Kragujevac Journal of Mathematics
Volume 46(5) (2022), Pages 711–719.

COMPUTING THE TOTAL VERTEX IRREGULARITY STRENGTH ASSOCIATED WITH ZERO DIVISOR GRAPH OF COMMUTATIVE RING

ALI AHMAD[1]

Abstract. Let R be a commutative ring and $Z(R)$ be the set of all zero divisors of R. $\Gamma(R)$ is said to be a zero divisor graph if $x, y \in V(\Gamma(R)) = Z(R)$ and $(x, y) \in E(\Gamma(R))$ if and only if $x.y = 0$. In this paper, we determine the total vertex irregularity strength of zero divisor graphs associated with the commutative rings $\mathbb{Z}_{p^2} \times Z_q$ for p, q prime numbers.

1. Introduction

Let $G = (V, E)$ be a simple graph, the weight of a vertex $x \in V(G)$ for an edge k-labeling $\varphi : E(G) \to \{1, 2, \ldots, k\}$ is $w_\varphi(x) = \sum_{xy \in E(G)} \varphi(xy)$. For all distinct vertices $x, y \in V(G)$, with $w_\varphi(x) \neq w_\varphi(y)$, an edge k-labeling φ is called a vertex irregular k-labeling of G. The minimum value of k for which G has an edge k-labeling φ with labels at most k is known as irregularity strength, $s(G)$, of a graph G. This labeling is also called irregular assignments and introduced by Chartrand et al. in [12]. For further results on irregularity strength, one can see [11, 13, 15] and a detailed survey [14].

Motivated by these papers, Bača et al. in [9] introduced an edge irregular total k-labeling and a vertex irregular total k-labeling. For a graph $G = (V, E)$, a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is defined to be a vertex irregular total k-labeling, if for every two distinct vertices $x, y \in V(G)$ is $wt_\varphi(x) \neq wt_\varphi(y)$, where the weight of a vertex $x \in E(G)$ is $wt_\varphi(x) = \varphi(x) + \sum_{z \in N(x)} \varphi(xz)$ and $N(x)$ is the

Key words and phrases. Total vertex irregularity strength, zero divisor graph, commutative ring.
2010 *Mathematics Subject Classification.* Primary: 05C78. Secondary: 05C25, 05C12.
DOI 10.46793/KgJMat2205.711A
Received: May 12, 2019.
Accepted: March 14, 2020.

set of neighbors of x. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, denoted by $\text{tvs}(G)$.

In [9] several exact values and bounds of tvs(G) were determined for different types of graphs. Among others, the authors proved the following theorem.

Theorem 1.1 ([9]). *Let G be a $(|V(G)|, |E(G)|)$-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{|V(G)| + \delta}{\Delta + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta - 2\delta + 1.$$

Przybylo [17] improved the results for sparse graphs and for graphs with large minimum degree. In the latter case the bounds $\text{tvs}(G) < \frac{32|V(G)|}{\delta} + 8$ in general and $\text{tvs}(G) < \frac{8|V(G)|}{r} + 1$ for r-regular $(|V(G)|, |E(G)|)$-graphs were proved to hold. Anholcer et al. [8] determined a new upper bound of the form

$$\text{tvs}(G) \leq \frac{3|V(G)|}{\delta} + 1. \tag{1.1}$$

Some results on total vertex irregularity strength can be found in [1–3, 16]. The main aim of this paper is to find an exact value of the total vertex irregularity strength of certain classes of zero divisor graph of commutative rings which is much closer to the lower bound in Theorem 1.1 than to the upper bound in (1.1).

2. Results and Discussion

Let R be a commutative ring and $Z(R)$ be the set of all zero divisors of R. $\Gamma(R)$ is said to be a zero divisor graph if $x, y \in V(\Gamma(R)) = Z(R)$ and $(x, y) \in E(\Gamma(R))$ if and only if $x.y = 0$. Beck [10] introduced the notion of zero divisor graph. Anderson and Livingston [6] proved that $G(R)$ is always connected if R is commutative. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [4, 5, 7].

Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ denotes the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and is defined as following. For $x \in \mathbb{Z}_{p^2}$ and $y \in \mathbb{Z}_q$, $(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$ if and only if $x \neq p, 2p, 3p, \ldots, (p-1)p$ and $y \neq 0$. Let $I = \{(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) : x \neq p, 2p, 3p, \ldots, (p-1)p \text{ and } y \neq 0\}$, then $|I| = (p^2 - p)(q - 1)$. The vertices of the set I are the non zero divisors of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Also $(0, 0) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ is a non zero divisor. Therefore, the total number of non zero divisors are: $|I| + 1 = (p^2 - p)(q - 1) + 1 = p^2 q - p^2 - pq + p + 1$. There are $p^2 q$ total vertices of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Hence, there are $p^2 q - (p^2 q - p^2 - pq + p + 1) = p^2 + pq - p - 1$ total number of zero divisors. This implies that the order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $p^2 + pq - p - 1$, i.e., $|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = p^2 + pq - p - 1$.

In the following theorem, we determine the lower bound of total vertex irregularity strength for zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$.

Lemma 2.1. *Let p, q be two prime numbers and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$, $p > q$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times Z_q$ with maximum degree $\Delta = \Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$ and minimum degree $\delta = \delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$. Then*

$$\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \geq \max\left\{\left\lceil\frac{p^2-p+q-1}{q}\right\rceil, p+q-2, \left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil, \left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil\right\}.$$

Proof. The order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $p^2 + pq - p - 1$, i.e.,

$$|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = p^2 + pq - p - 1.$$

The degree of each vertex $(u, v) \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$, is discussed as follows.

If $x = 0$ and $y \in \{1, 2, 3, \ldots, q-1\}$, then each such vertex $(0, y)$ is only adjacent to the vertices $(x', 0)$ for every $x' \in \{1, 2, 3, \ldots, p^2 - 1\}$. Hence, the degree of each vertex $(0, y)$ is $p^2 - 1$ and the number of vertices of type $(0, y)$, with $y \neq 0$ are $q - 1$. Similarly, the degree of each vertex of type $(x, 0)$, $x \neq 0, p, 2p, \ldots, (p-1)p$ is $q - 1$ and the number of vertices of type $(x, 0)$, with $x \neq 0, p, 2p, \ldots, (p-1)p$ are $p^2 - p$.

If $x = kp$, $1 \leq k \leq p-1$ and $y \in \{1, 2, 3, \ldots, q-1\}$, then each such vertex (x, y) is only adjacent to the vertices $(x', 0)$ for every $x' = kp$, $1 \leq k \leq p-1$. Hence, the degree of each vertex (x, y) is $p - 1$ and the number of vertices of this type is $(p-1)(q-1)$.

If $x = kp$, $1 \leq k \leq p-1$ and $y = 0$, then each such vertex $(x, 0)$ is adjacent to the vertices $(0, y'), (x', 0)$, with $x \neq x'$ and (x', y') for every $y' \in \{1, 2, 3, \ldots, q-1\}$ and $x' = kp$, $1 \leq k \leq p-1$. Hence, the degree of each vertex $(x, 0)$ is $(q-1) + (p-2) + (pq - p - q + 1) = pq - 2$.

Let V_a denotes the vertex partition of zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ of degree a and n_a denotes the number of vertices in the partition V_a. Therefore, $n_{q-1} = p^2 - p$, $n_{pq-2} = p - 1$, $n_{p^2-1} = q - 1$ and $n_{p-1} = (p-1)(q-1)$. As $p > q$, this implies that $\Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = p^2 - 1$ and $\delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = q - 1$.

To prove the lower bound consider the weights of the vertices. The smallest weight among all vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is at least $q - 1$, so the largest weight of vertex of degree $q - 1$ is at least $p^2 - p + q - 1$. Since the weight of any vertex of degree $q - 1$ is the sum of q positive integers, so at least one label is at least $\left\lceil\frac{p^2-p+q-1}{q}\right\rceil$.

The largest value among the weights of vertices of degree $q - 1$ and $p - 1$ is at least $p^2 + pq - 2p$ and this weight is the sum of at most p integers. Hence, the largest label contributing to this weight must be at least $\left\lceil\frac{p^2+pq-2p}{p}\right\rceil = p + q - 2$.

The largest value among the weights of vertices of degree $q - 1$, $p - 1$ and $pq - 2$ is at least $p^2 + pq - p - 1$ and this weight is the sum of at most $pq - 1$ integers. Hence, the largest label contributing to this weight must be at least $\left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil$.

If we consider all vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ then the lower bound

$$\left\lceil\frac{|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| + \delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))}{\Delta(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) + 1}\right\rceil = \left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil$$

follows from Theorem 1.1. This gives

$$\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \geq \max\left\{\left\lceil\frac{p^2-p+q-1}{q}\right\rceil, p+q-2, \left\lceil\frac{p^2+pq-p-1}{pq-1}\right\rceil, \left\lceil\frac{p^2+pq-p+q-2}{p^2}\right\rceil\right\}$$

and we are done. □

In the following theorems, we determine the total vertex irregularity strength of zero divisor graph $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ associative with commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_q$, for p, q prime numbers.

Theorem 2.1. *Let p be a prime number and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$, $p\geq 3$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_2$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) = \left\lceil\frac{p^2-p+1}{2}\right\rceil$.

Proof. Let $(x,y)\in\mathbb{Z}_{p^2}\times\mathbb{Z}_2$, with $(x,y)\neq(0,0)$, such that $x\in\mathbb{Z}_{p^2}$ and $y\in\mathbb{Z}_2=\{0,1\}$. For our convenient, we partitioned the vertices of type $(x,0)$ into two distinct partitions as: If $x\in\mathbb{Z}_{p^2}\setminus\{0,p,2p,3p,\dots,(p-1)p\}$, then we denotes such vertices as x_i and the number of these vertices is p^2-p, the degree of each vertex is 1. If $x\in\{p,2p,3p,\dots,(p-1)p\}$, then we denote such vertices as z_j, $1\leq j\leq p-1$. This implies that $(x,0)=(x_i,0)\cup(z_j,0)$ for $1\leq i\leq p^2-p$, $1\leq j\leq p-1$. The degree of each vertex of type $(z_j,0)$ and $(z_j,1)$ is $2p-2$ and $p-1$, respectively. The vertex $(0,1)$ is the only one vertex with degree p^2-1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$ are defined as:

$$\begin{aligned}
V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) =&\{(x_i,0):1\leq i\leq p^2-p\}\\
&\cup\{(z_j,0),(z_j,1):1\leq j\leq p-1\}\cup\{(0,1)\},\\
E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)) =&\{(x_i,0)(0,1):1\leq i\leq p^2-p\}\\
&\cup\{(z_j,0)(z_t,1),(z_j,0)(0,1):1\leq j,t\leq p-1\}\\
&\cup\{(z_j,0)(z_t,0):1\leq j,t\leq p-1, j\neq t\}.
\end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\geq\lceil\frac{p^2-p+1}{2}\rceil$. Put $k=\lceil\frac{p^2-p+1}{2}\rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi: V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\cup E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2))\to\{1,2,\dots,k\}$ as:

$$\varphi((0,1))=k,\quad \varphi((x_i,0))=\max\{1,i+1-k\},$$

for $1\leq i\leq p^2-p$ and $\varphi((z_j,0))=\varphi((z_j,1))=i$ for $1\leq j\leq p-1$, $\varphi((x_i,0)(0,1))=\min\{i,k\}$ for $1\leq i\leq p^2-p$, $\varphi((z_j,0)(0,1))=\varphi((z_j,0)(z_t,1))=k$, for $1\leq j,t\leq p-1$ and $\varphi((z_j,0)(z_t,0))=k$ for $1\leq j,t\leq p-1$, $j\neq t$.

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_2)$ are as follows:

$$\begin{aligned}
wt_\varphi((x_i,0)) =&i+1,\quad \text{for } 1\leq i\leq p^2-p,\\
wt_\varphi((z_j,1)) =&(p-1)k+i,\quad \text{for } 1\leq j\leq p-1,\\
wt_\varphi((z_j,0)) =&2k(p-1)+i,\quad \text{for } 1\leq j\leq p-1,
\end{aligned}$$

$$wt_\varphi((0,1)) = \frac{k(k+1)}{2} + (p^2-k)k.$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k = \lceil \frac{p^2-p+1}{2} \rceil$. Thus, the labeling φ is the desired vertex irregular total $\lceil \frac{p^2-p+1}{2} \rceil$-labeling. This completes the proof. □

Theorem 2.2. *Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$, $p > 3$, be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_3$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) = \left\lceil \frac{p^2-p+2}{3} \right\rceil$.

Proof. Let us consider the vertex partition of type $(x,0)$ as defined in the proof of Theorem 2.1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$ are defined as:

$$\begin{aligned} V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) =& \{(x_i,0) : 1 \le i \le p^2-p\} \cup \{(z_j,t) : 1 \le j \le p-1, 0 \le t \le 2\} \\ & \cup \{(0,1),(0,2)\}, \\ E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) =& \{(x_i,0)(0,1),(x_i,0)(0,2) : 1 \le i \le p^2-p\} \\ & \cup \{(z_j,0)(z_t,1),(z_j,0)(z_t,2),(z_j,0)(0,1),(z_j,0)(0,2) : \\ & 1 \le j,t \le p-1\} \cup \{(z_j,0)(z_t,0) : 1 \le j,t \le p-1, j \ne t\}. \end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \ge \lceil \frac{p^2-p+2}{3} \rceil$. Put $k = \lceil \frac{p^2-p+2}{3} \rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi : V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \cup E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)) \to \{1,2,\dots,k\}$ as $\varphi((x_i,0)) = \max\{1, i+2-2k\}$, for $1 \le i \le p^2-p$, $\varphi((0,1)) = k$, $\varphi((0,2)) = k-1$, $\varphi((z_j,0)) = \varphi((z_j,1)) = j$ for $1 \le j \le p-1$ and $\varphi((z_j,2)) = p+j-1$, $1 \le j \le p-1$.

For $1 \le i \le p^2-p$, $\varphi((0,1)(x_i,0)) = \min\{i,k\}$, $\varphi((0,2)(x_i,0)) = \min\{\max\{1, i+1-k\}, k\}$ and $\varphi((0,1)(z_j,0)) = \varphi((0,2)(z_j,0)) = \varphi((z_j,1)(z_j,0)) = \varphi((z_j,2)(z_j,0)) = \varphi((z_j,0)(z_s,0)) = k$, for $1 \le j \le p-1$, $j \ne s$.

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_3)$ are as follows:

$$\begin{aligned} wt_\varphi((x_i,0)) =& i+2, \quad \text{for } 1 \le i \le p^2-p, \\ wt_\varphi((z_j,t)) =& (p-1)(k+t-1)+j, \quad \text{for } 1 \le j \le p-1, 1 \le t \le 2, \\ wt_\varphi((z_j,0)) =& k(pq-q+1)+j, \quad \text{for } 1 \le j \le p-1, \\ wt_\varphi((0,1)) =& k\left(\frac{2p^2+1-k}{2}\right), \\ wt_\varphi((0,2)) =& k\left(\frac{2p^2+5-3k}{2}\right) - 2. \end{aligned}$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k = \lceil \frac{p^2-p+2}{3} \rceil$. Thus, the labeling φ is the desired vertex irregular total $\lceil \frac{p^2-p+2}{3} \rceil$-labeling. This completes the proof. □

Theorem 2.3. *Let $p > q > 3$, $(p-q)(p-1) \ge (q-1)^2$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = \left\lceil \frac{p^2-p+q-1}{q} \right\rceil$.

Proof. Let us consider the vertex partition of type $(x, 0)$ as defined in the proof of Theorem 2.1. The vertex set and edge set of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ are defined as:

$$\begin{aligned} V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) =& \{(x_i, 0) : 1 \leq i \leq p^2 - p\} \cup \{(z_j, t) : 1 \leq j \leq p-1, 0 \leq t \leq q-1\} \\ & \cup \{(0, t) : 1 \leq t \leq q-1\}, \\ E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) =& \{(x_i, 0)(0, t) : 1 \leq i \leq p^2 - p, 1 \leq t \leq q-1\} \\ & \cup \{(z_j, 0)(z_s, t), (z_j, 0)(0, t) : 1 \leq j, s \leq p-1, 1 \leq t \leq q-1\} \\ & \cup \{(z_j, 0)(z_s, 0) : 1 \leq j, s \leq p-1, j \neq s\}. \end{aligned}$$

According to Lemma 2.1, we have $\text{tvs}(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \geq \lceil \frac{p^2-p+q-1}{q} \rceil$ for $p > q > 3$, $(p-q)(p-1) \geq (q-1)^2$. Put $k = \lceil \frac{p^2-p+q-1}{q} \rceil$. It is enough to describe a suitable vertex irregular total k-labeling. We define a labeling $\varphi : V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \cup E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \to \{1, 2, \dots, k\}$ as: $\varphi((0, t)) = k + 1 - t$, $1 \leq t \leq q-1$, $\varphi((z_j, 0)) = j$ for $1 \leq j \leq p-1$, $\varphi((z_j, t)) = j + (t-1)(p-1)$ for $1 \leq j \leq p-1$, $1 \leq t \leq q-1$ and

$$\varphi((x_i, 0)) = \begin{cases} 1, & \text{for } 1 \leq i \leq (q-1)k - q + 2, \\ i + (q-1)(1-k), & \text{for } (q-1)k - q + 3 \leq i \leq p^2 - p, \end{cases}$$

$\varphi((z_j, 0)(0, t)) = k$ for $1 \leq j \leq p-1$, $1 \leq t \leq q-1$, $\varphi((z_j, 0)(z_s, 0)) = k$ for $1 \leq j, s \leq p-1$, $j \neq s$, $\varphi((z_j, 0)(z_s, t)) = k$ for $1 \leq j, s \leq p-1$, $1 \leq t \leq q-1$, $\varphi((0, 1)(x_i, 0)) = \min\{i, k\}$ for $1 \leq i \leq p^2 - p$ and for $2 \leq t \leq q-1$

$$\varphi((0, t)(x_i, 0)) = \begin{cases} 1, & \text{for } 1 \leq i \leq (t-1)(k-1) + 1, \\ i + (t-1)(1-k), & \text{for } (t-1)(k-1) + 2 \leq i \leq t(k-1) + 1, \\ k, & \text{for } t(k-1) + 2 \leq i \leq p^2 - p. \end{cases}$$

The weights of vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ are as follows:

$$\begin{aligned} wt_\varphi((x_i, 0)) =& i + q - 1, \quad \text{for } 1 \leq i \leq p^2 - p, \\ wt_\varphi((z_j, t)) =& j + (p-1)(t-1+k), \quad \text{for } 1 \leq t \leq q-1, 1 \leq j \leq p-1, \\ wt_\varphi((z_j, 0)) =& j + (pq-2)k, \quad \text{for } 1 \leq j \leq p-1, \\ wt_\varphi((0, 1)) =& \frac{k(2p^2 - k + 1)}{2}, \\ wt_\varphi((0, t)) =& \sum_{i=1}^{(t-1)(k-1)+1} 1 + \sum_{i=(t-1)(k-1)+2}^{t(k-1)+1} (i + (t-1)(1-k)), \\ & + (p^2 - tk + t - 1)k - 1 - t, \quad \text{for } 2 \leq t \leq q-1. \end{aligned}$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $k = \lceil \frac{p^2-p+q-1}{q} \rceil$. Thus the labeling φ is the desired vertex irregular total $\lceil \frac{p^2-p+q-1}{q} \rceil$-labeling. This completes the proof. $\square$

Theorem 2.4. *Let $p \geq q > 3$, $(p+q)(q+1) \geq p^2+4q-1$ and $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_q$. Then* $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) = p+q-2$.

Proof. For our convenient, we partitioned the vertices of the graph $\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)$ as:

$$\begin{aligned}
A &= \{(x,0) : x \in \mathbb{Z}_{p^2} \setminus \{0,p,2p,\dots,(p-1)p\}\} = \left\{a_i : 1 \leq i \leq p^2-p\right\},\\
B &= \{(x,y) : x = p,2p,\dots,(p-1)p \text{ and } y \in \mathbb{Z}_q \setminus \{0\}\}\\
&= \{b_j : 1 \leq j \leq (p-1)(q-1)\},\\
C &= \{(0,y) : y \in \mathbb{Z}_q \setminus \{0\}\} = \{c_t : 1 \leq t \leq q-1\},\\
D &= \{(x,0) : x = p,2p,\dots,(p-1)p\} = \{d_s : 1 \leq s \leq p-1\}.
\end{aligned}$$

This implies that $V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) = A\cup B\cup C\cup D$ and the edge set is

$$\begin{aligned}
E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) =& \{a_ic_t : 1 \leq i \leq p^2-p, 1 \leq t \leq q-1\}\\
&\cup \{c_td_s : 1 \leq s \leq p-1, 1 \leq t \leq q-1\}\\
&\cup \{d_sb_j : 1 \leq s \leq p-1, 1 \leq j \leq (p-1)(q-1)\}\\
&\cup \{d_sd_{s'} : 1 \leq s,\, s' \leq p-1, s \neq s'\}.
\end{aligned}$$

According to Lemma 2.1, we have $\operatorname{tvs}(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \geq p+q-2$, for $p \geq q > 3, (p+q)(q+1) \geq p^2+4q-1$. It is enough to describe a suitable vertex irregular total $(p+q-2)$-labeling. We define a labeling $\varphi : V(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \cup E(\Gamma(\mathbb{Z}_{p^2}\times\mathbb{Z}_q)) \to \{1,2,\dots,p+q-2\}$ as $\varphi(a_i) = 1$ for $1 \leq i \leq p^2-p$, $\varphi(c_t) = t$ for $1 \leq t \leq q-1$, $\varphi(d_s) = s$ for $1 \leq s \leq p-1$

$$\varphi(b_j) = \begin{cases} q, & \text{for } 1 \leq j \leq pq-2p-q+3,\\ j-pq+2p+2q-3, & \text{for } pq-2p-q+4 \leq j \leq (p-1)(q-1), \end{cases}$$

$\varphi(c_td_s) = \varphi(d_sd_{s'}) = p+q-2$ for $1 \leq t \leq q-1$, $1 \leq s,s' \leq p-1$, $s \neq s'$, $\varphi(c_1a_i) = \min\{i, p+q-2\}$ for $1 \leq i \leq p^2-p$, $\varphi(d_1b_j) = \min\{p+j, p+q-2\}$ for $1 \leq j \leq (p-1)(q-1)$. For $2 \leq t \leq q-1$

$$\varphi(c_ta_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq (t-1)(p+q-3)+1,\\ i-(t-1)(p+q-3), & \text{for } (t-1)(p+q-3)+2 \leq i \leq t(p+q-3)+1,\\ p+q-2, & \text{for } t(p+q-3)+2 \leq i \leq p^2-p. \end{cases}$$

For $2 \leq s \leq p-1$

$$\varphi(d_sb_j) = \begin{cases} p, & \text{for } 1 \leq j \leq (s-1)(q-3)+1,\\ p+j-(s-1)(q-3), & \text{for } (s-1)(q-3)+2 \leq j \leq s(q-3)+1,\\ p+q-2, & \text{for } s(q-3)+2 \leq j \leq (p-1)(q-1), \end{cases}$$

$$\begin{aligned}
wt_\varphi(a_i) =& i+q-1, \quad \text{for } 1 \leq i \leq p^2-p,\\
wt_\varphi(b_j) =& p^2-p+q-1+j, \quad \text{for } 1 \leq j \leq (p-1)(q-1),\\
wt_\varphi(c_1) =& \frac{p+q-2}{2}(2p^2-p-q+1)+1,
\end{aligned}$$

$$
\begin{aligned}
wt_\varphi(c_t) =& t + \sum_{i=1}^{(t-1)(p+q-3)+1} (1) + \sum_{i=(t-1)(p+q-3)+2}^{t(p+q-3)+1} (i-(t-1)(p+q-3)) \\
&+ \sum_{i=t(p+q-3)+2}^{p^2-p} (p+q-2), \quad \text{for } 2 \le t \le q-1, \\
wt_\varphi(d_1) =& \frac{p+q-2}{2}(3p+3q-7)+1-\frac{p(p+1)}{2}, \\
wt_\varphi(d_s) =& s + (p+q-2)(p+q-3) + \sum_{j=1}^{(s-1)(q-3)+1} p \\
&+ \sum_{j=(s-1)(q-3)+2}^{s(q-3)+1} (p+j-(s-1)(q-3)) \\
&+ \sum_{j=s(q-3)+2}^{(p-1)(q-1)} (p+q-2), \quad \text{for } 2 \le s \le p-1.
\end{aligned}
$$

One can see that the weights of vertices under the function φ receive distinct labels and the maximum label used on vertices and edges is $p+q-2$. Thus the labeling φ is the desired vertex irregular total $(p+q-2)$-labeling. This completes the proof. □

References

[1] A. Ahmad and M. Bača, *On vertex irregular total labelings*, Ars Combin. **112** (2013), 129–139.

[2] A. Ahmad, M. Bača and Y. Bashir, *Total vertex irregularity strength of certain classes of unicyclic graphs*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **57**(105) (2014), 147–152.

[3] A. Ahmad, E. T. Baskoro and M. Imran, *Total vertex irregularity strength of disjoint union of helm graphs*, Discuss. Math. Graph Theory **32**(3) (2012), 427–434.

[4] A. Ahmad and A. Haider, *Computing the radio labeling associated with zero divisor graph of a commutative ring*, UPB Scientific Bulletin, Series A **81**(1) (2019), 65–72.

[5] S. Akbari and A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra **274** (2004), 847–855.

[6] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of commutative ring*, J. Algebra **217** (1999), 434–447.

[7] D. F. Anderson and S. B. Mulay, *On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra **210**(2) (2008), 543–550.

[8] M. Anholcer, M. Kalkowski and J. Przybylo, *A new upper bound for the total vertex irregularity strength of graphs*, Discrete Math. **309** (2009), 6316–6317.

[9] M. Bača, S. Jendroľ, M. Miller and J. Ryan, *On irregular total labellings*, Discrete Math. **307** (2007), 1378–1388.

[10] I. Beck, *Coloring of a commutative ring*, J. Algebra **116** (1988), 208–226.

[11] T. Bohman and D. Kravitz, *On the irregularity strength of trees*, J. Graph Theory **45** (2004), 241–254.

[12] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, *Irregular networks*, Congr. Numer. **64** (1988), 187–192.

[13] A. Frieze, R. J. Gould, M. Karonski and F. Pfender, *On graph irregularity strength*, J. Graph Theory **41** (2002), 120–137.

[14] J. A. Gallian, *A Dynamic Survey of Graph Labeling*, Electron. J. Combin. **20** (2017), #DS6.

[15] T. Nierhoff, *A tight bound on the irregularity strength of graphs*, SIAM J. Discrete Math. **13** (2000), 313–323.
[16] Nurdin, E. T. Baskoro, A. N. M. Salman and N. N. Gaos, *On the total vertex irregularity strength of trees*, Discrete Math. **310** (2010), 3043–3048.
[17] J. Przybylo, *Linear bound on the irregularity strength and the total vertex irregularity strength of graphs*, SIAM J. Discrete Math. **23** (2009), 511–516.

[1]College of Computer Science & Information Technology,
Jazan University, Jazan, KSA
Email address: ahmadsms@gmail.com