ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 46, Number 6, 2022

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić
. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Belgrade : Donat Graf). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2206

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Email: krag_j_math@kg.ac.rs
Website: http://kjm.pmf.kg.ac.rs

Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: Donat Graf, Belgrade, Serbia
From 2021 the journal appears in one volume and six issues per annum.

Contents

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 841–855.

NONLINEAR SEQUENTIAL CAPUTO AND CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS IN BANACH SPACES

CHOUKRI DERBAZI[1]

Abstract. This paper is devoted to the existence of solutions for certain classes of nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet boundary conditions in Banach spaces. Moreover, our analysis is based on Darbo's fixed point theorem in conjunction with the technique of Hausdorff measure of noncompactness. An example is also presented to illustrate the effectiveness of the main results.

1. Introduction

Fractional calculus and fractional differential equations describe various phenomena in diverse areas of natural science such as physics, aerodynamics, biology, control theory, and chemistry; see for instance [26,29,31,39–41,43]. On the other hand, there are several definitions of fractional integrals and derivatives in the literature, but the most popular definitions are in the sense of the Riemann-Liouville and Caputo. However, there is another kind of fractional derivatives that appears in the literature due to Hadamard [23], which is known as Hadamard derivative and differs from the preceding ones in the sense that its definition involves logarithmic function of arbitrary exponent. Another significant aspect of Hadamard derivative is that its expression can be viewed as a generalization operator $\left(t\frac{\mathrm{d}}{\mathrm{dt}}\right)^n$ [6,23], whilst the Riemann-Liouville derivative is regarded as an extension of the classical differential operator $\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^n$. For some developments on the existence results of the Hadamard fractional differential equations, we

Key words and phrases. Caputo and Caputo-Hadamard fractional differential equations, Hausdorff measure of noncompactness, Darbo's fixed point theorem, Banach spaces.

2010 *Mathematics Subject Classification.* Primary: 34A08. Secondary: 26A33, 34G20.

DOI 10.46793/KgJMat2206.841D

Received: September 14, 2019.

Accepted: April 15, 2020.

can refer to [6,16,18,28,44]. In recent times, another derivative was proposed by modifying the Hadamard derivative with the Caputo one, known as Caputo-Hadamard derivative [24]. It is obtained from the Hadamard derivative by changing the order of its differentiation and integration. In addition, the main difference between the Caputo-Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo-Hadamard derivative of a constant is zero; another aspect is that the Cauchy problems for Caputo-Hadamard fractional differential equations contain initial conditions which can be physically interpretable, similarly to the case with Caputo fractional derivatives. From these points of view, it is imperative to study Caputo-Hadamard fractional calculus. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo-Hadamard fractional differential equations [7,8,17,27]. Moreover, it has been noticed that most of the above-mentioned work on the topic is based on the technique of nonlinear analysis such as Banach fixed point theorem, Schauder's fixed point theorem and Leray-Schauder nonlinear alternative, etc. But if compactness and Lipschitz condition are not satisfied these results cannot be used. Measure of noncompactness comes handy in such situations. For instance, the celebrated Darbo fixed point theorem and Mönch fixed point theorem are used by several authors with the end goal to establish existence results for nonlinear integral equations (see [1,2,4,13,15,21,45] and references therein).

In 2018, Tariboon et al. in [44], discussed the existence and uniqueness of solutions for two sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}\mathfrak{D}^{p}\left[{}^{H}\mathfrak{D}^{q}u(t)\right] = f(t,u(t)), \quad t \in (a,b), \\ a_1 u(a) + b_1 {}^{H}\mathfrak{D}^{q}u(a) = 0, \\ a_2 u(b) + b_2 {}^{H}\mathfrak{D}^{q}u(b) = 0, \end{cases}$$

and

$$\begin{cases} {}^{H}\mathfrak{D}^{q}\left[{}^{C}\mathfrak{D}^{p}u(t)\right] = f(t,u(t)), \quad t \in (a,b), \\ a_1 u(a) + b_1 {}^{C}\mathfrak{D}^{p}u(a) = 0, \\ a_2 u(b) + b_2 {}^{C}\mathfrak{D}^{p}u(b) = 0, \end{cases}$$

where ${}^{C}\mathfrak{D}^{p}$ and ${}^{H}\mathfrak{D}^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \leq 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$ and $a_i, b_i \in \mathbb{R}$, $i = 1, 2$.

Very recently, in [30], the authors considered the infinite system of second-order differential equations of the type

$$\begin{cases} t\frac{\mathrm{d}^2 v_j}{\mathrm{d}t^2} + \frac{\mathrm{d}v_j}{\mathrm{d}t} = f_i(t, v(t)), \quad t \in J := [1,T], \\ v_j(1) = v_j(T) = 0, \end{cases} \tag{1.1}$$

where $v(t) = \{v_j(t)\}_{j=1}^{\infty}$, in Banach sequence space ℓ^p, $p \geq 1$. The authors obtained the existence of solutions by using the Hausdorff measure of noncompactness and Darbo type fixed point theorem. Additionally, for more interesting details about infinite systems of differential equations or integral equations in some Banach sequence spaces we suggest some works [5, 32–38]. Moreover, the reader is advised to see the recent book [12] where several applications of the measure of noncompactness can be found.

No contributions exist, as far as we know, concerning nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations in Banach spaces. As a result, the goal of this paper is to enrich this academic area. So, in this paper, we mainly study the following boundary value problem of the form.

$$\begin{cases} {}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = f(t, u(t)), \quad 0 < p, q \leq 1,\, t \in J := [a, b], \\ u(a) = u(b) = \theta, \end{cases} \tag{1.2}$$

where ${}^{C}_{H}\mathfrak{D}^{p}u(t)$ and ${}^{C}\mathfrak{D}^{q}$ are the Caputo Hadamard and Caputo fractional derivatives of orders p and q, respectively, $0 < p, q \leq 1$, $f : [a, b] \times E \to E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$, and θ refers to the null vector in the space E.

The main motivation for the elaboration of this paper comes from the above highlighted articles on the existence of solutions of fractional differential equations. In addition, as in the Banach space (in general in any infinite-dimensional linear space) a closed and bounded set is not necessarily compact set, mere continuity of the function f does not guarantee the existence of a solution of differential equations. The arguments are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness to establish the existence of solution for (1.2). Obviously, BVP (1.2) is more general than the problems discussed in some recent literature (such as [30, 44]). Firstly, our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem. For example, if we take $a = p = q = 1$, $b = T$ and $E = \ell^{p}$, then the BVP (1.2) corresponds to the infinite system represented in (1.1). Secondly, the required conditions to prove the existence of solutions for the system (1.1) depend strongly on the chosen Banach space of sequences. This is because the formula for the Hausdorff MNC is, of course, different from one space to another. However, our conditions do not depend on the chosen Banach space.

Here is a brief outline of the paper. The next section provides the definitions and preliminary results that we will need to prove our main results. Then, we present the existence results in Section 3. In Section 4, we give an example to illustrate the obtained results. The last section concludes this paper.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $C(J,E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_\infty = \sup_{t\in J}\|u(t)\|.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \to E$, with the norm

$$\|u\|_1 = \int_a^b \|u(t)\|\mathrm{dt}.$$

Next, we define the Hausdorff measure of noncompactness and give some of its important properties.

Definition 2.1 ([11])**.** Let E be a Banach space and B a bounded subsets of E . Then Hausdorff measure of non-compactness of B is defined by

$$\chi(B) = \inf\Big\{\varepsilon > 0 : B \text{ has a finite cover by closed balls of radius } \varepsilon\Big\}.$$

To discuss the problem in this paper, we need the following lemmas.

Lemma 2.1. *Let $A, B \subset E$ be bounded. Then Hausdorff measure of non-compactness has the following properties:*

(1) $A \subset B \Rightarrow \chi(A) \le \chi(B)$;
(2) $\chi(A) = 0 \Leftrightarrow A$ is relatively compact;
(3) $\chi(A\cup B) = \max\{\chi(A), \chi(B)\}$;
(4) $\chi(A) = \chi(\overline{A}) = \chi(\mathrm{conv}(A))$, where $\overline{A}$ and conv$A$ represent the closure and the convex hull of A, respectively;
(5) $\chi(A + B) \le \chi(A) + \chi(B)$, where $A + B = \{x + y : x \in A, y \in B\}$;
(6) $\chi(\lambda A) \le |\lambda|\chi(A)$ for any $\lambda \in \mathbb{R}$,

For more details and the proof of these properties see [11].

Lemma 2.2 ([11])**.** *If $W \subseteq C(J,E)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous on J and*

$$\chi(W) = \sup_{t\in J}\chi(W(t)).$$

We call $B \subset L^1(J,E)$ uniformly integrable if there exists $\eta \in L^1(J,\mathbb{R}^+)$ such that

$$\|u(s)\| \le \eta(s), \quad \text{for all } u \in B \text{ and a.e. } s \in J.$$

Lemma 2.3 ([25])**.** *If $\{u_n\}_{n=1}^\infty \subset L^1(J,E)$ is uniformly integrable, then $\chi(\{u_n\}_{n=1}^\infty)$ is measurable, and*

$$\chi\left(\left\{\int_0^t u_n(s)\mathrm{ds}\right\}_{n=1}^\infty\right) \le 2\int_0^t \chi(\{u_n(s)\}_{n=1}^\infty)\mathrm{ds}.$$

Lemma 2.4 ([19])**.** *If W is bounded, then for each ε, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$, such that*

$$\chi(W) \le 2\chi(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

Definition 2.2 ([46])**.** A function $f : [a, b] \times E \to E$ is said to satisfy the Carathéodory conditions, if the following hold

- $f(t, u)$ is measurable with respect to t for $u \in E$;
- $f(t, u)$ is continuous with respect to $u \in E$ for $t \in J$.

Definition 2.3 ([10])**.** The mapping $\mathfrak{T} : \Omega \subset E \to E$ is said to be a χ-contraction, if there exists a positive constant $k < 1$ such that

$$\chi(\mathfrak{T}(W)) \leq k\chi(W),$$

for every bounded subset W of Ω.

A useful fixed point result for our goals is the following, proved in [11, 20].

Theorem 2.1 (Darbo and Sadovskii)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathfrak{T} : \Omega \to \Omega$ be a continuous operator. If $\mathfrak{T}$ is a χ-contraction, then $\mathfrak{T}$ has at least one fixed point.*

Let us recall some preliminary concepts of fractional calculus related to our work.

Definition 2.4 ([29])**.** The Riemann-Liouville fractional integral of order $p > 0$ of a function $u \in L^1([a, b])$ is defined by

$$^{RL}\mathfrak{J}^p u(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} u(s)\mathrm{d}s, \quad t > a,\, p > 0,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function $\Gamma(p) = \int_0^{+\infty} e^{-t} t^{p-1}\mathrm{d}t$, $p > 0$. Moreover, for $p = 0$, we set $^{RL}\mathfrak{J}^p u := u$.

Lemma 2.5 ([29])**.** *The following basic properties of the Riemann-Liouville integrals hold.*

(a) *The integral operator $^{RL}\mathfrak{J}^p$ is linear.*

(b) *The semigroup property of the fractional integration operator $^{RL}\mathfrak{J}^p$ is given by the following result*

$$^{RL}\mathfrak{J}^p(^{RL}\mathfrak{J}^q u(t)) = {}^{RL}\mathfrak{J}^{p+q} u(t), \quad p, q > 0,$$

holds at every point if $u \in C([a, b])$ and holds almost everywhere if $u \in L^1([a, b])$.

(c) *Commutativity*

$$^{RL}\mathfrak{J}^p(^{RL}\mathfrak{J}^q u(t)) = {}^{RL}\mathfrak{J}^q(^{RL}\mathfrak{J}^p u(t)), \quad p, q > 0.$$

(d) *The fractional integration operator $^{RL}\mathfrak{J}^p$ is bounded in $L^p[a, b]$, $1 \leq p \leq \infty$,*

$$\|^{RL}\mathfrak{J}^p u\|_{L^p} \leq \frac{1}{\Gamma(p+1)} \|u\|_{L^p}.$$

Example 2.1. The Riemann-Liouville fractional integral of the power function $(t - a)^q$, $p > 0$, $q > -1$

$$^{RL}\mathfrak{J}^p (t-a)^q = \frac{\Gamma(q+1)}{\Gamma(p+q+1)} (t-a)^{p+q}.$$

Definition 2.5 ([29,40])**.** The Caputo fractional derivative ${}^{C}\mathfrak{D}^{p}$ of order p of a function $u \in AC^n([a,b])$ is represented by

$$ {}^{C}\mathfrak{D}^{p}u(t) = \begin{cases} \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} u^{(n)}(s)\mathrm{d}s, & \text{if } p \notin \mathbb{N}, \\ u^{(n)}(t), & \text{if } p \in \mathbb{N}, \end{cases} $$

where $u^{(n)}(t) = \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}$, $p > 0$, $n = [p]+1$ and $[p]$ denotes the integer part of the real number p.

Example 2.2. The Caputo fractional derivative of order $n-1 < p < n$ for $(t-a)^q$ is given by

$$ {}^{C}\mathfrak{D}^{p}(t-a)^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q-p+1)}(t-a)^{q-p}, & q \in \mathbb{N} \text{ and } q \geq n \text{ or } q \notin \mathbb{N} \text{ and } q > n-1, \\ 0, & q \in \{0, \dots, n-1\}. \end{cases} \tag{2.1} $$

Lemma 2.6 ([29,40])**.** *Let $p > 0$ and $n = [p]+1$, then the differential equation*

$$ {}^{C}\mathfrak{D}^{p}u(t) = 0 $$

has solutions

$$ u(t) = \sum_{j=0}^{n-1} c_j (t-a)^j, \quad c_j \in \mathbb{R},\ j = 0, \dots, n-1. $$

Lemma 2.7 ([29,40])**.** *Let $p > q > 0$ and $u \in L^1([a,b])$. Then we have:*

(1) *the Caputo fractional derivative is linear;*
(2) *the Caputo fractional derivative obeys the following property:*

$$ {}^{RL}\mathfrak{J}^{p}\,{}^{C}\mathfrak{D}^{p}u(t) = u(t) + \sum_{j=0}^{n-1} c_j (t-a)^j, $$

for some $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, n-1$, where $n = [p]+1$;

(3) ${}^{C}\mathfrak{D}^{p}\,{}^{RL}\mathfrak{J}^{p}u(t) = u(t)$;
(4) ${}^{C}\mathfrak{D}^{q}\,{}^{RL}\mathfrak{J}^{p}u(t) = {}^{RL}\mathfrak{J}^{p-q}u(t)$.

Definition 2.6 ([29])**.** The Hadamard fractional integral of order $p > 0$ for a function $u \in L^1(J)$ is defined as

$$ \left({}^{H}\mathfrak{J}^{p}u\right)(t) = \frac{1}{\Gamma(p)} \int_a^t \left(\log \frac{t}{s}\right)^{p-1} u(s) \frac{\mathrm{d}s}{s}, \quad p > 0. $$

Set

$$ \delta = t\frac{\mathrm{d}}{\mathrm{d}t}, \quad p > 0,\ n = [p]+1, $$

where $[p]$ denotes the integer part of p. Define the space

$$ AC_\delta^n[a,b] := \left\{ u : [a,b] \to \mathbb{R} : \delta^{n-1}u(t) \in AC([a,b]) \right\}. $$

Definition 2.7 ([29]). The Hadamard fractional derivative of order $p > 0$ applied to the function $u \in AC_\delta^n[a,b]$ is defined as

$$\left({}^{H}\mathfrak{D}^{p} u\right)(t) = \delta^{n}\left({}^{H}\mathfrak{J}^{n-p} u\right)(t).$$

Definition 2.8 ([24, 29]). The Caputo-Hadamard fractional derivative of order $p > 0$ applied to the function $u \in AC_\delta^n[a,b]$ is defined as

$$\left({}^{C}_{H}\mathfrak{D}^{p} u\right)(t) = \left({}^{H}\mathfrak{J}^{n-p}\delta^{n} u\right)(t).$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 2.8 ([24, 29]). *Let $p > 0$, $r > 0$, $n = [p] + 1$ and $a > 0$, then the following relations hold :*

- $\left({}^{H}\mathfrak{J}^{p}\left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \frac{\Gamma(r)}{\Gamma(p+r)}\left(\log \frac{t}{a}\right)^{p+r-1}$;
-
$$\left({}^{C}_{H}\mathfrak{D}^{p}\left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-p)}\left(\log \frac{t}{a}\right)^{r-p-1}, & r > n, \\ 0, & r \in \{0, \dots, n-1\}. \end{cases}$$

Lemma 2.9 ([22, 29]). *Let $p > q > 0$ and $u \in AC_\delta^n[a,b]$. Then we have:*

- ${}^{H}\mathfrak{J}^{p}\, {}^{H}\mathfrak{J}^{q} u(t) = {}^{H}\mathfrak{J}^{p+q} u(t)$;
- ${}^{C}_{H}\mathfrak{D}^{p}\, {}^{H}\mathfrak{J}^{p} u(t) = u(t)$;
- ${}^{C}_{H}\mathfrak{D}^{q}\, {}^{H}\mathfrak{J}^{p} u(t) = {}^{H}\mathfrak{J}^{p-q} u(t)$.

Lemma 2.10 ([24, 29]). *Let $p \geq 0$ and $n = [p] + 1$. If $u \in AC_\delta^n[a,b]$, then the Caputo-Hadamard fractional differential equation*

$$\left({}^{C}_{H}\mathfrak{D}^{p} u\right)(t) = 0,$$

has a solution

$$u(t) = \sum_{j=0}^{n-1} c_j \left(\log \frac{t}{a}\right)^{j},$$

and the following formula holds:

$${}^{H}\mathfrak{J}^{p}\left({}^{C}_{H}\mathfrak{D}^{p} u(t)\right) = u(t) + \sum_{j=0}^{n-1} c_j \left(\log \frac{t}{a}\right)^{j},$$

where $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, n-1$.

Remark 2.1. Note that for an abstract function $u : J \to E$, the integrals which appear in the previous definitions are taken in Bochner's sense (see, for instance, [42]).

3. Main Results

Let us recall the definition and lemma of a solution for problem (1.2).

First of all, we define what we mean by a solution for the boundary value problem (1.2).

Definition 3.1. A function $u \in C(J, E)$ is said to be a solution of (1.2) if u satisfies the equation ${}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = f(t, u(t))$ a.e. on J and the condition $u(a) = u(b) = \theta$.

For the existence of solutions for the problem (1.2) we need the following lemma.

Lemma 3.1. *For a given $h \in C(J, \mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^{C}\mathfrak{D}^{q}\left[{}^{C}_{H}\mathfrak{D}^{p}u(t)\right] = h(t), \quad 0 < p, q \leq 1,\ t \in J := [a, b], \\ u(a) = u(b) = 0, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} u(t) =& {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(t) - \left(\frac{\log(t/a)}{\log(b/a)}\right)^{p} {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b) \\ =& \frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{t}\int_{a}^{s}\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}h(\tau)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}} \\ & - \left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{b}\int_{a}^{s}\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}h(\tau)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}. \end{aligned} \tag{3.2}$$

Proof. Taking the Riemann-Liouville fractional integral of order q to the first equation of (3.1), we get

$${}^{C}_{H}\mathfrak{D}^{p}u(t) = {}^{RL}\mathfrak{J}^{q}h(t) + k_0, \quad k_0 \in \mathbb{R}. \tag{3.3}$$

Again taking the Hadamard fractional integral of order p to the above equation, we obtain

$$u(t) = {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(t) + k_0\frac{(\log(t/a))^{p}}{\Gamma(p+1)} + k_1, \quad k_0 \in \mathbb{R}. \tag{3.4}$$

Substituting $t = a$ in (3.4) and applying the first boundary condition of (3.1), it follows that $k_1 = 0$. For $t = b$ in (3.4) and using the second boundary condition of (3.1), it yields

$$u(b) = 0 = {}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b) + k_0\frac{(\log(b/a))^{p}}{\Gamma(p+1)}. \tag{3.5}$$

By solving (3.5), we find that

$$k_0 = -\frac{\Gamma(p+1)}{(\log(b/a))^{p}}{}^{H}\mathfrak{J}^{p}\left({}^{RL}\mathfrak{J}^{q}h\right)(b). \tag{3.6}$$

Substituting the values of k_0 and k_1 into (3.4), we get the integral equation (3.2). The converse follows by the direct computation which completes the proof. $\square$

Now, we shall present our main result concerning the existence of solutions of problem (1.2). Let us introduce the following hypotheses.

(H1) The function $f : [a, b] \times E \longrightarrow E$ satisfies Carathéodory conditions.

(H2) There exists function $\psi \in L^{\infty}(J, \mathbb{R}_{+})$ such that

$$\|f(t, u(t))\| \leq \psi(t)(1 + \|u\|), \quad \text{for all } u \in C(J, E).$$

(H3) For each bounded set $W \subset E$ and each $t \in J$, the following inequality holds

$$\chi(f(t, W)) \leq \psi(t)\chi(W).$$

For computational convenience we put

$$\mathcal{M}_{\psi} = \frac{2\|\psi\|\,(b-a)^{q}\left(\log\frac{b}{a}\right)^{p}}{\Gamma(p+1)\Gamma(q+1)}. \tag{3.7}$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1.2)

Theorem 3.1. *Assume that the hypotheses* (H1)-(H3) *are satisfied. If*

$$4\mathcal{M}_{\psi} < 1, \tag{3.8}$$

then the problem (1.2) *has at least one solution defined on* J.

Proof. Consider the operator $\mathcal{N} : C(J, E) \to C(J, E)$ defined by:

$$\begin{aligned}\mathcal{N}u(t) =& \frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{t}\int_{a}^{s}\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}f(\tau, u(\tau))\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\ &-\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{b}\int_{a}^{s}\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}f(\tau, u(\tau))\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}.\end{aligned} \tag{3.9}$$

It is obvious that $\mathcal{N}$ is well defined due to (H1) and (H2). Then, fractional integral equation (3.2) can be written as the following operator equation

$$u = \mathcal{N}u. \tag{3.10}$$

Thus, the existence of a solution for (1.2) is equivalent to the existence of a fixed point for operator $\mathcal{N}$ which satisfies operator equation (3.10). Define a bounded closed convex set

$$B_{R} = \{w \in C(J, E) : \|w\|_{\infty} \leq R\},$$

with $R > 0$, such that

$$R \geq \frac{\mathcal{M}_{\psi}}{1 - \mathcal{M}_{\psi}}.$$

In order to satisfy the hypotheses of the Darbo fixed point theorem, we split the proof into four steps.

Step 1. The operator $\mathcal{N}$ maps the set B_R into itself. By the assumption (H2), we have

$$\|\mathcal{N}u(t)\| \leq \frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{t}\int_{a}^{s}\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau, u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}$$

$$
\begin{aligned}
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{\|\psi\|(1+\|u\|)}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\frac{\|\psi\|(1+\|u\|)}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

Also, note that

$$
\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\leq\frac{(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)},
$$

where we have used the fact that $(s-a)^q\leq(b-a)^q$ for $0<q\leq1$. Using the above arguments, we have

$$
\|\mathcal{N}u(t)\|\leq\|\psi\|(1+\|u\|)\frac{2\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\leq(1+R)\mathcal{M}_\psi\leq R.
$$

Thus, $\|\mathcal{N}u\|\leq R$. This proves that $\mathcal{N}$ transforms the ball B_R into itself.

Step 2. The operator $\mathcal{N}$ is continuous. Suppose that $\{u_n\}$ is a sequence such that $u_n\to u$ in B_R as $n\to\infty$. It is easy to see that $f(s,u_n(s))\to f(s,u(s))$ as $n\to+\infty$, due to the Carathéodory continuity of f. On the other hand taking (H2) into consideration we get

$$
\begin{aligned}
&\|\mathcal{N}u_n(t)-\mathcal{N}u(t)\|\\
\leq&\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^s\left(\log\frac{t}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u_n(\tau))-f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left(\frac{\log(t/a)}{\log(b/a)}\right)^{p}\frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\|f(\tau,u_n(\tau))-f(\tau,u(\tau))\|\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\\
\leq&\frac{2\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\|f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot))\|.
\end{aligned}
$$

By using the Lebesgue dominated convergence theorem, we know that

$$
\|\mathcal{N}u_n(t)-\mathcal{N}u(t)\|\to0\quad\text{as}\quad n\to+\infty,
$$

for any $t\in J$. Therefore, we get that

$$
\|\mathcal{N}u_n-\mathcal{N}u\|\to0\quad\text{as}\quad n\to+\infty,
$$

which implies the continuity of the operator $\mathcal{N}$.

Step 3. The operator $\mathcal{N}$ is equicontinuous. For any $a < t_1 < t_2 < b$ and $u \in B_R$, we get

$$\begin{aligned}
&\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{t_1} \int_a^s \left[\left(\log \frac{t_1}{s}\right)^{p-1} - \left(\log \frac{t_2}{s}\right)^{p-1} \right] (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \frac{1}{\Gamma(p)\Gamma(q)} \int_{t_1}^{t_2} \int_a^s \left(\log \frac{t_2}{s}\right)^{p-1} (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \frac{(\log(t_2/a))^p - (\log(t_1/a))^p}{\Gamma(p)\Gamma(q)\,(\log(b/a))^p} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
\leq & \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p)\Gamma(q+1)} \left[\int_a^{t_1} \left[\left(\log \frac{t_1}{s}\right)^{p-1} - \left(\log \frac{t_2}{s}\right)^{p-1} \right] \frac{\mathrm{ds}}{\mathrm{s}} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{p-1} \frac{\mathrm{ds}}{\mathrm{s}} \right] \\
&+ \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p)\Gamma(q+1)} \frac{(\log(t_2/a))^p - (\log(t_1/a))^p}{\Gamma(p)\,(\log(b/a))^p} \int_a^b \left(\log \frac{b}{s}\right)^{p-1} \frac{\mathrm{ds}}{\mathrm{s}} \\
\leq & 2 \frac{\|\psi\|(1+r)\,(b-a)^q}{\Gamma(p+1)\Gamma(q+1)} \left(\log \frac{t_2}{t_1}\right)^p .
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero independently of $u \in B_R$. Hence, we conclude that $\mathcal{N}(B_R) \subseteq C(J, E)$ is bounded and equicontinuous.
Step 4: Our aim in this step is to show that $\mathcal{N}$ is χ-contraction on B_R. For every bounded subset $W \subset B_R$ and $\varepsilon > 0$ using Lemma 2.4 and the properties of χ, there exist sequences $\{u_k\}_{k=1}^{\infty} \subset W$ such that

$$\begin{aligned}
&\chi\left(\mathcal{N}W(t)\right) \\
\leq & 2\chi \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, \{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&- \left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} f(\tau, \{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \Bigg\} + \varepsilon.
\end{aligned}$$

Next, by Lemma 2.3 and the properties of χ and (H3) we have

$$\begin{aligned}
&\chi\left(\mathcal{N}W(t)\right) \\
\leq & 4 \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} \chi(f(\tau, \{u_k(\tau)\}_{k=1}^{\infty})) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)} \int_a^b \int_a^s \left(\log \frac{b}{s}\right)^{p-1} (s-\tau)^{q-1} \chi(f(\tau, \{u_k(\tau)\}_{k=1}^{\infty})) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}} \Bigg\} + \varepsilon \\
\leq & 4 \Bigg\{ \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^s \left(\log \frac{t}{s}\right)^{p-1} (s-\tau)^{q-1} \psi(\tau) \chi(\{u_k(\tau)\}_{k=1}^{\infty}) \mathrm{d}\tau \frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}$$

$$+\left(\frac{\log(t/a)}{\log(b/a)}\right)^p \frac{1}{\Gamma(p)\Gamma(q)}\int_a^b\int_a^s\left(\log\frac{b}{s}\right)^{p-1}(s-\tau)^{q-1}\psi(\tau)\chi(\{u_k(\tau)\}_{k=1}^{\infty})\mathrm{d}\tau\frac{\mathrm{ds}}{\mathrm{s}}\Bigg\}+\varepsilon$$
$$\leq 4\frac{2\|\psi\|\,(b-a)^q\left(\log\frac{b}{a}\right)^p}{\Gamma(p+1)\Gamma(q+1)}\chi(B)+\varepsilon.$$

As the last inequality is true for every $\varepsilon > 0$ we infer

$$\chi(\mathcal{N}W)=\sup_{t\in J}\chi(\mathcal{N}W(t))\leq 4\mathcal{M}_\psi\chi(B).$$

Using the condition (3.8), we claim that $\mathcal{N}$ is a χ-contraction on B_R. By Theorem 2.1, there is a fixed point u of $\mathcal{N}$ on B_R, which is a solution of (1.2). This completes the proof. □

4. An Example

In this section we give an example to illustrate the usefulness of our main result. Let

$$E=c_0=\{u=(u_1,u_2,\dots,u_n,\dots):u_n\to 0\ (n\to\infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|u\|_\infty=\sup_{n\geq 1}|u_n|.$$

Example 4.1. Consider the following boundary value problem of a fractional differential posed in c_0:

$$\begin{cases}{}^{C}\mathcal{D}^{\frac{3}{4}}\left[{}^{C}_{H}\mathcal{D}^{\frac{7}{8}}u(t)\right]=f(t,u(t)), & 0<p,q\leq 1,\ t\in J:=\left[1,\frac{3}{2}\right],\\ u(1)=u\left(\frac{3}{2}\right)=0.\end{cases}\tag{4.1}$$

Note that this problem is a particular case of BVP (1.2), where

$$p=\frac{7}{8}=q=\frac{3}{4},\quad a=1,\quad b=\frac{3}{2},$$

and $f: J\times c_0\to c_0$ given by

$$f(t,u)=\left\{\frac{1}{(t^2+2)^2}\left(\frac{1}{n^2}+\sin|u_n|\right)\right\}_{n\geq 1},\quad \text{for } t\in J,\ u=\{u_n\}_{n\geq 1}\in c_0.$$

It is clear that condition (H1) holds, and as

$$\|f(t,u)\|=\left\|\frac{1}{(t^2+2)^2}\left(\frac{1}{n^2}+\sin|u_n|\right)\right\|\leq\frac{1}{(t^2+2)^2}(1+\|u\|)=\psi(t)(1+\|u\|).$$

Therefore, assumption (H2) of Theorem 3.1 is satisfied, with $\psi(t)=\frac{1}{(t^2+2)^2}$, $t\in J$.

On the other hand, for any bounded set $W\subset c_0$, we have

$$\chi(f(t,W))\leq\frac{1}{(t^2+2)^2}\chi(W),\quad \text{for each } t\in J.$$

Hence (H3) is satisfied.

We shall check that condition (3.8) is satisfied. Indeed, $4\mathcal{M}_\psi = 0.616 < 1$ and $(1+R)\mathcal{M}_\psi \leq R$. Thus,

$$R \geq \frac{\mathcal{M}_\psi}{1-\mathcal{M}_\psi} = 1.6041.$$

Then R can be chosen as $R = 2 > 1.6041$. Consequently, Theorem 3.1 implies that problem (4.1) has at least one solution $u \in C(J, c_0)$.

5. Conclusions

We have proved the existence of solutions for certain classes of nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet boundary conditions in a given Banach space. The problem is issued by applying Darbo's fixed point theorem combined with the technique of Hausdorff measure of noncompactness. We also provide an example to make our results clear.

References

[1] S. Abbas, M. Benchohra and J. Henderson, *Weak solutions for implicit fractional differential equations of Hadamard type*, Adv. Dyn. Syst. Appl. **11**(3) (2016), 1–13.

[2] S. Abbas, M. Benchohra, N. Hamidi and J. Henderson, *Caputo-Hadamard fractional differential equations in Banach spaces,* Fract. Calc. Appl. Anal. **21**(4) (2018), 1027–1045.

[3] Y. Adjabi, F. Jarad, D. Baleanu and T. Abdeljawad, *On Cauchy problems with Caputo Hadamard fractional derivatives*, Journal of Computational Analysis and Applications **21**(4) (2016), 661–681.

[4] A. Aghajani, E. Pourhadi and J. J. Trujillo, *Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces*, Fract. Calc. Appl. Anal. **16**(4) (2013), 962–977.

[5] A. Aghajani and E. Pourhadi, *Application of measure of noncompactness to ℓ_1-solvability of infinite systems of second order differential equations*, Bull. Belg. Math. Soc. Simon Stevin **22**(1) (2015), 105–118.

[6] B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017.

[7] A. Ardjouni and A. Djoudi, *Positive solutions for nonlinear Caputo-Hadamard fractional differential equations with integral boundary conditions,* Open J. Math. Anal. **3**(1) (2019), 62–69.

[8] Y. Arioua and N. Benhamidouche, *Boundary value problem for Caputo-Hadamard fractional differential equations*, Surv. Math. Appl. **12** (2017), 103–115.

[9] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis.* John Wiley & Sons, New York, 1984.

[10] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel 1997.

[11] J. Banaŝ and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[12] J. Banaŝ, M. Jleli, M. Mursaleen, B. Samet and C. Vetro, *Advances in nonlinear analysis via the concept of measure of noncompactness*, Springer, Singapore, 2017.

[13] M. Benchohra, J. Henderson and D. Seba, *Measure of noncompactness and fractional differential equations in Banach spaces*, Comm. Pure Appl. Math. **12**(4) (2008), 419–427.

[14] M. Benchohra, S. Hamani and S. K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. **71** (2009), 2391–2396.

[15] M. Benchohra and F.-Z. Mostefai, *Weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces*, Opuscula Math. **32**(1) (2012), 31–40.

[16] M. Benchohra, S. Bouriah and J. J. Nieto, *Existence of periodic solutions for nonlinear implicit Hadamard's fractional differential equations*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **112**(1) (2018), 25–35.

[17] W. Benhamida, S. Hamani and J. Henderson, *Boundary value problems for Caputo-Hadamard fractional differential equations,* Advances in the Theory of Nonlinear Analysis and its Applications **2** (2018), 138–145.

[18] W. Benhamida, J. R. Graef and S. Hamani, *Boundary value problems for Hadamard fractional differential equations with nonlocal multi-point boundary conditions*, Fract. Differ. Calc. **8**(1) (2018), 165–176.

[19] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, Israel J. Math. **108** (1998), 109–138.

[20] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Semin. Mat. Univ. Padovaa **24** (1955), 84–92.

[21] C. Derbazi, H. Hammouche and M. Benchohra, *Weak solutions for some nonlinear fractional differential equations with fractional integral boundary conditions in Banach spaces,* Journal of Nonlinear Functional Analysis **2019** (2019), 1-11.

[22] Y. Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, *On Caputo modification of the Hadamard fractional derivatives*, Adv. Difference Equ. **2014** (2014), 12 pages.

[23] J. Hadamard, *Essai sur l'etude des fonctions donnees par leur developpment de Taylor*, Journal de Mathématiques Pures et Appliquées **8** (1892), 101-186.

[24] F. Jarad, T. Abdeljawad and D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Difference Equ. **2012** (2012), 8 pages.

[25] H.-P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7**(12) (1983), 1351–1371.

[26] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co. Inc., River Edge, New Jersey, 2000.

[27] P. Karthikeyan and R. Arul, *Integral boundary value problems for implicit fractional differential equations involving Hadamard and Caputo-Hadamard fractional derivatives*, Kragujevac J. Math. **45**(3) (2021), 331–341.

[28] A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38**(6) (2001), 1191–1204.

[29] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V., Amsterdam, 2006.

[30] I. A. Malik and T. Jalal, *Application of measure of noncompactness to infinite systems of differential equations in ℓ_p spaces,* Rend. Circ. Mat. Palermo (2) (2019), DOI 10.1007/s12215-019-00411-6

[31] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[32] M. Mursaleen and S. A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces*, Nonlinear Anal. **75**(4) (2012), 2111–2115.

[33] M. Mursaleen and S. M. H. Rizvi, *Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators*, Proc. Amer. Math. Soc. **144**(10) (2016), 4279–4289.

[34] M. Mursaleen, B. Bilalov and S. M. H. Rizvi, *Applications of measures of noncompactness to infinite system of fractional differential equations*, Filomat **31**(11) (2017), 3421–3432.

[35] M. Mursaleen, S. M. H. Rizvi and B. Samet, *Solvability of a class of boundary value problems in the space of convergent sequences,* Appl. Anal. **97**(11) (2018), 1829–1845.

[36] M. Mursaleen and S. M. H. Rizvi, *Existence results for second order linear differential equations in Banach spaces*, Appl. Anal. Discrete Math. **12**(2) (2018), 481–492.

[37] R. Saadati, E. Pourhadi and M. Mursaleen, *Solvability of infinite systems of third-order differential equations in c_0 by Meir-Keeler condensing operators,* J. Fixed Point Theory Appl. **21**(2) (2019), 16 pages.

[38] B. Hazarika, R. Arab and M. Mursaleen, *Application measure of noncompactness and operator type contraction for solvability of an infinite system of differential equations in ℓ_p-space,* Filomat **33**(7) (2019), 2181–2189.

[39] K. B. Oldham, *Fractional differential equations in electrochemistry*, Advances in Engineering Software **41**(1) (2010), 9–12.

[40] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1993.

[41] J. Sabatier, O. P. Agrawal and J. A. T. Machado, *Advances in Fractional Calculus-Theoretical Developments and Applications in Physics and Engineering,* Springer, Dordrecht, 2007.

[42] S. Schwabik and G. Ye, *Topics in Banach Space integration*, Series in Real Analysis **10**, World Scientific Publishing Co. Pte. Ltd., Hackensack, New Jersey, 2005.

[43] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg & Higher Education Press, Beijing, 2010.

[44] J. Tariboon, A. Cuntavepanit, S. K. Ntouyas and W. Nithiarayaphaks, *Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations*, J. Funct. Spaces **2018**, Article ID 6974046, 8 pages.

[45] J. Wang, L. Lv and Y. Zhou, *Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces*, J. Appl. Math. Comput. **38** (2012), 209–224.

[46] E. Zeidler, *Nonlinear Functional Analysis and its Applications, part II/B: Nonlinear Monotone Operators*, Springer Verlag, New York, 1989.

[1]Laboratory of Mathematics and Applied Sciences,
University of Ghardaia, 47000, Algeria
Email address: choukriedp@yahoo.com

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 857–864.

REGULARITY OF SEMIGROUPS IN TERMS OF HYBRID IDEALS AND HYBRID BI-IDEALS

BALASUBRAMANIAN ELAVARASAN[1] AND YOUNG BAE JUN[2]

Abstract. In this paper, we establish some equivalent conditions for a semigroup to be regular and intra-regular, in terms of hybrid ideals and hybrid bi-ideals. We also characterize the left and right simple and the completely regular semigroups utilizing hybrid ideals and hybrid bi-ideals. We show that a semigroup $\mathcal{S}$ is left simple if and only if it is hybrid left simple. We also prove that if a semigroup $\mathcal{S}$ is intra-regular, then for each hybrid ideal $\tilde{\jmath}_\mu$ of $\mathcal{S}$, we have $\tilde{\jmath}_\mu(r_1r_2) = \tilde{\jmath}_\mu(r_2r_1)$ for all $r_1, r_2 \in \mathcal{S}$.

1. Introduction

The fuzzy set theory is applicable to many mathematical branches which was introduced by Zadeh [11]. The fuzzification of algebraic structures and the introduction of the notion of fuzzy subgroups was inspired by Rosenfeld [10]. The definition given by Rosenfeld was a epochal turning point for pure mathematicians. Inspired by these studies, many authors have pursued the field of fuzzy algebraic structure in many different areas such as groups, rings, modules, vector spaces and so on (see [4, 5, 8]). Molodtsov [9] introduced the soft set theory as a new mathematical tool to deal with uncertainties. Many researchers have rigorously pursued and have studied extensively the fundamentals of soft set theory in recent years.

The notion of hybrid structure was introduced by Jun, Song and Muhiuddin [3] as a parallel circuit of fuzzy and soft sets. The notion of hybrid structure was introduced into a set of parameters on a initial universe set and it was applied to BCK/BCI algebras and linear spaces. In the year 2017, S. Anis, M. Khan and Y. B. Jun [1] gave

Key words and phrases. Semigroup, ideals, regular, hybrid structure, hybrid ideal, hybrid bi-ideal and hybrid product.

2010 *Mathematics Subject Classification.* Primary: 20M12. Secondary: 20M17, 06D72.

DOI 10.46793/KgJMat2206.857E

Received: February 18, 2019.

Accepted: April 21, 2020.

the notions of hybrid sub-semigroups and hybrid left (resp. right) ideals in semigroups and obtained various properties. Several equivalent conditions for a semi-group hybrid bi-ideal to be a semi-group ideal were obtained by B. Elavarasan and Y. B. Jun [2] in their study of the notion of semi-group hybrid bi-ideals and they have also studied some of its important properties.

In this paper, we obtain some equivalent characterizations of a regular and intra-regular semigroup in terms of their hybrid ideals and hybrid bi-ideals. We present necessary definitions to be used in the sequel in Section 2 from the already available literature. Section 3 is devoted to major results of the characterization of the regular and intra-regular semigroup through the properties of their hybrid and hybrid bi-ideals.

2. Preliminaries

In this section, we collect some basic notions and results on semigroup and hybrid structures.

Let $\mathcal{S}$ be a semigroup. Let V and W be subsets of $\mathcal{S}$. Then the multiplication of V and W is defined as $VW = \{vw : v \in V \text{ and } w \in W\}$. A non-empty subset W of $\mathcal{S}$ is called a subsemigroup of $\mathcal{S}$ if $W^2 \subseteq W$. A subsemigroup W of $\mathcal{S}$ is called a left (resp. right) ideal of $\mathcal{S}$ if $\mathcal{S}W \subseteq W$ (resp. $W\mathcal{S} \subseteq W$). If W is both a left and right ideal of $\mathcal{S}$, then W is called a two-sided ideal or ideal of $\mathcal{S}$. Clearly for any $r \in \mathcal{S}$, $L(r) = \{r \cup \mathcal{S}r\}$ (resp. $R(r) = \{r \cup r\mathcal{S}\}$, $I(r) = \{r \cup r\mathcal{S} \cup \mathcal{S}r \cup \mathcal{S}r\mathcal{S}\}$ is a left ideal (resp. right ideal, ideal) of $\mathcal{S}$ generated by r in $\mathcal{S}$. A semigroup $\mathcal{S}$ is called regular if for each $d \in \mathcal{S}$, $d = dtd$ for some t in $\mathcal{S}$. A semigroup $\mathcal{S}$ is called intra-regular if for each $d \in \mathcal{S}$, $d = sd^2t$ for some $s, t \in \mathcal{S}$ [6]. A subsemigroup W of $\mathcal{S}$ is called a bi-ideal of $\mathcal{S}$ if $W\mathcal{S}W \subseteq W$. It is clear that for any $d \in \mathcal{S}$, $B(d) = \{d, d^2, d\mathcal{S}d\}$ is a bi-ideal of $\mathcal{S}$ generated by d in $\mathcal{S}$.

Throughout this paper, we denote $\mathcal{S}$ is a semigroup, J is a unit interval and $\mathscr{P}(U)$ is the power set of an initial universal set U.

Definition 2.1 ([1])**.** A hybrid structure in $\mathcal{S}$ over U is a mapping $\tilde{j}_\alpha := (\tilde{j}, \alpha) : \mathcal{S} \to \mathscr{P}(U) \times J$, $d \mapsto (\tilde{j}(d), \alpha(d))$, where $\tilde{j} : \mathcal{S} \to \mathscr{P}(U)$ and $\alpha : \mathcal{S} \to J$ are mappings.

For any semigroup $\mathcal{S}$, $HY(\mathcal{S})$ denote the set of all hybrid structures in $\mathcal{S}$ over U. We define an order $\ll$ in $HY(\mathcal{S})$ as follows.

For all $\tilde{j}_\nu, \tilde{k}_\mu \in HY(\mathcal{S})$, $\tilde{j}_\nu \ll \tilde{k}_\mu$ if and only if $\tilde{j} \mathrel{\tilde{\subseteq}} \tilde{k}$, $\nu \succeq \mu$, where $\tilde{j} \mathrel{\tilde{\subseteq}} \tilde{k}$ means that $\tilde{j}(d) \subseteq \tilde{k}(d)$ and $\nu \succeq \mu$ means that $\nu(d) \geq \mu(d)$ for all $d \in \mathcal{S}$.

For any $r_1, r_2 \in \mathcal{S}$, $\tilde{j}_\nu(r_1) = \tilde{k}_\mu(r_2)$ if and only if $\tilde{j}_\nu(r_1) \ll \tilde{k}_\mu(r_2)$ and $\tilde{k}_\mu(r_2) \ll \tilde{j}_\nu(r_1)$, where $\tilde{j}_\nu(r_1) \ll \tilde{k}_\mu(r_2)$ means that $\tilde{j}(r_1) \subseteq \tilde{k}(r_2)$ and $\nu(r_1) \geq \mu(r_2)$. Also, $\tilde{j}_\nu = \tilde{k}_\mu$ if and only if $\tilde{j}_\nu \ll \tilde{k}_\mu$ and $\tilde{k}_\mu \ll \tilde{j}_\nu$. Note that $(HY(\mathcal{S}), \ll)$ is a poset.

Remark 2.1. For a family of real numbers $\{a_i : i \in \alpha\}$, we define

$$\bigvee\{a_i : i \in \alpha\} := \begin{cases} \max\{a_i : i \in \alpha\}, & \text{if } \alpha \text{ is finite,} \\ \sup\{a_i : i \in \alpha\}, & \text{otherwise,} \end{cases}$$

and

$$\bigwedge\{a_i : i \in \alpha\} := \begin{cases} \min\{a_i : i \in \alpha\}, & \text{if } \alpha \text{ is finite,} \\ \inf\{a_i : i \in \alpha\}, & \text{otherwise.} \end{cases}$$

For real numbers r and k, we also use $r \vee k$ and $r \wedge k$ instead of $\bigvee\{r, k\}$ and $\bigwedge\{r, k\}$, respectively.

Definition 2.2 ([2])**.** A hybrid subsemigroup $\tilde{j}_\alpha$ in $\mathcal{S}$ over U is called a hybrid bi-ideal if for all $m, y, n \in \mathcal{S}$

(i) $\tilde{j}(myn) \supseteq \tilde{j}(m) \cap \tilde{j}(n)$;
(ii) $\alpha(myn) \leq \alpha(m) \vee \alpha(n)$.

Definition 2.3 ([1])**.** Let $\tilde{j}_\alpha \in HY(\mathcal{S})$. Then $\tilde{j}_\alpha$ is called a hybrid left (resp. right) ideal if for all $m, n \in \mathcal{S}$

(i) $\tilde{j}(mn) \supseteq \tilde{j}(n)$ (resp. $\tilde{j}(mn) \supseteq \tilde{j}(m)$);
(ii) $\alpha(mn) \leq \alpha(n)$ (resp. $\alpha(mn) \leq \alpha(m)$).

If $\tilde{j}_\alpha$ is both a hybrid left and hybrid right ideal of $\mathcal{S}$, then it is called a hybrid ideal of $\mathcal{S}$.

Definition 2.4. A semigroup $\mathcal{S}$ is called hybrid left (resp. right) duo over U if every hybrid left (resp. right) ideal of $\mathcal{S}$ over U is a hybrid ideal of $\mathcal{S}$ over U. $\mathcal{S}$ is called hybrid duo over U if it is both hybrid left and hybrid right duo over U.

Definition 2.5 ([1])**.** Let $\phi \neq B \subseteq \mathcal{S}$ and $\tilde{j}_\nu \in HY(\mathcal{S})$. Then the characteristic hybrid structure is denoted by $\chi_B(\tilde{j}_\nu) = (\chi_B(\tilde{j}), \chi_B(\nu))$, where $\chi_B(\tilde{j}) : \mathcal{S} \to \mathscr{P}(U)$,

$$d \mapsto \begin{cases} U, & \text{if } d \in B, \\ \phi, & \text{otherwise,} \end{cases}$$

and $\chi_B(\nu) : \mathcal{S} \to J$,

$$d \mapsto \begin{cases} 0, & \text{if } d \in B, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.6 ([1])**.** Let $\tilde{j}_\alpha, \tilde{k}_\beta \in HY(\mathcal{S})$.

(i) The hybrid product of $\tilde{j}_\alpha$ and $\tilde{k}_\beta$ in $\mathcal{S}$ is defined as $\tilde{j}_\alpha \odot \tilde{k}_\beta = (\tilde{j}\tilde{\circ}\tilde{k}, \alpha\tilde{\circ}\beta)$ in $\mathcal{S}$ over U, where

$$(\tilde{j}\tilde{\circ}\tilde{k})(d) = \begin{cases} \bigcup\limits_{d=r_1r_2} \{\tilde{j}(r_1) \cap \tilde{k}(r_2)\}, & \text{if exist } r_1, r_2 \in \mathcal{S} \text{ such that } d = r_1r_2, \\ \phi, & \text{otherwise,} \end{cases}$$

and

$$(\alpha\tilde{\circ}\beta)(d) = \begin{cases} \bigwedge\limits_{d=r_1r_2} \{\alpha(r_1) \vee \beta(r_2)\} & \text{if exist } r_1, r_2 \in \mathcal{S} \text{ such that } d = r_1r_2, \\ 1 & \text{otherwise,} \end{cases}$$

for all $d \in \mathcal{S}$.

(ii) The hybrid intersection of $\tilde{j}_\alpha$ and $\tilde{k}_\beta$, denoted by $\tilde{j}_\alpha \Cap \tilde{k}_\beta$, is defined to be a hybrid structure $\tilde{j}_\alpha \Cap \tilde{k}_\beta : \mathcal{S} \to \mathscr{P}(U) \times I$, $d \mapsto ((\tilde{j}\tilde{\cap}\tilde{k})(r_1), (\alpha \vee \beta)(r_1))$, where $\tilde{j}\tilde{\cap}\tilde{k} : \mathcal{S} \longrightarrow \mathscr{P}(U)$, $r_1 \mapsto \tilde{j}(r_1) \cap \tilde{k}(r_1)$ and $\alpha \vee \beta : \mathcal{S} \to I$, $r_1 \mapsto \alpha(r_1) \vee \beta(r_1)$.

3. Hybrid Structures in Regular and Intra-Regular Semigroups

Lemma 3.1. ([1, Lemma 3.11]). *For subsets M and N of $\mathcal{S}$, we have*
(*i*) $\chi_M(\tilde{j}_\nu) \Cap \chi_N(\tilde{j}_\nu) = \chi_{M\cap N}(\tilde{j}_\nu)$;
(*ii*) $\chi_M(\tilde{j}_\nu) \odot \chi_N(\tilde{j}_\nu) = \chi_{MN}(\tilde{j}_\nu)$.

Theorem 3.1. ([1, Proposition 3.21]). *If $\tilde{j}_\nu$ and $\tilde{k}_\mu$ are hybrid right ideal and hybrid left ideal of $\mathcal{S}$ over U, respectively, then $\tilde{j}_\nu \odot \tilde{k}_\mu \ll \tilde{j}_\nu \Cap \tilde{k}_\mu$.*

We now obtain theorems on the characterizations of a regular and intra-regular semigroup in terms of different hybrid ideals of semigroups.

Theorem 3.2. *For any $\mathcal{S}$, the following assertions are equivalent:*
(*i*) *$\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$;*
(*ii*) *$\mathcal{S}$ is regular and intra-regular.*

Proof. (*i*) $\Rightarrow$ (*ii*) If $\tilde{j}_\nu$ is a hybrid right ideal and $\tilde{k}_\mu$ is a hybrid left ideal of $\mathcal{S}$, then $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ and by Theorem 3.1, we can get $\tilde{j}_\nu \odot \tilde{k}_\mu \ll \tilde{j}_\nu \Cap \tilde{k}_\mu$. So, for any hybrid right ideal $\tilde{j}_\nu$ and hybrid left ideal $\tilde{k}_\mu$ of $\mathcal{S}$, we have $\tilde{j}_\nu \Cap \tilde{k}_\mu = \tilde{j}_\nu \odot \tilde{k}_\mu$.

Let E be a right ideal and D be a left ideal of $\mathcal{S}$ and $t \in E \cap D$. Then $\chi_E(\tilde{j}_\nu)(t) \Cap \chi_D(\tilde{j}_\nu)(t) = \chi_E(\tilde{j}_\nu)(t) \odot \chi_D(\tilde{j}_\nu)(t)$. By Lemma 3.1 (*i*), we have $\chi_{E\cap D}(\tilde{j}_\nu) = \chi_{ED}(\tilde{j}_\nu)$. Since $t \in E \cap D$ and $\chi_{E\cap D}(\tilde{j})(t) = U$, we have $\chi_{ED}(\tilde{j})(t) = U$, which implies $t \in ED$. Thus, $E \cap D \subseteq ED \subseteq E \cap D$ and hence $E \cap D = ED$. Therefore, $\mathcal{S}$ is regular. Also for $t \in \mathcal{S}$, we have $\chi_{B(t)}(\tilde{j}_\nu) \Cap \chi_{B(t)}(\tilde{j}_\nu) \ll \chi_{B(t)}(\tilde{j}_\nu) \odot \chi_{B(t)}(\tilde{j}_\nu)$. Again by Lemma 3.1 (*i*), we have $\chi_{B(t)}(\tilde{j}_\nu) \ll \chi_{B(t)B(t)}(\tilde{j}_\nu)$. Since $\chi_{B(t)}(\tilde{j})(t) = U$, we have $\chi_{B(t)B(t)}(\tilde{j})(t) = U$ which implies $t \in B(t)B(t)$, so $\mathcal{S}$ is intra-regular.

(*ii*) $\Rightarrow$ (*i*) Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$. Suppose $r_1 \in \mathcal{S}$. Then for some $x, y, z \in \mathcal{S}$, $r_1 = r_1xr_1 = r_1xr_1xr_1$ and $r_1 = yr_1^2z$, which imply $r_1 = r_1xyr_1^2zxr_1$. Now

$$\tilde{j}\tilde{\circ}\tilde{k} = \bigcup_{r_1=uv} \{\tilde{j}(u) \cap \tilde{k}(v)\} \supseteq \tilde{j}(r_1xyr_1) \cap \tilde{k}(r_1zxr_1) \supseteq \{\tilde{j}(r_1) \cap \tilde{j}(r_1)\} \cap \{\tilde{k}(r_1) \cap \tilde{k}(r_1)\}$$
$$= \tilde{j}(r_1) \cap \tilde{k}(r_1) = (\tilde{j} \cap \tilde{k})(r_1).$$

Also,

$$(\nu\tilde{\circ}\mu)(r_1) = \bigwedge_{r_1=uv} \{\nu(u) \vee \mu(v)\} \leq \nu(r_1xyr_1) \vee \mu(r_1zxr_1)$$
$$= \{\nu(r_1) \vee \nu(r_1)\} \vee \{\mu(r_1) \vee \mu(r_1)\} = \nu(r_1) \vee \mu(r_1) = (\nu \cap \mu)(r_1).$$

Therefore, $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$. □

Theorem 3.3. *For any $\mathcal{S}$, the below assertions are equivalent:*
(*i*) *$\tilde{j}_\nu \Cap \tilde{k}_\mu \ll (\tilde{j}_\nu \odot \tilde{k}_\mu) \Cap (\tilde{k}_\mu \odot \tilde{j}_\nu)$ for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$;*
(*ii*) *$\mathcal{S}$ is intra-regular and regular.*

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$. Then, by Theorem 3.2, we can have $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ and $\tilde{j}_\nu \Cap \tilde{k}_\mu = \tilde{k}_\mu \Cap \tilde{j}_\nu \ll \tilde{k}_\mu \odot \tilde{j}_\nu$, which imply $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll (\tilde{j}_\nu \odot \tilde{k}_\mu) \Cap (\tilde{k}_\mu \odot \tilde{j}_\nu)$.

$(ii) \Rightarrow (i)$ By (ii), $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu$ for every hybrid bi-ideals $\tilde{j}_\nu$ and $\tilde{k}_\mu$ of $\mathcal{S}$. By Theorem 3.2, we get $\mathcal{S}$ is regular and intra-regular. □

Theorem 3.4. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) $\tilde{i}_\nu \Cap \tilde{k}_\mu \ll \tilde{i}_\nu \odot \tilde{k}_\mu \odot \tilde{i}_\nu$ *for every hybrid bi-ideals* $\tilde{i}_\nu$ *and* $\tilde{k}_\mu$ *of* $\mathcal{S}$*;*

(ii) $\mathcal{S}$ *is intra-regular and regular.*

Proof. $(i) \Rightarrow (ii)$ Let $w \in \mathcal{S}$. Then by assumption and by Lemma 3.1, we can get $\chi_{B(w)}(\tilde{i}_\nu) \ll \chi_{B(w)}(\tilde{i}_\nu) \odot \chi_{B(w)}(\tilde{i}_\nu) \odot \chi_{B(w)}(\tilde{i}_\nu) = \chi_{B(w)B(w)B(w)}(\tilde{i}_\nu)$. Since $\chi_{B(w)}(\tilde{i})(w) = U$, $\chi_{B(w)B(w)B(w)}(\tilde{i})(w) = U$ implies $w \in B(w)B(w)B(w)$. Therefore, $\mathcal{S}$ is regular and intra-regular.

$(ii) \Rightarrow (i)$ Let $\tilde{j}_\nu$ and $\tilde{k}_\mu$ be hybrid bi-ideals of $\mathcal{S}$ and let $w \in \mathcal{S}$. Then there exist $r_1, r_2, r_3 \in \mathcal{S}$ such that $w = wr_1w = wr_1wr_1w$ and $w = r_2w^2r_3$, which imply $w = wr_1r_2w^2r_3r_1r_2w^2r_3r_1w = (wr_1r_2w)(wr_3r_1r_2w)(wr_3r_1w)$. Then

$$
\begin{aligned}
(\tilde{j}\tilde{\circ}\tilde{k}\tilde{\circ}\tilde{j})(w) = & \bigcup_{w=w_1w_2} \{\tilde{j}(w_1) \cap (\tilde{k}\tilde{\circ}\tilde{j})(w_2)\} \\
\supseteq & \tilde{j}(wr_1r_2w) \cap (\tilde{k}\tilde{\circ}\tilde{j})((wr_3r_1r_2w)(wr_3r_1w)) \\
\supseteq & \{\tilde{j}(w) \cap \tilde{j}(w)\} \cap \{\tilde{k}(wr_3r_1r_2w) \cap \tilde{j}(wr_3r_1w)\} \\
\supseteq & \tilde{j}(w) \cap \tilde{k}(w) \cap \tilde{j}(w) = \tilde{j}(w) \cap \tilde{k}(w) = (\tilde{j} \cap \tilde{k})(w).
\end{aligned}
$$

Also,

$$
\begin{aligned}
(\nu\tilde{\circ}\mu\tilde{\circ}\nu)(w) = & \bigwedge_{w=w_1w_2} \{\nu(w_1) \vee (\mu\tilde{\circ}\nu)(w_2)\} \\
\leq & \nu(wr_1r_2w) \vee (\mu\tilde{\circ}\nu)((wr_3r_1r_2w)(wr_3r_1w)) \\
\leq & \{\nu(w) \vee \nu(w)\} \vee \{\mu(wr_3r_1r_2w) \vee \nu(wr_3r_1w)\} \\
\leq & \bigvee\{\nu(w), \mu(w), \nu(w)\} \\
= & \nu(w) \vee \mu(w) = (\nu \cap \mu)(w).
\end{aligned}
$$

Therefore, $\tilde{j}_\nu \Cap \tilde{k}_\mu \ll \tilde{j}_\nu \odot \tilde{k}_\mu \odot \tilde{j}_\nu$. □

Theorem 3.5. ([1, Theorem 3.5]). *For $\varnothing \neq M \subset \mathcal{S}$, the following assertions are equivalent:*

(i) $\chi_M(\tilde{i}_\nu)$ *in* $\mathcal{S}$ *is hybrid right* (*resp. left*) *ideal;*

(ii) M *is a right* (*resp. left*) *ideal of* $\mathcal{S}$.

Theorem 3.6. ([2, Theorem 2.12]). *Let $\mathcal{S}$ be a regular right duo* (*resp. left duo, duo*) *semigroup. Then the following assertions are equivalent:*

(i) $\tilde{i}_\nu$ *is a hybrid left ideal* (*resp. right ideal*) *of* $\mathcal{S}$*;*

(ii) $\tilde{i}_\nu$ *is a hybrid bi-ideal of* $\mathcal{S}$.

Theorem 3.7. *For any $\mathcal{S}$, the following assertions are equivalent:*
(i) *if $\tilde{k}_\mu$ is a hybrid ideal of $\mathcal{S}$, then $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ for all $r \in \mathcal{S}$;*
(ii) *$\mathcal{S}$ is intra-regular.*

Proof. (i) $\Rightarrow$ (ii) Let $r \in \mathcal{S}$. Then $I(r^2)$ is an ideal of $\mathcal{S}$ and by Theorem 3.5, $\chi_{I(r^2)}(\tilde{k}_\mu)(r) = \chi_{I(r^2)}(\tilde{k}_\mu)(r^2)$. Since $r^2 \in I(r^2)$, we have $\chi_{I(r^2)}(\tilde{k})(r^2) = U$ implies that $\chi_{I(r^2)}(\tilde{k}_\mu)(r) = U$. Thus, $r \in I(r^2)$ and hence $\mathcal{S}$ is intra-regular.

(ii) $\Rightarrow$ (i) Let $r \in \mathcal{S}$. Then there exist $x, y \in \mathcal{S}$ such that $r = xr^2y$. Let $\tilde{k}_\mu$ be a hybrid ideal of $\mathcal{S}$. Then $\tilde{k}_\mu(r) \ll \tilde{k}_\mu(r^2)$. Also,

$$\tilde{k}(r) = \tilde{k}(xr^2y) = \tilde{k}((xrr)y) \supseteq \tilde{k}(xrr) \supseteq \tilde{k}(rr) = \tilde{k}(r^2)$$

and $\mu(r) = \mu(xr^2y) = \mu((xrr)y) \leq \mu(xrr) \leq \mu(rr) = \mu(r^2)$. Thus, $\tilde{k}_\mu(r^2) \ll \tilde{k}_\mu(r)$ and hence $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$. □

As a consequence of above two theorems, we have the following result.

Corollary 3.1. *If $\mathcal{S}$ is a regular duo, then the following assertions are equivalent:*
(i) *if $\tilde{k}_\nu$ is a hybrid bi-ideal in $\mathcal{S}$, then $\tilde{k}_\nu(r) = \tilde{k}_\nu(r^2)$ for all $r \in \mathcal{S}$;*
(ii) *$\mathcal{S}$ is intra-regular.*

Theorem 3.8. *If $\mathcal{S}$ is an intra-regular semigroup, then for every hybrid ideal $\tilde{k}_\nu$ in $\mathcal{S}$, we have $\tilde{k}_\nu(r_1r_2) = \tilde{k}_\nu(r_2r_1)$ for all $r_1, r_2 \in \mathcal{S}$.*

Proof. Let $\tilde{k}_\nu$ be a hybrid ideal in $\mathcal{S}$. Then by Theorem 3.7, we get $\tilde{k}_\nu(r_1r_2) = \tilde{k}_\nu((r_1r_2)^2)$ and $\tilde{k}_\nu(r_2r_1) = \tilde{k}_\nu((r_2r_1)^2)$. Now

$$\tilde{k}(r_1r_2) = \tilde{k}(r_1r_2r_1r_2) \supseteq \tilde{k}(r_1r_2r_1) \supseteq \tilde{k}(r_2r_1)$$

and $\tilde{k}(r_2r_1) = \tilde{k}(r_2r_1r_2r_1) \supseteq \tilde{k}(r_2r_1r_2) \supseteq \tilde{k}(r_1r_2)$. So, $\tilde{k}(r_1r_2) = \tilde{k}(r_2r_1)$. Also,

$$\nu(r_1r_2) = \nu((r_1r_2)^2) = \nu(r_1r_2r_1r_2) \leq \nu(r_2r_1)$$

and $\nu(r_2r_1) = \nu((r_2r_1)^2) = \nu(r_2r_1r_2r_1) \leq \nu(r_1r_2r_1) \leq \nu(r_1r_2)$, so $\nu(r_1r_2) = \nu(r_2r_1)$. Therefore, $\tilde{k}_\nu(ab) = \tilde{k}_\nu(r_2r_1)$. □

A semigroup $\mathcal{S}$ is called completely regular if for each $r \in \mathcal{S}$, there exists $t \in \mathcal{S}$ such that $r = rtr$ and $rt = tr$. $\mathcal{S}$ is called left (resp. right) regular if for each $r \in \mathcal{S}$ there exists $t \in \mathcal{S}$ such that $r = tr^2$ (resp. $r = r^2t$) [7].

Theorem 3.9. *For any $\mathcal{S}$, the following statements are equivalent:*
(i) *for every hybrid left ideal $\tilde{k}_\mu$ (resp. right ideal) of $\mathcal{S}$, we have $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ for all $r \in \mathcal{S}$;*
(ii) *$\mathcal{S}$ is left regular (resp. right regular).*

Proof. (i) $\Rightarrow$ (ii) Let $\tilde{k}_\mu$ be a hybrid left ideal in $\mathcal{S}$. Then for any $r \in \mathcal{S}$, we have $\chi_{L(r^2)}(\tilde{k}_\mu)(r) = \chi_{L(r^2)}(\tilde{k}_\mu)(r^2)$ by Theorem 3.5. Since $r^2 \in L(r^2)$, we have $\chi_{L(r^2)}(\tilde{k})(r^2) = U$ implies that $\chi_{L(r^2)}(\tilde{k}_\mu)(r) = U$. Thus, $r \in L(r^2)$ and hence $\mathcal{S}$ is left-regular.

$(ii) \Rightarrow (i)$ Let $s \in \mathcal{S}$. Then there exists $x \in \mathcal{S}$ such that $s = xs^2$. Let $\tilde{k}_\mu$ be a hybrid left ideal $\mathcal{S}$. Then $\tilde{k}(s) = \tilde{k}(xs^2) \supseteq \tilde{k}(s^2) \supseteq \tilde{k}(s)$ and $\mu(s) = \mu(xs^2) \leq \mu(s^2) \leq \mu(s)$. So, $\tilde{k}_\mu(s) = \tilde{k}_\mu(s^2)$. □

Corollary 3.2. *If $\mathcal{S}$ is regular left duo (resp. right duo), then the below assertions are equivalent:*

(i) *if $\tilde{k}_\mu$ is a hybrid bi-ideal of $\mathcal{S}$, then $\tilde{k}_\mu(r) = \tilde{k}_\mu(r^2)$ for all $r \in \mathcal{S}$;*

(ii) *$\mathcal{S}$ is right regular.*

Proof. It is evident from Theorem 3.6 and Theorem 3.9. □

Theorem 3.10. ([2, Theorem 2.9]). *Let $\phi \neq X \subseteq \mathcal{S}$. Then X is bi-ideal if and only if $\chi_X(\tilde{i}_\lambda)$ is hybrid bi-ideal.*

The equivalent conditions for complete regularity of a semigroup are given below.

Theorem 3.11. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) *for each hybrid bi-ideal $\tilde{j}_\mu$ in $\mathcal{S}$, we have $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$ for all $r \in \mathcal{S}$;*

(ii) *$\mathcal{S}$ is completely regular;*

(iii) *for every hybrid left ideal $\tilde{j}_\mu$ and hybrid right ideal $\tilde{k}_\lambda$ of $\mathcal{S}$, we have $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$ and $\tilde{k}_\lambda(r) = \tilde{k}_\lambda(r^2)$ for all $r \in \mathcal{S}$.*

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{j}_\mu$ be a hybrid bi-ideal of $\mathcal{S}$. Then for any $r \in \mathcal{S}$, by Theorem 3.10 we have $\chi_{B(r^2)}(\tilde{j}_\mu)(r) = \chi_{B(r^2)}(\tilde{j}_\mu)(r^2)$. Since $r^2 \in B(r^2)$, we have $\chi_{B(r^2)}(\tilde{j})(r^2) = U$ implies that $\chi_{B(r^2)}(\tilde{j}_\mu)(r) = U$. Thus, $r \in B(r^2)$ and hence $\mathcal{S}$ is completely regular.

$(ii) \Rightarrow (i)$ Let $r \in \mathcal{S}$. Then there exists $q \in \mathcal{S}$ such that $r = r^2qr^2$. Let $\tilde{j}_\mu$ be a hybrid bi-ideal in $\mathcal{S}$. Then $\tilde{j}(r) = \tilde{j}(r^2qr^2) = \tilde{j}(r(rqr)r) \supseteq \tilde{j}(r) \cap \tilde{j}(r) = \tilde{j}(r)$ and $\mu(r) = \mu(r^2qr^2) = \mu(r(rqr)r) \leq \mu(r) \vee \mu(r) = \mu(r)$. Therefore, $\tilde{j}_\mu(r) = \tilde{j}_\mu(r^2)$.

$(ii) \Leftrightarrow (iii)$ It is evident from Theorem 3.9. □

A semigroup $\mathcal{S}$ is said to be right (resp. left) simple if $\mathcal{S}$ has no proper right (resp. left) ideals of $\mathcal{S}$. A semigroup $\mathcal{S}$ is said to be simple if $\mathcal{S}$ has no proper ideals.

A hybrid structure $\tilde{j}_\nu$ in $\mathcal{S}$ is called a constant function if $\tilde{j} : \mathcal{S} \to \mathscr{P}(U)$ and $\nu : \mathcal{S} \to J$ are constant mappings.

A semigroup $\mathcal{S}$ is called hybrid right (resp. left) simple if every hybrid right (resp. left) ideal of $\mathcal{S}$ over U is a constant function, and hybrid simple if every hybrid ideal of $\mathcal{S}$ is a constant function.

Theorem 3.12. *For any $\mathcal{S}$, the following assertions are equivalent:*

(i) *$\mathcal{S}$ is hybrid left simple (resp. hybrid right simple, hybrid simple);*

(ii) *$\mathcal{S}$ is left simple (resp. right simple, simple).*

Proof. $(i) \Rightarrow (ii)$ Suppose $\mathcal{S}$ is hybrid left simple and let D be a left ideal of $\mathcal{S}$. Then $\chi_D(\tilde{k}_\mu)$ is hybrid left ideal. Since $D \neq \phi$, the constant value is U. So, every element of $\mathcal{S}$ is in D. Thus, $\mathcal{S} = D$ and hence $\mathcal{S}$ is left simple.

$(ii) \Rightarrow (i)$ Suppose $\mathcal{S}$ is hybrid left simple. Then for any $w_1, w_2 \in \mathcal{S}$, we have $\mathcal{S}w_1 = \mathcal{S} = \mathcal{S}w_2$ implies $w_2 = xw_1$ and $w_1 = yw_2$ for some $x, y \in \mathcal{S}$. Let $\tilde{k}_\mu$ be a hybrid left ideal of $\mathcal{S}$. Then $\tilde{k}(w_1) = \tilde{k}(yw_2) \supseteq \tilde{k}(w_2) = \tilde{k}(xw_1) \supseteq \tilde{k}(w_1)$ implies $\tilde{k}(w_1) = \tilde{k}(w_2)$. Also, $\mu(w_1) = \mu(yw_2) \leq \mu(w_2) = \mu(xw_1) \leq \mu(w_1)$ implies $\mu(w_1) = \mu(w_2)$. Thus, $\tilde{k}_\mu$ is a constant function and hence $\mathcal{S}$ is hybrid left simple. □

Theorem 3.13. *Let $\mathcal{S}$ be a hybrid left (resp. right) simple semigroup. Then every hybrid bi-ideal is hybrid right ideal (resp. hybrid left ideal).*

Proof. Let $\tilde{k}_\mu$ be any hybrid bi-ideal and $u, v \in \mathcal{S}$. Then $\mathcal{S}u = \mathcal{S}$ and there exists $k \in \mathcal{S}$ such that $v = ku$, so $uv = uku$. Now $\tilde{k}(uv) = \tilde{k}(uku) \supseteq \tilde{k}(u) \cap \tilde{k}(u) = \tilde{k}(u)$ and $\mu(uv) = \mu(uku) \leq \mu(u)$. Therefore, $\tilde{k}_\mu$ is hybrid right ideal. □

Acknowledgements. The authors express their heartfelt gratitude to the referees for valuable comments and suggestions which significantly improve the paper.

References

[1] S. Anis, M. Khan and Y. B. Jun, *Hybrid ideals in semigroups*, Cogent Mathematics **4**(1) (2017), Paper ID 1352117, 12 pages.
[2] B. Elavarasan and Y. B. Jun, *On hybrid ideals and hybrid bi-ideals in semigroups*, Iran. J. Math. Sci. Inform. (to appear).
[3] Y. B. Jun, S. Z. Sang, and G. Muhiuddin, *Hybrid structures and applications*, Annals of Communications in Mathematics **1**(1) (2018), 11–25.
[4] N. Kehayopulu and M. Tsingelis, *Fuzzy bi-ideals in ordered semigroups*, Inform. Sci. **171** (2005), 13–28.
[5] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems **5** (1981), 203–215.
[6] S. Lajos, *A note on intra-regular semigroups*, Proc. Japan Acad. **39**(9) (1963), 626–627.
[7] M. Mitrović, *Regular subsets of semigroups related to their idempotents*, Semigroup Forum **70** (2005), 356–360.
[8] T. K. Mukherjee and M. K. Sen, *Prime fuzzy ideals in rings*, Fuzzy Sets and Systems **32** (1989), 337–341.
[9] D. Molodtsov, *Soft set theory - First results*, Comput. Math. Appl. **37** (1999), 19–31.
[10] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
[11] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

[1]Department of Mathematics,
Karunya Institute of Technology and Sciences,
Coimbatore - 641 114, India
Email address: belavarasan@gmail.com

[2]Department of Mathematics,
Gyeongsang National University,
Chinju 660-701, Korea
Email address: skywine@gmail.com

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 865–882.

ON GENERALIZED LAGRANGE-BASED APOSTOL-TYPE AND RELATED POLYNOMIALS

WASEEM A. KHAN[1]

Abstract. In this article, we introduce a new class of generalized polynomials, ascribed to the new families of generating functions and identities concerning Lagrange, Hermite, Miller-Lee, and Laguerre polynomials and of their associated forms. It is shown that the proposed method allows the derivation of sum rules involving products of generalized polynomials and addition theorems. We develop a point of view based on generating relations, exploited in the past, to study some aspects of the theory of special functions. The possibility of extending the results to include generating functions involving products of Lagrange-based unified Apostol-type and other polynomials is finally analyzed.

1. Introduction

The Lagrange polynomials in several variables, which are known as the Chan-Chyan-Srivastava polynomials [2] are defined by means of the following generating function

$$\prod_{j=1}^{r}(1-x_jt)^{-\alpha_j}=\sum_{n=0}^{\infty}g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)t^n, \tag{1.1}$$

where $\alpha_j\in\mathbb{C}$, $j=1,\ldots,r$, $|t|<\min\{|x_1|^{-1},\ldots,|x_r|^{-1}\}$, and are represented by

$$\begin{aligned}g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,x_2,\ldots,x_r)=&\sum_{k_{r-1=0}}^{n}\cdots\sum_{k_2}^{k_3}\sum_{k_1}^{k_2}(\alpha_1)_{k_1}\\&\times(\alpha_2)_{k_2-k_1}\cdots(\alpha_{r-1})_{k_{r-1}-k_{r-2}}(\alpha_r)_{n-k_{r-1}}\end{aligned}$$

Key words and phrases. Lagrange polynomials, Lagrange-Hermite polynomials, Lagrange-based unified Apostol type polynomials, Miller-Lee polynomials, Laguerre polynomials.

2010 *Mathematics Subject Classification.* Primary: 11B68, 33C45.

DOI 10.46793/KgJMat2206.865K

Received: January 20, 2020.

Accepted: May 06, 2020.

$$\times \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!}\frac{x_r^{n-k_r-1}}{(n-k_r-1)!}, \tag{1.2}$$

where $(\lambda)_0 := 1$ and $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$, $n \in \mathbb{N} := \{1,2,3,\dots\}$.

Altin and Erkus [1] presented a multivariable extension of the so called Lagrange-Hermite polynomials generated by (see [1, page 239, (1.2)]):

$$\prod_{j=1}^{r}(1-x_jt^j)^{-\alpha_j} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots x_r)t^n, \tag{1.3}$$

where $\alpha_j \in \mathbb{C}$, $j=1,\dots,r$, $|t| < \min\{|x_1|^{-1}, |x_2|^{-\frac{1}{2}},\dots,|x_r|^{-\frac{1}{r}}\}$ and

$$h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{k_1+2k_2+\cdots+rk_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}.$$

The special case when $r=2$ in (1.3) is essentially a case which corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli et al. [3]

$$(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n. \tag{1.4}$$

The multi-variable (Erkus-Srivastava) polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ defined by the following generating function, (see [5, page 268, (3))]:

$$\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j} = \sum_{n=0}^{\infty} U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)t^n, \tag{1.5}$$

where $\alpha_j \in \mathbb{C}$, $j=1,\cdots,r$, $l_j \in \mathbb{N}$, $j=1,\dots,r$, $|t| < \min\{|x_1|^{-1},\dots,|x_r|^{-1}\}$, are a unification (and generalization) of several known families of multivariable polynomials including (for example) the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ defined by (1.1) (see, for details, [5]). It is evident that the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ and the Lagrange-Hermite polynomials $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ follow as the special cases of the Erkus-Srivastava polynomials $U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ when $l_j=1$, $j=1,\dots,r$.

The generating function (1.5) yields the following explicit representation (see [5, page 268, (4)]):

$$U_{n,l_1,\dots,l_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) = \sum_{l_1k_1+\cdots+l_rk_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!},$$

which, in the special case when $l_j=1$, $j=1,\dots,r$, corresponds to (1.2).

Recently, Ozarslan [14] introduced the following unification of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Explicitly Ozarslan studied the following generating function:

$$f_{a,b}^{(\alpha)}(x;t,a,b) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{1.6}$$

where

$$\left|t + b\ln\left(\frac{\beta}{\alpha}\right)\right| < 2\pi, \quad k \in \mathbb{N}_0, \quad a, b \in \mathbb{R}\backslash\{0\}, \quad \alpha, \beta \in \mathbb{C}.$$

For $\alpha = 1$ in (1.6), we get

$$f_{a,b}(x; t, a, b) = \frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}(x; k, a, b)\frac{t^n}{n!}, \tag{1.7}$$

where

$$\left|t + b\ln\left(\frac{\beta}{\alpha}\right)\right| < 2\pi, \quad k \in \mathbb{N}_0, \quad a, b \in \mathbb{R}\backslash\{0\}, \quad \alpha, \beta \in \mathbb{C}.$$

From (1.6) and (1.7), we have

$$P^{(1)}_{n,\beta}(x; k, a, b) = P_{n,\beta}(x; k, a, b), \quad n \in \mathbb{N},$$

which is defined by Ozden and Simsek [16]. Now Ozden et al. [15] introduced many properties of these polynomials. We give some specific special cases.

1. By substituting $a = b = k = 1$ and $\beta = \lambda$ into (1.6), one has the Apostol-Bernoulli polynomials $P^{(\alpha)}_{n,\beta}(x; 1, 1, 1) = P^{(\alpha)}_{n,\lambda}(x; 1, 1, 1)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B^{(\alpha)}_n(x; \lambda)\frac{t^n}{n!}, \quad |t + \log\lambda| < 2\pi, \tag{1.8}$$

(see for details [6–13] and also, the references cited in each of these earlier works).

For $\lambda = \alpha = 1$ in (1.8), the result reduces to

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi,$$

where $B_n(x)$ denotes the classical Bernoulli polynomials (see from example [17, 18], see also the references cited in each of these earlier works).

2. If we substitute $b = 1$, $k = 0$, $a = -1$ and $\beta = \lambda$ into (1.6), we have the Apostol-Euler polynomials $P^{(\alpha)}_{n,\lambda}(x; 0, -1, 1) = E^{(\alpha)}_n(x, \lambda)$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E^{(\alpha)}_n(x; \lambda)\frac{t^n}{n!}, \quad |t + \log\lambda| < \pi, \tag{1.9}$$

(see for details [6–13] and also the references cited in each of these earlier works).

For $\lambda = \alpha = 1$ in (1.9), the result reduces to

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

where $E_n(x)$ denotes the classical Euler polynomials (see from example [14–18] and also the references cited in each of these earlier works).

3. By substituting $b = \alpha = 1$, $k = 1$, $a = -1$ and $\beta = \lambda$ into (1.6), one has the Apostol-Genocchi polynomials $P^{(1)}_{n,\beta}(x; 1, -1, 1) = \frac{1}{2}G_n(x; \lambda)$, which is defined by means of the following generating function

$$\frac{2t}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!}, \quad |t + \log \lambda| < \pi,$$

(see for details [6–18] and also the references cited in each of these earlier works).

4. By substituting $x = 0$ in the generating function (1.6), we obtain the corresponding unification of the generating functions of Bernoulli, Euler and Genocchi numbers of higher order. Thus, we have

$$P^{(\alpha)}_{n,\beta}(0; k, a, b) = P^{(\alpha)}_{n,\beta}(k, a, b), \quad n \in \mathbb{N}.$$

The generalized Stirling numbers of the second kinds $S(n, \nu, a, b, \beta)$ of order ν are defined in [16] as follows:

$$\sum_{n=0}^{\infty} S(n, \nu, a, b, \beta) \frac{t^n}{n!} = \frac{(\beta^b e^t - a^b)^\nu}{\nu!}. \tag{1.10}$$

On setting $\beta = \lambda$, $a = b = 1$, (1.10) reduces to

$$\sum_{n=0}^{\infty} S(n, \nu, \lambda) \frac{t^n}{n!} = \frac{(\lambda e^t - 1)^\nu}{\nu!}.$$

The outline of this paper is as follows. In Section 2, we introduce the Lagrange-based unified Apostol-type polynomials and investigate some properties. In Section 3, we introduce Miller-Lee polynomials and derive some relationship between Lagrange-based unified Apostol-type polynomials. In Section 4, we introduce Laguerre polynomials and obtain some properties of Laguerre and Lagrange-based unified Apostol-type polynomials.

2. Lagrange-Based Unified Apostol-type Polynomials

In this section, we connect the Lagrange polynomials in several variables with Hermite and Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. The resulting formulae allow a considerable unification of various special results that appear in the literature.

Definition 2.1. The Lagrange-based unified Apostol-type polynomials $T^{(\alpha_1,\cdots,\alpha_r)}_{n,\beta,k}(x|x_1, \ldots x_r; a, b)$ in several variables by means of the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt} \prod_{j=1}^{r} (1 - x_j t^j)^{-\alpha_j} = \sum_{n=0}^{\infty} T^{(\alpha_1,\cdots,\alpha_r)}_{n,\beta,k}(x|x_1, \ldots x_r; a, b) t^n, \tag{2.1}$$

which, for ordinary case $r = 2$, (2.1) reduces to the Lagrange-based unified Apostol-type Hermite polynomials

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.2}$$

In particular, when $x_1 = x_2 = 0$, (2.2) reduces to unified Apostol-type polynomials $Y_{n,\beta}(x;k,a,b)$ defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}(x;k,a,b)\frac{t^n}{n!}.$$

The Lagrange-based unified Apostol-type Hermite polynomials of order α are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.3}$$

Note that

$${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} h_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{Y_{n-m,\beta}^{(\alpha)}(x;k,a,b)}{(n-m)!}.$$

On setting $\alpha = x = 0$ in (2.3), the result reduces to (1.4). For $\alpha = 1$, (2.3) reduces to (2.2).

The Lagrange-based unified Apostol-type polynomials of order α are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_Yg_{n.\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n. \tag{2.4}$$

Thus, we have

$${}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} g_m^{(\alpha_1,\alpha_2)}(x_1,x_2)\frac{Y_{n-m,\beta}^{(\alpha)}(x;k,a,b)}{(n-m)!}.$$

On taking $x = 0$ in (2.4), the result reduces to

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} (1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2;k,a,b)t^n,$$

where

$${}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(0|x_1,x_2;k,a,b) = {}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x_1,x_2;k,a,b).$$

Remark 2.1. By substituting $a = b = k = 1$ and $\beta = \lambda$ in (2.3), we get the generalized Lagrange-based Apostol-Bernoulli polynomials by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_BH_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n. \tag{2.5}$$

For $\lambda = 1$ in (2.5), the result reduces to the known result of Khan and Pathan [8] as follows

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_BH_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n.$$

Remark 2.2. If we substitute $b = \alpha = 1$, $k = 0$, $a = -1$ and $\beta = \lambda$ into (2.3), we get the generalized Lagrange-based Apostol-Euler polynomials by means of the following generating function

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_EH_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n.$$

Remark 2.3. By substituting $b = \alpha = 1$, $k = 1$, $a = -\frac{1}{2}$ and $\beta = \frac{\lambda}{2}$ into (2.3), we obtain the generalized Lagrange-based Apostol-Genocchi polynomials by means of the following generating function

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_GH_{n,\lambda}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n.$$

Theorem 2.1. *The following summation formula for Lagrange-Hermite polynomials $h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)$ holds true:*

$$h_n^{(\alpha_1,\alpha_2)}(x_1,x_2) = \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{(-x)^m}{m!}. \tag{2.6}$$

Proof. For $\alpha = 0$ in (2.3), we have

$$\begin{aligned} e^{-xt}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n =&(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} \\ &\times \sum_{m=0}^{\infty}\frac{(-x)^mt^m}{m!}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \\ =&\sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n. \end{aligned}$$

Replacing n by $n-m$ in L. H. S, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{(-x)^m}{m!}t^n = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n.$$

Comparing the coefficients of t^n on both sides, we get (2.6). □

Theorem 2.2. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ holds true:*

$${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x+y|x_1,x_2;k,a,b) = \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{y^m}{m!}. \tag{2.7}$$

Proof. From (2.3), we have

$$\begin{aligned}&\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n\\&=\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}e^{yt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\sum_{m=0}^{\infty} y^m\frac{t^m}{m!}.\end{aligned}$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t^n on both sides, we get the result (2.7). □

Theorem 2.3. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ holds true:*

$$\begin{aligned}&{}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1+\alpha_2,\alpha_3+\alpha_4)}(x+y|x_1,x_2;k,a,b)\\&=\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_3)}(x|x_1,x_2;k,a,b)\,{}_HY_{m,\beta}^{(\gamma|\alpha_2,\alpha_4)}(y|x_1,x_2;k,a,b).\end{aligned}\tag{2.8}$$

Proof. By using definition (2.3), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1+\alpha_2,\alpha_3+\alpha_4)}(x+y|x_1,x_2;k,a,b)t^n\\&=\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha+\gamma} e^{(x+y)t}(1-x_1t)^{-\alpha_1-\alpha_2}(1-x_2t^2)^{-\alpha_3-\alpha_4}\\&=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\sum_{m=0}^{\infty} {}_HY_{m,\beta}^{(\gamma|\alpha_3,\alpha_4)}(y|x_1,x_2;k,a,b)t^m.\end{aligned}$$

Replacing n by $n-m$ in above equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_3)}(x|x_1,x_2;k,a,b)\,{}_HY_{m,\beta}^{(\gamma|\alpha_2,\alpha_4)}(y|x_1,x_2;k,a,b)\right)t^n.\end{aligned}$$

Comparing the coefficients of t^non both sides, we get the result (2.8). □

Theorem 2.4. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ holds true:*

$${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)=\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)\frac{z^m}{m!}.\tag{2.9}$$

Proof. By exploiting the generating function (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-z)t} e^{zt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} z^m \frac{t^m}{m!}.$$

Replacing n by $n-m$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x-z|x_1,x_2;k,a,b)\frac{z^m}{m!}\right)t^n.$$

Comparing the coefficients of t^n on both sides, we get the result (2.9). □

Theorem 2.5. *The following summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$\begin{aligned}{}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b) &= \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2;k,a,b)\\ &\times \frac{{}_HY_{m,\beta}^{(\gamma|0,0)}(x-z;k,a,b)}{m!}.\end{aligned} \tag{2.10}$$

Proof. Going back to the generating function (2.3), we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{zt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\gamma} e^{(x-z)t}$$
$$= \sum_{n=0}^{\infty} {}_HY_{n}^{(\alpha|\alpha_1,\alpha_2)}(z|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} \frac{{}_HY_{m,\beta}^{(\gamma|0,0)}(x-z;k,a,b)t^m}{m!}.$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t^n on both sides, we get the result (2.10). □

Theorem 2.6. *The following implicit summation formula for the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$ {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b) = \sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)h_m^{(\beta_1,\beta_2)}(x_1,x_2). \tag{2.11}$$

Proof. Using definition (2.3), we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b)t^n$$

$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1-\beta_1}(1-x_2t^2)^{-\alpha_2-\beta_2}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{m=0}^{\infty} h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^m.$$

Replacing n by $n-m$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+\beta_1,\alpha_2+\beta_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)h_m^{(\beta_1,\beta_2)}(x_1,x_2)t^n.$$

Comparing the coefficients of t^n on both sides, we get the result (2.11). □

Theorem 2.7. *There is the following relation between the Apostol-type Stirling numbers of second kind and Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$*:*

$$a^{b\alpha}\alpha!\sum_{r=0}^{n} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)}{r!}$$
$$=\begin{cases} 0, & \text{for } n<k\alpha,\\ 2^{(1-k)\alpha}h_{n-k\alpha}^{(\alpha_1,\alpha_2)}(x|x_1,x_2), & \text{for } n\ge k\alpha, \end{cases} \tag{2.12}$$

with $\alpha\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ *and* $k\in\mathbb{N}$ *fixed.*

Proof. By using equation (2.3) and (1.10), we have

$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=\frac{2^{(1-k)\alpha}t^{k\alpha}}{a^{b\alpha}\left(\left(\frac{\beta}{a}\right)^b e^t-1\right)^{\alpha}}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=\frac{2^{(1-k)\alpha}t^{k\alpha}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{a^{b\alpha}\alpha!\sum_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{r!}},$$
$$\left(\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\right)\left(a^{b\alpha}\alpha!\sum_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{r!}\right)$$
$$=2^{(1-k)\alpha}t^{k\alpha}\sum_{n=0}^{\infty}h_n^{(\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},$$

$$a^{b\alpha}\alpha!\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}{}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{1}{r!}\right)t^n$$
$$=2^{(1-k)\alpha}\sum_{n=0}^{\infty}h_{n-k\alpha}^{(\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{(n-k\alpha)!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result (2.12). □

Theorem 2.8. *There is the following relation between the Apostol-type Stirling numbers of second kind and Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$*:*

$$\gamma!\sum_{r=0}^{n}{}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S(r,\gamma,a,b,\beta)}{r!}$$
$$=\begin{cases}0, & \text{for } n<k\gamma,\\ 2^{(1-k)\gamma}{}_HY_{n-k\gamma,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b), & \text{for } n\geq k\gamma,\end{cases} \tag{2.13}$$

with $\gamma\in\mathbb{N}_0$ *and* $k\in\mathbb{N}$ *fixed.*

Proof. From (2.3) and (1.10), we have

$$\sum_{n=0}^{\infty}{}_HY_{n,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n$$
$$=\left(\frac{2^{1-k}t^k}{\beta^be^t-a^b}\right)^{\alpha-\gamma}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=\left(\frac{2^{1-k}t^k}{\beta^be^t-a^b}\right)^{\alpha}e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\left(\frac{\beta^be^t-a^b}{2^{1-k}t^k}\right)^{\gamma},$$
$$2^{(1-k)\gamma}\sum_{n=0}^{\infty}{}_HY_{n,\beta}^{(\alpha-\gamma|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^{n+k\gamma}$$
$$=\gamma!\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}{}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{S(r,\gamma,a,b,\beta)}{r!}\right)t^n.$$

Comparing the coefficients of t^n on both sides, we get the desired result (2.13). □

Theorem 2.9. *The following implicit summation formula involving the Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *and Lagrange-based unified Apostol-type polynomials* ${}_Yg_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *holds true:*

$$\sum_{m=0}^{n}{}_HY_{n-m,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{(\gamma)_my^m}{m!}=\sum_{m=0}^{n}{}_Yg_{n-m,\beta}^{(\alpha|\gamma,\alpha_2)}(x|y,x_2;k,a,b)\frac{x_1^m(\alpha_1)_m}{m!}. \tag{2.14}$$

Proof. We first start with the generating function (2.3). On multiplying both the sides by $(1-yt)^{-\gamma}$ and interpreting the result using (2.4) and series expansion of $(1-yt)^{-\gamma}$, we get the required result (2.14). □

3. Lagrange-Based Unified Apostol-Type Miller-Lee Polynomials

The definitions (2.3) and (2.4) can be exploited in a number of ways and provide a useful tool to frame known and new generating functions in the following way.

As a first example, we set $\alpha = \alpha_2 = 0$, $\alpha_1 = m+1$, $x_1 = 1$ in (2.3) to get

$$e^{xt}(1-t)^{-m-1} = \sum_{n=0}^{\infty} G_n^{(m)}(x)t^n, \quad |t|<1,$$

where $G_n^{(m)}(x)$ are called the Miller-Lee polynomials (see [4, page 21, (1.11)]).

Another example is the definition of Lagrange-based Apostol-type Hermite-Miller-Lee polynomials ${}_YHG_{n,\beta}^{(m,\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ given by the following generating function

(3.1)
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\frac{(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} {}_YHG_{n,\beta}^{(m,\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)\frac{t^n}{n!},$$

which for $\alpha = 0$ reduces to

$$\frac{(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}}e^{xt} = \sum_{n=0}^{\infty} {}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},$$

where ${}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)$ are called the Lagrange-based Hermite-Miller-Lee polynomials.

Putting $\alpha_1 = \alpha_2 = 0$ into (3.1) gives

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} \frac{e^{xt}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{3.2}$$

where ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ are called the Apostol-type Miller-Lee polynomials.

Theorem 3.1. *The following relationship between Lagrange-based unified Apostol-type Hermite polynomials ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$, Apostol-type Miller-Lee polynomials ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ and Miller-Lee polynomials $G_n^{(m)}(x)$ holds true:*

$$\begin{aligned} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b) &= \sum_{r=0}^{n}\binom{n}{r} Y_{n-r,\beta}^{(\alpha)}(k,a,b)G_r^{(m)}(x) \\ &= n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!}\,{}_HY_{n-2r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b). \end{aligned} \tag{3.3}$$

Proof. For $x_1 = 1$ and $\alpha_1 = m+1$ in (2.3) and using (3.2), we have

$$\begin{aligned} \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-t)^{-m-1} &= \sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!} \\ &= (1-x_2t^2)^{\alpha_2}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n, \end{aligned}$$

which on using binomial expansion takes the form

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} G_r^{(m)}(x)t^r = \sum_{r=0}^{\infty}\frac{(-\alpha_2)_r(x_2)^r t^{2r}}{r!} \times \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Replacing n by $n-r$ in above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)G_r^{(m)}(x)\frac{t^n}{(n-r)!}\\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!}{}_HY_{n-2r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.\end{aligned}$$

Finally, comparing the coefficients of t^n, we get (3.3). □

Remark 3.1. Equation (3.3) is obviously a series representation of the Apostol-type Miller-Lee polynomials ${}_YG_{n,\beta}^{(m,\alpha)}(x;k,a,b)$ linking Lagrange-based unified Apostol-type Hermite polynomials and Miller-Lee polynomials.

Theorem 3.2. *The following relationship holds true:*

(3.4)

$${}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b) = \sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r.$$

Proof. On replacing x by $x+y$ and α_1 by α_1+m+1, respectively in (2.3), we have

$$\begin{aligned}&\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+y)t}(1-x_1t)^{-m-1}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\ &= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n,\end{aligned}$$

which can be written as

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)t^n\sum_{r=0}^{\infty} G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^r\\ &= \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Now replacing n by $n-r$ in the left hand side of the above equation, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2;k,a,b)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^n\\ &= \sum_{n=0}^{\infty} {}_HY_{n}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Finally, comparing the coefficients of t^n, we get the result (3.4). □

Theorem 3.3. *The following relationship holds true:*

$$\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)_H G_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{r!}{(n-r)!}$$
$$=\sum_{r=0}^{n}(\alpha_1)_r x_1^r {}_H Y_{n-r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b). \tag{3.5}$$

Proof. For $\alpha_1 = m+1$ and $x_1 = 1$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1-t)^{-m-1}(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_H Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Multiplying both sides by $(1-x_1t)^{-\alpha_1}$, we have

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_H G_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)t^r$$
$$=7(1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_H Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n$$
$$=\sum_{r=0}^{\infty}(\alpha_1)_r x_1^r\frac{t^r}{r!}\sum_{n=0}^{\infty} {}_H Y_{n,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)t^n.$$

Now replacing n by $n-r$ in the above equation, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} Y_{n-r,\beta}^{(\alpha)}(k,a,b)_H G_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{(n-r)!}$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha_1)_r x_1^r {}_H Y_{n-r,\beta}^{(\alpha|m+1,\alpha_2)}(x|1,x_2;k,a,b)\frac{t^n}{r!}.$$

Comparing the coefficients of t^n on both sides, we get the result (3.5). □

4. Lagrange-Based Unified Apostol-Type Laguerre Polynomials

In this section, we shall be interested in the connection between the Lagrange-based unified Apostol-type Hermite polynomials ${}_H Y_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ and Laguerre polynomials $L_n^{(m)}(x)$.

For $x_2 = 0$, $x_1 = -1$, $\alpha_1 = -m$ and $\alpha_2 = 0$ in equation (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}(1+t)^m = \sum_{n=0}^{\infty} {}_L Y_{n,\beta}^{(\alpha|m)}(x;k,a,b)\frac{t^n}{n!}, \tag{4.1}$$

where ${}_H Y_{n,\beta}^{(\alpha|-m,0)}(x|-1,0;k,a,b) = {}_L Y_{n,\beta}^{(\alpha|m)}(x;k,a,b)$ are called the generalized Laguerre-based unified Apostol-type polynomials.

When $\alpha = 0$ in (4.1), ${}_Y L_{n,\beta}^{(\alpha|m)}(x;k,a,b)$ reduces to ordinary Laguerre polynomials $L_n^{(m)}(x)$ as follows (see [19])

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right).$$

Theorem 4.1. *The following relationship between Lagrange-based unified Apostol-type Hermite polynomials* ${}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds true:*

$$\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)L_r^{(m-r)}(y)$$
$$=\sum_{r=0}^{n}(\alpha)_r x_1^r {}_HY_{n-r,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)\frac{1}{r!}. \tag{4.2}$$

Proof. Replacing x by $x+y$ and setting $x_1=-1$, $\alpha_1=-m$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m(1-x_2t^2)^{-\alpha_2}=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n.$$

Multiplying both sides $(1-x_1t)^{-\alpha_1}$, we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}$$
$$=(1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n,$$
$$\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n\sum_{r=0}^{\infty}L_r^{(m-r)}(y)t^r$$
$$=\sum_{r=0}^{\infty}\frac{(\alpha)_r(x_1)^r t^r}{r!}\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)t^n.$$

Replacing n by $n-r$ in above equation, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HY_{n-r,\beta}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)L_r^{(m-r)}(y)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha)_r x_1^r {}_HY_{n-r,\beta}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2;k,a,b)\frac{t^n}{r!}.$$

Comparing the coefficients of t^n on both sides, we get (4.2). □

Theorem 4.2. *The following relationship holds true:*

$$\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_k^{(m-k)}(y)\frac{1}{(n-k)!}={}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b). \tag{4.3}$$

Proof. Replacing x by $x+y$ and setting $x_1=-1$, $\alpha_1=-m$ and $\alpha_2=0$ in equation (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{(x+y)t}(1+t)^m=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n,$$
$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\sum_{k=0}^{\infty}L_k^{(m-k)}(y)t^k=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_k^{(m-k)}(y)\frac{t^n}{(n-k)!} = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b)t^n.$$

Comparing the coefficients of t^n on both sides, we get (4.3). □

Theorem 4.3. *The following relationship holds true:*

$$\begin{aligned}&\sum_{k=0}^{n} {}_HY_{n-k,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)(-x_1)^k L_k^{(m-k)}(y/x_1)\\ &={}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b).\end{aligned}\tag{4.4}$$

Proof. Replacing α_1 by $-m+\alpha_1$ and $x \to x-y$ in (2.3), we have

$$\begin{aligned}&\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-y)t}(1-x_1t)^{m-\alpha_1}(1-x_2t^2)^{-\alpha_2}\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n,\\ &\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)t^n \sum_{k=0}^{\infty}(-x_1)^k t^k L_k^{(m-k)}(y/x_1)\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_HY_{n-k,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2;k,a,b)(-x_1)^k L_k^{(m-k)}(y/x_1)t^n\\ &=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2;k,a,b)t^n.\end{aligned}$$

Comparing the coefficients of t^n on both sides, we get (4.4). □

Theorem 4.4. *The following relationship holds true:*

$$\sum_{k=0}^{n} L_k^{(m-k)}(y)\frac{Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)}{(n-k)!} = {}_HY_{n,\beta}^{(\alpha|-m,0)}(x+y|-1,0;k,a,b).\tag{4.5}$$

Proof. For $x_1=-1$, $\alpha_1=-m$, $\alpha_2=0$, $x \longrightarrow x-y$ in (2.3), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-y)t}(1+t)^m = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n,$$

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\sum_{k=0}^{\infty} L_k^{(m-k)}(-y)t^k = \sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n.$$

Replacing n by $n-k$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} Y_{n-k,\beta}^{(\alpha)}(x;k,a,b)L_k^{(m-k)}(-y)\frac{t^n}{(n-k)!}=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha|-m,0)}(x-y|-1,0;k,a,b)t^n.$$

Finally, replacing y by $-y$ and comparing the coefficients of t^n, we get (4.5). $\square$

Theorem 4.5. *The following relationship holds true:*

(4.6)

$$\sum_{r=0}^{n}\binom{n}{r} {}_LY_{n-r,\beta}^{(\alpha|m)}(x;k,a,b) {}_LY_{r,\beta}^{(\gamma|k)}(y;k,a,b)=n! {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b).$$

Proof. Replacing α by $\alpha+\gamma$, $x\to x+y$ and setting $x_1=-1$, $\alpha_1=-m-k$, $\alpha_2=0$ in (2.4), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha+\gamma} e^{(x+y)t}(1+t)^{m+k}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n,$$
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1+t)^m\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\gamma} e^{yt}(1+t)^k$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(-x-y|-1,x_2;k,a,b)t^n,$$
$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha} e^{xt}(1+t)^m\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\gamma} e^{yt}(1+t)^k$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,x_2;k,a,b)t^n,$$

which leads directly to

$$\sum_{n=0}^{\infty} {}_LY_{n,\beta}^{(\alpha|m)}(x;k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_LY_{r,\beta}^{(\gamma|k)}(y;k,a,b)\frac{t^r}{r!}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n.$$

Replacing n by $n-r$ in the left hand side of the above equation, we have

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} {}_LY_{n-r,\beta}^{(\alpha|m)}(x;k,a,b) {}_LY_{r,\beta}^{(\gamma|k)}(y;k,a,b)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty} {}_HY_{n,\beta}^{(\alpha+\gamma|-m-k,0)}(x+y|-1,0;k,a,b)t^n.$$

Now comparing the coefficients of t^n, we get (4.6). $\square$

Acknowledgements. The author would like to thank Prince Mohammad Bin Fahd University for providing facilities and support.

References

1. A. Altin and E. Erkus, *On a multivariable extension of the Lagrange-Hermite polynomials*, Integral Transforms Spec. Funct. **17** (2006), 239–244.
2. W. Ch. C. Chan, Ch. J. Chyan and H. M. Srivastava, *The Lagrange polynomials in several variables*, Integral Transforms Spec. Funct. **12**(2) (2001), 139–148.
3. G. Dattoil, P. E. Ricci and C. Cesarano, *The Lagrange polynomials the associated generalizations, and the umbral calculus*, Integral Transforms Spec. Funct. **14** (2003), 181–186.
4. G. Dattoli, S. Lorenzutta and D. Sacchetti, *Integral representations of new families of polynomials*, Italian Journal of Pure and Applied Mathematics **15** (2004), 19–28.
5. E. Erkus and H. M. Srivastava, *A unified presentation of some families of multivariable polynomials*, Integral Transforms Spec. Funct. **17** (2006), 267–273.
6. W. A. Khan, *Some properties of Hermite-based Apostol-type polynomials*, Kyungpook Math. J. **55** (2015), 597–614.
7. W. A. Khan and H. Haroon, *Some symmetric identities for the generalized Bernoulli, Euler and Genocchi polynomials associated with Hermite polynomials*, Springer Plus **5** (2016), Article ID 1920.
8. W. A. Khan and M. A. Pathan, *On generalized Lagrange Hermite-Bernoulli and related polynomials*, Acta Comm. Univ. Tart. Math. **23**(2) (2019), 211–224.
9. V. Kurt, *Some symmetry identities for the Apostol-type polynomials related to multiple alternating sums*, Adv. Difference Equ. **2013**(1) (2013), Article ID 32.
10. V. Kurt, *Some symmetry identities for the unified Apostol-type polynomials and multiple power sums*, Filomat **30**(3) (2016), 583–592.
11. Q. M. Luo and H. M. Srivastava, *Some series identities involving the Apostol-type and related polynomials*, Comp. Math. Appl. **62** (2011), 3591–3602.
12. Q. M. Luo, *The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order*, Integral Trans. Spec. Func. **20** (2009), 377–391.
13. Q. M. Luo, *The multiplication formulas for Apostol-type polynomials and multiple alternating sums*, Math. Notes **91** (2012), 46–57.
14. M. A. Ozarslan, *Unified Apostol-Bernoulli, Euler and Genocchi polynomials*, Comp. Math. Appl. **62** (2011), 2482–2462.
15. H. Ozden, Y. Simsek and H. M. Srivastava, *A unified presentation of the generating function of the generalized Bernoulli, Euler and Genocchi polynomials*, Comp. Math. Appl. **60** (2010), 2779–2789.
16. H. Ozden and Y. Simsek, *Modification and unification of the Apostol-type numbers and polynomials*, Appl. Math. Comp. **235** (2014), 338–351.
17. M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-based polynomials*, Acta Univ. Apulensis Math. Inform. **13** (2014), 113–136.
18. M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials*, Mediterr. J. Math. **12** (2015), 679–695.
19. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1984.

[1]Department of Mathematics and Natural Sciences,
Prince Mohammad Bin Fahd University, P.O Box 1664,
Al Khobar 31952, Saudi Arabia
Email address: wkhan1@pmu.edu.sa

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 883–894.

ON THE COMPOSITION OF CONDITIONAL EXPECTATION AND MULTIPLICATION OPERATORS

YOUSEF ESTAREMI[1,2]

Abstract. In this paper, first we provide some necessary and sufficient conditions for quasi- normality and quasi- hyponormality of weighted conditional type operators. And then the spectrum, residual spectrum, point spectrum and spectral radius of weighted conditional type operators are computed. As an application, we give an equivalent conditions for weighted conditional type operators to be quasinilpotent. Also, some examples are provided to illustrate concrete applications of the main results.

1. Introduction and Preliminaries

This paper is about an important operator in statistics and analysis, that is called conditional expectation. Theory of conditional type operators is one of important arguments in the connection of operator theory and measure theory. By the projection theorem, the conditional expectation $E(X)$ is the best mean square predictor of X in range E, also in analysis it is proved that lots of operators are of the form E and of the form of combinations of E and multiplications operators. Conditional expectations have been studied in an operator theoretic setting, by, for example in [8], S.-T. C. Moy characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded, P. G. Dodds, C. B. Huijsmans and B. De Pagter [1], extended these characterizations to the setting of function ideals and vector lattices and J. Herron presented some assertions about the operator EM_u on L^p spaces in [6]. Also, some results about multiplication conditional expectation operators can be found in [5, 7]. In [3] we investigated some classic properties of multiplication conditional expectation operators M_wEM_u on L^p spaces.

Key words and phrases. Conditional expectation, spectrum, point spectrum, spectral radius.
2010 *Mathematics Subject Classification.* Primary: 47B38. Secondary:47B40.
DOI 10.46793/KgJMat2206.883E
Received: October 14, 2019.
Accepted: May 09, 2020.

Let (X, Σ, μ) be a σ-finite measure space. For any sub-algebra $\mathcal{A} \subseteq \Sigma$, the L^p-space $L^p(X, \mathcal{A}, \mu_{|_\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$, where $1 \leq p < \infty$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f) = \{x \in X : f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

The notion of conditional expectation plays an important role throughout the paper and so we recall the definition and some elementary properties for the reader's covenience. Let (X, Σ, μ) be a σ-finite measure space and $\mathcal{A} \subseteq \Sigma$ be a σ-finite subalgebra of Σ. For all non-negative Σ-measurable functions like f as well as for all $f \in L^p(\Sigma)$, we denote by $E^{\mathcal{A}}f$, the (μ-a.e.) unique $\mathcal{A}$-measurable function with the property that

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \text{for all } A \in \mathcal{A}.$$

The existence of $E^{\mathcal{A}}(f)$ is a consequence of the Radon-Nikodym theorem. The function $E^{\mathcal{A}}(f)$ is called conditional expectation of f with respect to $\mathcal{A}$. As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. This operator will play a major role in our work. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [10]. We list here some useful properties of conditional expectation operator:

- if g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$;
- $|E(f)|^p \leq E(|f|^p)$ for all $f \in L^p(\Sigma)$;
- if $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$;
- $|E(fg)| \leq (E(|f|^p))^{\frac{1}{p}}(E(|g|^{p'}))^{\frac{1}{p'}}$, where $p^{-1} + p'^{-1} = 1$ (Hölder inequality);
- for each $f \geq 0$, $S(f) \subseteq S(E(f))$.

For a measurable function $w \in L^0(\Sigma)$, the operator $M_w : L^0(\Sigma) \to L^0(\Sigma)$ with $M_w(f) = w.f$, for every $f \in L^0(\Sigma)$, is called Multiplication operator. Now we give a definition for weighted conditional type operators on L^p-spaces.

Definition 1.1. Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-subalgebra of Σ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite. Let E be the conditional expectation operator on $L^p(\Sigma)$ relative to $\mathcal{A}$. If $w, u \in L^0(\Sigma)$ such that uf is conditionable and $wE(uf) \in L^p(\Sigma)$, for all $f \in L^p(\Sigma)$, then the corresponding weighted conditional type operator is the linear transformation $T : L^p(\Sigma) \to L^p(\Sigma)$ defined by $f \to wE(uf)$.

In this paper we will be concerned with characterizing weighted conditional expectation type operators on $L^p(\Sigma)$, provide necessary and sufficient conditions for quasi-normality and quasi-hyponormality, computing the spectrum, residual spectrum, point spectrum and spectral radius. The results of [1] state that our results are valid for a large class of linear operators.

2. Quasi-Normality and Quasi-Hyponormality

In this section first we reminisce some properties of weighted conditional type operators that we proved in [3]. Also, we give some examples of conditional expectation operator. I recall that throughout this paper we assume that (X, Σ, μ) and $(X, \mathcal{A}, \mu_{\mathcal{A}})$ are σ-finite measure spaces.

Let $T = M_w E M_u$ be a bounded operator on $L^2(\Sigma)$ and let $p \in (0, \infty)$. Then

$$(T^*T)^p = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p} E M_u, \tag{2.1}$$

$$(TT^*)^p = M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p} E M_{\bar{w}}, \tag{2.2}$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

Let $\mathcal{H}$ be a Hilbert spaces and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is called quasi-normal if the equation $T(T^*T) = (T^*T)T$ holds, in which T^* is the adjoint of T. Also, for $M > 0$, the operator $T \in \mathcal{B}(\mathcal{H})$ is called M-quasi-hyponormal if $M^2 {T^*}^2 T^2 - (T^*T)^2 \geq 0$. Specially T is called quasi-hyponormal if T is 1-quasi-hyponormal. This definition comes from [11]. There is another concept with almost the same names but different definition in [4]. There is a small difference between them as you see, M-quasi-hyponormal (capital letter M) and k-quasi-hyponormal (small letter k).

In the next theorem we give some necessary and sufficient conditions for quasi-normality of weighted conditional type operators.

Theorem 2.1. *Let $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$. Then we have the foolowing.*

(a) If $\bar{u}E(uw) = wE(|u|^2)$ on G, then T is quasi-normal.

(b) If T is quasi-normal, then $E(|w|^2)E(uw)|E(u)|^2 = E(|u|^2)E(|w|^2)E(u)E(w)$.

Proof. (a) By using (2.1) and (2.2) easily we can obtain that

$$T(T^*T) = M_{wE(|u|^2)E(|w|^2)} E M_u, \quad (T^*T)T = M_{E(|w|^2)E(uw)\bar{u}} E M_u.$$

So, for every $f \in L^2(\Sigma)$ we have

$$\begin{aligned} &\langle T(T^*T)f - (T^*T)Tf, f\rangle \\ &= \int_X (E(|u|^2)E(|w|^2)wE(uf)\bar{f} - E(|w|^2)E(uw)\bar{u}E(uf)\bar{f})d\mu. \end{aligned}$$

This implies that, if $\bar{u}E(uw) = wE(|u|^2)$ on G, then T is quasi-normal.

(b) If T is quasi-normal then, for all $f \in L^2(\Sigma)$ we have

$$\begin{aligned} \langle T(T^*T)f - (T^*T)Tf, f\rangle &= \int_X (E(|u|^2)E(|w|^2)wE(uf)\bar{f} \\ &\quad - E(|w|^2)E(uw)\bar{u}E(uf)\bar{f})d\mu \\ &= \int_X (E(|u|^2)E(|w|^2)E(uf)E(w\bar{f}) \\ &\quad - E(|w|^2)E(uw)|E(uf)|^2)d\mu \\ &= 0. \end{aligned}$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A (E(|u|^2)E(|w|^2)E(u)E(w) - E(|w|^2)E(uw)|E(u)|^2)d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$E(|u|^2)E(|w|^2)E(u)E(w) - E(|w|^2)E(uw)|E(u)|^2 = 0. \qquad \square$$

Now we provide some necessary and sufficient conditions for M-quasi-hyponormality of weighted conditional type operators.

Theorem 2.2. *Let $T = M_w E M_u \in \mathcal{B}(L^2(\Sigma))$. Then we have the following.*
(a) If $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$, then T is M-quasi-hyponormal.
(b) If T is M-quasi-hyponormal, then $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$ on $S(E(u))$.

Proof. (a) Direct computation shows that for all $f \in L^2(\Sigma)$

$$M^2T^{*^2}T^2(f) - (T^*T)^2(f) = M^2\bar{u}|E(uw)|^2E(|w|^2)E(uf) - \bar{u}(E(|w|^2))^2E(|u|^2)E(uf).$$

If $M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$, then for all $f \in L^2(\Sigma)$

$$\begin{aligned}&\langle M^2T^{*^2}T^2(f) - (T^*T)^2(f), f\rangle \\ &= \int_X (M^2\bar{u}|E(uw)|^2E(|w|^2)E(uf)\bar{f} - \bar{u}(E(|w|^2))^2E(|u|^2)E(uf)\bar{f})d\mu \\ &= \int_X (M^2|E(uw)|^2E(|w|^2)|E(uf)|^2 - (E(|w|^2))^2E(|u|^2)|E(uf)|^2)d\mu \geq 0,\end{aligned}$$

so T is M-quasi-hyponormal.

(b) If T is M-quasi-hyponormal, then for all $f \in L^2(\Sigma)$ we have

$$\int_X (M^2|E(uw)|^2E(|w|^2)|E(uf)|^2 - (E(|w|^2))^2E(|u|^2)|E(uf)|^2)d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A (M^2|E(uw)|^2E(|w|^2)|E(u)|^2 - (E(|w|^2))^2E(|u|^2)|E(u)|^2)d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$M^2|E(uw)|^2E(|w|^2) - E(|u|^2)(E(|w|^2))^2 \geq 0$$

on $S(E(u))$. $\square$

Hence, we get the next corollary.

Corollary 2.1. *Let $T = EM_u$ and $S(E(u)) = X$. Then*
(a) T is M-quasi hyponormal if and only if $M^2|E(u)|^2 - E(|u|^2) \geq 0$;
(b) T is quasi hyponormal if and only if $u \in L^0(\mathcal{A})$;
(c) T is quasi-normal if and only if $u \in L^0(\mathcal{A})$.

Now we provide an example for Theorem 2.2 in which we have a weighted conditional type operator is M-quasi-hyponormal, but the inequality in (a) does not hold.

Example 2.1. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} = \langle\{(-a, a) : 0 \leq a \leq 1\}\rangle$ (σ-algebra generated by symmetric intervals). Then

$$E^{\mathcal{A}}(f)(x) = \frac{f(x) + f(-x)}{2}, \quad x \in X,$$

whenever $E^{\mathcal{A}}(f)$ is defined. Let $w(x) = 1$, $u(x) = x + 1$ for $x \in X$. Then easily we get that $E(u)(x) = 1$ and $E(|u|^2)(x) = x^2 + 1$. Hence, we have $|E(u)(x)|^2 < E(|u|^2)(x)$ for all $x \in X \setminus \{0\}$. Also, for $f \in L^p(X)$ and $M \geq \sqrt{2}$ we have

$$\int_X (M^2|E(uf)|^2 - E(|u|^2)|E(uf)|^2)d\mu = \int_X (M^2 - E(|u|^2))|E(uf)|^2 d\mu \geq 0,$$

this implies that the operator M_wEM_u is M-quasi-hyponormal.

3. The Spectrum

In this section we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_r(T)$, $r(T)$ the spectrum of T, the point spectrum of T, the joint point spectrum of T, the residual spectrum, the spectral radius of T, respectively for $T \in \mathcal{B}(X)$ in which X is a Banach space. The spectrum of an operator T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T, if there is a unit vector x satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. The residual spectrum of T is equal to

$$\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ exists and } \overline{R(T)} \subsetneq X\}.$$

Also, the spectral radius of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. For more information one can see [4].

If A is a unital algebra and $a, b \in A$, then it is well known that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$, $\sigma_p(ab) \setminus \{0\} = \sigma_p(ba) \setminus \{0\}$, $\sigma_{jp}(ab) \setminus \{0\} = \sigma_{jp}(ba) \setminus \{0\}$ and $\sigma_r(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. J. Herron showed that if $EM_u : L^p(\Sigma) \to L^p(\Sigma)$ is bounded, then $\sigma(EM_u) = \operatorname{ess\,range}(E(u)) \cup \{0\}$[6]. By means of the above mentioned properties of weighted conditional expectation type operators we have the following theorems.

Theorem 3.1. *Let $\mathcal{A} \subsetneq \Sigma$ and $EM_u : L^p(\Sigma) \to L^p(\mathcal{A})$ be bounded, for $1 \leq p \leq \infty$. Then*

- $\sigma_p(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(\{x \in X | E(u)(x) = \lambda\}) > 0\}$;
- $\rho(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : (\exists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\}$,
- $\sigma(EM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : (\nexists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\}$;
- $\sigma_r(EM_u) \setminus \{0\} = \begin{cases} \emptyset, & 1 \leq p < \infty, \\ B_\infty, & p = \infty, \end{cases}$

  in which $B_\infty = \{\lambda \in \mathbb{C} : (\nexists \epsilon > 0)$ *such that* $|\lambda - E(u)| \geq \epsilon$ *a.e*$\} \setminus \sigma_p(EM_u)$.

Proof. Point spectrum. Let $A_\lambda = \{x \in X : E(u)(x) = \lambda\}$, for $\lambda \in \mathbb{C}$. Suppose that $\mu(A_\lambda) > 0$. Since $\mathcal{A}$ is σ-finite, there exists an $\mathcal{A}$-measurable subset B of A_λ such that $0 < \mu(B) < \infty$ and $f = \chi_B \in L^p(\mathcal{A}) \subseteq L^p(\Sigma)$. Now

$$EM_u(f) - \lambda f = E(u)\chi_B - \lambda\chi_B = 0.$$

This implies that $\lambda \in \sigma_p(EM_u)$.

If there exists $f \in L^p(\Sigma)$ such that $f\chi_C \neq 0$ μ-a.e, for $C \in \Sigma$ of positive measure and $E(uf) = \lambda f$ for $\lambda \in \mathbb{C}$, which means that f is $\mathcal{A}$-measurable. Therefore $E(uf) = E(u)f = \lambda f$ and $(E(u) - \lambda)f = 0$. This implies that $C \subseteq A_\lambda$ and so $\mu(A_\lambda) > 0$.

Resolvent set. Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda - E(u)| > \epsilon$, a.e, for some $\epsilon > 0$. We show that $EM_u - \lambda I$ is invertible. If $\lambda f - E(uf) = 0$, then f is $\mathcal{A}$-measurable and so, $(\lambda - E(u))f = 0$. This implies that $f = 0$ a.e, therefore $\lambda I - EM_u$ is injective. Now we show that $EM_u - \lambda I$ is surjective. Let $g \in L^p(\Sigma)$. We can write

$$g = g - E(g) + E(g), \quad g_2 = g - E(g), \quad g_1 = E(g).$$

Then $g_1 \in L^p(\mathcal{A})$ and $g_2 \in L^p(\Sigma)$, $E(g_2) = 0$. Let

$$f_1 = \frac{\lambda g_1 + E(u(g_2))}{\lambda(E(u) - \lambda)}, \quad f_2 = -\frac{g_2}{\lambda}.$$

Since $|E(u) - \lambda| \geq \varepsilon$ a.e for some $\varepsilon > 0$, then $\|\frac{1}{E(u)-\lambda}\|_\infty \leq \frac{1}{\varepsilon}$. So, $f_2 \in L^p(\mathcal{A})$, $f_1 \in L^p(\Sigma)$ and $f := f_1 + f_2 \in L^p(\Sigma)$. Now, we show that $T(f) - \lambda f = g$. We have

$$E(uf) = E(u\left(\frac{\lambda g_1 + E(ug_2) - g_2(E(u) - \lambda)}{\lambda(E(u) - \lambda)}\right) = \frac{g_1 E(u) + E(ug_2)}{E(u) - \lambda}.$$

So,

$$\begin{aligned}(EM_u - \lambda I)f &= \frac{g_1 E(u) + E(ug_2)}{E(u) - \lambda} - \lambda\frac{\lambda g_1 + E(ug_2) - g_2(E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= \frac{g_1(E(u) - \lambda) + g_2(E(u) - \lambda)}{E(u) - \lambda} \\ &= g_1 + g_2 = g.\end{aligned}$$

This implies that $EM_u - \lambda I$ is invertible and so $\lambda \in \rho(EM_u)$. Hence

$$\rho(EM_u) \supseteq \{\lambda \in \mathbb{C} \setminus \{0\} : (\exists\epsilon > 0) \text{ such that } |\lambda - E(u)| \geq \epsilon \text{ a.e}\}.$$

Conversely, let $\lambda \in \rho(EM_u)$, then $\lambda I - EM_u$ has an inverse operator. Define the linear transformation L on $L^p(\Sigma)$ as follows

$$Lf = \frac{E(uf) - f(E(u) - \lambda)}{\lambda(E(u) - \lambda)}, \quad f \in L^p(\Sigma).$$

If there exists $\epsilon > 0$ such that $|\lambda - E(u)| > \epsilon$ a.e then $\|\frac{1}{E(u)-\lambda}\|_\infty \leq \frac{1}{\varepsilon}$. So,

$$\|Sf\|_p \leq \left\|\frac{E(uf)}{\lambda(E(u) - \lambda)}\right\|_p + \left\|\frac{f}{\lambda}\right\|_p \leq \left(\frac{\|EM_u\|}{\lambda\varepsilon} + \frac{1}{\lambda}\right)\|f\|_p.$$

Thus, L is bounded on $L^p(\Sigma)$. If L is bounded on $L^p(\Sigma)$, then for $f \in L^p(\mathcal{A}) \subseteq L^p(\Sigma)$, $Lf = \alpha f = M_\alpha f$, where $\alpha = \frac{1}{Eu-\lambda}$. Thus, multiplication operator M_α is bounded on $L^p(\mathcal{A})$. This implies that $\alpha \in L^\infty(\mathcal{A})$ and so there exists some $\varepsilon > 0$ such that $E(u) - \lambda = \frac{1}{\alpha} \geq \varepsilon$ a.e. Also, we have $L \circ (EM_u - \lambda I) = I$. Indeed for each $f \in L^p(\Sigma)$ we have

$$\begin{aligned} L \circ (EM_u - \lambda I)(f) &= L(E(uf) - \lambda f) \\ &= \frac{E[u(E(uf) - \lambda f)] - [E(uf) - \lambda f](E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= \frac{\lambda f(E(u) - \lambda)}{\lambda(E(u) - \lambda)} \\ &= f. \end{aligned}$$

Thus, $(EM_u - \lambda I)^{-1} = L$ and so L have to be bounded. Hence there exists some $\varepsilon > 0$ such that $E(u) - \lambda = \frac{1}{\alpha} \geq \varepsilon$ a.e.

Residual spectrum. Let $\lambda \in \mathbb{C} \setminus (\sigma_p(EM_u) \cup \rho(EM_u))$. So, $\mu(\{x \in X : E(u)(x) = \lambda\}) = 0$, but on the other hand $\mu(\{x \in X : |E(u)(x) - \lambda| > \epsilon\}) > 0$ for every $\epsilon > 0$. We wish to determine if the range of $\lambda I - EM_u$, i.e., the domain of $(\lambda I - EM_u)^{-1}$, is dense. Set, for each $n \in \mathbb{N}$,

$$E_n = \left\{ x \in X : |E(u)(x) - \lambda| \geq \frac{1}{n} \right\}.$$

The range of $\lambda I - EM_u$ contains $\{\chi_{E_n} f : f \in L^p(\mathcal{A}), n \in \mathbb{N}\}$, because, for every $f \in L^p(\mathcal{A})$, $(\lambda I - EM_u)(\frac{1}{\lambda - E(u)} \chi_{E_n} f) = \chi_{E_n} f$. Furthermore, $\chi_{E_n} f$ converges pointwise almost everywhere to f as $n \to \infty$. So, if $1 \leq p < \infty$, the Lebesgue dominated convergence theorem implies that $\chi_{E_n} f$ converges in $L^p(X, \mathcal{A}, \mu)$ to f as $n \to \infty$. Thus the range of $\lambda I - EM_u$ is dense in $L^p(X, \mathcal{A}, \mu)$ if $1 \leq p < \infty$. On the other hand, if $p = \infty$, the constant function 1 is not in the closure of the range of $\lambda I - EM_u$ because, for every $0 \neq f \in L^p(X, \mathcal{A}, \mu)$, there is $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $|\lambda - E(u)| \leq \frac{1}{2\|f\|_\infty}$ on A and hence $|1 - (\lambda - E(u))f)| \geq \frac{1}{2}$ on A. Thus, the proof is completed. $\square$

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The unique polar decomposition of bounded operator $T = M_w EM_u$, were given in [3], is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),$$

for all $f\in L^2(\Sigma)$. Also, the Aluthge transformation of $T=M_wEM_u$ is

$$\widehat{T}(f)=\frac{\chi_S E(uw)}{E(|u|^2)}\bar{u}E(uf),\quad f\in L^2(\Sigma).$$

Now, by using Theorem 2.1 we get the next theorem for M_wEM_u.

Theorem 3.2. *Let $T=M_wEM_u:L^p(\Sigma)\to L^p(\Sigma)$, for $1\le p\le\infty$ and*

$$E(|u|^p),E(|w|^p)\in L^\infty(\mathcal{A}).$$

Then we have the following.

(a) $\sigma(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)$ *such that* $|\lambda-E(uw)|\ge\epsilon$ *a.e*$\}\setminus\{0\}$.

(b) *If $S\cap G=X$ and $p=2$, then*

$$\sigma(M_wEM_u)=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)\ \textit{such that}\ |\lambda-E(uw)|\ge\epsilon\ a.e\},$$

where $S=S(E(|u|^2))$ and $G=S(E(|w|^2))$.

(c) $\sigma_p(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}\setminus\{0\}:\mu(A_{\lambda,w})>0\}$, *where* $A_{\lambda,w}=\{x\in X:E(uw)(x)=\lambda\}$.

(d) $\rho(M_wEM_u)\setminus\{0\}=\{\lambda\in\mathbb{C}:(\exists\epsilon>0)$ *such that* $|\lambda-E(uw)|\ge\epsilon$ *a.e*$\}\setminus\{0\}$.

(e) $\sigma_r(M_wEM_u)\setminus\{0\}=\begin{cases}\emptyset, & 1\le p<\infty,\\ ER(E(uw))\setminus\{0\}\cup\sigma_p(M_wEM_u), & p=\infty,\end{cases}$ *in which* $ER(E(uw))=\{\lambda\in\mathbb{C}:(\nexists\epsilon>0)$ *such that* $|\lambda-E(uw)|\ge\epsilon$ *a.e*$\}$.

Proof. (a) Since

$$\sigma(M_wEM_u)\setminus\{0\}=\sigma(EM_uM_w)\setminus\{0\}=\sigma(EM_{uw})\setminus\{0\}=\operatorname{ess\,range}(E(uw))\setminus\{0\},$$

then we have

$$\sigma(M_wEM_u)\setminus\{0\}=\operatorname{ess\,range}(E(uw))\setminus\{0\}.$$

(b) We know that $\sigma(EM_{uw})=\operatorname{ess\,range}(E(uw))$. So, we have to prove that $0\notin\sigma(EM_{uw})$ if and only if $0\notin\sigma(M_wEM_u)$ by (a).

Let $0\notin\sigma(EM_{uw})$. Then EM_{uw} is surjective and so $\mathcal{A}=\Sigma$. Thus, $E=I$. So, $0\notin\sigma(M_wEM_u)$.

Conversely, we know that the polar decomposition of $M_wEM_u=U|M_wEM_u|$ is as follow

$$|M_wEM_u|(f)=\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S\bar{u}E(uf)$$

and

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),$$

for all $f\in L^2(\Sigma)$.

If $0 \notin \sigma(M_wEM_u)$, then $|M_wEM_u|$ is invertible and U is unitary [page 73, [4]]. Therefore, $U^*U = UU^* = I$. The equation $UU^* = I$ implies that $w \in L^0(\mathcal{A}_{S\cap G})$, where $\mathcal{A}_{S\cap G} = \{A \cap S \cap G : A \in \mathcal{A}\}$. Since $S \cap G = X$, then $w \in L^0(\mathcal{A})$. Hence, $0 \notin \sigma(M_wEM_u) = \sigma(EM_{uw})$.

(c) By Theorem 3.1 we have

$$\sigma_p(M_wEM_u) \setminus \{0\} = \sigma_p(EM_uM_w) \setminus \{0\} = \sigma_p(EM_{uw}) \setminus \{0\}.$$

So,

$$\sigma_p(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\}.$$

The proof of (d) and (e) are as the same as (c). □

If the converse of conditional-type Hölder inequality is satisfied for w and u, then we get that the joint point spectrum and point spectrum of M_wEM_u are equal. Hence we have the next remark.

Remark 3.1. If $M_wEM_u : L^2(\Sigma) \to L^2(\Sigma)$ and $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then the following hold.

(a) $\sigma_{jp}(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \mu(A_{\lambda,w}) > 0\}$.

(b) If $S \cap G = X$, then

$$\sigma_{jp}(M_wEM_u) = \operatorname{ess\,range}(E(uw)) = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\}.$$

By Theorem 3.2 we have the following corollary.

Corollary 3.1. *For $T = M_wEM_u : L^p(\Sigma) \to L^p(\Sigma)$ and $1 \leq p \leq \infty$ we have $r(T) = \|E(uw)\|_\infty$.*

Recall that an operator T is quasinilpotent if $\sigma(T) = \{0\}$. In light of Theorem 3.2, we have the following proposition.

Proposition 3.1. *If $T = M_wEM_u : L^p(\Sigma) \to L^p(\Sigma)$, $1 \leq p \leq \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$, then the following are equivalent:*

(a) *M_wEM_u is quasinilpotent;*

(b) *$E(uw) \equiv 0$;*

(c) *$M_{wE(uw)}EM_u \equiv 0$.*

Proof. ($a \Leftrightarrow b$) Since $\sigma(M_wEM_u) \setminus \{0\} = \operatorname{ess\,range}(E(uw)) \setminus \{0\}$, it follows that $\sigma(M_wEM_u) = 0$ if and only if $E(uw) = 0$ almost every where.

($b \Leftrightarrow c$) As we investigated the norm of weighted conditional type operators in [3], we have

$$\|M_{wE(uw)}EM_u\| = \|E(uw)(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}}\|_\infty$$

and

$$|E(uw)| \leq (E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}},$$

it follows that $E(uw) = 0$, a.e, if and only if $E(uw)(E(|w|^p))^{\frac{1}{p}}(E(|u|^{p'}))^{\frac{1}{p'}} = 0$, a.e, if and only if $\|M_{wE(uw)}EM_u\| = 0$. □

Now, we give some example of conditional expectation.

Example 3.1. (a) Let $X = \mathbb{N} \cup \{0\}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{x\}) = \frac{e^{-\theta}\theta^x}{x!}$, for each $x \in X$ and $\theta \geq 0$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B = \{\emptyset, X, \{0\}, X_1 = \{1, 3, 5, 7, 9, \dots\}, X_2 = \{2, 4, 6, 8, \dots\},\}$ of $\mathbb{N}$. Note that, $\mathcal{A}$ is a sub-σ-algebra of Σ and each of element of $\mathcal{A}$ is an $\mathcal{A}$-atom. Thus, the conditional expectation of any $f \in \mathcal{D}(E)$ relative to $\mathcal{A}$ is constant on $\mathcal{A}$-atoms. Hence, there exist scalars a_1, a_2, a_3 such that

$$E(f) = a_1\chi_0 + a_2\chi_{X_1} + a_3\chi_{X_2}.$$

So,

$$E(f)(0) = a_1, \quad E(f)(2n-1) = a_2, \quad E(f)(2n) = a_3,$$

for all $n \in \mathbb{N}$. By definition of conditional expectation with respect to $\mathcal{A}$, we have

$$f(0)\mu(\{0\}) = \int_{\{0\}} f d\mu = \int_{\{0\}} E(f) d\mu = a_1\mu(\{0\}),$$

so $a_1 = f(0)$. Also,

$$\sum_{n\in\mathbb{N}} f(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!} = \int_{X_1} f d\mu = \int_{X_1} E(f) d\mu = a_2\mu(X_2) = a_2 \sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}.$$

So,

$$a_2 = \frac{\sum_{n\in\mathbb{N}} f(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}.$$

By the same method we have

$$a_3 = \frac{\sum_{n\in\mathbb{N}} f(2n)\frac{e^{-\theta}\theta^{2n}}{(2n)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n}}{(2n)!}}.$$

If we set $f(x) = x$, then $E(f)$ is a special function as follows:

$$E(f) = \theta\coth(\theta)\chi_{X_1} + \frac{\cosh(\theta) - 1}{\cosh(\theta)}\chi_{X_2}.$$

Also, if u and w are real functions on X such that M_wEM_u is bounded on l^p, then by Theorem 3.2 we have

$$\begin{aligned}&\sigma(M_wEM_u)\\ &= \left\{u(0)w(0), \frac{\sum_{n\in\mathbb{N}} u(2n-1)w(2n-1)\frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n-1}}{(2n-1)!}}, \frac{\sum_{n\in\mathbb{N}} u(2n)w(2n)\frac{e^{-\theta}\theta^{2n}}{(2n)!}}{\sum_{n\in\mathbb{N}} \frac{e^{-\theta}\theta^{2n}}{(2n)!}}\right\}.\end{aligned}$$

(b) Let $X = \mathbb{N}$, $\mathcal{G} = 2^{\mathbb{N}}$ and let $\mu(\{x\}) = pq^{x-1}$ for each $x \in X$, $0 \leq p \leq 1$ and $q = 1 - p$. Elementary calculations show that μ is a probability measure on $\mathcal{G}$. Let $\mathcal{A}$ be the σ-algebra generated by the partition $B = \{X_1 = \{3n : n \geq 1\}, X_1^c\}$ of X. So, for every $f \in \mathcal{D}(E^{\mathcal{A}})$

$$E(f) = \alpha_1\chi_{X_1} + \alpha_2\chi_{X_1^c},$$

and direct computations show that

$$\alpha_1 = \frac{\sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}$$

and

$$\alpha_2 = \frac{\sum_{n\geq 1} f(n)pq^{n-1} - \sum_{n\geq 1} f(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}}.$$

For example, if we set $f(x) = x$, then $E(f)$ is a special function as follows

$$\alpha_1 = \frac{3}{1-q^3}, \quad \alpha_2 = \frac{1+q^6-3q^4+4q^3-3q^2}{(1-q^2)(1-q^3)}.$$

So, if u and w are real functions on X such that M_wEM_u is bounded on l^p, then by Theorem 3.2 we have

$$\sigma(M_wEM_u) = \left\{ \frac{\sum_{n\geq 1} u(3n)w(2n)pq^{3n-1}}{\sum_{n\geq 1} pq^{3n-1}}, \frac{\sum_{n\geq 1} u(n)w(n)pq^{n-1} - \sum_{n\geq 1} u(3n)w(3n)pq^{3n-1}}{\sum_{n\geq 1} pq^{n-1} - \sum_{n\geq 1} pq^{3n-1}} \right\}.$$

(c) Let $X = [0,1)$, Σ is σ-algebra of Lebesgue measurable subsets of X, μ is the Lebesgue measure on X. Let $s : [0,1) \to [0,1)$ be defined by $s(x) = x + \frac{1}{4}(\text{mod } 1)$. Let $\mathcal{B} = \{E \in \Sigma : s(E) = E\}$. In this case

$$E^{\mathcal{B}}(f)(x) = \frac{f(x) + f(s(x)) + f(s^2(x)) + f(s^3(x))}{4},$$

where s^j denotes the jth iteration of s. Also, $|f| \leq 3E^{\mathcal{B}}(|f|)$ a.e. Hence, the operator EM_u is bounded on $L^p([0,1))$ if and only if $u \in L^\infty([0,1))$.

(d) Let $X = [0,a] \times [0,a]$ for $a > 0$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0,a] : A \text{ is a Lebesgue set in } [0,a]\}$. Then, for each $f \in \mathcal{D}(E)$, $(Ef)(x,y) = \int_0^a f(x,t)dt$, which is independent of the second coordinate. For example, if we set $a = 1$, $w(x,y) = 1$ and $u(x,y) = e^{(x+y)}$, then $E(u)(x,y) = e^x - e^{x+1}$ and M_wEM_u is bounded. Therefore, by Theorem 3.2 $\sigma(M_wEM_u) = [e - e^2, 1 - e]$.

References

[1] P. G. Dodds, C. B. Huijsmans and B. De Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.

[2] Y. Estaremi, *Unbounded weighted conditional expectation operators*, Complex Anal. Oper. Theory **10** (2016), 567–580.

[3] Y. Estaremi and M. R. Jabbarzadeh, *Weighted Lambert type operators on L^p-spaces*, Oper. Matrices **1** (2013), 101–116.

[4] T. Furuta, *Invitation to Linear Operators*, Taylor and Francis Group, London, New York, 2001.

[5] J. J. Grobler and B. de Pagter, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory **48** (2002), 15–40.

[6] J. Herron, *Weighted conditional expectation operators*, Oper. Matrices **1** (2011), 107–118.

[7] A. Lambert, *L^p multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.

[8] S.-T. Chen and Moy, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math. **4** (1954), 47–63.

[9] G. J. Murphy, *C^*- Algebras and Operator Theory*, Academic Press, Boston, 1990.

[10] M. M. Rao, *Conditional Measure and Applications*, Marcel Dekker, New York, 1993.

[11] P. R. Suri and N. Singh, *M-quasi-hyponormal composition operators*, Int. J. Math. Math. Sci. **3** (1987) 621–623.

[12] T. Yamazaki, *An expression of spectral radius via Aluthge transformation*, Proc. Amer. Math. Soc. **4** (2001), 1131–1137.

[1]Department of Mathematics,
Payame Noor University,
Tehran, Iran

[2]Department of Mathematics and Computer Sciences,
Golestan University,
Gorgan, Iran
Email address: yestaremi@pnu.ac.ir
Email address: estaremi@gmail.com

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 895–903.

TOTAL ABSOLUTE DIFFERENCE EDGE IRREGULARITY STRENGTH OF GRAPHS

R. RAMALAKSHMI[1] AND KM. KATHIRESAN[2]

Abstract. We introduce a new graph characteristic, the total absolute difference edge irregularity strength. We obtain the estimation on the total absolute difference edge irregularity strength and determine the precise values for some families of graphs.

1. Introduction

Throughout this paper, G is a simple graph, V and E are the sets of vertices and edges of G, with cardinalities $|V|$ and $|E|$ respectively. A labeling of a graph is a map that carries graph elements to the numbers.

A labeling is called a vertex labeling, an edge labeling or a total labeling, if the domain of the map is the vertex set, the edge set, or the union of vertex and edge sets respectively. Baca et al. in [2] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labeling. For a graph $G = (V(G), E(G))$, the weight of an edge $e = e_1e_2$ under a total labeling ξ is $wt_\xi(e) = \xi(e_1) + \xi(e) + \xi(e_2)$. For a graph G we define a labeling $\xi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an edge irregular total k-labeling of a graph G if for every two different edges xy and $x'y'$ of G one has $wt_\xi(xy) \neq wt_\xi(x'y')$. The total edge irregular strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. In [2], we can find that

$$tes(G) \geq \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\},$$

Key words and phrases. Edge irregularity strength, total absolute difference edge irregularity strength.

2010 *Mathematics Subject Classification.* Primary: 05C78.

DOI 10.46793/KgJMat2206.895R

Received: October 03, 2019.

Accepted: May 15, 2020.

where $\Delta(G)$ is the maximum degree of G, and also there are determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs. Recently Ivanco and Jendrol [3] proved that for any tree T

$$tes(T) = \max\left\{\left\lceil\frac{|E\,(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta\,(G)+1}{2}\right\rceil\right\}.$$

Moreover, they posed a conjecture that for an arbitrary graph G different from K_5 and the maximum degree $\Delta(G)$

$$tes(G) = \max\left\{\left\lceil\frac{|E\,(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta\,(G)+1}{2}\right\rceil\right\}.$$

The Ivanco and Jendrol's conjecture has been verified for complete graphs and complete bipartite graphs in [4], for categorical product of cycle and path in [1] and [6], for corona product of paths with some graphs in [5].

A graceful labeling of a graph $G = (V, E)$ with $|V|$ vertices and $|E|$ edges is a one-to-one mapping Ψ of the vertex set $V(G)$ into the set $\{0, 1, 2, \ldots, |E|\}$ with the following property: If we define, for any edge $e = uv \in E(G)$, the value $\Psi'(e) = |\Psi(u) - \Psi(v)|$ then Ψ' is a one-to-one mapping of the set $E(G)$ onto the set $\{1, 2, \ldots, |E|\}$. Motivated by the total edge irregularity strength of a graph and motivated by the graceful labeling, we introduce and investigate the total absolute difference edge irregularity strength of graphs to reduce the edge weights.

A total labeling ξ is defined to be an edge irregular total absolute difference k-labeling of the graph G if for every two different edges e and f of G there is $wt(e) \neq wt(f)$ where weight of an edge $e = xy$ is defined as $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. The minimum k for which the graph G has an edge irregular total absolute difference labeling is called the total absolute difference edge irregularity strength of the graph G, $tades(G)$. The main aim of this paper is to obtain estimations on the parameter $tades$ and determine the precise values of $tades$ for some families of graphs.

2. Main Results

The following result shows that the absolute difference edge irregularity strength is defined for all graphs.

Theorem 2.1. *Let $G = (V, E)$ be a graph with vertex set V and a non-empty edge set E. Then* $\left\lceil\frac{|E|}{2}\right\rceil \leq tades(G) \leq |E| + 1$.

Proof. To get the upper bound we label each vertex of G with label 1 and the edges of G consecutively with labels $2, 3, \ldots, |E|+1$. Then $wt(e)$ are consecutively $0, 1, \ldots, |E|-1$ and the weights for any two distinct edges e and f are distinct.

To get the lower bound, let ξ be an optimal labeling with respect to the tades(G). The weight of the heaviest edge implies that $|\xi(e) - \xi(x) - \xi(y)| \geq |E| - 1$.

That is, $\xi(e)-\xi(x)-\xi(y) \geq |E|-1$ if $\xi(e) > \xi(x)+\xi(y)$, $\xi(x)+\xi(y)-\xi(e) \geq |E|-1$. If $\xi(e) < \xi(x)+\xi(y)$, then

$$\xi(x)+\xi(y)-\xi(e) \geq |E|-1,$$

which implies

$$\xi(x)+\xi(y) \geq |E|-1+\xi(e) \geq |E|-1+1 = |E|.$$

That is,

$$\xi(x)+\xi(y) \geq |E|.$$

So at least one label is at least $\left\lceil \frac{|E|}{2} \right\rceil$.

If $\xi(e) > \xi(x)+\xi(y)$, then $\xi(e)-\xi(x)-\xi(y) \geq |E|-1$.

Suppose $\xi(e) < \left\lceil \frac{|E|}{2} \right\rceil$, then

$$-\xi(x)-\xi(y) \geq |E|-1-\xi(e) > |E|-1-\left\lceil \frac{|E|}{2} \right\rceil > \left\lfloor \frac{|E|}{2} \right\rfloor - 1.$$

That is,

$$\xi(x)+\xi(y) < 1-\left\lfloor \frac{|E|}{2} \right\rfloor = 0,$$

which is not possible. Hence,

$$\xi(e) \geq \left\lceil \frac{|E|}{2} \right\rceil.$$

That is,

$$\left\lceil \frac{|E|}{2} \right\rceil \leq tades(G) \leq |E|+1. \qquad \square$$

The lower bound in the Theorem 2.1 is tight as can be seen from the following theorem.

Theorem 2.2. *Let P_n be a path on $n \geq 4$ vertices. Then $tades(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$.*

Proof. From the Theorem 2.1 we have $tades(P_n) \geq \left\lceil \frac{n-1}{2} \right\rceil$. So, it is enough to prove that

$$tades(P_n) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

Let P_n be the path $v_1e_1v_2e_2v_3, \ldots, v_{n-1}e_{n-1}v_n$, $n \geq 4$. Now define a mapping $\xi : V \cup E \to \left\{1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}$ by $\xi(v_1) = 1$ and $\xi(v_i) = \left\lceil \frac{i-1}{2} \right\rceil$ for $2 \leq i \leq n$, $\xi(e_1) = 2$ and $\xi(e_i) = 1$ for $2 \leq i \leq n-1$. Now,

$$\max\left\{\{\xi(v)|v \in V(P_n)\} \cup \{\xi(e)|e \in E(P_n)\}\right\} = \left\lceil \frac{n-1}{2} \right\rceil$$

and the edge weights are given by

$$wt(e_1) = |\xi(e_1)-\xi(v_1)-\xi(v_2)| = |2-1-1| = 0,$$

for $2 \leq i \leq n-1$,

$$wt(e_i) = |\xi(e_i) - \xi(v_i) - \xi(v_{i+1})| = i - 1.$$

Hence, the weights are distinct. Therefore, $tades(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$. □

Observation 1. We observe that $tades(P_3) = 2$, $tades(P_2) = 1$.

The upper bound in the Theorem 2.1 is not sharp. If we utilize the maximum degree $\Delta = \Delta(G)$ of the graph G, we obtain the following result.

Theorem 2.3. *Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$. Then $tades(G) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil$.*

Proof. Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$. Let x be a vertex of G with maximum degree Δ in G. Let $e_j = xu_j$ be the edges incident with the vertex x, $1 \leq j \leq \Delta$. Assume to the contrary that, $tades(G) < \left\lceil \frac{\Delta+1}{2} \right\rceil$. Suppose ξ is an optimal total labeling of G. Then $wt(e_j)$ is either $\xi(e_j) - \xi(x) - \xi(u_j)$ or $-\xi(e_j) + \xi(x) + \xi(u_j)$ for $1 \leq j \leq \Delta$. Among the Δ edges, let i denote the number of edges that have weight $-\xi(e_j) + \xi(x) + \xi(u_j)$. Then $0 \leq i \leq \Delta$. Suppose $i = 0$, then all the edges have weight $\xi(e_j) - \xi(x) - \xi(u_j)$. Then $wt(e_j) = \xi(e_j) - \xi(x) - \xi(u_j)$ for $1 \leq j \leq \Delta$. Since $\xi(x) \geq 1$ and $\xi(u_j) \geq 1$, we have $-\xi(x) \leq -1$ and $-\xi(u_j) \leq -1$. Therefore, $wt(e_j) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor - 2$. Then we have at most $\left\lfloor \frac{\Delta}{2} \right\rfloor - 1$ distinct weights, but we need at least Δ weights. Therefore, $i = 0$ is not possible. Among the edges e_j, $1 \leq j \leq \Delta$, there is an edge $e_k = xu_k$ with weight $wt(e_k) = -\xi(e_k) + \xi(x) + \xi(u_k)$. That is, $\xi(x) + \xi(u_k) \leq \xi(e_k) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$. That is, $\xi(x) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor - 1$. Then the possible values of $\xi(x)$ are $1, 2, \ldots, \left\lfloor \frac{\Delta}{2} \right\rfloor - 1$. Now, fix the value $\left\lfloor \frac{\Delta}{2} \right\rfloor - h$ to $\xi(x)$ for some $1 \leq h \leq \left\lfloor \frac{\Delta}{2} \right\rfloor - 1$. Then

$$wt(e_j) = -\xi(e_j) + \xi(x) + \xi(u_j) = -\xi(e_j) + \left\lfloor \frac{\Delta}{2} \right\rfloor - h + \xi(u_j).$$

Suppose $-\xi(e_j) + \left\lfloor \frac{\Delta}{2} \right\rfloor - h + \xi(u_j) \geq \Delta - h$, then $\xi(u_j) \geq \left\lceil \frac{\Delta}{2} \right\rceil + \xi(e_j) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil + 1$, which is a contradiction to our assumption. Therefore, $-\xi(e_j) + \left\lfloor \frac{\Delta}{2} \right\rfloor - h + \xi(u_j) < \Delta - h$. Then we have at most $\Delta - 1$ distinct weights, but we need at least Δ weights. Therefore, $i \neq 0$ is also not possible. Therefore, $tades(G) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil$. □

The lower bound in the Theorem 2.3 is tight as can be seen from the following theorem.

Theorem 2.4. *Let $S_n = K_{1,n}$ be a star on $n+1$ vertices, $n > 2$. Then $tades(S_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. From the Theorem 2.3

$$tades(S_n) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Let the vertices of S_n be $\{u, v_1, v_2, \dots, v_n\}$ where u is a vertex of maximum degree. Let $e_i = uv_i$, $1 \le i \le n$, be the edges of the star S_n. Now define the labeling $\xi : V \cup E \to \left\{1, 2, \dots, \left\lceil \frac{n+1}{2} \right\rceil\right\}$ by $\xi(u) = \left\lfloor \frac{n-1}{2} \right\rfloor$. Then

$$\xi(v_i) = \begin{cases} i, & \text{if } 1 \le i < \left\lceil \frac{n+1}{2} \right\rceil, \\ \lceil \frac{n+1}{2} \rceil, & \text{if } \lceil \frac{n+1}{2} \rceil \le i \le n, \end{cases}$$

$$\xi(uv_i) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } 1 \le i \le \lceil \frac{n+1}{2} \rceil, \\ 2\lceil \frac{n+1}{2} \rceil - i, & \text{if } \lceil \frac{n+1}{2} \rceil < i \le n,\ n \text{ is an odd integer}, \\ 2\lceil \frac{n+1}{2} \rceil - i - 1, & \text{if } \lceil \frac{n+1}{2} \rceil < i \le n,\ n \text{ is an even integer}. \end{cases}$$

Now,

$$\max\{\{\xi(v) \mid v \in V(S_n)\} \cup \{\xi(e) \mid e \in E(S_n)\}\} = \left\lceil \frac{n+1}{2} \right\rceil.$$

Also, $0 \le wt(uv_i) \le \left\lceil \frac{n+1}{2} \right\rceil - 1$ for $1 \le i \le \left\lceil \frac{n+1}{2} \right\rceil$

$$wt(uv_{\lceil \frac{n+1}{2} \rceil}) = \left\lceil \frac{n+1}{2} \right\rceil - 1,$$

$\left\lceil \frac{n+1}{2} \right\rceil \le wt(uv_i) \le n - 1$ for $\left\lceil \frac{n+1}{2} \right\rceil < i \le n$. Hence, the edge weights are distinct. Therefore, $tades(S_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

□

Observation 2. We observe that $tades(S_1) = 1$, $tades(S_2) = 2$.

In the next theorem, we discuss a technique to determine tades for some families of graphs.

Theorem 2.5. *Let G be a graph and $\phi : V(G) \to \{0, 1\}$ be a mapping and let $E_i(\phi) = \{xy \in E(G) \,|\phi(x) + \phi(y) = i\}$ for $i \in \{0, 1, 2\}$. If $|E_0(\phi)| \le k - 1$, $|E_1(\phi)| \le k - 1$, $|E_2(\phi)| = k - 1$ and $|E| \le 2k$, then G has a total edge-irregular absolute difference k-labeling.*

Proof. Let G be a graph and $\phi : V(G) \to \{0, 1\}$ be a mapping and

$$wt_\phi(e) = \phi(u) + \phi(v),$$

where u and v are end vertices of e. Let $E_i(\phi) = \{xy \in E(G) \mid \phi(x) + \phi(y) = i\}$ for $i = 0, 1, 2$. Suppose that $|E_0(\phi)| \le k - 1, |E_1(\phi)| \le k - 1, |E_2(\phi)| = k - 1$. Let $E_0(\phi) = \{e_1, e_2, \dots, e_{r_0}\}$, $E_1(\phi) = \{e'_1, e'_2, \dots, e'_{r_1}\}$ and $E_2(\phi) = \{e''_1, e''_2, \dots, e''_{k-1}\}$. Then define a mapping ξ_1 from $V(G)$ into the set of positive integers by $\xi_1(x) = k^{\phi(x)}$ if $x \in V(G)$. Under the labeling ϕ, there are r_0 edges with weight 0, r_1 edges with weight 1 and $k - 1$ edges with weight 2. Then under the vertex labling ξ_1, there are r_0 edges with weight 2, r_1 edges with weight $k + 1$ and $k - 1$ edges with weight $2k$. Now define a labeling ξ from $V(G) \cup E(G)$ into the set of positive integers as follows: $\xi(x) = \xi_1(x)$ for all $x \in V(G)$,

$$\xi(e_i) = \begin{cases} i, & \text{if } i = 1, 2, \\ i + 1, & \text{if } 3 \le i \le r_0, \end{cases}$$

$\xi(e_i') = i$ if $1 \le i \le r_1$, $\xi(e_i'') = i$ if $1 \le i \le k-1$. Then

$$\begin{aligned}\{wt_\xi(e_i) \mid 1 \le i \le r_0\} =& \{0, 1, 2, \ldots, r_0 - 1\},\\ \{wt_\xi(e_i') \mid 1 \le i \le r_1\} =& \{k, k-1, k-2, \ldots, k - r_1 + 1\},\\ \{wt_\xi(e_i'') \mid 1 \le i \le k-1\} =& \{2k-1, 2k-2, \ldots, k+1\}.\end{aligned}$$

Since $|E| \le 2k$, we have

$$r_0 + r_1 + k - 1 \le 2k.$$

That is $r_0 + r_1 \le k + 1$. That is, $r_0 - 1 \le k - r_1 < 1 + k - r_1$. Hence, the edge weights are distinct. Therefore, the graph G has a total edge-irregular absolute difference k-labeling. □

We determine the *tades* for the graphs C_n, S_n and F_n using the Theorems 2.1 and 2.5.

Theorem 2.6. *For $n \ge 3$, $tades(C_n) = \left\lceil \frac{n}{2} \right\rceil$.*

Proof. For $n = 3, 4, 5$, from the labeling given in the Figure 1, we get the required result.

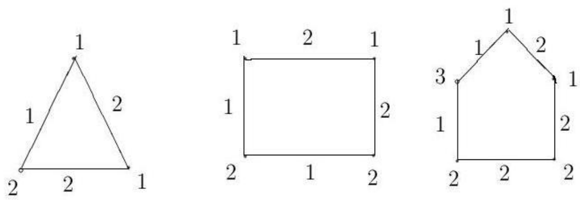

FIGURE 1. Tades for C_n, $n = 3, 4, 5$

For $n > 5$, the proof is as follows. From the Theorem 2.1, we have

$$tades(C_n) \ge \left\lceil \frac{n}{2} \right\rceil.$$

The vertex set of C_n is $\{u_i | 1 \le i \le n\}$ and the edge set of C_n is $\{u_i u_{i+1} | 1 \le i \le n-1\}$. Now, define the labeling $\phi : V(C_n) \to \{0, 1\}$ by

$$\phi(u_i) = \begin{cases} 1, & \text{if } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ 0, & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

Then

$$\begin{aligned} E_0 =& \left\{ u_{\lceil \frac{n}{2} \rceil + 1} u_{\lceil \frac{n}{2} \rceil + 2}, u_{\lceil \frac{n}{2} \rceil + 2} u_{\lceil \frac{n}{2} \rceil + 3}, \ldots, u_{n-1} u_n \right\}, \\ E_1 =& \left\{ u_n u_1, u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1} \right\} \end{aligned}$$

and

$$E_2 = \left\{ u_1 u_2, u_2 u_3, \ldots, u_{\lceil \frac{n}{2} \rceil - 1} u_{\lceil \frac{n}{2} \rceil} \right\}.$$

That is,

$$|E_0| = \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq \left\lceil \frac{n}{2} \right\rceil - 1,$$
$$|E_1| = 2 \leq \left\lceil \frac{n}{2} \right\rceil - 1$$

and

$$|E_2| = \left\lceil \frac{n}{2} \right\rceil - 1.$$

Take $k = \left\lceil \frac{n}{2} \right\rceil$. Then, by Theorem 2.5, C_n has a total edge-irregular absolute difference k-labeling. Hence, $tades(C_n) = \left\lceil \frac{n}{2} \right\rceil$. ☐

Theorem 2.7. *Let $\Im_n$ denote the sun graph on $2n$ vertices. Then $tades(\Im_n) = n$ for $n > 3$.*

Proof. The vertex set of $\Im_n$ is $V(\Im_n) = \{u_i \mid 1 \leq i \leq n\} \cup \{u'_i \mid 1 \leq i \leq n\}$ and the edge set of $\Im_n$ is $E(\Im_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_i u'_i \mid 1 \leq i \leq n\}$. Then $|E| = 2n$. From the Theorem 2.1, we have $tades(\Im_n) \geq n$. Now, define the labeling $\phi : V(\Im_n) \to \{0, 1\}$ by

$$\phi(u_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ 0, & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, \end{cases}$$

$$\phi(u'_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ 0, & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Then

$$E_0 = \begin{cases} \{u_i u_{i+1} \mid (\frac{n}{2}) + 1 \leq i \leq n-1\} \cup \{u_i u'_i \mid (\frac{n}{2}) + 1 \leq i \leq n\}, & \text{if } n \text{ is even,} \\ \{u_i u_{i+1} \mid \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-1\} \cup \{u_i u'_i \mid \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\}, & \text{if } n \text{ is odd,} \end{cases}$$

$$E_1 = \begin{cases} \{u_1 u_n, u_{\frac{n}{2}} u_{\frac{n}{2}+1}\}, & \text{if } n \text{ is even,} \\ \{u_1 u_n, u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1}, u_{\lceil \frac{n}{2} \rceil} u'_{\lceil \frac{n}{2} \rceil}\}, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$E_2 = \left\{ u_i u_{i+1} \middle| 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1 \right\} \cup \left\{ u_i u'_i \middle| 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Here

$$|E_0| = \begin{cases} n-1, & \text{if } n \text{ is even,} \\ n-2, & \text{if } n \text{ is odd,} \end{cases}$$

$$|E_1| = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$|E_2| = n - 1.$$

Then, by Theorem 2.5, $\Im_n$ has a total edge-irregular absolute difference k-labeling. Hence, $tades(\Im_n) = n$. ☐

Theorem 2.8. *Let $\boldsymbol{F}_n$ be the fan graph on $2n+1$ vertices, then $tades(\boldsymbol{F}_n) = \left\lceil \frac{3n}{2} \right\rceil$ for an odd integer n.*

Proof. Let $k = \left\lceil \frac{3n}{2} \right\rceil = \frac{3n+1}{2}$ for an odd integer n. The vertex set of $\boldsymbol{F}_n$ is

$$V(\boldsymbol{F}_n) = \{u, v_1, v_2, \dots, v_{2n}\}$$

and the edge set of $\boldsymbol{F}_n$ is

$$E(\boldsymbol{F}_n) = \{uv_i \mid 1 \leq i \leq 2n\} \cup \{v_{2i+1}v_{2i+2} \mid 0 \leq i \leq n-1\}.$$

From the Theorem 2.1, we have $tades(\boldsymbol{F}_n) \geq \lceil \frac{3n}{2} \rceil$. Now, define the labeling $\phi : V(\boldsymbol{F}_n) \to \{0,1\}$ by $\phi(u) = 1$

$$\phi(v_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq n, \\ 0, & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

Then

$$\begin{aligned} E_0 =& \left\{ v_{2i+1}v_{2i+2} \,\middle|\, \frac{n+1}{2} \leq i \leq n-1 \right\}, \\ E_1 =& \{uv_i \mid n+1 \leq i \leq 2n\} \cup \{v_n v_{n+1}\} \end{aligned}$$

and

$$\begin{aligned} E_2 =& \{uv_i \mid 1 \leq i \leq n\} \cup \left\{ v_{2i+1}v_{2i+2} \,\middle|\, 0 \leq i \leq \frac{n-3}{2} \right\}, \\ |E_0| =& \frac{n-1}{2} \leq k-1, \\ |E_1| =& n+1 \leq k-1 \end{aligned}$$

and

$$|E_2| = \frac{3n+1}{2} - 1 = k-1.$$

Then, by Theorem 2.5, $\boldsymbol{F}_n$ has a total edge-irregular absolute difference k-labeling. Hence, $tades(\boldsymbol{F}_n) = \left\lceil \frac{3n}{2} \right\rceil$. ☐

3. Open Problem and Conjectures

Problem. Determine $tades(\boldsymbol{F}_n)$ when n is even.

From our experience on this labeling, we propose the following conjectures.

Conjectures:

- for every tree T of maximum degree Δ on p vertices,

$$tades(T) = \max \left\{ \left\lceil \frac{p}{2} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil \right\};$$

- for any graph G, $tes(G) \leq tades(G)$.

Acknowledgements. We thank anonymous reviewer for the valuable comments on an earlier version of the manuscript.

References

[1] A. Ahmad and M. Baca, *Edge irregular total labeling of certain family of graphs*, AKCE Int. J. Graphs Comb. **6**(1) (2009), 21–29.

[2] M. Baca, S. Jendrol, M. Miller and J. Ryan, *On irregular total labeling*, Discrete Math. **307** (2007), 1378–1388.

[3] J. Ivanco and S. Jendrol, *Total edge irregularity strength for trees*, Discuss. Math. Graph Theory **26** (2006), 449–456.

[4] S. Jendrol, J. Miskuf and R. Sotak, *Total edge irregularity strength of complete graphs and complete bipartite graphs*, Discrete Math. **310**(2010), 400–407.

[5] Nurdin, A. N. M. Salman and E. T. Baskoro, *The total edge-irregular strengths of the corona product of paths with some graphs*, J. Combin. Math. Combin. Comput. **65** (2008), 163–175.

[6] M. K. Siddiqui, *On total edge irregularity strength of categorical product of cycle and path*, AKCE Int. J. Graphs Comb. **9**(1) (2012), 43–52.

[1]Department of Mathematics,
Rajapalayam Rajus' College,
Rajapalayam-626 117, Tamilnadu,
India
Email address: ramimani20@gmail.com

[2]Centre for Graph Theory,
Ayya Nadar Janaki Ammal College,
Sivakasi - 626 124, Tamilnadu
India
Email address: kathir2esan@yahoo.com

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 905–917.

STABILITY OF CAUCHY-JENSEN TYPE FUNCTIONAL EQUATION IN $(2,\alpha)$-BANACH SPACES

KHALED YAHYA NAIF SAYAR[1,2] AND AMAL BERGAM[1]

Abstract. In this paper, we investigate some stability and hyperstability results for the following Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

in $(2,\alpha)$-Banach spaces using Brzdęk and Ciepliński's fixed point approach.

1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and the set of real numbers by $\mathbb{R}$. By $\mathbb{N}_m$, $m\in\mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

Let $\mathbb{R}_+=[0,\infty)$ be the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B and we use the notation E_0 for the set $E\backslash\{0\}$.

The method of the proof of the main result corresponds to some observations in [12] and the main tool in it is a fixed point. The problem of the stability of functional equations was first raised by Ulam [30]. This included the following question concerning the stability of group homomorphisms.

Let $(G_1,*_1)$ be a group and let $(G_2,*_2)$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*_1 y),h(x)*_2 h(y))<\delta,$$

Key words and phrases. Stability, hyperstability, $(2,\alpha)$-Banach space, Cauchy-Jensen functional equation.

2010 *Mathematics Subject Classification.* Primary: 39B82. Secondary: 39B52, 47H10.

DOI 10.46793/KgJMat2206.905S

Received: December 02, 2019.

Accepted: May 18, 2020.

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$, with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

Hyers [19] provided the first partial answer to Ulam's question and obtained the result of stability where G_1 and G_2 are Banach spaces.

Aoki [5], Bourgin [7] considered the problem of stability with unbounded Cauchy differences. Later, Rassias [25, 26] used a direct method to prove a generalization of Hyers result (cf. Theorem 1.1).

The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$T(x + y) = T(x) + T(y).$$

Theorem 1.1. *Let E_1 be a normed space, E_2 be a Banach space and $f : E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \theta\Big(\|x\|^p + \|y\|^p\Big), \tag{1.1}$$

for some $\theta \ge 0$, for some $p \in \mathbb{R}$, with $p \ne 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{|2 - 2^p|}\|x\|^p, \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

It is due to Aoki [5] (for $0 < p < 1$, see also [24]), Gajda [17] (for $p > 1$) and Rassias [26] (for $p < 0$, see also [27, page 326] and [7]). Also, Brzdęk [8] showed that estimation (1.2) is optimal for $p \ge 0$ in the general case. Recently, Brzdęk [10] showed that Theorem 1.1 can be significantly improved. Namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive, and the assumption of completeness of E_2 is not necessary.

Regrettably, if we restrict the domain of f, this result will not remain valid (see the further detail in [14]). Nowadays, a lot of papers concerning the stability and the hyperstability of the functional equation in various spaces have been appeared (see in [1, 2, 4, 9, 11, 22, 28, 29] and references therein).

Let us recall first (see, for instance, [16]) some definitions.

We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. By a linear 2-normed space, we mean a pair $(X, \|\cdot, \cdot\|)$ such that X is at least a two-dimensional real linear space and

$$\|\cdot, \cdot\| : X \times X \to \mathbb{R}_+$$

is a function satisfying the following conditions:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\| = \|y, x\|$ for $x, y \in X$;
(c) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x, y, z \in X$;
(d) $\|\lambda x, y\| = |\lambda|\|x, y\|$, $\lambda \in \mathbb{R}$, and $x, y \in X$.

A generalized version of a linear 2-normed spaces is the $(2,\alpha)$-normed space defined in the following manner.

Definition 1.2. Let α be a fixed real number with $0 < \alpha \leq 1$, and let X be a linear space over K with dim $X > 1$. A function

$$\|\cdot, \cdot\|_\alpha : X \cdot X \to \mathbb{R}$$

is called a $(2,\alpha)$-norm on X if and only if it satisfies the following conditions:
(a) $\|x, y\|_\alpha = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\|_\alpha = \|y, x\|_\alpha$ for $x, y \in X$;
(c) $\|x, y + z\|_\alpha \leq \|x, y\|_\alpha + \|x, z\|_\alpha$ for $x, y, z \in X$;
(d) $\|\beta x, y\|_\alpha = |\beta|^\alpha \|x, y\|_\alpha$ for $\beta \in \mathbb{R}$ and $x, y \in X$.
The pair $(X, \|\cdot, \cdot\|_\alpha)$ is called a $(2,\alpha)$-normed space.

Example 1.1. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in E = \mathbb{R}^2$, the Euclidean $(2,\alpha)$-norm $\|x, y\|_\alpha$ is defined by

$$\|x, y\|_\alpha = |x_1 y_2 - x_2 y_1|^\alpha ,$$

where α is a fixed real number with $0 < \alpha \leq 1$.

Definition 1.3. A sequence $\{x_k\}$ in a $(2,\alpha)$-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\|_\alpha = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \to x$, with $k \to \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4. A sequence $\{x_k\}$ in a $(2,\alpha)$-normed space X is said to be a *Cauchy sequence* with respect to the $(2,\alpha)$-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\|_\alpha = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the $(2,\alpha)$-norm. Any complete $(2,\alpha)$-normed space is said to be a $(2,\alpha)$-Banach space.

Next, it is easily seen that we have the following property.

Lemma 1.1. *If X is a linear $(2,\alpha)$-normed space, $x, y_1, y_2 \in X$, y_1, y_2 are linearly independent, and $\|x, y_1\|_\alpha = \|x, y_2\|_\alpha = 0$, then $x = 0$.*

Let us yet recall a lemma from [23].

Lemma 1.2. *If X is a linear $(2,\alpha)$-normed space and $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence of elements of X, then*

$$\lim_{n\to\infty}\|x_n, y\|_\alpha = \|\lim_{n\to\infty} x_n, y\|_\alpha = 0, \quad y \in X.$$

Let E, Y be normed spaces. A function $f : E \to Y$ is Cauchy-Jensen provided it satisfies the functional equation

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x), \tag{1.3}$$

and we can say that $f : E \to Y$ is Cauchy-Jensen on E_0 if it satisfies (1.3) for all $x, y \in E_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$. Recently, interesting results concerning the Cauchy-Jensen functional equation (1.3) have been obtained in [3,6,18,20,21].

In 2018, Brzdęk and Ciepliński [12] proved a new fixed point theorem in 2-Banach spaces and showed its applications to the Ulam stability of some single-variable equations and the most important functional equation in several variables. And they extended the fixed point result to the n-normed spaces in [13].

The main purpose of this paper is to establish the stability result concerning the functional equation (1.3) in $(2,\alpha)$-Banach spaces using fixed point theorem which was prove by Brzdęk and Ciepliński [12]. Before approaching our main results, we present the fixed point theorem concerning $(2,\alpha)$-Banach spaces which is given in [15]. To present it, we use the following three hypotheses.

(H1) E is a nonempty set, $\left(Y, \|\cdot,\cdot\|_\alpha\right)$ is a $(2,\alpha)$-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}$, $f_i : E \to E$, $g_i : Y_0 \to Y_0$, and $L_i : E \times Y_0 \to \mathbb{R}_+$ for $i = 1, \ldots, j$.

(H2) $\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\right\|_\alpha \leq \sum_{i=1}^{j} L_i(x,y)\left\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g_i(y)\right\|_\alpha,$$

for all $\xi, \mu \in Y^E$, $x \in E$, $y \in Y_0$.

(H3) $\Lambda : \mathbb{R}_+^{E\times Y_0} \to \mathbb{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta\big(f_i(x), g_i(y)\big), \quad \delta \in \mathbb{R}_+^{E\times Y_0},\ x \in E,\ y \in Y_0.$$

Theorem 1.2 ([15])**.** *Let hypotheses* (H1)-(H3) *hold and functions $\varepsilon : E \times Y_0 \to \mathbb{R}_+$ and $\varphi : E \to Y$ fulfill the following two conditions:*

$$\left\|\mathcal{T}\varphi(x) - \varphi(x), y\right\|_\alpha \leq \varepsilon(x,y), \quad x \in E,\ y \in Y_0,$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty}\left(\Lambda^n\varepsilon\right)(x,y) < \infty, \quad x \in E,\ y \in Y_0.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ for which

$$\left\|\varphi(x) - \psi(x), y\right\|_\alpha \leq \varepsilon^*(x,y), \quad x \in E,\ y \in Y_0.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n\varphi)(x), \quad x \in E.$$

2. Main Results

In this section, we prove some stability results for the Cauchy-Jensen equation (1.3) in $(2,\alpha)$-Banach spaces by using Theorem 1.2. In what follows $(Y, \|\cdot,\cdot\|_\alpha)$ is a real $(2,\alpha)$-Banach space.

Theorem 2.1. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number, with $0 < \alpha \leq 1$, Y_0 be a subset of Y containing two linearly independent vectors and $h_1, h_2 : E_0 \times Y_0 \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \{n \in \mathbb{N} \colon b_n := \lambda_1(2+n)\lambda_2(2+n) + \lambda_1(1+n)\lambda_2(1+n) < 1\} \neq \emptyset,$$

where

$$\lambda_i(n) := \inf\{t \in \mathbb{R}_+ \colon h_i(nx,z) \leq t\, h_i(x,z),\, x \in E_0, z \in Y_0\},$$

for all $n \in \mathbb{N}$, where $i = 1,2$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), z \right\|_\alpha \leq h_1(x,z)h_2(y,z), \tag{2.1}$$

for all $x, y \in E_0$, $z \in Y_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, then there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\left\| f(x) - F(x), z \right\|_\alpha \leq \lambda_0 h_1(x,z)h_2(x,z), \tag{2.2}$$

for all $x \in E_0$, $z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_1(2+n)\lambda_2(n)}{1-b_n} \right\}.$$

Proof. Replacing x by $(2+m)x$ and y by mx, where $x \in E_0$ and $m \in \mathbb{N}$, in inequality (2.1), we get

$$\left\| f\Big((2+m)x\Big) - f((1+m)x) - f(x), z \right\|_\alpha \leq h_1((2+m)x,z)h_2(mx,z), \tag{2.3}$$

for all $x \in E_0$, $z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m\xi(x) := \xi\Big((2+m)x\Big) - \xi((1+m)x), \quad \xi \in Y^{E_0},\, x \in E_0.$$

Further put

$$\varepsilon_m(x,z) := h_1((2+m)x,z)h_2(mx,z), \quad x \in E_0,\, z \in Y_0, \tag{2.4}$$

and observe that

$$\varepsilon_m(x,z) = h_1((2+m)x,z)h_2(mx,z) \leq \lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z), \tag{2.5}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathbb{N}$. Then the inequality (2.3) takes the form

$$\left\| \mathcal{T}_m f(x) - f(x), z \right\|_\alpha \leq \varepsilon_m(x,z), \quad x \in E_0,\, z \in Y_0.$$

Furthermore, for every $x \in E_0$, $z \in Y^{E_0}$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned}\left\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\right\|_\alpha &= \Big\|\xi\Big((2+m)x\Big) - \xi((1+m)x) \\ &\quad - \mu\Big((2+m)x\Big) + \mu((1+m)x), z\Big\|_\alpha \\ &\leq \left\|(\xi-\mu)\Big((2+m)x\Big), z\right\|_\alpha + \left\|(\xi-\mu)((1+m)x), z\right\|_\alpha.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0\times Y_0} \to \mathbb{R}_+^{E_0\times Y_0}$ by

$$\Lambda_m\delta(x,z) := \delta\Big((2+m)x, z\Big) + \delta((1+m)x, z), \quad \delta \in \mathbb{R}_+^{E_0\times Y_0},\ x \in E_0,\ z \in Y_0.$$

For each $m \in \mathbb{N}$ the above operator has the form described in (H2) with $f_1(x) = (2+m)x$, $f_2(x) = (1+m)x$, $g_1(z) = g_2(z) = z$ and $L_1(x) = L_2(x) = 1$ for all $x \in E_0$. By mathematical induction on $n \in \mathbb{N}_0$, we prove that

$$(\Lambda_m^n\varepsilon_m)(x,z) \leq \lambda_1(2+m)\lambda_2(m)b_m^n h_1(x,z)h_2(x,z), \tag{2.6}$$

for all $x \in E_0$ and $z \in Y_0$, where

$$b_m = \lambda_1(2+m)\lambda_2(2+m) + \lambda_1(1+m)\lambda_2(1+m).$$

From (2.4) and (2.5), we obtain that the inequality (2.6) holds for $n = 0$. Next, we will assume that (2.6) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,z) &= \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right) \\ &= (\Lambda_m^k\varepsilon_m)\Big((2+m)x, z\Big) + (\Lambda_m^k\varepsilon_m)((1+m)x, z) \\ &\leq \lambda_1(2+m)\lambda_2(m)b_m^k h_1((2+m)x,z)h_2((2+m)x,z) \\ &\quad + \lambda_1(2+m)\lambda_2(m)b_m^k h_1((1+m)x,z)h_2((1+m)x,z) \\ &= \lambda_1(2+m)\lambda_2(m)b_m^{k+1}h_1(x,z)h_2(x,z)\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. This shows that (2.6) holds for $n = k+1$. Now we can conclude that the inequality (2.6) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z) \\ &\leq \sum_{n=0}^{\infty}\lambda_1(2+m)\lambda_2(m)b_m^n h_1(x,z)h_2(x,z) \\ &= \frac{\lambda_1(2+m)\lambda_2(m)}{1-b_m}h_1(x,z)h_2(x,z) < \infty,\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1.2 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\Big(\mathcal{T}_m^n f\Big)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\Big\|f(x) - F_m(x), z\Big\|_\alpha \le \frac{\lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z)}{1-b_m}, \quad x \in E_0,\ z \in Y_0,\ m \in \mathcal{U}. \tag{2.7}$$

To prove that F_m satisfies the functional equation (1.3), just prove the following inequality

$$\Big\|(\mathcal{T}_m^n f)\Big(\frac{x+y}{2}\Big) + (\mathcal{T}_m^n f)\Big(\frac{x-y}{2}\Big) - (\mathcal{T}_m^n f)(x), z\Big\|_\alpha \le b_m^n h_1(x,z)h_2(y,z), \tag{2.8}$$

for every $x, y \in E_0$, $z \in Y_0$, $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ is just (2.1), take $k \in \mathbb{N}$ and assume that (2.8) holds for $n = k$. Then, for each $x, y \in E_0$, $z \in Y_0$ and $m \in \mathcal{U}$, we have

$$\begin{aligned}
&\Big\|(\mathcal{T}_m^{k+1} f)\Big(\frac{x+y}{2}\Big) + (\mathcal{T}_m^{k+1} f)\Big(\frac{x-y}{2}\Big) - (\mathcal{T}_m^{k+1} f)(x), z\Big\|_\alpha \\
=&\Big\|\mathcal{T}_m^k f\Big((2+m)\Big(\frac{x+y}{2}\Big)\Big) - \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x+y}{2}\Big)\Big) \\
&+ \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x-y}{2}\Big)\Big) - \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x-y}{2}\Big)\Big) \\
&- \mathcal{T}_m^k f\Big((2+m)x\Big) + \mathcal{T}_m^k f((1+m)x), z\Big\|_\alpha \\
\le&\Big\|\mathcal{T}_m^k f\Big((2+m)\Big(\frac{x+y}{2}\Big)\Big) + \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x-y}{2}\Big)\Big) \\
&- \mathcal{T}_m^k f\Big((2+m)x\Big), z\Big\|_\alpha + \Big\|\mathcal{T}_m^k f\Big((1+m)\Big(\frac{x+y}{2}\Big)\Big) \\
&+ \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x-y}{2}\Big)\Big) - \mathcal{T}_m^k f((1+m)x), z\Big\|_\alpha \\
\le& b_m^k h_1\Big((2+m)x, z\Big)h_2\Big((2+m)y, z\Big) + b_m^k h_1\Big((1+m)x, z\Big)h_2\Big((1+m)y, z\Big) \\
=& b_m^{k+1} h_1(x,z)h_2(y,z).
\end{aligned}$$

Thus, by using the mathematical induction on $n \in \mathbb{N}_0$, we have shown that (2.8) holds for all $x, y \in E_0$, $z \in Y_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.8), we obtain the equality

$$F_m\Big(\frac{x+y}{2}\Big) + F_m\Big(\frac{x-y}{2}\Big) = F_m(x),$$

for all $x, y \in E_0$, such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, $m \in \mathcal{U}$. This implies that $F_m : E \to Y$, defined in this way, is a solution of the equation

$$F(x) = F\Big((2+m)x\Big) - F((1+m)x), \quad x \in E_0,\ m \in \mathcal{U}. \tag{2.9}$$

Next, we will prove that each Cauchy-Jensen function $F : E \to Y$ satisfying the inequality

$$\Big\|f(x) - F(x), z\Big\|_\alpha \le L\, h_1(x,z)h_2(x,z), \quad x \in E_0,\ z \in Y_0 \tag{2.10}$$

with some $L > 0$, is equal to F_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F : E \to Y$ satisfying (2.10). From (2.7), for each $x \in E$, we get

$$\begin{aligned}\Big\|F(x) - F_{m_0}(x), z\Big\|_\alpha \leq& \Big\|F(x) - f(x), z\Big\|_\alpha + \Big\|f(x) - F_{m_0}(x), z\Big\|_\alpha \\ \leq& L\ h_1(x, z)h_2(x, z) + \varepsilon^*_{m_0}(x, z) \\ \leq& L_0\ h_1(x, z)h_2(x, z) \sum_{n=0}^{\infty} b^n_{m_0}, \end{aligned} \tag{2.11}$$

where $L_0 := (1-b_{m_0})L + \lambda_1(m_0)\lambda_2(m_0) > 0$ and we exclude the case that $h_1(x, z) \equiv 0$ or $h_2(x, z) \equiv 0$, which is trivial. Observe that F and F_{m_0} are solutions to equation (2.9) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\Big\|F(x) - F_{m_0}(x), z\Big\|_\alpha \leq L_0\ h_1(x, z)h_2(x, z) \sum_{n=j}^{\infty} b^n_{m_0}, \qquad x \in E_0,\ z \in Y_0. \tag{2.12}$$

The case $j = 0$ is exactly (2.11). We fix $k \in \mathbb{N}$ and assume that (2.12) holds for $j = k$. Then, in view of (2.11), for each $x \in E_0$, $z \in Y_0$, we get

$$\begin{aligned}\Big\|F(x) - F_{m_0}(x), z\Big\|_\alpha =& \Big\|F\Big((2 + m_0)x\Big) - F((1 + m_0)x) \\ & - F_{m_0}\Big((2 + m_0)x\Big) + F_{m_0}((1 + m_0)x), z\Big\|_\alpha \\ \leq& \Big\|F\Big((2 + m_0)x\Big) - F_{m_0}\Big((2 + m_0)x\Big), z\Big\|_\alpha \\ & + \Big\|F((1 + m_0)x) - F_{m_0}((1 + m_0)x), z\Big\|_\alpha \\ \leq& L_0\ h_1\Big((2 + m_0)x, z\Big)h_2\Big((2 + m_0)x, z\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ & + L_0\ h_1\Big((1 + m_0)x, z\Big)h_2\Big((1 + m_0)x, z\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ =& L_0\ \Big(h_1\Big((2 + m_0)x, z\Big)h_2\Big((2 + m_0)x, z\Big) \\ & + h_1\Big((1 + m_0)x, z\Big)h_2\Big((1 + m_0)x, z\Big)\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ \leq& L_0\ b_{m_0}h_1(x, z)h_2(x, z) \sum_{n=k}^{\infty} b^n_{m_0} \\ =& L_0\ h_1(x, z)h_2(x, z) \sum_{n=k+1}^{\infty} b^n_{m_0}. \end{aligned}$$

This shows that (2.12) holds for $j = k+1$. Now we can conclude that the inequality (2.12) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (2.12), we get

$$F = F_{m_0}. \tag{2.13}$$

Thus, we have also proved that $F_m = F_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (2.7)) yields

$$\left\|f(x) - F_{m_0}(x), z\right\|_\alpha \le \frac{\lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z)}{1-b_m},$$

for all $x \in E_0, z \in Y_0,\ m \in \mathcal{U}$. This implies (2.2) with $F = F_{m_0}$ and (2.13) confirms the uniqueness of F. $\square$

Theorem 2.2. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \le 1$, Y_0 be a subset of Y containing two linearly independent vectors and $h : E_0 \times Y_0 \to \mathbb{R}_+$ be a functions such that*

$$\mathcal{U} := \{n \in \mathbb{N} \colon \beta_n := \lambda(2+n) + \lambda(1+n) < 1\} \neq \emptyset,$$

where

$$\lambda(n) := \inf\left\{t \in \mathbb{R}_+ \colon h(nx,z) \le t\, h(x,z),\ x \in E_0,\ z \in Y_0\right\},$$

for all $n \in \mathbb{N}$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), z\right\|_\alpha \le h(x,z) + h(y,z), \tag{2.14}$$

for all $x, y \in E_0$, $z \in Y_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$. Then there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\left\|f(x) - F(x), z\right\|_\alpha \le \lambda_0 h(x,z),$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}}\left\{\frac{\lambda(2+n)+\lambda(n)}{1-\lambda(2+n)-\lambda(1+n)}\right\}.$$

Proof. Replacing x with $(2+m)x$ and y with mx, where $x \in E_0$ and $m \in \mathbb{N}$, in inequality (2.14), we get

$$\left\|f\Big((2+m)x\Big) - f\Big((1+m)x\Big) - f(x), z\right\|_\alpha \le h((2+m)x,z) + h(mx,z), \tag{2.15}$$

for all $x \in E_0$, $z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m\xi(x) := \xi\Big((2+m)x\Big) - \xi\Big((1+m)x\Big), \quad \xi \in Y^{E_0},\ x \in E_0.$$

Further put

$$\varepsilon_m(x,z) := h((2+m)x,z) + h(mx,z), \quad x \in E_0,\ z \in Y_0, \tag{2.16}$$

and observe that

$$\varepsilon_m(x,z) = \Big(h((2+m)x,z) + h(mx,z)\Big) \le (\lambda(2+m)+\lambda(m))h(x,z), \quad m \in \mathbb{N}. \tag{2.17}$$

Then the inequality (2.15) takes the form

$$\left\|\mathcal{T}_m f(x) - f(x), z\right\|_\alpha \le \varepsilon_m(x,z), \quad x \in E_0,\ z \in Y_0.$$

Furthermore, for every $x \in E_0, z \in Y_0$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\Big\|_\alpha =& \Big\|\xi\Big((2+m)x\Big) - \xi\Big((1+m)x\Big) \\ & - \mu\Big((2+m)x\Big) + \mu\Big((1+m)x\Big), z\Big\|_\alpha \\ \leq & \Big\|(\xi-\mu)\Big((2+m)x\Big), z\Big\|_\alpha + \Big\|(\xi-\mu)\Big((1+m)x\Big), z\Big\|_\alpha.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0\times Y_0} \to \mathbb{R}_+^{E_0\times Y_0}$ by

$$\Lambda_m\delta(x,z) := \delta\Big((2+m)x, z\Big) + \delta\Big((1+m)x, z\Big), \quad \delta \in \mathbb{R}_+^{E_0\times Y_0},\ x \in E_0,\ z \in Y_0.$$

For each $m \in \mathbb{N}$ the above operator has the form described in (H2) with $f_1(x) = (2+m)x$, $f_2(x) = (1+m)x$, $g_1(z) = g_2(z) = z$ and $L_1(x) = L_2(x) = 1$ for all $x \in X$. By mathematical induction on $n \in \mathbb{N}_0$, we prove that

$$(\Lambda_m^n \varepsilon_m)(x,z) \leq (\lambda(2+m) + \lambda(m))\beta_m^n h(x,z), \tag{2.18}$$

for all $x \in E_0$ and $z \in Y_0$, where

$$\beta_m := \lambda(2+m) + \lambda(1+m).$$

From (2.16) and (2.17), we obtain that the inequality (2.18) holds for $n = 0$. Next, we will assume that (2.18) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,z) =& \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right) \\ =& (\Lambda_m^k\varepsilon_m)\Big((2+m)x, z\Big) + (\Lambda_m^k\varepsilon_m)\Big((1+m)x, z\Big) \\ \leq & \Big((\lambda(2+m) + \lambda(m))\beta_m^k h((2+m)x, z) \\ & + (\lambda(2+m) + \lambda(m))\beta_m^k h((1+m)x, z)\Big) \\ =& (\lambda(2+m) + \lambda(m))\beta_m^{k+1} h(x,z),\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. This shows that (2.18) holds for $n = k+1$. Now we can conclude that the inequality (2.18) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z) \\ &\leq \sum_{n=0}^{\infty}(\lambda(2+m) + \lambda(m))\beta_m^n h(x,z) \\ &= \frac{(\lambda(2+m) + \lambda(m))h(x,z)}{(1-\beta_m)} < \infty,\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1.2 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\Big(\mathcal{T}_m^n f\Big)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\Big\|f(x) - F_m(x), z\Big\|_\alpha \leq \frac{(\lambda(2+m)+\lambda(m))h(x,z)}{(1-\beta_m)}, \quad x \in E_0,\ z \in Y_0,\ m \in \mathcal{U}.$$

By a similar method in the proof of Theorem 2.1, we show that

$$\Big\|(\mathcal{T}_m^n f)\big(x+y\big) + (\mathcal{T}_m^n f)\big(x-y\big) - (\mathcal{T}_m^n f)(x), z\Big\|_\alpha \leq \beta_m^n(h(x,z)+h(y,z)),$$

for every $x, y \in E_0, z \in Y_0, n \in \mathbb{N}_0$ and $m \in \mathcal{U}$. Also, the remaining reasonings are analogous as in the proof of that theorem. $\square$

3. Applications

According to above theorems, we can obtain the following corollaries for the hyperstability results of the Cauchy-Jensen equation (1.3) in $(2,\alpha)$-Banach spaces.

Corollary 3.1. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \leq 1$, Y_0 be a subset of Y containing two linearly independent vectors and h_1, h_2, and $\mathcal{U}$ be as in Theorem 2.1. Assume that*

$$\tag{3.1} \begin{cases} \lim_{n\to\infty} \lambda_1(2+n)\lambda_2(n) = 0, \\ \lim_{n\to\infty} \lambda_1(2+n)\lambda_2(2+n) = \lim_{n\to\infty} \lambda_1(1+n)\lambda_2(1+n) = 0. \end{cases}$$

Then every function $f : E \to Y$ satisfying (2.1) is a solution of (1.3) on E_0.

Proof. Suppose that $f : E \to Y$ satisfies (2.1). Then, by Theorem 2.1, there exists a function $F : E \to Y$ satisfying (1.3) and

$$\|f(x) - F(x), z\|_\alpha \leq \lambda_0 h_1(x,z)h_2(x,z),$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{\frac{\lambda_1(2+n)\lambda_2(n)}{1-b_n}\right\}.$$

By (3.1), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) = f(x),$$

for all $x, y \in E_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, which implies that f satisfies the functional equation (1.3) on E_0. $\square$

Corollary 3.2. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \leq 1$, Y_0 be a subset of Y containing two linearly independent vectors and h_1 and $\mathcal{U}$ be as in Theorem 2.2. Assume that*

$$\tag{3.2} \begin{cases} \lim_{n\to\infty}(\lambda_1(2+n) + \lambda_2(n)) = 0, \\ \lim_{n\to\infty}(\lambda_1(2+n) + \lambda_2(1+n)) = 0. \end{cases}$$

Then every function $f : E \to Y$ satisfying (2.14) is a solution of (1.3) on E_0.

Proof. Suppose that $f : E \to Y$ satisfies (2.14). Then, by Theorem 2.2, there exists a function $F : E \to Y$ satisfying (1.3) and

$$\|f(x) - F(x), z\|_\alpha \leq \lambda_0 h(x, z)$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n \in \mathcal{U}} \left\{ \frac{\lambda_1(2+n) + \lambda_2(n)}{1 - \beta_n} \right\}.$$

By (3.2), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) = f(x),$$

for all $x, y \in E_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, which implies that f satisfies the functional equation (1.3) on E_0. □

References

[1] L. Aiemsomboon and W. Sintunavarat, *On new stability results for generalized Cauchy functional equations on groups by using Brzdęk's fixed point theorem*, J. Fixed Point Theory Appl. **18** (2016), 45–59.

[2] L. Aiemsomboon and W. Sintunavarat, *On generalized hyperstability of a general linear equation*, Acta Math. Hungar. **149** (2016), 413–422.

[3] M Almahalebi, *Non-Archimedean hyperstability of a Cauchy-Jensen type functional equation*, J. Class. Anal. **11**(2) (2017), 159–170.

[4] M. Almahalebi and A. Chahbi, *Approximate solution of p-radical functional equation in 2-Banach spaces*, Acta Math. Sci. **39**(2) (2019), 551–566.

[5] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[6] C. Baak, *Cauchy-Rassias stability of Cauchy-Jensen additive mappings in Banach spaces*, Acta Math. Sin. (Engl. Ser.) **22**(6) (2006), 1789–1796.

[7] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[8] J. Brzdęk, *A note on stability of additive mappings*, in: T. M. Rassias and J. Tabor, (Eds.) *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press, Palm Harbor, 1994, 19–22.

[9] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory Appl. **2013**(9) (2013), Article ID 285.

[10] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141**(1-2) (2013), 58–67.

[11] J. Brzdęk and K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal. **74** (2011), 6861–6867.

[12] J. Brzdęk and K. Ciepliński, *On a fixed point theorm in 2-Banach spaces and some of its applications*, Acta Math. Sci. **38**(2) (2018), 377–390.

[13] J. Brzdęk and K. Ciepliński, *A fixed point theorm in n-Banach spaces and Ulam stability*, J. Math. Anal. Appl. **470** (2019), 632–646.

[14] J. Brzdęk, W. Fechner, M. S. Moslehian and J. Sikorska, *Recent developments of the conditional stability of the homomorphism equation*, Banach J. Math. Anal. **9** (2015), 278–327.

[15] E. El-hady, *On stability of the functional equation of p-Wright affine functions in $(2, \alpha)$-Banach spaces*, J. Egyptian Math. Soc.**27** (2019), Article ID 21.

[16] R. W. Freese and Y. J. Cho, *Geometry of Linear* 2*-Normed Spaces*, Nova Science Publishers Inc., Hauppauge, NY, 2001.
[17] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.
[18] M. E. Gordji, H. Khodaei and M. Kamar, *Stability of Cauchy-Jensen type functional equation in generalized fuzzy normed spaces*, Comput. Math. Appl. **62** (2011), 2950–2960.
[19] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
[20] K.-W. Jun, H.-M. Kim and J. M. Rassias, *Extended Hyers-Ulam stability for Cauchy-Jensen mappings*, J. Diference Equ. Appl. **13** (2007), 1139–1153.
[21] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. (2007), Article ID 50175, 15 pages.
[22] C. Park and J. M. Rassias, *Stability of the Jensen-type functional equation in C^*-algebras: a fixed point approach*, Abs. Appl. Anal. (2009), Article ID 360432, 17 pages.
[23] W. G. Park, *Approximate additive mappings in* 2*-Banach spaces and related topics*, J. Math. Anal. Appl. **376** (2011), 193–202.
[24] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[25] Th. M. Rassias, *Problem* 16*,* 2*. Report of the* 27*th international symposium on functional equations*, Aequationes Math. **39** (1990), 292–293.
[26] Th. M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl. **158** (1991), 106–113.
[27] Th. M. Rassias and P. Semrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
[28] K. Y. N. Sayar and A. Bergam, *Some hyperstability results for a Cauchy-Jensen type functional equation in* 2*-Banach spaces*, Proyecciones **39**(1)(2020), 73–89.
[29] K. Y. N. Sayar and A. Bergam, *Approximate solutions of a quadratic functional equation in* 2*-Banach spaces using fixed point theorem*, J. Fixed Point Theory Appl. **22** (2020), Article ID 3.
[30] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John-Wiley & Sons Inc., New York, 1964.

[1]MAE2D Laboratory, Polydisciplinary Faculty of Larache,
Abdelmalek Essaadi University,
Morocco

[2]Department of Mathematics,
University of Sana'a,
Sana'a, Yemen
Email address: `khaledsayar@gmail.com, ysayarkhalid@uae.ac.ma`
Email address: `bergamamal11@gmail.com`

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 919–928.

k-TYPE BI-NULL CARTAN SLANT HELICES IN $\mathbb{R}_2^6$

ALI UÇUM[1*] AND KAZIM İLARSLAN[2]

Abstract. In the present paper, we give the notion of k-type bi-null Cartan slant helices in $\mathbb{R}_2^6$, where $k \in \{1,2,3,4,5,6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k-type slant helices in terms of their curvature functions.

1. Introduction

The notion of a slant helix is introduced by Izumiya and Takeuchi [4]. A curve γ with non-zero curvature is called a slant helix in Euclidean 3-space $\mathbb{R}^3$ if the principal normal line of γ makes a constant angle with a fixed vector in $\mathbb{R}^3$. Some characterizations of such curves were presented in [1, 5, 6, 9].

Further, k-type slant helices emerged and attracted attention of researchers. Ergüt et al. ([3]) studied k-slant helices in Minkowski 3-space $\mathbb{R}_1^3$. The curves of such type were studied in Minkowski space-time $\mathbb{R}_1^4$ by some researchers in [2, 7].

Bi-null Cartan curves in $\mathbb{R}_2^n$ are defined in [8]. Some characterizations of bi-null Cartan curves in terms of their curvature functions in $\mathbb{R}_2^n$ for $n \geq 6$ are also given in [8]. The necessary and the sufficient conditions for bi-null curves to be k-type slant helices in semi-Euclidean spaces $\mathbb{R}_3^6$ and $\mathbb{R}_2^5$ are given in [10, 11].

On the other hand, the third named author of this paper gave the notion of bi-null Cartan curves in semi-Euclidean spaces $\mathbb{R}_2^n$ of index 2, together with the unique Cartan frame and the Cartan curvatures. He discussed some properties of bi-null Cartan curves in terms of the Cartan curvatures in the case where $n \geq 6$ ([8]). In $\mathbb{R}_2^5$ and $\mathbb{R}_3^6$, we define k-type bi-null slant helices and we give the necessary and sufficient

Key words and phrases. k-Slant helix, bi-null Cartan curves, semi-Euclidean space, Cartan curvatures, Frenet equations.

2010 *Mathematics Subject Classification.* Primary: 53B30. Secondary: 53A04.

DOI 10.46793/KgJMat2206.919U

Received: April 08, 2020.

Accepted: May 21, 2020.

conditions for bi-null curves to be k-type slant helices in terms of their curvature functions ([10, 11]).

In this paper, we give the notion of k-type bi-null Cartan slant helices in $\mathbb{R}^6_2$, where $k \in \{1, 2, 3, 4, 5, 6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k-type slant helices in terms of their curvature functions.

2. Priliminaries

In this section, following [8], we recall the Frenet equations for bi-null Cartan curves in $\mathbb{R}^6_2$. Let $\mathbb{R}^6_2$ be the 6-dimensional semi-Euclidean space of index 2 with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^6_2$. Recall that a vector $v \in \mathbb{R}^6_2 \backslash \{0\}$ can be spacelike if $\langle v, v \rangle > 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|\langle v, v \rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w \rangle = 0$.

We say that a curve $\gamma(t)$ in $\mathbb{R}^6_2$ is a bi-null curve if span$\{\gamma'(t), \gamma''(t)\}$ is an isotropic 2-plane for all t. That is,

$$\langle \gamma'(t), \gamma'(t) \rangle = \langle \gamma'(t), \gamma''(t) \rangle = \langle \gamma''(t), \gamma''(t) \rangle = 0,$$

and $\{\gamma'(t), \gamma''(t)\}$ are linearly independent. The condition is independent of the choice of the parameter.

We say that a bi-null curve $\gamma(t)$ in $\mathbb{R}^6_2$ is parametrized by the bi-null arc if $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle = 1$. If a bi-null curve $\gamma(t)$ in $\mathbb{R}^6_2$ satisfies $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle \neq 0$, then by the anti-isometry of $\mathbb{R}^6_2$, we may assume that $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle > 0$ and we can see that

$$u(t) = \int_{t_0}^{t} \left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle^{1/6} dt$$

becomes the bi-null arc parameter.

Let us say that a bi-null curve $\gamma(t)$ in $\mathbb{R}^6_2$ with $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle > 0$ is a bi-null Cartan curve if $\left\{ \gamma'(t), \gamma''(t), \gamma^{(3)}(t), \gamma^{(4)}(t), \gamma^{(5)}(t) \right\}$ are linearly independent for any t.

For a bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}^6_2$ with bi-null arc parameter t, there exists a unique pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ such that

$$\begin{aligned} \gamma' &= L_1, \quad L_1' = L_2, \quad L_2' = W_1, \\ W_1' &= -k_0 L_2 - N_2, \\ N_2' &= -k_1 L_1 - N_1 + k_0 W_1, \\ N_1' &= k_1 L_2 + k_2 W_2, \\ W_2' &= -k_2 L_1, \end{aligned} \tag{2.1}$$

where N_1, N_2 are null, $\langle L_1, N_1\rangle = \langle L_2, N_2\rangle = 1$, $\{L_1, N_1\}$, $\{L_2, N_2\}$ and $\{W_1, W_2\}$ are mutually orthogonal, $\{W_1, W_2\}$ are orthonormal of signature $(+,+)$, and $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is positively oriented.

We say that the pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is the Cartan frame and the functions $\{k_0, k_1, k_2\}$ are the Cartan curvatures of γ.

3. k-Type Bi-Null Cartan Slant Helices

Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ with bi-null arc parameter t and Cartan frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$. Let us set $V_1 = L_1$, $V_2 = L_2$, $V_3 = N_1$, $V_4 = N_2$, $V_5 = W_1$ and $V_6 = W_2$. Then we give the following definition.

Definition 3.1. A bi-null Cartan curve γ in $\mathbb{R}^6_2$ with Cartan frame $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ is called a k-type bi-null Cartan slant helix if there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that the following holds

$$\langle V_k, U\rangle = \text{constant}, \quad \text{where } k \in \{1,2,3,4,5,6\}.$$

Firstly, we consider 1-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.1. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 1-type bi-null Cartan slant helix if and only if k_1 and k_2 satisfy the following*

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0. \tag{3.1}$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle L_1, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.2}$$

Taking derivative of the equation (3.2) with respect to t and using equations (2.1), we get

$$\langle L_2, U\rangle = 0, \quad \langle W_1, U\rangle = 0, \quad \langle N_2, U\rangle = 0. \tag{3.3}$$

Then using (3.3), we can write U as follows

$$U = \lambda_1 L_1 + cN_1 + \lambda_2 W_2, \tag{3.4}$$

where λ_1 and λ_2 are differentiable functions. Taking derivative of the equation (3.4) with respect to t and using equations (2.1), we have

$$0 = (\lambda_1' - \lambda_2 k_2) L_1 + (\lambda_1 + ck_1) L_2 + (ck_2 + \lambda_2') W_2,$$

which implies that

$$\begin{cases} \lambda_1' - \lambda_2 k_2 = 0, \\ \lambda_1 + ck_1 = 0, \\ ck_2 + \lambda_2' = 0. \end{cases} \tag{3.5}$$

From (3.5), we find that $c \neq 0$ and

$$\lambda_1 = -ck_1, \quad \lambda_2 = -c\frac{k_1'}{k_2}, \quad k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

Conversely, assume that k_1 and k_2 satisfy

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

For $c \neq 0$, choosing the vector U as

$$U = -ck_1 L_1 + cN_1 - (ck_1'/k_2)\, W_2, \tag{3.6}$$

we get $U' = 0$ and $\langle L_1, U\rangle = c$ (constant). Thus, $\gamma(t)$ is a 1-type bi-null Cartan slant helix. □

Example 3.1. The following curvature functions satisfy (3.1):
(*i*) $k_1 = t^2/2$, $k_2 = 1$;
(*ii*) $k_1 = 2t^2$, $k_2 = -2$.

Corollary 3.1. *The axis of a* 1*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}^6_2$ *with* k_0, k_1 *and non-zero* k_2*, is given by*

$$U = -ck_1 L_1 + cN_1 - (ck_1'/k_2)\, W_2, \tag{3.7}$$

where $c \in \mathbb{R}/\{0\}$.

Corollary 3.2. *There exists no* 1*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}^6_2$ *with* k_0, k_1 *and non-zero* k_2*, whose axis* U *satisfies* $\langle L_1, U\rangle = 0$.

Corollary 3.3. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}^6_2$ *parametrized by bi-null arc* t *with* k_0, k_1 *and non-zero* k_2. *Assume that* $k_1' \neq 0$. *Then* $\gamma(t)$ *is a* 1*-type bi-null Cartan slant helix if and only if* k_1 *and* k_2 *satisfy that*

$$-2k_1 + \left(\frac{k_1'}{k_2}\right)^2 = constant. \tag{3.8}$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Using Corollary 3.1 and that U is constant, we obtain

$$-2k_1 + \left(\frac{k_1'}{k_2}\right)^2 = \text{constant}.$$

Conversely, assume that the relation (3.8) holds. Then taking derivative of the equation (3.8) with respect to t, we get

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

Thus from Theorem 3.1, we find that $\gamma(t)$ is a 1-type bi-null Cartan slant helix. □

Corollary 3.4. *Let $\gamma(t)$ be a 1-type bi-null Cartan slant helix in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then we get*

$$k_1 = \frac{1}{2}\left(\int k_2(t)\,dt + a\right)^2 + b,$$

where $a, b \in \mathbb{R}$.

Secondly, we consider 2-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.2. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 2-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$k_2(ct+a) + \left[\frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right)\right]' = 0, \tag{3.9}$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Proof. Assume that $\gamma(t)$ is a 2-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle L_2, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.10}$$

Taking derivative of the equation (3.10) with respect to t and using equations (2.1), we get $\langle W_1, U\rangle = 0$. Then we can write the vector U as follows

$$U = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 N_1 + cN_2 + \lambda_4 W_2, \tag{3.11}$$

where λ_i is a differentiable function ($i \in \{1, 2, 3, 4\}$). Differentiating the equation (3.11) with respect to t and using equations (2.1), we get

$$\begin{aligned} 0 = &(\lambda_1' - ck_1 - \lambda_4 k_2) L_1 + (\lambda_1 + \lambda_2' + \lambda_3 k_1) L_2 + (\lambda_3' - c) N_1 \\ &+ (\lambda_2 + ck_0) W_1 + (\lambda_3 k_2 + \lambda_4') W_2, \end{aligned}$$

which implies that

$$\begin{cases} \lambda_1' - ck_1 - \lambda_4 k_2 = 0, \\ \lambda_1 + \lambda_2' + \lambda_3 k_1 = 0, \\ \lambda_3' - c = 0, \\ \lambda_2 + ck_0 = 0, \\ \lambda_3 k_2 + \lambda_4' = 0. \end{cases} \tag{3.12}$$

Solving (3.12), we get

$$\begin{aligned} \lambda_1 &= ck_0' - k_1(ct+a), \quad \lambda_2 = -ck_0, \quad \lambda_3 = ct + a, \\ \lambda_4 &= \frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right) \end{aligned}$$

and

$$k_2(ct+a) + \left[\frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right)\right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Conversely, assume that the following relation holds

$$k_2 (ct + a) + \left[\frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a) \right) \right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$. Then choosing the vector U as follows

$$\begin{aligned} U = {} & \left(ck_0' - k_1 (ct + a) \right) L_1 - ck_0 L_2 + (ct + a) N_1 + cN_2 \\ & + \frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a) \right) W_2, \end{aligned}$$

we get $U' = 0$ and $\langle L_2, U \rangle = c$ (constant). Thus, $\gamma(t)$ is a 2-type bi-null Cartan slant helix. □

Example 3.2. The following curvature functions satisfy (3.9):

(i) $c = 0$, $a = 1$, $k_0 = t^2$, $k_1 = t^2/2$, $k_2 = 1$;

(ii) $c = 1$, $a = 0$, $k_0 = t$, $k_1 = t^2/8$, $k_2 = 1$.

Corollary 3.5. *The axis of a* 2*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}_2^6$ *with* k_0, k_1 *and* $k_2 \neq 0$, *is given by*

$$\begin{aligned} U = {} & \left(ck_0' - k_1 (ct + a) \right) L_1 - ck_0 L_2 + (ct + a) N_1 + cN_2 \\ & + \frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a) \right) W_2, \end{aligned}$$

where $a, c \in \mathbb{R}$ *and* $(a, c) \neq (0, 0)$.

Corollary 3.6. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}_2^6$ *parametrized by bi-null arc* t *with* k_0, k_1 *and* $k_2 \neq 0$. *If* $\gamma(t)$ *is a* 1*-type bi-null Cartan slant helix then* $\gamma(t)$ *is a* 2*-type bi-null Cartan slant helix whose principal normal* L_2 *is orthogonal to the axis* U *of the helix.*

Thirdly, we consider 3-type bi-null Cartan slant helices in $\mathbb{R}_2^6$.

Theorem 3.3. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}_2^6$ *parametrized by bi-null arc* t *with* k_0, k_1 *and non-zero* k_2. *Then* $\gamma(t)$ *is a* 3*-type bi-null Cartan slant helix if and only if* k_0, k_1 *and* k_2 *satisfy*

$$\lambda_2^{(4)} + 2k_0 \lambda_2'' + k_0' \lambda_2' - \lambda_2 k_1 = c \tag{3.13}$$

and

$$k_2 \lambda_2 - \left(\frac{k_1 \lambda_2'}{k_2} \right)' = 0, \tag{3.14}$$

where $c \in \mathbb{R}$ *and* λ_2 *is a differentiable function which is not identically zero.*

Proof. Assume that $\gamma(t)$ is a 3-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$\langle N_1, U \rangle = c, \quad c \in \mathbb{R}. \tag{3.15}$$

Then we can write U as follows

$$U = cL_1 + \lambda_1 L_2 + \lambda_2 N_1 + \lambda_3 N_2 + \lambda_4 W_1 + \lambda_5 W_2, \tag{3.16}$$

where λ_i is a differentiable function. Taking derivative of the equation (3.16) with respect to t and using equations (2.1), we get

$$\begin{aligned} 0 = &\left(-k_1\lambda_3 - \lambda_5 k_2\right) L_1 + \left(c + \lambda_1' + \lambda_2 k_1 - \lambda_4 k_0\right) L_2 + \left(\lambda_2' - \lambda_3\right) N_1 \\ &+ \left(\lambda_3' - \lambda_4\right) N_2 + \left(\lambda_1 + k_0\lambda_3 + \lambda_4'\right) W_1 + \left(k_2\lambda_2 + \lambda_5'\right) W_2, \end{aligned}$$

which implies that

$$\left\{ \begin{aligned} -k_1\lambda_3 - \lambda_5 k_2 &= 0, \\ c + \lambda_1' + \lambda_2 k_1 - \lambda_4 k_0 &= 0, \\ \lambda_2' - \lambda_3 &= 0, \\ \lambda_3' - \lambda_4 &= 0, \\ \lambda_1 + k_0\lambda_3 + \lambda_4' &= 0, \\ k_2\lambda_2 + \lambda_5' &= 0. \end{aligned} \right. \tag{3.17}$$

By (3.17) , λ_2 cannot be identically zero. Solving (3.17), we get

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2 k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$.

Conversely, assume that the following relation holds

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2 k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero. Then choosing the vector U as follows

$$U = cL_1 - \left(k_0\lambda_2' + \lambda_2^{(3)}\right) L_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 - \frac{k_1}{k_2}\lambda_2' W_2,$$

we get $U' = 0$ and $\langle N_1, U\rangle = c$ (constant). Thus, $\gamma(t)$ is a 3-type bi-null Cartan slant helix. □

Example 3.3. The following curvature functions satisfy the equations (3.13) and (3.14):
(*i*) $\lambda_2 = t$, $k_0 = -t + (t^4/8)$, $k_1 = t^2/2$, $k_2 = 1$, $c = -1$;
(*ii*) $\lambda_2 = t^2$, $k_0 = t^6/128$, $k_1 = t^4/8$, $k_2 = t$, $c = 0$.

Corollary 3.7. *The axis of a* 3*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}_2^6$ *with* k_0*,* k_1 *and* $k_2 \neq 0$*, is given by*

$$U = cL_1 - \left(k_0\lambda_2' + \lambda_2^{(3)}\right) L_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 - \frac{k_1}{k_2}\lambda_2' W_2,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 4-type bi-null Cartan slant helices in $\mathbb{R}^6_2$. In the following, we omit the proofs.

Theorem 3.4. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 4-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$k_2\lambda_2 + \left[\frac{1}{k_2}\left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2'\right)\right]' = 0 \tag{3.18}$$

and

$$c + k_0\lambda_2' + \lambda_2^{(3)} = 0, \tag{3.19}$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero.

Example 3.4. The following curvature functions satisfy the equations (3.18) and (3.19):
(i) $\lambda_2 = t$, $k_0 = -c$, $k_1 = t^2/8$, $k_2 = 1$;
(ii) $\lambda_2 = 1$, $k_0 = t$, $k_1 = t^2/2$, $k_2 = 1$, $c = 0$.

Corollary 3.8. *The axis of a 4-type bi-null Cartan slant helix $\gamma(t)$ in $\mathbb{R}^6_2$ with k_0, k_1 and $k_2 \neq 0$, is given by*

$$\begin{aligned} U = {} & (k_0\lambda_2'' - k_1\lambda_2) L_1 + cL_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 \\ & + \frac{1}{k_2}\left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2'\right) W_2, \end{aligned}$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 5-type bi-null Cartan slant helices in $\mathbb{R}^6_2$. In the following, we omit the proofs.

Theorem 3.5. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 5-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$\lambda' + k_2\left(c\frac{t^2}{2} + a_1 t + a_2\right) = 0,$$

where

$$\lambda = \frac{1}{k_2}\left(3ck_0' + (k_0'' - 2k_1)(ct + a_1) - k_1'\left(\frac{ct^2}{2} + a_1 t + a_2\right)\right),$$

$c, a_1, a_2 \in \mathbb{R}$ and $(c, a_1, a_2) \neq (0,0,0)$.

Example 3.5. The following curvature functions satisfy the above equations:
(i) $c = a_1 = 0$, $a_2 = 1$, $k_0 = t^2$, $k_1 = -t^2/2$, $k_2 = 1$;
(ii) $c = a_2 = 0$, $a_1 = 1$, $k_0 = 1$, $k_1 = (t^2/8) + (1/2t^2)$, $k_2 = 1$.

Corollary 3.9. *The axis of a* 5*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}_2^6$*, with* k_0*,* k_1 *and* $k_2 \neq 0$*, is given by*

$$\begin{aligned} U = &\left(2ck_0 + k_0'(ct+a_1) - k_1\left(\frac{ct^2}{2} + a_1t + a_2\right)\right)L_1 - k_0(ct+a_1)L_2 \\ &+ \left(\frac{ct^2}{2} + a_1t + a_2\right)N_1 + (ct+a_1)N_2 + cW_1 \\ &+ \frac{1}{k_2}\left(3ck_0' + (k_0'' - 2k_1)(ct+a_1) - k_1'\left(\frac{ct^2}{2} + a_1t + a_2\right)\right)W_2, \end{aligned}$$

where $c, a_1, a_2 \in \mathbb{R}$ *and* $(c, a_1, a_2) \neq (0,0,0)$*.*

Corollary 3.10. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}_2^6$ *parametrized by bi-null arc* t *with* k_0*,* k_1 *and* $k_2 \neq 0$*. If* $\gamma(t)$ *is a* 2*-type bi-null Cartan slant helix then* $\gamma(t)$ *is a* 5*-type bi-null Cartan slant helix such that* W_1 *is orthogonal to the axis* U *of the helix.*

Lastly, we consider 6-type bi-null Cartan slant helices in $\mathbb{R}_2^6$.

Theorem 3.6. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}_2^6$ *parametrized by bi-null arc* t *with* k_0*,* k_1*,* k_2*. If* $\gamma(t)$ *is a* 6*-type bi-null Cartan slant helix, then it lies in some hyperplane of index* 2*.*

Proof. Assume that $\gamma(t)$ is a 6-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1, k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}_2^6$ such that

$$\langle W_2, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.20}$$

Taking derivative of the equation (3.20) with respect to t and using equations (2.1), we get

$$k_2\langle L_1, U\rangle = 0. \tag{3.21}$$

Assume that $k_2 \neq 0$. Then we have

$$\langle L_1, U\rangle = \langle L_2, U\rangle = \langle W_1, U\rangle = \langle N_2, U\rangle = \langle N_1, U\rangle = \langle W_2, U\rangle = 0,$$

and U is zero, which is a contradiction. Thus, $k_2 = 0$ which means that $\gamma(t)$ lies in some hyperplane of index 2. □

Remark 3.1. Any bi-null Cartan curve in $\mathbb{R}_2^6$ with $k_2 = 0$ is 6-type bi-null Cartan helix, since W_2 is a constant vector, so it trivially makes constant angle with any fixed direction.

References

[1] A. Ali, *Position vectors of slant helices in Euclidean* 3*-space*, J. Egyptian Math. Soc. **20** (2012), 1–6.

[2] A. T. Ali, R. Lopez and M. Turgut, *k-Type partially null and pseudo null slant helices in Minkowski* 4*-space*, Math. Commun. **17** (2012), 93–103.

[3] M. Ergüt, H. B. Öztekin and S. Aykurt, *Non-null k-slant helices and their spherical indicatrices in Minkowski* 3*-space*, Journal of Advanced Research in Dynamical and Control Systems **2** (2010), 1–12.

[4] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces*, Turkish J. Math. **28** (2004), 531–537.

[5] L. Kula, N. Ekmekçi, Y. Yaylı and K. İlarslan, *Characterizations of slant helices in Euclidean* 3*-space*, Turkish J. Math. **34** (2010), 261–273.

[6] L. Kula and Y. Yaylı, *On slant helix and its spherical indicatrix*, Appl. Math. Comput. **169** (2005), 600–607.

[7] E. Nešović, E. B. Koç Öztürk and U. Öztürk, *k-Type null slant helices in Minkowski space-time*, Math. Commun. **20** (2015), 83–95.

[8] M. Sakaki, *Bi-null Cartan curves in semi-Euclidean spaces of index* 2, Beitr. Algebra Geom. **53**(2) (2012), 421–436.

[9] M. Turgut and S. Yılmaz, *Characterizations of some special helices in* $\mathbb{E}^4$, Scientia Magna **4** (2008), 51–55.

[10] A. Uçum, K. İlarslan and M. Sakaki, *k-Type bi-null slant helices in* $\mathbb{R}^5_2$, J. Geom. **108** (2017), 913–924.

[11] A. Uçum, K. İlarslan and M. Sakaki, *k-Type bi-null Cartan slant helices in* $\mathbb{R}^6_3$, Southeast Asian Bull. Math. **42** (2018), 937–946.

[1]Kirikkale University,
Faculty of Sciences and Arts,
Department of Mathematics, Kirikkale-Turkey
Email address: aliucum05@gmail.com

[2]Kirikkale University,
Faculty of Sciences and Arts,
Department of Mathematics, Kirikkale-Turkey
Email address: kilarslan@yahoo.com

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 929–942.

NUMERICAL SOLUTION OF SHRÖDINGER EQUATIONS BASED ON THE MESHLESS METHODS

MOJTABA RAHIMI[1], SEYED MEHDI KARBASSI[1],
AND MOHAMMAD REZA HOOSHMANDASL[2*]

Abstract. In this work, two-dimensional time-dependent quantum equation problems are studied. We introduce a numerical algorithm for solving the two-dimensional nonlinear complex quantum system with MLS and FDM methods. An efficient and accurate computational algorithm based on both, the moving least squares (MLS) and the finite difference (FDM) methods is proposed for solving it. The results demonstrate that the proposed algorithm is a robust algorithm with good accuracy. This is developed on MLS and FDM methods using numerical simulation for solving these kind of problems.

1. Introduction

Recently, some researchers have considered several types of quantum equations such as two-dimensional time-dependent Shrödinger equations mainly used for modeling several physical phenomena. These types of equations appear in many science and engineering problems. Very interesting problems in quantum physics consist of multi-particle systems that can be modeled by the multi-particle Shrödinger equations. These equations are important in many different fields of science such as wave propagation, relativistic quantum mechanics, quantum field theory and mathematical physics [2, 8, 10, 12, 16, 18, 19].

There are various achievements on the numerical solution of partial differential equations (PDEs). Many numerical algorithms have been developed for the solution of partial differential equations such as meshless methods, finite difference methods, differential quadrature methods, radial basis functions and collocation base methods.

Key words and phrases. Time-dependent quantum equations, moving least squares (MLS) method, finite difference scheme (FDM).

2010 *Mathematics Subject Classification.* Primary: 41A45. Secondary: 41A25, 65M06.

DOI 10.46793/KgJMat2206.929R

Received: December 12, 2019.

Accepted: May 22, 2020.

However, the meshless methods are generally used as mathematical tools for solving system of differential equations. These numerical approximation methods are suitable for solving ordinary and partial differential equations. The simple structure of these methods are implemented in algorithms to solve these types of differential equations [1, 3–5, 7–9, 13, 14, 17].

The main object of this paper is to present an efficient numerical algorithm based on the MLS method to solve the following 2D time-dependent Shrödinger equation of the form:

$$-iu_t(x,y,t) = (u_{xx}(x,y,t) + u_{yy}(x,y,t))u(x,y,t) + V_e(x,y,t)u(x,y,t), \tag{1.1}$$

where $(x,y,t) \in [a,b] \times [c,d] \times [0,T]$. The initial condition for the above equation is

$$u(x,y,0) = g_1(x,y), \tag{1.2}$$

while the boundary conditions are:

$$\begin{aligned} u(a,y,t) &= h_1(y,t), u(b,y,t) = h_2(y,t),\\ u(x,c,t) &= h_3(x,t), u(x,d,t) = h_2(x,t). \end{aligned} \tag{1.3}$$

2. The MLS Approximation

Consider the 2-D unknown function $u(x,y)$ and randomly located nodes (x_i, y_j), $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Then we define

$$z_k = (x_i, y_j),$$

where

$$i = \begin{cases} \frac{k}{M}, & M|k, \\ k - \left[\frac{k}{M}\right] M, & \text{otherwise}, \end{cases}$$

and

$$j = \begin{cases} M, & M|k, \\ \left[\frac{k}{M}\right] + 1, & \text{otherwise}. \end{cases}$$

In other words we have $k = (i-1)m + j$. To approximate $u(x,y)$ by MLS method, we can write

$$u(x,y) = \sum_{j=1}^{m} p_j(x,y)a_j(x,y) \triangleq \mathbf{p}^T(x,y)\mathbf{a}(x,y), \quad \text{for all } (x,y) \in [a,b] \times [c,d], \tag{2.1}$$

where $a_j(x,y)$ are the unknown coefficients and $\mathbf{p}^T(x,y) = [\ p_1(x,y) \ \cdots \ p_m(x,y)\ ]$ here $p_j(x,y)$ are the basis polynomial functions. For example, the linear basis is $\mathbf{p}^T(x,y) = [\ 1 \ \ x \ \ y\ ]$ and the quadratic basis is $\mathbf{p}^T(x,y) = [\ 1 \ \ x \ \ y \ \ x^2 \ \ y^2 \ \ xy\ ]$. The unknown coefficients $a_j(x,y)$ can be determined by MLS method. In this method,

the main concept is minimizing the weighted error of the exact values and approximations of the function. The weighted error function is defined as
(2.2)

$$\mathcal{S}(x,y)=\sum_{k=1}^{N^2} w_k(x,y)\left(u^h(x_i,y_j)-u_k\right)^2=\sum_{k=1}^{N^2} w_k(x,y)\Big(\mathbf{p}^T(x_i,y_j)\mathbf{a}(x,y)-u_k\Big)^2.$$

In weighted error function, (x_i,y_j), $i=1,2,\dots,N$, $j=1,2,\dots,N$, are the nodes and $w_k(x_i,y_i)$, $k=1,2,\dots,N^2$, are weighting functions associated with the nodes (x_i,y_j), $i=1,2,\dots,N$, $j=1,2,\dots,N$. Here, the Gaussian weight function is preferred rather than other popular weighted functions. In k^{th} node, we use the Gaussian weight function as [11]:

$$w_k(x,y)=\begin{cases}\dfrac{\exp\left[-(\frac{d_k}{\alpha})^2\right]-\exp[-(\frac{h_k}{\alpha})^2]}{1-\exp\left[-(\frac{h_k}{\alpha})^2\right]}, & 0\le d_k<h_k,\\ 0, & d_k\ge h_k,\end{cases}$$

where $d_k=\|(x,y)-(x_i,y_j)\|$, α is a constant used for controlling the shape of the weight function and h_k denotes support domain size of the node (x_i,y_j).

In other words, an equivalent form of the Gaussian weight function can be used:

$$w_k(x,y)=\begin{cases}\dfrac{\exp\left(-s_k^2r_k^2\right)-\exp(-s_k^2)}{1-\exp(-s_k^2)}, & 0\le r_k<1,\\ 0, & r_k\ge 1,\end{cases}\tag{2.3}$$

where

$$r_k=\frac{\|(x,y)-(x_i,y_j)\|}{h_k},\qquad s_k=\frac{h_k}{\alpha}.$$

To minimize $\mathcal{S}$ in (2.2), with respect to $\mathbf{a}(x,y)$ we require that

$$\frac{\partial\mathcal{S}}{\partial\mathbf{a}}=0,$$

which yields the following equations:

$$\begin{aligned}\sum_{k=1}^{N^2} w_k(x,y)2p_1(x_i,y_j)[\mathbf{p}^T(x_i,y_j)\mathbf{a}(x,y)-u_k]&=0,\\ \sum_{k=1}^{N^2} w_k(x,y)2p_2(x_i,y_j)[\mathbf{p}^T(x_i,y_j)\mathbf{a}(x,y)-u_k]&=0,\\ &\vdots\\ \sum_{k=1}^{N^2} w_k(x,y)2p_m(x_i,y_j)[\mathbf{p}^T(x_i,y_j)\mathbf{a}(x,y)-u_k]&=0.\end{aligned}$$

The above equations can be formulated as

$$\sum_{k=1}^{N^2} w_k(x,y)\begin{pmatrix} p_1(x_i,y_j) \\ \vdots \\ p_m(x_i,y_j) \end{pmatrix} [p_1(x_i,y_j),\dots,p_m(x_i,y_j)] \begin{pmatrix} a_1(x,y) \\ \vdots \\ a_m(x,y) \end{pmatrix} \tag{2.4}$$

$$= \sum_{k=1}^{N^2} w_k(x,y)\begin{pmatrix} p_1(x_i,y_j) \\ \vdots \\ p_m(x_i,y_j) \end{pmatrix} u_k.$$

The matrices $\mathbf{C}(x,y)$, $\mathbf{D}(x,y)$ and column vector $\mathbf{u}$ are defined as follows:

$$\mathbf{C}(x,y) = \sum_{k=1}^{N^2} w_k(x,y)\begin{pmatrix} p_1(x_i,y_j) \\ \vdots \\ p_m(x_i,y_j) \end{pmatrix} [p_1(x_i,y_j),\dots,p_m(x_i,y_j)], \tag{2.5}$$

$$\mathbf{D}(x,y) = \sum_{k=1}^{N^2} w_k(x,y)\begin{pmatrix} p_1(x_i,y_j) \\ \vdots \\ p_1(x_i,y_j) \end{pmatrix},$$

$$\mathbf{u} = [u_1,\dots,u_{N^2}]^T.$$

Then (2.4) may be re-written in the following compact form:

$$\mathbf{C}(x,y)\mathbf{a}(x,y) = \mathbf{D}(x,y)\mathbf{u},$$

therefore, we have

$$\mathbf{a}(x,y) = \mathbf{C}^{-1}(x,y)\mathbf{D}(x,y)\mathbf{u}.$$

After computing $\mathbf{a}(x,y)$ in the above equation and substituting it into (2.1), the MLS can be approximated as follows:

$$u(x,y) = \mathbf{p}^T(x,y)\mathbf{C}^{-1}(x,y)\mathbf{D}(x,y) \ = \mathbf{\Phi}^T(x,y)\mathbf{u} = \sum_{k=1}^{N^2} \phi_k(x,y)u_k,$$

where

$$\mathbf{\Phi}^T(x,y) = [\ \phi_1(x,y) \quad \cdots \quad \phi_{N^2}(x)\ ] = \mathbf{p}^T(x,y)\mathbf{C}^{-1}(x,y)\mathbf{D}(x,y) \tag{2.6}$$

and

$$\phi_k(x,y) = \mathbf{p}^T(x,y)[\mathbf{C}^{-1}(x,y)w_k(x,y)\mathbf{p}(x_i,y_j).$$

The function $\phi_k(x,y)$ is commonly known as the shape function of the nodal point (x_i,y_j) in the MLS approximation. In this section, we need to compute derivatives of $\phi_k(x,y)$ and $\mathbf{C}^{-1}(x,y)$. If $\mathbf{F}$ is a nonsingular matrix then, we have $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$. Thus, its differentiation with respect to x gives $\mathbf{F}_x^{-1}\mathbf{F} + \mathbf{F}^{-1}\mathbf{F}_x = 0$ and $\mathbf{F}_x^{-1} = -\mathbf{F}^{-1}\mathbf{F}_x\mathbf{F}^{-1}$. We can write

$$\mathbf{F}_{xx}^{-1} = -\left(\mathbf{F}_x^{-1}\mathbf{F}_x\mathbf{F}^{-1} + \mathbf{F}^{-1}\mathbf{F}_{xx}\mathbf{F}^{-1} + \mathbf{F}^{-1}\mathbf{F}_x\mathbf{F}_x^{-1}\right).$$

The above equations can be written in a simpler form:

$$\mathbf{F}_{xx}^{-1} = 2\left(\mathbf{F}^{-1}\mathbf{F}_x\mathbf{F}^{-1}\mathbf{F}_x\mathbf{F}^{-1}\right) - \mathbf{F}^{-1}\mathbf{F}_{xx}\mathbf{F}^{-1}.$$

Now, the first derivative of $\phi_j(x,y)$ with respect to x is obtained as

$$\begin{aligned}\mathbf{\Phi}_x^T &= \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}_x^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_x\\ &= \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_x,\end{aligned}$$

also the second derivative of $\phi_j(x,y)$ with respect to x is obtained as

$$\begin{aligned}\mathbf{\Phi}_{xx}^T =&\mathbf{p}_{xx}^T\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{D}_x - \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D}\\ &+ \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_{xx}\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D}\\ &- \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D}_x + \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{D}_x + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_{xx} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D}_x.\end{aligned}$$

We can write these equations in a simpler form:

$$\begin{aligned}\mathbf{\Phi}_{xx}^T =&\mathbf{p}_{xx}^T\mathbf{C}^{-1}\mathbf{D} + 2(\mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D}_x)\\ &- \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_{xx}\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_{xx} - 2(\mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{C}_x\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}_x^T\mathbf{C}^{-1}\mathbf{D}_x).\end{aligned} \tag{2.7}$$

In a similar method, we can obtain the first and second derivatives of $\phi_j(x,y)$ with respect to y as follows:

$$\mathbf{\Phi}_y^T =\mathbf{p}_y^T\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_y\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_y$$

and

$$\begin{aligned}\mathbf{\Phi}_{yy}^T =&\mathbf{p}_{yy}^T\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_{yy}\mathbf{C}^{-1}\mathbf{D} + \mathbf{p}^T\mathbf{C}^{-1}\mathbf{D}_{yy} + 2(\mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_y\mathbf{C}^{-1}\mathbf{C}_y\mathbf{C}^{-1}\mathbf{D})\\ &- 2(\mathbf{p}^T\mathbf{C}^{-1}\mathbf{C}_y\mathbf{C}^{-1}\mathbf{D}_y + \mathbf{p}_y^T\mathbf{C}^{-1}\mathbf{C}_y\mathbf{C}^{-1}\mathbf{D} - \mathbf{p}_y^T\mathbf{C}^{-1}\mathbf{D}_y).\end{aligned} \tag{2.8}$$

3. Discretization of Shrödinger Equation

Consider the Shrödinger equation (1.1) with initial condition (1.2) and boundary condition (1.3). This can be discretized by the following $\theta-$weighted plan [15]:

$$\begin{aligned}-i\frac{u(x,y,t+dt)-u(x,y,t)}{dt} =&\theta(\partial_{xx}+\partial_{yy}+V_e(x,y,t+dt))u(x,y,t+dt)\\ &+(1-\theta)(\partial_{xx}+\partial_{yy}+V_e(x,y,t))u(x,y,t).\end{aligned}$$

Then

$$\begin{aligned}&(-i-dt\theta(\partial_{xx}+\partial_{yy}+V_e(x,y,t+dt)))u(x,y,t+dt)\\ =&(-i+(1-\theta)dt(\partial_{xx}+\partial_{yy}+V_e(x,y,t)))u(x,y,t).\end{aligned} \tag{3.1}$$

Consider N^2 points (x_i,y_j), where $x_i = a+(i-1)h \in [a,b]$, $y_j = c+(j-1)k \in [c,d]$ and $i,j = 1,\dots,N$, such that $x_1 = a$, $x_N = b$, $y_1 = c$, $y_N = d$, $h = \frac{b-a}{N-1}$ and $k = \frac{d-c}{N-1}$. In addition, in the time interval $[0,T]$, the grid points are $t^n = ndt$, $n = 0,1,2,\dots,M$, where $M = \frac{T}{dt}$. We apply the finite difference method to discretize the variable-order time fractional derivative by replacement t^{n+1} into (3.1). Then we have:

$$(-i - \theta dt(\partial_{xx} + \partial_{yy} + V_e^{n+1}))u^{n+1} = (-i + (1-\theta)dt(\partial_{xx} + \partial_{yy} + V_e^{n}))u^{n}. \tag{3.2}$$

Now, by using the MLS shape functions we can approximate $u^n(x)$ as follows:

$$u^n(x,y) = \sum_{k=1}^{N^2} \mu_k^n \varphi_k(x,y), \tag{3.3}$$

where $\varphi_k(x,y)$, $k = 1, 2, \ldots, N^2$, are the shape functions for the MLS approximation and μ_k^n, $k = 1, 2, \ldots, N^2$, are unknown coefficients, to be determined. Thus, to determine the values of coefficients μ_k^n, $k = 1, 2, \ldots, N^2$, we use N^2 collocation points of $u^n(x,y)$ as:

$$u^n(x_i, y_j) = \sum_{k=1}^{N^2} \mu_k^n \varphi_k(x_i, y_j), \tag{3.4}$$

where $i, j = 1, 2, \ldots, N$. Rewriting (3.4) in a compact form, we have:

$$[\mathbf{u}]^n = \mathbf{L}[\mu]^n,$$

where $[\mathbf{u}]^n = [\begin{array}{cccc} u_1^n & u_2^n & \cdots & u_{N^2}^n \end{array}]^T$, $[\mu]^n = [\begin{array}{cccc} \mu_1^n & \mu_2^n & \cdots & \mu_{N^2}^n \end{array}]^T$ and $\mathbf{L}$ is an $N^2 \times N^2$ matrix given by:

$$\mathbf{L} = [l_{ij}] = \begin{pmatrix} \varphi_{1,(1,1)} & \cdots & \varphi_{N^2,(1,1)} \\ \vdots & \ddots & \vdots \\ \varphi_{1,(N,N)} & \cdots & \varphi_{N^2,(N,N)} \end{pmatrix}, \tag{3.5}$$

where $\varphi_{k,(i,j)} = \varphi_k(x_i, y_j)$.

Assuming q internal points and $N - q$ boundary points, then matrix $\mathbf{L}$ can be separated into $\mathbf{L} = \mathbf{L1} + \mathbf{L2}$ in which the entries of $\mathbf{L1}$ and $\mathbf{L2}$ are:

$$\begin{aligned} \mathbf{L1} &= [l_{ij}^{(1)}] = \begin{cases} L_{ij}, & 1 \le i \le q,\ 1 \le j \le N, \\ 0, & \text{elsewhere}, \end{cases} \\ \mathbf{L2} &= [l_{ij}^{(2)}] = \begin{cases} L_{ij}, & q+1 \le i \le N,\ 1 \le j \le N, \\ 0, & \text{elsewhere}. \end{cases} \end{aligned} \tag{3.6}$$

Using (3.3), we can write u_{xx} and u_{yy} as follows:

$$u_{xx}^n(x,y) = \sum_{k=1}^{N^2} \mu_k^n \frac{\partial^2 \varphi_k(x,y)}{\partial x^2} = \sum_{k=1}^{N^2} \mu_k^n \psi_k(x,y), \tag{3.7}$$

$$u_{yy}^n(x,y) = \sum_{k=1}^{N^2} \mu_k^n \frac{\partial^2 \varphi_k(x,y)}{\partial y^2} = \sum_{k=1}^{N^2} \mu_k^n \eta_k(x,y), \tag{3.8}$$

where $\psi_k(x,y)$ and $\eta_k(x,y)$ for $k=1,2,\ldots,N^2$, are obtained by (2.7) and (2.8). By taking the collocation points in (3.7) and (3.8), we obtain:

$$u^n_{xx}(x_i,y_j)=\sum_{k=1}^{N^2}\mu^n_k\psi_k(x_i,y_j),\quad i,j=2,3,\ldots,N-1,$$
$$u^n_{yy}(x_i,y_j)=\sum_{k=1}^{N^2}\mu^n_k\eta_k(x_i,y_j),\quad i,j=2,3,\ldots,N-1,$$

which can be rephrased as:

$$[\mathbf{u}_{xx}]^n=\mathbf{K}[\mu]^n, \tag{3.9}$$
$$[\mathbf{u}_{yy}]^n=\mathbf{H}[\mu]^n, \tag{3.10}$$

where

$$\mathbf{K}=[k_{ij}]=\begin{pmatrix}\psi_{11} & \cdots & \psi_{(N)1}\\ \vdots & \ddots & \vdots\\ \psi_{1(q)} & \cdots & \psi_{(N)(q)}\\ 0 & \cdots & 0\end{pmatrix}, \tag{3.11}$$

$$\mathbf{H}=[h_{ij}]=\begin{pmatrix}\eta_{11} & \cdots & \eta_{(N)1}\\ \vdots & \ddots & \vdots\\ \eta_{1(q)} & \cdots & \eta_{(N)(q)}\\ 0 & \cdots & 0\end{pmatrix}, \tag{3.12}$$

in which $\eta_{ji}=\eta_j(x_i,y_i)$ and $\psi_{ji}=\psi_j(x_i,y_i)$.

Now, by replacing (3.9) and (3.10) into (3.2) together with (1.2), (1.3) and using the collocation points, the following matrix form is obtained:

$$\mathbf{M}[\mu]^{n+1}=\mathbf{N}[\mu]^n+[\mathbf{G}]^{n+1}, \tag{3.13}$$

where

$$\mathbf{M}=-i\mathbf{L1}+\theta dt([\mathbf{K}+\mathbf{H}]-\theta dt\mathbf{V}^n_e*\mathbf{L1}+\mathbf{L2}, \tag{3.14}$$
$$\mathbf{N}=(1-\theta)dt[\mathbf{K}+\mathbf{H}]+(1-\theta)dt\mathbf{V^n_e}*\mathbf{L1}+\mathbf{L2},$$
$$[\mathbf{G}]^{n+1}=[\begin{matrix}0 & 0 & \cdots & g^{n+1}_{q+1}\cdots\cdots g^{n+1}_N\end{matrix}]^T. \tag{3.15}$$

The symbol $*$ means that each component of the left vector is multiplied to all the components of corresponding row of right matrix. For solving the system, $[\mu]^{n+1}$ can be computed by recursive relation in (3.13) for $n=1,2,\ldots,M$. First we must compute $[\mu]^0$. Also, it should be noted that according to (3.13) for $n=0$, we have:

$$\mathbf{M}[\mu]^1=\mathbf{N}[\mu]^0+[\mathbf{G}]^1.$$

We can write an algorithm for this approach.

TABLE 1. Absolute errors obtained in Example 5.1 for $N = 8$, $m = 10$ and $dt = 0.01$.

t	Real part(error)	Imaginary part(error)	MAX(error)
0.1	7.748E-7	4.299E-8	7.775E-7
0.3	7.451E-7	2.223E-7	7.776E-7
0.5	6.861E-7	3.659E-7	7.775E-7
0.7	5.998E-7	4.949E-7	7.780E-7
1	4.266E-7	4.949E-7	7.774E-7

4. Proposed Algorithm

The object of this algorithm is designed to solve the Shrödinger equation.
Input: N, m, T(final time), dt(step length) and the functions $V_e(x, y, t)$, $h_1(y, t)$, $h_2(y, t)$, $h_3(x, t)$, $h_4(x, t)$, $g(x, y)$.
Step 1: Define X, Y vectors of grid points in (x, y) coordinates and $P(x, y)$ vectors of the basis function.
Step 2: Define $w(x, y)$ by (2.3).
Step 3: Compute matrices C and D by (2.5).
Step 4: Compute Φ by (2.6), and Φ_{xx} and Φ_{yy} by (2.7) and (2.8).
Step 5: Compute matrices L, L_1, L_2, K and H by (3.5), (3.6), (3.11) and (3.12).
Step 6: Discrete the Shrödinger equation to (3.2).
Step 7: Compute matrices M, N and vector G by (3.14).
Step 8: Put $\mathbf{M}[\mu]^{n+1} = \mathbf{N}[\mu]^n + [\mathbf{G}]^{n+1}$.
Step 9: Compute $[\mu]^0 = L^{-1}g_1(X, Y)$.
Step 10: Substitute $[\mu]^0$ in the above matrix equation to obtain other $[\mu]^i$, $i = 0, \dots, n$.
Output: The approximate solution $u^n(x, y) = \sum_{j=1}^{N} \mu_j^n \varphi_j(x, y)$.

5. Numerical Examples

In this section, some numerical examples are presented with their exact solutions, to demonstrate the performance and validity of the proposed method.

Example 5.1. Consider the quantum equation as following

$$-iu_t(x, y, t) = (u_{xx}(x, y, t) + u_{yy}(x, y, t))u(x, y, t) + V_e(x, y, t)u(x, y, t),$$

in the region $(x, y) \in [0, 1]$ with $V_e(x, y, t) = 1 - \frac{60}{x^2} - \frac{60}{y^2}$.

The initial condition is $u(x, y, 0) = x^5y^5$ and the boundary conditions are

$$u(0, y, t) = 0, \quad u(1, y, t) = y^2e^{it},$$
$$u(x, 0, t) = 0, \quad u(x, 1, t) = x^2e^{it}.$$

The exact solution of this equation is $u(x, y, t) = x^5y^5e^{it}$.

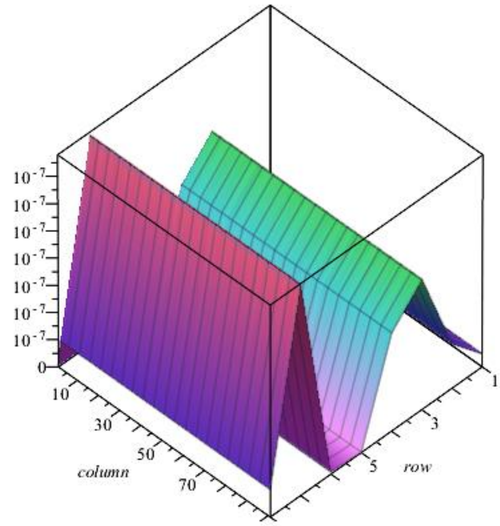

FIGURE 1. The plot of absolute error for Example 5.1.

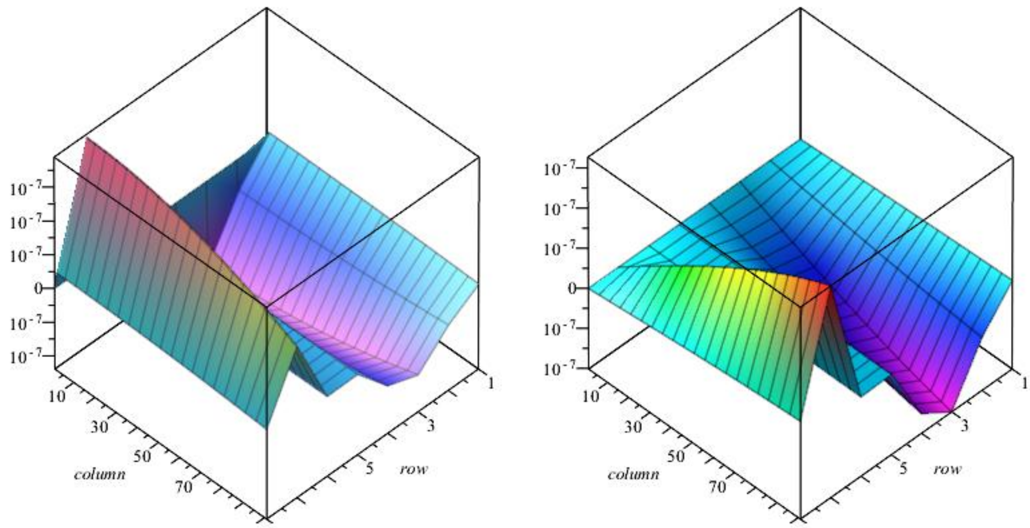

FIGURE 2. The plots of the approximate results (real part in left side) and (imaginary part in right side) for Example 5.1.

We solved the above problem by proposed algorithm for $N = 8$, $m = 10$ and $dt = 0.01$. The plot of the absolute errors is shown in Figure 1 and plots of real and imaginary parts of absolute errors are shown in Figure 2. Approximate results of absolute errors and their real and imaginary parts for different t are presented in Table 1.

From Figures 1–2 and Table 1 it can be observed that the proposed algorithm is very efficient and accurate.

TABLE 2. Absolute errors obtained in Example 5.2 for $N = 8$, $m = 12$ and $dt = 0.01$.

t	Real part	Imaginary part	MAX(error)
0.1	3.212E-4	2.733E-7	3.221E-4
0.3	3.214E-4	8.692E-7	3.214E-4
0.5	2.828E-4	1.430E-6	2.828E-4
0.7	2.819E-4	1.935E-6	2.811E-4
1	2.817E-4	2.254E-6	2.817E-4

Example 5.2. Let us consider the quantum equation as following

$$-iu_t(x,y,t) = (u_{xx}(x,y,t) + u_{yy}(x,y,t))u(x,y,t) + V_e(x,y,t)u(x,y,t),$$

in the region $(x, y) \in [0, 1]$ with $V_e(x, y, t) = 1 + x^2 + y^2$.

The initial condition $u(x, y, 0) = \sin(xy)$, and boundary conditions are

$$u(0,y,t) = 0, \quad u(1,y,t) = \sin(y)e^{it},$$
$$u(x,0,t) = 0, \quad u(x,1,t) = \sin(x)e^{it}.$$

The exact solution of the above equation is $u(x, y, t) = \sin(xy)e^{it}$.

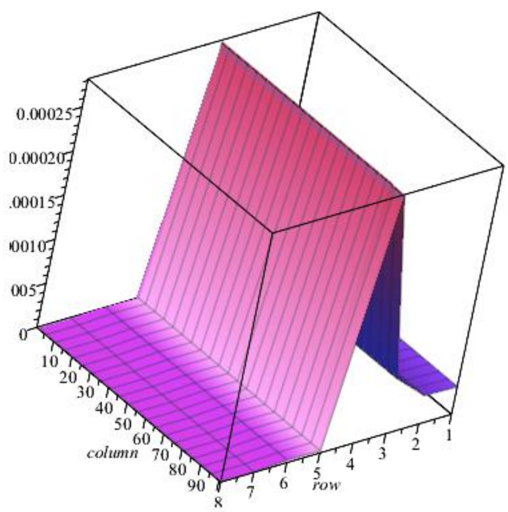

FIGURE 3. The plot of absolute error for Example 5.2.

We solved the above problem by proposed algorithm for $N = 8$, $m = 12$ and $dt = 0.01$. The plot of the absolute errors is shown in Figure 3 and plots of real and imaginary parts of absolute errors are shown in Figure 4. Approximate results of absolute errors and their real and imaginary parts for different t are presented in Table 2. From Figures 3–4 and Table 2 it can be observed that the proposed algorithm is very efficient and accurate.

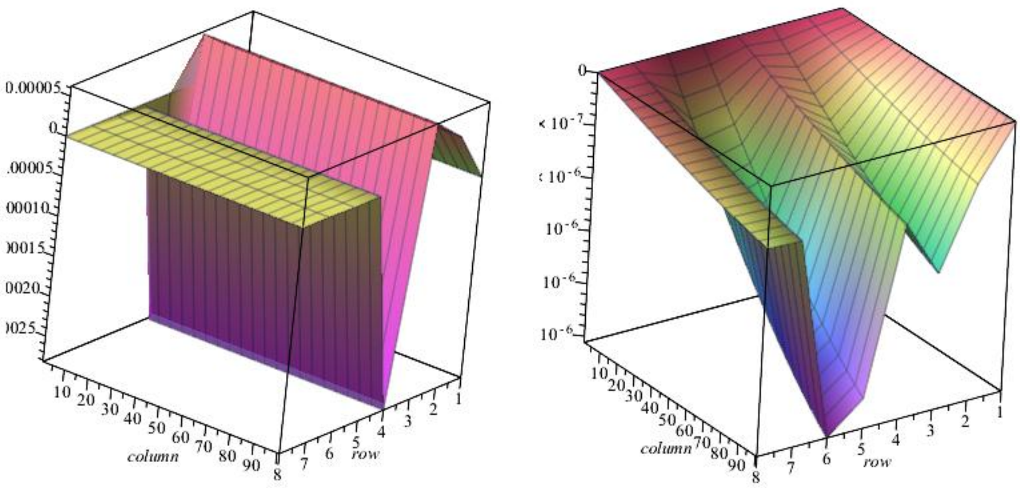

FIGURE 4. The plots of the approximate results (real part in left side) and (imaginary part in right side) for Example 5.2.

TABLE 3. Absolute errors obtained in Example 5.3 for $N = 8$, $m = 10$ and $dt = 0.01$.

t	Real part	Imaginary part	MAX(error)
0.1	8.895E-6	8.027E-6	8.932E-6
0.3	8.553E-6	2.255E-6	8.932E-6
0.5	7.881E-6	4.203E-6	8.932E-6
0.7	6.888E-6	6.585E-6	8.932E-6
1	4.901E-6	7.467E-6	8.832E-6

Example 5.3. Let us consider the quantum equation as following

$$-iu_t(x,y,t) = (u_{xx}(x,y,t) + u_{yy}(x,y,t))u(x,y,t) + V_e(x,y,t)u(x,y,t),$$

in the region $(x,y) \in [0,1]$ with $V_e(x,y,t) = 1 - \frac{2}{x^2} - \frac{2}{y^2}$.

The initial condition is $u(x,y,0) = x^2y^2$, and boundary conditions are

$$u(0,y,t) = 0, \quad u(1,y,t) = y^2e^{it},$$
$$u(x,0,t) = 0, \quad u(x,1,t) = x^2e^{it}.$$

The exact solution of the this example is $u(x,y,t) = x^2y^2e^{it}$.

We solved the above problem by proposed algorithm for $N = 8$, $m = 10$ and $dt = 0.01$. The plot of the absolute errors is shown in Figure 5 and plots of real and imaginary parts of absolute errors are shown in Figure 6. Approximation results of absolute errors and their real and imaginary parts for different t are presented in Table 3.

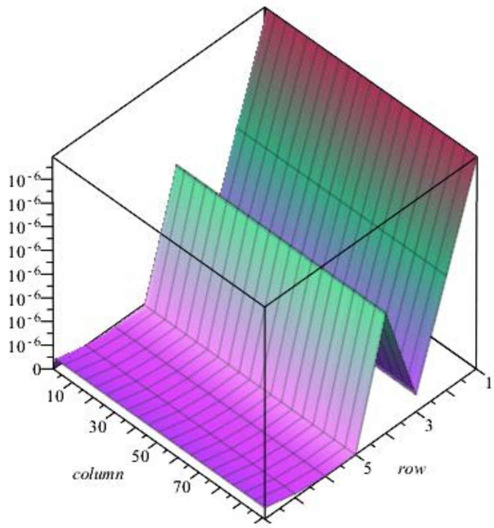

FIGURE 5. The plot of absolute error for Example 5.3.

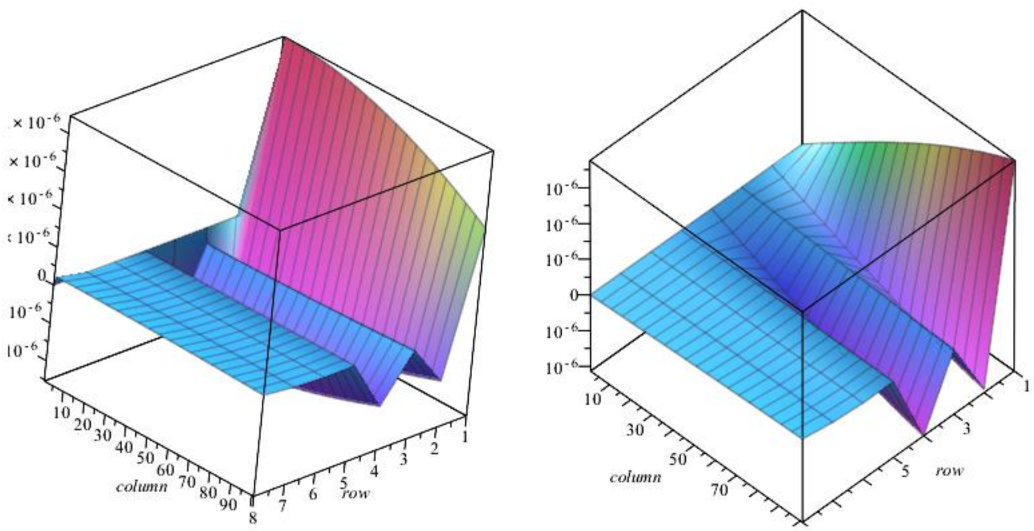

FIGURE 6. The plots of the approximate results (real part in left side) and (imaginary part in right side) for Example 5.3.

Dehghan and Shokri have solved this problem for $dx = dy = 0.1$ and $dt = 0.001$ [6]. By comparing the results of the two methods, it is observed that the proposed algorithm in this work is much better than [6]. In this example, the number of nodes (N) mentioned in [6] was 100 and in the proposed method is $N = 10$. From Figures 5-6 and Table 3 it can be observed that the proposed algorithm is very efficient and accurate.

6. Conclusion

In this paper, based on the moving least squares method (MLS) and the finite difference method (FDM) an algorithm was proposed for solving Shrödinger equation. For this purpose, first we discretized the Shrödinger equation by FDM and then applied the MLS method to obtain a numerical algorithm for solving the partial differential equation thus obtained. To verify the results, three numerical examples were presented. The merit of our approach is the applicability and accuracy the algorithm provides.

References

[1] J. E. Akin, *Finite Elements for Analysis and Design: Computational Mathematics and Applications Series*, Academic Press, San Diego, 2014.

[2] A. Arnold, *Numerically absorbing boundary conditions for quantum evolution equations*, VLSI Design **6** (1998), 313–319.

[3] J. C. Cortés, L. Jódar and L. Villafuerte, *Mean square numerical solution of random differential equations: facts and possibilities*, Comput. Math. Appl. **53** (2007), 1098–1106.

[4] J. C. Cortés, L. Jódar and L. Villafuerte, *Numerical solution of random differential equations: a mean square approach*, Math. Comput. Model. **45** (2007), 757–765.

[5] M. Dehghan and D. Mirzaei, *Meshless local Petrov-Galerkin (MLPG) method for the unsteady magnetohydrodynamic (MHD) flow through pipe with arbitrary wall conductivity*, Appl. Numer. Math. **59** (2009), 1043–1058.

[6] M. Dehghan and A. Shokri, *A numerical method for two-dimensional Schrödinger equation using collocation and radial basis functions*, Comput. Math. Appl. **54** (2007), 136–146.

[7] M. Ehler, *Shrinkage rules for variational minimization problems and applications to analytical ultracentrifugation*, J. Inverse Ill-Posed Probl. **19** (2011), 593–614.

[8] J. C. Kalita, P. Chhabra and S. Kumar, *A semi-discrete higher order compact scheme for the unsteady two-dimensional Schrödinger equation*, J. Comput. Appl. Math. **197** (2006), 141–149.

[9] M. Khodabin, K. Maleknejad, M. Rostami and M. Nouri, *Numerical solution of stochastic differential equations by second order Runge–Kutta methods*, Math. Comput. Model. **53** (2011), 1910–1920.

[10] L. Kong, J. Hong, F. Fu and J. Chen, *Symplectic structure-preserving integrators for the two-dimensional Gross–Pitaevskii equation for BEC*, J. Comput. Appl. Math. **235** (2011), 4937–4948.

[11] G. Li, X. Jin and N. Alum, *Meshless methods for numerical solution of partial differential equations*, in: *Handbook of Materials Modeling*, Springer, Dordrecht, 2005, 2447–2474.

[12] H. Li, C. Jiang and Z. Lv, *A Galerkin energy-preserving method for two dimensional nonlinear Schrödinger equation*, Appl. Math. Comput. **324** (2018), 16–27.

[13] H. Li, Y. Wang and Q. Sheng, *An energy-preserving Crank-Nicolson Galerkin method for Hamiltonian partial differential equations*, Numer. Methods for Partial Differential Equations **32** (2016), 1485–1504.

[14] H. Lin and S. Atluri, *The meshless local Petrov-Galerkin (MLPG) method for solving incompressible Navier-Stokes equations*, CMES-Computer Modeling in Engineering and Sciences **2** (2001), 117–142.

[15] A. Mardani, M. Hooshmandasl, M. Hosseini and M. Heydari, *Moving least squares (MLS) method for the nonlinear hyperbolic telegraph equation with variable coefficients*, Int. J. Comput. Methods **14** (2017), Paper ID 1750026.

[16] M. N. Oğuztöreli, *Time-Lag Control Systems*, Academic Press, London, 1966.

[17] M. Shibahara and S. Atluri, *The meshless local Petrov-Galerkin method for the analysis of heat conduction due to a moving heat source, in welding*, International Journal of Thermal Sciences **50** (2011), 984–992.

[18] W. Thirring, *Classical Mathematical Physics: Dynamical Systems and Field Theories*, Springer Science & Business Media, New York, 2013.

[19] T. Wang, B. Guo and Q. Xu, *Fourth-order compact and energy conservative difference schemes for the nonlinear Schrödinger equation in two dimensions*, J. Comput. Phys. **243** (2013), 382–399.

[1]Department of Mathematical Sciences,
Yazd University,
Yazd, Iran
Email address: rahimimojtaba1358@gmail.com
Email address: smkarbassi@yazd.ac.ir

[2]Department of Computer Science,
Yazd University,
Yazd, Iran
Email address: hooshmandasl@yazd.ac.ir

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 943–957.

MULTIVALUED FG-CONTRACTION MAPPINGS ON DIRECTED GRAPHS

HEMANT KUMAR NASHINE[1,2] AND ZORAN KADELBURG[3]

Abstract. In this paper, we study generalized $\mathbb{FG}$-contraction conditions for a pair of mappings defined on a family of subsets of a metric space endowed with a directed graph, and discuss coincidence and common fixed point results relaxing the continuity of mappings. The given notions and results are exemplified by suitable models. We apply our results to the problem of existence of solutions of a Fredholm integral inclusion.

1. Introduction and Preliminaries

Fixed point theory plays an important role not only in solving problems arising in science and technology but also other problems that come in various parts of life, by converting the problem into operator form. In the last decades, various approaches and techniques have been applied to get the solution. In particular, the concept of graph theory has been applied by Jachymski and Jozwik [10] to obtain fixed points of mappings acting on metric spaces equipped with directed graph. Gwozdz-Lukawska and Jachymski [9] discussed such problems for finite families of mappings.

We start recalling the terminology given in Jachymski [11].

Let (X, d) be a metric space and let Δ denote the diagonal of $X \times X$. Let $\mathcal{G} = (v_{\mathcal{G}}, e_{\mathcal{G}})$ be a directed graph where the set $v_{\mathcal{G}}$ of its vertices coincides with X, and the set $e_{\mathcal{G}}$ of edges contains all loops, that is, $\Delta \subseteq e_{\mathcal{G}}$. In addition, assume that the graph $\mathcal{G}$ has no parallel edges. The triplet $(X, d, \mathcal{G})$ is then called a directed graph metric space.

Key words and phrases. Multivalued contraction, metric space with directed graph, Fredholm integral inclusion.

2020 *Mathematics Subject Classification*. Primary: 47H10. Secondary: 54H25.

DOI 10.46793/KgJMat2206.943N

Received: April 28, 2020.

Accepted: May 23, 2020.

If u and v are vertices of $\mathcal{G}$, then a path in $\mathcal{G}$ from u to v is a finite sequence $\{u_i\}$, $i \in \{0, 1, 2, \ldots, k\}$, of vertices such that $u_0 = u$, $u_k = v$ and $(u_{i-1}, u_i) \in e_{\mathcal{G}}$ for $i \in \{1, 2, \ldots, k\}$.

Recall that a graph $\mathcal{G}$ is called connected if there is a directed path between any two vertices, and it is called weakly connected if $\overline{\mathcal{G}}$ is connected, where $\overline{\mathcal{G}}$ denotes the undirected graph obtained from $\mathcal{G}$ by ignoring the direction of edges.

Fixed point results for single-valued mappings on directed graph metric spaces were first obtained by Jachymski in [11] and further generalized by various researchers. Some multivalued results of this kind were given by Abbas et al. in [1–3]. We recall some basic notions.

Let (X, d) be a metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. The Pompeiu-Hausdorff metric induced by d is the mapping $H : CB(X) \times CB(X) \to [0, +\infty)$ defined by

$$H(Z, W) = \max\{\sup_{v \in W} d(v, Z), \sup_{u \in Z} d(u, W)\},$$

for $Z, W \in CB(X)$, where $d(u, W) = \inf\{d(u, v) : v \in W\}$.

Lemma 1.1 ([14])**.** *Let (X, d) be a metric space. If $Z, W \in CB(X)$ are such that $H(Z, W) < \epsilon$, then for each $v \in Z$ there exists an element $u \in W$ such that $d(v, u) < \epsilon$.*

Definition 1.1 ([1])**.** Let $(X, d, \mathcal{G})$ be a directed graph metric space and let Z and W be two nonempty subsets of X. Then we say that:

(a) there is an edge between Z and W if there is an edge between some $u \in Z$ and $v \in W$;
(b) there is a path between Z and W if there is a path between some $u \in Z$ and $v \in W$;
(c) the graph $\mathcal{G}$ is said to be set-transitive if, for all $Z, W, V \in CB(X)$, whenever there is a path between Z and W and there is a path between W and V, then there is a path between Z and V.

Definition 1.2 ([12])**.** Let $P, Q : CB(X) \to CB(X)$ be two multivalued mappings. A set $Z \in CB(X)$ is said to be a coincidence point of P and Q, if $P(Z) = Q(Z)$. A set $Z \in CB(X)$ is said to be a fixed point of P if $P(Z) = Z$. The maps P, Q are said to be weakly compatible if they commute at their coincidence points.

We will denote by $Coin(P, Q)$ the set of all coincidence points of P and Q and by $Fix(P)$ the set of all fixed points of P.

A collection $\Lambda \subset CB(X)$ is said to be complete if for any sets $Z, W \in \Lambda$, there is an edge between Z and W.

Recall [1] that the space $(X, d, \mathcal{G})$ is said to have property (P^*), if for any sequence $\{Z_n\}$ in $CB(X)$ with $Z_n \to Z$ as $n \to +\infty$ (in the sense of Pompeiu-Hausdorff metric), the existence of an edge between Z_n and Z_{n+1} for each $n \in \mathbb{N}$, implies that there is a subsequence $\{Z_{n_k}\}$ of $\{Z_n\}$ with an edge between Z_{n_k} and Z for $k \in \mathbb{N}$.

The aim of this paper is to prove some coincidence and common fixed point results for a pair of (not necessarily continuous) multivalued generalized graphic $\mathbb{FG}$-contractive mappings defined on the family of closed and bounded subsets of a directed graph metric space. These results extend and strengthen various comparable results in the existing literature (see, e.g., [1–7, 14, 19]). Application to Fredholm-type integral inclusions is presented. For basic notions in metrical fixed point theory see, e.g., [8, 13].

2. Generalized Graphic $\mathbb{FG}$-Contractions

Parvaneh et al. [16] introduced and used the following classes of functions, modifying Wardowski's approach in [20].

$\mathfrak{F}$ is the set of all functions $\mathbb{F} : \mathbb{R}^+ \to \mathbb{R}$ such that

($\mathfrak{F}_1$) $\mathbb{F}$ is strictly increasing;

($\mathfrak{F}_2$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to+\infty} \xi_n = 0$ if and only if $\lim_{n\to+\infty} \mathbb{F}(\xi_n) = -\infty$.

$\mathfrak{G}_\beta$ is the set of pairs (G, β), where $G : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0, +\infty) \to [0, 1)$, such that

($\mathfrak{G}_{\beta 1}$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\limsup_{n\to+\infty} G(\xi_n) \geq 0$ if and only if $\limsup_{n\to+\infty} \xi_n \geq 1$;

($\mathfrak{G}_{\beta 2}$) for each sequence $\{\xi_n\} \subseteq [0, +\infty)$, $\limsup_{n\to+\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to+\infty} \xi_n = 0$;

($\mathfrak{G}_{\beta 3}$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\sum_{n=1}^{+\infty} G(\beta(\xi_n)) = -\infty$.

Definition 2.1. Let $(X, d, \mathcal{G})$ be a directed graph metric space. The pair (P, Q) of maps $P, Q : CB(X) \to CB(X)$ is said to be a generalized graphic $\mathbb{F}G$-contraction if

(i) for every $Z \in CB(X)$ there is a path between Z and $P(Z)$, as well as between $Q(Z)$ and Z, and

(ii) there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ such that for all $Z, W \in v_\mathcal{G}$, with a path between them and $P(Z) \neq P(W)$,

$$\mathbb{F}(H(P(Z), P(W))) \leq \mathbb{F}(\Theta(Z, W)) + G(\beta(\Theta(Z, W))) \tag{2.1}$$

holds, where

$$\Theta(Z, W) = \max \left\{ \begin{array}{l} H(Q(Z), Q(W)), H(P(Z), Q(Z)), H(P(W), Q(W)), \\ \frac{1}{2}[H(P(Z), Q(W)) + H(P(W), Q(Z))] \end{array} \right\}.$$

Theorem 2.1. *Let $(X, d, \mathcal{G})$ be a directed graph metric space and $P, Q : CB(X) \to CB(X)$ be a pair of mappings. Assume the following conditions hold:*

(i) *$P(CB(X)) \subseteq Q(CB(X))$;*

(ii) *the graph $\mathcal{G}$ is set-transitive;*

(iii) *(P, Q) is a generalized graphic $\mathbb{FG}$-contraction pair;*

(iv) *$Q(CB(X))$ is a complete subspace of $(CB(X), H)$, and*

(v) *$\mathcal{G}$ is weakly connected and property (P^*) holds.*

Then $Coin(P, Q) \neq \emptyset$.

Proof. Let $Z_0 \in CB(X)$ be arbitrary. Using (i), choose $Z_1 \in CB(X)$ such that $P(Z_0) = Q(Z_1)$. Proceeding in this way, if $Z_n \in CB(X)$ is chosen, we choose $Z_{n+1} \in CB(X)$ such that $P(Z_n) = Q(Z_{n+1})$ for $n \in \mathbb{N}$. Since there is a path between Z_n and $P(Z_n)$ and there is a path between $P(Z_n) = Q(Z_{n+1})$ and Z_{n+1}, it follows by (ii) that there is a path between Z_n and Z_{n+1} for each $n \in \mathbb{N}$.

Assume that $P(Z_n) \neq P(Z_{n+1})$ for all $n \in \mathbb{N}$. (If not, then $P(Z_n) = P(Z_{n+1})$ is true for some n, which implies that $Q(Z_{n+1}) = P(Z_n)$, and hence $Z_{n+1} \in Coin(P, Q)$.)

As there is a path between Z_n and Z_{n+1}, due to (iii), we have that

$$\begin{aligned}\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) &= \mathbb{F}(H(P(Z_n), P(Z_{n+1}))) \\ &\leq \mathbb{F}(\Theta(Z_n, Z_{n+1})) + G(\beta(\Theta(Z_n, Z_{n+1}))),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\Theta(Z_n, Z_{n+1}) &= \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(P(Z_n), Q(Z_n)), \\ H(P(Z_{n+1}), Q(Z_{n+1})), \\ \frac{1}{2}[H(P(Z_n), Q(Z_{n+1})) + H(P(Z_{n+1}), Q(Z_n))] \end{array}\right\} \\ &= \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_n)), \\ H(Q(Z_{n+2}), Q(Z_{n+1})), \\ \frac{1}{2}[H(Q(Z_{n+1}), Q(Z_{n+1})) + H(Q(Z_{n+2}), Q(Z_n))] \end{array}\right\} \\ &\leq \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2})), \\ \frac{1}{2}[H(Q(Z_{n+2}), Q(Z_{n+1})) + H(Q(Z_{n+1}), Q(Z_n))] \end{array}\right\} \\ &= \max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) \\ \leq&\mathbb{F}(\max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\}) \\ &+ G(\beta(\max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\})) \\ \leq&\mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) + G(\beta(H(Q(Z_n), Q(Z_{n+1})))),\end{aligned}$$

that is,

$$\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) \leq \mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) + G(\beta(H(Q(Z_n), Q(Z_{n+1})))),$$

for all $n \in \mathbb{N}$. We conclude that

$$\begin{aligned}&\mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) \\ \leq&\mathbb{F}(H(Q(Z_{n-1}), Q(Z_n))) + G(\beta(\Theta(Z_{n-1}, Z_n))) \\ \leq&\mathbb{F}(H(Q(Z_{n-2}), Q(Z_{n-1}))) + G(\beta(\Theta(Z_{n-1}, Z_n))) + G(\beta(\Theta(Z_{n-2}, Z_{n-1}))) \\ &\vdots \\ \leq&\mathbb{F}(H(Q(Z_0), Q(Z_1))) + \sum_{i=1}^{n} G(\beta(\Theta(Z_{n-1}, Z_n))),\end{aligned}$$

that is,

$$\mathbb{F}(H(Q(Z_n),Q(Z_{n+1}))) \leq \mathbb{F}(H(Q(Z_0),Q(Z_1))) + \sum_{i=1}^{n} G(\beta(\Theta(Z_{n-1},Z_n))),$$

for all $n \in \mathbb{N}$. By the properties of $(G,\beta) \in \mathfrak{G}_\beta$, $\lim_{n\to+\infty} \mathbb{F}(H(Q(Z_n),Q(Z_{n+1}))) = -\infty$ and by $(\mathfrak{F}_2)$, $\lim_{n\to+\infty} H(Q(Z_n),Q(Z_{n+1})) = 0$.

Next we claim that the sequence $\{Q(Z_n)\}$ is a Cauchy one. Suppose the contrary, which means that there exists an $\epsilon > 0$ and two increasing sequences of integers $\{p(\ell)\}$ and $\{q(\ell)\}$, $q(\ell) > p(\ell) > \ell$ such that $H(Q(Z_{p(\ell)}),Q(Z_{q(\ell)}))$, $H(Q(Z_{q(\ell)+1}),Q(Z_{p(\ell)-1}))$ and $H(Q(Z_{q(\ell)}),\, Q(Z_{p(\ell)-1}))$ tend to ϵ as $\ell \to +\infty$. Due to (iii) with $Z = Z_{p(\ell)-1}$ and $W = Z_{q(\ell)}$, we have

$$\begin{aligned}\mathbb{F}(H(Q(Z_{p(\ell)}),Q(Z_{q(\ell)+1}))) =&\mathbb{F}(H(P(Z_{p(\ell)-1}),P(Z_{q(\ell)}))) \\ \leq&\mathbb{F}(\Theta(Z_{p(\ell)-1},Z_{q(\ell)})) + G(\beta(\Theta(Z_{p(\ell)-1},Z_{q(\ell)}))),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}&\Theta(Z_{p(\ell)-1},Z_{q(\ell)}) \\ &= \max\left\{\begin{array}{l} H(Q(Z_{p(\ell)-1}),Q(Z_{q(\ell)})), H(P(Z_{p(\ell)-1}),Q(Z_{p(\ell)-1})), \\ H(P(Z_{q(\ell)}),Q(Z_{q(\ell)})), \\ \frac{1}{2}[H(P(Z_{p(\ell)-1}),Q(Z_{q(\ell)})) + H(P(Z_{q(\ell)}),Q(Z_{p(\ell)-1}))] \end{array}\right\}. \\ &= \max\left\{\begin{array}{l} H(Q(Z_{p(\ell)-1}),Q(Z_{q(\ell)})), H(Q(Z_{p(\ell)}),Q(Z_{p(\ell)-1})), \\ H(Q(Z_{q(\ell)+1}),Q(Z_{q(\ell)})), \\ \frac{1}{2}[H(Q(Z_{p(\ell)}),Q(Z_{q(\ell)})) + H(Q(Z_{q(\ell)+1}),Q(Z_{p(\ell)-1}))] \end{array}\right\}.\end{aligned} \tag{2.4}$$

Taking the limit as $\ell \to +\infty$ in (2.4), we have

$$\lim_{k\to+\infty} \Theta(Z_{p(\ell)-1},Z_{q(\ell)}) = \max\{\epsilon,0,0,\tfrac{1}{2}(\epsilon+\epsilon)\} = \epsilon.$$

Taking the limit as $\ell \to +\infty$ in (2.3) we get

$$\begin{aligned}F(\epsilon) &\leq F(\limsup_{\ell\to+\infty} H(Q(Z_{p(l)}),Q(Z_{q(\ell)+1})) \\ &\leq \limsup_{\ell\to+\infty} \mathbb{F}(\Theta(Z_{p(\ell)-1},Z_{q(\ell)})) + \limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1},Z_{q(\ell)}))), \\ &\leq \mathbb{F}(\epsilon) + \limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1},Z_{q(\ell)}))),\end{aligned}$$

which further implies

$$\limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1},Z_{q(\ell)}))) \geq 0.$$

Using the properties of functions $\mathbb{G}$ and β, we get $\limsup_{\ell\to+\infty} \beta(\Theta(Z_{p(\ell)-1},Z_{q(\ell)})) = 1$ and $\lim_{\ell\to+\infty} \Theta(Z_{p(\ell)-1},Z_{q(\ell)}) = 0$, which is in contradiction with $\epsilon > 0$. Hence $\{Q(Z_n)\}$ is a Cauchy sequence in $Q(CB(X))$. Due to (iv), $Q(Z_n) \to D$ as $n \to +\infty$ for some $D \in CB(X)$. In addition, $Q(C) = D$ for some $C \in CB(X)$.

We argue that $P(C) = Q(C)$. If not, then, since there is a path between $Q(Z_{n+1})$ and $Q(Z_n)$ by the property (P^*), there exists a subsequence $\{Q(Z_{n_k+1})\}$ of $\{Q(Z_{n+1})\}$

such that there is a path between $Q(C)$ and $Q(Z_{n_k+1})$ for every $k \in \mathbb{N}$. As there is a path between C and $Q(C)$ and there is a path between $Q(Z_{n_k+1}) = P(Z_{n_k})$ and Z_{n_k}, we have that there is a path between C and Z_{n_k}. Using condition (iii), we get that

$$\begin{aligned}\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) &= \mathbb{F}(H(P(C), P(Z_{n_k}))) \\ &\leq \mathbb{F}(\Theta(C, Z_{n_k})) + G(\beta(\Theta(C, Z_{n_k}))),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}&\Theta(C, Z_{n_k}) \\ &= \max\left\{\begin{array}{l} H(Q(C), Q(Z_{n_k})), H(P(C), Q(C)), H(P(Z_{n_k}), Q(Z_{n_k})), \\ \frac{1}{2}[H(P(C), Q(Z_{n_k})) + H(P(Z_{n_k}), Q(C))] \end{array}\right\} \\ &= \max\left\{\begin{array}{l} H(C, Q(Z_{n_k})), H(P(C), Q(C)), H(Q(Z_{n_k+1}), Q(Z_{n_k})), \\ \frac{1}{2}[H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))] \end{array}\right\}.\end{aligned}$$

There are the following four possibilities.

- $\Theta(C, Z_{n_k}) = H(Q(C), Q(Z_{n_k}))$. From (2.5),

$$\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) = \mathbb{F}(H(P(C), P(Z_{n_k}))) + G(\beta(H(Q(C), Q(Z_{n_k})))).$$

Passing to the upper limit as $k \to +\infty$ gives

$$\mathbb{F}(H(P(C), Q(C))) \leq \mathbb{F}(H(Q(C), Q(C))) + G(\beta(H(Q(C), Q(C)))),$$

which is a contradiction.

- When $\Theta(C, Z_{n_k}) = H(P(C), Q(C))$, then

$$\mathbb{F}(H(P(C), Q(C))) \leq \mathbb{F}(H(P(C), Q(C))) + G(\beta(H(P(C), Q(C)))).$$

Therefore, $G(\beta(H(P(C), Q(C)))) \geq 0$, which implies that $\beta(H(P(C), Q(C)) \geq 1$, which is a contradiction.

- $\Theta(C, Z_{n_k}) = H(Q(Z_{n_k+1}), Q(Z_{n_k}))$. From (2.5),

$$\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) = \mathbb{F}(H(Q(Z_{n_k+1}), Q(Z_{n_k}))) + G(\beta(H(Q(Z_{n_k+1}), Q(Z_{n_k})))).$$

Passing to the upper limit as $k \to +\infty$, we have

$$\mathbb{F}(H(P(C), Q(C))) \leq \mathbb{F}(H(Q(C), Q(C))) + G(\beta(H(Q(C), Q(C)))),$$

which is a contradiction.

- $\Theta(C, Z_{n_k}) = \dfrac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}$. From (2.5),

$$\begin{aligned}\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) =& \mathbb{F}\left(\frac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}\right) \\ &+ G\left(\beta\left(\frac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}\right)\right).\end{aligned}$$

Passing to the upper limit as $k \to +\infty$ gives

$$\begin{aligned}\mathbb{F}(H(P(C),Q(C))) \leq& \mathbb{F}\left(\frac{H(P(C),Q(C))+H(Q(C),Q(C))}{2}\right)\\ &+G\left(\beta\left(\frac{H(P(C),Q(C))+H(Q(C),Q(C))}{2}\right)\right)\\ =&\mathbb{F}\left(\frac{H(P(C),Q(C))}{2}\right)+G\left(\beta\left(\frac{H(P(C),Q(C))}{2}\right)\right)\\ <&\mathbb{F}\left(H(P(C),Q(C))\right)+G\left(\beta\left(\frac{H(P(C),Q(C))}{2}\right)\right),\end{aligned}$$

by the properties of $(G,\beta) \in \mathfrak{G}_\beta$, which is a contradiction.

Thus, in all cases we have $P(C) = Q(C)$, that is, $C \in Coin(P,Q)$. □

Theorem 2.2. *Let all of the conditions of Theorem* 2.1 *be satisfied. Then the following hold.*

(1) *If $Coin(P,Q)$ is a complete subgraph of X, then $H(P(C),P(D)) = 0$ for all $C,D \in Coin(P,Q)$.*
(2) *If, moreover, P and Q are weakly compatible, then they have a unique common fixed point in $CB(X)$.*
(3) *$Fix(P) \cap Fix(Q)$ is a complete subgraph of X if and only if P and Q have a unique common fixed point in $CB(X)$.*

Proof. Following the proof of Theorem 2.1, $Coin(P,Q) \neq \emptyset$.

In order to show (1), suppose that $C,D \in Coin(P,Q)$. Assume on contrary that $H(P(C),P(D)) \neq 0$. Due to (iii),

$$\mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(\Theta(C,D)) + G(\beta(\Theta(C,D))), \tag{2.6}$$

where

$$\begin{aligned}\Theta(C,D) &= \max\left\{\begin{array}{l} H(Q(C),Q(D)), H(P(C),Q(C)), H(P(D),Q(D)),\\ \frac{1}{2}[H(P(C),Q(D))+H(P(D),Q(C))]\end{array}\right\}\\ &= \max\left\{\begin{array}{l} H(P(C),P(D)), H(P(C),P(C)), H(P(D),Q(D)),\\ \frac{1}{2}H(P(C),P(D))+H(P(D),P(C))\end{array}\right\}\\ &= H(P(C),P(D)).\end{aligned}$$

Thus,

$$\mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(H(P(C),P(D))) + G(\beta(H(P(C),P(D))))$$

by the properties of $(G,\beta) \in \mathfrak{G}_\beta$, a contradiction. Hence, we have derived that (1) holds.

In order to show (2), we start with $Fix(Q) \cap Fix(P) \neq \emptyset$. If $Y = P(C) = Q(C)$, then we have $Q(Y) = QP(C) = PQ(C) = P(Y)$, which shows that $Y \in Coin(P,Q)$. Thus, $H(P(C),P(Y)) = 0$ (by (1)). Hence $Y = P(Y) = Q(Y)$, that is, $Y \in$

$Fix(P) \cap Fix(Q)$. As $Coin(P, Q)$ contains exactly one element, the same is true for $Fix(P) \cap Fix(Q)$.

Finally, we show (3). Assume that $Fix(P) \cap Fix(Q)$ is a complete subgraph of X. In order to show that it contains only one element assume that there exist $C, D \in Fix(P) \cap Fix(Q)$ with $C \neq D$. By the assumption, there exists an edge between C and D. Due to (iii),

$$\mathbb{F}(H(C,D)) = \mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(\Theta(C,D)) + G(\beta(\Theta(C,D))),$$

where

$$\begin{aligned}\Theta(C,D) &= \max\left\{\begin{array}{l} H(Q(C),Q(D)), H(P(C),Q(C)), H(P(D),Q(D)), \\ \frac{1}{2}H(P(C),Q(D)) + H(P(D),Q(C)) \end{array}\right\} \\ &= \max\left\{H(C,D), H(C,C), H(D,D), \frac{1}{2}[H(C,D)+H(D,C)]\right\} \\ &= H(C,D).\end{aligned}$$

Thus,

$$\mathbb{F}(H(C,D)) \leq \mathbb{F}(H(C,D)) + G(\beta(H(C,D))),$$

which is a contradiction. Hence, $C = D$. Conversely, if $Fix(P) \cap Fix(Q)$ contains exactly one element, then since $e_{\mathcal{G}} \supseteq \Delta$, $Fix(P) \cap Fix(Q)$ is a complete subgraph of X. □

Remark 2.1. (a) Taking $\mathbb{F}(\xi) = G(\xi) = \ln \xi$ and $\beta(\xi) = k \in (0,1)$ (obviously $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$), Theorems 2 and 3, as well as Corollary 1 from the paper [3] follow as special cases of our results. In particular, all the results mentioned in Remark 1 of [3] can also be considered as corollaries of our results.

(b) Several other results can be obtained from Theorems 2.1 and 2.2 by taking various other possible choices for functions $\mathbb{F}$, G and β. We formulate just that, taking $F(\xi) = -1/\sqrt{\xi}$ and $G(\xi) = \ln \xi$, the condition (2.1) reduces to

$$H(P(Z),P(W)) \leq \frac{\Theta(Z,W)}{\left[1 - \sqrt{\Theta(Z,W)}\,\ln\beta(\Theta(Z,W))\right]^2}.$$

In particular, taking $\beta(\xi) = \text{const} \in (0,1)$ and denoting $\ln\beta = -\tau < 0$, the previous condition further reduces to

$$H(P(Z),P(W)) \leq \frac{\Theta(Z,W)}{\left[1 + \tau\sqrt{\Theta(Z,W}\right]^2}. \tag{2.7}$$

Example 2.1. Let $X = \{\alpha, \beta, \gamma\} = v_{\mathcal{G}}$, $e_{\mathcal{G}} = \{(\alpha,\alpha), (\beta,\beta), (\gamma,\gamma), (\alpha,\beta), (\alpha,\gamma), (\beta,\gamma)\}$ and $d : X \times X \to [0, +\infty)$ be defined by

$$\begin{aligned} d(\alpha,\beta) &= \frac{1}{3}, \quad d(\alpha,\gamma) = d(\beta,\gamma) = \frac{3}{4}, \\ d(u,u) &= 0 \text{ for } u \in X \quad \text{ and } \quad d(u,v) = d(v,u) \text{ for } u, v \in X. \end{aligned}$$

Then $(X, d, \mathcal{G})$ is a directed graph metric space. Consider the following mappings $P, Q : CB(X) \to CB(X)$:

$$P(Z) = \begin{cases} \{\alpha\}, & \text{if } Z \subseteq \{\alpha, \beta\}, \\ \{\alpha, \beta\}, & \text{otherwise;} \end{cases} \quad Q(Z) = \begin{cases} \{\alpha\}, & \text{if } Z = \{\alpha\}, \\ \{\alpha, \beta\}, & \text{if } Z \in \{\{\beta\}, \{\alpha, \beta\}\}, \\ \{\alpha, \beta, \gamma\}, & \text{otherwise.} \end{cases}$$

Concerning conditions (i)-(v) of Theorem 2.1, just condition (iii) has to be checked in cases when $P(Z) \neq P(W)$. We will use version (2.7) with $\tau = \frac{1}{\sqrt{3}}$. The only possible such cases are when $Z \subseteq \{\alpha, \beta\}$, $W \ni \gamma$ or vice versa (they are symmetric, so just the first one will be considered).

In this case, $P(Z) = \{\alpha\}$, $P(W) = \{\alpha, \beta\}$, $Q(Z) = \begin{cases} \{\alpha\}, & \text{if } \beta \notin Z, \\ \{\alpha, \beta\}, & \text{if } \beta \in Z \end{cases}$, $Q(W) = \{\alpha, \beta, \gamma\}$. Hence, $H(P(Z), P(W)) = \frac{1}{3}$, $\Theta(Z, W) = H(P(W), Q(W)) = \frac{3}{4}$, and thus

$$\frac{\Theta(Z, W)}{[1 + \tau\sqrt{\Theta(Z, W})]^2} = \frac{\frac{3}{4}}{\left[1 + \frac{1}{\sqrt{3}}\sqrt{\frac{3}{4}}\right]^2} = \frac{1}{3} = H(P(Z), P(W)),$$

and condition (2.7) holds true.

Hence, by Theorem 2.1, $Coin(P, Q) \neq \emptyset$ (in fact, $\{\alpha\}$ is the unique coincidence point of P and Q). Since also conditions of Theorem 2.2 are satisfied, this is also the unique common fixed point of these mappings.

Remark 2.2. The presented example is a simplified version of Example 1 from the paper [3]. In a similar way, a simplified version of Example 2 from this paper can be constructed, showing that it is not necessary that the graph $(v_{\mathcal{G}}, e_{\mathcal{G}})$ be complete in order to obtain conclusions using results of this kind.

If $Q =$ (identity map on $CB(X)$) in (2.2), then we have the following consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let $(X, d, \mathcal{G})$ be a set-transitive directed graph metric space. Assume that $P : CB(X) \to CB(X)$ satisfies the following:*

(a) *there is a path between Z and $P(Z)$ for each Z in $CB(X)$;*
(b) *there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ and for all $Z, W \in X$ such that there is a path between Z and W, and $P(Z) \neq P(W)$,*

$$\mathbb{F}(H(P(Z), P(W))) \leq \mathbb{F}(\Theta(Z, W)) + G(\beta(\Theta(Z, W))),$$

holds, where

$$\Theta(Z, W) = \max \left\{ \begin{array}{c} H(Z, W), H(Z, P(Z)), H(W, P(W)), \\ \frac{1}{2}[H(Z, P(W)) + H(W, P(Z))] \end{array} \right\};$$

(c) *$\mathcal{G}$ is weakly connected and property (P^*) holds.*

Then we have the following conclusions:

(i) *P has a fixed point;*
(ii) *if $Fix(P)$ is a complete subgraph of X, then $H(C, D) = 0$, for all $C, D \in Fix(P)$;*
(iii) *$Fix(P)$ is a complete subgraph of $\mathcal{G}$ if and only if $Fix(P)$ has exactly one element.*

Assuming that the mappings P and Q are defined just on the subset of $CB(X)$ containing all singleton subsets of X (which is equivalent to assuming that they are defined on X), we obtain the following corollary of Theorems 2.1 and 2.2.

Corollary 2.2. *Let $(X, d, \mathcal{G})$ be a set-transitive directed graph metric space and $P, Q : X \to CB(X)$ be a pair of mappings. Assume the following conditions hold:*
(i) *$P(X) \subseteq Q(X)$;*
(ii) *for every $u \in X$, there is a path between $\{u\}$ and Pu, as well as between Qu and $\{u\}$;*
(iii) *there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ such that for all $u, v \in X$ such that there is a path between $\{u\}$ and $\{v\}$ and $Pu \neq Pv$,*
$$\mathbb{F}(H(Pu, Pv)) \leq \mathbb{F}(\Theta(u, v)) + G(\beta(\Theta(u, v)))$$
holds, where
$$\Theta(u, v) = \max \left\{ \begin{array}{l} H(Qu, Qv), H(Pu, Qu), H(Pv, Qv), \\ \frac{1}{2}[H(Pu, Qv) + H(Pv, Qu)] \end{array} \right\};$$
(iv) *$\mathcal{G}$ is weakly connected and property (P^*) holds, and*
(v) *$Q(X)$ is a complete subspace of $(CB(X), H)$.*

Then there exists $u \in X$ such that $Pu = Qu$. Moreover,
(1) *if $Coin(P, Q)$ is a complete subgraph of X, then $H(Pu, Pv) = 0$ for all $u, v \in Coin(P, Q)$.*
(2) *if $Coin(P, Q)$ is a complete subgraph of X and P and Q are weakly compatible, then $Fix(P) \cap Fix(Q)$ contains exactly one element;*
(3) *$Fix(P) \cap Fix(Q)$ is a complete subgraph of X if and only if it contains exactly one element.*

Finally, assuming that the mapping P is defined just on X, we obtain the following from Corollary 2.1.

Corollary 2.3. *Let $(X, d, \mathcal{G})$ be a set-transitive complete directed graph metric space and $P : X \to CB(X)$ be a mapping. Assume the following conditions hold.*
(a) *There is a path between $\{u\}$ and Pu for each $u \in X$.*
(b) *There exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ so that for all $u, v \in X$ such that there is a path between them, and $Pu \neq Pv$,*
$$\mathbb{F}(H(Pu, Pv)) \leq \mathbb{F}(\Theta(u, v)) + G(\beta(\Theta(u, v)))$$

holds, where

$$\Theta(u,v) = \max\left\{d(u,v), \delta(u,Pu), \delta(v,Pv), \frac{1}{2}[\delta(u,Pv)+\delta(v,Pu)]\right\}. \tag{2.8}$$

(c) *The graph $\mathcal{G}$ is weakly connected and satisfies the property (P^*).*

Then we have the following conclusions.

(i) *There is a point $u \in X$ such that $Pu = \{u\}$.*

(ii) *If $Fix(P)$ is a complete subgraph of X, then it contains exactly one element.*

Here, in (2.8), for $u \in X$ and $Z \subseteq X$,

$$\delta(u,Z) = \sup_{v\in Z} d(u,v) = H(\{u\},Z).$$

3. Application

In this section we are going to apply the obtained results to the problem of existence of solutions for a Fredholm-type integral inclusion. Problems of this kind were treated by several researchers, see, e.g., [15, 17, 18].

Consider the integral inclusion

$$v(t) \in \gamma(t) + \int_a^b M(t,s,v(s))\,ds, \quad t \in J = [a,b], \tag{3.1}$$

where $\gamma \in X = C[a,b]$ is a given function, $M\colon J\times J\times\mathbb{R} \to CB(\mathbb{R})$ is a given set-valued mapping and $v \in X$ is the unknown function. Here, $X = C[a,b]$ is the standard Banach space of continuous real functions with the maximum norm. We will consider the space X as endowed with the partial order $\preceq$ introduced by

$$u \preceq v \iff u(t) \le v(t), \quad \text{for all } t \in J,$$

where $u,v \in X$. We will say that u and v are comparable if $u \preceq v$ or $v \preceq u$ holds.

Consider the following assumptions.

(I) For each $v \in X$, the mapping $M_v : J^2 \to CB(\mathbb{R})$, given by $M_v(t,s) = M(t,s,v(s))$, is continuous.

(II) For fixed $v \in X$ and for any sequence $\{m_n\}$ with $m_n(t,s) \in M_v(t,s)$, there exists a subsequence $\{m_{n_i}\}$ of $\{m_n\}$ such that $\{m_{n_i}\}$ converges for all $t,s \in J$ towards a function m with $m(t,s) \in M_v(t,s)$ as $i \to +\infty$ and, moreover, for every $t \in J$, $\int_a^b m_{n_i}(t,s)\,ds \to \int_a^b m(t,s)\,ds$, as $i \to +\infty$.

(III) For every $v \in X$ there is a function m_v, such that $m_v(t,s) \in M_v(t,s)$ for $t,s \in J$ and

$$v(t) \le \gamma(t) + \int_a^b m_v(t,s)\,ds, \quad t \in J.$$

(IV) There exists $\tau > 0$ such that for all comparable $u,v \in X$ and for all m_u, m_v with $m_u(t,s) \in M_u(t,s)$ and $m_v(t,s) \in M_v(t,s)$ for $t,s \in J$,

$$|m_u(t,s) - m_v(t,s)| \le \frac{1}{b-a}\frac{|u(t)-v(t)|}{[1+\tau\sqrt{|u(t)-v(t)|}]^2} \tag{3.2}$$

holds for all $t, s \in J$.

Theorem 3.1. *Let the assumptions* (I)–(IV) *hold. Then the integral inclusion* (3.1) *has a solution in* X.

Proof. Let $P\colon X \to CB(X)$ be the operator given by

$$Pv = \left\{ u \in X : u(t) \in \gamma(t) + \int_a^b M(t, s, v(s))\, ds,\ t \in [a, b] \right\}.$$

Obviously, $v \in X$ is a solution of the inclusion (3.1) if and only if v is a fixed point of operator P.

We first check that the operator P is well-defined. Indeed, let $v \in X$ be arbitrary. By (I), the set-valued operator $M_v\colon J^2 \to CB(\mathbb{R})$ is continuous (w.r.t. Pompeiu-Hausdorff metric on $CB(\mathbb{R})$). From the Michael's selection theorem, it follows that there exists a continuous function $m_v : J^2 \to \mathbb{R}$ such that $m_v(t, s) \in M_v(t, s)$ for each $(t, s) \in J^2$. Hence, the function $u(t) = \gamma(t) + \int_a^b m_v(t, s)\, ds$ belongs to Pv, i.e., $Pv \neq \emptyset$. Since γ and M_v are continuous on J, resp. J^2, their ranges are bounded and hence Pv is bounded.

Let $v \in X$ be fixed, $\{v_n\}$ be a sequence in Pv and $v_n \to v \in X$. Then, there exists a sequence of functions $\{m_n\}$ such that $m_n(t, s) \in M_v(t, s)$ for $t, s \in J$, and

$$v_n(t) = \gamma(t) + \int_a^b m_n(t, s)\, ds, \quad t \in J.$$

By hypothesis (II), there exists a subsequence $\{m_{n_i}\}$ of $\{m_n\}$ such that $\{m_{n_i}\}$ converges for all $t, s \in J$ towards a function m as $i \to +\infty$, and, for every $t \in J, \int_a^b m_{n_i}(t, s)\, ds \to \int_a^b m(t, s)\, ds$, as $i \to +\infty$. As $M_v(t, s)$ is closed for all $t, s \in J$, then $m(t, s) \in M_v(t, s)$ for all $t, s \in J^2$. Besides,

$$v(t) = \lim_{i \to +\infty} v_{n_i}(t) = \gamma(t) + \int_a^b m(t, s)\, ds, \quad t \in J.$$

Thus, $v \in Pv$ and we have proved that images of P are closed subsets of X.

Hence, $P : X \to CB(X)$.

Consider the graph $\mathcal{G}$ with $v_{\mathcal{G}} = X$ and $e_{\mathcal{G}} = \{(u, v) \in X^2 : u \preceq v\}$. We have to check the conditions of Corollary 2.3.

(a) The assumption (III) assures that there is a path between $\{v\}$ and Pv for each $v \in X$.

(b) To see that P is a generalized graphic $\mathbb{FG}$-contraction, let $u, v \in X$ be comparable. Then, using the assumption (IV) and the fact that the function $\xi \mapsto \frac{\xi}{[1+\tau\sqrt{\xi}]^2}$

is increasing (which is easy to check), we get that

$$\begin{aligned}
\sup_{\varphi\in Pu} d(\varphi, Pv) &= \sup_{\varphi\in Pu}\inf_{\chi\in Pv} d(\varphi,\chi)\\
&= \sup_{\varphi\in Pu}\inf_{\chi\in Pv}\max_{t\in J}|\varphi(t)-\chi(t)|\\
&= \sup_{m_u\in M_u}\inf_{m_v\in M_v}\max_{t\in J}\left|\int_a^b [m_u(t,s)-m_v(t,s)]\,ds\right|\\
&\le \sup_{m_u\in M_u}\inf_{m_v\in M_v}\max_{t\in J}\int_a^b |m_u(t,s)-m_v(t,s)|\,ds\\
&\le \frac{1}{b-a}\max_{t\in J}\int_a^b \frac{|u(t)-v(t)|}{[1+\tau\sqrt{|u(t)-v(t)|}]^2}\,ds\\
&\le \frac{1}{b-a}\cdot\frac{\max_{t\in J}|u(t)-v(t)|}{[1+\tau\sqrt{\max_{t\in J}|u(t)-v(t)|}]^2}\int_a^b ds\\
&= \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2}.
\end{aligned}$$

Similarly, one can see that

$$\sup_{\chi\in Pv} d(\chi, Pu) \le \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2}.$$

Therefore, we have

$$H(Pu,Pv)\le \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2}\le \frac{\Theta(u,v)}{[1+\tau\sqrt{\Theta(u,v)}]^2}.$$

Taking $\mathbb{F}(\xi) = -\frac{1}{\sqrt{\xi}}$, $G(\xi)=\ln\xi$ and $\beta(\xi)=e^{-\tau}\in(0,1)$, we get that P is a generalized graphic $\mathbb{F}\mathbb{G}$-contraction (see the inequality (2.7)).

(c) Let $\{u_n\}$ be a sequence in X with $u_n\to u$ as $n\to+\infty$ and let $u_n\preceq u_{n+1}$ for each $n\in\mathbb{N}$. Then obviously $u_n\preceq u$, i.e., $(u_n,u)\in e_{\mathcal{G}}$ holds for all $n\in\mathbb{N}$.

Thus, P satisfies all the conditions of Corollary 2.3, and so P has a fixed point, that is, the integral inclusion (3.1) has a solution in $X=C[a,b]$. □

Remark 3.1. Using other possibilities for $\mathbb{F}\in\mathfrak{F}$ an $(G,\beta)\in G_\beta$, the inequality (3.2) in the contractive condition (IV) of Theorem 3.1 can take various other forms. For example, taking $\mathbb{F}(\xi)=G(\xi)=\ln\xi$ and $\beta\colon[0,+\infty)\to[0,1)$ satisfying $(G_{\beta 2})$ and $(G_{\beta 3})$ (with $G=\ln$), (3.2) is replaced by

$$|m_u(t,s)-m_v(t,s)|\le\frac{1}{b-a}|u(t)-v(t)|\cdot\beta(|u(t)-v(t)|),\quad t,s\in J.$$

In particular, taking $\beta(\xi) = \frac{k}{b-a}$, with $k \in (0,1)$, we get that the following inequality is required:

$$|m_u(t,s) - m_v(t,s)| \leq \frac{k}{b-a}|u(t) - v(t)|, \quad t, s \in J. \tag{3.3}$$

Acknowledgement

We are grateful to the learned referee for useful suggestions which helped us to improve the text in several places.

References

[1] M. Abbas, M. R. Alfuraidan, A. R. Khan and T. Nazir, *Fixed point results for set-contractions on metric spaces with a directed graph*, Fixed Point Theory Appl. **2015**(14) (2015), 1–9.
[2] M. Abbas, T. Nazir, T. A. Lampert and S. Radenović, *Common fixed points of set-valued F-contraction mappings on domain of sets endowed with directed graph*, Comp. Appl. Math. **36** (2017), 1607–1622.
[3] M. Abbas, T. Nazir, B. Popović and S. Radenović, *On weakly commuting set-valued mappings on a domain of sets endowed with directed graph*, Results Math. **71** (2017), 1277–1295.
[4] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Some fixed point results on a metric space with a graph*, Topology Appl. **159** (2012), 659–663.
[5] I. Beg and A. Butt, *Fixed point of set-valued graph contractive mappings*, J. Inequal. Appl. **2013**(52) (2013), 1–7.
[6] F. Bojor, *Fixed point of φ-contraction in metric spaces endowed with a graph*, An. Univ. Craiova Ser. Mat. Inform. Ser. **37** (2010), 85–92.
[7] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. **75** (2012), 3895–3901.
[8] Lj. Ćirić, *Some Recent Results in Metrical Fixed Point Theory*, University of Belgrade, Beograd, 2003.
[9] G. Gwozdz-Lukawska and J. Jachymski, *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl. **356** (2009), 453–463.
[10] J. Jachymski and I. Jozwik, *Nonlinear contractive conditions: a comparison and related problems*, Banach Center Publ. **77** (2007), 123–146.
[11] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2008), 1359–1373.
[12] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4** (1996), 199–215.
[13] E. Malkowsky and V. Rakočević, *Advanced Functional Analysis*, CRS Press, Taylor and Francis Group, Boca Raton, FL, 2019.
[14] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
[15] H. K. Nashine, R. P. Agarwal and Z. Kadelburg, *Solution to Fredholm integral inclusions via (F, δ_b)-contractions*, Open Math. **14** (2016), 1053–1064.
[16] V. Parvaneh, N. Hussain and Z. Kadelburg, *Generalized Wardowski type fixed point theorems via α-admissible FG-contractions in b-metric spaces*, Acta Math. Sci. **36B** (2016), 1445–1456.
[17] A. Petrusel, *Integral inclusions. Fixed point approaches*, Annales Societatis Mathematicae Polonae. Seria I. Commentationes Mathematicae **40** (2000), 147–158.
[18] A. Petrusel, G. Petrusel and J-C. Yao, *A study of a system of operator inclusions via a fixed point approach and applications to functional-differential inclusions*, Carpathian J. Math. **32** (2016), 349–361.

[19] J. Vujaković, S. Mitrović, S. Radenović and M. Pavlović, *On recent results concerning F-contraction in generalized metric spaces*, Mathematics **8** (2020), DOI 10.3390/math8050767

[20] D. Wardowski, *Fixed points of new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012**(94) (2012), 1–6.

[1]Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology,
Vellore-632014, TN, India

[2]Department of Pure and Applied Mathematics, Faculty of Science,
University of Johannesburg,
P O Box 524, Auckland Park, 2006, South Africa
Email address: drhknashine@gmail.com
Email address: hemant.nashine@vit.ac.in

[3]Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, 11000 Beograd, Serbia
Email address: kadelbur@matf.bg.ac.rs

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 959–968.

A GENERAL APPROACH TO CHAIN CONDITION IN BL-ALGEBRAS

JAMAL KAZEMIASL[1], FARHAD KHAKSAR HAGHANI[2*], AND SHAHRAM HEIDARIAN[2]

Abstract. In this paper, we present a general definition of the notion of Noetherian and Artinian BL-algebra and present a more comprehensive insight at the chain conditions in BL-algebras. We derive some theorems which generalize the existence results. We give an axiomatic approach to the notion of being Noetherian and Artinian, which is also applicable to other algebraic structures. We use a theoretical approach to define arithmetic notion that is also possible for other algebraic devices. In this study, we only focus to BL-algebras.

1. Introduction

Motamed and Moghaderi [5] introduced the notion of Noetherian and Artinian BL-algebras and provided some results on the subject, which are analogues to the results of the Noetherian and Artinian modules. O. Zahiri [8] defined the notion of length for a filter in BL-algebras and derived some new relations between Noetherian and Artinian BL-algebras. Zhan and Meng [9] defined another type of chain conditions in terms of the ideals of a BL-algebra, and called BL-algebras satisfying in the relevant conditions, co-Noetherian and co-Artinian BL-algebras. They also proved some results on co-Noetherian and co-Artinian BL-algebras which are analogues to the results of the Noetherian and Artinian modules. In this paper, we provide a more general definition of the chain condition in BL-algebras that can be defined in any other algebraic structure, but we limit ourselves to BL-algebras for simplicity. We provide some theorems on this general definition which generalize the results in [5] and [9].

Key words and phrases. Artinian (Noetherian) BL-algebra, $\mathcal{PF}$ BL-algebra, Maximal element, Complete family.

2010 *Mathematics Subject Classification.* Primary: 03G99. Secondary: 06D99, 08A99.

DOI 10.46793/KgJMat2206.959K

Received: December 06, 2019.

Accepted: June 01, 2020.

The structure of the paper is as follows. In Section 2, we recall some definitions and results about BL-algebras which will be used in the sequel. In Section 3, we define the notion of complete family, structural, multiplicative and Noetherian (Artinian) BL-algebras with respect to a family of subsets of a BL-algebra. We also obtain some results about the relation between Noetherian, Artinian, finitely generated, maximal (minimal) element, multiplicative, onto BL-homomorphism and one to one BL-homomorphism BL-algebras.

2. Preliminaries

In this section, we recall and review some definitions and results, corresponding to co-Noetherian, Noetherian (Artinian) BL-algebras, which will be used throughout of the paper.

An algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of the type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if for all $a, b, c \in A$ satisfies the following axioms:

$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

$(BL2)$ $(A, \odot, 1)$ is a commutative monoid;

$(BL3)$ $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$;

$(BL4)$ $a \wedge b = a \odot (a \rightarrow b)$;

$(BL5)$ $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

We will denote $\overline{x} = x \rightarrow 0$ and $x^{--} = (\overline{x})^{-}$ for all $x \in A$.

Examples of BL-algebras [2] are t-algebras, $([0, 1], \wedge, \vee, \odot, \rightarrow, 0, 1)$, where $([0, 1], \wedge, \vee, 0, 1)$ is the usual lattice on $[0, 1]$ and $\odot$ is a continuous t-norm, whereas $\rightarrow$ is the corresponding residuum.

Throughout of this paper by A, we denote the universe of a BL-algebra. A BL-algebra is nontrivial if $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We denote the set of natural numbers by $\mathbb{N}$ and define a^0=1 and $a^n = a^{n-1} \odot a$, for $n \in \mathbb{N} \backslash \{0\}$. Hájek [2] defined a filter of a BL-algebra A to be a nonempty subset F of A such that (i) if $a, b \in F$ implies $a \odot b \in F$, (ii) if $a \in F, a \leq b$ then $b \in F$. E. Turunen [6] defined a deductive system of a BL-algebra A to be a nonempty subset D of A such that (i) $1 \in D$ and (ii) $x \in D$ and $x \rightarrow y \in D$ implies $y \in D$. Note that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A [6]. Let F be a filter of a BL-algebra A, then F is a proper filter if $F \neq A$. A proper filter P of A is called a prime filter of A if for all $x, y \in A, x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter P of A is Prime if and only if P can not be expressed as an intersection of two filters properly containing P or equivalently, for all $x, y \in A$, either $x \rightarrow y \in P$ or $y \rightarrow x \in P$ [6].

If F, G and P are filters of A, then P is a prime filter of A if and only if $F \bigcap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$.

In [6], it can be seen that a proper filter M of A is a maximal filter of A if and only if for all $x \notin M$, there exists $n \in \mathbb{N}$ such that $(x^n)^{-} \in M$. Every maximal filter of A is a prime filter of A [6]. The set of all filters, prime filters and maximal filters of a BL-algebra A are denote by $\mathcal{F}(A)$, $\mathrm{Spec}(A)$ and $\mathrm{Max}(A)$, respectively. The filter of A generated by X is denoted by $\langle X\rangle$, where $X \subseteq A$, in which $\langle\emptyset\rangle=\{1\}$ and $\langle X\rangle = \{a \in A : x_1 \odot x_2 \odot \cdots \odot x_n \leq a$, for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X\}$ [6]. $F \in F(A)$ is called a finitely generated filter, if $F = \langle x_1, \ldots, x_n\rangle$, for some $x_1, \ldots, x_n \in A$ and $n \in \mathbb{N}$. For $F \in F(A)$ and $x \in A\backslash F$, define $F\langle x\rangle = \langle F \cup \{x\}\rangle$ or equally $F\langle x\rangle = \{a \in A : a \geq f \odot x^n$ for some $f \in F$ and $n \geq 1\}$.

Definition 2.1 ([7]). Let A and B be two BL-algebras. A map $f : A \to B$ defined on A, is called a BL-homomorphism if for all $x, y \in A, f(x \to y) = f(x) \to f(y)$, $f(x \odot y) = f(x) \odot f(y)$ and $f(0_A) = 0_B$. We also define $\ker(f) = \{a \in A : f(a) = 1\}$ and $f(A) = \{f(a) : a \in A\}$.

Definition 2.2 ([5]). A BL-algebra A is called Noetherian (Artinian), if for every increasing (decreasing) chain of its filters $F_1 \subseteq F_2 \subseteq \cdots (F_1 \supseteq F_2 \supseteq \cdots)$, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$.

Definition 2.3 ([2,4]). Let A be a BL-algebra. A nonempty subset $I \subseteq A$ is called an ideal of A, if the following conditions hold:
(i) $0 \in I$;
(ii) if $x \in I$ and $(x^- \to y^-)^- \in I$, then $y \in I$.

Definition 2.4 ([9]). A BL-algebra A is said to be co-Noetherian with respect to ideals if every ideal of A is finitely generated. We say that A satisfies the ascending chain condition with respect to ideals if for every ascending chain sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of A, there exists $n \in \mathbb{N}$ such that $I_i = I_n$ for all $i \geq n$.

Definition 2.5 ([1,5]). Let A and B be BL-algebras. Then for every $a, c \in A$ and $b, d \in B$, we define the product of two BL-algebras which is clearly a BL-algebra, as follows:
$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$;
$(a, b) \vee (c, d) = (a \vee c, b \vee d)$;
$(a, b) \to (c, d) = (a \to c, b \to d)$;
$(a, b) \odot (c, d) = (a \odot c, b \odot d)$;
$(a, b) \leq (c, d) \Leftrightarrow (a \leq c, b \leq d)$.

3. Main Concepts and Results

In this section, regarding to the definitions of co-Noetherian, Noetherian (Artinian), multiplicative and $P\mathcal{F}$ BL-algebras and using related mentioned theorems, we derive some new results of finitely generated, maximal (minimal) element, onto BL-homomorphism, one-to-one BL-homomorphism, in a Noetherian (Artinian) BL-algebras.

Definition 3.1. Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A. A is said to be Noetherian with respect to family $\mathcal{F}$, if for any chain of elements of $\mathcal{F}$, $F_1 \subseteq F_2 \subset \cdots$, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$. We may similarly define Artinian BL-algebras.

Example 3.1. We know that every finite BL-algebra A is Noetherian and Artinian [5]. Therefore, if $\mathcal{F}$ is a family of subsets of A, since $\mathcal{F}$ is finite, so A is Noetherian (Artinian) with respect to family $\mathcal{F}$.

Theorem 3.1. *Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A. Then A is Noetherian* (*Artinian*) *with respect to $\mathcal{F}$ if and only if any set of elements of $\mathcal{F}$ has a maximal* (*minimal*) *element.*

Proof. Let A be a Noetherian BL-algebra with respect to $\mathcal{F}$ and S be a nonempty set of elements of $\mathcal{F}$ which does not have a maximal element, then, there exists $F_1 \in \mathcal{F}$. Since S does not have a maximal element, there is $F_2 \in S$ such that $F_1 \subset F_2$. By continuing this procedure, we obtain the following increasing chain of elements of $\mathcal{F}$: $F_1 \subset F_2 \subset \cdots$, which is a contradiction. So S has a maximal element.

Conversely, let $F_1 \subseteq F_2 \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$ and put $S = \{F_i : i \in \mathbb{N}\}$. Since S is nonempty, then it has a maximal element F_n. Thus $F_i = F_n$ for all $i \geq n$ and A is Noetherian with respect to $\mathcal{F}$ (Artinian case can be treated similarly). $\square$

Definition 3.2. Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A which is closed under intersection operation (that is intersection of any number of elements of $\mathcal{F}$ is also an element of $\mathcal{F}$). If $B \subseteq A$, then the set generated by B in $\mathcal{F}$ is defined as the intersection of all elements of $\mathcal{F}$ containing B and denoted by $\langle B \rangle$, i.e.,

$$\langle B \rangle = \bigcap_{B \subseteq F}^{F \in \mathcal{F}} F,$$

$\langle B \rangle$ is said to be finitely generated if there exists a set $C \subseteq A$ such that $\langle B \rangle = \langle C \rangle$ and C is finite.

Example 3.2. Let $A = \{0, a, b, c, 1\}$, with $0 < c < a < 1$ and $0 < c < b < 1$. For every $x, y \in A$, we define the operations "$\odot$" and "$\rightarrow$" as follows:

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

,

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

.

Then it is easy to see that $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra [3]. If we consider $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ and $B = \{b\} \subseteq A$, where $F_1 = \{0, a\}$, $F_2 = \{0, a, b\}$, $F_3 = \{0, a, b, 1\}$ and $F_4 = \{0, a, b, c, 1\}$, then

$$\langle \{b\} \rangle = \bigcap_{\{b\} \subseteq F_i}^{F_i \in \mathcal{F}} F_i = F_2 \cap F_3 \cap F_4 = \{0, a, b\}.$$

Definition 3.3. Let $\mathcal{F}$ be a family of subsets of BL-algebra A. $\mathcal{F}$ is said to be complete if for any subset B of A, $\langle B\rangle$ is nonempty.

Example 3.3. Let $A = \{0, a, 1\}$. For every $x, y \in A$, we define the operations "$\odot$" and "$\rightarrow$" as follows:

$$\begin{array}{c|ccc} \odot & 0 & a & 1 \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & a & a \\ 1 & 0 & a & 1 \end{array}, \qquad \begin{array}{c|ccc} \rightarrow & 0 & a & 1 \\ \hline 0 & 1 & 1 & 1 \\ a & 0 & 1 & 1 \\ 1 & 0 & a & 1 \end{array}.$$

Then it is easy to see that $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra.

Let $\mathcal{F} = \{\{a\}, \{0, a\}, \{0, a, 1\}\}$, then $\mathcal{F}$ is complete, since for every element of the powerset of A, we have: $\langle\{0\}\rangle = \{0, a\} \cap \{0, a, 1\} = \{0, a\}$, $\langle\{a\}\rangle = \langle\emptyset\rangle = \{a\} \cap \{0, a\} \cap \{0, a, 1\} = \{a\}$, $\langle\{0, a\}\rangle = \{0, a\} \cap \{0, a, 1\} = \{0, a\}$, $\langle\{0, 1\}\rangle = \langle\{a, 1\}\rangle = \langle\{0, a, 1\}\rangle = \{0, a, 1\}$, $\langle\{1\}\rangle = \{0, a, 1\} = \{1\}$.

Definition 3.4. Let A be a BL-algebra and $\mathcal{F}$ be a complete family which is closed under intersection. Then $\mathcal{F}$ is said to be closed under chain union if for any chain $F_1 \subseteq F_2 \subseteq \cdots$, of elements of $\mathcal{F}$, $\bigcup_{i=1}^{\infty} F_i$ also belongs to $\mathcal{F}$.

Example 3.4. (i) The family $\mathcal{F}$ in Example 3.2, is closed under chain union.

(ii) If we define the operations "$\odot$" and "$\rightarrow$" on $A = [0, 1]$ (real unit interval) by $x \odot y = \min\{x, y\}$ and

$$x \rightarrow y = \begin{cases} 1, & x \leq y, \\ y, & x > y, \end{cases}$$

then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra (Gödel structure) [7]. Now, the family $\mathcal{F} = \{(0, 1], F_n = [\frac{1}{n}, 1]_{n\geq 1}\} = \{(0, 1], F_1, F_2, F_3, \ldots, F_n, \ldots\}$, is closed under chain union.

Theorem 3.2. *Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A which is closed under intersection and chain union. Then A is Noetherian with respect to $\mathcal{F}$ if and only if any element of $\mathcal{F}$ is finitely generated.*

Proof. Set $S = \{G \in \mathcal{F} : G \subseteq F,\ G \text{ is finitely generated}\}$. Since F is nonempty, then it has an element x and $\langle x\rangle \in S$. By Theorem 3.1, it has a maximal element F_1. By definition of S, $F_1 \subseteq F$ and $F_1 = \langle x_1, \ldots, x_n\rangle$ for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in A$. Since F is not finitely generated so $F_1 \subset F$ and there exists $x \in F\backslash F_1$. We also have $\langle x_1, \ldots, x_n, x\rangle \subseteq F$ and $\langle x_1, \ldots, x_n, x\rangle \in S$ which contradicts the maximality of F_1, i.e., F is finitely generated.

Conversely, let any element of $\mathcal{F}$ be finitely generated and $F_1 \subseteq F_2 \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$. We set $F = F_1 \cup F_2 \cup \cdots$, thus $\mathcal{F}$ is finitely generated by definition and $F = \langle x_1, \ldots, x_n\rangle$ for $x_i \in A$. Now, by chaining condition, there exists $m \in \mathbb{N}$ such that $x_1, \ldots, x_n \in F_m$ and so $F = \langle x_1, \ldots, x_n\rangle \subseteq F_m \subseteq F$. Thus, $F_m = F_i$ for $i \geq m$ and A is Noetherian with respect to $\mathcal{F}$.

If $\mathcal{F}$ is a family of all filters of the BL-algebra A, then we obtain the concept of Noetherian and Artinian BL-algebra, which is introduced by Motamed and Moghaderi [5]. When $\mathcal{F}$ is a family of all ideals of the BL-algebra A, we obtain the concept of co-Noetherian and co-Artinian BL-algebra which is introduced in [9]. □

Definition 3.5. Let $\mathcal{A}$ be the family of all BL-algebras. Then $\mathcal{F}$ is said to be a complete family for elements of $\mathcal{A}$ if for every element A of $\mathcal{A}$ there exists $F_1 \in \mathcal{F}$ such that F_1 is a complete family for A and is closed under intersection.

Definition 3.6. Let $\mathcal{F}$ be a complete family for all BL-algebras, then $\mathcal{F}$ is said to be structural if it has the following property.

If A_1 and A_2 are two BL-algebras and $\varphi : A_1 \to A_2$ is an onto BL-homomorphism, then $\varphi^{-1}(F)$ is also an element of $\mathcal{F}$, for every $F \in \mathcal{F}$, $F \subseteq A_2$.

Theorem 3.3. *Let $\mathcal{F}$ be a structural family for the family of the all BL-algebras. If A_1, A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is an onto BL-homomorphism and A_1 is Noetherian with respect to $\mathcal{F}$, then A_2 is also Noetherian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \subseteq F_2 \subseteq \cdots$, be a chain of elements of $\mathcal{F}$ for A_2. Then, by Definition 3.6, $\psi^{-1}(F_1) \subseteq \psi^{-1}(F_2) \subseteq \cdots$, is a chain of elements of $\mathcal{F}$ for A_1. A_1 is Noetherian with respect to family $\mathcal{F}$, then there exists $n \in \mathbb{N}$ such that $\psi^{-1}(F_i) = \psi^{-1}(F_n)$ for all $i \geq n$. Since ψ is an onto BL-homomorphism, then $F_i = F_n$ for all $i \geq n$. Hence, A_2 is also Noetherian with respect to $\mathcal{F}$. □

Definition 3.7. Let $\mathcal{F}$ be a complete family for BL-algebra A which is closed under intersection and $F \in \mathcal{F}$. Then F is said to be cyclic if there exists $a \in A$ such that $F = \langle a \rangle$. If any $F \in \mathcal{F}$ is cyclic, then A is called principal with respect to $\mathcal{F}$ which is denoted by $P\mathcal{F}$-BL.

Example 3.5. We consider BL-algebra A and collection $\mathcal{F}$ in Example 3.3, then $\{a\} = \langle a \rangle = \langle \{a\} \rangle$, i.e., $\{a\}$ is cyclic.

Theorem 3.4. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A such that any element $F \in \mathcal{F}$ which is generated by two elements, is cyclic. If A is Noetherian with respect to family $\mathcal{F}$, then A is $P\mathcal{F}$-BL.*

Proof. Let $F \in \mathcal{F}$, then by Theorem 3.2, F is finitely generated. So $F = \langle x_1, \ldots, x_n \rangle$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in A$. We proceed by mathematical induction. From induction hypothesis, there is $b \in A$ such that $\langle x_1, \ldots, x_{n-1} \rangle = \langle b \rangle$. Thus, $\langle x_1, \ldots, x_n \rangle = \langle b, x_n \rangle$. So by hypothesis, there exists $a \in A$ such that $\langle b, x_n \rangle = \langle a \rangle$. Therefore, $F = \langle a \rangle$, i.e., F is cyclic. □

Theorem 3.5. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A which is closed under intersection. If A is $P\mathcal{F}$-BL and $\mathcal{F}$ is closed under chain union, then A is Noetherian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \subseteq F_2 \subseteq \cdots$, be a chain of elements of $\mathcal{F}$. Then $F = F_1 \cup F_2 \cup \cdots$, is also an element of $\mathcal{F}$. Since A is $P\mathcal{F}$-BL, then $F = \langle a \rangle$ for some $a \in A$. So, there exists $t \in \mathbb{N}$ such that $a \in F_t$. Thus, $F_i = F_t = F$ for all $i \geq t$ and A is Noetherian with respect to $\mathcal{F}$. $\square$

Theorem 3.6. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A such that any chain of finitely generated elements of $\mathcal{F}$ is stopping. Then A is Noetherian with respect to $\mathcal{F}$.*

Proof. We assume that A is not Noetherian with respect to $\mathcal{F}$. By Theorem 3.2, there is an element $F \in \mathcal{F}$ which is not finitely generated. Thus, there is $a_1 \in F$ such that $\langle a_1 \rangle \subsetneq F$. So, there exists $a_2 \in F \backslash \langle a_1 \rangle$ such that $\langle a_1, a_2 \rangle \subsetneq F$. By continuing this procedure, we come to a proper increasing chain of finitely generated elements of $\mathcal{F}$ ($\langle a_1 \rangle \subsetneq \langle a_1, a_2 \rangle \subsetneq \cdots$), which is a contradiction. Therefore, A is a Noetherian BL-algebra with respect to $\mathcal{F}$. $\square$

Definition 3.8. Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$. Then $\mathcal{F}$ is said to be multiplicative if for every two BL-algebras A and B, the following properties hold.

(i) If $F, G \in \mathcal{F}$, $F \subseteq A$ and $G \subseteq B$, then $F \times G \in \mathcal{F}$, where $F \times G \subseteq A \times B$.

(ii) If $F \times G$ is an element of $\mathcal{F}$ which is a subset of $A \times B$, then F and G are also elements of $\mathcal{F}$.

Theorem 3.7. *Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$. If $\mathcal{F}$ is multiplicative, then A_1 and A_2 are Noetherian with respect to $\mathcal{F}$ if and only if $A_1 \times A_2$ is Noetherian with respect to $\mathcal{F}$.*

Proof. Let $A_1 \times A_2$ be Noetherian with respect to $\mathcal{F}$. If $F_1 \subseteq F_2 \subseteq \cdots$, is a chain for A_1, then by multiplicity of $\mathcal{F}$, $F_1 \times \langle a \rangle \subseteq F_2 \times \langle a \rangle \subseteq \cdots$, is a chain for $A_1 \times A_2$, for any $a \in A_2$. Since $A_1 \times A_2$ is Noetherian, there exists $n \in \mathbb{N}$ such that $F_i \times \langle a \rangle = F_n \times \langle a \rangle$ for all $i \geq n$. Therefore, $F_i = F_n$ for all $i \geq n$ and A_1 is Noetherian with respect to $\mathcal{F}$. Similarly, we may prove that A_2 is Noetherian with respect to $\mathcal{F}$.

Conversely, let A_1 and A_2 be Noetherian BL-algebras with respect to $\mathcal{F}$ and $F_1 \times G_1 \subseteq F_2 \times G_2 \subseteq \cdots$, be a chain for $A_1 \times A_2$. Then by hypothesis, $F_1 \subseteq F_2 \subseteq \cdots$, and $G_1 \subseteq G_2 \subseteq \cdots$ are chains for A_1 and A_2, respectively. So, there exist $n, m \in \mathbb{N}$ such that $F_i = F_n$, $G_j = G_m$ for all $i \geq n$ and $j \geq m$. We set $k = \max\{n, m\}$, then $F_i \times G_i = F_k \times G_k$ for all $i \geq k$. Therefore, $A_1 \times A_2$ is Noetherian with respect to $\mathcal{F}$ (Artinian case can be treated similarly). $\square$

Corollary 3.1. *Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$ which is multiplicative. If $A_1, A_2, \ldots, A_n$ are BL-algebras, then $A_1, A_2, \ldots, A_n$ are Noetherian (Artinian) with respect to $\mathcal{F}$ if and only if $A_1 \times A_2 \times \cdots \times A_n$ is Noetherian (Artinian) with respect to $\mathcal{F}$.*

Proof. Let $A_1 \times A_2 \times \cdots \times A_n$ be Noetherian with respect to $\mathcal{F}$. We complete the proof by induction on n. For $n = 1$, the induction holds. If $n = 2$, by Theorem 3.7, it is true. Let for $n = k$ it is true, i.e., if $A_1 \times A_2 \times \cdots \times A_k$ is Noetherian with respect to $\mathcal{F}$, then $A_1, A_2, \ldots, A_k$ are Noetherian with respect to $\mathcal{F}$. Let $B = A_1 \times A_2 \times \cdots \times A_k$, so $A_1 \times A_2 \times \cdots \times A_k \times A_{k+1} = B \times A_{k+1}$. By Theorem 3.7, $B \times A_{k+1}$ is Noetherian with respect to $\mathcal{F}$, thus B and A_{k+1} are Noetherian with respect to $\mathcal{F}$. Therefore, $A_1, A_2, \ldots, A_k, A_{k+1}$ are Noetherian with respect to $\mathcal{F}$.

Conversely, it is clear by induction on n and applying Theorem 3.7 (Artinian case with respect to $\mathcal{F}$ can be treated similarly). □

Theorem 3.8. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A_1 and A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is a BL-homomorphism and A_1 is Noetherian (Artinian) with respect to $\mathcal{F}$, then $\psi\ (A_1)$ is also Noetherian (Artinian) with respect to $\mathcal{F}$.*

Proof. Let $\psi(F_1) \subseteq \psi(F_2) \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$ for $\psi\ (A_1)$. Since A_1 is Noetherian with respect to family $\mathcal{F}$, and $F_1 \subseteq F_2 \subseteq \cdots$, is an increasing chain of elements of $\mathcal{F}$ for A_1, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$. Then $\psi\ (F_i) = \psi(F_n)$ for all $i \geq n$. So, $\psi\ (A_1)$ is Noetherian with respect to $\mathcal{F}$. Similarly, we may prove that if A_1 is Artinian with respect to $\mathcal{F}$, then so is $\psi\ (A_1)$. □

Theorem 3.9. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A_1 and A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is an onto BL-homomorphism and A_1 is Artinian with respect to $\mathcal{F}$, then A_2 is also Artinian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \supseteq F_2 \supseteq \cdots$, be a decreasing chain of elements of $\mathcal{F}$ for A_2. Then, by Definition 3.6, $\psi^{-1}(F_1) \supseteq \psi^{-1}(F_2) \supseteq \cdots$, is a decreasing chain of elements of $\mathcal{F}$ for A_1. Since A_1 is Noetherian with respect to family $\mathcal{F}$, there exists $n \in \mathbb{N}$ such that $\psi^{-1}\ (F_i) = \psi^{-1}(F_n)$ for all $i \geq n$. By the fact that ψ is an onto BL-homomorphism, so $\psi\ (\psi^{-1}(F_i)) = \psi(\psi^{-1}(F_n))$ for all $i \geq n$. Hence, $F_i = F_n$ for all $i \geq n$, A_2 is Artinian with respect to $\mathcal{F}$. □

Theorem 3.10. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A is a BL-algebra, $\psi : A \to A$ is an onto BL-homomorphism and A is Noetherian with respect to $\mathcal{F}$, then ψ is an one-to-one BL-homomorphism.*

Proof. Let $\ker(\psi) \subseteq \ker(\psi^2) \subseteq \cdots$, be a chain of elements of $\mathcal{F}$ for A. Since A is Noetherian with respect to family $\mathcal{F}$, there exists $n \in \mathbb{N}$ such that $\ker(\psi^i) = \ker(\psi^n)$ for all $i \geq n$. Suppose $x \in \ker(\psi)$, then $\psi(x) = 1$. Since ψ and ψ^n are onto BL-homomorphisms, there exists $a \in A$ such that $x = \psi^n(a)$, so $\psi(x) = \psi^{n+1}(a) = 1$, i.e., $a \in \ker(\psi^{n+1}) = \ker(\psi^n)$. This means that $x = \psi^n(a) = 1$. Therefore, $\ker(\psi) = 1$ and ψ is an one-to-one BL-homomorphism. □

Theorem 3.11. *Let $\mathcal{F}$ be a structural family for the family of the all BL-algebras. If A is a BL-algebra, $\psi : A \to A$ is an one to one BL-homomorphism and A is Artinian with respect to $\mathcal{F}$, then ψ is an onto BL-homomorphism.*

Proof. Suppose ψ is not an onto BL-homomorphism, i.e., $A \supset \psi(A)$. Since ψ is one to one, so $\psi(A) \supset \psi^2(A)$. We also have $\psi^{n-1}(A) \supset \psi^n(\mathrm{A})$ for all $n \geq 2$, i.e., $A \supset \psi(A) \supset \psi^2(A) \supset \cdots \supset \psi^n(A) \supset \cdots$ is a decreasing chain of elements of $\mathcal{F}$. This chain is not stationary, because, if there exists $k \in \mathbb{N}$ such that $\psi^{k+1}(A) = \psi^k(A)$, then by the injectivity of ψ, there exists a map $\varphi : A \to A$, $\varphi(\psi(A)) = I_A$, thus $\varphi(\psi^{k+1}(A)) = \varphi(\psi^k(A))$, i.e., $\psi^k(A) = \psi^{k-1}(A)$. By continuing this procedure, we get $\psi(A) = A$, which is a contradiction. Therefore, the chain is not stationary and hence A is not Artinian BL-algebra, which is a contradiction with hypothesis. Therefore, $A = \psi(A)$ and ψ is an onto BL-homomorphism. □

4. Conclusion

In BL-algebras (indeed in any algebraic structure), the results of chain conditions can be defined. Chain conditions are defined to study the properties of an algebraic structure. However, we must note that we can prove similar results for chain conditions. It is considerable that these results can be formulated in general structure to have a general approach to chain conditions. We may also define chain conditions with respect to partial order relation in our future work.

Acknowledgements. The authors are grateful to the editor and referees for their valuable comments and suggestions which improved the paper.

References

[1] S. A. Borumand and S. Motamed, *Some resultrs in BL-algebras*, MLQ Math. Log. Q.**55** (2009), 649–658.

[2] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publisher, Dordrecht, 1998.

[3] R. T. Khorami and S. A. Borumand, *Nodal filters in BL-algebras*, Journal of Intelligent & Fuzzy Systems **28** (2015), 1159–1167.

[4] B. L. Meng and X. L. Xin, *Prime ideals and Gödel ideals of BL-algebras*, Journal of Advances in Mathematics **99** (2015), 2989–3005.

[5] S. Motamed and J. Moghadari, *Noetherian and Artinian BL-algebras*, Soft Computing **16** (2012), 1989–1994.

[6] E. Turunen, *BL-algebras of basic fuzzy logic*, Mathware and Soft Computing **6** (1999), 49–61.

[7] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica-Verlag, Heidelberg, 1999.

[8] O. Zahiri, *Chain conditions in BL-algebras*, Soft Computing **18** (2014), 419–426.

[9] J. H. Zhan and B. L. Meng, *Some resulte in co-Noetherian BL-algebras*, International Journal of Scientific and Innovative Mathematical Research **3** (2015), 18–25.

[1]Department of Mathematics,
Bandar Imam Khomeini branch,
Payame Noor University,
Bandar Imam Khomeini, Iran
Email address: Kazemiasl.j@gmail.com

[2]Department of Mathematics,
Shahrekord branch,
Islamic Azad University,
Shahrekord, Iran.
Email address: haghani1351@yahoo.com

Email address: heidarianshm@gmail.com

*Corresponding Author

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 46(6) (2022), PAGES 969–979.

SOME EXTENSIONS OF A THEOREM OF PAUL TURÁN CONCERNING POLYNOMIALS

N. A. RATHER[1], ISHFAQ DAR[1], AND A. IQBAL[1]

ABSTRACT. In this paper, certain new results concerning the maximum modulus of the polar derivative of a polynomial with restricted zeros are obtained. These estimates strengthen some well known inequalities for polynomial due to Turán, Dubinin and others.

1. INTRODUCTION

In scientific disciplines like physics, engineering, computer science, biology, physical chemistry, economics, and other applied areas, experimental observations and investigations when translated into mathematical language are called mathematical models. The solution of these models could lead to problems of estimating how large or small the maximum modulus of the derivative of an algebraic polynomial can be in terms of the maximum modulus of that polynomial. Bounds for such type of problems are of some practical importance. Since, there are no closed formulae for precise evaluation of these bounds and whatever is available in literature is in the form of approximations. However for practical purposes, nobody ever needs exacts bounds and mathematicians must only indicate methods for obtaining approximate bounds. These approximate bounds, when computed efficiently, are quite satisfactory for the needs of investigators and scientists. Therefore there is always a desire to look for better and improved bounds than those available in literature. It is this aspiration of obtaining more refined and revamped bounds that has inspired our work in this article. In this paper, we have generalized and refined some well known results concerning the polynomials due to Turán [16], Dubinin [6] and others. To begin with let $\mathcal{P}_n$ denote the linear space of all polynomials of the form $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree $n \geq 1$

Key words and phrases. Polynomials, polar derivative, inequalities in the complex domain.
2010 *Mathematics Subject Classification.* Primary: 30A10. Secondary: 30C10, 30D15.
DOI 10.46793/KgJMat2206.969R
Received: July 26, 2019.
Accepted: June 03, 2020.

and let $P'(z)$ be the derivative of $P(z)$. Then concerning the lower bound for the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ for class of polynomials $P \in \mathcal{P}_n$ not vanishing outside unit disc, Turán [16] showed that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality in inequality (1.1) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$. As an extension of (1.1), Govil [8] proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.2}$$

In literature, there exist several generalizations and extensions of (1.1) and (1.2) (see [1, 2, 4, 5, 13–15]). Dubinin [6] refined inequality (1.1) by proving that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right) \max_{|z|=1} |P(z)|. \tag{1.3}$$

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

uniformly for $|z| \leq R$, $R > 0$.

A. Aziz [1], Aziz and Rather ([4, 5]) obtained several sharp estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other things they extended inequality (1.2) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha P(z)| \geq n\left(\frac{|\alpha| - k}{1+k^n}\right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

2. Main Results

In this paper, we obtain certain refinements and generalizations of inequalities (1.1), (1.2), (1.3) and (1.4). We first prove the following result.

Theorem 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$, with $|\alpha| \geq k$*

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \geq &\frac{|\alpha| - k}{1+k^n}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|\phi(k)}{k}\right) \\ &+ \psi(k)|na_0 + \alpha a_1|,\end{aligned} \tag{2.1}$$

where $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ *or* $\frac{(k-1)^2}{2}$ *and* $\psi(k) = 1 - \frac{1}{k^2}$ *or* $1 - \frac{1}{k}$ *according as* $n > 2$ *or* $n = 2$.

Remark 2.1. Since all the zeros of $P(z)$ lie in $|z| \leq k$, $k \geq 1$, it follows that $\sqrt{|a_0|} \leq \sqrt{k^n|a_n|}$. In view of this, inequalities (2.1) refines inequality (1.4).

If we divide the two sides of inequality (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 2.1. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq k$, $k \geq 1$, *then*

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n}\left(n + \frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\left(\max_{|z|=1}|P(z)| + \frac{|a_{n-1}|}{k}\phi(k)\right) + \psi(k)|a_1|, \tag{2.2}$$

where $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

The result is best possible and equality in inequality (2.2) holds for $P(z) = z^n + k^n$.

Remark 2.2. As before, it can be easily seen that inequality (2.2) refines inequality (1.2). Further for $k = 1$, inequality (2.2) reduces to inequality (1.3).

Next, we present the following result which is generalisation of Theorem 2.1 and in particular, includes refinement of inequality (1.2) as a special case.

Theorem 2.2. *If all the zeros of* $P \in \mathcal{P}_n$ *lie in* $|z| \leq k$, $k \geq 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geq k$, $0 \leq l < 1$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \geq{}& \frac{n}{1+k^n}\left\{(|\alpha| - k)\max_{|z|=1}|P(z)| + (|\alpha| + 1/k^{n-1})lm\right\} \\ &+ \frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n\max_{|z|=1}|P(z)| - lm\right) \\ &+ \frac{(|\alpha|-k)|a_{n-1}|\phi(k)}{k(1+k^n)}\left(n + \frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right) \\ &+ \psi(k)|na_0 + \alpha a_1|,\end{aligned} \tag{2.3}$$

where $m = \min_{|z|=k}|P(z)|$, $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

If we divide both sides of inequality (2.3) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 2.2. *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k$, $k \geq 1$, then for $0 \leq l < 1$*

$$\begin{aligned}\max_{|z|=1} |P'(z)| \geq & \frac{n}{1+k^n}\left\{\max_{|z|=1} |P(z)| + lm\right\} + \psi(k)|a_1| \\ & + \frac{1}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n \max_{|z|=1} |P(z)| - lm\right) \\ & + \frac{|a_{n-1}|}{k(1+k^n)}\left(n + \frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\phi(k),\end{aligned} \tag{2.4}$$

where $m = \min_{|z|=k} |P(z)|$, $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 - \frac{1}{k^2}$ or $1 - \frac{1}{k}$ according as $n > 2$ or $n = 2$.

Remark 2.3. For $l = 0$, Corollary 2.2 reduces to Corollary 2.1 and for $k = 1$, inequality (2.4) refines inequality (1.3).

3. Lemmas

We need the following lemmas for the proof of our theorems. The first lemma is due to Dubinin [6].

Lemma 3.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then*

$$|P'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{|a_n|}-\sqrt{|a_0|}}{\sqrt{|a_n|}}\right)|P(z)|, \quad \textit{for } |z| = 1. \tag{3.1}$$

The next lemma is special case of a result due to Aziz and Rather [3, 4].

Lemma 3.2. *If $P \in \mathcal{P}_n$ and $P(z)$ has its all zeros in $|z| \leq 1$, then for $|z| = 1$*

$$|Q'(z)| \leq |P'(z)|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Lemma 3.3. If all the zeros of $P \in \mathcal{P}_n$ lie in a circular region C and w is any zero of $D_\alpha P(z)$, the polar derivative of $P(z)$, then at most one of the points w and α may lie outside C.

The above lemma is due to Laguerre (see [10]). The following lemma is due to Frappier, Rahman and Ruscheweyh [7].

Lemma 3.4. *If $P(z)$ is a polynomial of degree at most $n \geq 1$, then for $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \quad \textit{if } n \geq 2, \tag{3.2}$$

and

$$\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R-1)|P(0)|, \quad \textit{if } n = 1. \tag{3.3}$$

Next lemma is the famous result of P. D. Lax [9].

Lemma 3.5. *If $P \in \mathcal{P}_n$ does not vanish in $|z| < 1$, then*

$$|P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|, \quad \text{for } |z| = 1.$$

We also need the following lemma.

Lemma 3.6. *If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree $n \ge 2$ having no zero in $|z| < 1$, then for $R \ge 1$*

$$\max_{|z|=R} |P(z)| \le \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\tfrac{R^n-1}{n} - \tfrac{R^{n-2}-1}{n-2}\right) |P'(0)|, \quad \text{if } n > 2, \tag{3.4}$$

and

$$\max_{|z|=R} |P(z)| \le \frac{R^2 + 1}{2} \max_{|z|=1} |P(z)| - \frac{(R-1)^2}{2} |P'(0)|, \quad \text{if } n = 2. \tag{3.5}$$

Proof of Lemma 3.6. For each θ, $0 \le \theta < 2\pi$, we have

$$P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta}) dt, \tag{3.6}$$

which gives with the help of (3.2) of Lemma 3.4 and Lemma 3.5 for $n > 2$

$$\begin{aligned}
|P(Re^{i\theta}) - P(e^{i\theta})| \le& \int_1^R |P'(te^{i\theta})| dt \\
\le& \frac{n}{2} \int_1^R t^{n-1} dt \max_{|z|=1} |P(z)| - \int_1^R (t^{n-1} - t^{n-3}) dt |P'(0)| \\
=& \frac{R^n - 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |P'(0)|.
\end{aligned}$$

Consequently for $n > 2$ and $0 \le \theta < 2\pi$, we have

$$\begin{aligned}
|P(Re^{i\theta})| &\le |P(Re^{i\theta}) - P(e^{i\theta})| + |P(e^{i\theta})| \\
&\le \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |P'(0)|,
\end{aligned}$$

which immediately leads to (3.4). Similarly we can prove inequality (3.5) by using inequality (3.3) of Lemma 3.4. This proves Lemma 3.6. □

Finally we require the following lemma.

Lemma 3.7. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, where $k \ge 1$, then for $0 \le l < 1$*

$$\begin{aligned}
\max_{|z|=k} |P(z)| \ge& \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)| + l \left(\frac{k^n - 1}{k^n + 1}\right) \min_{|z|=k} |P(z)| \\
&+ \frac{2k^{n-1}|a_{n-1}|}{k^n + 1} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2}\right), \quad \text{if } n > 2,
\end{aligned} \tag{3.7}$$

and

(3.8)

$$\max_{|z|=k}|P(z)| \geq \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| + l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2|a_1|}{k^2+1}, \quad \text{if } n=2.$$

Proof of Lemma 3.7. Since all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k$, $k \geq 1$, therefore, all the zeros of $g(z) = P(kz)$ lie in $|z| \leq 1$ and hence all the zeros of $f(z) = z^n\overline{g(1/\overline{z})} = z^n\overline{P(k/\overline{z})}$ lie in $|z| \geq 1$. Moreover, $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$, so that

$$m|z^n| \leq |f(z)|, \quad \text{for } |z|=1.$$

We show that for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, $f(z) + \lambda m z^n \neq 0$ in $|z| < 1$. This is trivially true if $m = 0$. Henceforth we suppose that $m \neq 0$, so that all the zeros of $f(z)$ lie in $|z| > 1$. By the maximum modulus theorem

$$m|z^n| < |f(z)|, \quad \text{for } |z|<1. \tag{3.9}$$

Now if there is point $z = z_0$ with $|z_0| < 1$, such that $f(z_0) + \lambda m z_0^n = 0$, then

$$|f(z_0)| = |\lambda||z_0^n|m < |z_0^n|m,$$

a contradiction to inequality (3.9). Hence, it follows that the polynomial $T(z) = f(z) + \lambda m z^n$ does not vanish in $|z| < 1$. Applying inequality (3.4) of Lemma 3.6 to the polynomial $T(z)$, with $R = k \geq 1$ and $n > 2$, we get for $|z| = 1$,

$$|f(kz) + \lambda k^n m z^n| \leq \frac{k^n+1}{2}|f(z) + \lambda m z^n| - \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)|f'(0)|.$$

Which implies, for $n > 2$,

$$|f(kz) + \lambda m k^n z^n| \leq \frac{k^n+1}{2}\Big(|f(z)| + |\lambda m|\Big) - \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)|f'(0)|. \tag{3.10}$$

Choosing argument of λ suitably in the left hand side of inequality (3.10), we get for $n > 2$

$$|f(kz)| + |\lambda| m k^n \leq \frac{k^n+1}{2}\Big(|f(z)| + |\lambda| m\Big) - \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)k^{n-1}|a_{n-1}|.$$

Replacing $f(z)$ by $z^n\overline{P(k/\overline{z})}$, we obtain for $n > 2$ and $|z| = 1$

$$\begin{aligned} k^n\max_{|z|=1}|P(z)| + |\lambda| m k^n \leq & \frac{k^n+1}{2}\left(\max_{|z|=k}|P(z)| + |\lambda| m\right) \\ & - \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)k^{n-1}|a_{n-1}|, \end{aligned}$$

which on simplification yields inequality (3.7). In a similar manner we can prove inequality (3.8) by applying inequality (3.5) of Lemma 3.6 instead of inequality (3.4) to the polynomial $T(z)$. This proves Lemma 3.7. $\square$

4. **Proof of the Theorems**

Proof of Theorem 2.1. Let $f(z) = P(kz)$. Since $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, therefore, $f \in \mathcal{P}_n$ and $f(z)$ has all its zeros in $|z| \leq 1$. If $Q(z) = z^n\overline{f(1/\overline{z})}$, then it is easy to verify that

$$|Q'(z)| = |nf(z) - zf'(z)|, \quad \text{for } |z| = 1. \tag{4.1}$$

Combining (4.1) with Lemma 3.2, we get

$$|f'(z)| \geq |nf(z) - zf'(z)|, \quad \text{for } |z| = 1. \tag{4.2}$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have for $|z| = 1$,

$$|D_{\alpha/k}f(z)| = |nf(z) + (\alpha/k - z)f'(z)| \geq |\alpha/k||f'(z)| - |nf(z) - zf'(z)|,$$

which gives with the help of (4.2)

$$|D_{\alpha/k}f(z)| \geq \left(\frac{|\alpha| - k}{k}\right)|f'(z)|. \tag{4.3}$$

Consequently,

$$\max_{|z|=k}|D_\alpha P(z)| \geq (|\alpha| - k)\max_{|z|=k}|P'(z)|. \tag{4.4}$$

Again since all the zeros of $f(z) = P(kz)$ lie in $|z| \leq 1$, therefore, using Lemma 3.1, we have

$$|f'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)|f(z)|, \quad \text{for } |z| = 1.$$

Replacing $f(z)$ by $P(kz)$, we obtain

$$k\max_{|z|=1}|P'(kz)| \geq \frac{1}{2}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\max_{|z|=1}|P(kz)|,$$

which implies

$$\max_{|z|=k}|P'(z)| \geq \frac{1}{2k}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\max_{|z|=k}|P(z)|. \tag{4.5}$$

Combining inequality (4.4) and inequality (4.5), we have

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{2k}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\max_{|z|=k}|P(z)|. \tag{4.6}$$

Further since $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$, using inequality (3.2) of Lemma 3.4, we have for $n > 2$

$$\max_{|z|=R}|D_\alpha P(z)| \leq R^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (R^{n-1} - R^{n-3})|na_0 + \alpha a_1|.$$

Using this inequality and inequality (3.7) of Lemma 3.7 with $l = 0$ and $R = k \geq 1$ in (4.6), we have for $n > 2$

$$\begin{aligned}
&k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1|\\
\geq&\frac{(|\alpha|-k)}{2k}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\\
&\times\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\},
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \geq&\frac{|\alpha|-k}{1+k^n}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\\
&\times\left\{\max_{|z|=1}|P(z)|+\frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}\\
&+\left(1-1/k^2\right)|na_0+\alpha a_1|,\quad \text{if } n>2.
\end{aligned}$$

The above inequality is equivalent to the inequality (2.1) for $n > 2$. For $n = 2$, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.6). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. By hypothesis $P \in \mathcal{P}_n$ has all zeros in $|z| \leq k$, $k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and result follows from Theorem 2.1. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, $k \geq 1$, so that $m > 0$. Now if $f(z) = P(kz)$, then $f \in \mathcal{P}_n$ and $f(z)$ has all zeros in $|z| < 1$ and $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$. This implies

$$m \leq |f(z)|,\quad \text{for } |z| = 1.$$

By the Rouche's Theorem, we conclude that for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the polynomial $g(z) = f(z) - \lambda m z^n$ has all zeros in $|z| < 1$. Applying inequality (4.3) to the polynomial $g(z)$, it follows for $|z| = 1$ and $|\alpha| \geq k$

$$|D_{\alpha/k}g(z)| \geq \left(\frac{|\alpha|-k}{k}\right)|g'(z)|.$$

Since all the zeros of $g(z)$ lie in $|z| < 1$, using Lemma 3.1, we obtain for $|z| = 1$ and $|\alpha| \geq k$

$$|D_{\alpha/k}g(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{|k^n a_n-\lambda m|}-\sqrt{|a_0|}}{\sqrt{|k^n a_n-\lambda m|}}\right)|g(z)|.$$

Using the fact that the function $S(x) = \frac{x-\sqrt{|a_0|}}{x}$, $x > 0$, is non-decreasing function of x and $|k^n a_n - \lambda m| \geq k^n|a_n| - |\lambda|m > 0$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$

$$|D_{\alpha/k}g(z)| \geq \frac{1}{2}\Big(\frac{|\alpha|-k}{k}\Big)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|g(z)|. \tag{4.7}$$

Replacing $g(z)$ by $f(z) - \lambda m z^n$ in (4.7), we get for $|z| = 1$ and $|\alpha| \geq k$

$$\begin{aligned} &\left|D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}\right| \\ \geq &\frac{1}{2}\Big(\frac{|\alpha|-k}{k}\Big)\Big(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\Big)\big(|f(z) - \lambda m|\big). \end{aligned} \tag{4.8}$$

Since all the zeros of $f(z) - \lambda m z^n = g(z)$ lie in $|z| < 1$ and $|\alpha/k| \geq 1$, it follows by Lemma 3.3 that all the zeros of

$$D_{\alpha/k}(f(z) - m\lambda z^n) = D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}$$

lie in $|z| < 1$. This implies that

$$\Big|D_{\alpha/k}f(z)\Big| \geq \frac{nm|\alpha|}{k}|z|^{n-1}, \quad \text{for } |z| \geq 1. \tag{4.9}$$

In view of this inequality, choosing argument of λ in the left hand side of inequality (4.8) such that

$$\left|D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}\right| = \Big|D_{\alpha/k}f(z)\Big| - \frac{nm|\alpha||\lambda|}{k}, \quad \text{for } |z| = 1,$$

we get for $|z| = 1$ and $|\alpha| \geq k$

$$\begin{aligned} &\Big|D_{\alpha/k}f(z)\Big| - \frac{nm|\alpha||\lambda|}{k} \\ \geq &\frac{1}{2}\Big(\frac{|\alpha|-k}{k}\Big)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)\big(|f(z)| - |\lambda|m\big), \end{aligned}$$

which on simplification yields

$$\begin{aligned} \Big|D_{\alpha/k}f(z)\Big| \geq &\frac{1}{2}\Big(\frac{|\alpha|-k}{k}\Big)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|f(z)| \\ &- \frac{1}{2}\Big(\frac{|\alpha|-k}{k}\Big)\left(\frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|\lambda|m + \frac{n}{2}\Big(\frac{|\alpha|+k}{k}\Big)|\lambda|m. \end{aligned}$$

This implies for $|z|=1$ and $|\alpha|\geq k$

(4.10)

$$\begin{aligned}\max_{|z|=k}|D_\alpha P(z)|\geq&\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{k^n|a_n|-|\lambda|m}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-|\lambda|m}}\right)\max_{|z|=k}|P(z)|\\&-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{\sqrt{k^n|a_n|-|\lambda|m}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-|\lambda|m}}\right)|\lambda|m+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m.\end{aligned}$$

Moreover, since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, applying inequality (3.2) of Lemma 3.4 and inequality (3.7) of Lemma 3.7 with $R=k\geq 1$, we obtain for $|\alpha|\geq k$, $0\leq l<1$ and $|z|=1$

$$\begin{aligned}&k^{n-1}\max_{|z|=1}|D_\alpha P(z)|-(k^{n-1}-k^{n-3})|na_0+\alpha a_1|\\&\geq\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\\&\times\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+\frac{k^n-1}{k^n+1}lm+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}\\&-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)lm+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)lm,\quad\text{if } n>2.\end{aligned}$$

Equivalently, we have for $|\alpha|\geq k,\ 0\leq l<1$ and $|z|=1$

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)|\geq&\frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)|+(|\alpha|+1/k^{n-1})lm\right\}\\&+\frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n\max_{|z|=1}|P(z)|-lm\right)\\&+\frac{(|\alpha|-k)|a_{n-1}|}{k(1+k^n)}\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\\&\times\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)+(1-1/k^2)|na_0+\alpha a_1|,\quad\text{if } n>2.\end{aligned}$$

That proves the inequality (2.3) for $n>2$. For the case $n=2$, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.10). This completes the proof of Theorem 2.2. □

Acknowledgements. The authors wish to thank the referee, for the careful reading of the paper and for the helpful suggestions and comments.

References

[1] A. Aziz, *Inequalities for the derivative of a polynomial*, Proc. Amer. Math. Soc. **89** (1983), 259–266.

[2] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivatives*, J. Approx. Theory **54** (1998), 306–313.

[3] A. Aziz and N. A. Rather, *Some Zygmund type L^q inequalities for polynomials*, J. Math. Anal. Appl. **289** (2004), 14–29.

[4] A. Aziz and N. A. Rather, *Inequalities for the polar derivative of a polynomial with restricted zeros*, Math. Balkanica **17** (2003), 15–28.

[5] A. Aziz and N. A. Rather, *A refinement of a theorem of Paul Turán concerning polynomials*, Math. Inequal. Appl. **1** (1998), 231–238.

[6] V. N. Dubinin, *Distortion theorems for polynomials on a circle*, Sb. Math. **191**(12) (2000), 1797–1807.

[7] C. Frappier, Q. I. Rahman and St. Ruscheweyh, *New inequalities for polynomials*, Trans. Amer. Math. Soc. **288** (1985), 69–99.

[8] N. K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc. **41** (1973), 543–546.

[9] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[10] M. Marden, *Geometry of Polynomials*, Amer. Math. Soc., Math. Surveys **3**, Providence, 1966.

[11] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros,* World scientific Publishing Co., Singapore, 1994.

[12] G. Pólya and G. Szegö, *Problems and Theorems in Analysis, II*, Springer-Verlag, Berlin, New York, 1976.

[13] N. A. Rather and S. Gulzar, *Generalization of an inequality involving maximum moduli of a polynomial and its polar derivative*, Nonlinear Functional Analysis and Applications **19** (2014), 213–221.

[14] N. A. Rather and S. Gulzar, *Refinements of some inequalities concerning the polar derivative of a polynomial*, Funct. Approx. Comment. Math. **51** (2014), 269–283.

[15] S. Gulzar and N. A. Rather, *Inequalities concerning the polar derivative of a polynomial*, Bull. Malays. Math. Sci. Soc. **40** (2017), 1691–1700.

[16] P. Turán, *Über die ableitung von polynomen*, Compos. Math. **7** (1939), 89–95.

[1]Department of Mathematics,
University of Kashmir,
Hazratbal Srinagar–19006, J&K, India
Email address: dr.narather@gmail.com
Email address: ishfaq619@gmail.com
Email address: itz.a.iqbal@gmail.com

Kragujevac Journal of Mathematics
Volume 46(6) (2022), Pages 981–996.

SHIFTED GEGENBAUER-GAUSS COLLOCATION METHOD FOR SOLVING FRACTIONAL NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAYS

R. M. HAFEZ[1*] AND Y. H. YOUSSRI[2]

Abstract. In this paper, the shifted Gegenbauer-Gauss collocation (SGGC) method is applied to fractional neutral functional-differential equations with proportional delays. The technique we have used is based on shifted Gegenbauer polynomials and Gauss quadrature integration. The shifted Gegenbauer-Gauss method reduces solving the generalized fractional pantograph equation fractional neutral functional-differential equations to a system of algebraic equations. Reasonable numerical results are obtained by selecting few shifted Gegenbauer-Gauss collocation points. Numerical results demonstrate its accuracy, and versatility of the proposed techniques.

1. Introduction

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695. A history of the development of fractional differential operator can be founded in [26, 27]. One of the most recent works on the subject of the fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order, is the book of Podlubny [31], which deals principally with fractional equations. Today, there are many works on fractional calculus (see for example [11, 36]).

For the past three centuries, this subject has been dealt with by the mathematicians and only in the last few years, this was pulled to several (applied) fields of engineering, science and economics [11]. However, the number of scientific and engineering problems

Key words and phrases. Neutral fractional functional-differential equations, proportional delay, collocation method, shifted Gegenbauer-Gauss quadrature, shifted Gegenbauer polynomials.

2010 *Mathematics Subject Classification.* Primary: 34K40. Secondary: 65N35, 33C45.

DOI 10.46793/KgJMat2206.981H

Received: March 06, 2017.

Accepted: June 03, 2020.

involving fractional calculus is already very large and still growing and perhaps the fractional calculus will be the calculus of the twenty-first century. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of fractional derivatives. Fractional differentials and integral provide more accurate models of systems under considerations. Many authors have demonstrated applications of fractional calculus in the nonlinear oscillation of earthquakes [15], fluid-dynamic traffic model [16], to model frequency dependent damping behavior of many viscoelastic materials [4, 5], continuum and statistical mechanics [23], colored noise [24], solid mechanics [35], economics [6], bioengineering [20–22], anomalous transport [25], and dynamics of interfaces between nanoparticles and substrates [10].

The analytic results on the existence and uniqueness of solutions of the fractional differential equations have been investigated by many authors (see, for examples [31, 36]).

Spectral methods (see, for instance [14,30,33,38]) are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite-difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. Collocation methods [8, 9] have become increasingly popular for solving differential equations, also they are very useful in providing highly accurate solutions to nonlinear differential equations.

Neutral functional-differential equations play an important role in the mathematical modeling of several phenomena. It is well known that most of delay differential equations cannot be solved exactly. Therefore, numerical methods would be presented and developed to get approximate solutions of these equations. In this direction, Bhrawy et al. [3] proposed a new spectral collocation scheme based upon the generalized Laguerre polynomials and Gauss quadrature integration for solving generalized fractional pantograph equations. Anapali et al. [29] investigated a Taylor collocation method, which is based on collocation method for solving fractional pantograph equation. Rahimkhani et al. [28] defined a new functions called generalized fractional-order Bernoulli wavelet functions based on the Bernoulli wavelets to obtain the numerical solution of fractional order pantograph differential equations in a large interval. Furthermore, Yang and Huang [39] proposed a spectral Jacobi-collocation approximation for fractional order integrodifferential equations of Volterra type with pantograph delay. Recently, in [13], Ghasemi et al. proposed an approximate solution of a class of nonlinear fractional-order delay differential equation using Hilbert function space.

Our fundamental goal of this paper is to develop a suitable way to approximate the neutral fractional functional-differential equations with proportional delays on the interval $(0, L)$ using the shifted Gegenbauer polynomials, we propose the spectral shifted Gegenbauer-Gauss collocation (SGGC) method to find the solution $u_N(x)$. The shifted Gegenbauer spectral collocation (SGGC) approximation, which is more reliable, is employed to obtain approximate solution of neutral fractional functional-differential equations with proportional delays of order ν $(m-1<\nu<m)$ and m initial conditions. For suitable collocation points we use the $(N-m+1)$ nodes of the shifted Gegenbauer-Gauss interpolation on $(0, L)$. These equations together with initial conditions generate $(N+1)$ algebraic equations which can be solved. Finally, the accuracy of the proposed methods are demonstrated by test problems, numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

This paper is organized as follows. In the subsequent section, we present some definitions and properties of fractional calculus theory. In Section 3 we give an overview of shifted Gegenbauer polynomials and their relevant properties needed hereafter, and in Section 4, the way of constructing the collocation technique for neutral fractional functional-differential equations with proportional delays is described using the shifted Gegenbauer polynomials. In Section 6, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms.

2. Basic Definitions

The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator [36]. We give some definitions and properties of fractional derivatives.

Definition 2.1. The Riemann-Liouville fractional integral operator of order ν, $\nu>0$, is defined as

$$\begin{aligned} J^{\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \quad \nu>0,\, x>0, \\ J^0 f(x) &= f(x). \end{aligned}$$

Definition 2.2. The Caputo fractional derivative of order ν is defined as

$$D^{\nu} f(x) = J^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m}{dt^m} f(t)dt,$$

$m-1<\nu<m$, $x>0$, D^m is the classical differential operator of order m.

For the Caputo derivative we have

$$D^{\nu} C = 0, \quad C \text{ is a constant,} \tag{2.1}$$

$$D^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}\, x^{\beta-\nu}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \nu \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \nu \rfloor, \end{cases} \tag{2.2}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions, respectively, while $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

The Caputo's fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^{\nu}(\lambda f(x)+\mu g(x))=\lambda D^{\nu}f(x)+\mu D^{\nu}g(x), \tag{2.3}$$

where λ and μ are constants.

Remark 2.1. The reason for using the fractional derivative in the Caputo sense is that it is mathematically rigorous than the Riemann-Liouville sense (one can be referred to [19, 32]). Caputo definition is advantageous in many disciplines such as applied science and engineering [18]. Moreover, properties of the Caputo derivative are useful in translating the higher fractional-order differential systems into lower ones [17]. Some comparisons between Caputo and Riemann-Liouville operators can be found in [27]. It is a future work for us to study the Riemann-Liouville derivative in relation to solve some physical models.

3. Shifted Gegenbauer Polynomials Interpolation

In this section, we detail the properties of shifted Gegenbauer polynomials that will be used to construct the method.

3.1. **Shifted Gegenbauer polynomials.**

The Gegenbauer polynomial $C_i^{\alpha}(z)$, of degree $i\in\mathbb{Z}^+$, and associated with the parameter $\alpha>-\frac{1}{2}$, is a real-valued function, which appears as an eigensolution to a singular Sturm-Liouville problem in the finite domain $[-1,1]$ [37]. It is a Jacobi polynomial, $P_i^{(\alpha,\beta)}(x)$, with $\alpha=\beta$, and can be standardized so that

$$C_i^{\alpha}(z)=\frac{i!\Gamma(\alpha+\frac{1}{2})}{\Gamma(i+\alpha+\frac{1}{2})}P_i^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(z),\quad i=0,1,2,\dots$$

There exist useful relations between Legendre polynomials $L_i(z)$ and Chebyshev polynomials of the first kind and second kind, $T_i(z),\ U_i(z)$, respectively, and the Gegenbauer polynomials $C_i^{\alpha}(z)$ as [37],

$$L_i(z)=C^{\frac{1}{2}}(z),\quad U_i(z)=C^{1}(z),$$

and

$$T_i(z)=\frac{i}{2}\lim_{\alpha\to 0}\alpha^{-1}C_i^{\alpha}(z),\quad i\geq 1.$$

The well-known Gegenbauer polynomials can be determined with the aid of the following recurrence formulae:

$$(i+1)C_{i+1}^{\alpha}(z)=2z(\alpha+i)C_i^{\alpha}(z)-(2\alpha+i-1)C_{i-1}^{\alpha}(z),\quad i=1,2,\dots,$$

where $C_0^{\alpha}(z)=1$ and $C_1^{\alpha}(z)=2\alpha z$. The weight function for the Gegenbauer polynomials is the even function $w^{\alpha}(z)=(1-z^2)^{\alpha-\frac{1}{2}}$. The Gegenbauer polynomials form a complete orthogonal basis polynomials in $L^2_{w^\alpha}[-1,1]$ and their orthogonality relation is given by the following weighted inner product:

$$(C_j^\alpha, C_i^\alpha)_{w^\alpha} = \int_{-1}^{1} C_j^\alpha(z) C_i^\alpha(z) w^\alpha(z) dz = \gamma_i^\alpha \delta_{ji},$$

where

$$\gamma_i^\alpha = \frac{2^{1-2\alpha}\ \pi\ \Gamma(i+2\alpha)}{\Gamma(i+1)\ (i+\alpha)\ \Gamma^2(\alpha)}$$

is the normalization factor and δ_{ji} is the Kronecker delta function. We denote the zeroes of the Gegenbauer polynomial $C_{i+1}^\alpha(z)$ (also called Gegenbauer-Gauss nodes) by z_k^α, $k = 0,\ldots,i$. We also denote their corresponding Christoffel numbers by ϖ_k^α, $k = 0,\ldots,i$, and define them by the following relation:

$$(\varpi_k^\alpha)^{-1} = \sum_{j=0}^{i} (\gamma_j^\alpha)^{-1} (C_j^\alpha(z_k))^2, \quad k = 0,1,2,\ldots,i.$$

In order to use these polynomials on the interval $t \in [0,L]$ we defined the so-called shifted Gegenbauer polynomials by introducing the change of variable $z = \frac{2}{L}t - 1$. Let the shifted Gegenbauer polynomials $C_i^\alpha(\frac{2}{L}t - 1)$ be denoted by $G_i^\lambda(t)$. Then $G_i^\alpha(t)$ can be obtained as follows:

$$(i+1)G_{i+1}^\alpha(t) = 2\left(\frac{2}{L}t - 1\right)(\alpha+i)G_i^\alpha(t) - (2\alpha+i-1)G_{i-1}^\alpha(t), \quad i = 1,2,\ldots,$$

and

$$D^q G_i^\alpha(t) = \left(\frac{4}{L}\right)^q (\alpha)_q G_{i-q}^{\alpha+q}(t), \tag{3.1}$$

where $G_0^\alpha(t) = 1$ and $G_1^\alpha(t) = 2\alpha(\frac{2}{L}t - 1)$. The analytic form of the shifted Gegenbauer polynomials $G_i^\alpha(t)$ of degree i is given by

$$G_i^\alpha(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\ (2\alpha)_{i+k}}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k}\ t^k, \quad 0 \le t \le L, \tag{3.2}$$

where $(\theta)_i$ is the Pochhammer symbol, means $\theta(\theta+1)(\theta+2)\cdots(\theta+i-1)$ for $i \ge 1$ and $(\theta)_0 = 1$. Note that the values

$$G_i^\alpha(0) = (-1)^i \frac{\Gamma(i+2\alpha)}{i!\ \Gamma(2\alpha)} \quad \text{and} \quad G_i^\alpha(L) = \frac{\Gamma(i+2\alpha)}{i!\ \Gamma(2\alpha)}, \tag{3.3}$$

are fulfilled at the endpoints. Moreover, the relations (3.1) and (3.3) imply that the special value

$$D^q G_i^\alpha(0) = \frac{(-1)^{i-q} 2^{2q} (\alpha)_q \Gamma(i+2\alpha+q)}{L^q (i-q)! \Gamma(2\alpha+2q)}, \tag{3.4}$$

will be of important use later. The orthogonality condition is

$$\int_0^L G_j^\alpha(t) G_k^\alpha(t) w_L^\alpha(t) dt = \gamma_{L,k}^\alpha \delta_{jk},$$

where $w_L^\alpha(t)=(Lt-t^2)^{\alpha-\frac{1}{2}}$ is the weight function for the shifted Gegenbauer polynomials and $\gamma_{L,k}^\alpha=\frac{L^{2\alpha}\ \pi\ \Gamma(k+2\alpha)}{2^{4\alpha-1}\ \Gamma(k+1)\ (k+\alpha)\ \Gamma^2(\alpha)}$ is the normalization factor.

If $f(t)$ is a polynomial of degree n, then it may be expressed in terms of shifted Gegenbauer polynomials as

$$u(t)=\sum_{j=0}^{n} b_j G_j^\alpha(t),$$

where the coefficients b_j are given by

$$b_j=\frac{1}{\gamma_{L,j}^\alpha}\int_0^L u(t)G_j^\alpha(t)w_L^\alpha(t)dt, \quad j=0,1,2,\dots \tag{3.5}$$

We denote the zeroes of the shifted Gegenbauer polynomial $G_{i+1}^\alpha(t)$ by $t_{L,k}^\alpha$, $k=0,1,2,\dots,i$. We also denote their corresponding Christoffel numbers by $\varpi_{L,k}^\alpha$, $k=0,1,2,\dots,i$.

Clearly,

$$\begin{aligned} t_{L,k}^\alpha =& \frac{L}{2}(t_k^\alpha+1), \quad k=0,\dots,i,\\ \varpi_{L,k}^\alpha =& \left(\frac{L}{2}\right)^{2\alpha}\varpi_k^\alpha, \quad k=0,\dots,i. \end{aligned}$$

If we denote by $\mathbb{P}_i$, the space of all polynomials of degree at most i, $i\in\mathbb{Z}^+$, then for any $\phi\in\mathbb{P}_{i+1}$

$$\begin{aligned} \int_0^L \phi(t)w_L^\alpha dt =& \left(\frac{L}{2}\right)^{2\alpha}\int_{-1}^{1}\phi\left(\frac{L}{2}(t+1)\right)w^\alpha(t)dt=\left(\frac{L}{2}\right)^{2\alpha}\sum_{j=0}^{n}\varpi_j^\alpha\phi\left(\frac{L}{2}(t_j^\alpha+1)\right)\\ =& \sum_{j=0}^{n}\varpi_{L,j}^\alpha\phi(t_{L,j}^\alpha). \end{aligned}$$

Lemma 3.1. *The qth derivative of $G_k^\alpha(t)$ can be written as*

$$D^qG_k^\alpha(t)=\sum_{\substack{m=0\\(k+m-q)\ \ even}}^{k} C_q(k,m,\alpha)G_m^\alpha(t),$$

where

$$\begin{aligned} C_q(k,m,\alpha)=&\frac{2^{2q}\ k!}{L^q(q-1)!\ \Gamma(k+2\lambda)}\\ &\times\frac{(m+\alpha)\ \Gamma(m+2\alpha)\ \left(\frac{k-m+q-2}{2}\right)!\ \Gamma\left(\frac{k+m+q+2\alpha}{2}\right)}{m!\ \left(\frac{k-q-m}{2}\right)!\ \Gamma\left(\frac{k+m-q+2\alpha+2}{2}\right)}, \quad k\geq q. \end{aligned}$$

(For the proof see Doha [12]).

3.2. The fractional derivatives of $G_i^{\alpha}(t)$.

Lemma 3.2. *Let $G_i^{\alpha}(t)$ be a shifted Gegenbauer polynomial then*

$$D^{(\nu)}G^{\alpha}{}_i(t)=0, \quad i=0,1,2,\dots,\lceil\nu\rceil-1,\ \nu>0.$$

Proof. Using (2.1)–(2.3) in (3.2) the lemma can be proved. □

Theorem 3.1. *The fractional derivative of order ν in the Caputo sense for the shifted Gegenbauer polynomials is given by*

$$D^{(\nu)}G^{\alpha}{}_i(t)=\sum_{j=0}^{\infty}S_\nu(i,j,\alpha)G_j^{\alpha}(t), \quad i=\lceil\nu\rceil,\lceil\nu\rceil+1\dots, \tag{3.6}$$

where

$$S_\nu(i,j,\alpha)=\sum_{k=\lceil\nu\rceil}^{i}\theta_{i,j,k},$$

and

$$\begin{aligned}\theta_{i,j,k}=&\frac{(-1)^{i-k}\ 2^{4\alpha-1}\ \Gamma(j+1)\ (j+\alpha)\Gamma^2(\alpha)\ (2\alpha)_{i+k}}{\pi\ \Gamma(j+2\alpha)\ (i-k)!\ (\alpha+\frac{1}{2})_k\ \Gamma(k-\nu+1)}\\ &\times\sum_{l=0}^{j}\frac{(-1)^{j-l}\ L^{-\nu}\ (2\alpha)_{j+l}\ \Gamma(\alpha+\frac{1}{2})\ \Gamma(k+l+\alpha-\nu+\frac{1}{2})}{l!\ (j-l)!\ (\alpha+\frac{1}{2})_l\ \Gamma(2\alpha+k+l-\nu+1)}, \quad j=0,1,\dots,\end{aligned}$$

where $D^{(\nu)}$ is the fractional derivative of order ν in the Caputo sense.

Proof. The analytic form of the shifted Gegenbauer polynomials $G_i^{\alpha}(t)$ of degree i is given by (3.2). Using (2.2), (2.3) and (3.2) we have

(3.7)

$$\begin{aligned}D^{(\nu)}G_i^{\alpha}(t)&=\sum_{k=0}^{i}\frac{(-1)^{i-k}\ (2\alpha)_{i+k}}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k}\ D^{\nu}t^k\\ &=\sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^{i-k}\ (2\alpha)_{i+k}\ \Gamma(k+1)}{L^k\ k!\ (i-k)!\ (\alpha+\frac{1}{2})_k\ \Gamma(k-\nu+1)}\ t^{k-\nu}, \quad i=\lceil\nu\rceil,\ \lceil\nu\rceil+1,\dots\end{aligned}$$

Now approximate $t^{k-\nu}$ by $(N+1)$ terms of shifted Gegenbauer series, so we have

$$t^{k-\nu}\simeq\sum_{j=0}^{N}b_{k,j}G_j^{\alpha}(t), \tag{3.8}$$

where $b_{k,j}$ is given from (3.5) with $u(t)=t^{k-\nu}$ and

$$\begin{aligned}b_{k,j}=&\frac{2^{4\alpha-1}\ \Gamma(j+1)\ (j+\alpha)\ \Gamma^2(\alpha)}{\pi\ \Gamma(j+2\alpha)}\\ &\times\sum_{l=0}^{j}\frac{(-1)^{j-l}\ L^{k-\nu}\ (2\alpha)_{j+l}\ \Gamma(\alpha+\frac{1}{2})\ \Gamma(k+l+\alpha-\nu+\frac{1}{2})}{l!\ (j-l)!\ (\alpha+\frac{1}{2})_l\ \Gamma(2\alpha+k+l-\nu+1)}, \quad j=0,1,\dots\end{aligned}$$

Employing (3.7)–(4.1), yields

$$D^{\nu}G_i^{\alpha}(t) = \sum_{j=0}^{\infty} S_{\nu}(i,j,\alpha)G_j^{\alpha}(t), \quad i = \lceil \nu \rceil,\ \lceil \nu \rceil + 1, \ldots, \tag{3.9}$$

where $S_{\nu}(i,j,\alpha)$ is given as in (3.6) and this proves the theorem. □

Remark 3.1. It is to be noted here that if $\nu = q \in \mathbb{N}$, then Lemma 3.1 may be obtained as a special case of Theorem 3.1.

4. Shifted Gegenbauer Polynomials Interpolation Approximation

In this section, we use the shifted Gegenbauer-Gauss collocation method to solve numerically the following model problem:

$$(u(t) + a(t)u(p_m t))^{(\nu)} = \beta u(t) + \sum_{n=0}^{m-1} b_n(t) D^{\gamma_n} u(p_n t) + f(t), \quad t \geq 0, \tag{4.1}$$

with the initial conditions

$$\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1. \tag{4.2}$$

Here, a and b_n, $n = 0, 1, \ldots, m-1$, are given analytical functions, $m-1 < \nu \leq m$, $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \nu$ and β, p_n, c_{in}, λ_i denote given constants with $0 < p_n < 1$, $n = 0, 1, \ldots, m$.

By using the shifted Gegenbauer-Gauss collocation method, we can approximate the fractional neutral functional-differential equations with proportional delays, without any artificial boundary and variable transformation. Let us first introduce some basic notation that will be used in the sequel. We set

$$S_N(0, L) = \text{Span}\{G_0^{\alpha}(t), G_1^{\alpha}(t), \ldots, G_N^{\alpha}(t)\},$$

and we define the discrete inner product and norm as follows:

$$(u, v)_{w_{L,N}^{\alpha}} = \sum_{j=0}^{N} u(t_{L,j}^{\alpha})\, v(t_{L,j}^{\alpha})\, \varpi_{L,j}^{\alpha}, \quad \|u\|_{w_{L,N}^{\alpha}} = \sqrt{(u,u)_{w_{L,N}^{\alpha}}},$$

where $t_{L,j}^{\alpha}$ and $\varpi_{L,j}^{\alpha}$ are the nodes and the corresponding weights of the shifted Gegenbauer-Gauss quadrature formula on the interval $(0, L)$, respectively. Obviously,

$$(u, v)_{w_{L,N}^{\alpha}} = (u, v)_{w_L^{\alpha}}, \quad \text{for all } u, v \in S_{2N+1}.$$

Thus, for any $u \in S_N(0, L)$, the norms $\|u\|_{w_{L,N}^{\alpha}}$ and $\|u\|_{w_L^{\alpha}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{G_T^{\alpha}}$ the shifted Gegenbauer-Gauss interpolation,

$$I_N^{G_T^{\alpha}} u(t_{L,k}^{\alpha}) = u(t_{L,k}^{\alpha}), \quad 0 \leq k \leq N.$$

The shifted Gegenbauer-Gauss collocation method for solving (4.1) and (4.2) is to seek $u_N(t) \in S_N(0,L)$, such that

$$D^{\nu}(u(t^{\alpha}_{L,k}) + a(t^{\alpha}_{L,k})u(p_m t^{\alpha}_{L,k})) = \beta u(t^{\alpha}_{L,k}) + \sum_{n=0}^{m-1} b_n(t^{\alpha}_{L,k}) D^{\gamma_n} u(p_n t^{\alpha}_{L,k}) + f(t^{\alpha}_{L,k}),$$

where $k = 0, 1, \ldots, N-m$, $\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i$, $i = 0, 1, \ldots, m-1$.

We now derive the algorithm for solving (4.1) and (4.2). To do this, let

$$u_N(t) = \sum_{h=0}^{N} a_h G^{\alpha}_h(t), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^T. \tag{4.3}$$

We first approximate $D^{\nu}u(t)$ and $D^{\gamma_n}u(t)$, as (4.3). By substituting these approximation in (4.1), we get

$$\begin{aligned}\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t) + a(t)\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t)\right)^{(\nu)} =& \beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t) + \sum_{n=0}^{m-1}\sum_{h=0}^{N} a_h b_n(t) D^{\gamma_n} G^{\alpha}_h(p_n t)\\ &+ f(t).\end{aligned} \tag{4.4}$$

Making use of (3.6), we deduce that

$$\begin{aligned}\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t) + a(t)\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t)\right)^{(\nu)} =& \beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t)\\ &+ \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M} a_h b_n(t) S_{\gamma_n}(h,f) G^{\alpha}_f(p_n t) + f(t).\end{aligned} \tag{4.5}$$

Also, by substituting (4.3) in (4.2) we obtain

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M} a_{in} D^{(n)} G^{\alpha}_f(0) = \lambda_i. \tag{4.6}$$

Now, we collocate (4.5) at the $(N-m+1)$ shifted Gegenbauer-Gauss interpolation points, yields

$$\begin{aligned}&\left(\sum_{h=0}^{N} a_h G^{\alpha}_h(t^{\alpha}_{L,k}) + a(t^{\alpha}_{L,k})\sum_{h=0}^{N} a_h G^{\alpha}_h(p_m t^{\alpha}_{L,k})\right)^{(\nu)}\\ =&\beta \sum_{h=0}^{N} a_h G^{\alpha}_h(t^{\alpha}_{L,k}) + \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M} a_h b_n(t^{\alpha}_{L,k}) S_{\gamma_n}(h,f) G^{\alpha}_f(p_n t^{\alpha}_{L,k}) + f(t^{\alpha}_{L,k}).\end{aligned} \tag{4.7}$$

Next (4.6), after using (3.4), can be written as

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M} (-1)^{f-n} c_{in} \frac{2^{2n}\ (\alpha)_n\ \Gamma(f+2\alpha+n)}{L^n\ (f-n)!\ \Gamma(2\alpha+2n)} = \lambda_i. \tag{4.8}$$

Finally, (4.7), with relation (4.8) generate $(N+1)$ set of algebraic equations which can be solved for the unknown coefficients a_j, $j = 0, 1, 2, \dots, N$, by using any standard solver technique.

5. Convergence and Error Analysis

This section is dedicated to investigating the convergence and error analysis of the suggested Gegenbauer expansion. In this regard, we follow Abd-Elhameed and Youssri [2], the following two theorems are stated and proved. In what follows by the notation $A \lesssim B$, we means that there exists a generic constant Υ such that $A \leq \Upsilon B$. The following three lemmas are of important use in sequel.

Lemma 5.1 ([1]).

$$\int G_j^{\alpha}(t) w_L^{\alpha}(t) dt = -\frac{\alpha \; 2^{1-2\alpha} \; L^{2\alpha}}{j(j+\alpha)} G_{j-1}^{\alpha+1}(t).$$

Lemma 5.2 ([10]). *For $\alpha > 0$ one have:*

$$|G_j^{\alpha}(t)| \lesssim \frac{(Lt - t^2)^{\frac{-\alpha}{2}}}{j^{\frac{3}{2}}}.$$

Lemma 5.3 ([34]).

$$\lim_{n \to \infty} \frac{\Gamma(n+\alpha)}{(n-1)! \; n^{\alpha}} = 1.$$

Theorem 5.1. *If $u(t)$ is expanded in a series of shifted Gegenbauer polynomial, has a bounded first derivative, then we have the following estimate: for all $\alpha > 0$ and $j > 1 :\ |b_j| \lesssim j^{-2\alpha - \frac{3}{2}}$.*

Proof. Integration by parts on the right hand side of (3.5) and based on Lemma 5.1 we have

$$|b_j| \lesssim \frac{|\int_0^L u'(t) \; G_{j-1}^{\alpha+1}(t) \; dt|}{j^2 \; \gamma_{L,j}^{\alpha}}. \tag{5.1}$$

Based on Lemma 5.2 and by hypothesis of the theorem we have

$$\left| \int_0^L u'(t) \; G_{j-1}^{\alpha+1}(t) \; dt \right| \lesssim \frac{M}{j^{\frac{3}{2}}}, \tag{5.2}$$

where M is the upper bound of $u'(t)$. Application of Lemma 5.3 will yield

$$|\gamma_{L,j}^{\alpha}| = \mathcal{O}(j^{2\alpha-2}). \tag{5.3}$$

Joining (5.1), (5.2) and (5.3), we have

$$b_j \lesssim j^{-\frac{3}{2} - 2\alpha},$$

which completes the proof of the theorem. $\square$

Theorem 5.2. *If* $u(t) = \sum_{j=0}^{\infty} b_j\, G_j^{\alpha}(t)$, $u_N(t) = \sum_{j=0}^{N} b_j\, G_j^{\alpha}(t)$ *are the exact and approximate solutions of* (4.1), *respectively and* $u(t)$ *satisfies the hypothesis of Theorem* 5.1, *then we have the following error estimate:*

$$\|u - u_N\| w_{L,N}^{\alpha} \lesssim 1/N^{2+\alpha}.$$

Proof.

$$\begin{aligned}\| u - u_N \|^2_{w^{\alpha}_{L,N}} &= \int_0^L (u-u_N)^2 w_L^{\alpha}\, dt \\ &= \left(\sum_{j=N+1}^{\infty} b_j\, G_j^{\alpha},\ \sum_{j=N+1}^{\infty} b_j\, G_j^{\alpha} \right)_{w_L^{\alpha}} \\ &= \sum_{j=N+1}^{\infty} b_j^2\, \gamma_{L,j}^{\alpha}.\end{aligned}$$

Now, based on the estimate in Theorem 5.1 and the estimate in (5.3), we get

$$\| u - u_N \|^2_{w^{\alpha}_{L,N}} \lesssim \sum_{j=N+1}^{\infty} j^{-5-2\alpha} \lesssim N^{-4-2\alpha},$$

which completes the proof of the theorem. □

6. Numerical results

In order to show the effectiveness of shifted Gegenbauer-Gauss collocation method for solving fractional neutral functional-differential equations with proportional delays, we present some numerical examples. The absolute errors in the given tables are the values of $|u(x) - u_N(x)|$ at selected points.

Example 6.1 ([7]). Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{1}{2}}(t) = -u(t) + \frac{1}{3} u\left(\frac{t}{4}\right) + \frac{1}{2} u^{\frac{1}{2}}\left(\frac{t}{4}\right) + g(t), \quad u(0) = 1,\ t \in [0,5], \tag{6.1}$$

where

$$g(t) = -\frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-x)^{-\frac{1}{2}} \sin(x)\, dx + \cos(t) - \frac{1}{3} \cos\left(\frac{t}{4}\right) + \frac{1}{2\Gamma(\frac{1}{2})} \int_0^t (t-x)^{-\frac{1}{2}} \sin\left(\frac{x}{4}\right) dx,$$

and the exact solution is given by $u(t) = \cos(t)$.

In Table 1, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with different values of α at $N = 22$. The outcomes are contrasted with the outcome of the modifed generalized Laguerre-Gauss collocation (MGLC) method [7]. It is clear from this table that, the solutions got by our technique are superior in examination with modifed generalized Laguerre-Gauss collocation scheme [7]. In order to compare the present method with the analytic solution, the resulting graph of (6.1) is shown in Figure 1.

TABLE 1. Comparison of the absolute errors at $N = 22$ for Example 6.1.

x	MGLC method [7]			Our method		
	$\alpha = 1$, $\beta = 2$	$\alpha = 4$, $\beta = 3$	$\alpha = 5$, $\beta = 5$	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = \frac{3}{2}$
0	$7.993.10^{-15}$	$0.000.10^{-00}$	$2.220.10^{-16}$	$2.220.10^{-16}$	$0.00.10^{-00}$	$0.00.10^{-00}$
0.5	$5.164.10^{-4}$	$7.041.10^{-4}$	$1.746.10^{-4}$	$1.663.10^{-5}$	$2.466.10^{-5}$	$3.757.10^{-5}$
1	$5.066.10^{-4}$	$5.763.10^{-4}$	$1.416.10^{-4}$	$1.592.10^{-5}$	$2.466.10^{-5}$	$3.757.10^{-5}$
1.5	$3.521.10^{-4}$	$4.93.10^{-4}$	$1.257.10^{-4}$	$4.765.10^{-6}$	$1.804.10^{-5}$	$3.067.10^{-5}$
2	$2.793.10^{-4}$	$4.268.10^{-4}$	$1.059.10^{-4}$	$3.517.10^{-7}$	$1.684.10^{-5}$	$2.572.10^{-5}$
2.5	$4.480.10^{-4}$	$3.862.10^{-4}$	$8.856.10^{-5}$	$1.276.10^{-5}$	$1.684.10^{-5}$	$2.440.10^{-5}$
3	$2.269.10^{-4}$	$3.437.10^{-4}$	$8.429.10^{-5}$	$1.585.10^{-5}$	$1.694.10^{-5}$	$2.264.10^{-5}$
3.5	$1.998.10^{-4}$	$3.113.10^{-4}$	$7.249.10^{-5}$	$5.960.10^{-8}$	$1.066.10^{-5}$	$1.873.10^{-5}$
4	$5.164.10^{-4}$	$3.037.10^{-4}$	$6.271.10^{-5}$	$1.364.10^{-5}$	$5.364.10^{-6}$	$1.522.10^{-5}$
4.5	$1.141.10^{-3}$	$2.841.10^{-4}$	$1.159.10^{-4}$	$3.051.10^{-5}$	$4.649.10^{-6}$	$1.657.10^{-5}$

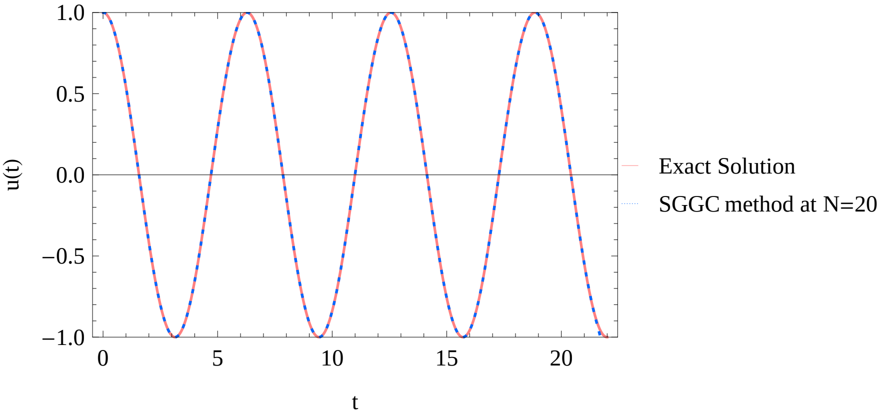

FIGURE 1. Graph of exact solution and approximate solution for $\alpha = \frac{5}{2}$ at $N = 20$ for Example 6.1.

Example 6.2. Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{1}{2}}(t) = -u(t) + \frac{1}{4}u\left(\frac{t}{3}\right) + \frac{1}{3}u^{\frac{1}{2}}\left(\frac{t}{3}\right) + g(t), \quad u(0) = 0,\, t \in [0,1], \tag{6.2}$$

where

$$g(t) = \frac{\Gamma(q+1)}{\Gamma(q+\frac{1}{2})}t^{q-\frac{1}{2}} + t^q - \frac{1}{4}\left(\frac{t}{3}\right)^q - \frac{1}{3}\frac{\Gamma(q+1)}{\Gamma(q+\frac{1}{2})}\left(\frac{t}{3}\right)^{q-\frac{1}{2}},$$

and the exact solution is given by $u(t) = t^q$, $q \geq \lceil \frac{1}{2} \rceil$.

In Table 2, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with several values of t, q and at $N = 16$.

TABLE 2. Absolute errors using SGGC method at $N = 16$ for Example 6.2.

t	$q = 1$	$q = 1.3$	$q = 1.5$	$q = 1.7$	$q = 1.9$
0.1	$1.259.10^{-4}$	$1.570.10^{-4}$	$7.967.10^{-5}$	$2.889.10^{-5}$	$4.584.10^{-6}$
0.2	$8.676.10^{-5}$	$1.784.10^{-4}$	$1.070.10^{-4}$	$4.379.10^{-5}$	$9.350.10^{-6}$
0.3	$3.542.10^{-4}$	$8.152.10^{-5}$	$3.702.10^{-5}$	$1.451.10^{-5}$	$1.102.10^{-6}$
0.4	$1.037.10^{-4}$	$2.138.10^{-5}$	$3.047.10^{-6}$	$1.577.10^{-6}$	$1.465.10^{-6}$
0.5	$3.993.10^{-4}$	$8.114.10^{-5}$	$5.435.10^{-5}$	$2.070.10^{-5}$	$5.960.10^{-6}$
0.6	$2.855.10^{-4}$	$1.182.10^{-4}$	$7.906.10^{-5}$	$3.327.10^{-5}$	$8.494.10^{-6}$
0.7	$2.324.10^{-4}$	$5.947.10^{-5}$	$3.263.10^{-5}$	$1.503.10^{-5}$	$2.164.10^{-6}$
0.8	$5.255.10^{-4}$	$3.819.10^{-5}$	$9.847.10^{-6}$	$4.389.10^{-6}$	$1.710.10^{-6}$
0.9	$2.723.10^{-4}$	$4.922.10^{-6}$	$1.289.10^{-5}$	$6.527.10^{-6}$	$3.031.10^{-6}$
1.0	$1.124.10^{-4}$	$5.547.10^{-5}$	$3.204.10^{-5}$	$1.146.10^{-5}$	$2.783.10^{-6}$

TABLE 3. Absolute errors using SGGC method at $N = 16$ for Example 6.3.

t	$\alpha = \frac{1}{2}$	$\alpha = 1$	$\alpha = \frac{3}{2}$
0.0	$4.609.10^{-16}$	$7.844.10^{-16}$	$2.387.10^{-17}$
0.1	$6.161.10^{-6}$	$8.530.10^{-6}$	$1.075.10^{-5}$
0.2	$3.563.10^{-5}$	$4.959.10^{-5}$	$6.287.10^{-5}$
0.3	$9.834.10^{-5}$	$1.380.10^{-4}$	$1.758.10^{-4}$
0.4	$2.028.10^{-4}$	$2.857.10^{-4}$	$3.651.10^{-4}$
0.5	$3.579.10^{-3}$	$5.054.10^{-4}$	$6.457.10^{-4}$
0.6	$5.777.10^{-4}$	$8.090.10^{-4}$	$1.032.10^{-3}$
0.7	$8.657.10^{-4}$	$1.208.10^{-3}$	$1.541.10^{-3}$
0.8	$1.227.10^{-3}$	$1.714.10^{-3}$	$2.186.10^{-3}$
0.9	$1.671.10^{-3}$	$2.338.10^{-3}$	$2.981.10^{-3}$
1.0	$2.071.10^{-3}$	$2.971.10^{-3}$	$3.834.10^{-3}$

Example 6.3. Consider the following fractional neutral functional-differential equation with proportional delay

$$u^{\frac{5}{2}}(t) = u(t) + u^{\frac{1}{2}}\left(\frac{t}{3}\right) + u^{\frac{3}{2}}\left(\frac{t}{4}\right) + u^{\frac{5}{2}}\left(\frac{t}{5}\right) + g(t), \quad t \in [0,1], \tag{6.3}$$

subject to

$$u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0,$$

where

$$g(t) = \frac{32}{\sqrt{\pi}} t^{\frac{3}{2}} - t^4 - \frac{128}{945\sqrt{3\pi}} t^{\frac{7}{2}} - \frac{2}{5\sqrt{\pi}} t^{\frac{5}{2}} - \frac{32}{5\sqrt{5\pi}} x^{\frac{3}{2}},$$

and the exact solution is given by $u(t) = t^4$.

In Table 3, we list the absolute errors obtained by the shifted Gegenbauer-Gauss collocation method, with several values of α and at $N = 16$. Meanwhile, Figure 1

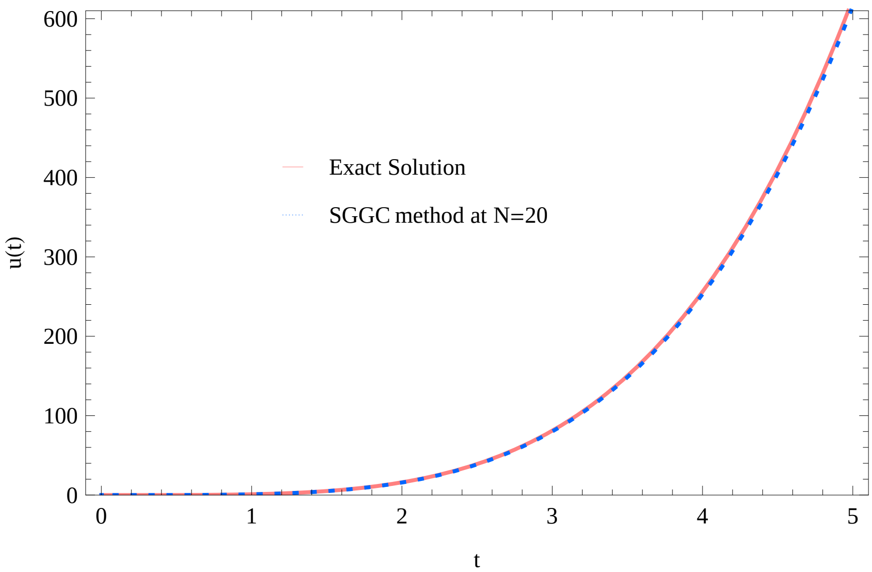

FIGURE 2. Graph of exact solution and approximate solution for $\alpha = \frac{5}{2}$ at $N = 20$ for Example 6.3.

presents the SGGC solution with $\alpha = \frac{5}{2}$ at $N = 20$ and exact solution, which are found to be in excellent agreement.

Acknowledgements. The authors are very much grateful to the editor and anonymous reviewers for their valuable comments and careful reading of the manuscript.

REFERENCES

[1] W. M. Abd-Elhameed and Y. H. Youssri, *New ultraspherical wavelets spectral solutions for fractional Riccati differential equations*, Abstr. Appl. Anal. **2014** (2014), 8 pages.

[2] W. M. Abd-Elhameed and Y. H. Youssri, *New spectral solutions of multi-term fractional-order initial value problems with error analysis*, CMES-Comp. Model. Eng. Sci. **105** (2015), 375–398.

[3] A. H. Bhrawy, A. A. AL-Zahrani, Y. A. Alhamed and D. Baleanu, *A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations*, Rom. Journ. Phys. **59** (2014), 646–657.

[4] R. L. Bagley and P. J. Torvik, *A theoritical asis for the appliation of fractional calculus to viscoelastiity*, J. Rheology **27** (1983), 201–210.

[5] R. L. Bagley and P. J. Torvik, *Fractional calculus in the transient analysis of viscoelastially damped struturesy*, AIAA J. **23** (1985), 918–925.

[6] R. Baillie, *Long memory processes and fractional integration in econometrices*, J. Econometrices **73** (1996), 5–59.

[7] A. H. Bhrawy, A. Al Zahrani, D. Baleanu and Y. Alhamed, *A modified generalized laguerre-gauss collocation method for fractional neutral functional-differential equations on the half-line*, Abstr. Appl. Anal. **2014** (2014), 7 pages.

[8] E. H. Doha, A. H. Bhrawy, D. Baleanu and R. M. Hafez, *Efficient Jacobi-Gauss collocation method for solving initial value problems of Bratu type*, Comp. Math. Math. Phys. **53** (2013), 1292–1302.

[9] E. H. Doha, A. H. Bhrawy, D. Baleanu and R. M. Hafez, *A new Jacobi rational-Gauss collocation method for numerical solution of generalized pantograph equations*, Appl. Numer. Math. **77** (2014), 43–54.

[10] T. S. Chow, *Fractional dynamics of interfaces etween soft-nanoparticls and rough sustrates*, Phys. Lett. A **342** (2005), 148–155.

[11] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer, New York, 2008.

[12] E. H. Doha, *The coefficients of differentiated expansions and derivatives of ultraspherical polynomials*, J. Comp. Math. Appl. **19** (1991), 115–122.

[13] M. Ghasemi, M. Fardi and R. K. Ghaziani, *Numerical solution of nonlinear delay differential equations of fractional order in reproducing kernel Hilbert space*, Appl. Math. Comput. **268** (2015), 815–831.

[14] B. Fornberg, *A practical guide to pseudospectral methods*, Cambridge University Press, Cambridge, 1998.

[15] J. H. He, *Nonlinear oscillation with fractional derivatives and its applications* in: *International Conference on Vibrating Engineering'98*, Dalia, Chinan, 1998.

[16] J. H. He, *Some applications of nonlinear fractional differential equations and thier approximations bull*, Sci. Technol. **15** (1999), 86–90.

[17] M. Ishteva, *Properties and applications of the Caputo fractional operator*, PhD thesis, Department of Mathematics, Universitat Karlsruhe (TH), Sola, Bulgaria, 2005.

[18] C. Li and W. Deng, *Remarks on fractional derivatives*, Appl. Math. Comput. **187** (2007), 777–784.

[19] C. Li and Z. Zhao, *Introduction to fractional integrability and differentiability*, The European Physical Journal Special Topics **193** (2011), 5–26.

[20] R. Magin, *Frational calculus in bioengineering*, Crit. Rev. Biomed. Eng. **32** (2004), 1–104.

[21] R. Magin, *Frational calculus in bioengineering-part* 2, Crit. Rev. Biomed. Eng. **32** (2004), 105–193.

[22] R. Magin, *Frational calculus in bioengineering-part* 3, Crit. Rev. Biomed. Eng. **32** (2004), 194–377.

[23] F. Mainardi, *Fractional calculus: some basic prolems in continuum and statical mechanis*, in: A. Carpinteri and F. Mainardi (Eds.), *Fratals and Fractional Calculus in Continuum Mechanics*, Springer, Verlag, New York, 1997, 291–348.

[24] B. Mandelbort, *Some noises with $1/f$ spectrum, a bridge between direct current and white noise*, IEEE Trans. Inform. Theory **13** (1967), 289–298.

[25] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: Recent developments in the description of anomalos transport by fractional dyuamics*, J. Phys. A **37** (2004), 161–208.

[26] K. S. Miller and B. Ross, *An Introduaction to the Frational Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[27] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[28] P. Rahimkhani, Y. Ordokhani and E. Babolian, *Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet*, J. Comput. Appl. Math. **309** (2017), 493–510.

[29] A. Anapali, Y. Öztürk and M. Gülsu, *Numerical approach for solving fractional pantograph equation*, IJCA **113** (2015), 45–52.

[30] R. Peyret, *Spectral Methods for Incompressible Viscous Flow*, New York, Springer, 2002.

[31] I. Podluny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[32] C. Li D. Qian and Y. Chen, *On Riemann-Liouville and Caputo derivatives*, Discrete Dynamics in Nature and Society **2011** (2011), 15 pages.

[33] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Fluid Dynamics*, New York, Springer, 1988.

[34] E. D. Rainville, *Special functions*, Chelsea Pub Co., New York, 1971.
[35] Y. A. Rossikhin and M. V. Shitikova, *Applications of fractional calculus to dynamic prolems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev. **50** (1997), 15–67.
[36] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego, 2006.
[37] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Providence, Rhode Island, 1975.
[38] L. N. Trefethen, *Spectral Methods in MATLAB*, Philadelphia, PA: SIAM, 2000.
[39] Y. Yang and Y. Huang, *Spectral-collocation methods for fractional pantograph delay-integrodifferential equations*, Adv. Math. Phys. **2013** (2013), 14 pages.

[1]Department of Mathematics,
Faculty of Education, Matrouh University,
Matrouh, Egypt
Email address: r_mhafez@yahoo.com

[2]Department of Mathematics,
Faculty of Science, Cairo University,
Giza 12613, Egypt
Email address: youssri@sci.cu.edu.eg

[*]Corresponding Author

KRAGUJEVAC JOURNAL OF MATHEMATICS

About this Journal

The *Kragujevac Journal of Mathematics* (KJM) is an international journal devoted to research concerning all aspects of mathematics. The journal's policy is to motivate authors to publish original research that represents a significant contribution and is of broad interest to the fields of pure and applied mathematics. All published papers are reviewed and final versions are freely available online upon receipt. Volumes are compiled and published and hard copies are available for purchase. From 2018 the journal appears in one volume and four issues per annum: in March, June, September and December. From 2021 the journal appears in one volume and six issues per annum: in February, April, June, August, October and December.

During the period 1980–1999 (volumes 1–21) the journal appeared under the name *Zbornik radova Prirodno–matematičkog fakulteta Kragujevac* (Collection of Scientific Papers from the Faculty of Science, Kragujevac), after which two separate journals—the *Kragujevac Journal of Mathematics* and the *Kragujevac Journal of Science*—were formed.

Instructions for Authors

The journal's acceptance criteria are originality, significance, and clarity of presentation. The submitted contributions must be written in English and be typeset in TEX or LATEX using the journal's defined style (please refer to the Information for Authors section of the journal's website `http://kjm.pmf.kg.ac.rs`). Papers should be submitted using the online system located on the journal's website by creating an account and following the submission instructions (the same account allows the paper's progress to be monitored). For additional information please contact the Editorial Board via e-mail (`krag_j_math@kg.ac.rs`).