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LOWER BOUNDS FOR ENERGY OF MATRICES AND ENERGY OF REGULAR GRAPHS

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Abstract. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The energy of A, denoted by $\mathcal{E}(A)$, is defined as $|\lambda_1| + \cdots + |\lambda_n|$. We prove that if A is non-zero and $|\lambda_1| \geq \cdots \geq |\lambda_n|$, then

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1\leq i,j\leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}. \tag{0.1}$$

In particular, we show that $\Psi(A)\mathcal{E}(A) \geq \sum_{1\leq i,j\leq n} a_{ij}^2$, where $\Psi(A)$ is the maximum value of the sequence $\sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \ldots, \sum_{j=1}^n |a_{nj}|$. The energy of a simple graph G, denoted by $\mathcal{E}(G)$, is defined as the energy of its adjacency matrix. As an application of inequality (0.1) we show that if G is a t-regular graph ($t \neq 0$) of order n with no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \geq \frac{2tn}{t+1}$ and the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K_{t+1}^\star$.

1. Introduction

Throughout this paper the matrices are complex and the graphs are simple (that is graphs are finite and undirected, without loops and multiple edges). The *conjugate transpose* of a complex matrix A is denoted by A^*. We recall that a *Hermitian* matrix (or *self-adjoint* matrix) is a complex square matrix that is equal to its own *conjugate transpose*. If A is a real matrix, then A is Hermitian if and only if A is symmetric. It is well known that the eigenvalues of Hermitian matrices (in particular, the eigenvalues of real symmetric matrices) are real. A complex square matrix A is called *normal* if it commutes with its conjugate transpose, that is $AA^* = A^*A$. For example, every

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real symmetric matrix is normal. For every complex square matrix A, the *trace* of A, denoted by $tr(A)$, is defined to be the sum of the entries on the main diagonal of A. The *energy* of a square complex matrix A, denoted by $\mathcal{E}(A)$, is defined as the sum of the absolute values of its eigenvalues. In other words, if A is an $n \times n$ complex matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$\mathcal{E}(A) = |\lambda_1| + \cdots + |\lambda_n|. \tag{1.1}$$

Nikiforov [9] defined the energy of any complex matrix A by considering the *singular values*. This definition of energy of matrices coincides with the previous definition of energy of matrices if and only if the matrix is normal [1].

Let $G = (V(G), E(G))$ be a simple graph. The *order* of G denotes the number of vertices of G. For two vertices u and v by $e = uv$ we mean the edge e between u and v. For a vertex v of G, the *degree* of v is the number of edges incident with v. A k–regular graph is a graph such that every vertex of that has degree k. Let $B \subseteq V(G)$ $(B \subseteq E(G))$. By $G \setminus B$ we mean the graph that obtained from G by removing the vertices of B (the edges of B). The *complement* of G, denoted by $\overline{G}$, is the simple graph with vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. For two disjoint graphs G_1 and G_2, the *disjoint union* of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. A *matching* in G is a set of edges of G without common vertices. A *perfect matching* of G is a matching in which every vertex of G is incident to exactly one edge of the matching. The *edgeless graph* (*empty graph*), the *complete graph* and the *cycle* of order n, are denoted by $\overline{K_n}$, K_n and C_n, respectively. The *complete bipartite graph* with part sizes m and n is denoted by $K_{m,n}$. Let $t \geq 0$ be an integer and M be a perfect matching of $K_{t+1,t+1}$. By $K^\star_{t+1}$ we mean the t–regular graph $K_{t+1,t+1} \setminus M$. The graph $K^\star_{t+1}$ is called the *crown* graph of order $2t + 2$. For example $K^\star_1 = 2K_1$, $K^\star_2 = 2K_2$ and $K^\star_3 = C_6$.

Let G be a simple graph with vertex set $\{v_1, \dots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix such that the (i, j)-entry is 1 if v_i and v_j are adjacent, and otherwise is 0. Since $A(G)$ is symmetric, all of the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. By *Spec*(G) we mean the multiset of all eigenvalues of G. The *energy* of G, denoted by $\mathcal{E}(G)$, is defined as the energy of the adjacency matrix of G. In other words, the energy of G is the sum of the absolute values of all eigenvalues of G. More precisely, $\mathcal{E}(G) = |\lambda_1| + \cdots + |\lambda_n|$, where $\mathrm{Spec}(G) = \{\lambda_1, \dots, \lambda_n\}$. The energy of graphs was defined by Ivan Gutman in 1978. For example, since the eigenvalues of the complete graph K_n are $n - 1$ (with multiplicity 1) and -1 (with multiplicity $n - 1$), so $\mathcal{E}(K_n) = 2n - 2$. See [4, 5] for more details. Many papers are devoted to studying the properties of the spectra of adjacency matrix, in particular studying the energy of graphs. For instance see [1–20] and the references therein. There are many other matrices associated to graphs such as *Laplacian matrix*, *signless Laplacian matrix* [20] and *distance matrix* [18]. For instance the Laplacian matrix of a graph G, denoted by $L(G)$, is defined as $D(G) - A(G)$, where $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of G.

We note that the definition (1.1) for energy of matrices is significant for square real symmetric matrices whose trace are equal to zero. In other words, this definition maybe not notable for matrices with non-zero trace. For instance consider the Laplacian matrix of a graph G of order n. It is well known that the eigenvalues of Laplacian matrix of graphs are real and non-negative. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of the Laplacian matrix of G (in fact $\mu_n = 0$). By definition (1.1), the energy of $L(G)$ is $\mathcal{E}(L(G)) = \mu_1 + \cdots + \mu_n = tr(L(G)) = 2m$, where m is the number of edges of G. We remark that in [7] the authors define the Laplacian energy of graphs in another way.

In this paper first we obtain a new lower bound for energy of real symmetric matrices. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. If $A = 0$, then clearly $\mathcal{E}(A) = 0$. We show that if $A \neq 0$, then

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1 \leq i,j \leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}, \tag{1.2}$$

where $|\lambda_1| \geq \cdots \geq |\lambda_n|$. By studying the lower bound (1.2) we obtain a simple lower bound for energy of matrices. Let $\Psi(A)$ be the maximum value of the sequence of real numbers $\sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \ldots, \sum_{j=1}^{n} |a_{nj}|$. In other words, $\Psi(A)$ is the maximum value of the sum of the absolute values of the entries of rows of A. We prove that if $A \neq 0$, then

$$\mathcal{E}(A) \geq \frac{\sum_{1 \leq i,j \leq n} a_{ij}^2}{\Psi(A)}.$$

Finally we study the energy of regular graphs. Let G be a t–regular graph of order n and $t \neq 0$. In [6] (see also [4]) it was shown that $\mathcal{E}(G) \geq n$. By applying the lower bound (1.2) we improve this result and prove that if G has no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \geq \frac{2tn}{t+1}$. In addition we show that the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K_{t+1}^{\star}$.

2. Energy of Matrices

In this section we obtain some lower bounds for the energy of matrices. At first similar to Lemma 1 of [19] we prove the inequality (1.2).

Theorem 2.1. *Let $n \geq 2$ be an integer and $A = [a_{ij}] \neq 0$ be an $n \times n$ real symmetric matrix. Assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then*

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1 \leq i,j \leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|}.$$

Moreover, the equality holds if and only if for some $r \in \{1, \ldots, n\}$, $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$.

Proof. We note that for every $j \in \{1, \ldots, n\}$, $|\lambda_1| \geq |\lambda_j| \geq |\lambda_n|$. Thus, for $j = 1, \ldots, n$, $(|\lambda_1| - |\lambda_j|)(|\lambda_j| - |\lambda_n|) \geq 0$. In addition, the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. On the other hand

$$|\lambda_j|(|\lambda_1| + |\lambda_n|) - (|\lambda_j|^2 + |\lambda_1||\lambda_n|) = (|\lambda_1| - |\lambda_j|)(|\lambda_j| - |\lambda_n|).$$

Hence, $|\lambda_j|(|\lambda_1| + |\lambda_n|) - (|\lambda_j|^2 + |\lambda_1||\lambda_n|) \geq 0$ and the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. So, for every $j \in \{1, \ldots, n\}$

$$|\lambda_j|(|\lambda_1| + |\lambda_n|) \geq |\lambda_j|^2 + |\lambda_1||\lambda_n|, \tag{2.1}$$

and the equality holds if and only if $|\lambda_j| = |\lambda_1|$ or $|\lambda_j| = |\lambda_n|$. Now by summing the sides of (2.1) for $j = 1, \ldots, n$, we find that

$$(|\lambda_1| + |\lambda_n|)\Big(|\lambda_1| + \cdots + |\lambda_n|\Big) \geq |\lambda_1|^2 + \cdots + |\lambda_n|^2 + n|\lambda_1||\lambda_n|, \tag{2.2}$$

and the equality holds if and only if for some $r \in \{1, \ldots, n\}$, $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$. On the other hand $\mathcal{E}(A) = |\lambda_1| + \cdots + |\lambda_n|$ and

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 = \lambda_1^2 + \cdots + \lambda_n^2 = tr(A^2) = \sum_{1 \leq i,j \leq n} a_{ij}^2. \tag{2.3}$$

We note that because of A is symmetric, we have $tr(A^2) = \sum_{1 \leq i,j \leq n} a_{ij}^2$. Since $A \neq 0$, clearly $\sum_{1 \leq i,j \leq n} a_{ij}^2 \neq 0$. Thus, by (2.3), A has at least one non-zero eigenvalue. Thus, $|\lambda_1| > 0$. So, $|\lambda_1| + |\lambda_n| \neq 0$. Now by dividing the sides of (2.2) by $|\lambda_1| + |\lambda_n|$ and using (2.3) the result follows. $\square$

Remark 2.1. We note that in Theorem 2.1 the equality holds for some family of matrices. For example for diagonal matrices such as $\mathrm{diag}(a, \ldots, a, b, \ldots, b)$, where a and b are real. Since the eigenvalues of the complete bipartite graph $K_{p,q}$ are $-\sqrt{pq}$ (with multiplicity 1), 0 (with multiplicity $p + q - 2$) and $\sqrt{pq}$ (with multiplicity 1), the adjacency matrix of $K_{p,q}$ also satisfying in the equality of Theorem 2.1.

We are interested in to obtain a suitable estimation for the lower bound of Theorem 2.1 in terms of the entries of the matrix. First we prove the following lemma.

Lemma 2.1. *Let a and b be some positive real numbers. Let α, β, x and y be some non-negative real numbers such that $\beta \geq y \geq \sqrt{\frac{a}{b}} \geq x \geq \alpha$. Then*

$$\frac{a + bxy}{x + y} \geq \frac{a + b\alpha\beta}{\alpha + \beta},$$

and the equality holds if and only if $x = \alpha = \sqrt{\frac{a}{b}}$ or $x = \beta = \sqrt{\frac{a}{b}}$ or $y = \beta = \sqrt{\frac{a}{b}}$ or $x = \alpha$ and $y = \beta$.

Proof. Let d be a positive real number and $f_d(t) = \frac{a+bdt}{d+t}$ be the one-variable function on t, where $t \geq 0$. So the derivative of $f_d(t)$ with respect to t is

$$f_d'(t) = \frac{bd^2 - a}{(d + t)^2}.$$

This shows that if $d > \sqrt{\frac{a}{b}}$, then $f_d(t)$ is strictly increasing on the interval $[0, \infty)$ and if $d < \sqrt{\frac{a}{b}}$, then $f_d(t)$ is strictly decreasing on the interval $[0, \infty)$. We note that if $d = \sqrt{\frac{a}{b}}$, then for every $t \geq 0$, $f_d(t) = \frac{a}{d} = \sqrt{ab}$.

Since $y \leq \beta$ and $f_x(t)$ is strictly decreasing on the interval $[0, \infty)$, if $x < \sqrt{\frac{a}{b}}$,

$$f_x(y) \geq f_x(\beta) \quad (\text{if} \beta > y \text{ and } x \neq \sqrt{\frac{a}{b}}, \text{ then } f_x(y) > f_x(\beta)). \tag{2.4}$$

On the other hand, since $x \geq \alpha$ and $f_\beta(t)$ is strictly increasing on the interval $[0, \infty)$, if $\beta > \sqrt{\frac{a}{b}}$,

$$f_\beta(x) \geq f_\beta(\alpha) \quad (\text{if } x > \alpha \text{ and } \beta \neq \sqrt{\frac{a}{b}}, \text{ then } f_\beta(x) > f_\beta(\alpha)). \tag{2.5}$$

Since $f_x(\beta) = f_\beta(x)$, (2.4) and (2.5) show that $f_x(y) \geq f_\beta(\alpha)$. In other words, we obtain that $\frac{a+bxy}{x+y} \geq \frac{a+b\alpha\beta}{\alpha+\beta}$.

Now we consider the equality. Assume that $\frac{a+bxy}{x+y} = \frac{a+b\alpha\beta}{\alpha+\beta}$. So, $f_x(y) = f_\beta(\alpha)$. Hence, by (2.4) and (2.5) we find that $f_x(y) = f_x(\beta)$ and $f_\beta(x) = f_\beta(\alpha)$. Using (2.4) and (2.5) one can easily obtain the result. $\square$

Let $A = [a_{ij}]$ be a complex $n \times n$ matrix, where $n \geq 1$ be an integer. As we mentioned before, $\Psi(A)$ denotes the maximum value of the sequence of real numbers $\sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \ldots, \sum_{j=1}^{n} |a_{nj}|$. We need the following result.

Theorem 2.2 ([8]). *Let A be a complex square matrix and λ be an eigenvalue of A. Then $|\lambda| \leq \Psi(A)$.*

Now we obtain a lower bound for the energy of matrices in terms of their entries.

Theorem 2.3. *Let $n \geq 2$ be an integer and $A = [a_{ij}] \neq 0$ be an $n \times n$ real symmetric matrix. Then*

$$\mathcal{E}(A) \geq \frac{\sum_{1 \leq i,j \leq n} a_{ij}^2}{\Psi(A)}.$$

Proof. Assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. We note that the eigenvalues of A are real. Since A is symmetric, $tr(A^2) = \sum_{1 \leq i,j \leq n} a_{ij}^2$. On the other hand $\lambda_1^2 + \cdots + \lambda_n^2 = tr(A^2)$ and

$$n|\lambda_n|^2 \leq |\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq n|\lambda_1|^2. \tag{2.6}$$

Therefore,

$$n|\lambda_n|^2 \leq \sum_{1 \leq i,j \leq n} a_{ij}^2 \leq n|\lambda_1|^2. \tag{2.7}$$

Hence,

$$|\lambda_n| \leq \sqrt{\frac{\sum_{1 \leq i,j \leq n} a_{ij}^2}{n}} \leq |\lambda_1|. \tag{2.8}$$

Let $\alpha = 0$, $\beta = \Psi(A)$, $a = \sum_{1\leq i,j\leq n} a_{ij}^2$, $b = n$, $x = |\lambda_n|$ and $y = |\lambda_1|$. We note that since $A \neq 0$, $a > 0$. Using Theorem 2.2 and (2.8) we deduce that

$$\beta \geq y \geq \sqrt{\frac{a}{b}} \geq x \geq \alpha.$$

Thus by applying Theorem 2.1 and Lemma 2.1 we obtain that

$$\mathcal{E}(A) \geq \frac{a + bxy}{x + y} \geq \frac{a + b\alpha\beta}{\alpha + \beta} = \frac{a}{\beta} = \frac{\sum_{1\leq i,j\leq n} a_{ij}^2}{\Psi(A)}.$$

This completes the proof. □

3. Energy of Regular Graphs

In [6] it was proved that if G is a t-regular graph of order n where $t \neq 0$, then $\mathcal{E}(G) \geq n$. In this section by applying Theorem 2.1 we improve this result and show that if G has no eigenvalue in the interval $(-1, 1)$, then $\mathcal{E}(G) \geq \frac{2tn}{t+1}$. Two examples of this kind of regular graphs are the cycle C_6 (with spectrum $\{2, 1, 1, -1, -1, -2\}$) and the *Petersen* graph (with spectrum $\{3, 1, 1, 1, 1, 1, -2, -2, -2, -2\}$). First we recall some results.

Theorem 3.1 ([2]). *Let G be a graph and $\rho(G)$ be the largest eigenvalue (the spectral radius) of G. Then the following hold:*

(i) *if G is connected, then the multiplicity of $\rho(G)$ is one;*
(ii) *for every eigenvalue λ of G, $|\lambda| \leq \rho(G)$.*

Theorem 3.2. [2] *Let G be a graph. Then the following hold:*

(i) *G is bipartite if and only if for every eigenvalue λ of G, also $-\lambda$ is an eigenvalue of G, with the same multiplicity.*
(ii) *If G is connected with largest eigenvalue θ, then G is bipartite if and only if $-\theta$ is an eigenvalue of G.*

Lemma 3.1 ([19]). *Let H be a connected t-regular graph where $t \geq 2$. Assume that*

$$\mathrm{Spec}(H) = \{t, \underbrace{1, \ldots, 1}_{b}, \underbrace{-1, \ldots, -1}_{c}\},$$

where b and c are some non-negative integers. Then H is the complete graph K_{t+1}.

Lemma 3.2 ([19]). *Let H be a connected bipartite t-regular graph where $t \geq 2$. Assume that*

$$\mathrm{Spec}(H) = \{t, \underbrace{1, \ldots, 1}_{b}, \underbrace{-1, \ldots, -1}_{c}, -t\},$$

where b and c are some non-negative integers. Then H is the crown graph $K_{t+1}^{\star}$.

Lemma 3.3 ([19]). *Let $t \geq 0$ be an integer. Then*

$$\mathrm{Spec}(K_{t+1}^{\star}) = \{t, \underbrace{1, \ldots, 1}_{t}, \underbrace{-1, \ldots, -1}_{t}, -t\}.$$

Now we prove the main result of this section.

Theorem 3.3. *Let G be a t–regular graph of order n where $t \neq 0$. Suppose that G has no eigenvalue in the interval $(-1, 1)$. Then*

$$\mathcal{E}(G) \geq \frac{2tn}{t+1}. \tag{3.1}$$

In particular, if $t \geq 2$, then $\mathcal{E}(G) \geq \frac{4n}{3}$. Moreover in (3.1) the equality holds if and only if every connected component of G is the complete graph K_{t+1} or the crown graph $K^{\star}_{t+1}$.

Proof. Note that if H is a 0-regular graph, then $\mathcal{E}(H) = 0$. Let $A = A(G) = [a_{ij}]$ be the adjacency matrix of G. Assume that $\lambda_1, \dots, \lambda_n$ be the eigenvalues of G (the eigenvalues of A) such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Thus, $|\lambda_1| \geq |\lambda_i|$, for $i = 1, \dots, n$. Since $Aj = tj$ (j is the vector of size n such that all of its entries are equal to 1), t is one of the eigenvalues of G. Hence, $|\lambda_1| \geq t$. On the other hand $\Psi(A) = t$. So, by Theorem 2.2, $|\lambda_1| \leq t$. Thus, $|\lambda_1| = t$. In fact t is the largest eigenvalue of G. Since G has no eigenvalue in the interval $(-1, 1)$, $|\lambda_n| \geq 1$. As we see in the proof of Theorem 2.3

$$|\lambda_n| \leq \sqrt{\frac{\sum_{1\leq i,j\leq n} a_{ij}^2}{n}} \leq |\lambda_1|. \tag{3.2}$$

Let $\alpha = 1$, $\beta = t$, $a = \sum_{1\leq i,j\leq n} a_{ij}^2$, $b = n$, $x = |\lambda_n|$ and $y = |\lambda_1|$. Since $t \neq 0$, $A \neq 0$. Therefore, $a > 0$. In fact $a = nt$. By (3.2) we find that

$$\beta \geq y \geq \sqrt{\frac{a}{b}} \geq x \geq \alpha.$$

Now, by applying Theorem 2.1 and Lemma 2.1 ,we find that

$$\mathcal{E}(G) = \mathcal{E}(A) \geq \frac{a + bxy}{x + y} \geq \frac{a + b\alpha\beta}{\alpha + \beta} = \frac{2tn}{t+1}. \tag{3.3}$$

Hence,

$$\mathcal{E}(G) \geq \frac{2tn}{t+1}. \tag{3.4}$$

If $t \geq 2$, then $\frac{2t}{t+1} \geq \frac{4}{3}$ and so (3.4) implies that $\mathcal{E}(G) \geq \frac{4n}{3}$.

Now we investigate the equality of (3.4). We note that for every disjoint graphs G_1 and G_2, $\mathcal{E}(G_1 \cup G_2) = \mathcal{E}(G_1) + \mathcal{E}(G_2)$. Since $\mathcal{E}(K_{t+1}) = 2t$ and $\mathcal{E}(K^{\star}_{t+1}) = 4t$ (by Lemma 3.3), it is easy to check that if $G = pK_{t+1} \cup qK^{\star}_{t+1}$, where p and q are some non-negative integers, then the equality holds. Hence, it remains to consider the converse. Thus, assume that $t \neq 0$ and G is a t-regular graph of order n such that G has no eigenvalue in the interval $(-1, 1)$ and $\mathcal{E}(G) = \frac{2tn}{t+1}$. Using (3.3) we obtain that

$$\mathcal{E}(A) = \frac{a + bxy}{x + y} \quad \text{and} \quad \frac{a + bxy}{x + y} = \frac{a + b\alpha\beta}{\alpha + \beta}. \tag{3.5}$$

Thus, the equality hold in Theorem 2.1 and in Lemma 2.1. By Theorem 2.1, there exists $r \in \{1, \ldots, n\}$ such that $|\lambda_1| = \cdots = |\lambda_r|$ and $|\lambda_{r+1}| = \cdots = |\lambda_n|$ (we note that $|\lambda_1| = t$). On the other hand, by Lemma 2.1, we deduce that $|\lambda_1| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = 1$. If $|\lambda_1| = \sqrt{\frac{a}{b}}$ or $|\lambda_n| = \sqrt{\frac{a}{b}}$, then by (2.6) we obtain that $|\lambda_1| = \cdots = |\lambda_n|$. By combining these conditions, we find that there are two following cases.

(I) $|\lambda_1| = \cdots = |\lambda_n| = t$. Hence, every eigenvalue of G is t or $-t$. By the fact that $\lambda_1^2 + \cdots + \lambda_n^2 = 2m$, where m is the number of edges of G, we conclude that $nt^2 = nt$. Thus, $t = 1$. Since G is t–regular, this shows that every connected component of G is K_2.

(II) $|\lambda_1| = \cdots = |\lambda_r| = t$ and $|\lambda_{r+1}| = \cdots = |\lambda_n| = 1$. Thus, every eigenvalue of G is t or $-t$ or 1 or -1. If $t = 1$, then every connected component of G is K_2. Thus assume that $t \geq 2$. Let H be a connected component of G. Since H is t-regular, t is the largest eigenvalue of H (we note that since H is connected, by the first part of Theorem 3.1, the multiplicity of t as an eigenvalue of H is one). First suppose that H is bipartite. Thus, by the first part of Theorem 3.2, $-t$ is also one of the eigenvalues of H with multiplicity one. Thus $\mathrm{Spec}(H)$ is consist of one t, one $-t$ and the other elements are 1 or -1. Thus, by Lemma 3.2, H is $K_{t+1}^{\star}$. Now assume that H is not bipartite. Since t is the largest eigenvalue of H, by the second part of Theorem 3.2, $-t$ is not an eigenvalue of H. Therefore, $\mathrm{Spec}(H)$ is consist of one t and the other elements are 1 or -1. Hence, by Lemma 3.1, H is K_{t+1}. The proof is complete. □

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