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A GENERAL APPROACH TO CHAIN CONDITION IN BL-ALGEBRAS

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Abstract. In this paper, we present a general definition of the notion of Noetherian and Artinian BL-algebra and present a more comprehensive insight at the chain conditions in BL-algebras. We derive some theorems which generalize the existence results. We give an axiomatic approach to the notion of being Noetherian and Artinian, which is also applicable to other algebraic structures. We use a theoretical approach to define arithmetic notion that is also possible for other algebraic devices. In this study, we only focus to BL-algebras.

1. Introduction

Motamed and Moghaderi [5] introduced the notion of Noetherian and Artinian BL-algebras and provided some results on the subject, which are analogues to the results of the Noetherian and Artinian modules. O. Zahiri [8] defined the notion of length for a filter in BL-algebras and derived some new relations between Noetherian and Artinian BL-algebras. Zhan and Meng [9] defined another type of chain conditions in terms of the ideals of a BL-algebra, and called BL-algebras satisfying in the relevant conditions, co-Noetherian and co-Artinian BL-algebras. They also proved some results on co-Noetherian and co-Artinian BL-algebras which are analogues to the results of the Noetherian and Artinian modules. In this paper, we provide a more general definition of the chain condition in BL-algebras that can be defined in any other algebraic structure, but we limit ourselves to BL-algebras for simplicity. We provide some theorems on this general definition which generalize the results in [5] and [9].

Key words and phrases. Artinian (Noetherian) BL-algebra, $\mathcal{PF}$ BL-algebra, Maximal element, Complete family.

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The structure of the paper is as follows. In Section 2, we recall some definitions and results about BL-algebras which will be used in the sequel. In Section 3, we define the notion of complete family, structural, multiplicative and Noetherian (Artinian) BL-algebras with respect to a family of subsets of a BL-algebra. We also obtain some results about the relation between Noetherian, Artinian, finitely generated, maximal (minimal) element, multiplicative, onto BL-homomorphism and one to one BL-homomorphism BL-algebras.

2. Preliminaries

In this section, we recall and review some definitions and results, corresponding to co-Noetherian, Noetherian (Artinian) BL-algebras, which will be used throughout of the paper.

An algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of the type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if for all $a, b, c \in A$ satisfies the following axioms:

$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

$(BL2)$ $(A, \odot, 1)$ is a commutative monoid;

$(BL3)$ $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$;

$(BL4)$ $a \wedge b = a \odot (a \rightarrow b)$;

$(BL5)$ $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

We will denote $\overline{x} = x \rightarrow 0$ and $x^{--} = (\overline{x})^{-}$ for all $x \in A$.

Examples of BL-algebras [2] are t-algebras, $([0, 1], \wedge, \vee, \odot, \rightarrow, 0, 1)$, where $([0, 1], \wedge, \vee, 0, 1)$ is the usual lattice on $[0, 1]$ and $\odot$ is a continuous t-norm, whereas $\rightarrow$ is the corresponding residuum.

Throughout of this paper by A, we denote the universe of a BL-algebra. A BL-algebra is nontrivial if $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We denote the set of natural numbers by $\mathbb{N}$ and define a^0=1 and $a^n = a^{n-1} \odot a$, for $n \in \mathbb{N}\backslash\{0\}$. Hájek [2] defined a filter of a BL-algebra A to be a nonempty subset F of A such that (i) if $a, b \in F$ implies $a \odot b \in F$, (ii) if $a \in F, a \leq b$ then $b \in F$. E. Turunen [6] defined a deductive system of a BL-algebra A to be a nonempty subset D of A such that (i) $1 \in D$ and (ii) $x \in D$ and $x \rightarrow y \in D$ implies $y \in D$. Note that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A [6]. Let F be a filter of a BL-algebra A, then F is a proper filter if $F \neq A$. A proper filter P of A is called a prime filter of A if for all $x, y \in A, x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter P of A is Prime if and only if P can not be expressed as an intersection of two filters properly containing P or equivalently, for all $x, y \in A$, either $x \rightarrow y \in P$ or $y \rightarrow x \in P$ [6].

If F, G and P are filters of A, then P is a prime filter of A if and only if $F \bigcap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$.

In [6], it can be seen that a proper filter M of A is a maximal filter of A if and only if for all $x \notin M$, there exists $n \in \mathbb{N}$ such that $(x^n)^{-} \in M$. Every maximal filter of A is a prime filter of A [6]. The set of all filters, prime filters and maximal filters of a BL-algebra A are denote by $\mathcal{F}(A)$, $\text{Spec}(A)$ and $\text{Max}(A)$, respectively. The filter of A generated by X is denoted by $\langle X\rangle$, where $X \subseteq A$, in which $\langle\emptyset\rangle$=$\{1\}$ and $\langle X\rangle = \{a \in A : x_1 \odot x_2 \odot \cdots \odot x_n \leq a$, for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X\}$ [6]. $F \in \mathcal{F}(A)$ is called a finitely generated filter, if $F = \langle x_1, \ldots, x_n\rangle$, for some $x_{1,} \ldots, x_n \in A$ and $n \in \mathbb{N}$. For $F \in \mathcal{F}(A)$ and $x \in A\backslash F$, define $F\langle x\rangle = \langle F \cup \{x\}\rangle$ or equally $F\langle x\rangle = \{a \in A : a \geq f \odot x^n$ for some $f \in F$ and $n \geq 1\}$.

Definition 2.1 ([7])**.** Let A and B be two BL-algebras. A map $f : A \to B$ defined on A, is called a BL-homomorphism if for all $x, y \in A, f(x \to y) = f(x) \to f(y)$, $f(x \odot y) = f(x) \odot f(y)$ and $f(0_A) = 0_B$. We also define $\ker(f) = \{a \in A : f(a) = 1\}$ and $f(A) = \{f(a) : a \in A\}$.

Definition 2.2 ([5])**.** A BL-algebra A is called Noetherian (Artinian), if for every increasing (decreasing) chain of its filters $F_1 \subseteq F_2 \subseteq \cdots (F_1 \supseteq F_2 \supseteq \cdots)$, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$.

Definition 2.3 ([2,4])**.** Let A be a BL-algebra. A nonempty subset $I \subseteq A$ is called an ideal of A, if the following conditions hold:
(i) $0 \in I$;
(ii) if $x \in I$ and $(x^- \to y^-)^- \in I$, then $y \in I$.

Definition 2.4 ([9])**.** A BL-algebra A is said to be co-Noetherian with respect to ideals if every ideal of A is finitely generated. We say that A satisfies the ascending chain condition with respect to ideals if for every ascending chain sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of A, there exists $n \in \mathbb{N}$ such that $I_i = I_n$ for all $i \geq n$.

Definition 2.5 ([1,5])**.** Let A and B be BL-algebras. Then for every $a, c \in A$ and $b, d \in B$, we define the product of two BL-algebras which is clearly a BL-algebra, as follows:
$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$;
$(a, b) \vee (c, d) = (a \vee c, b \vee d)$;
$(a, b) \to (c, d) = (a \to c, b \to d)$;
$(a, b) \odot (c, d) = (a \odot c, b \odot d)$;
$(a, b) \leq (c, d) \Leftrightarrow (a \leq c, b \leq d)$.

3. Main Concepts and Results

In this section, regarding to the definitions of co-Noetherian, Noetherian (Artinian), multiplicative and $P\mathcal{F}$ BL-algebras and using related mentioned theorems, we derive some new results of finitely generated, maximal (minimal) element, onto BL-homomorphism, one-to-one BL-homomorphism, in a Noetherian (Artinian) BL-algebras.

Definition 3.1. Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A. A is said to be Noetherian with respect to family $\mathcal{F}$, if for any chain of elements of $\mathcal{F}$, $F_1 \subseteq F_2 \subset \cdots$, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$. We may similarly define Artinian BL-algebras.

Example 3.1. We know that every finite BL-algebra A is Noetherian and Artinian [5]. Therefore, if $\mathcal{F}$ is a family of subsets of A, since $\mathcal{F}$ is finite, so A is Noetherian (Artinian) with respect to family $\mathcal{F}$.

Theorem 3.1. *Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A. Then A is Noetherian* (*Artinian*) *with respect to $\mathcal{F}$ if and only if any set of elements of $\mathcal{F}$ has a maximal* (*minimal*) *element.*

Proof. Let A be a Noetherian BL-algebra with respect to $\mathcal{F}$ and S be a nonempty set of elements of $\mathcal{F}$ which does not have a maximal element, then, there exists $F_1 \in \mathcal{F}$. Since S does not have a maximal element, there is $F_2 \in S$ such that $F_1 \subset F_2$. By continuing this procedure, we obtain the following increasing chain of elements of $\mathcal{F}$: $F_1 \subset F_2 \subset \cdots$, which is a contradiction. So S has a maximal element.

Conversely, let $F_1 \subseteq F_2 \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$ and put $S = \{F_i : i \in \mathbb{N}\}$. Since S is nonempty, then it has a maximal element F_n. Thus $F_i = F_n$ for all $i \geq n$ and A is Noetherian with respect to $\mathcal{F}$ (Artinian case can be treated similarly). □

Definition 3.2. Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A which is closed under intersection operation (that is intersection of any number of elements of $\mathcal{F}$ is also an element of $\mathcal{F}$). If $B \subseteq A$, then the set generated by B in $\mathcal{F}$ is defined as the intersection of all elements of $\mathcal{F}$ containing B and denoted by $\langle B\rangle$, i.e.,

$$\langle B\rangle = \bigcap_{B\subseteq F}^{F\in\mathcal{F}} F,$$

$\langle B\rangle$ is said to be finitely generated if there exists a set $C \subseteq A$ such that $\langle B\rangle = \langle C\rangle$ and C is finite.

Example 3.2. Let $A = \{0, a, b, c, 1\}$, with $0 < c < a < 1$ and $0 < c < b < 1$. For every $x, y \in A$, we define the operations "$\odot$" and "$\rightarrow$" as follows:

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

,

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

.

Then it is easy to see that $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra [3]. If we consider $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ and $B = \{b\} \subseteq A$, where $F_1 = \{0, a\}$, $F_2 = \{0, a, b\}$, $F_3 = \{0, a, b, 1\}$ and $F_4 = \{0, a, b, c, 1\}$, then

$$\langle\{b\}\rangle = \bigcap_{\{b\}\subseteq F_i}^{F_i\in\mathcal{F}} F_i = F_2 \cap F_3 \cap F_4 = \{0, a, b\}.$$

Definition 3.3. Let $\mathcal{F}$ be a family of subsets of BL-algebra A. $\mathcal{F}$ is said to be complete if for any subset B of A, $\langle B\rangle$ is nonempty.

Example 3.3. Let $A = \{0, a, 1\}$. For every $x, y \in A$, we define the operations "$\odot$" and "$\to$" as follows:

$$\begin{array}{c|ccc} \odot & 0 & a & 1 \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & a & a \\ 1 & 0 & a & 1 \end{array}\ , \qquad \begin{array}{c|ccc} \to & 0 & a & 1 \\ \hline 0 & 1 & 1 & 1 \\ a & 0 & 1 & 1 \\ 1 & 0 & a & 1 \end{array}\ .$$

Then it is easy to see that $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra.

Let $\mathcal{F} = \{\{a\}, \{0, a\}, \{0, a, 1\}\}$, then $\mathcal{F}$ is complete, since for every element of the powerset of A, we have: $\langle\{0\}\rangle = \{0, a\} \cap \{0, a, 1\} = \{0, a\}$, $\langle\{a\}\rangle = \langle\emptyset\rangle = \{a\} \cap \{0, a\} \cap \{0, a, 1\} = \{a\}$, $\langle\{0, a\}\rangle = \{0, a\} \cap \{0, a, 1\} = \{0, a\}$, $\langle\{0, 1\}\rangle = \langle\{a, 1\}\rangle = \langle\{0, a, 1\}\rangle = \{0, a, 1\}$, $\langle\{1\}\rangle = \{0, a, 1\} = \{1\}$.

Definition 3.4. Let A be a BL-algebra and $\mathcal{F}$ be a complete family which is closed under intersection. Then $\mathcal{F}$ is said to be closed under chain union if for any chain $F_1 \subseteq F_2 \subseteq \cdots$, of elements of $\mathcal{F}$, $\bigcup_{i=1}^{\infty} F_i$ also belongs to $\mathcal{F}$.

Example 3.4. (i) The family $\mathcal{F}$ in Example 3.2, is closed under chain union.

(ii) If we define the operations "$\odot$" and "$\to$" on $A = [0, 1]$ (real unit interval) by $x \odot y = \min\{x, y\}$ and

$$x \to y = \begin{cases} 1, & x \le y, \\ y, & x > y, \end{cases}$$

then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra (Gödel structure) [7]. Now, the family $\mathcal{F} = \{(0, 1], F_n = [\frac{1}{n}, 1]_{n\ge1}\} = \{(0, 1], F_1, F_2, F_3, \ldots, F_n, \ldots\}$, is closed under chain union.

Theorem 3.2. *Let A be a BL-algebra and $\mathcal{F}$ be a family of subsets of A which is closed under intersection and chain union. Then A is Noetherian with respect to $\mathcal{F}$ if and only if any element of $\mathcal{F}$ is finitely generated.*

Proof. Set $S = \{G \in \mathcal{F} : G \subseteq F,\ G \text{ is finitely generated}\}$. Since F is nonempty, then it has an element x and $\langle x\rangle \in S$. By Theorem 3.1, it has a maximal element F_1. By definition of S, $F_1 \subseteq F$ and $F_1 = \langle x_1, \ldots, x_n\rangle$ for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in A$. Since F is not finitely generated so $F_1 \subset F$ and there exists $x \in F\backslash F_1$. We also have $\langle x_1, \ldots, x_n, x\rangle \subseteq F$ and $\langle x_1, \ldots, x_n, x\rangle \in S$ which contradicts the maximality of F_1, i.e., F is finitely generated.

Conversely, let any element of $\mathcal{F}$ be finitely generated and $F_1 \subseteq F_2 \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$. We set $F = F_1 \cup F_2 \cup \cdots$, thus $\mathcal{F}$ is finitely generated by definition and $F = \langle x_1, \ldots, x_n\rangle$ for $x_i \in A$. Now, by chaining condition, there exists $m \in \mathbb{N}$ such that $x_1, \ldots, x_n \in F_m$ and so $F = \langle x_1, \ldots, x_n\rangle \subseteq F_m \subseteq F$. Thus, $F_m = F_i$ for $i \ge m$ and A is Noetherian with respect to $\mathcal{F}$.

If $\mathcal{F}$ is a family of all filters of the BL-algebra A, then we obtain the concept of Noetherian and Artinian BL-algebra, which is introduced by Motamed and Moghaderi [5]. When $\mathcal{F}$ is a family of all ideals of the BL-algebra A, we obtain the concept of co-Noetherian and co-Artinian BL-algebra which is introduced in [9]. $\square$

Definition 3.5. Let $\mathcal{A}$ be the family of all BL-algebras. Then $\mathcal{F}$ is said to be a complete family for elements of $\mathcal{A}$ if for every element A of $\mathcal{A}$ there exists $F_1 \in \mathcal{F}$ such that F_1 is a complete family for A and is closed under intersection.

Definition 3.6. Let $\mathcal{F}$ be a complete family for all BL-algebras, then $\mathcal{F}$ is said to be structural if it has the following property.

If A_1 and A_2 are two BL-algebras and $\varphi : A_1 \to A_2$ is an onto BL-homomorphism, then $\varphi^{-1}(F)$ is also an element of $\mathcal{F}$, for every $F \in \mathcal{F}$, $F \subseteq A_2$.

Theorem 3.3. *Let $\mathcal{F}$ be a structural family for the family of the all BL-algebras. If A_1, A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is an onto BL-homomorphism and A_1 is Noetherian with respect to $\mathcal{F}$, then A_2 is also Noetherian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \subseteq F_2 \subseteq \cdots$, be a chain of elements of $\mathcal{F}$ for A_2. Then, by Definition 3.6, $\psi^{-1}(F_1) \subseteq \psi^{-1}(F_2) \subseteq \cdots$, is a chain of elements of $\mathcal{F}$ for A_1. A_1 is Noetherian with respect to family $\mathcal{F}$, then there exists $n \in \mathbb{N}$ such that $\psi^{-1}(F_i) = \psi^{-1}(F_n)$ for all $i \geq n$. Since ψ is an onto BL-homomorphism, then $F_i = F_n$ for all $i \geq n$. Hence, A_2 is also Noetherian with respect to $\mathcal{F}$. $\square$

Definition 3.7. Let $\mathcal{F}$ be a complete family for BL-algebra A which is closed under intersection and $F \in \mathcal{F}$. Then F is said to be cyclic if there exists $a \in A$ such that $F = \langle a \rangle$. If any $F \in \mathcal{F}$ is cyclic, then A is called principal with respect to $\mathcal{F}$ which is denoted by $P\mathcal{F}$-BL.

Example 3.5. We consider BL-algebra A and collection $\mathcal{F}$ in Example 3.3, then $\{a\} = \langle a \rangle = \langle \{a\} \rangle$, i.e., $\{a\}$ is cyclic.

Theorem 3.4. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A such that any element $F \in \mathcal{F}$ which is generated by two elements, is cyclic. If A is Noetherian with respect to family $\mathcal{F}$, then A is $P\mathcal{F}$-BL.*

Proof. Let $F \in \mathcal{F}$, then by Theorem 3.2, F is finitely generated. So $F = \langle x_1, \dots, x_n \rangle$ for some $n \in \mathbb{N}$ and $x_1, \dots, x_n \in A$. We proceed by mathematical induction. From induction hypothesis, there is $b \in A$ such that $\langle x_1, \dots, x_{n-1} \rangle = \langle b \rangle$. Thus, $\langle x_1, \dots, x_n \rangle = \langle b, x_n \rangle$. So by hypothesis, there exists $a \in A$ such that $\langle b, x_n \rangle = \langle a \rangle$. Therefore, $F = \langle a \rangle$, i.e., F is cyclic. $\square$

Theorem 3.5. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A which is closed under intersection. If A is $P\mathcal{F}$-BL and $\mathcal{F}$ is closed under chain union, then A is Noetherian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \subseteq F_2 \subseteq \cdots$, be a chain of elements of $\mathcal{F}$. Then $F = F_1 \cup F_2 \cup \cdots$, is also an element of $\mathcal{F}$. Since A is $P\mathcal{F}$-BL, then $F = \langle a \rangle$ for some $a \in A$. So, there exists $t \in \mathbb{N}$ such that $a \in F_t$. Thus, $F_i = F_t = F$ for all $i \geq t$ and A is Noetherian with respect to $\mathcal{F}$. $\square$

Theorem 3.6. *Let A be a BL-algebra and $\mathcal{F}$ be a complete family for A such that any chain of finitely generated elements of $\mathcal{F}$ is stopping. Then A is Noetherian with respect to $\mathcal{F}$.*

Proof. We assume that A is not Noetherian with respect to $\mathcal{F}$. By Theorem 3.2, there is an element $F \in \mathcal{F}$ which is not finitely generated. Thus, there is $a_1 \in F$ such that $\langle a_1 \rangle \subsetneq F$. So, there exists $a_2 \in F \backslash \langle a_1 \rangle$ such that $\langle a_1, a_2 \rangle \subsetneq F$. By continuing this procedure, we come to a proper increasing chain of finitely generated elements of $\mathcal{F}$ ($\langle a_1 \rangle \subsetneq \langle a_1, a_2 \rangle \subsetneq \cdots$), which is a contradiction. Therefore, A is a Noetherian BL-algebra with respect to $\mathcal{F}$. $\square$

Definition 3.8. Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$. Then $\mathcal{F}$ is said to be multiplicative if for every two BL-algebras A and B, the following properties hold.

(i) If $F, G \in \mathcal{F}$, $F \subseteq A$ and $G \subseteq B$, then $F \times G \in \mathcal{F}$, where $F \times G \subseteq A \times B$.

(ii) If $F \times G$ is an element of $\mathcal{F}$ which is a subset of $A \times B$, then F and G are also elements of $\mathcal{F}$.

Theorem 3.7. *Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$. If $\mathcal{F}$ is multiplicative, then A_1 and A_2 are Noetherian with respect to $\mathcal{F}$ if and only if $A_1 \times A_2$ is Noetherian with respect to $\mathcal{F}$.*

Proof. Let $A_1 \times A_2$ be Noetherian with respect to $\mathcal{F}$. If $F_1 \subseteq F_2 \subseteq \cdots$, is a chain for A_1, then by multiplicity of $\mathcal{F}$, $F_1 \times \langle a \rangle \subseteq F_2 \times \langle a \rangle \subseteq \cdots$, is a chain for $A_1 \times A_2$, for any $a \in A_2$. Since $A_1 \times A_2$ is Noetherian, there exists $n \in \mathbb{N}$ such that $F_i \times \langle a \rangle = F_n \times \langle a \rangle$ for all $i \geq n$. Therefore, $F_i = F_n$ for all $i \geq n$ and A_1 is Noetherian with respect to $\mathcal{F}$. Similarly, we may prove that A_2 is Noetherian with respect to $\mathcal{F}$.

Conversely, let A_1 and A_2 be Noetherian BL-algebras with respect to $\mathcal{F}$ and $F_1 \times G_1 \subseteq F_2 \times G_2 \subseteq \cdots$, be a chain for $A_1 \times A_2$. Then by hypothesis, $F_1 \subseteq F_2 \subseteq \cdots$, and $G_1 \subseteq G_2 \subseteq \cdots$ are chains for A_1 and A_2, respectively. So, there exist $n, m \in \mathbb{N}$ such that $F_i = F_n$, $G_j = G_m$ for all $i \geq n$ and $j \geq m$. We set $k = \max\{n, m\}$, then $F_i \times G_i = F_k \times G_k$ for all $i \geq k$. Therefore, $A_1 \times A_2$ is Noetherian with respect to $\mathcal{F}$ (Artinian case can be treated similarly). $\square$

Corollary 3.1. *Let $\mathcal{A}$ be a family of all BL-algebras and $\mathcal{F}$ be a complete family for $\mathcal{A}$ which is multiplicative. If $A_1, A_2, \ldots, A_n$ are BL-algebras, then $A_1, A_2, \ldots, A_n$ are Noetherian (Artinian) with respect to $\mathcal{F}$ if and only if $A_1 \times A_2 \times \cdots \times A_n$ is Noetherian (Artinian) with respect to $\mathcal{F}$.*

Proof. Let $A_1 \times A_2 \times \cdots \times A_n$ be Noetherian with respect to $\mathcal{F}$. We complete the proof by induction on n. For $n = 1$, the induction holds. If $n = 2$, by Theorem 3.7, it is true. Let for $n = k$ it is true, i.e., if $A_1 \times A_2 \times \cdots \times A_k$ is Noetherian with respect to $\mathcal{F}$, then $A_1, A_2, \ldots, A_k$ are Noetherian with respect to $\mathcal{F}$. Let $B = A_1 \times A_2 \times \cdots \times A_k$, so $A_1 \times A_2 \times \cdots \times A_k \times A_{k+1} = B \times A_{k+1}$. By Theorem 3.7, $B \times A_{k+1}$ is Noetherian with respect to $\mathcal{F}$, thus B and A_{k+1} are Noetherian with respect to $\mathcal{F}$. Therefore, $A_1, A_2, \ldots, A_k, A_{k+1}$ are Noetherian with respect to $\mathcal{F}$.

Conversely, it is clear by induction on n and applying Theorem 3.7 (Artinian case with respect to $\mathcal{F}$ can be treated similarly). $\square$

Theorem 3.8. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A_1 and A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is a BL-homomorphism and A_1 is Noetherian (Artinian) with respect to $\mathcal{F}$, then $\psi\ (A_1)$ is also Noetherian (Artinian) with respect to $\mathcal{F}$.*

Proof. Let $\psi(F_1) \subseteq \psi(F_2) \subseteq \cdots$, be an increasing chain of elements of $\mathcal{F}$ for $\psi\ (A_1)$. Since A_1 is Noetherian with respect to family $\mathcal{F}$, and $F_1 \subseteq F_2 \subseteq \cdots$, is an increasing chain of elements of $\mathcal{F}$ for A_1, there exists $n \in \mathbb{N}$ such that $F_i = F_n$ for all $i \geq n$. Then $\psi\ (F_i) = \psi(F_n)$ for all $i \geq n$. So, $\psi\ (A_1)$ is Noetherian with respect to $\mathcal{F}$. Similarly, we may prove that if A_1 is Artinian with respect to $\mathcal{F}$, then so is $\psi\ (A_1)$. $\square$

Theorem 3.9. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A_1 and A_2 are two BL-algebras, $\psi : A_1 \to A_2$ is an onto BL-homomorphism and A_1 is Artinian with respect to $\mathcal{F}$, then A_2 is also Artinian with respect to $\mathcal{F}$.*

Proof. Let $F_1 \supseteq F_2 \supseteq \cdots$, be a decreasing chain of elements of $\mathcal{F}$ for A_2. Then, by Definition 3.6, $\psi^{-1}(F_1) \supseteq \psi^{-1}(F_2) \supseteq \cdots$, is a decreasing chain of elements of $\mathcal{F}$ for A_1. Since A_1 is Noetherian with respect to family $\mathcal{F}$, there exists $n \in \mathbb{N}$ such that $\psi^{-1}\ (F_i) = \psi^{-1}(F_n)$ for all $i \geq n$. By the fact that ψ is an onto BL-homomorphism, so $\psi\ (\psi^{-1}(F_i)) = \psi(\psi^{-1}(F_n))$ for all $i \geq n$. Hence, $F_i = F_n$ for all $i \geq n$, A_2 is Artinian with respect to $\mathcal{F}$. $\square$

Theorem 3.10. *Let $\mathcal{F}$ be a structural family for the family of all BL-algebras. If A is a BL-algebra, $\psi : A \to A$ is an onto BL-homomorphism and A is Noetherian with respect to $\mathcal{F}$, then ψ is an one-to-one BL-homomorphism.*

Proof. Let $\ker(\psi) \subseteq \ker(\psi^2) \subseteq \cdots$, be a chain of elements of $\mathcal{F}$ for A. Since A is Noetherian with respect to family $\mathcal{F}$, there exists $n \in \mathbb{N}$ such that $\ker(\psi^i) = \ker(\psi^n)$ for all $i \geq n$. Suppose $x \in \ker(\psi)$, then $\psi(x) = 1$. Since ψ and ψ^n are onto BL-homomorphisms, there exists $a \in A$ such that $x = \psi^n(a)$, so $\psi(x) = \psi^{n+1}(a) = 1$, i.e., $a \in \ker(\psi^{n+1}) = \ker(\psi^n)$. This means that $x = \psi^n(a) = 1$. Therefore, $\ker(\psi) = 1$ and ψ is an one-to-one BL-homomorphism. $\square$

Theorem 3.11. *Let $\mathcal{F}$ be a structural family for the family of the all BL-algebras. If A is a BL-algebra, $\psi : A \to A$ is an one to one BL-homomorphism and A is Artinian with respect to $\mathcal{F}$, then ψ is an onto BL-homomorphism.*

Proof. Suppose ψ is not an onto BL-homomorphism, i.e., $A \supset \psi(A)$. Since ψ is one to one, so $\psi(A) \supset \psi^2(A)$. We also have $\psi^{n-1}(A) \supset \psi^n(\mathrm{A})$ for all $n \geq 2$, i.e., $A \supset \psi(A) \supset \psi^2(A) \supset \cdots \supset \psi^n(A) \supset \cdots$ is a decreasing chain of elements of $\mathcal{F}$. This chain is not stationary, because, if there exists $k \in \mathbb{N}$ such that $\psi^{k+1}(A) = \psi^k(A)$, then by the injectivity of ψ, there exists a map $\varphi : A \to A$, $\varphi(\psi(A)) = I_A$, thus $\varphi(\psi^{k+1}(A)) = \varphi(\psi^k(A))$, i.e., $\psi^k(A) = \psi^{k-1}(A)$. By continuing this procedure, we get $\psi(A) = A$, which is a contradiction. Therefore, the chain is not stationary and hence A is not Artinian BL-algebra, which is a contradiction with hypothesis. Therefore, $A = \psi(A)$ and ψ is an onto BL-homomorphism. □

4. Conclusion

In BL-algebras (indeed in any algebraic structure), the results of chain conditions can be defined. Chain conditions are defined to study the properties of an algebraic structure. However, we must note that we can prove similar results for chain conditions. It is considerable that these results can be formulated in general structure to have a general approach to chain conditions. We may also define chain conditions with respect to partial order relation in our future work.

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