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ENTIRE FUNCTION SHARING ENTIRE FUNCTION WITH ITS FIRST DERIVATIVE

SUJOY MAJUMDER[1] AND JEET SARKAR[2]

Abstract. In this paper, we use the idea of normal family to investigate the problem of entire function that share entire function with its first derivative.

1. Introduction, Definitions and Results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $|z|<r$, the poles are counted with their multiplicities. We call the quantity

$$N(r,\infty;f)=\int\limits_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

as the integrated counting function or simply the counting function of poles of f and

$$m(r,\infty;f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(re^{i\theta})|d\theta$$

as the proximity function of poles of f, where $\log^+ x=\log x$, if $x\geq 1$ and $\log^+ x=0$, if $0\leq x<1$.

We use the notation $T(r,f)$ for the sum $m(r,\infty;f)+N(r,\infty;f)$ and it is called the Nevanlinna characteristic function of f. We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $\frac{S(r,f)}{T(r,f)}\to 0$ as $r\to\infty$ except possibly a set of finite linear measure.

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For $a \in \mathbb{C}$, we write $N(r,a;f) = N(r,\infty;\frac{1}{f-a})$ and $m(r,a;f) = m(r,\infty;\frac{1}{f-a})$.

Again we denote by $\overline{n}(r,a;f)$ the number of distinct a points of f lying in $|z| < r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r,a;f) = \int_0^r \frac{\overline{n}(t,a;f) - \overline{n}(0,a;f)}{t} dt + \overline{n}(0,a;f) \log r$$

denotes the reduced counting function of a points of f (see, e.g., [6, 15]).

A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$, i.e., if $T(r,a) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ and Q be a polynomial or a finite complex number. If $g(z) - Q(z) = 0$ whenever $f(z) - Q(z) = 0$, we write $f = Q \Rightarrow g = Q$.

Let f and g be two non-constant meromorphic functions. Let a be a small function with respect to both f and g. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities then we say that f and g share a with CM (counting multiplicities) and if we do not consider the multiplicities then we say that f and g share a with IM (ignoring multiplicities).

We recall that the order $\rho(f)$ of meromorphic function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [13]).

Rubel and Yang [12] were the first authors to study the entire functions that share values with their derivatives. In 1977, they proved the following important result.

Theorem A ([12])**.** *Let a and b be complex numbers such that $b \neq a$ and let f be a non-constant entire function. If f and f' share the values a and b CM, then $f \equiv f'$.*

In 1979, Mues and Steinmetz [11] generalized Theorem A from sharing values CM to IM and obtained the following result.

Theorem B ([11])**.** *Let a and b be complex numbers such that $b \neq a$ and f a non-constant entire function. If f and f' share the values a and b IM, then $f \equiv f'$.*

In 1983, Gundersen [4] improved Theorem A from entire function to meromorphic function and obtained the following result.

Theorem C ([4])**.** *Let f be a non-constant meromorphic function, a and b two distinct finite values. If f and f' share the values a and b CM, then $f \equiv f'$.*

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. Let f be a non-constant entire function. Suppose

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a} = c, \tag{1.1}$$

for some non-zero constant c.

By the solutions of the differential equations

$$\frac{f'-a}{f-a} = e^{z^n} \quad \text{and} \quad \frac{f'-a}{f-a} = e^{e^z}, \tag{1.2}$$

we see that when $\rho_1(f)$ is a positive integer or infinite, the conjecture does not hold.

Conjecture A for the case $a = 0$ had been proved by Brück [1]. In the same paper Brück [1] proved that the growth restriction on f is not necessary when $N(r,0;f') = S(r,f)$.

Gundersen and Yang [5] proved that Conjecture A is true when f is of finite order. Further Chen and Shon [3] proved that Conjecture A is also true when f is of infinite order with $\rho_1(f) < \frac{1}{2}$. Recently Cao [2] proved that Brück conjecture is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open.

Since then, shared value problems, especially the case of f and $f^{(k)}$, where $k \in \mathbb{N}$ sharing one value or small function have undergone various extensions and improvements (see [15]).

Now it is interesting to know what happens if f is replaced by f^n in Conjecture A. From (1.2), we see that Conjecture A does not hold when $n = 1$. Thus, we have to discuss the problem only when $n \geq 2$.

Yang and Zhang [14] proved that Conjecture A holds for the function f^n without imposing the order restriction on f if n is relatively large. Actually they proved the following result.

Theorem D ([14])**.** *Let f be a non-constant entire function, $n \in \mathbb{N} \setminus \{1,2,\ldots,6\}$ and $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2009, Lü, Xu and Chen [8] improved Theorem D in the following manner.

Theorem E ([8])**.** *Let $a(\not\equiv 0)$ be a polynomial and $n \in \mathbb{N} \setminus \{1\}$, f a transcendental entire function and $F = f^n$. If F and F' share a CM, then conclusion of Theorem D holds.*

In 2011, Lü [9] further improved Theorem E as follows.

Theorem F ([9])**.** Let f be a transcendental meromorphic function with finitely many poles, $n \in \mathbb{N} \setminus \{1\}$ and $\alpha = Pe^Q(\not\equiv \alpha')$ an entire function such that the order of α is less than that of f, where P, Q are two polynomials. If f^n and $(f^n)'$ share α CM, then conclusion of Theorem D holds.

Remark 1.1. If Q is a constant, then Theorem F still holds without the assumption that $\rho(\alpha) < \rho(f)$.

In 2014, Zhang, Kang and Liao [17] improved Theorem F in a different direction as follows.

Theorem G ([17])**.** *Let f be a transcendental entire function, $a = a(z)(\not\equiv 0, \infty)$ a small function of f such that order of a is less than that of f and $n \in \mathbb{N} \setminus \{1\}$. If f^n and $(f^n)'$ share a CM, then conclusion of Theorem D holds.*

Naturally, one can ask whether the conclusion of Theorem E still holds if F and F' share a CM is replaced by share a IM. In 2015, Lü and Yi [10] gave an affirmative answer and obtained the following result.

Theorem H ([10])**.** *Let $a(\not\equiv 0)$ be a polynomial and $n \in \mathbb{N} \setminus \{1\}$. Let f be a transcendental entire function and $F = f^n$. If F and F' share a IM, then conclusion of Theorem D holds.*

We now emerge the following question as an open problem.
Question 1. What happens if F and F' share a CM is replaced by share Pe^Q IM, where $P(\not\equiv 0)$ and Q are polynomials in Theorem E?
In the paper we prove the following result that answer the above question.

Theorem 1.1. *Let f be a transcendental entire function and $n \in \mathbb{N} \setminus \{1\}$. Let $\alpha = Pe^Q(\not\equiv \alpha')$, where $P(\not\equiv 0)$ and Q are polynomials such that $2\rho(\alpha) < \rho(f)$. If f^n and $(f^n)'$ share α IM, then conclusion of Theorem D holds.*

Remark 1.2. If Q is a constant, then Theorem 1.1 still holds without the assumption that $2\rho(\alpha) < \rho(f)$. Also from Theorem 1.1, it is clear that Theorem 1.1 is the generalization of Theorem H.

2. Lemmas

In this section we present the lemmas which will be needed in the sequel.

Lemma 2.1 ([8])**.** *Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc Δ. If $a_n \to a$, $|a| < 1$ and $f_n^{\#}(a_n) \to \infty$, then there exist*

(a) *a subsequence of* f_n (*which we still write as* $\{f_n\}$);
(b) *points* $z_n \to z_0$, $|z_0| < 1$;
(c) *positive numbers* $\rho_n \to 0$,

such that $f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ *locally uniformly, where* g *is a non-constant meromorphic* (*entire*) *function on* $\mathbb{C}$ *such that*

$$\rho_n \leq \frac{M}{f_n^{\#}(a_n)},$$

where M *is a constant which is independent of* n.

Lemma 2.2 ([16])**.** *Let* f *be a meromorphic function in the complex plane and* $\rho(f) > 2$. *Then for each* $0 < \mu < \frac{\rho(f)-2}{2}$, *there exist points* $a_n \to \infty$, $n \to \infty$, *such that*

$$\lim_{n\to\infty} \frac{f^{\#}(a_n)}{|a_n|^{\mu}} = +\infty.$$

Lemma 2.3 ([7])**.** *Let* f *be a meromorphic function of infinite order on* $\mathbb{C}$. *Then there exist points* $z_n \to \infty$ *such that for every* $N > 0$, $f^{\#}(z_n) > |z_n|^N$, *if* n *is sufficiently large.*

3. Proof of the Theorem 1.1

Proof. Let $F = \frac{f^n}{\alpha}$ and $G = \frac{(f^n)'}{\alpha}$. Now we consider following two cases.

Case 1. Suppose $\rho(f) < +\infty$. Clearly $\rho(\alpha) = \deg(Q)$ and $\rho(f) = \rho(f^n)$. Since $\rho(\alpha) < \rho(f)$, we have $\rho(\alpha) < \rho(f^n)$. Note that $\rho\left(\frac{f^n}{\alpha}\right) \leq \max\{\rho(f^n), \rho(\alpha)\} = \rho(f^n)$. Since $\rho(\alpha) < \rho(f^n)$, it follow that $\rho(f^n) = \rho\left(\frac{f^n}{\alpha}\alpha\right) \leq \max\left\{\rho\left(\frac{f^n}{\alpha}\right), \rho(\alpha)\right\} = \rho\left(\frac{f^n}{\alpha}\right)$. Consequently, $\rho(f^n) = \rho\left(\frac{f^n}{\alpha}\right) = \rho(F)$. Therefore,

$$\rho(f) = \rho\Big(f^n\Big) = \rho\Big(\frac{f^n}{\alpha}\Big) = \rho(F) < +\infty.$$

Since $\rho((f^n)') = \rho(f^n) < +\infty$, we have $\rho(G) \leq \max\{\rho((f^n)'), \rho(\alpha)\} < +\infty$. Following two sub-cases are immediately.

Sub-case 1.1. Suppose Q is a constant. In that case α reduces to a polynomial. Then by Theorem H, we have $F \equiv G$, i.e., $f^n \equiv (f^n)'$ and so $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.

Sub-case 1.2. Suppose Q is non-constant. Let $\mu_1 = 2\deg(Q) \geq 2$ and $\mu_2 = \frac{\mu_1 - 2}{2}$. Since $\mu_1 < \rho(f)$, we have $0 \leq \mu_2 < \frac{\rho(f)-2}{2}$. Let $0 < \varepsilon < \frac{\rho(f)-\mu_1}{2}$. Then $0 \leq \mu_2 < \mu_2 + \varepsilon < \frac{\rho(f)-2}{2}$. Let $\mu = \mu_2 + \varepsilon$. Now by Lemma 2.2, for $0 < \mu < \frac{\rho(f)-2}{2}$, there exists a sequence $\{w_n\}_n$ such that $w_n \to \infty$, $n \to \infty$, and

$$\lim_{n\to\infty} \frac{F^{\#}(w_n)}{|w_n|^{\mu}} = +\infty. \tag{3.1}$$

Since P is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \geq r_1$, we have

$$0 \leftarrow \left|\frac{P'(z)}{P(z)}\right| \leq \frac{M_1}{|z|} < 1, \quad P(z) \neq 0.$$

Let $r > r_1$ and $D = \{z : |z| \geq r\}$. Then F is analytic in D. Since $w_n \to \infty$ as $n \to \infty$, without loss of generality we may assume that $|w_n| \geq r + 1$ for all n. Let $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f^n(w_n + z)}{\alpha(w_n + z)}.$$

Since $|w_n+z| \geq |w_n|-|z|$, it follows that $w_n+z \in D$ for all $z \in D_1$. Also, since $F(z)$ is analytic in D, it follows that $F_n(z)$ is analytic in D_1 for all n. Thus, we have structured a family $(F_n)_n$ of holomorphic functions. Note that $F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now it follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$. Let $a_n = 0$ for all n and $a = 0$. Then $a_n \to a$ and $|a| < 1$. Also, $F_n^{\#}(a_n) = F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now we apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$, if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ such that $|z_n| < r < 1$, $z_n \to 0$, $\rho_n \to 0$ and that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{f^n(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{3.2}$$

converges locally and uniformly in $\mathbb{C}$, where $g(\zeta)$ is a non-constant entire function. By Hurwitz's theorem, we conclude that zeros of g are of multiplicities at least n. Also,

$$\rho_n \leq \frac{M}{F_n^{\#}(a_n)} = \frac{M}{F^{\#}(w_n)}, \tag{3.3}$$

for a positive number M. Now from (3.1) and (3.3), we deduce that

$$\rho_n \leq \frac{M}{F^{\#}(w_n)} \leq M_1|w_n|^{-\mu}, \tag{3.4}$$

for sufficiently large values of n, where M_1 is a positive constant.

Also from (3.2), we see that

$$\begin{aligned} \rho_n\frac{(f^n)'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} &= g_n'(\zeta) + \rho_n\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha^2(w_n + z_n + \rho_n\zeta)}f^n(w_n + z_n + \rho_n\zeta) \\ &= g_n'(\zeta) + \rho_n\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)}g_n(\zeta). \end{aligned} \tag{3.5}$$

Note that

$$\frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} = \frac{P'(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} + Q'(w_n + z_n + \rho_n\zeta). \tag{3.6}$$

Observe that $\frac{P'(w_n+z_n+\rho_n\zeta)}{P(w_n+z_n+\rho_n\zeta)} \to 0$ as $n \to \infty$. Let $s = \deg(Q')$. Since $2\deg(Q) \leq \mu_1$, it follows that $0 \leq s \leq \mu_2 < \mu$. Therefore, from (3.4), we have

$$\lim_{n\to\infty} \rho_n|w_n|^s \leq \lim_{n\to\infty} M_1|w_n|^{s-\mu} = 0. \tag{3.7}$$

Note that $|Q'(w_n + z_n + \rho_n\zeta)| = O(|w_n|^s)$ and so from (3.7), we have

$$\rho_n|Q'(w_n + z_n + \rho_n\zeta)| = O(\rho_n|w_n|^s) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.8}$$

From (3.6) and (3.8), we have

$$\rho_n \frac{\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Now from (3.2), (3.5) and (3.9), we observe that

$$\rho_n \frac{(f^n)'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g'(\zeta). \tag{3.10}$$

Clearly $g'(z) \not\equiv 0$, for otherwise $g(z)$ would be a polynomial of degree at most 1 and so $g(z)$ could not have zero of multiplicity at least $n(\geq 2)$.

Firstly we claim that $g = 1 \Rightarrow g' = 0$. Suppose that $g(\eta_0) = 1$. Then by Hurwitz's theorem there exists a sequence $(\eta_n)_n$, $\eta_n \to \eta_0$ such that (for sufficiently large n)

$$g_n(\eta_n) = \frac{f^n(w_n + z_n + \rho_n\eta_n)}{\alpha(w_n + z_n + \rho_n\eta_n)} = 1,$$

i.e., $f^n(w_n + z_n + \rho_n\eta_n) = \alpha(w_n + z_n + \rho_n\eta_n)$. By the given condition, we have

$$(f^n)'(w_n + z_n + \rho_n\eta_n) = \alpha(w_n + z_n + \rho_n\eta_n). \tag{3.11}$$

Now from (3.10) and (3.11), we see that

$$g'(\eta_0) = \lim_{n\to\infty} g'(\eta_n) = \lim_{n\to\infty} \rho_n \frac{(f^n)'(w_n + z_n + \rho_n\eta_n)}{\alpha(w_n + z_n + \rho_n\eta_n)} = \lim_{n\to\infty} \rho_n = 0.$$

Thus, $g = 1 \Rightarrow g' = 0$. Finally we want to prove that $g' = 0 \Rightarrow g = 1$. Now from (3.10), we see that

$$\rho_n \frac{(f^n)'(w_n + z_n + \rho_n\zeta) - \alpha(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g'(\zeta). \tag{3.12}$$

Suppose that $g'(\xi_0) = 0$. Then by (3.12) and Hurwitz's theorem, there exists a sequence $(\xi_n)_n$, $\xi_n \to \xi_0$ such that (for sufficiently large n) $(f^n)'(w_n + z_n + \rho_n\xi_n) = \alpha(w_n + z_n + \rho_n\xi_n)$. By the given condition, we have

$$f^n(w_n + z_n + \rho_n\xi_n) = \alpha(w_n + z_n + \rho_n\xi_n).$$

Therefore, from (3.2), we have

$$g(\xi_0) = \lim_{n\to\infty} \frac{f^n(w_n + z_n + \rho_n\xi_n)}{\alpha(w_n + z_n + \rho_n\xi_n)} = 1.$$

Thus $g' = 0 \Rightarrow g = 1$. As a result we have (1) $g = 0 \Rightarrow g' = 0$ and (2) $g = 1 \Leftrightarrow g' = 0$. From (1) and (2), one can easily deduce that $g \neq 0$. Also from (2), we see that zeros of $g-1$ are of multiplicities at least 2. Now by the second fundamental theorem, we have

$$\begin{aligned} T(r,g) \leq& \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;g)+S(r,g) \leq \frac{1}{2}\,N(r,1;g)+S(r,g)\\ \leq& \frac{1}{2}\,T(r,g)+S(r,g), \end{aligned}$$

which is a contradiction.

Case 2. Suppose $\rho(f)=+\infty$. Then $\rho(f^n)=+\infty$. Since $\rho(\alpha)<+\infty$, it follows that $\rho(F)=+\infty$. Now by Lemma 2.3, there exist $\{w_n\}_n$ satisfying $w_n\to\infty$, $n\to\infty$, such that for every $N>0$,

$$F^{\#}(w_n)>|w_n|^N, \tag{3.13}$$

if n is sufficiently large. Then from (3.3) and (3.13), we deduce for every $N>0$ that

$$\rho_n<M|w_n|^{-N}, \tag{3.14}$$

if n is sufficiently large. If we take $N>s$, then from (3.14) we deduce that $\lim_{n\to\infty}\rho_n|w_n|^s=0$ and so (3.9) holds. We omit the proof since the proof of Case 2 can be carried out in the line of proof of Sub-case 1.2. □

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[1]Department of Mathematics,
Raiganj University,
Raiganj, West Bengal-733134, India
Email address: sm05math@gmail.com

[2]Department of Mathematics,
Raiganj University,
Raiganj, West Bengal-733134, India
Email address: sarkar.jeet05@gmail.com