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MULTIVALUED FG-CONTRACTION MAPPINGS ON DIRECTED GRAPHS

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Abstract. In this paper, we study generalized $\mathbb{FG}$-contraction conditions for a pair of mappings defined on a family of subsets of a metric space endowed with a directed graph, and discuss coincidence and common fixed point results relaxing the continuity of mappings. The given notions and results are exemplified by suitable models. We apply our results to the problem of existence of solutions of a Fredholm integral inclusion.

1. Introduction and Preliminaries

Fixed point theory plays an important role not only in solving problems arising in science and technology but also other problems that come in various parts of life, by converting the problem into operator form. In the last decades, various approaches and techniques have been applied to get the solution. In particular, the concept of graph theory has been applied by Jachymski and Jozwik [10] to obtain fixed points of mappings acting on metric spaces equipped with directed graph. Gwozdz-Lukawska and Jachymski [9] discussed such problems for finite families of mappings.

We start recalling the terminology given in Jachymski [11].

Let (X, d) be a metric space and let Δ denote the diagonal of $X \times X$. Let $\mathcal{G} = (v_{\mathcal{G}}, e_{\mathcal{G}})$ be a directed graph where the set $v_{\mathcal{G}}$ of its vertices coincides with X, and the set $e_{\mathcal{G}}$ of edges contains all loops, that is, $\Delta \subseteq e_{\mathcal{G}}$. In addition, assume that the graph $\mathcal{G}$ has no parallel edges. The triplet $(X, d, \mathcal{G})$ is then called a directed graph metric space.

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If u and v are vertices of $\mathcal{G}$, then a path in $\mathcal{G}$ from u to v is a finite sequence $\{u_i\}$, $i \in \{0, 1, 2, \ldots, k\}$, of vertices such that $u_0 = u$, $u_k = v$ and $(u_{i-1}, u_i) \in e_{\mathcal{G}}$ for $i \in \{1, 2, \ldots, k\}$.

Recall that a graph $\mathcal{G}$ is called connected if there is a directed path between any two vertices, and it is called weakly connected if $\overline{\mathcal{G}}$ is connected, where $\overline{\mathcal{G}}$ denotes the undirected graph obtained from $\mathcal{G}$ by ignoring the direction of edges.

Fixed point results for single-valued mappings on directed graph metric spaces were first obtained by Jachymski in [11] and further generalized by various researchers. Some multivalued results of this kind were given by Abbas et al. in [1–3]. We recall some basic notions.

Let (X, d) be a metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. The Pompeiu-Hausdorff metric induced by d is the mapping $H : CB(X) \times CB(X) \to [0, +\infty)$ defined by

$$H(Z, W) = \max\{\sup_{v \in W} d(v, Z), \sup_{u \in Z} d(u, W)\},$$

for $Z, W \in CB(X)$, where $d(u, W) = \inf\{d(u, v) : v \in W\}$.

Lemma 1.1 ([14]). *Let (X, d) be a metric space. If $Z, W \in CB(X)$ are such that $H(Z, W) < \epsilon$, then for each $v \in Z$ there exists an element $u \in W$ such that $d(v, u) < \epsilon$.*

Definition 1.1 ([1]). Let $(X, d, \mathcal{G})$ be a directed graph metric space and let Z and W be two nonempty subsets of X. Then we say that:

(a) there is an edge between Z and W if there is an edge between some $u \in Z$ and $v \in W$;
(b) there is a path between Z and W if there is a path between some $u \in Z$ and $v \in W$;
(c) the graph $\mathcal{G}$ is said to be set-transitive if, for all $Z, W, V \in CB(X)$, whenever there is a path between Z and W and there is a path between W and V, then there is a path between Z and V.

Definition 1.2 ([12]). Let $P, Q : CB(X) \to CB(X)$ be two multivalued mappings. A set $Z \in CB(X)$ is said to be a coincidence point of P and Q, if $P(Z) = Q(Z)$. A set $Z \in CB(X)$ is said to be a fixed point of P if $P(Z) = Z$. The maps P, Q are said to be weakly compatible if they commute at their coincidence points.

We will denote by $Coin(P, Q)$ the set of all coincidence points of P and Q and by $Fix(P)$ the set of all fixed points of P.

A collection $\Lambda \subset CB(X)$ is said to be complete if for any sets $Z, W \in \Lambda$, there is an edge between Z and W.

Recall [1] that the space $(X, d, \mathcal{G})$ is said to have property (P^*), if for any sequence $\{Z_n\}$ in $CB(X)$ with $Z_n \to Z$ as $n \to +\infty$ (in the sense of Pompeiu-Hausdorff metric), the existence of an edge between Z_n and Z_{n+1} for each $n \in \mathbb{N}$, implies that there is a subsequence $\{Z_{n_k}\}$ of $\{Z_n\}$ with an edge between Z_{n_k} and Z for $k \in \mathbb{N}$.

The aim of this paper is to prove some coincidence and common fixed point results for a pair of (not necessarily continuous) multivalued generalized graphic $\mathbb{F}\mathbb{G}$-contractive mappings defined on the family of closed and bounded subsets of a directed graph metric space. These results extend and strengthen various comparable results in the existing literature (see, e.g., [1–7, 14, 19]). Application to Fredholm-type integral inclusions is presented. For basic notions in metrical fixed point theory see, e.g., [8, 13].

2. Generalized Graphic $\mathbb{F}\mathbb{G}$-Contractions

Parvaneh et al. [16] introduced and used the following classes of functions, modifying Wardowski's approach in [20].

$\mathfrak{F}$ is the set of all functions $\mathbb{F} : \mathbb{R}^+ \to \mathbb{R}$ such that

($\mathfrak{F}_1$) $\mathbb{F}$ is strictly increasing;

($\mathfrak{F}_2$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to+\infty} \xi_n = 0$ if and only if $\lim_{n\to+\infty} \mathbb{F}(\xi_n) = -\infty$.

$\mathfrak{G}_\beta$ is the set of pairs (G, β), where $G : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0, +\infty) \to [0, 1)$, such that

($\mathfrak{G}_{\beta 1}$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\limsup_{n\to+\infty} G(\xi_n) \geq 0$ if and only if $\limsup_{n\to+\infty} \xi_n \geq 1$;

($\mathfrak{G}_{\beta 2}$) for each sequence $\{\xi_n\} \subseteq [0, +\infty)$, $\limsup_{n\to+\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to+\infty} \xi_n = 0$;

($\mathfrak{G}_{\beta 3}$) for each sequence $\{\xi_n\} \subseteq \mathbb{R}^+$, $\sum_{n=1}^{+\infty} G(\beta(\xi_n)) = -\infty$.

Definition 2.1. Let $(X, d, \mathcal{G})$ be a directed graph metric space. The pair (P, Q) of maps $P, Q : CB(X) \to CB(X)$ is said to be a generalized graphic $\mathbb{F}G$-contraction if

(i) for every $Z \in CB(X)$ there is a path between Z and $P(Z)$, as well as between $Q(Z)$ and Z, and

(ii) there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ such that for all $Z, W \in v_{\mathcal{G}}$, with a path between them and $P(Z) \neq P(W)$,

$$\mathbb{F}(H(P(Z), P(W))) \leq \mathbb{F}(\Theta(Z, W)) + G(\beta(\Theta(Z, W))) \tag{2.1}$$

holds, where

$$\Theta(Z, W) = \max \left\{ \begin{array}{l} H(Q(Z), Q(W)), H(P(Z), Q(Z)), H(P(W), Q(W)), \\ \frac{1}{2}[H(P(Z), Q(W)) + H(P(W), Q(Z))] \end{array} \right\}.$$

Theorem 2.1. *Let $(X, d, \mathcal{G})$ be a directed graph metric space and $P, Q : CB(X) \to CB(X)$ be a pair of mappings. Assume the following conditions hold:*

(i) *$P(CB(X)) \subseteq Q(CB(X))$;*

(ii) *the graph $\mathcal{G}$ is set-transitive;*

(iii) *(P, Q) is a generalized graphic $\mathbb{F}\mathbb{G}$-contraction pair;*

(iv) *$Q(CB(X))$ is a complete subspace of $(CB(X), H)$, and*

(v) *$\mathcal{G}$ is weakly connected and property (P^*) holds.*

Then $Coin(P, Q) \neq \emptyset$.

Proof. Let $Z_0 \in CB(X)$ be arbitrary. Using (i), choose $Z_1 \in CB(X)$ such that $P(Z_0) = Q(Z_1)$. Proceeding in this way, if $Z_n \in CB(X)$ is chosen, we choose $Z_{n+1} \in CB(X)$ such that $P(Z_n) = Q(Z_{n+1})$ for $n \in \mathbb{N}$. Since there is a path between Z_n and $P(Z_n)$ and there is a path between $P(Z_n) = Q(Z_{n+1})$ and Z_{n+1}, it follows by (ii) that there is a path between Z_n and Z_{n+1} for each $n \in \mathbb{N}$.

Assume that $P(Z_n) \neq P(Z_{n+1})$ for all $n \in \mathbb{N}$. (If not, then $P(Z_n) = P(Z_{n+1})$ is true for some n, which implies that $Q(Z_{n+1}) = P(Z_n)$, and hence $Z_{n+1} \in Coin(P, Q)$.)

As there is a path between Z_n and Z_{n+1}, due to (iii), we have that

$$\begin{aligned}\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) &= \mathbb{F}(H(P(Z_n), P(Z_{n+1}))) \\ &\leq \mathbb{F}(\Theta(Z_n, Z_{n+1})) + G(\beta(\Theta(Z_n, Z_{n+1}))),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\Theta(Z_n, Z_{n+1}) &= \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(P(Z_n), Q(Z_n)), \\ H(P(Z_{n+1}), Q(Z_{n+1})), \\ \frac{1}{2}[H(P(Z_n), Q(Z_{n+1})) + H(P(Z_{n+1}), Q(Z_n))] \end{array}\right\} \\ &= \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_n)), \\ H(Q(Z_{n+2}), Q(Z_{n+1})), \\ \frac{1}{2}[H(Q(Z_{n+1}), Q(Z_{n+1})) + H(Q(Z_{n+2}), Q(Z_n))] \end{array}\right\} \\ &\leq \max\left\{\begin{array}{l} H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2})), \\ \frac{1}{2}[H(Q(Z_{n+2}), Q(Z_{n+1})) + H(Q(Z_{n+1}), Q(Z_n))] \end{array}\right\} \\ &= \max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) \\ \leq&\mathbb{F}(\max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\}) \\ &+ G(\beta(\max\{H(Q(Z_n), Q(Z_{n+1})), H(Q(Z_{n+1}), Q(Z_{n+2}))\})) \\ \leq&\mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) + G(\beta(H(Q(Z_n), Q(Z_{n+1})))),\end{aligned}$$

that is,

$$\mathbb{F}(H(Q(Z_{n+1}), Q(Z_{n+2}))) \leq \mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) + G(\beta(H(Q(Z_n), Q(Z_{n+1})))),$$

for all $n \in \mathbb{N}$. We conclude that

$$\begin{aligned}&\mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) \\ \leq&\mathbb{F}(H(Q(Z_{n-1}), Q(Z_n))) + G(\beta(\Theta(Z_{n-1}, Z_n))) \\ \leq&\mathbb{F}(H(Q(Z_{n-2}), Q(Z_{n-1}))) + G(\beta(\Theta(Z_{n-1}, Z_n))) + G(\beta(\Theta(Z_{n-2}, Z_{n-1}))) \\ &\vdots \\ \leq&\mathbb{F}(H(Q(Z_0), Q(Z_1))) + \sum_{i=1}^{n} G(\beta(\Theta(Z_{n-1}, Z_n))),\end{aligned}$$

that is,

$$\mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) \leq \mathbb{F}(H(Q(Z_0), Q(Z_1))) + \sum_{i=1}^{n} G(\beta(\Theta(Z_{n-1}, Z_n))),$$

for all $n \in \mathbb{N}$. By the properties of $(G, \beta) \in \mathfrak{G}_\beta$, $\lim_{n\to+\infty} \mathbb{F}(H(Q(Z_n), Q(Z_{n+1}))) = -\infty$ and by $(\mathfrak{F}_2)$, $\lim_{n\to+\infty} H(Q(Z_n), Q(Z_{n+1})) = 0$.

Next we claim that the sequence $\{Q(Z_n)\}$ is a Cauchy one. Suppose the contrary, which means that there exists an $\epsilon > 0$ and two increasing sequences of integers $\{p(\ell)\}$ and $\{q(\ell)\}$, $q(\ell) > p(\ell) > \ell$ such that $H(Q(Z_{p(\ell)}), Q(Z_{q(\ell)}))$, $H(Q(Z_{q(\ell)+1}), Q(Z_{p(\ell)-1}))$ and $H(Q(Z_{q(\ell)}), Q(Z_{p(\ell)-1}))$ tend to ϵ as $\ell \to +\infty$. Due to (iii) with $Z = Z_{p(\ell)-1}$ and $W = Z_{q(\ell)}$, we have

$$\begin{aligned} \mathbb{F}(H(Q(Z_{p(\ell)}), Q(Z_{q(\ell)+1}))) &= \mathbb{F}(H(P(Z_{p(\ell)-1}), P(Z_{q(\ell)}))) \\ &\leq \mathbb{F}(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)})) + G(\beta(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)}))), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &\Theta(Z_{p(\ell)-1}, Z_{q(\ell)}) \\ &= \max\left\{ \begin{array}{l} H(Q(Z_{p(\ell)-1}), Q(Z_{q(\ell)})), H(P(Z_{p(\ell)-1}), Q(Z_{p(\ell)-1})), \\ H(P(Z_{q(\ell)}), Q(Z_{q(\ell)})), \\ \frac{1}{2}[H(P(Z_{p(\ell)-1}), Q(Z_{q(\ell)})) + H(P(Z_{q(\ell)}), Q(Z_{p(\ell)-1}))] \end{array} \right\}. \\ &= \max\left\{ \begin{array}{l} H(Q(Z_{p(\ell)-1}), Q(Z_{q(\ell)})), H(Q(Z_{p(\ell)}), Q(Z_{p(\ell)-1})), \\ H(Q(Z_{q(\ell)+1}), Q(Z_{q(\ell)})), \\ \frac{1}{2}[H(Q(Z_{p(\ell)}), Q(Z_{q(\ell)})) + H(Q(Z_{q(\ell)+1}), Q(Z_{p(\ell)-1}))] \end{array} \right\}. \end{aligned} \tag{2.4}$$

Taking the limit as $\ell \to +\infty$ in (2.4), we have

$$\lim_{k\to+\infty} \Theta(Z_{p(\ell)-1}, Z_{q(\ell)}) = \max\{\epsilon, 0, 0, \tfrac{1}{2}(\epsilon + \epsilon)\} = \epsilon.$$

Taking the limit as $\ell \to +\infty$ in (2.3) we get

$$\begin{aligned} F(\epsilon) &\leq F(\limsup_{\ell\to+\infty} H(Q(Z_{p(l)}), Q(Z_{q(\ell)+1})) \\ &\leq \limsup_{\ell\to+\infty} \mathbb{F}(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)})) + \limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)}))), \\ &\leq \mathbb{F}(\epsilon) + \limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)}))), \end{aligned}$$

which further implies

$$\limsup_{\ell\to+\infty} \mathbb{G}(\beta(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)}))) \geq 0.$$

Using the properties of functions $\mathbb{G}$ and β, we get $\limsup_{\ell\to+\infty} \beta(\Theta(Z_{p(\ell)-1}, Z_{q(\ell)})) = 1$ and $\lim_{\ell\to+\infty} \Theta(Z_{p(\ell)-1}, Z_{q(\ell)}) = 0$, which is in contradiction with $\epsilon > 0$. Hence $\{Q(Z_n)\}$ is a Cauchy sequence in $Q(CB(X))$. Due to (iv), $Q(Z_n) \to D$ as $n \to +\infty$ for some $D \in CB(X)$. In addition, $Q(C) = D$ for some $C \in CB(X)$.

We argue that $P(C) = Q(C)$. If not, then, since there is a path between $Q(Z_{n+1})$ and $Q(Z_n)$ by the property (P^*), there exists a subsequence $\{Q(Z_{n_k+1})\}$ of $\{Q(Z_{n+1})\}$

such that there is a path between $Q(C)$ and $Q(Z_{n_k+1})$ for every $k \in \mathbb{N}$. As there is a path between C and $Q(C)$ and there is a path between $Q(Z_{n_k+1}) = P(Z_{n_k})$ and Z_{n_k}, we have that there is a path between C and Z_{n_k}. Using condition (iii), we get that

$$\begin{aligned}\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) &= \mathbb{F}(H(P(C), P(Z_{n_k}))) \\ &\le \mathbb{F}(\Theta(C, Z_{n_k})) + G(\beta(\Theta(C, Z_{n_k}))),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}&\Theta(C, Z_{n_k}) \\ &= \max\left\{ \begin{array}{l} H(Q(C), Q(Z_{n_k})), H(P(C), Q(C)), H(P(Z_{n_k}), Q(Z_{n_k})), \\ \frac{1}{2}[H(P(C), Q(Z_{n_k})) + H(P(Z_{n_k}), Q(C))] \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} H(C, Q(Z_{n_k})), H(P(C), Q(C)), H(Q(Z_{n_k+1}), Q(Z_{n_k})), \\ \frac{1}{2}[H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))] \end{array} \right\}.\end{aligned}$$

There are the following four possibilities.

- $\Theta(C, Z_{n_k}) = H(Q(C), Q(Z_{n_k}))$. From (2.5),

$$\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) = \mathbb{F}(H(P(C), P(Z_{n_k}))) + G(\beta(H(Q(C), Q(Z_{n_k})))).$$

Passing to the upper limit as $k \to +\infty$ gives

$$\mathbb{F}(H(P(C), Q(C))) \le \mathbb{F}(H(Q(C), Q(C))) + G(\beta(H(Q(C), Q(C)))),$$

which is a contradiction.

- When $\Theta(C, Z_{n_k}) = H(P(C), Q(C))$, then

$$\mathbb{F}(H(P(C), Q(C))) \le \mathbb{F}(H(P(C), Q(C))) + G(\beta(H(P(C), Q(C)))).$$

Therefore, $G(\beta(H(P(C), Q(C)))) \ge 0$, which implies that $\beta(H(P(C), Q(C)) \ge 1$, which is a contradiction.

- $\Theta(C, Z_{n_k}) = H(Q(Z_{n_k+1}), Q(Z_{n_k}))$. From (2.5),

$$\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) = \mathbb{F}(H(Q(Z_{n_k+1}), Q(Z_{n_k}))) + G(\beta(H(Q(Z_{n_k+1}), Q(Z_{n_k})))).$$

Passing to the upper limit as $k \to +\infty$, we have

$$\mathbb{F}(H(P(C), Q(C))) \le \mathbb{F}(H(Q(C), Q(C))) + G(\beta(H(Q(C), Q(C)))),$$

which is a contradiction.

- $\Theta(C, Z_{n_k}) = \dfrac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}$. From (2.5),

$$\begin{aligned}\mathbb{F}(H(P(C), Q(Z_{n_k+1}))) =&\mathbb{F}\left(\frac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}\right) \\ &+ G\left(\beta\left(\frac{H(P(C), Q(Z_{n_k})) + H(Q(Z_{n_k+1}), Q(C))}{2}\right)\right).\end{aligned}$$

Passing to the upper limit as $k \to +\infty$ gives

$$\begin{aligned}\mathbb{F}(H(P(C),Q(C))) \leq & \mathbb{F}\left(\frac{H(P(C),Q(C))+H(Q(C),Q(C))}{2}\right. \\ & \left. + G\left(\beta\left(\frac{H(P(C),Q(C))+H(Q(C),Q(C))}{2}\right)\right)\right. \\ = & \mathbb{F}\left(\frac{H(P(C),Q(C))}{2}\right)+G\left(\beta\left(\frac{H(P(C),Q(C))}{2}\right)\right) \\ < & \mathbb{F}\left(H(P(C),Q(C))\right)+G\left(\beta\left(\frac{H(P(C),Q(C))}{2}\right)\right),\end{aligned}$$

by the properties of $(G,\beta) \in \mathfrak{G}_\beta$, which is a contradiction.

Thus, in all cases we have $P(C) = Q(C)$, that is, $C \in Coin(P,Q)$. ☐

Theorem 2.2. *Let all of the conditions of Theorem* 2.1 *be satisfied. Then the following hold.*

(1) *If $Coin(P,Q)$ is a complete subgraph of X, then $H(P(C),P(D)) = 0$ for all $C,D \in Coin(P,Q)$.*
(2) *If, moreover, P and Q are weakly compatible, then they have a unique common fixed point in $CB(X)$.*
(3) *$Fix(P) \cap Fix(Q)$ is a complete subgraph of X if and only if P and Q have a unique common fixed point in $CB(X)$.*

Proof. Following the proof of Theorem 2.1, $Coin(P,Q) \neq \emptyset$.

In order to show (1), suppose that $C,D \in Coin(P,Q)$. Assume on contrary that $H(P(C),P(D)) \neq 0$. Due to (iii),

$$\mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(\Theta(C,D)) + G(\beta(\Theta(C,D))), \tag{2.6}$$

where

$$\begin{aligned}\Theta(C,D) &= \max\left\{\begin{array}{l} H(Q(C),Q(D)),H(P(C),Q(C)),H(P(D),Q(D)), \\ \frac{1}{2}[H(P(C),Q(D))+H(P(D),Q(C))] \end{array}\right\} \\ &= \max\left\{\begin{array}{l} H(P(C),P(D)),H(P(C),P(C)),H(P(D),Q(D)), \\ \frac{1}{2}H(P(C),P(D))+H(P(D),P(C)) \end{array}\right\} \\ &= H(P(C),P(D)).\end{aligned}$$

Thus,

$$\mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(H(P(C),P(D))) + G(\beta(H(P(C),P(D))))$$

by the properties of $(G,\beta) \in \mathfrak{G}_\beta$, a contradiction. Hence, we have derived that (1) holds.

In order to show (2), we start with $Fix(Q) \cap Fix(P) \neq \emptyset$. If $Y = P(C) = Q(C)$, then we have $Q(Y) = QP(C) = PQ(C) = P(Y)$, which shows that $Y \in Coin(P,Q)$. Thus, $H(P(C),P(Y)) = 0$ (by (1)). Hence $Y = P(Y) = Q(Y)$, that is, $Y \in$

$Fix(P) \cap Fix(Q)$. As $Coin(P,Q)$ contains exactly one element, the same is true for $Fix(P) \cap Fix(Q)$.

Finally, we show (3). Assume that $Fix(P) \cap Fix(Q)$ is a complete subgraph of X. In order to show that it contains only one element assume that there exist $C, D \in Fix(P) \cap Fix(Q)$ with $C \neq D$. By the assumption, there exists an edge between C and D. Due to (iii),

$$\mathbb{F}(H(C,D)) = \mathbb{F}(H(P(C),P(D))) \leq \mathbb{F}(\Theta(C,D)) + G(\beta(\Theta(C,D))),$$

where

$$\begin{aligned}\Theta(C,D) &= \max\left\{\begin{array}{l} H(Q(C),Q(D)), H(P(C),Q(C)), H(P(D),Q(D)), \\ \frac{1}{2}H(P(C),Q(D)) + H(P(D),Q(C)) \end{array}\right\} \\ &= \max\left\{H(C,D), H(C,C), H(D,D), \frac{1}{2}[H(C,D)+H(D,C)]\right\} \\ &= H(C,D).\end{aligned}$$

Thus,

$$\mathbb{F}(H(C,D)) \leq \mathbb{F}(H(C,D)) + G(\beta(H(C,D))),$$

which is a contradiction. Hence, $C = D$. Conversely, if $Fix(P) \cap Fix(Q)$ contains exactly one element, then since $e_{\mathcal{G}} \supseteq \Delta$, $Fix(P) \cap Fix(Q)$ is a complete subgraph of X. □

Remark 2.1. (a) Taking $\mathbb{F}(\xi) = G(\xi) = \ln \xi$ and $\beta(\xi) = k \in (0,1)$ (obviously $\mathbb{F} \in \mathfrak{F}$ and $(G,\beta) \in \mathfrak{G}_\beta$), Theorems 2 and 3, as well as Corollary 1 from the paper [3] follow as special cases of our results. In particular, all the results mentioned in Remark 1 of [3] can also be considered as corollaries of our results.

(b) Several other results can be obtained from Theorems 2.1 and 2.2 by taking various other possible choices for functions $\mathbb{F}$, G and β. We formulate just that, taking $F(\xi) = -1/\sqrt{\xi}$ and $G(\xi) = \ln \xi$, the condition (2.1) reduces to

$$H(P(Z),P(W)) \leq \frac{\Theta(Z,W)}{\left[1 - \sqrt{\Theta(Z,W)}\,\ln \beta(\Theta(Z,W))\right]^2}.$$

In particular, taking $\beta(\xi) = \text{const} \in (0,1)$ and denoting $\ln \beta = -\tau < 0$, the previous condition further reduces to

$$H(P(Z),P(W)) \leq \frac{\Theta(Z,W)}{\left[1 + \tau\sqrt{\Theta(Z,W}\right]^2}. \tag{2.7}$$

Example 2.1. Let $X = \{\alpha, \beta, \gamma\} = v_{\mathcal{G}}$, $e_{\mathcal{G}} = \{(\alpha,\alpha), (\beta,\beta), (\gamma,\gamma), (\alpha,\beta), (\alpha,\gamma), (\beta,\gamma)\}$ and $d : X \times X \to [0,+\infty)$ be defined by

$$\begin{aligned} d(\alpha,\beta) =& \frac{1}{3}, \quad d(\alpha,\gamma) = d(\beta,\gamma) = \frac{3}{4}, \\ d(u,u) =& 0 \text{ for } u \in X \quad \text{and} \quad d(u,v) = d(v,u) \text{ for } u,v \in X. \end{aligned}$$

Then $(X, d, \mathcal{G})$ is a directed graph metric space. Consider the following mappings $P, Q : CB(X) \to CB(X)$:

$$P(Z) = \begin{cases} \{\alpha\}, & \text{if } Z \subseteq \{\alpha, \beta\}, \\ \{\alpha, \beta\}, & \text{otherwise;} \end{cases} \quad Q(Z) = \begin{cases} \{\alpha\}, & \text{if } Z = \{\alpha\}, \\ \{\alpha, \beta\}, & \text{if } Z \in \{\{\beta\}, \{\alpha, \beta\}\}, \\ \{\alpha, \beta, \gamma\}, & \text{otherwise.} \end{cases}$$

Concerning conditions (i)-(v) of Theorem 2.1, just condition (iii) has to be checked in cases when $P(Z) \neq P(W)$. We will use version (2.7) with $\tau = \frac{1}{\sqrt{3}}$. The only possible such cases are when $Z \subseteq \{\alpha, \beta\}$, $W \ni \gamma$ or vice versa (they are symmetric, so just the first one will be considered).

In this case, $P(Z) = \{\alpha\}$, $P(W) = \{\alpha, \beta\}$, $Q(Z) = \begin{cases} \{\alpha\}, & \text{if } \beta \notin Z, \\ \{\alpha, \beta\}, & \text{if } \beta \in Z \end{cases}$, $Q(W) = \{\alpha, \beta, \gamma\}$. Hence, $H(P(Z), P(W)) = \frac{1}{3}$, $\Theta(Z, W) = H(P(W), Q(W)) = \frac{3}{4}$, and thus

$$\frac{\Theta(Z, W)}{[1 + \tau\sqrt{\Theta(Z, W})]^2} = \frac{\frac{3}{4}}{\left[1 + \frac{1}{\sqrt{3}}\sqrt{\frac{3}{4}}\right]^2} = \frac{1}{3} = H(P(Z), P(W)),$$

and condition (2.7) holds true.

Hence, by Theorem 2.1, $Coin(P, Q) \neq \emptyset$ (in fact, $\{\alpha\}$ is the unique coincidence point of P and Q). Since also conditions of Theorem 2.2 are satisfied, this is also the unique common fixed point of these mappings.

Remark 2.2. The presented example is a simplified version of Example 1 from the paper [3]. In a similar way, a simplified version of Example 2 from this paper can be constructed, showing that it is not necessary that the graph $(v_{\mathcal{G}}, e_{\mathcal{G}})$ be complete in order to obtain conclusions using results of this kind.

If $Q =$ (identity map on $CB(X)$) in (2.2), then we have the following consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let $(X, d, \mathcal{G})$ be a set-transitive directed graph metric space. Assume that $P : CB(X) \to CB(X)$ satisfies the following:*

(a) *there is a path between Z and $P(Z)$ for each Z in $CB(X)$;*
(b) *there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ and for all $Z, W \in X$ such that there is a path between Z and W, and $P(Z) \neq P(W)$,*
$$\mathbb{F}(H(P(Z), P(W))) \leq \mathbb{F}(\Theta(Z, W)) + G(\beta(\Theta(Z, W))),$$
holds, where
$$\Theta(Z, W) = \max \left\{ \begin{array}{c} H(Z, W), H(Z, P(Z)), H(W, P(W)), \\ \frac{1}{2}[H(Z, P(W)) + H(W, P(Z))] \end{array} \right\};$$
(c) *$\mathcal{G}$ is weakly connected and property (P^*) holds.*

Then we have the following conclusions:

(i) *P has a fixed point;*
(ii) *if $Fix(P)$ is a complete subgraph of X, then $H(C,D) = 0$, for all $C, D \in Fix(P)$;*
(iii) *$Fix(P)$ is a complete subgraph of $\mathcal{G}$ if and only if $Fix(P)$ has exactly one element.*

Assuming that the mappings P and Q are defined just on the subset of $CB(X)$ containing all singleton subsets of X (which is equivalent to assuming that they are defined on X), we obtain the following corollary of Theorems 2.1 and 2.2.

Corollary 2.2. *Let $(X, d, \mathcal{G})$ be a set-transitive directed graph metric space and $P, Q : X \to CB(X)$ be a pair of mappings. Assume the following conditions hold:*
(i) *$P(X) \subseteq Q(X)$;*
(ii) *for every $u \in X$, there is a path between $\{u\}$ and Pu, as well as between Qu and $\{u\}$;*
(iii) *there exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ such that for all $u, v \in X$ such that there is a path between $\{u\}$ and $\{v\}$ and $Pu \neq Pv$,*

$$\mathbb{F}(H(Pu, Pv)) \leq \mathbb{F}(\Theta(u, v)) + G(\beta(\Theta(u, v)))$$

holds, where

$$\Theta(u,v) = \max\left\{ \begin{array}{l} H(Qu,Qv), H(Pu,Qu), H(Pv,Qv), \\ \frac{1}{2}[H(Pu,Qv) + H(Pv,Qu)] \end{array} \right\};$$

(iv) *$\mathcal{G}$ is weakly connected and property (P^*) holds, and*
(v) *$Q(X)$ is a complete subspace of $(CB(X), H)$.*

Then there exists $u \in X$ such that $Pu = Qu$. Moreover,
(1) *if $Coin(P, Q)$ is a complete subgraph of X, then $H(Pu, Pv) = 0$ for all $u, v \in Coin(P, Q)$.*
(2) *if $Coin(P, Q)$ is a complete subgraph of X and P and Q are weakly compatible, then $Fix(P) \cap Fix(Q)$ contains exactly one element;*
(3) *$Fix(P) \cap Fix(Q)$ is a complete subgraph of X if and only if it contains exactly one element.*

Finally, assuming that the mapping P is defined just on X, we obtain the following from Corollary 2.1.

Corollary 2.3. *Let $(X, d, \mathcal{G})$ be a set-transitive complete directed graph metric space and $P : X \to CB(X)$ be a mapping. Assume the following conditions hold.*
(a) *There is a path between $\{u\}$ and Pu for each $u \in X$.*
(b) *There exist $\mathbb{F} \in \mathfrak{F}$ and $(G, \beta) \in \mathfrak{G}_\beta$ so that for all $u, v \in X$ such that there is a path between them, and $Pu \neq Pv$,*

$$\mathbb{F}(H(Pu, Pv)) \leq \mathbb{F}(\Theta(u, v)) + G(\beta(\Theta(u, v)))$$

holds, where

$$\Theta(u,v)=\max\left\{d(u,v),\delta(u,Pu),\delta(v,Pv),\frac{1}{2}[\delta(u,Pv)+\delta(v,Pu)]\right\}. \tag{2.8}$$

(c) *The graph* $\mathcal{G}$ *is weakly connected and satisfies the property* (P^*).

Then we have the following conclusions.

(i) *There is a point* $u\in X$ *such that* $Pu=\{u\}$.

(ii) *If* $Fix(P)$ *is a complete subgraph of* X*, then it contains exactly one element.*

Here, in (2.8), for $u\in X$ and $Z\subseteq X$,

$$\delta(u,Z)=\sup_{v\in Z}d(u,v)=H(\{u\},Z).$$

3. Application

In this section we are going to apply the obtained results to the problem of existence of solutions for a Fredholm-type integral inclusion. Problems of this kind were treated by several researchers, see, e.g., [15, 17, 18].

Consider the integral inclusion

$$v(t)\in\gamma(t)+\int_a^b M(t,s,v(s))\,ds,\quad t\in J=[a,b], \tag{3.1}$$

where $\gamma\in X=C[a,b]$ is a given function, $M\colon J\times J\times\mathbb{R}\to CB(\mathbb{R})$ is a given set-valued mapping and $v\in X$ is the unknown function. Here, $X=C[a,b]$ is the standard Banach space of continuous real functions with the maximum norm. We will consider the space X as endowed with the partial order $\preceq$ introduced by

$$u\preceq v \iff u(t)\le v(t),\quad \text{for all } t\in J,$$

where $u,v\in X$. We will say that u and v are comparable if $u\preceq v$ or $v\preceq u$ holds.

Consider the following assumptions.

(I) For each $v\in X$, the mapping $M_v : J^2\to CB(\mathbb{R})$, given by $M_v(t,s)=M(t,s,v(s))$, is continuous.

(II) For fixed $v\in X$ and for any sequence $\{m_n\}$ with $m_n(t,s)\in M_v(t,s)$, there exists a subsequence $\{m_{n_i}\}$ of $\{m_n\}$ such that $\{m_{n_i}\}$ converges for all $t,s\in J$ towards a function m with $m(t,s)\in M_v(t,s)$ as $i\to+\infty$ and, moreover, for every $t\in J$, $\int_a^b m_{n_i}(t,s)\,ds\to\int_a^b m(t,s)\,ds$, as $i\to+\infty$.

(III) For every $v\in X$ there is a function m_v, such that $m_v(t,s)\in M_v(t,s)$ for $t,s\in J$ and

$$v(t)\le\gamma(t)+\int_a^b m_v(t,s)\,ds,\quad t\in J.$$

(IV) There exists $\tau>0$ such that for all comparable $u,v\in X$ and for all m_u,m_v with $m_u(t,s)\in M_u(t,s)$ and $m_v(t,s)\in M_v(t,s)$ for $t,s\in J$,

$$|m_u(t,s)-m_v(t,s)|\le\frac{1}{b-a}\frac{|u(t)-v(t)|}{[1+\tau\sqrt{|u(t)-v(t)|}]^2} \tag{3.2}$$

holds for all $t, s \in J$.

Theorem 3.1. *Let the assumptions* (I)–(IV) *hold. Then the integral inclusion* (3.1) *has a solution in* X.

Proof. Let $P\colon X \to CB(X)$ be the operator given by

$$Pv = \left\{ u \in X : u(t) \in \gamma(t) + \int_a^b M(t, s, v(s))\, ds,\ t \in [a, b] \right\}.$$

Obviously, $v \in X$ is a solution of the inclusion (3.1) if and only if v is a fixed point of operator P.

We first check that the operator P is well-defined. Indeed, let $v \in X$ be arbitrary. By (I), the set-valued operator $M_v\colon J^2 \to CB(\mathbb{R})$ is continuous (w.r.t. Pompeiu-Hausdorff metric on $CB(\mathbb{R})$). From the Michael's selection theorem, it follows that there exists a continuous function $m_v : J^2 \to \mathbb{R}$ such that $m_v(t, s) \in M_v(t, s)$ for each $(t, s) \in J^2$. Hence, the function $u(t) = \gamma(t) + \int_a^b m_v(t, s)\, ds$ belongs to Pv, i.e., $Pv \neq \emptyset$. Since γ and M_v are continuous on J, resp. J^2, their ranges are bounded and hence Pv is bounded.

Let $v \in X$ be fixed, $\{v_n\}$ be a sequence in Pv and $v_n \to v \in X$. Then, there exists a sequence of functions $\{m_n\}$ such that $m_n(t, s) \in M_v(t, s)$ for $t, s \in J$, and

$$v_n(t) = \gamma(t) + \int_a^b m_n(t, s)\, ds, \quad t \in J.$$

By hypothesis (II), there exists a subsequence $\{m_{n_i}\}$ of $\{m_n\}$ such that $\{m_{n_i}\}$ converges for all $t, s \in J$ towards a function m as $i \to +\infty$, and, for every $t \in J, \int_a^b m_{n_i}(t, s)\, ds \to \int_a^b m(t, s)\, ds$, as $i \to +\infty$. As $M_v(t, s)$ is closed for all $t, s \in J$, then $m(t, s) \in M_v(t, s)$ for all $t, s \in J^2$. Besides,

$$v(t) = \lim_{i \to +\infty} v_{n_i}(t) = \gamma(t) + \int_a^b m(t, s)\, ds, \quad t \in J.$$

Thus, $v \in Pv$ and we have proved that images of P are closed subsets of X.

Hence, $P : X \to CB(X)$.

Consider the graph $\mathcal{G}$ with $v_{\mathcal{G}} = X$ and $e_{\mathcal{G}} = \{(u, v) \in X^2 : u \preceq v\}$. We have to check the conditions of Corollary 2.3.

(a) The assumption (III) assures that there is a path between $\{v\}$ and Pv for each $v \in X$.

(b) To see that P is a generalized graphic $\mathbb{FG}$-contraction, let $u, v \in X$ be comparable. Then, using the assumption (IV) and the fact that the function $\xi \mapsto \frac{\xi}{[1+\tau\sqrt{\xi}]^2}$

is increasing (which is easy to check), we get that

$$\begin{aligned}
\sup_{\varphi\in Pu} d(\varphi, Pv) &= \sup_{\varphi\in Pu}\inf_{\chi\in Pv} d(\varphi,\chi)\\
&= \sup_{\varphi\in Pu}\inf_{\chi\in Pv}\max_{t\in J}|\varphi(t)-\chi(t)|\\
&= \sup_{m_u\in M_u}\inf_{m_v\in M_v}\max_{t\in J}\left|\int_a^b [m_u(t,s)-m_v(t,s)]\,ds\right|\\
&\le \sup_{m_u\in M_u}\inf_{m_v\in M_v}\max_{t\in J}\int_a^b |m_u(t,s)-m_v(t,s)|\,ds\\
&\le \frac{1}{b-a}\max_{t\in J}\int_a^b \frac{|u(t)-v(t)|}{[1+\tau\sqrt{|u(t)-v(t)|}]^2}\,ds\\
&\le \frac{1}{b-a}\cdot\frac{\max_{t\in J}|u(t)-v(t)|}{[1+\tau\sqrt{\max_{t\in J}|u(t)-v(t)|}]^2}\int_a^b ds\\
&= \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2}.
\end{aligned}$$

Similarly, one can see that

$$\sup_{\chi\in Pv} d(\chi, Pu) \le \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2}.$$

Therefore, we have

$$H(Pu,Pv) \le \frac{d(u,v)}{[1+\tau\sqrt{d(u,v)}]^2} \le \frac{\Theta(u,v)}{[1+\tau\sqrt{\Theta(u,v)}]^2}.$$

Taking $\mathbb{F}(\xi) = -\frac{1}{\sqrt{\xi}}$, $G(\xi) = \ln\xi$ and $\beta(\xi) = e^{-\tau} \in (0,1)$, we get that P is a generalized graphic $\mathbb{F}\mathbb{G}$-contraction (see the inequality (2.7)).

(c) Let $\{u_n\}$ be a sequence in X with $u_n \to u$ as $n \to +\infty$ and let $u_n \preceq u_{n+1}$ for each $n \in \mathbb{N}$. Then obviously $u_n \preceq u$, i.e., $(u_n, u) \in e_{\mathcal{G}}$ holds for all $n \in \mathbb{N}$.

Thus, P satisfies all the conditions of Corollary 2.3, and so P has a fixed point, that is, the integral inclusion (3.1) has a solution in $X = C[a,b]$. □

Remark 3.1. Using other possibilities for $\mathbb{F} \in \mathfrak{F}$ an $(G,\beta) \in G_\beta$, the inequality (3.2) in the contractive condition (IV) of Theorem 3.1 can take various other forms. For example, taking $\mathbb{F}(\xi) = G(\xi) = \ln\xi$ and $\beta\colon [0,+\infty) \to [0,1)$ satisfying $(G_{\beta 2})$ and $(G_{\beta 3})$ (with $G = \ln$), (3.2) is replaced by

$$|m_u(t,s)-m_v(t,s)| \le \frac{1}{b-a}|u(t)-v(t)|\cdot\beta(|u(t)-v(t)|), \quad t,s\in J.$$

In particular, taking $\beta(\xi) = \frac{k}{b-a}$, with $k \in (0,1)$, we get that the following inequality is required:

$$|m_u(t,s) - m_v(t,s)| \leq \frac{k}{b-a}|u(t) - v(t)|, \quad t, s \in J. \tag{3.3}$$

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