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EXISTENCE AND STABILITY RESULTS OF A NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper deals with the stability results for solution of a fractional integro-differential problem with integral conditions. Using the Krasnoselskii's, Banach fixed point theorems, we proof the existence and uniqueness results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system. The results are illustrated by an example.

1. Introduction and Formulation of the problem

So far, similar to the simplest case-solution of a system of linear ordinary differential equations, the fractional derivative is not explicitly presented, and therefore it makes sense to consider for $t \in [0,1]$, $0 < \alpha, \beta < 1$, the problem for the system

$$\begin{cases} {}^C D_{0^+}^{\alpha+\beta} u(t) = h(t, u(t)) + I_{0^+}^{\alpha} f(t, u(t)) + \int_0^t K(t, s, u(s)) ds, \\ u(0) = b \int\limits_0^{\eta} u(s) ds, \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where b is a real constant, $0 < \alpha + \beta \leq 1$, ${}^C D_{0^+}^{\alpha+\beta}$ is the Caputo fractional derivative of order $\alpha + \beta$, $I_{0^+}^{\alpha}$ denotes the left sided Riemann-Liouville fractional integral of order α and f, h, K defined as

$$\begin{aligned} &f : [0,1] \times X \to X, \\ &h : [0,1] \times X \to X, \\ &K : [0,1] \times [0,1] \times X \to X, \end{aligned} \tag{1.2}$$

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are an appropriate functions satisfying some conditions which will be stated later. X here is a Banach space. It is also interesting to study solution to fractional integro-differential problem with integral conditions, which will allow a generalized stability. The fractional differential equation

$$D_*^\alpha x(t) = f(t, x(t)), \quad \alpha \in \mathbb{R},\ 0 < \alpha < 1, \tag{1.3}$$

was considered in [4, 5, 8] and results related to the existence and uniqueness for solution, with some analytical properties and useful inequalities, were obtained. Next, it is shown in [9] that, in a real n-dimensional Euclidean space, the local and global solutions exist for the following Cauchy problem

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = f(t, u(t)) + \int_{t_0}^t K(t, s, u(s))ds, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, $f \in C\Big([0,1] \times \mathbb{R}^n, \mathbb{R}^n\Big)$, $K \in C\Big([0,1] \times [0,1] \times \mathbb{R}^n, \mathbb{R}^n\Big)$ and ${}^C D_{0+}^\alpha$ is the Caputo fractional operator.

A class of abstract delayed fractional neutral integro-differential equations was introduced in [11]

$$\begin{cases} D_t^\alpha N(x_t) = AN(x_t) + \int_0^t B(t-s)N(x_s)ds + f(t, x_{\rho(t,x_t)}), \\ x_0 = \phi, \quad x'(0) = 0, \quad \alpha \in (1, 2), \end{cases} \tag{1.5}$$

Using the Leray-Schauder alternative fixed point theorem, the existence results were obtained (for more details, please see [10]). Recently, much attention has been paid to the study of differential equations with fractional derivatives [2, 3], mainly to the questions of the existence and stability for a fractional order differential equation with non-conjugate Riemann-Stieltjes Integro-multipoint boundary conditions.

Note that in [3], the authors introduced and studied a related problem. Precisely the authors studied the existence for the following problem

$$\begin{cases} {}^C D_{0+}^p \{{}^C D_{0+}^q x(t) + f(t, x(t))\} = g(t, x(t)), \quad t \in [0, 1], \\ x(0) = \sum_{j=1}^{j=m} \beta_j x(\sigma_j), \\ bx(1) = a \int_0^1 x(s) dH(s) + \sum_{i=1}^{i=n} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \end{cases} \tag{1.6}$$

where

$$0 < \sigma_j < \xi_i < \eta_i < 1, \quad 0 < p, q < 1, \quad \beta_j, \alpha_i \in \mathbb{R}, \quad i = 1, 2, \dots n,\ j = 1, 2, \dots, m.$$

${}^C D_{0+}^p$ is the Caputo fractional derivative of order p, f, g, are given continuous functions. By using a classical tools of fixed point theory, the existence and uniqueness results were obtained. On an arbitrary domain, in [2], the authors study the existence and stability results for a fractional order differential equation with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions by using a new tools on function analysis.

Here we focused our study on the question of existence and uniqueness in section 3. Section 4 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. The paper is ended by an example illustrating our results.

2. Notations and Notions Preliminaries

In the present section, we present some notations, definitions and auxiliary lemmas concerning fractional calculus and fixed point theorems. Let $J = [0,1]$, X is Banach space equipped with the norm $\|\cdot\|$ and $C(J,X)$, $C^n(J,X)$ denotes respectively the Banach spaces of all continuous bounded functions and n times continuously differentiable functions on J. In addition, we define the norm $\|g\| = \max\{|g(t)| : t \in J\}$ for any continuous function $g : J \to X$.

Definition 2.1 ([1,6])**.** Let $\alpha > 0$ and $g : J \to X$. The left sided Riemann-Liouville fractional integral of order α of a function g is defined by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in J.$$

Definition 2.2 ([1,7])**.** Let $n-1 < \alpha < n$, $n \in \mathbb{N}^{\star}$, and $g \in C^n(J,X)$. The left sided Caputo fractional derivative of order α of a function g is given by

$${}^C D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds = I_{0^+}^{n-\alpha} \frac{d^n}{dt^n} g(t), \quad t \in J,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 ([1,7])**.** *For real numbers $\alpha, \beta > 0$ and appropriate function g, we have for all $t \in J$:*
1) $I_{0^+}^{\alpha} I_{0^+}^{\beta} g(t) = I_{0^+}^{\beta} I_{0^+}^{\alpha} g(t) = I_{0^+}^{\alpha+\beta} g(t)$;
2) $I_{0^+}^{\alpha} {}^C D_{0^+}^{\alpha} g(t) = g(t) - g(0)$, $0 < \alpha < 1$;
3) ${}^C D_{0^+}^{\alpha} I_{0^+}^{\alpha} g(t) = g(t)$.

Lemma 2.2 (Banach fixed point theorem, [12])**.** *Let U be a non-empty complete metric space and $T : U \to U$ is contraction mapping. Then, there exists a unique point $u \in U$ such that $T(u) = u$.*

Lemma 2.3 (Krasnoselskii fixed point theorem, [12])**.** *Let E be bounded, closed and convex subset in a Banach space X. If $T_1, T_2 : E \to E$ are two applications satisfying the following conditions:*
1) *$T_1 x + T_2 y \in E$ for every $x, y \in E$;*
2) *T_1 is a contraction;*
3) *T_2 is compact and continuous.*

Then there exists $z \in E$ such that $T_1 z + T_2 z = z$.

Before presenting our main results, we need the following auxiliary lemma.

Lemma 2.4. *Let $0 < \alpha + \beta < 1$ and $b \neq \frac{1}{\eta}$. Assume that h, f and K are three continuous functions. If $u \in C(J,X)$, then u is solution of (1.1) if and only if u*

satisfies the integral equation

$$
\begin{aligned}
u(t) =&\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds \\
&+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\bigg] d\tau.
\end{aligned}
\tag{2.1}
$$

Proof. Let $u \in C(J,X)$ be a solution of (1.1). Firstly, we show that u is solution of integral equation (2.1). By Lemma 2.1, we obtain

$$
I_{0^+}^{\alpha+\beta C} D_{0^+}^{\alpha+\beta} u(t) = u(t) - u(0). \tag{2.2}
$$

In addition, from equation in (1.1) and Definition 2.1, and use the assumption 1) of Lemma 2.1 we have

$$
\begin{aligned}
I_{0^+}^{\alpha+\beta C} D_{0^+}^{\alpha+\beta} u(t) =& I_{0^+}^{\alpha+\beta}\bigg(h(t,u(t)) + \int_0^t K(t,s,u(s))ds + I_{0^+}^{\alpha} f(t,u(t))\bigg) ds \\
=& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds.
\end{aligned}
\tag{2.3}
$$

By substituting (2.3) in (2.2) with nonlocal condition in problem (2.1), we get the following integral equation:

$$
\begin{aligned}
u(t) =& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\bigg] ds + u(0).
\end{aligned}
\tag{2.4}
$$

From integral boundary condition of our problem with using Fubini's thorem and after some computations, we get:

$$
\begin{aligned}
u(0) =& b\int_0^\eta u(s)ds \\
=& b\int_0^\eta \bigg[\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg(h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\bigg) d\tau\bigg] ds + b\eta u(0) \\
=& b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(\tau,u(\tau))d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&+ b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau ds \\
&+ b\int_0^\eta \int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma d\tau ds + b\eta u(0) \\
=& b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds h(\tau,u(\tau))d\tau \\
&+ b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau \\
&+ b\int_0^\eta \int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma d\tau + b\eta u(0),
\end{aligned}
$$

that is

$$
\begin{aligned}
u(0) =& \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Bigg]d\tau. \qquad (2.5)
\end{aligned}
$$

Finally, by substituting (2.5) in (2.4) we find (2.1).

Conversely, from Lemma 2.1 and by applying the operator ${}^{C}D_{0^+}^{\alpha+\beta}$ on both sides of (2.1), we find

$$
\begin{aligned}
{}^{C}D_{0^+}^{\alpha+\beta}u(t) =& {}^{C}D_{0^+}^{\alpha+\beta} I_{0^+}^{\alpha+\beta}\Bigg[h(t,u(t)) + \int_0^t K(t,s,u(s))ds + I_{0^+}^{\alpha} f(t,u(t))\Bigg] + {}^{C}D_{0^+}^{\alpha+\beta}u(0). \\
=& h(t,u(t)) + I_{0^+}^{\alpha} f(t,u(t)) + \int_0^t K(t,s,u(s))ds, \qquad (2.6)
\end{aligned}
$$

this means that u satisfies the equation in problem (1.1). Furthermore, by substituting t by 0 in integral equation (2.1), we have clearly that the integral boundary condition in (1.1) holds. Therefore, u is solution of problem (1.1), which completes the proof. □

3. Existence Results

In order to prove the existence and uniqueness of solution for the problem (1.1) in $C([0,1],X)$, we use two fixed point theorem.

Firstly, we transform the system (1.1) into fixed point problem as $u = Tu$, where $T : C(J,X) \to C(J,X)$ is an operator defined by following

$$
\begin{aligned}
Tu(t) =& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\Bigg]ds \\
&+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma
\end{aligned}
$$

$$+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\bigg]d\tau. \tag{3.1}$$

In order to simplify the computations, we offer the following notations

$$\begin{aligned}\Delta =&\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\\ &+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+2)}\\ &+\frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\end{aligned} \tag{3.2}$$

and

$$\Delta_1=\frac{|b|}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}+\frac{\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]. \tag{3.3}$$

3.1. **Existence result by Krasnoselskii's fixed point.**

Theorem 3.1. *Let $h,f:[0,1]\times X\to X$ and $K:[0,1]\times[0,1]\times X\longrightarrow X$ be continuous functions satisfying the following.*

$(H1)$ *The inequalities*

$$\begin{aligned}\|h(t,u(t))-h(t,v(t))\|\le&L_1\|u(t)-v(t)\|,\quad t\in[0,1],\ u,v\in X,\\ \|f(t,u(t))-f(t,v(t))\|\le&L_2\|u(t)-v(t)\|,\quad t\in[0,1],\ u,v\in X,\\ \|K(t,s,u(s))-K(t,s,v(s))\|\le&L_3\|u(s)-v(s)\|,\quad (t,s)\in G,\ u,v\in X,\end{aligned}$$

hold, where $L_1,L_2,L_3\ge 0$, with $L=\max\{L_1,L_2,L_3\}$ and $G=\{(t,s):0\le s\le t\le 1\}$.

$(H2)$ *There exist three functions $\mu_1,\mu_2,\mu_3\in L^\infty([0,1],\mathbb{R}^+)$ such that*

$$\begin{aligned}\|h(t,u(t))\|\le&\mu_1(t)\|u(t)\|,\quad t\in[0,1],\ u\in X,\\ \|f(t,u(t))\|\le&\mu_2(t)\|u(t)\|,\quad t\in[0,1],\ u\in X,\\ \|K(t,s,u(s))\|\le&\mu_3(t)\|u(s)\|,\quad (t,s)\in G,\ u\in X.\end{aligned}$$

If $\Delta\le 1$ and $L\Delta_1\le 1$, then the problem (1.1) has at least one solution on $[0,1]$.

Proof. For any function $u\in C(J,X)$ we define the norm

$$\|u\|_1=\max\{e^{-t}\|u(t)\|:t\in[0,1]\},$$

and consider the closed ball

$$B_r=\{u\in C(J,X):\|u\|_1\le r\}.$$

Next, let us define the operators T_1,T_2 on B_r as follows

$$T_1u(t)=\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[h(s,u(s))+\int_0^s K(s,\tau,u(\tau))d\tau$$

$$+\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\Big]ds \tag{3.4}$$

and

$$\begin{aligned} T_2u(t) =& \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Big]d\tau. \end{aligned} \tag{3.5}$$

For $u, v \in B_r$, $t \in [0,1]$ and by the assumption $(H2)$, we find

$$\begin{aligned} \|T_1u(t)+T_2v(t)\| \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|h(s,u(s))\| + \int_0^s \|K(s,\tau,u(\tau))\|d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau))\|d\tau\Big]ds \\ &+ \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\|h(\tau,v(\tau))\| + \int_0^\tau \|K(\tau,\sigma,v(\sigma))\|d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,v(\sigma))\|d\sigma\Big]d\tau \\ \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\mu_1(s)\|u(s)\| + \int_0^s \mu_3(s)\|u(\tau)\|d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mu_2(\tau)\|u(\tau)\|d\tau\Big]ds \\ &+ \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\mu_1(\tau)\|v(\tau)\| + \int_0^\tau \mu_3(\tau)\|v(\sigma)\|d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\mu_2(\sigma)\|v(\sigma)\|d\sigma\Big]d\tau \\ \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1 e^s + \|\mu_3\|_{L^\infty}\|u\|_1(e^s-1) \\ &+ \|\mu_2\|_{L^\infty}\|u\|_1\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}e^\tau d\tau\Big]ds \\ &+ \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\|\mu_1\|_{L^\infty}\|v\|_1 e^\tau + \|\mu_3\|_{L^\infty}\|v\|_1(e^\tau-1) \\ &+ \|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}e^\sigma d\sigma\Big]d\tau. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\|T_1u+T_2v\|_1 \leq &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\\
&+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t}d\tau\Bigg]ds\\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|\mu_1\|_{L^\infty}\|v\|_1\frac{e^\tau}{e^t}+\|\mu_3\|_{L^\infty}\|v\|_1\frac{(e^\tau-1)}{e^t}\\
&+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau\frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\sigma}{e^t}d\sigma\Bigg]d\tau.\\
\leq & r\Bigg[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta+1}s^\alpha ds\\
&+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+1)}\\
&+\frac{|b|\|\mu_2\|_{L^\infty}}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\int_0^\eta(\eta-\tau)^{\alpha+\beta}\tau^\alpha d\tau\Bigg]\\
= & r\Bigg[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\\
&+\frac{|b|}{|1-b\eta|}\Bigg(\frac{\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\\
&+\frac{\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\\
= & r\Delta\leq r. \qquad (3.6)
\end{aligned}
$$

This implies that $(T_1u+T_2v)\in B_r$. Here we used the computations

$$
\begin{aligned}
\int_0^1(1-s)^{\alpha+\beta}s^\alpha ds =&\beta(\alpha+1,\alpha+\beta),\\
\int_0^\eta(\eta-\tau)^{\alpha+\beta}\tau^\alpha ds =&\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1),
\end{aligned}
$$

and the estimations: $\frac{e^s}{e^t}\leq 1$, $\frac{e^\tau}{e^t}\leq 1$, $\frac{e^\sigma}{e^t}\leq 1$. Now, we establish that T_2 is a contraction mapping. For $u,v\in X$ and $t\in[0,1]$, we have

$$
\begin{aligned}
\|T_2u(t)-T_2v(t)\| \leq&\frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[\|h(\tau,u(\tau))-h(\tau,v(\tau))\|\\
&+\int_0^\tau\|K(\tau,\sigma,u(\sigma))-K(\tau,\sigma,v(\sigma))\|d\sigma\\
&+\int_0^\tau\frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\|f(\sigma,u(\sigma))-f(\sigma,v(\sigma))\|d\sigma\Bigg]d\tau
\end{aligned}
$$

$$\begin{aligned}
\leq&\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\bigg[L_1\|u-v\|_1 e^\tau+\int_0^\tau L_3\|u-v\|_1 e^\sigma d\sigma\\
&+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L_2\|u-v\|_1 e^\sigma d\sigma\bigg]d\tau\\
\leq&\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\bigg[L\|u-v\|_1 e^\tau+L\|u-v\|_1(e^\tau-1)\\
&+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L\|u-v\|_1 e^\sigma d\sigma\bigg]d\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|T_2u-T_2v\|_1\leq&\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\bigg[L\|u-v\|_1\frac{e^\tau}{e^t}+L\|u-v\|_1\frac{(e^\tau-1)}{e^t}\\
&+\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L\|u-v\|_1\frac{e^\sigma}{e^t}d\sigma\bigg]d\tau\\
\leq&\frac{|b|L}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}+\frac{\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]\|u-v\|_1.
\end{aligned}$$

Then since $L\Delta_1\leq 1$, T_2 is a contraction mapping. The continuity of the functions h, f and K implies that the operator T_1 is continuous. Also, $T_1B_r\subset B_r$, for each $u\in B_r$,, i.e., T_1 is uniformly bounded on B_r as

$$\begin{aligned}
\|(T_1u)(t)\|\leq&\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[\|h(s,u(s))\|+\int_0^s\|K(s,\tau,u(\tau))\|d\tau\\
&+\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|f(\tau,u(\tau))\|d\tau\bigg]ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|T_1u\|_1\leq&\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\bigg[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\\
&+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t}d\tau\bigg]ds\\
\leq&r\left[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\right]\\
\leq&r\Delta.\\
\leq&r. \qquad (3.7)
\end{aligned}$$

Finally, we will show that $(\overline{T_1}B_r)$ is equi-continuous. For this end, we define

$$\begin{aligned}
\overline{h}=&\sup_{(s,u)\in[0,1]\times B_r}\|h(s,u)\|,\\
\overline{f}=&\sup_{(s,u)\in[0,1]\times B_r}\|f(s,u)\|,
\end{aligned}$$

$$\overline{K}=\sup_{(s,\tau,u)\in G\times B_r}\int_0^s\|K(t,s,u)\|d\tau.$$

Let for any $u\in B_r$ and for each $t_1,t_2\in[0,1]$ with $t_1\le t_2$, we have:

$$\begin{aligned}
&\|(T_1u)(t_2)-(T_1u)(t_1)\|\\
\le&\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\Bigg[\|h(s,u(s))\|+\int_0^s\|K(s,\tau,u(\tau))\|d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\|f(\tau,u(\tau))\|d\tau\Bigg]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\right]\Bigg[\|h(s,u(s))\|\\
&+\int_0^s\|K(s,\tau,u(\tau))\|d\tau+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\|f(\tau,u(\tau))\|d\tau\Bigg]ds\\
\le&\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\left[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}d\tau\right]ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\right]\left[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}d\tau\right]ds\\
\le&\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\left[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\right]ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\right]\left[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\right]ds\\
=&\frac{1}{\Gamma(\alpha+\beta+1)}\left[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\right]\left[2(t_2-t_1)^{\alpha+\beta}+t_1^{\alpha+\beta}-t_2^{\alpha+\beta}\right].
\end{aligned}$$

The RHS of the last inequality is independent of u and tends to zero when $|t_2-t_1|\to0$, this means that $|T_1u(t_2)-T_1u(t_1)|\to0$, which implies that $(\overline{T_1B_r})$ is equi-continuous, then T_1 is relatively compact on B_r. Hence by Arzela-Ascoli theorem, T_1 is compact on B_r. Now, all hypothesis of Theorem 3.2 hold, therefore the operator T_1+T_2 has a fixed point on B_r. So the problem (1.1) has at least one solution on $[0,1]$. This proves the theorem. □

3.2. **Existence and uniqueness result.**

Theorem 3.2. *Assume that* $(H1)$ *holds. If* $L\Delta<1$, *then the BVP* (1.1) *has a unique solution on* $[0,1]$.

Proof. Define $M=\max\{M_1,M_2,M_3\}$, where M_1,M_2,M_3 are positive numbers such that:

$$M_1=\sup_{t\in[0,1]}\|h(t,0)\|,\quad M_2=\sup_{t\in[0,1]}\|f(t,0)\|,\quad M_3=\sup_{(t,s)\in G}\|K(t,s,0)\|.$$

We fix $r \geq \frac{M\Delta}{1-L\Delta}$ and we consider

$$D_r = \{x \in C([0,1], X) : \|u\| \leq r\}.$$

Then, in view of the assumption $(H1)$, we have

$$\begin{aligned}\|h(t,u(t))\| =& \|h(t,u(t)) - h(t,0) + h(t,0)\| \leq \|h(t,u(t)) - h(t,0)\| + \|h(t,0)\| \\ \leq& L_1\|u\| + M_1, \\ \|f(t,u(t))\| \leq& L_2\|u\| + M_2,\end{aligned}$$

and

$$\|K(t,s,u(s))\| \leq L_3\|u\| + M_3.$$

First step. We show that $TD_r \subset D_r$. For each $t \in [0,1]$ and for any $u \in D_r$

$$\begin{aligned}\|(Tu)(t)\| \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \bigg[\|h(s,u(s))\| + \int_0^s \|K(s,\tau,u(\tau))\| d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|f(\tau,u(\tau))\| d\tau\bigg] ds \\ &+ \frac{|b|}{|1-b\eta|} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \bigg[\|h(\tau,v(\tau))\| + \int_0^\tau \|K(\tau,\sigma,v(\sigma))\| d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \|f(\sigma,v(\sigma))\| d\sigma\bigg] d\tau \\ \leq& (Lr+M)\Delta \\ \leq& r.\end{aligned}$$

Hence, $TD_r \subset D_r$.

Second step. We shall show that $T : D_r \to D_r$ is a contraction. From the assumption $(H1)$ we have for any $u, v \in D_r$ and for each $t \in [0,1]$

$$\begin{aligned}&\|(Tu)(t) - (Tv)(t)\| \\ \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \bigg[\|h(s,u(s)) - h(s,v(s))\| \qquad (3.8) \\ &+ \int_0^s \|K(s,\tau,u(\tau)) - K(s,\tau,v(\tau))\| d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|f(\tau,u(\tau)) - f(\tau,v(\tau))\| d\tau\bigg] ds \\ &+ \frac{|b|}{|1-b\eta|} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \bigg[\|h(\tau,u(\tau)) - h(\tau,v(\tau))\| \qquad (3.9) \\ &+ \int_0^\tau \|K(\tau,\sigma,u(\sigma)) - K(\tau,\sigma,v(\sigma))\| d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \|f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))\| d\sigma\bigg] d\tau\end{aligned}$$

$$\leq L\Delta\|u-v\|.$$

Since $L\Delta < 1$, it follows that T is a contraction. All assumptions of Lemma 2.2 are satisfied, then there exists $u \in C(J, X)$ such that $Tu = u$, which is the unique solution of the problem (1.1) in $C(J, X)$. □

4. Generalized Ulam Stabilities

The aim is to discus the Ulam stability for (1.1), by using the integration

$$\begin{aligned}
v(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[h(s,v(s)) + \int_0^s K(s,\tau,v(\tau))d\tau \\
&+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,v(\tau))d\tau\Bigg]ds \\
&+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[h(\tau,v(\tau)) + \int_0^\tau K(\tau,\sigma,v(\sigma))d\sigma \\
&+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,v(\sigma))d\sigma\Bigg]d\tau.
\end{aligned} \tag{4.1}$$

Here $v \in C([0,1], X)$ possess a fractional derivative of order $\alpha+\beta$, where $0 < \alpha+\beta < 1$ and

$$f, h : [0,1] \times X \to X,$$

and

$$K : [0,1] \times [0,1] \times X \to X,$$

are continuous functions. Then we define the nonlinear continuous operator

$$P : C([0,1], X) \to C([0,1], X),$$

as follows

$$Pv(t) = {}^C D^{\alpha+\beta} v(t) - I_{0+}^{\alpha} f(t,v(t)) - h(t,v(t)) - \int_0^t K(t,s,v(s))ds.$$

Definition 4.1. For each $\epsilon > 0$ and for each solution v of (1.1), such that

$$\|Pv\| \leq \epsilon, \tag{4.2}$$

the problem (1.1), is said to be Ulam-Hyers stable if we can find a positive real number ν and a solution $u \in C([0,1], X)$ of (1.1), satisfying the inequality

$$\|u - v\| \leq \nu\epsilon^*, \tag{4.3}$$

where ϵ^* is a positive real number depending on ϵ.

Definition 4.2. Let $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each solution v of (1.1), we can find a solution $u \in C([0,1], X)$ of (1.1) such that

$$\|u(t) - v(t)\| \leq m(\epsilon), \quad t \in [0,1]. \tag{4.4}$$

Then the problem (1.1), is said to be generalized Ulam-Hyers stable.

Definition 4.3. For each $\epsilon > 0$ and for each solution v of (1.1), the problem (1.1) is called Ulam-Hyers-Rassias stable with respect to $\theta \in C([0,1], \mathbb{R}^+)$ if

$$\|Pv(t)\| \leq \epsilon\theta(t), \quad t \in [0,1], \tag{4.5}$$

and there exist a real number $\nu > 0$ and a solution $v \in C([0,1], X)$ of (1.1) such that

$$\|u(t) - v(t\| \leq \nu\epsilon_*\theta(t), \quad t \in [0,1], \tag{4.6}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Under assumption* $(H1)$ *in Theorem* 3.1, *with* $L\Delta < 1$. *The problem* (1.1) *is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C([0,1], X)$ be a solution of (1.1), satisfying (2.1) in the sense of Theorem 3.2. Let v be any solution satisfying (4.2). Lemma 2.4 implies the equivalence between the operators P and $T - Id$ (where Id is the identity operator) for every solution $v \in C([0,1], X)$ of (1.1) satisfying $L\Delta < 1$. Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned}\|v(t) - u(t\| &= \|v(t) - Tv(t) + Tv(t) - u(t)\| = \|v(t) - Tv(t) + Tv(t) - Tu(t)\| \\ &\leq \|Tv(t) - Tu(t)\| + |Tv(t) - v(t)\| \leq L\Delta\|u - v\| + \epsilon,\end{aligned}$$

because $L\Delta < 1$ and $\epsilon > 0$, we find

$$\|u - v\| \leq \frac{\epsilon}{1 - L\Delta}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-L\Delta}$ and $\nu = 1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m(\epsilon) = \frac{\epsilon}{1-L\Delta}$. □

Theorem 4.2. *Assume that* $(H1)$ *holds with* $L < \Delta^{-1}$, *and there exists a function* $\theta \in C([0,1], \mathbb{R}^+)$ *satisfying the condition* (4.5). *Then the problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* θ.

Proof. We have from the proof of Theorem 4.1,

$$\|u(t) - v(t\| \leq \epsilon_*\theta(t), \quad t \in [0,1],$$

where $\epsilon_* = \frac{\epsilon}{1-L\Delta}$. This completes the proof. □

Example 4.1. Consider the following fractional integro-differential problem

$$\begin{cases} {}^C D_{0^+}^{\frac{2}{5}} u(t) = h(t, u(t)) + I_{0^+}^{\alpha} f(t, u(t)) + \int_0^t K(t, s, u(s))ds, & t \in [0,1], \\ u(0) = 3\int_0^{\frac{1}{5}} u(s)ds, & 0 < \eta < 1, \end{cases} \tag{4.7}$$

where $\alpha = \beta = \frac{1}{5}$, $b = 3$, $\eta = \frac{1}{5}$. By the above, we find that $\Delta = 0.4602$, $\Delta_1 = 4.3755$. To illustrate our results in Theorem 3.1 and Theorem 4.1, we take for $u, v \in X = \mathbb{R}^+$ and $t \in [0,1]$ the following continuous functions:

$$h(t, u(t)) = \frac{(2-t)u(t)}{60}, \quad f(t, u(t)) = \frac{3-t^2}{72}u(t), \quad K(t, s, u(s)) = \frac{e^{-(s+t)}}{64}u(s).$$

Note that we can find

$$L_1 = \frac{1}{20}, \quad L_2 = \frac{1}{18}, \quad L_3 = \frac{1}{64},$$

Moreover,

$$\mu_1(t) = \frac{2-t}{60}, \quad \mu_2(t) = \frac{3-t^2}{72}, \quad \mu_3(t) = \frac{e^{-t}}{64}.$$

Obviously,

$$\|\mu_1\|_{L_\infty} = \frac{1}{30}, \quad \|\mu_2\|_{L_\infty} = \frac{1}{24}, \quad \|\mu_3\|_{L_\infty} = \frac{1}{64},$$

and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{18}.$$

Then, we get

$$L\Delta_1 = 0.2431 < 1, \quad \Delta = 0.3229 < 1.$$

All assumptions of Theorem 3.1 are satisfied. Hence, there exists at least one solution for the problem (4.7) on $[0,1]$.

By take the same functions, we result the assumption

$$L\Delta = 0.0179 < 1,$$

then there exists a unique solution of (4.7) on $[0,1]$.

In order to illustrate our stability result, we consider the same above example:

$$L = \frac{1}{18}, \quad L\Delta_1 = 0.2431$$

This implies that the system (4.7) is Ulam-Hyers stable, then it is generalized Ulam-Hyers stable. It is Ulam-Hyers-Rassias stable if there exists a continuous and positive function.

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