to y) \le a \to (x \to y) = f_a(x \to y),$$

which proves (1).

(2) and (3) are straightforward by (2.6) and (2.3), respectively.

(4) Using (2.1) and (2.5), we have $x \to y \le (a \to x) \to (a \to y) = f_a(x) \to f_a(y)$. Suppose that $x \to y \le f_a^k(x) \to f_a^k(y)$ for $k \in \mathbb{N}$. Then

$$x \to y \le f_a^k(x) \to f_a^k(y) \le f_a(f_a^k(x)) \to f_a(f_a^k(y)) = f_a^{k+1}(x) \to f_a^{k+1}(y),$$

and so $x \to y \le f_a^n(x) \to f_a^n(y)$ by mathematical induction.

(5) If $y \in \mathrm{Im}(f_a^2)$, then $y = f_a^2(x) = f_a(f_a(x))$ for some $x \in X$ and so $y \in \mathrm{Im}(f_a)$, which shows that $\mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a)$. Repeating this process induces

$$\mathrm{Im}(f_a^n) \subseteq \cdots \subseteq \mathrm{Im}(f_a^2) \subseteq \mathrm{Im}(f_a).$$

By the similar way to the proof of (5), we have (6), (7) and (8).

(9) Let $x \in \mathrm{Fix}(f_a^n) \cap \ker(f_a^n)$. Then $x = f_a^n(x) = 1$. Hence, $\mathrm{Fix}(f_a^n) \cap \ker(f_a^n) = \{1\}$ for any $n \in \mathbb{N}$.

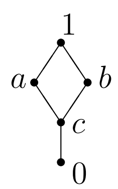
(10) Let $a \in X$ with $a \odot a = a$. Then

$$f_a^2(x) = a \to (a \to x) = (a \odot a) \to x = a \to x = f_a(x),$$

for all $x \in X$ and so $f_a^2 = f_a$. ☐

We pose a question as follows. Given a left mapping f_a on X, is the subset $\mathrm{Fix}(f_a)$ of X a filter of X? But the following example shows that the answer is negative.

Example 3.3. Let $Y = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram.



Then $(Y, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on Y by Table 5. Then $(Y, \wedge, \sim, 1)$ is an equality algebra which is not

TABLE 5. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

commutative, and the implication $(\rightarrow)$ is given by Table 6. We know that the map

TABLE 6. Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

$f_b : Y \rightarrow Y$ given by $f_b(0) = 0$, $f_b(a) = f_b(c) = a$ and $f_b(b) = f_b(1) = 1$ is a left mapping on Y. Then $\mathrm{Fix}(f_b) = \{0, a, 1\}$, which is not a filter of Y.

Proposition 3.2. *Given a left mapping f_a on X, the following statements are equivalent.*

(1) $(\forall x, y \in X)(\forall n \in \mathbb{N})$ $(y^n \rightarrow x = y^{n+1} \rightarrow x)$, *where*

$$y^n \rightarrow x = \underbrace{y \rightarrow (y \rightarrow \cdots (y}_{n \text{ times}} \rightarrow x)).$$

(2) $(\forall n \in \mathbb{N})(\mathrm{Im}(f_a^n) = \mathrm{Fix}(f_a^n))$.
(3) $(\forall n \in \mathbb{N})(f_a^n = f_a^{n+1})$.

Proof. (1) $\Rightarrow$ (2) By Proposition 3.1 (8), we have $\mathrm{Fix}(f_a^n) \subseteq \mathrm{Im}(f_a^n)$. If $y \in \mathrm{Im}(f_a^n)$, then there exists $x \in X$ such that $f_a^n(x) = y$. By (1), we get

$$y = f_a^n(x) = f_a^{n+1}(x) = \cdots = f_a^{2n}(x) = f_a^n(f_a^n(x)) = f_a^n(y).$$

Hence, $y \in \mathrm{Fix}(f_a^n)$, and so $\mathrm{Im}(f_a^n) \subseteq \mathrm{Fix}(f_a^n)$. Therefore, $\mathrm{Im}(f_a^n) = \mathrm{Fix}(f_a^n)$.

(2) $\Rightarrow$ (1) Let $x, y \in X$. It is clear that $y^n \rightarrow x \leq y^{n+1} \rightarrow x$. On the other hand,

$$(y^{n+1} \rightarrow x) \rightarrow (y^n \rightarrow x) = f_y^{n+1}(x) \rightarrow f_y^n(x) = f_y^n(f_y(x)) \rightarrow f_y^n(x).$$

Then $f_y(x) \in \mathrm{Im}(f_y^n) = \mathrm{Fix}(f_y^n)$ and so $f_y^n(f_y(x)) = f_y(x)$. Since $f_y(x) \leq f_y^n(x)$, we have

$$(y^{n+1} \to x) \to (y^n \to x) = f_y(x) \to (f_y^n(x)) = 1,$$

and so $(y^{n+1} \to x) \leq (y^n \to x)$. Therefore, $y^n \to x = y^{n+1} \to x$.

(1) $\Leftrightarrow$ (3) The proof is clear. □

Corollary 3.2. *In a positive implicative equality algebra X, the conditions* (2) *and* (3) *of Proposition* 3.2 *are always valid.*

Definition 3.3. Let X and Y be equality algebras. A mapping $f : X \to Y$ is called a *$\to$-homomorphism* if $f(a \to x) = f(a) \to f(x)$ for all $a, x \in X$.

By a $\to$-endomorphism on X we mean a $\to$-homomorphism from X to X. It is clear that the left mapping f_1 on X is a $\to$-homomorphism.

Example 3.4. Let X be the equality algebra as in Example 3.1 and Y be the equality algebra as in Example 3.3. We define a mapping $f : X \to Y$ by $f(0) = f(b) = 0$ and $f(a) = f(1) = 1$. Then f is a $\to$-homomorphism.

Theorem 3.2. *Let $(X, \wedge_X, \sim_X, 1_X)$ and $(Y, \wedge_Y, \sim_Y, 1_Y)$ be equality algebras. Then every homomorphism from X to Y is a $\to$-homomorphism.*

Proof. Let $f : X \to Y$ be a homomorphism. Then

$$\begin{aligned} f(x \to_X y) &= f((x \wedge_X y) \sim_X x) \\ &= f(x \wedge_X y) \sim_Y f(x) \\ &= (f(x) \wedge_Y f(y)) \sim_Y f(x) \\ &= f(x) \to_Y f(y), \end{aligned}$$

for all $x, y \in X$. Hence, f is a $\to$-homomorphism. □

The following example shows that a left mapping is not a $\to$-endomorphism.

Example 3.5. The left mapping f_a in Example 3.2 is not a $\to$-endomorphism since

$$1 = f_a(c) = f_a(d \to 0) \neq f_a(d) \to f_a(0) = (a \to d) \to (a \to 0) = 1 \to d = d.$$

We provide a condition for a left mapping to be a $\to$-endomorphism, and consider a characterization of a positive implicative equality algebra by using the notion of left mapping.

Theorem 3.3. *An equality algebra X is a positive implicative if and only if every left mapping on X is a $\to$-endomorphism of X.*

Proof. Let X be a positive implicative equality algebra and $f_a : X \to X$ be a left mapping on X where $a \in X$. Then

$$f_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = f_a(x) \to f_a(y),$$

for all $x, y \in X$, and so f_a is a $\to$-endomorphism of X.

Conversely, assume that every left mapping on X is a $\to$-endomorphism of X. Let f_a be a left mapping on X for each $a \in X$. Then f_a is a $\to$-endomorphism of X and so

$$a \to (x \to y) = f_a(x \to y) = f_a(x) \to f_a(y) = (a \to x) \to (a \to y).$$

Therefore, X is a positive implicative equality algebra. $\square$

Corollary 3.3. *Let f_a be a left mapping on X. If $f_a^2 = f_a$, then f_a is a $\to$-endomorphism.*

Corollary 3.4. *If f_a is a $\to$-endomorphism on X, then $f_a^n = f_a^{n+1}$ for any $n \in \mathbb{N}$.*

Theorem 3.4. *Let X be an &-equality algebra. Then $\mathcal{L}(X)$ is a commutative monoid under the composition of mappings with the zero element f_1.*

Proof. For any $f_a, f_b, f_c \in \mathcal{L}(X)$, where $a, b, c \in X$, we have

$$(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(b \to x) = a \to (b \to x) = (a \odot b) \to x = f_{a\odot b}(x),$$

for all $x \in X$. Hence, $\mathcal{L}(X)$ is closed under the operation $\circ$. Also, we have

$$\begin{aligned}(f_a \circ (f_b \circ f_c))(x) &= f_a(f_{(b\odot c)}(x)) = f_{(a\odot(b\odot c))}(x) = f_{((a\odot b)\odot c)}(x)\\ &= f_{(a\odot b)} \circ f_c(x) = ((f_a \circ f_b) \circ f_c)(x),\\ (f_a \circ f_b)(x) &= f_a(b \to x) = a \to (b \to x) = b \to (a \to x) = f_b(a \to x)\\ &= (f_b \circ f_a)(x),\end{aligned}$$

and $(f_a \circ f_1)(x) = f_{a\odot 1}(x) = f_a(x)$ for all $x \in X$. Therefore, $\mathcal{L}(X)$ is a commutative monoid. $\square$

Theorem 3.5. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then $\mathrm{Im}(f_a)$, $\mathrm{Fix}(f_a)$ and $\ker(f_a)$ are closed under the operation $\to$.*

Proof. If $x, y \in \mathrm{Im}(f_a)$, then there exist $u, v \in X$ such that $f_a(u) = x$ and $f_a(v) = y$. It follows that

$$x \to y = f_a(u) \to f_a(v) = (a \to u) \to (a \to v) = a \to (u \to v) = f_a(u \to v) \in \mathrm{Im}(f_a).$$

Thus, $\mathrm{Im}(f_a)$ is closed under $\to$. Let $x, y \in \ker(f_a)$. Then $f_a(x) = 1 = f_a(y)$ and thus

$$f_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = f_a(x) \to f_a(y) = 1.$$

Hence, $x \to y \in \ker(f_a)$ and so $\ker(f_a)$ is closed under $\to$. Let $x, y \in \mathrm{Fix}(f_a)$. Then $f_a(x) = x$ and $f_a(y) = y$. Thus,

$$x \to y = f_a(x) \to f_a(y) = (a \to x) \to (a \to y) = a \to (x \to y) = f_a(x \to y),$$

and so $x \to y \in \mathrm{Fix}(f_a)$. Hence, $\mathrm{Fix}(f_a)$ is closed under $\to$. $\square$

Using mathematical induction, we have the following corollary.

Corollary 3.5. *In a positive implicative equality algebra X, if f_a is a left mapping on X for $a \in X$, then $\mathrm{Im}(f_a^n)$, $\mathrm{Fix}(f_a^n)$ and $\ker(f_a^n)$ are closed under the operation $\to$ for all $n \in \mathbb{N}$.*

We define an order "$\leq$" and equality "$=$" on $\mathcal{L}(X)$ as follows.

$$f_a \leq f_b \Leftrightarrow f_a(x) \leq f_b(x) \text{ for all } x \in X,$$
$$f_a = f_b \Leftrightarrow f_a \leq f_b \;\&\; f_b \leq f_a,$$

for all $f_a, f_b \in \mathcal{L}(X)$.

Proposition 3.3. *If X is a positive implicative equality algebra, then the following assertions are true in $\mathcal{L}(X)$:*

(1) $f_a \circ f_b = f_b \circ f_a$;
(2) $f_a \circ f_a = f_a$;
(3) $f_1 \circ f_a = f_a = f_a \circ f_1$;
(4) $a \leq b \Rightarrow f_b \leq f_a,\ f_a \circ f_b = f_a$.

Proof. (1) Let $a, b, x \in X$. Then by (2.1), it is clear that

$$f_a \circ f_b(x) = f_a(f_b(x)) = a \to (b \to x) = b \to (a \to x) = f_b(f_a(x)) = f_b \circ f_a(x).$$

(2) Let $a, x \in X$. Since X is a positive equality algebra, we get that

$$f_a \circ f_a(x) = f_a(f_a(x)) = a \to (a \to x) = a \to x = f_a(x).$$

(3) The proof is clear.

(4) Let $a, b \in X$ such that $a \leq b$. Then for any $x \in X$, by (2.6), we get $f_b(x) = b \to x \leq a \to x = f_a(x)$. Moreover, since X is positive implicative, we have

$$f_a \circ f_b(x) = a \to (b \to x) = (a \to b) \to (a \to x) = 1 \to (a \to x) = a \to x = f_a.$$

This completes the proof. □

Let $\mathrm{End}_{\to}(X)$ denote the set of all left mappings on X which is a $\to$-homomorphism, that is,

$$\mathrm{End}_{\to}(X) = \{f_a \in \mathcal{L}(X) \mid f_a \text{ is a } \to\text{-homomorphism}\}.$$

Theorem 3.6. *If X is a positive implicative &-equality algebra, then* $(\mathrm{End}_{\to}(X), \circ)$ *is a commutative semigroup with the zero element f_1.*

Proof. Let $x \in X$. Since X is an &-equality algebra, we get

$$(f_a \circ f_b)(x) = a \to (b \to x) = (a \odot b) \to x = f_{a \odot b}(x),$$

and so, $f_{a \odot b}(x) \in \mathcal{L}(X)$. Since X is a positive implicative equality algebra, we have

$$\begin{aligned}(f_a \circ f_b)(x \to y) &= f_{a \odot b}(x \to y) = (a \odot b) \to (x \to y)\\ &= ((a \odot b) \to x) \to ((a \odot b) \to y)\\ &= (f_a \circ f_b)(x) \to (f_a \circ f_b)(y).\end{aligned}$$

Let $f_a, f_b, f_c \in \mathrm{End}_{\to}(X)$. Since X is an &-equality algebra, we have

$$(f_a \circ (f_b \circ f_c))(x) = f_{a \odot (b \odot c)}(x) = f_{(a \odot b) \odot c}(x) = ((f_a \circ f_b) \circ f_c)(x).$$

Also $f_a \circ f_b = f_b \circ f_a$ and $f_a \circ f_1 = f_1$ by Proposition 3.3. Therefore, $(\mathrm{End}_{\to}(X), \circ)$ is a commutative semigroup with the zero element f_1. □

4. Right Mappings

In this section, we introduce the notion of right mapping and investigate some properties of it. Also, we prove that kernel of g_a^2 is a filter of X. Finally we show that the set of all right mappings on positive implicative equality algebra is a dual BCK-algebra.

Definition 4.1. Given a fixed element a in an equality algebra X, we define a self-mapping g_a of X by

$$g_a : X \to X, \quad x \mapsto x \to a, \tag{4.1}$$

and we say that g_a is a *right mapping* on X.

Let $\mathcal{R}(X)$ denote the set of all right mappings on an equality algebra X.

Example 4.1. Let X be the equality algebra as in Example 3.1. Then define a self mapping $g_a : X \to X$ by $g_a(0) = g_a(a) = 1$ and $g_a(b) = g_a(1) = a$. It is routine to verify that g_a is a right mapping on X.

Proposition 4.1. *Every right mapping g_β on X, where β is any element of X, satisfies the following conditions:*

1. $(\forall a \in X)(g_a(a) = 1, g_a(1) = a)$;
2. $(\forall a, b \in X)(g_a(1) \leq g_a(b))$;
3. *If X is bounded, then $g_a(0) = 1$ and $g_0(a) = \neg a$ for all $a \in X$;*
4. $(\forall x \in X)(g_1(x) = 1)$;
5. $(\forall a, x, y \in X)(x \leq y \Rightarrow g_a(y) \leq g_a(x))$.

Proof. Straightforward. □

Proposition 4.2. *For any right mapping g_β on X where β is any element of X, we have the following assertions.*

1. *If X is a commutative equality algebra, then $g_a^2(x) = g_x^2(a)$ for all $x, a \in X$.*
2. *For any natural number $n \in \mathbb{N}$ and $a \in X$, we have*
$$g_a^n = \begin{cases} g_a & n \text{ is odd,} \\ g_a^2 & n \text{ is even.} \end{cases}$$
3. $g_a^2(x) \to g_a(y) = g_a^2(y) \to g_a(x)$ *for any $a, x, y \in X$.*
4. $y \to g_a^2(x) = g_a(x) \to g_a(y)$ *and* $g_a^2(x) \to g_a^2(y) = x \to g_a^2(y)$ *for any $a, x, y \in X$.*
5. $g_a^2(x) = 1$ *if and only if $f_x(a) = a$, where f_x is a left mapping on X.*
6. *The mapping g_a^2 is isotone.*

Proof. (1) Since X is a commutative equality algebra, we have

$$g_a^2(x) = (x \to a) \to a = (a \to x) \to x = g_x^2(a),$$

for all $a, x \in X$.

(2) Let $x, a \in X$ and $n \in \mathbb{N}$. Suppose $n = 4$. Then

$$g_a^4(x) = (((x \to a) \to a) \to a) \to a = (x \to a) \to a = g_a^2(x),$$

by (2.7). By the similar way, we can prove that $g_a^n(x) = g_a^2(x)$ for any even number $n \in \mathbb{N}$. Now, if $n = 3$, then

$$g_a^3(x) = ((x \to a) \to a) \to a = x \to a = g_a(x),$$

by (2.7). By the similar way, we can prove that $g_a^n(x) = g_a(x)$ for any odd number $n \in \mathbb{N}$.

(3) Let $a, x, y \in X$. Then

$$\begin{aligned} g_a^2(x) \to g_a(y) &= ((x \to a) \to a) \to (y \to a) \\ &= y \to (((x \to a) \to a) \to a) \\ &= y \to (x \to a) \\ &= y \to g_a(x), \end{aligned}$$

by (2.7). By the similar way, we can prove that $g_a^2(y) \to g_a(x) = y \to g_a(x)$. Hence, $g_a^2(x) \to g_a(y) = g_a^2(y) \to g_a(x)$.

(4) Let $x, y, a \in X$. Then

$$y \to g_a^2(x) = y \to ((x \to a) \to a) = (x \to a) \to (y \to a) = g_a(x) \to g_a(y),$$

by (2.7). Also, we have

$$\begin{aligned} g_a^2(x) \to g_a^2(y) &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= g_a(y) \to g_a(x) = x \to g_a^2(y). \end{aligned}$$

(5) and (6) are straightforward. □

Theorem 4.1. *For any right mapping g_a on X, the following are equivalent.*

(1) *g_a^2 is a $\to$-endomorphism.*
(2) *$g_a^2(x \to y) = x \to g_a^2(y)$ for all $x, y \in X$.*
(3) *$g_a^2(x \to y) = g_a(y) \to g_a(x)$ for all $x, y \in X$.*

Proof. (1) $\Rightarrow$ (2). Let g_a^2 be a $\to$-endomorphism and $x, y \in X$. Then

$$\begin{aligned} g_a^2(x \to y) &= g_a^2(x) \to g_a^2(y) \\ &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= x \to ((y \to a) \to a) \\ &= x \to g_a^2(y), \end{aligned}$$

by (2.7).

(2) $\Rightarrow$ (3). For any $x, y \in X$ we have

$$\begin{aligned} g_a^2(x \to y) =& x \to g_a^2(y) = x \to ((y \to a) \to a) = (y \to a) \to (x \to a) \\ =& g_a(y) \to g_a(x), \end{aligned}$$

by (2).

(3) $\Rightarrow$ (1). For any $a, x, y \in X$ we have

$$\begin{aligned} g_a^2(x) \to g_a^2(y) &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) \\ &= g_a(y) \to g_a(x) \\ &= g_a^2(x \to y), \end{aligned}$$

by (2.7) and (3). Therefore, g_a^2 is a $\to$-endomorphism on X. $\square$

Theorem 4.2. *For any right mapping g_a on X, the following are equivalent.*

(1) g_a^2 *is an identity map.*
(2) g_a *is an injective map.*
(3) g_a *is a surjective map.*

Proof. (1) $\Rightarrow$ (2). Let g_a^2 be an identity map. Let $x, y \in X$ be such that $g_a(x) = g_a(y)$. Then $x \to a = y \to a$ and so

$$x = g_a^2(x) = (x \to a) \to a = (y \to a) \to a = g_a^2(y) = y.$$

Hence, g_a is an injective map on X.

(2) $\Rightarrow$ (3). For any $x, y \in X$, we have $g_a((x \to a) \to a) = g_a(x)$ by (2.7). Since g_a is an injective map on X, it follows that $(x \to a) \to a = x$. Moreover, we know that $\mathrm{Im}(g_a) \subseteq X$. Let $y \in X$. Then $g_a(y \to a) = (y \to a) \to a = y$ and so $y \in \mathrm{Im}(g_a)$. Hence, $X = \mathrm{Im}(g_a)$. Therefore, g_a is a surjective map on X.

(3) $\Rightarrow$ (1). Using (2.4), we have $x \leq (x \to a) \to a = g_a^2(x)$ for any $x \in X$. Since g_a is a surjective map, for any $y \in X$, there exists $x \in X$ such that $g_a(x) = y$, i.e., $x \to a = y$. It follows from (2.1) and (2.7) that

$$g_a^2(y) \to y = ((y \to a) \to a) \to (x \to a) = x \to (y \to a) = y \to y = 1,$$

that is, $g_a^2(y) \leq y$ for all $y \in X$. Hence, $g_a^2(y) = y$ for all $y \in X$ and therefore g_a^2 is an identity map. $\square$

Corollary 4.1. *For any right mapping g_β on X where β is any element of X, the following are equivalent.*

(1) g_a^2 *is an injective map for all $a \in X$.*
(2) g_a^2 *is an identity map for all $a \in X$.*
(3) g_a^2 *is a surjective map for all $a \in X$.*

Proof. By Theorem 4.2 and Proposition 4.2 (2), the proof is clear. $\square$

Theorem 4.3. *For any right map g_a on X, the set* $\ker(g_a^2) = \{x \in X \mid g_a^2(x) = 1\}$ *is a filter of X.*

Proof. Let $a \in X$. Since $g_a^2(1) = (1 \to a) \to a = a \to a = 1$, we get that $1 \in \ker(g_a^2)$. Let $x, y \in X$ be such that $x, x \to y \in \ker(g_a^2)$. Then $g_a^2(x) = g_a^2(x \to y) = 1$. It follows from (2.1), (2.5) and (2.7) that

$$\begin{aligned} g_a^2(y) &= (y \to a) \to a \\ &= 1 \to ((y \to a) \to a) \\ &= (((x \to y) \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to ((((x \to y) \to a) \to a) \to a) \\ &= (y \to a) \to ((x \to y) \to a) \\ &= (x \to y) \to ((y \to a) \to a) \\ &= (x \to y) \to (1 \to ((y \to a) \to a)) \\ &= (x \to y) \to (((x \to a) \to a) \to ((y \to a) \to a)) \\ &= (x \to y) \to ((y \to a) \to (((x \to a) \to a) \to a)) \\ &= (x \to y) \to ((y \to a) \to (x \to a)) \\ &= (x \to y) \to (x \to ((y \to a) \to a)) \\ &\geq y \to ((y \to a) \to a) \\ &= (y \to a) \to (y \to a) \\ &= 1 \end{aligned}$$

Hence, $g_a^2(y) = 1$, and so $y \in \ker(g_a^2)$. Therefore, $\ker(g_a^2)$ is a filter of X. □

Corollary 4.2. *For any right map g_a^{2k} on X, the set $\ker(g_a^{2k})$ is a filter of X, where k is any natural number.*

Proposition 4.3. *Let g_β be a right mapping on X where β is any element of X. If F and G are filters of X such that $F \cap G = \{1\}$, then $g_x^2(y) = g_y^2(x) = 1$ for all $x \in F$ and $y \in G$.*

Proof. Let F and G be filters of X such that $F \cap G = \{1\}$. Suppose $x \in F$ and $y \in G$. Since $x \leq (x \to y) \to y$ and $y \leq (x \to y) \to y$ by (2.1), (2.2) and (2.4), we have $(x \to y) \to y \in F \cap G = \{1\}$ and so $g_y^2(x) = (x \to y) \to y = 1$. By the similar way we can prove that $g_x^2(y) = 1$. □

Let X be a positive implicative equality algebra. We define the implication "$\hookrightarrow$" on $\mathcal{R}(X)$ as follows:

$$\hookrightarrow : \mathcal{R}(X) \times \mathcal{R}(X) \to \mathcal{R}(X), \quad (g_a, g_b) \mapsto g_a(x) \to g_b(x).$$

Using the positive implicativity of X, we have

$$(g_a \hookrightarrow g_b)(x) = g_a(x) \to g_b(x) = (x \to a) \to (x \to b) = x \to (a \to b) = g_{a \to b}(x),$$

and so $g_a \to g_b \in \mathcal{R}(X)$.

Theorem 4.4. *If X is a positive implicative equality algebra, then $(\mathcal{R}(X), \hookrightarrow, g_1)$ is a dual BCK-algebra (see [3] for the notion of dual BCK-algebra).*

Proof. Let $g_a, g_b, g_c \in \mathcal{R}(X)$. Then

$$\begin{aligned}
&((g_b \hookrightarrow g_c) \hookrightarrow ((g_c \hookrightarrow g_a) \hookrightarrow (g_b \hookrightarrow g_a)))(x)\\
=&(g_b(x) \to g_c(x)) \to ((g_c(x) \to g_a(x)) \to (g_b(x) \to g_a(x)))\\
=&((x \to b) \to (x \to c)) \to (((x \to c) \to (x \to a)) \to ((x \to b) \to (x \to a)))\\
=&(x \to (b \to c)) \to ((x \to (c \to a)) \to (x \to (b \to a)))\\
=&(x \to (b \to c)) \to (x \to ((c \to a) \to (b \to a)))\\
=&x \to ((b \to c) \to ((c \to a) \to (b \to a)))\\
=&x \to 1 = g_1(x)
\end{aligned}$$

and

$$\begin{aligned}
&(g_b \hookrightarrow ((g_b \hookrightarrow g_a) \hookrightarrow g_a))(x) = g_b(x) \to ((g_b(x) \to g_a(x)) \to g_a(x))\\
=&(x \to b) \to (((x \to b) \to (x \to a)) \to (x \to a))\\
=&(x \to b) \to ((x \to (b \to a)) \to (x \to a))\\
=&(x \to b) \to (x \to ((b \to a) \to a))\\
=&x \to (b \to ((b \to a) \to a))\\
=&x \to ((b \to a) \to (b \to a))\\
=&x \to 1 = g_1(x),
\end{aligned}$$

for all $x \in X$ by (2.1), (2.2), (2.5) and (3.1). Thus,

$$(g_b \hookrightarrow g_c) \hookrightarrow ((g_c \hookrightarrow g_a) \hookrightarrow (g_b \hookrightarrow g_a)) = g_1,$$

and $g_b \hookrightarrow ((g_b \hookrightarrow g_a) \hookrightarrow g_a) = g_1$. Since

$$(g_a \hookrightarrow g_a)(x) = g_a(x) \to g_a(x) = (x \to a) \to (x \to a) = 1 = x \to 1 = g_1(x)$$

and

$$\begin{aligned}
(g_a \hookrightarrow g_1)(x) &= g_a(x) \to g_1(x) = (x \to a) \to (x \to 1)\\
&= x \to (a \to 1) = x \to 1 = g_1(x),
\end{aligned}$$

for all $x \in X$, we have $g_a \hookrightarrow g_a = g_1$ and $g_a \hookrightarrow g_1 = g_1$. Assume that $g_a \to g_b = g_1$ and $g_b \to g_a = g_1$. Then

$$(x \to a) \to (x \to b) = g_a(x) \to g_b(x) = (g_a \hookrightarrow g_b)(x) = g_1(x) = x \to 1 = 1$$

and

$$(x \to b) \to (x \to a) = g_b(x) \to g_a(x) = (g_b \hookrightarrow g_a)(x) = g_1(x) = x \to 1 = 1,$$

for all $x \in X$. It follows that $g_a(x) = x \to a = x \to b = g_b(x)$ for all $x \in X$. Hence, $g_a = g_b$. Therefore, $(\mathcal{R}(X), \hookrightarrow, g_1)$ is a dual BCK-algebra. □

Define an order "$\leq$" on $\mathcal{R}(X)$ as follows:

$$(\forall g_a, g_b \in \mathcal{R}(X))(g_a \leq g_b \Leftrightarrow (g_a \hookrightarrow g_b)(x) = g_1(x) \text{ for all } x \in X.$$

It is clear that if X is a positive implicative equality algebra, then $(\mathcal{R}(X), \leq)$ is a partially ordered set.

Proposition 4.4. *If X is a positive implicative equality algebra, then the following assertions are true in $\mathcal{R}(X)$:*

(1) $g_a \hookrightarrow g_b \leq (g_b \hookrightarrow g_c) \hookrightarrow (g_a \hookrightarrow g_c)$;
(2) $g_a \leq (g_a \hookrightarrow g_b) \hookrightarrow g_b$;
(3) $g_a \leq g_a$;
(4) $g_a \leq g_b$ *and* $g_b \leq g_a$ *imply* $g_a = g_b$;
(5) $g_a \leq g_1$;
(6) $f_a \leq f_b \Rightarrow f_b \hookrightarrow f_c \leq f_a \hookrightarrow f_c,\ f_c \hookrightarrow f_a \leq f_c \hookrightarrow f_b$;
(7) $f_a \hookrightarrow (f_b \hookrightarrow f_c) = f_b \hookrightarrow (f_a \hookrightarrow f_c)$;
(8) $f_a \leq f_b \hookrightarrow f_c \Rightarrow f_b \leq f_a \hookrightarrow f_c$;
(9) $f_a \hookrightarrow f_b \leq (f_c \hookrightarrow f_a) \hookrightarrow (f_c \hookrightarrow f_b)$;
(10) $f_a \leq f_b \hookrightarrow f_a$.

Proof. It is easy by routine calculations. □

5. Conclusions and Future Works

In this paper, the notion of (right) left mapping on equality algebras is introduced, some properties of it are investigated and it is proved that the set of all right mappings on positive implicative equality algebra makes a dual BCK-algebra. Also, we studied that under which condition the kernel of (right) left mapping is a filter. The notion of $\rightarrow$-endomorphism is introduced and it is proved that the set of all $\rightarrow$-endomorphisms on equality algebra is a commutative semigroup with zero element. Moreover, the relation between left mapping and $\rightarrow$-endomorphism and a characterization of positive implicative equality algebra are investigated.

In future work, by using the notion of (right) left mapping on equality algeras and the set of fixed point of that, we can introduce the notion of (right) left stabilizer on equality algebra and by using this notion we can define a basis of a topology on equality algebra. Also, we can introduce the notion of derivation on equality algebra and extend it.

Acknowledgments.
The authors are very indebted to the editor and anonymous referees for their careful reading and valuable suggestions which helped to improve the readability of the paper.

This research is supported by a grant of National Natural Science Foundation of China (11971384).

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