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(ω, c)-ALMOST PERIODIC DISTRIBUTIONS

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Abstract. The aim of this work is the introduction of (w, c)-almost periodicity (resp. asymptotic (w, c)-almost periodicity) in distributions spaces. The characterizations and main properties of these distributions are given. We also study the existence of distributional (w, c)-almost periodic solutions of linear differential systems.

1. Introduction

The theory of almost periodicity was introduced by H. Bohr around 1925 and generalized by many other authors, see [3, 5].

The (ω, c)-almost periodicity of continuous functions and their Stepanov generalizations is introduced and studied recently by M. T. Khalladi, M. Kostić, A. Rahmani and D. Velinov.

Almost periodic distributions extending the classical Bohr and Stepanoff almost periodic functions are due to L. Schwartz, see [9]. Asymptotic almost periodicity of Schwartz distributions was introduced by I. Cioransescu [4].

This work is aimed to introduce and investigate (ω, c)-almost periodicity (resp. asymptotic (w, c)-almost periodicity) in the setting of Schwartz-Sobolev distributions.

The paper is organized as follows. In the second section, we recall the concept of (w, c)-almost periodicity which is a generalization of the classical notion of almost periodicity and give some of their fundamental properties. Next, we introduce the

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space $L^p_{w,c}$ of (w,c)-Lebesgue functions with exponent p, and then, in a similar way to L. Schwartz's work [9], we define the functional space $\mathcal{D}_{L^p_{w,c}}$ of all infinitely differentiable functions belonging to the space $L^p_{w,c}$ as well as each of their derivatives. Some properties of these spaces of (w,c)-functions are given. At the end of this section, we introduce the space of (w,c)-smooth almost periodic functions and analyze their basic properties. The third section is devoted to the study of (w,c)-almost periodic distributions (resp. asymptotically (w,c)-almost periodic distributions) by first defining the space $\mathcal{D}'_{L^p_{w,c}}$ as topological dual of $\mathcal{D}_{L^q_{w,c}}$, $\frac{1}{p}+\frac{1}{q}=1$. In particular, we study the space $\mathcal{B}'_{w,c}$ of (w,c)-bounded distributions. This space provides a general framework for our investigation of generalized (ω,c)-almost periodicity. We also give some characterizations of (w,c)-almost periodic distributions and their main properties. Finally, we apply our abstract theoretical results in the study of the existence of distributional (w,c)-almost periodic solutions of linear differential systems. Throughout the paper, we consider functions and distributions defined on the whole space of real numbers $\mathbb{R}$.

2. Smooth (w,c)-Almost Periodic Functions

In this section, we introduce the space of smooth (w,c)-almost periodic functions and investigate some of their basic properties. Denote by AP the well-known space of Bohr almost periodic functions on $\mathbb{R}$. We recall the definition and some properties of the space $AP_{w,c}$ of (ω,c)- almost periodic functions.

In the sequel we will use the following notations:

$$\varphi_{w,c}\left(\cdot\right)=c^{-\frac{(\cdot)}{w}}\varphi\left(\cdot\right),\ \varphi\in\mathcal{C}^{\infty}\text{ or }L^{p},1\leq p\leq+\infty,\ \text{ and }T_{w,c}=c^{-\frac{(\cdot)}{w}}T,\ T\in\mathcal{D}',\tag{2.1}$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

Definition 2.1. Let $c\in\mathbb{C}\backslash\{0\}$ and $w>0$. A complex-valued function f defined and continuous on $\mathbb{R}$ is called (w,c)-almost periodic if and only if $f_{\omega,c}\in AP$. Denote by $AP_{w,c}$ the set of all such functions.

When $c=1$ and $w>0$ arbitrary, $AP_{w,c}=AP$, the space of Bohr almost periodic functions.

The space $AP_{\omega,c}$ is a vector space together with the usual operations of addition and pointwise multiplication with scalars.

Some properties of (w,c)-almost periodic functions are summarized in the following proposition.

Proposition 2.1. (i) *The space $AP_{w,c}$ endowed with the (w,c)-norm*

$$\|f\|_{w,c}=\sup_{t\in\mathbb{R}}\left|f_{w,c}\left(t\right)\right|$$

is a Banach space.

(ii) *If $f\in AP_{w,c}$, then $\widetilde{f}\left(\cdot\right)=f\left(-\cdot\right)\in AP_{w,1/c}$.*

(iii) *If $w>0,c\in\mathbb{C}\backslash\left\{0\right\}$ such that $|c|=1$ and if $f\in AP_{w,c}$ such that $\inf_{x\in\mathbb{R}}\left|f\left(x\right)\right|>0$, then $1/f\in AP_{w,1/c}$.*

(iv) If $f \in AP_{w,c}$ *and* $g_{w,c} \in L^1$, *then* $f * g \in AP_{w,c}$.

To construct the (w, c)-smooth almost periodic functions, we need to introduce some new functional spaces. Let $p \in [1, +\infty]$ and f a complex valued measurable function on $\mathbb{R}$.

We say that f is a (w, c)-Lebesgue function with exponent p, if

$$\left(\int_{\mathbb{R}} |f_{w,c}(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \quad \text{for } 1 \leq p < +\infty,$$

and

$$\sup_{t \in \mathbb{R}} |f_{w,c}(t)| < \infty, \quad \text{for } p = +\infty.$$

We denote by $L^p_{w,c}$ the set of (w, c)-Lebesgue functions with exponent p, i.e.,

$$L^p_{w,c} := \{ f : \mathbb{R} \to \mathbb{C} \text{ measurable}, \; f_{w,c} \in L^p \}.$$

When $c = 1$, $L^p_{w,c} := L^p$ is the classical Lebesgue space over $\mathbb{R}$.

Proposition 2.2. *The space* $L^p_{w,c}$ *endowed with the* (w, c)*-norm*

$$\|f\|_{L^p_{w,c}} := \|f_{w,c}\|_{L^p}, \quad \textit{for } 1 \leq p < +\infty,$$

and

$$\|f\|_{L^\infty_{w,c}} := \|f\|_{w,c}, \quad \textit{for } p = +\infty,$$

is a Banach space.

Proposition 2.3. $\mathcal{D}$ *is dense in* $L^p_{w,c}$, $1 \leq p < \infty$.

Proof. Since $\mathcal{D}$ is dense in the space $\mathcal{C}_c$ of continuous functions with compact support it suffices to show that $\mathcal{C}_c$ is dense in $L^p_{w,c}$ for $1 \leq p < \infty$.

Let S be the set of all simple measurable functions s, with complex values, defined on $\mathbb{R}$ and such that

$$\text{mes}\{t : s(t) \neq 0\} < \infty.$$

First, it is clear that S is dense in $L^p_{w,c}$ for $1 \leq p < \infty$. Indeed, as $c^{-\frac{t}{w}} s \in L^p$, then $S \subset L^p_{w,c}$. Suppose $f \in L^p_{w,c}$ is positive and define the sequence $(s_n)_n$ such that $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, and for each $t \in \mathbb{R}$, $s_n(t) \to f(t)$ when $n \to +\infty$. Then $(f - s_n)_{w,c} = c^{-\frac{t}{w}} (f - s_n) \in L^p$, hence $s_n \in S$. Furthermore, since

$$\left| c^{-\frac{t}{w}} (f - s_n) \right|^p \leq f^p,$$

Lebesgue's dominated convergence theorem shows that

$$\left\| (f - s_n)_{w,c} \right\|_{L^p} = \left\| c^{-\frac{t}{w}} (f - s_n) \right\|_{L^p} \to 0,$$

when $n \to +\infty$. Hence, $\|f - s_n\|_{L^p_{w,c}} \to 0$ when $n \to +\infty$. On the other hand, by Lusin's theorem, for $s \in S$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}_c$ such that $g(t) = s(t)$, except on a set of measure less than ε and $|g| \leq \|s\|_{\infty}$ and since s takes only a finite number of values, there exists a constant $C > 0$ which depends on c and w such that

$$\left\|(g-s)_{w,c}\right\|_{L^p} = \left(\int_{\mathbb{R}} \left|c^{-\frac{t}{w}}\left(g\left(t\right)-s\left(t\right)\right)\right|^p dt\right)^{\frac{1}{p}} \leq 2C\varepsilon^{\frac{1}{p}}\left\|s\right\|_{\infty}.$$

The density of S in $L^p_{w,c}$ completes the proof. ☐

We define

$$\mathcal{D}_{L^p_{w,c}} := \left\{\varphi \in \mathcal{C}^{\infty} : \varphi_{w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{Z}_+\right\}.$$

When $c = 1$, we get $\mathcal{D}_{L^p_{w,c}} := \mathcal{D}_{L^p}$. Moreover, it is easy to show that the space $\mathcal{D}_{L^p_{w,c}}$, $1 \leq p \leq \infty$, endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,p;w,c} := \sum_{j\leq k}\left\|(\varphi_{w,c})^{(j)}\right\|_{L^p}, \quad k \in \mathbb{Z}_+,$$

is a Fréchet subspace of $\mathcal{C}^{\infty}$.

Proposition 2.4. *Let $1 \leq p \leq \infty$. If $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$.*

Proof. Let $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$. Then $\varphi_{2w,c} \in \mathcal{D}_{L^p}$ and $\psi_{2w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{Z}_+$. So, $\varphi^{(j)}_{2w,c} \in L^p$ and $\psi^{(j)}_{2w,c} \in L^p$. By Leibniz's rule, we obtain

$$\left(\left(\varphi\psi\right)_{w,c}\right)^{(j)} = \left(c^{-\frac{t}{2w}}\varphi c^{-\frac{t}{2w}}\psi\right)^{(j)} = \left(\varphi_{2w,c}\psi_{2w,c}\right)^{(j)} = \sum_{i=1}^{j}\binom{i}{j}\varphi^{(i)}_{2w,c}\psi^{(j-i)}_{2w,c} \in L^p. \tag{2.2}$$

This shows that$(\varphi\psi)_{w,c} \in \mathcal{D}_{L^p}$. Hence, $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$. ☐

The following result shows that the family of norms $|\cdot|_{k,p;w,c}$ is submultiplicative.

Proposition 2.5. *Let $1 \leq p \leq \infty$. If $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then for all $k \in \mathbb{Z}_+$, there exists $C_k > 0$ such that*

$$|\varphi\psi|_{k,p;w,c} \leq C_k\,|\varphi|_{k,p;2w,c}\cdot|\psi|_{k,p;2w,c}.$$

Proof. Let $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$. By Proposition 2.2-Proposition 2.4, we have

$$\begin{aligned}\sum_{j\leq k}\left\|\left(\left(\varphi\psi\right)_{w,c}\right)^{(j)}\right\|_{L^p} &= \sum_{j\leq k}\left\|\sum_{i=1}^{j}\binom{i}{j}\left(\varphi_{2w,c}\right)^{(i)}\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p}\\ &\leq \sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\varphi_{2w,c}\right)^{(i)}\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p}\\ &\leq \sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\varphi_{2w,c}\right)^{(i)}\right\|_{L^p}\sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\left\|\left(\psi_{2w,c}\right)^{(j-i)}\right\|_{L^p}.\end{aligned}$$

So, there exists $C_k = \left(\sum_{j\leq k}\sum_{i=1}^{j}\binom{i}{j}\right)^2 > 0$ such that

$$|\varphi\psi|_{k,p;w,c} \leq C_k\,|\varphi|_{k,p;2w,c}\cdot|\psi|_{k,p;2w,c}.$$

☐

For $1 \leq p < \infty$, we have $\mathcal{D} \subset \mathcal{D}_{L^p_{w,c}} \subset \mathcal{D}_{L^\infty_{w,c}}$. Moreover, we have the following result.

Proposition 2.6. *For $1 \leq p < \infty$, the space $\mathcal{D}$ is dense in $\mathcal{D}_{L^p_{w,c}}$.*

Proof. It follows from the fact that $\mathcal{D}_{L^p_{w,c}} \subset L^p_{w,c}$ and the density of $\mathcal{D}$ in $L^p_{w,c}$, see Proposition 2.3. □

The space $\mathcal{D}$ is not dense in $\mathcal{D}_{L^\infty_{w,c}}$, we then define $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ as the subspace of all functions in $\mathcal{D}_{L^\infty_{w,c}}$ which vanish at infinity with all their derivatives. This space is the closure of the space $\mathcal{D}_{L^\infty_{w,c}}$ in $\mathcal{D}$. It is clear that $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$, hence it is a Fréchet space. Moreover, it is easy to check the following properties on the structure of $\mathcal{D}_{L^p_{w,c}}$.

Proposition 2.7. *For $1 \leq p < \infty$, we have $\mathcal{D}_{L^p_{w,c}} \hookrightarrow \dot{\mathcal{D}}_{L^\infty_{w,c}} \hookrightarrow \mathcal{D}_{L^\infty_{w,c}}$, with continuous embedding.*

Recall also the following space of smooth almost periodic functions introduced by L. Schwartz

$$\mathcal{B}_{ap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP,\ j \in \mathbb{Z}_+ \right\}.$$

We have the following properties of $\mathcal{B}_{ap}$.

Proposition 2.8. (i) $\mathcal{B}_{ap} = AP \cap \mathcal{D}_{L^\infty}$.
(ii) $\mathcal{B}_{ap}$ is a closed differential subalgebra of $\mathcal{D}_{L^\infty}$.
(iii) If $f \in L^1$ and $\varphi \in \mathcal{B}_{ap}$, then $f * \varphi \in \mathcal{B}_{ap}$.

Proof. See [9]. □

Now, we can introduce the space of smooth (w, c)-almost periodic functions.

Definition 2.2. The space of smooth (w, c)-almost periodic functions on $\mathbb{R}$, is defined by

$$\mathcal{B}_{AP_{w,c}} := \left\{ \varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi_{w,c} \in \mathcal{B}_{ap},\ j \in \mathbb{Z}_+ \right\}.$$

We endow $\mathcal{B}_{AP_{w,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{w,c}}$. Some properties of $\mathcal{B}_{AP_{w,c}}$ are given in the following.

Proposition 2.9. (i) $\mathcal{B}_{AP_{w,c}} = AP_{w,c} \cap \mathcal{D}_{L^\infty_{w,c}}$.
(ii) $\mathcal{B}_{AP_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$.
(iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \in \mathcal{B}_{AP_{w,c}}$.

Proof. (i) Obvious.
(ii) It follows from (i) and the completeness of $\left(AP, \|\cdot\|_\infty \right)$.
(iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $f_{w,c} \in L^1$ and $\varphi_{w,c} \in \mathcal{B}_{ap}$. From Proposition 2.8, we have $f_{w,c} * \varphi_{w,c} \in \mathcal{B}_{ap}$, hence $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \right) \in \mathcal{B}_{ap}$, which shows that $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \in \mathcal{B}_{AP_{w,c}}$. □

Corollary 2.1. *If* $f \in \mathcal{D}_{L^{\infty}_{w,c}}$ *and* $c^{\frac{t}{w}}\left(f_{w,c} * \varphi_{w,c}\right) \in AP_{w,c}$, $\varphi \in \mathcal{D}$, *then* $f \in \mathcal{B}_{AP_{w,c}}$.

Remark 2.1. It is clear that $\mathcal{B}_{AP_{w,c}} \subset AP_{w,c} \cap \mathcal{C}^{\infty}$, whereas the converse inclusion is not true. Indeed, the function

$$f(t) = 2^{-t}\sqrt{2+\cos t+\cos\sqrt{2}t}$$

is an element of $AP_{w,c} \cap \mathcal{C}^{\infty}$, with $c = 2$ and $w = 1$. However,

$$f'(t) = 2^{-t}\left(\frac{-\sin t-\sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} - \ln 2\sqrt{2+\cos t+\cos\sqrt{2}t}\right)$$

is not bounded, because $\inf\limits_{t\in\mathbb{R}}\left(2+\cos t+\cos\sqrt{2}t\right) = 0$ and therefore

$$\frac{-\sin t-\sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} \notin AP,$$

hence $f \notin \mathcal{B}_{AP_{w,c}}$.

3. (w,c)-Almost Periodic Distributions

This section deals with the concept of (w,c)-almost periodicity in the setting of Sobolev-Schwartz distributions. For this we need to introduce the so-called space of $L^p_{w,c}$-distributions, $1 \leq p \leq \infty$. We first recall the space of L^p-distributions, $1 \leq p \leq \infty$, which has been introduced for the first time by L. Schwartz in [9]. L. Schwartz has introduced the space $\mathcal{D}'_{L^p}$ as topological dual of $\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$. These spaces is related to Sobolev spaces. For more details, see [1] and [9].

Definition 3.1. Let $1 < p \leq \infty$, the space $\mathcal{D}'_{L^p}$ is the topological dual of $\mathcal{D}_{L^q}$, where $\frac{1}{p}+\frac{1}{q}=1$. An element of $\mathcal{D}'_{L^\infty}$ is called a bounded distribution.

Theorem 3.1. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent.*
(*i*) $T \in \mathcal{D}'_{L^p}$.
(*ii*) $T * \varphi \in L^p$, $\varphi \in \mathcal{D}$.
(*iii*) *There exist* $k \in \mathbb{Z}_+$ *and* $\left(f_j\right)_{0\leq j\leq k} \subset L^p : T = \sum\limits_{j=0}^{k} f_j^{(j)}$.

Proof. See [1] or [9]. □

Thanks to the density of the space $\mathcal{D}$ in $\mathcal{D}_{L^p_{w,c}}$, $1 \leq p < \infty$, (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$), we have that the space $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) is a normal space of distributions, i.e., the elements of topological dual of $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) can be identified with continuous linear forms on $\mathcal{D}$.

Definition 3.2. For $1 < p \leq \infty$, we denote by $\mathcal{D}'_{L^p_{w,c}}$ the topological dual of $\mathcal{D}_{L^q_{w,c}}$, where $\frac{1}{p}+\frac{1}{q}=1$.

The following spaces of $L^p_{w,c}$-distributions are needed to define and study the (w,c)-almost periodicity of distributions.

Definition 3.3. (i) The topological dual of $\mathcal{D}_{L^1_{w,c}}$, denoted by $\mathcal{B}'_{w,c}$, is called the space of (w,c)-bounded distributions.

(ii) The topological dual of $\dot{\mathcal{D}}_{L^\infty_{w,c}}$, denoted by $\mathcal{D}'_{L^1_{w,c}}$, is called the space of (w,c)-integrable distributions.

By applying Theorem 3.1, we can easily show the following characterizations of $L^p_{w,c}$-distributions.

Theorem 3.2. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent.*

(i) $T \in \mathcal{D}'_{L^p_{w,c}}$.

(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in L^p_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0 \le j \le k} \subset L^p_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$, *where* $\left((f_{w,c})_j\right)_{0\le j\le k} = \left(c^{-\frac{t}{w}} f_j\right)_{0\le j\le k}$.

Remark 3.1. As a consequence of Theorem 3.2, we have that $T \in \mathcal{D}'_{L^p_{w,c}}$ if and only if $T_{w,c} \in \mathcal{D}'_{L^p}$.

Returning to the notation (2.1), we recall that a distribution $T \in \mathcal{D}'$ is zero on an open subset V of $\mathbb{R}$ if

$$\langle T, \varphi \rangle = 0, \quad \varphi \in \mathcal{D}(V),$$

and that two distributions $T, S \in \mathcal{D}'$ coincide on V if $T - S = 0$ on V.

Lemma 3.1. *Let $f \in \mathcal{C}^\infty$ and $T \in \mathcal{D}'$. If $fT = 0$, then $T = 0$ on the set $G = \{x \in \mathbb{R} : f(x) \neq 0\}$.*

Proof. Let $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subset G$. Then we have

$$\langle T, \varphi \rangle = \left\langle T, f\frac{\varphi}{f} \right\rangle = \left\langle fT, \frac{\varphi}{f} \right\rangle = 0,$$

because $\frac{\varphi}{f} \in \mathcal{D}$ and by hypothesis $fT = 0$. □

Proposition 3.1. *Let $T \in \mathcal{D}'$. Then $T \in \mathcal{D}'_{L^p_{w,c}}$, $1 \le p \le \infty$, if and only if there exists $S \in \mathcal{D}'_{L^p}$, $1 \le p \le \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.*

Proof. $(\Longrightarrow)$ If $T \in \mathcal{D}'_{L^p_{w,c}}$, then we have (see Remark 3.1) $T_{w,c} = c^{-\frac{t}{w}} T \in \mathcal{D}'_{L^p}$. So, there exists $S \in \mathcal{D}'_{L^p}$ such that $c^{-\frac{t}{w}} T - S = 0$ in $\mathcal{D}'_{L^p}$, i.e., $c^{-\frac{t}{w}} \left(T - c^{\frac{t}{w}} S\right) = 0$ in $\mathcal{D}'_{L^p}$. By applying Lemma 3.1, it follows that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.

$(\Longleftarrow)$ Suppose that $T \in \mathcal{D}'$ and there exists $S \in \mathcal{D}'_{L^p}$, $1 \le p \le \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$, then $c^{-\frac{t}{w}} T = S \in \mathcal{D}'_{L^p}$, hence $T \in \mathcal{D}'_{L^p_{w,c}}$. □

Recall that the space $\mathcal{B}'_{ap}$ of almost periodic distributions which was introduced and studied by L. Schwartz is based on the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h, denoted by $\tau_h T$, is defined as:

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle , \quad \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) = \varphi(x+h)$.

The following result gives the basic characterizations of Schwartz almost periodic distributions.

Theorem 3.3. *For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent.*

(i) *The set $\{\tau_h T : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.*

(ii) $T * \varphi \in AP$, $\varphi \in \mathcal{D}$.

(iii) *There exist $k \in \mathbb{Z}_+$ and $(f_j)_{0\leq j\leq k} \subset AP : T = \sum_{j=0}^{k} f_j^{(j)}$.*

Proof. See [9]. □

The following proposition summarizes the main properties of $\mathcal{B}'_{ap}$.

Proposition 3.2. (i) *If $T \in \mathcal{B}'_{ap}$, then $T^{(j)} \in \mathcal{B}'_{ap}$, $j \in \mathbb{Z}_+$.*

(ii) $\mathcal{B}_{ap} \times \mathcal{B}'_{ap} \subset \mathcal{B}'_{ap}$.

(iii) $\mathcal{B}'_{ap} * \mathcal{D}'_{L^1} \subset \mathcal{B}'_{ap}$.

Proof. See [9]. □

Now we will introduce the following concept.

Definition 3.4. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w,c)-almost periodic if and only if $T_{w,c} \in \mathcal{B}'_{ap}$, i.e., the set $\{\tau_h T_{w,c} : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of (w,c)-almost periodic distributions is denoted by $\mathcal{B}'_{AP_{w,c}}$.

Example 3.1. (i) The associated distribution of a (w,c)-almost periodic function (resp. Stepanov (p,w,c)-almost periodic function) is a (w,c)-almost periodic distribution, i.e.,

$$AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}} \quad (\text{resp. } S^p AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}}).$$

(ii) When $c = 1$ it follows that $\mathcal{B}'_{AP_{w,c}} := \mathcal{B}'_{ap}$.

Characterizations of (w,c)-almost periodic distributions are given in the following theorem.

Theorem 3.4. *Let $T \in \mathcal{B}'_{w,c}$. Then the following statements are equivalent.*

(i) $T \in \mathcal{B}'_{AP_{w,c}}$.

(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in AP_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) *There exist $k \in \mathbb{Z}_+$ and $(f_j)_{0\leq j\leq k} \subset AP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$, where $\left((f_{w,c})_j\right)_{0\leq j\leq k} = \left(c^{-\frac{t}{w}} f_j\right)_{0\leq j\leq k}$.*

Proof. Since for every $T \in \mathcal{B}'_{AP_{w,c}}$, we have $T_{w,c} \in \mathcal{B}'_{ap}$. Hence, the result follows immediately from Theorem 3.3. □

The main propreties of $\mathcal{B}'_{AP_{w,c}}$ are given in the following proposition.

Proposition 3.3. (i) *If* $T \in \mathcal{B}'_{AP_{w,c}}$, *then* $c^{\frac{t}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{AP_{w,c}}$, $j \in \mathbb{Z}_+$.

(ii) *If* $\varphi \in \mathcal{B}_{AP_{w,c}}$ *and* $T \in \mathcal{B}'_{AP_{w,c}}$, *then* $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.

(iii) *If* $T \in \mathcal{B}'_{AP_{w,c}}$ *and* $S \in \mathcal{D}'_{L^1_{w,c}}$, *then* $c^{\frac{t}{w}} (T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$.

Proof. (i) Obvious.

(ii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c} \in \mathcal{B}_{ap}$ and $T_{w,c} \in \mathcal{B}'_{ap}$. From Proposition 3.2(ii), we get $\varphi_{w,c}T_{w,c} \in \mathcal{B}'_{ap}$ and therefore $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (\varphi_{w,c}T_{w,c})\right) \in \mathcal{B}'_{ap}$, which gives $c^{\frac{t}{w}} (\varphi_{w,c}T_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$. Hence $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.

(iii) Let $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$. Then $T_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{D}'_{L^1}$. According to Proposition 3.2 (iii), we have $T_{w,c} * S_{w,c} \in \mathcal{B}'_{ap}$ and $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (T_{w,c} * S_{w,c})\right) \in \mathcal{B}'_{ap}$. Hence, $c^{\frac{t}{w}} (T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$. □

The following result shows that $\mathcal{B}_{AP_{w,c}}$ is dense in $\mathcal{B}'_{AP_{w,c}}$.

Proposition 3.4. *Let* $T \in \mathcal{B}'_{w,c}$. *Then* $T \in \mathcal{B}'_{AP_{w,c}}$ *if and only if there exists* $(\varphi_n)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{AP_{w,c}}$ *such that* $\lim\limits_{n\to+\infty} \varphi_n = T$ *in* $\mathcal{B}'_{w,c}$.

Proof. If $T \in \mathcal{B}'_{AP_{w,c}}$, then $T_{w,c} \in \mathcal{B}'_{ap}$ and from the density of $\mathcal{B}_{ap}$ in $\mathcal{B}'_{ap}$ there exists $(\psi_n)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{ap}$ such that

$$\lim_{n\to+\infty} \psi_n = T_{w,c} \text{ in } \mathcal{D}'_{L^\infty}.$$

This is equivalent to

$$c^{\frac{t}{w}} \lim_{n\to+\infty} \psi_n = \lim_{n\to+\infty} \left(c^{\frac{t}{w}} \psi_n\right) = c^{\frac{t}{w}} T_{w,c} = T \text{ in } \mathcal{B}'_{w,c}.$$

Hence, there exists $(\varphi_n)_{n\in\mathbb{Z}_+} = \left(c^{\frac{t}{w}} \psi_n\right)_{n\in\mathbb{Z}_+} \subset \mathcal{B}_{AP_{w,c}}$ such that

$$\lim_{n\to+\infty} \varphi_n = T \text{ in } \mathcal{B}'_{w,c}.$$ □

Now we will introduce the concept of asymptotic (w, c)-almost periodicity of distributions. M. Fréchet introduced the space $AAP(\mathbb{R}_+)$ of classical asymptotically almost periodic functions in [6] and proved the main properties of these functions. The space $AAP_{w,c}(\mathbb{R}_+)$ of asymptotically (w, c)-almost periodic functions were introduced recently by M. T. Khalladi, M. Kostić, A. Rahmani and D. Velinov. Asymptotically almost periodic Schwartz distributions have been introduced and studied by I. Cioranescu in [4]. We recall the definition and some properties of asymptotically almost periodic Schwartz distributions.

Definition 3.5. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called vanishing at infinity if
$$\lim_{h \to +\infty} \langle \tau_{-h} T, \varphi \rangle = 0 \quad \text{in } \mathbb{C},\ \varphi \in \mathcal{D}.$$
Denote by $\mathcal{B}'_{0+}$ the space of bounded distributions vanishing at infinity.

Definition 3.6. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called asymptotically almost periodic if there exist $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$. The space of asymptotically almost periodic Schwartz distributions is denoted by $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

Proposition 3.5. *If $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, then the decomposition $T = R + S$ on $\mathbb{R}_+$ is unique in $\mathcal{D}'_{L^\infty}$.*

Proof. See [4]. □

Set $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \operatorname{supp} \varphi \subset \mathbb{R}_+\}$. Then we have the following characterization of space $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

Theorem 3.5. *Let $T \in \mathcal{D}'_{L^\infty}$. Then the following assertions are equivalent.*

(i) $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) $T * \check{\varphi} \in AAP(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) = \varphi(-x)$.

(iii) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0 \le j \le k} \subset AAP(\mathbb{R}_+) : T = \sum_{j=0}^{k} f_j^{(j)}$ *on* $\mathbb{R}_+$.

Proof. See [4]. □

Asymptotic (w, c)-almost periodicity of distributions is introduced in the following.

Definition 3.7. Let $c \in \mathbb{C}$, $|c| \ge 1$ and $w > 0$. Then a distribution $T \in \mathcal{B}'_{w,c}$ is said asymptotically (w, c)-almost periodic if and only if $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. The space of asymptotically (w, c)-almost periodic distributions is denoted by $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

Remark 3.2. (i) When $c = 1$ it follows that $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+) := \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) The associated distribution of an asymptotically (w, c)-almost periodic function (resp. asymptotically Stepanov (p, w, c)-almost periodic function) is asymptotically (w, c)-almost periodic distribution.

Now let us define the space $\left(\mathcal{B}'_{w,c}\right)_{0+}$ of (w, c)-bounded distributions vanishing at infinity as follows.

Definition 3.8. Let $c \in \mathbb{C}$, $|c| \ge 1$ and $w > 0$. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c)-bounded distribution vanishing at infinity if and only if $T_{w,c} \in \mathcal{B}'_{0+}$.

We have the following result.

Theorem 3.6. *Let $c \in \mathbb{C}$, $|c| \ge 1$, $w > 0$ and $T \in \mathcal{B}'_{w,c}$. Then $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ if and only if there exist $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that*
$$T = R + S \quad \text{on } \mathbb{R}_+. \tag{3.1}$$

Proof. ($\Longrightarrow$) Let $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. Then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$ and by Definition 3.6, there exist $P \in \mathcal{B}'_{ap}$ and $Q \in \mathcal{B}'_{0+}$ such that $T_{w,c} = P + Q$ on $\mathbb{R}_+$. On the other hand, we have

$$\begin{aligned} T_{w,c} = c^{-\frac{t}{w}} T = P + Q &\Longrightarrow \left\langle c^{-\frac{t}{w}} T, \varphi \right\rangle = \langle P, \varphi \rangle + \langle Q, \varphi \rangle, \quad \varphi \in \mathcal{D} \\ &\Longrightarrow \langle T, \psi \rangle = \left\langle c^{\frac{t}{w}} P, \psi \right\rangle + \left\langle c^{\frac{t}{w}} Q, \psi \right\rangle, \quad \psi = c^{-\frac{t}{w}} \varphi \in \mathcal{D}. \end{aligned}$$

Thus, there exist $R = c^{\frac{t}{w}} P \in \mathcal{B}'_{AP_{w,c}}$ and $S = c^{\frac{t}{w}} Q \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$.

($\Longleftarrow$) If there exist $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R+S$ on $\mathbb{R}_+$, then $c^{-\frac{t}{w}} T = c^{-\frac{t}{w}} R + c^{-\frac{t}{w}} S$ on $\mathbb{R}_+$, i.e., $T_{w,c} = R_{w,c} + S_{w,c}$ on $\mathbb{R}_+$, where $R_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{B}'_{0+}$. Hence, $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, which shows that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. ☐

Proposition 3.6. *The decomposition* (3.1) *is unique in* $\mathcal{B}'_{w,c}$.

Proof. Suppose that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ is such that $T = R + S$ on $\mathbb{R}_+$, where $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$. Then the result follows from the proof of the implication ($\Longleftarrow$) of Theorem 3.6 and the uniqueness of the decomposition of asymptotically almost periodic distributions. ☐

Some characterizations of asymptotically (w, c)-almost periodic distributions are given in the following result.

Theorem 3.7. *Let* $c \in \mathbb{C}$, $|c| \geq 1$, $w > 0$ *and* $T \in \mathcal{B}'_{w,c}$. *The following assertions are equivalent.*

(*i*) $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

(*ii*) $c^{\frac{t}{w}} \left(T_{w,c} * \check{\varphi}\right) \in AAP_{w,c}(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) = \varphi(-x)$.

(*iii*) *There exist* $k \in \mathbb{Z}_+$ *and* $(f_j)_{0 \leq j \leq k} \subset AAP_{w,c}(\mathbb{R}_+) : T = c^{\frac{t}{w}} \sum\limits_{j=0}^{k} (f_{w,c})_j^{(j)}$ *on* $\mathbb{R}_+$, *where* $\left((f_{w,c})_j\right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j\right)_{0 \leq j \leq k}$.

Proof. It is clear that if $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$, then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. Thus, by applying Theorem 3.5 we obtain the result. ☐

4. Linear Differential Equations in $\mathcal{B}'_{AP_{w,c}}$

In this section we will study the existence of distributional (w, c)-almost periodic solutions of the following system of linear ordinary differential equations

$$T' = AT + S, \tag{4.1}$$

where $A = (a_{ij})_{1 \leq i,j \leq k}$, $k \in \mathbb{N}$, is a given square matrix of complex numbers, $S = (S_i)_{1 \leq i \leq k} \in (\mathcal{D}')^k$ is a vector distribution and $T = (T_i)_{1 \leq i \leq k}$ is the unknown vector distribution.

First, consider the system (4.1) with $S \in (AP)^k$ and let us recall the following result.

Theorem 4.1. *If the matrix A has no eigenvalues with real part zero, then for any $S \in (AP)^k$, there exists a unique solution $T \in (AP)^k$ of the system* (4.1).

Proof. See [5]. □

Let I_k be the unit matrix of order k. The following result gives the (w,c)-almost periodicity of the solution (if it exists) of the system (4.1).

Theorem 4.2. *Let $S \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. If the matrix $A - \frac{\log c}{w} I_k$ has no eigenvalues with real part zero, then the system* (4.1) *admits a unique solution $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ which is an (w,c)-almost periodic vector distribution.*

Proof. Let $\varphi \in \mathcal{D}$. We have

$$c^{-\frac{t}{w}} T' * \varphi = \left(c^{-\frac{t}{w}} T * \varphi\right)' + \frac{\log c}{w} c^{-\frac{t}{w}} T * \varphi. \tag{4.2}$$

On the other hand, if $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ satisfies system (4.1), then

$$c^{-\frac{t}{w}} T' * \varphi = A c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi.$$

So from (4.2), we have

$$\left(c^{-\frac{t}{w}} T * \varphi\right)' = \left(A - \frac{\log c}{w} I_k\right) c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi,$$

i.e.,

$$(T_{w,c} * \varphi)' = \left(A - \frac{\log c}{w} I_k\right) (T_{w,c} * \varphi) + S_{w,c} * \varphi, \tag{4.3}$$

where

$$T_{w,c} * \varphi = \left((T_{w,c})_i * \varphi\right)_{1 \le i \le k} = \left(\left(c^{-\frac{t}{w}} T_i\right) * \varphi\right)_{1 \le i \le k}$$

and

$$S_{w,c} * \varphi = \left((S_{w,c})_i * \varphi\right)_{1 \le i \le k} = \left(\left(c^{-\frac{t}{w}} S_i\right) * \varphi\right)_{1 \le i \le k}.$$

Then the system (4.3) is equivalent in $(\mathcal{C}^\infty)^k$ to the following system of differential equations

$$P' = BP + Q,$$

with $B = A - \frac{\log c}{w} I_k$, $P = T_{w,c} * \varphi \in (\mathcal{C}^\infty)^k$ and $Q = S_{w,c} * \varphi \in (AP)^k$. According to Theorem 4.1, it follows that there exists a unique bounded solution P which is almost periodic. Therefore, $(T_{w,c})_i * \varphi \in AP$, $1 \le i \le k$, $\varphi \in \mathcal{D}$, hence $c^{\frac{t}{w}} \left((T_{w,c})_i * \varphi\right) \in AP_{w,c}$, $1 \le i \le k$, $\varphi \in \mathcal{D}$. Thus, according to Theorem 3.4, we get $(T_i)_{1 \le i \le k} \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. □

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b-GENERALIZED SKEW DERIVATIONS ON MULTILINEAR POLYNOMIALS

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Abstract. Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and F, G be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $P \notin Z(R)$ such that

$$\big[P, [F(f(r)), f(r)]\big] = [G(f(r)), f(r)],$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:
(1) there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;
(2) there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax+\lambda x+xa$, $G(x) = bx+\mu x+xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

1. Introduction

Throughout the article R always denotes an associative ring with the center $Z(R)$, U denotes the Utumi quotient ring of ring R. The definition and axiomatic formulation of Utumi quotient ring U can be found in [5] and [10]. We notice that U is a prime ring with unity and $Z(U) = C$ is called the extended centroid of ring R. The extended centroid C is a field. For $x, y \in R$, the commutator of x and y is $xy - yx$ and it is denoted by $[x, y]$. Sometimes commutator of x and y is called Lie product of x and y. Let $S \subseteq R$, a function f on R is called centralizing (or commuting) function on S if $[f(s), s] \in Z(R)$ (or $[f(s), s] = 0$) for all $s \in S$. In this direction, Divinsky [17] studied the commuting automorphism on rings. More precisely, it is proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Mayne [9] generalized this result and proved that if R is

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a prime ring with a nontrivial centralizing automorphism then R is a commutative integral domain. Further, Posner [8] studied the centralizing derivations on prime rings. More precisely, he proved that there does not exist any non zero centralizing derivation on non commutative prime ring. This was the starting point for the research by several authors. By derivation, we mean an additive mapping d on R such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$, define a mapping f on R such that $f(x) = [a, x]$ for all $x \in R$. Here, we notice that f is a derivation on R. This kind of derivation is called an inner derivation induced by an element a. Derivation is called outer if it not an inner.

Brešar [13] extended the Posner's [8] result by taking two derivations and proved that if d and δ are two derivations of R with atleast one derivation is non zero, such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then R is commutative. Notable work has been done by several mathematicians to generalize these results on some appropriate subsets of prime ring R. Natural question will arise that what will happen if we replace x with multilinear polynomial in Posner's theorem [8] as well as Brešar's theorem [13] and in this direction many results have been done. One of these results in this direction is given by De Filippis and Wei [27] for skew derivation on multilinear polynomial. Note that an additive mapping d on R is said to be skew derivation associated with an automorphism α if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. It is natural to ask that what will happen if derivation replaced by generalized derivation. The notion of generalized derivation introduced by Brešar in [12] which is a generalization of derivation. An additive mapping F is said to be a generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Note that if R is a prime or a semiprime ring then the derivation d is uniquely determined by F and d is called the associated derivation of F.

Argaç and De Filippis [16] have given the partial generalization of Posner's theorem [8]. More precisely, they describe the structure of additive mapping satisfying the identity $F(f(r))f(r) - f(r)G(f(r)) = 0$ for all $r \in R^n$, where f is a multilinear polynomial and F, G are two generalized derivations on prime ring R. In 2018, Tiwari [19] studied the commuting generalized derivations on prime ring, which is generalization of the work of Argaç and De Filippis [16]. The generalization of Posner's theorem for generalized derivation on multilinear polynomial in [26] (where further generalization can be found in [1, 2, 20, 21]) is given below.

Let K be a commutative ring with unity, R be a prime algebra over K and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K, not central valued on R. Suppose that d is a non zero derivation and F is a non zero generalized derivation of R such that $d([F(f(r)), f(r)]) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;
(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.

2. b-Generalized Skew Derivation

Generalizations of derivations and generalized derivations are b-generalized derivations and b-generalized skew derivations. The definition of b-generalized derivation is given below which is from [14]. Let R be a prime ring and U be its Utumi ring of quotient. Let $b \in U$.

Definition 2.1. An additive mapping $F : R \to U$ is called b-generalized derivation of R if $F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$, where $d : R \to U$ is an additive map.

In [14] Košan and Lee proved that if R is a prime ring and $b \neq 0$ then the associated map d must be a derivation of R. Here, we see that a 1-generalized derivation is a generalized derivation. For some $a, b, c \in U$, define a map $F : R \to U$ as $F(x) = ax + bxc$ for all $x \in R$. This is a b-generalized derivation which is called b-generalized inner derivation.

Let α be an automorphism on R. This α is said to be an inner automorphism of R if there exists an invertible element $p \in U$ such that $\alpha(x) = pxp^{-1}$ for all $x \in R$ otherwise it is called outer automorphism. An additive mapping F on R is called generalized skew derivation associated with an automorphism α if there exists a skew derivation d on R such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$. Note that a skew derivation on R associated with an automorphism α is an additive mapping such that $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. A skew derivation associated with the identity automorphism is a derivation and generalized skew derivation associated with identity automorphism is a generalized derivation.

Let α be an inner automorphism on R, that is, $\alpha(x) = pxp^{-1}$ for some $p \in U$ and for all $x \in R$. Now by definition of generalized skew derivation associated with this α, we have $F(xy) = F(x)y + pxp^{-1}d(y)$ for all $x, y \in R$. If d is a skew inner derivation associated with same α, then we know that $d(x) = ax - \alpha(x)a = ax - pxp^{-1}a$. Thus we have $F(xy) = F(x)y + pxp^{-1}(ay - pyp^{-1}a)$, which implies that $F(xy) = F(x)y + pxp^{-1}ay - pxp^{-1}pyp^{-1}a = F(x)y + px\{p^{-1}ay - yp^{-1}a\}$. This gives that $F(xy) = F(x)y + pxd(y)$, where $d(y) = [p^{-1}a, y]$ for all $y \in R$, is an inner derivation induced by $p^{-1}a$. This implies that it is a p-generalized derivation on R. Thus, if α is an inner automorphism on R, then every generalized skew derivation on R is a b-generalized derivation.

The following definition given by De Filippis and Wei [28] is a generalization of above.

Definition 2.2. Let R be an associative ring, $b \in U$, $d : R \to R$ a linear mapping and α be an automorphism of R. A linear mapping $F : R \to R$ is said to be b-generalized skew derivation of R associated with an automorphism α if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$.

As par the above definition b-generalized skew derivations cover the concepts of derivations, generalized derivations, skew derivations, generalized skew derivations and b-generalized derivations. In the same article it is proved that if $b \neq 0$ and R

is a prime ring then the associated additive mapping d becomes a skew derivation associated with the same automorphism α. Further, it is proved that F can be extended to U and it assumes the form $F(x) = ax + bd(x)$, where $a \in U$.

Recently, Liu [6] generalized the result of Posner [8] by taking b-generalized derivation with Engel conditions on prime ring R.

More recently, Sharma et al. [18] studied an identity related to generalized derivations on prime ring with multilinear polynomial over C. More precisely, they proved the following.

Suppose that R is a prime ring of characteristic different from 2 with Utumi quotient ring U and $f(x_1, \ldots, x_n)$ is a non central multilinear polynomial over C. Let F and G be two generalized derivations of R and d a non zero derivation of R such that

$$d([F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = [G(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)],$$

for all $r_1, \ldots, r_n \in R$, then one of the following holds:

(a) there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;
(b) there exist $a, b \in U$ and $\lambda, \mu \in C$ such that $F(x) = ax + \lambda x + xa$, $G(x) = bx + \mu x + xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

Motivated by above result we prove our main theorem. In this case we take d to be an inner derivation and F, G are b-generalized skew derivations. More precisely, the statement of our main theorem is the following.

Theorem 2.1 (Main Theorem). *Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and F, G be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $P \notin Z(R)$ such that*

$$\Big[P, [F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]\Big] = [G(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)],$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*
(2) *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax+\lambda x+xa$, $G(x) = bx+\mu x+xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

The following corollaries are immediate consequence of our main result.

Corollary 2.1. *Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and G be a b-generalized skew derivation on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. If*

$$[G(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0,$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) *there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in R$;*

(2) *there exist* $a \in U$, $\lambda \in C$ *such that* $G(x) = ax + \lambda x + xa$ *for all* $x \in R$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

Corollary 2.2. *Let* R *be a prime ring of characteristic different from* 2 *with the center* $Z(R)$ *and* F *be a b-generalized skew derivations on* R. *Let* U *be Utumi quotient ring of* R *with the extended centroid* C *and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C *which is not central valued on* R. *If*

$$[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in Z(R),$$

for all $r = (r_1, \ldots, r_n) \in R^n$, *then one of the following holds:*

(1) *there exists* $\lambda \in C$ *such that* $F(x) = \lambda x$ *for all* $x \in R$;
(2) *there exist* $a \in U$, $\lambda \in C$ *such that* $F(x) = ax + \lambda x + xa$ *for all* $x \in R$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

If we take $F = d$, a skew derivation, then we get the following.

Corollary 2.3. *Let* R *be a prime ring of characteristic different from* 2 *and* d *be a skew derivation on* R *such that* $[d(x), x] \in Z(R)$ *for all* $x \in R$, *then either* $d = 0$ *or* R *is a commutative ring.*

Let α be any automorphism, then $\alpha - 1$ is a skew derivation. From above corollary we get $[(\alpha-1)(x), x] \in Z(R)$ which implies either R is commutative or α is an identity automorphism. Therefore we state the result of Mayne [9].

Corollary 2.4. *Let* R *be a prime ring of characteristic different from* 2 *and* α *be an automorphism on* R *such that* $[\alpha(x), x] \in Z(R)$ *for all* $x \in R$, *then either* α *is an identity automorphism or* R *is a commutative ring.*

3. Preliminaries and Notations

Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C. Then $f(x_1, \ldots, x_n)$ has the following form:

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},$$

where $\gamma_\sigma \in C$ and S_n be the symmetric group of n symbols.

If d is a skew derivation associated with an automorphism α then

$$\begin{aligned} d(\gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}) =& d(\gamma_\sigma) x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} \\ &+ \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)} \end{aligned}$$

and therefore

$$\begin{aligned} d(f(x_1, \ldots, x_n)) =& f^d(x_1, \ldots, x_n) \\ &+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)}, \end{aligned} \tag{3.1}$$

where $f^d(x_1, \ldots, x_n)$ is a multilinear polynomial originated from $f(x_1, \ldots, x_n)$ after replacing each coefficients γ_σ with $d(\gamma_\sigma)$. Similarly, we use $f^\alpha(x_1, \ldots, x_n)$ to denote a multilinear polynomial originated from $f(x_1, \ldots, x_n)$ after replacing each coefficients γ_σ with $\alpha(\gamma_\sigma)$. Let SD denotes the set of all skew derivations and SD_{in} denotes the set of all skew inner derivations of R.

Further, we will frequently use some important theory of generalized polynomial identities and differential identities. We recall some of the remarks.

Remark 3.1. If I is a two-sided ideal of R then R, I and U satisfy the same differential identities [23].

Remark 3.2. If I is a two-sided ideal of R then R, I and U satisfy the same generalized polynomial identities with coefficients in U [5].

Remark 3.3. Let R be a prime ring and $\alpha \in \mathrm{Aut}(R)$ be an outer automorphism of R. If $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates [29].

Remark 3.4. If $f(x_i, d(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, d is an outer skew derivation and α is an outer automorphism of R then R also satisfies the generalized polynomial identity $f(x_i, y_i, z_i)$, where x_i, y_i, z_i are distinct indeterminates ([4, Theorem 1], also see [29]).

Remark 3.5. If d is a non zero skew derivation of R, then the associated automorphism α is unique [11].

Remark 3.6. From [4] we can state the following result. Let R be a prime ring, d a non zero skew derivation on R and I a non zero ideal of R. If I satisfies the skew differential polynomial identity

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0,$$

for any $r_1, \ldots, r_n \in I$ then either

(i) I satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, x_1, \ldots, x_n) = 0$ or
(ii) d is skew U-inner.

Remark 3.7. Let R be a prime ring. Suppose $\sum_{i=1}^{n} a_i x b_i + \sum_{j=1}^{m} c_j x q_j = 0$ for all $x \in R$, where $a_i, b_i, c_j, q_j \in U$, $1 \le i \le n$ and $1 \le j \le m$. If $a_1, \ldots, a_n$ are C-independent, then each b_i is C-dependent on $q_1, \ldots, q_m$. Similarly, if $b_1, \ldots, b_n$ are C-independent, then each a_i is C-dependent on $c_1, \ldots, c_m$ (see [24, Lemma 1]).

4. F and G be b-Generalized Skew Inner Derivations

In this section, we study the situation when F and G are b-generalized skew inner derivations of R. Let $F(x) = ax + b\alpha(x)u$ and $G(x) = cx + b\alpha(x)v$ for all $x \in R$ and for some $a, b, c, u, v \in U$. Then we prove the following proposition.

Proposition 4.1. *Let R be a prime ring of characteristic different from 2, U be Utumi ring of quotient of R with the extended centroid C. Suppose F and G are b-generalized skew derivations defined as $F(x) = ax + bpxp^{-1}u$ and $G(x) = cx + bpxp^{-1}v$ for all $x \in R$ and for some $a, b, c, u, v, p \in U$ with invertible p. Let $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over C. If $P \in R$ be non central such that*

$$\Big[P, [F(f(r)), f(r)]\Big] = [G(f(r)), f(r)],$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following conditions holds:

1. *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*
2. *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax+\lambda x+xa$, $G(x) = bx+\mu x+xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

For proof of this proposition, we need the following.

Lemma 4.1. ([26, Lemma 1]). *Let C be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are non scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that each matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all non zero entries.*

Proposition 4.2. *Let $R = M_k(C)$, $k \geq 2$, be the ring of all $k \times k$ matrices over the infinite field C with characteristic different from 2. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2 + b'xq'x - Pxax - Pxqxq' - ax^2P - qxq'xP + xaxP + xqxc' - qxq''x - xcx - xqxq'' = 0$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then either q or q' or P is central.*

Proof. By our assumption

$$\begin{aligned} &a'f(r)^2 + b'f(r)q'f(r) - Pf(r)af(r) - Pf(r)qf(r)q' - af(r)^2P - qf(r)q'f(r)P \\ &+f(r)af(r)P + f(r)qf(r)c' - qf(r)q''f(r) - f(r)cf(r) - f(r)qf(r)q'' = 0, \end{aligned} \tag{4.1}$$

for all $r = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \in R$. We shall prove this result by contradiction. Suppose that $q \notin C$, $q' \notin C$ and $P \notin C$. Then by Lemma 4.1 there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(q)$, $\phi(q')$ and $\phi(P)$ have all non zero entries. Clearly $\phi(q)$, $\phi(q')$, $\phi(P)$, $\phi(a)$, $\phi(a')$, $\phi(b')$, $\phi(c)$, $\phi(c')$ and $\phi(q'')$ must satisfy the condition (4.1).

Let e_{ij} be the matrix whose (i, j)-entry is 1 and rest entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [23] (see also [25]), there exist $s_1, \ldots, s_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(s_1, \ldots, s_n) = \gamma e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Since ϕ is an automorphism, without loss of generality we write (4.1) after replacing $f(r_1, \ldots, r_n) = e_{ij}$

$$\begin{aligned} &a'e_{ij}^2 + b'e_{ij}q'e_{ij} - Pe_{ij}ae_{ij} - Pe_{ij}qe_{ij}q' - ae_{ij}^2P - qe_{ij}q'e_{ij}P \\ &+e_{ij}ae_{ij}P + e_{ij}qe_{ij}c' - qe_{ij}q''e_{ij} - e_{ij}ce_{ij} - e_{ij}qe_{ij}q'' = 0. \end{aligned}$$

It implies that

$$(4.2)\qquad \begin{aligned} &b'e_{ij}q'e_{ij} - Pe_{ij}ae_{ij} - Pe_{ij}qe_{ij}q' - qe_{ij}q'e_{ij}P \\ &+e_{ij}ae_{ij}P + e_{ij}qe_{ij}c' - qe_{ij}q''e_{ij} - e_{ij}ce_{ij} - e_{ij}qe_{ij}q'' = 0. \end{aligned}$$

Left and right multiplying by e_{ij} in (4.2), we obtain

$$-e_{ij}Pe_{ij}qe_{ij}q'e_{ij} - e_{ij}qe_{ij}q'e_{ij}Pe_{ij} = 0.$$

From this we have $2(P)_{ji}(q)_{ji}(q')_{ji}e_{ij} = 0$ or get $(P)_{ji}(q)_{ji}(q')_{ji}e_{ij} = 0$, since $\mathrm{char}(R) \neq 2$. It gives that either $(P)_{ji} = 0$ or $(q)_{ji} = 0$ or $(q')_{ji} = 0$, a contradiction, since P, q and q' have all non zero entries. Thus, we conclude that either q or q' or P is central. □

Proposition 4.3. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all matrices over the field C with characteristic different from 2 and $f(x_1, \ldots, x_n)$ a non central multilinear polynomial over C. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2 + b'xq'x - Pxax - Pxqxq' - ax^2P - qxq'xP + xaxP + xqxc' - qxq''x - xcx - xqxq'' = 0$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then either q or q' or P is central.*

Proof. The conclusions follow from Proposition 4.2 in the case of infinite field C. Now we assume that C is a finite field. Suppose that K is an infinite extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_n)$ such that

$$(4.3)\qquad \begin{aligned} Q(r_1, \ldots, r_n) =& a'f(r_1, \ldots, r_n)^2 + b'f(r_1, \ldots, r_n)q'f(r_1, \ldots, r_n) \\ &- Pf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) - Pf(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)q' \\ &- af(r_1, \ldots, r_n)^2P - qf(r_1, \ldots, r_n)q'f(r_1, \ldots, r_n)P \\ &+ f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)P + f(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)c' \\ &- qf(r_1, \ldots, r_n)q''f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)cf(r_1, \ldots, r_n) \\ &- f(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)q'' \end{aligned}$$

is a generalized polynomial identity for R. It is a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Hence the complete linearization of $Q(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ in $2n$ indeterminates, moreover $\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^nQ(r_1, \ldots, r_n)$. It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for both R and $\overline{R}$. By assumption $\mathrm{char}(R) \neq 2$ we obtain $Q(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 4.3. □

Lemma 4.2. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, which is not central valued on R. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2+b'xq'x-Pxax-Pxqxq'-ax^2P-qxq'xP+xaxP+xqxc'-qxq''x-xcx-xqxq'' = 0$*

for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in R, then either q or q' or P is central.

Proof. We shall prove this by contradiction. Suppose that none of q, q' and P is in C. By hypothesis, we have

$$\begin{aligned} h(x_1,\dots,x_n) =& a'f(x_1,\dots,x_n)^2 + b'f(x_1,\dots,x_n)q'f(x_1,\dots,x_n) \\ & - Pf(x_1,\dots,x_n)af(x_1,\dots,x_n) - Pf(x_1,\dots,x_n)qf(x_1,\dots,x_n)q' \\ & - af(x_1,\dots,x_n)^2P - qf(x_1,\dots,x_n)q'f(x_1,\dots,x_n)P \\ & + f(x_1,\dots,x_n)af(x_1,\dots,x_n)P + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)c' \\ & - qf(x_1,\dots,x_n)q''f(x_1,\dots,x_n) - f(x_1,\dots,x_n)cf(x_1,\dots,x_n) \\ & - f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q'', \end{aligned} \tag{4.4}$$

for all $x_1,\dots,x_n \in R$. Since R and U satisfy same generalized polynomial identity (GPI) (see [5]), U satisfies $h(x_1,\dots,x_n) = 0_T$. Suppose that $h(x_1,\dots,x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1,\dots,x_n\}$, the free product of U and $C\{x_1,\dots,x_n\}$, the free C-algebra in non commuting indeterminates $x_1,\dots,x_n$. Then $h(x_1,\dots,x_n)$ is zero element in $T = U *_C C\{x_1,\dots,x_n\}$. Since neither q nor q' nor P is central, hence the term

$$-Pf(x_1,\dots,x_n)qf(x_1,\dots,x_n)q' - qf(x_1,\dots,x_n)q'f(x_1,\dots,x_n)P$$

appears nontrivially in $h(x_1,\dots,x_n)$. Thus, U satisfies

$$-Pf(x_1,\dots,x_n)qf(x_1,\dots,x_n)q' - qf(x_1,\dots,x_n)q'f(x_1,\dots,x_n)P = 0_T.$$

Since $P \notin C$, hence it implies that $Pf(x_1,\dots,x_n)qf(x_1,\dots,x_n)q' = 0$. This gives a contradiction, i.e., we have either $P \in C$ or $q' \in C$ or $q \in C$.

Next, suppose that $h(x_1,\dots,x_n)$ is a non trivial GPI for U. In case C is infinite, we have $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [22, Theorem 2.5 and Theorem 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in R$. By Martindale's theorem [30], R is then a primitive ring with non zero socle $\mathrm{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [15, page 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1,\dots,r_n)$ is not central valued on R, R must be non commutative and so $m \geq 2$. In this case, by Proposition 4.3, we get that either $P \in C$ or $q' \in C$ or $q \in C$, a contradiction.

Next we suppose that V is infinite dimensional over C. By Martindale's theorem [30, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since we have assumed that neither P nor q nor q' is in the center. Then there exist $h_1, h_2, h_3 \in \mathrm{soc}(R)$ such that $[P, h_1] \neq 0$, $[q, h_2] \neq 0$ and

$[q', h_3] \neq 0$. By Litoff's Theorem [7], there exists an idempotent $e \in \text{soc}(R)$ such that $Ph_1, h_1P, qh_2, h_2q, q'h_3, h_3q', h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned}
&e\Big\{a'f(ex_1e,\ldots,ex_ne)^2 + b'f(ex_1e,\ldots,ex_ne)q'f(ex_1e,\ldots,ex_ne)\\
&- Pf(ex_1e,\ldots,ex_ne)af(ex_1e,\ldots,ex_ne) - Pf(ex_1e,\ldots,ex_ne)qf(ex_1e,\ldots,ex_ne)q'\\
&- af(ex_1e,\ldots,ex_ne)^2P - qf(ex_1e,\ldots,ex_ne)q'f(ex_1e,\ldots,ex_ne)P\\
&+ f(ex_1e,\ldots,ex_ne)af(ex_1e,\ldots,ex_ne)P + f(ex_1e,\ldots,ex_ne)qf(ex_1e,\ldots,ex_ne)c'\\
&- qf(ex_1e,\ldots,ex_ne)q''f(ex_1e,\ldots,ex_ne) - f(ex_1e,\ldots,ex_ne)cf(ex_1e,\ldots,ex_ne)\\
&- f(ex_1e,\ldots,ex_ne)qf(ex_1e,\ldots,ex_ne)q''\Big\}e,
\end{aligned}$$

the subring eRe satisfies

$$\begin{aligned}
&\Big\{ea'ef(x_1,\ldots,x_n)^2 + eb'ef(x_1,\ldots,x_n)eq'ef(x_1,\ldots,x_n)\\
&- ePef(x_1,\ldots,x_n)eaef(x_1,\ldots,x_n) - ePef(x_1,\ldots,x_n)eqef(x_1,\ldots,x_n)eq'e\\
&- eaef(x_1,\ldots,x_n)^2ePe - eqef(x_1,\ldots,x_n)eq'ef(x_1,\ldots,x_n)ePe\\
&+ f(x_1,\ldots,x_n)eaef(x_1,\ldots,x_n)ePe + f(x_1,\ldots,x_n)eqef(x_1,\ldots,x_n)ec'e\\
&- eqef(x_1,\ldots,x_n)eq''ef(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)ecef(x_1,\ldots,x_n)\\
&- f(x_1,\ldots,x_n)eqef(x_1,\ldots,x_n)eq''e\Big\}.
\end{aligned}$$

Then by the above finite dimensional case, either ePe or eqe or $eq'e$ is central element of eRe. Thus either $Ph_1 = (ePe)h_1 = h_1ePe = h_1P$ or $qh_2 = (eqe)h_2 = h_2(eqe) = h_2q$ or $q'h_3 = (eq'e)h_3 = h_3(eq'e) = h_3q'$, a contradiction. □

Lemma 4.3. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let F and G are mappings defined as $F(x) = ax + bxu$, $G(x) = cx + bxv$ for some $a, b, c, u, v \in R$. Let $P \in R$ be non central such that $\Big[P, [F(x), x]\Big] = [G(x), x]$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\ldots,x_n)$ in R, then either b is central or u is central.*

Proof. By applying similar argument as we have used in Lemma 4.2, we get our desired result. □

Remark 4.1. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let $P, p, q \in R$ and P non central be such that $\Big[P, [p, x]\Big] = [q, x]$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\ldots,x_n)$ in R, then p and q are central.

Proof. Similar as proof of Lemma 4.2. □

Remark 4.2. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let $a\in R$ be such that $af(r_1,\dots,r_n)\in C$ for all $r_1,\dots,r_n\in R$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in R, then $a=0$.

Proof. By applying similar argument as we have used in Lemma 4.2 we get $a\in C$. If $a\neq 0$ then from $af(x_1,\dots,x_n)\in C$, we get $f(x_1,\dots,x_n)\in C$ for all $x_1,\dots,x_n\in R$, a contradiction. Therefore, we must have $a=0$. □

Now we are in position to prove Proposition 4.1.

Proof of Proposition 4.1. From Lemma 4.2, we get either $P\in C$ or $bp\in C$ or $p^{-1}u\in C$. Since $P\notin C$, we shall study following cases.

Case-I. If $bp\in C$ then $F(x)=ax+xbu$ and $G(x)=cx+xbv$ are generalized inner derivations. By [18, Lemma 3.6] we get our conclusions.

Case-II. If $p^{-1}u\in C$ then $F(x)=(a+bu)x=u'x$, $G(x)=cx+bpxp^{-1}v=cx+qxq''$, where $u'=a+bu, q=bp, q''=p^{-1}v$. Then from $\Big[P,[F(x),x]\Big]=[G(x),x]$, R satisfies the generalized polynomial identity $\theta(x_1,\dots,x_n)$ which can be written as

$$\begin{aligned}\theta(x_1,\dots,x_n)=&Pu'f(x_1,\dots,x_n)^2-Pf(x_1,\dots,x_n)u'f(x_1,\dots,x_n)\\&-u'f(x_1,\dots,x_n)^2P+f(x_1,\dots,x_n)u'f(x_1,\dots,x_n)P\\&-cf(x_1,\dots,x_n)^2-qf(x_1,\dots,x_n)q''f(x_1,\dots,x_n)\\&+f(x_1,\dots,x_n)cf(x_1,\dots,x_n)+f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q''.\end{aligned}\tag{4.5}$$

If $\theta(x_1,\dots,x_n)$ is a trivial generalized polynomial identity for R then each of the following is a trivial generalized polynomial identity for R:

- $Pu'f(x_1,\dots,x_n)^2-Pf(x_1,\dots,x_n)u'f(x_1,\dots,x_n)-u'f(x_1,\dots,x_n)^2P$ $+f(x_1,\dots,x_n)u'f(x_1,\dots,x_n)P$;
- $-cf(x_1,\dots,x_n)^2+f(x_1,\dots,x_n)cf(x_1,\dots,x_n)$;
- $-qf(x_1,\dots,x_n)q''f(x_1,\dots,x_n)+f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q''$.

Therefore, we must have $u'\in C$, $c\in C$ and $q,q''\in C$. In this case we get our conclusion.

If $\theta(x_1,\dots,x_n)$ is a non trivial generalized polynomial identity for R then by Matindale's theorem [30] U is a primitive ring having non zero socle with the field C as its associated division ring. By [15, page 35] U is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing non zero linear transformations of finite rank. Assume first that $\dim_C V=k\geq 2$ is a finite positive integer, then $U\cong M_k(C)$ and the conclusion follows from Proposition 4.3.

Now suppose that $\dim_C V=\infty$. Then the set $f(U)=\{f(r_1,\dots,r_n): r_i\in U\}$ is dense on U, see [31, Lemma 2]. By the fact that $\theta(x_1,\dots,x_n)=0$ is a generalized polynomial identity for U, therefore U satisfies the generalized polynomial identity

$$Pu'x^2-Pxu'x-u'x^2P+xu'xP=cx^2+qxq''x-xcx-xqxq''.\tag{4.6}$$

In (4.6) replace x by $x+1$ we get

$$Pu'x - Pxu' - u'xP + xu'P - cx - qq''x + xc + xqq'' = 0, \tag{4.7}$$

for all $x \in U$. Replace x by xy in the expression (4.7) we get

$$Pu'xy - Pxyu' - u'xyP + xyu'P - cxy - qq''xy + xyc + xyqq'' = 0, \tag{4.8}$$

for all $x, y \in U$. Now multiply from right side by y in expression (4.7) we get

$$Pu'xy - Pxu'y - u'xPy + xu'Py - cxy - qq''xy + xcy + xqq''y = 0. \tag{4.9}$$

Comparing (4.8) and (4.9) we get

$$Px[u', y] + u'x[P, y] + x[-u'P - c - qq'', y] = 0, \tag{4.10}$$

for all $x, y \in U$. By Remark 3.7 either $u' \in C$ or there exist λ_y, μ_y depending on y such that $[P, y] = \lambda_y[u', y]$ and $[-u'P - c - qq'', y] = \mu_y[u', y]$. If $u' \in C$ then by [3, Main theorem] we get our conclusions. If $u' \notin C$ then there is some $y_0 \in U$ such that $[u', y_0] \neq 0$. Therefore, we have $[P, y_0] = \lambda_{y_0}[u', y_0]$ and $[-u'P-c-qq'', y_0] = \mu_{y_0}[u', y_0]$. Substituting these values in (4.10) we get

$$Px[u', y_0] + u'x\lambda_{y_0}[u', y_0] + x\mu_{y_0}[u', y_0] = (P + u'\lambda_{y_0} + \mu_{y_0})x[u', y_0] = 0,$$

by primeness of U we get $P + u'\lambda_{y_0} + \mu_{y_0} = 0$. We note that $\lambda_{y_0} \neq 0$ otherwise $P \in C$, a contradiction. Substituting the value of P in (4.10) we get

$$2\lambda_{y_0}u'x[u', y] + x[-\lambda_{y_0}u'^2 - c - qq'', y] = 0.$$

Again by Remark 3.7 there exists η_y depending on y such that $[-\lambda_{y_0}u'^2 - c - qq'', y] = \eta_y[u', y]$. Since $u' \notin C$ there is some y_0' such that $[u', y_0'] \neq 0$. For fixed $\eta_{y_0'}$ we have $[-\lambda_{y_0}u'^2 - c - qq'', y_0'] = \eta_{y_0'}[u', y_0']$. Thus, we get $(2\lambda_{y_0}u' + \eta_{y_0'})x[u', y_0'] = 0$ for all $x \in U$. The primeness of U gives $2\lambda_{y_0}u' + \eta_{y_0'} = 0$. Since char$R \neq 2$ and $\lambda_{y_0} \neq 0$ we get $u' \in C$, a contradiction. □

Proposition 4.4. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R. Suppose that F and G are b-generalized skew derivations associated with an outer automorphism α defined as $F(x) = ax + b\alpha(x)u$, $G(x) = cx + b\alpha(x)v$ for all $x \in f(R)$ and for some $a, b, c, u, v \in R$. Let $P \in R$ be non central element of R such that $\Big[P, [F(f(r)), f(r)]\Big] = [G(f(r)), f(r)]$ for all $f(r) \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then one of the following holds:*

1. *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*
2. *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax + \lambda x + xa$, $G(x) = bx + \mu x + xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Proof. From the given hypothesis we get

$$\begin{aligned}&\Big[P, [af(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))u, f(x_1, \ldots, x_n)]\Big] \\ =&[cf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))v, f(x_1, \ldots, x_n)],\end{aligned}$$

for all $x_1, \dots, x_n \in R$. Since R and U satisfy the same polynomial identity we get

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + b\alpha(f(x_1, \dots, x_n))u, f(x_1, \dots, x_n)]\Big]\\ =&[cf(x_1, \dots, x_n) + b\alpha(f(x_1, \dots, x_n))v, f(x_1, \dots, x_n)],\end{aligned}$$

for all $x_1, \dots, x_n \in U$. By Remark 3.3 above expression becomes

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + bf^{\alpha}(y_1, \dots, y_n)u, f(x_1, \dots, x_n)]\Big]\\ =&[cf(x_1, \dots, x_n) + bf^{\alpha}(y_1, \dots, y_n)v, f(x_1, \dots, x_n)],\end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in U$. In particular, U satisfies the blended component

$$\Big[P, [af(x_1, \dots, x_n), f(x_1, \dots, x_n)]\Big] = [cf(x_1, \dots, x_n), f(x_1, \dots, x_n)].$$

Now result follows from Proposition 4.1 by taking $F(x) = ax$ and $G(x) = cx$. □

5. Proof of the Main Theorem

We can write $F(x) = ax + bd(x)$, $G(x) = cx + b\delta(x)$ for all $x \in R$ and for some $a, b, c \in U$, where d, δ are skew derivations on R. If d and δ both are skew inner derivations on R then by Proposition 4.1 and Proposition 4.4, we get our conclusions. If $b = 0$ then also we get our conclusions from Proposition 4.1. So assume $b \neq 0$. Now we assume that both are not skew inner derivations. We shall study the following cases.

Case-I. Let d be skew inner and δ be outer. In this case we write $F(x) = ax + b\alpha(x)u$ and $G(x) = cx + b\delta(x)$. From given hypothesis we get

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + b\alpha(f(x_1, \dots, x_n))u, f(x_1, \dots, x_n)]\Big]\\ =&[cf(x_1, \dots, x_n) + b\delta(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)].\end{aligned} \tag{5.1}$$

Substituting the value of $\delta(f(x_1, \dots, x_n))$ from (3.1) in equation (5.1), we get

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + b\alpha(f(x_1, \dots, x_n))u, f(x_1, \dots, x_n)]\Big]\\ =&\Big[cf(x_1, \dots, x_n) + bf^{\delta}(x_1, \dots, x_n)\\ &+ b\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)} \dots x_{\sigma(n)}, f(x_1, \dots, x_n)\Big].\end{aligned} \tag{5.2}$$

Since δ is outer, by using Remark 3.6 in above expression, we get

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + b\alpha(f(x_1, \dots, x_n))u, f(x_1, \dots, x_n)]\Big]\\ =&\Bigg[cf(x_1, \dots, x_n) + bf^{\delta}(x_1, \dots, x_n)\\ &+ b\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)} \dots x_{\sigma(n)}, f(x_1, \dots, x_n)\Bigg].\end{aligned} \tag{5.3}$$

In particular U satisfies the blended component

$$\left[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1,\dots,x_n)\right]. \tag{5.4}$$

Suppose α is an inner automorphism. In (5.4) replace $y_{\sigma(1)} = x_{\sigma(1)}$ and $y_{\sigma(i)} = 0$ for all $i > 1$, we get

$$\Big[bf(x_1,\dots,x_n), f(x_1,\dots,x_n)\Big] = 0.$$

Conclusions follow from the inner case.

Suppose α is an outer automorphism, then for $y_{\sigma(n)} = \alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)}) = t_{\sigma(i)}$ for all i and $y_{\sigma(i)} = 0$ for $i < n$ in (5.4) we get

$$\Big[bf^\alpha(t_1,\dots,t_n), f(x_1,\dots,x_n)\Big] = 0,$$

for all $t_1,\dots,t_n,x_1,\dots,x_n \in U$. By Remark 4.1 we get $bf^\alpha(t_1,\dots,t_n) \in C$ for all $t_1,\dots,t_n \in U$ and by Remark 4.2 we get $b = 0$, a contradiction.

Case-II. Now we assume that d is an outer derivation and δ is a skew inner derivation then we write $F(x) = ax + bd(x)$ and $G(x) = cx + b\alpha(x)v$. Then our hypothesis becomes

$$\begin{aligned}&\Big[P, [af(x_1,\dots,x_n) + bd(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)]\Big]\\ =&[cf(x_1,\dots,x_n) + b\alpha(f(x_1,\dots,x_n))v, f(x_1,\dots,x_n)].\end{aligned} \tag{5.5}$$

We substitute the value of $d(f(x_1,\dots,x_n))$ from (3.1) in above equation, we get that U satisfies

$$\begin{aligned}&\Bigg[P, \Big[af(x_1,\dots,x_n) + bf^d(x_1,\dots,x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1,\dots,x_n)\Big]\Bigg]\\ =&[cf(x_1,\dots,x_n) + b\alpha(f(x_1,\dots,x_n))v, f(x_1,\dots,x_n)].\end{aligned}$$

Since d is outer derivation, by using Remark 3.6 in above expression, we get

$$\begin{aligned}&\Bigg[P, \Big[af(x_1,\dots,x_n) + bf^d(x_1,\dots,x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1,\dots,x_n)\Big]\Bigg]\\ =&[cf(x_1,\dots,x_n) + b\alpha(f(x_1,\dots,x_n))v, f(x_1,\dots,x_n)],\end{aligned}$$

where $d(x_i) = y_i$. In particular, U satisfies the blended component

$$\left[P, \left[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1,\dots,x_n)\right]\right]. \tag{5.6}$$

Suppose α is an inner automorphism. Replace $y_{\sigma(1)} = x_{\sigma(1)}$ and $y_{\sigma(i)} = 0$ for all $i > 1$ in (5.6) we get $\Big[P, [bf(x_1, \ldots, x_n), f(x_1, \ldots, x_n)]\Big] = 0$ for all $x_1, \ldots, x_n \in U$. Conclusions follow from inner case.

Now suppose α is an outer automorphism, then for $y_{\sigma(n)} = \alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)}) = t_{\sigma(i)}$ for all i and $y_{\sigma(i)} = 0$ for $i < n$ in (5.6) we get

$$\Big[P, [bf^{\alpha}(t_1, \ldots, t_n), f(x_1, \ldots, x_n)]\Big] = 0,$$

for all $t_1, \ldots, t_n, x_1, \ldots, x_n \in U$. From Remark 4.1 we get $bf^{\alpha}(t_1, \ldots, t_n) \in C$ for all $t_1, \ldots, t_n \in U$ and by Remark 4.2 we get $b = 0$, a contradiction.

Case-III. Now we suppose that none of d and δ are skew inner derivations. In this case we write $F(x) = ax + bd(x)$, $G(x) = cx + b\delta(x)$, where d and δ both are outer derivations. Now we have the following two subcases.

d and δ be C-Linearly Independent Modulo SD_{in}

In this case from our hypothesis, U satisfies

$$\begin{aligned}&\Big[P, [af(x_1, \ldots, x_n) + bd(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]\Big]\\ &= [cx + b\delta(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)].\end{aligned} \tag{5.7}$$

We substitute the value of $d(f(x_1, \ldots, x_n))$ and $\delta(f(x_1, \ldots, x_n))$ from (3.1) and use Remark 3.6 to (5.7) then U satisfies

$$\begin{aligned}&\Bigg[P, \Big[af(x_1, \ldots, x_n) + bf^{d}(x_1, \ldots, x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Big]\Bigg]\\ =&\Bigg[cf(x_1, \ldots, x_n) + bf^{\delta}(x_1, \ldots, x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg],\end{aligned}$$

where $y_{\sigma(j+1)} = d(x_{\sigma(j+1)})$ and $z_{\sigma(j+1)} = \delta(x_{\sigma(j+1)})$. In particular, U satisfies the blended component

$$\Bigg[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg]. \tag{5.8}$$

Suppose α is an inner automorphism. Replace $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for all $i > 1$ in expression (5.8) we get $[bf(x_1, \ldots, x_n), f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in U$. Conclusions follow from inner case.

Now suppose α is an outer automorphism. Then for $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for $i > 1$ in (5.8) we get

$$[bf^{\alpha}(t_1, \ldots, t_n), f(x_1, \ldots, x_n)] = 0, \tag{5.9}$$

for all $t_1, \ldots, t_n, x_1, \ldots, x_n \in U$. By Remark 4.1 we get $bf^{\alpha}(t_1, \ldots, t_n) \in C$ for all $t_1, \ldots, t_n \in U$. By Remark 4.2 we get $b = 0$, a contradiction.

d and δ be C-Linearly Dependent Modulo SD_{in}

Since d and δ be C-linearly dependent modulo SD_{in} there are some $\lambda, \mu \in C$, $q' \in U$ such that $\lambda d(x) + \mu\delta(x) = q'x - \alpha(x)q'$ for all $x \in R$.

If $\lambda = 0$ and $\mu \neq 0$ then $\delta(x) = qx - \alpha(x)q$, where $q = \mu^{-1}q'$ is a skew inner derivation, a contradiction.

If $\lambda \neq 0$ and $\mu = 0$ then $d(x) = qx - \alpha(x)q$, where $q = \lambda^{-1}q'$ is a skew inner derivation, a contradiction.

Suppose $\lambda \neq 0$ and $\mu \neq 0$ and we write $d(x) = \beta\delta(x) + qx - \alpha(x)q$, where $\beta = -\lambda^{-1}\mu$, $q = \lambda^{-1}q'$. Now from our hypothesis we have

$$\Big[P, [af(x_1, \ldots, x_n) + b\beta\delta(f(x_1, \ldots, x_n)) + bqf(x_1, \ldots, x_n) - b\alpha(f(x_1, \ldots, x_n))q, f(x_1, \ldots, x_n)]\Big] = [cf(x_1, \ldots, x_n) + b\delta(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)].$$

Substituting the value of $\delta(f(x_1, \ldots, x_n))$ from (3.1) in above expression we get

$$\begin{aligned} &\Bigg[P, \Bigg[af(x_1, \ldots, x_n) + b\beta f^{\delta}(x_1, \ldots, x_n) \\ &+ b\beta \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)} + bqf(x_1, \ldots, x_n) \\ &- b\alpha(f(x_1, \ldots, x_n))q, f(x_1, \ldots, x_n)\Bigg]\Bigg] \\ =&\Bigg[cf(x_1, \ldots, x_n) + bf^{\delta}(x_1, \ldots, x_n) \\ &+ b \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg], \end{aligned} \tag{5.10}$$

where $z_{\sigma(j+1)} = \delta(x_{\sigma(j+1)})$. In particular, U satisfies the blended component

$$\begin{aligned} &\Bigg[P, \Big[b\beta \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Big]\Bigg] \\ =&\Bigg[b \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg]. \end{aligned} \tag{5.11}$$

Suppose α is an inner automorphism. Replacing $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for $i > 1$ in (5.11) we get

$$\Big[P, [b\beta f(x_1,\ldots,x_n), f(x_1,\ldots,x_n)]\Big] = [bf(x_1,\ldots,x_n), f(x_1,\ldots,x_n)],$$

for all $x_1,\ldots,x_n \in U$. Conclusions follow from inner case.

Suppose α is an outer automorphism. Replace $z_{\sigma(n)} = \alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)}) = t_{\sigma(i)}$ for all i and $z_{\sigma(i)} = 0$ for $i < n$ in (5.11) we get

$$\Big[P, [b\beta f^{\alpha}(t_1,\ldots,t_n), f(x_1,\ldots,x_n)]\Big] = [bf^{\alpha}(t_1,\ldots,t_n), f(x_1,\ldots,x_n)],$$

for all $t_1,\ldots,t_n, x_1,\ldots,x_n \in U$. By Remark 4.1 we get $b\beta f^{\alpha}(t_1,\ldots,t_n) \in C$ and $bf^{\alpha}(t_1,\ldots,t_n) \in C$ for all $t_1,\ldots,t_n \in U$. In both cases $bf^{\alpha}(t_1,\ldots,t_n) \in C$ for all $t_1,\ldots,t_n$, since $0 \neq \beta \in C$. By Remark 4.2 we get $b = 0$, a contradiction.

Similarly, if we consider $\delta(x) = \beta d(x) + qx - \alpha(x)q$ for all $x \in R$ then we get a contradiction.

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(FUZZY) FILTERS OF SHEFFER STROKE BL-ALGEBRAS

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Abstract. In this study, some (fuzzy) filters of a Sheffer stroke BL-algebra and its properties are presented. To show a relationship between a filter and a fuzzy filter of Sheffer stroke BL-algebra, we prove that f is a fuzzy (ultra) filter of C if and only if f_p is either empty or a (ultra) filter of C for each $p \in [0,1]$, and it is satisfied for $p = f(1)$ and for the characteristic function of a nonempty subset of a Sheffer stroke BL-algebra.

1. Introduction

The idea of fuzzy set theory as well as fuzzy logic was propounded by Lotfi Zadeh ([20, 21]). The interest in foundations of fuzzy logic has been rapidly proceeding recently and many new algebras playing the role of the structures of truth values have been introduced.

The most important task of artificial intelligence is to make computers which simulate human behaviors. The classical logic deals with certain information while nonclassical logic such as many valued logics and fuzzy logic engages in uncertainty, or fuzziness and randomness. Since fuzziness and randomness are closely related to human's intelligence and behaviors, the fuzzy theory using in many various areas from science to technology plays an important role in improving artificial intelligence.

Filters have fundamental importance in algebra and play significant role in studying fuzzy logics. From logical point of view, they correspond to sets of provable formulas. Besides, they have a variety of some applications in logic and topology. Different approaches of fuzzy filters have been investigated by many authors ([6, 9, 10, 18]).

Petr Hájek introduced the axiom system of basic logic (BL) for fuzzy propositional logic and defined the class of BL-algebras [5]. He presented filters and prime filters

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on this algebraic structure and gave the completeness proof of basic logic by using these prime filters [5]. Since the filters and fuzzy filters have an important place in the logical algebra theory, Boolean, positive implicative, maximal, prime, proper filters and implicative deductive systems on BL-algebras are researched ([6, 8]). In recent times, Liu and Li studied on fuzzy filters on a BL-algebra ([9, 10]), and Saeid et al. analyzed some kinds of filters, open problems on fuzzy filters and double complemeted elements of BL-algebras ([1–3]). Also, Xueling et al. generalized fuzzy filters of BL-algebras [18] and Zhan et al. examined on their some types [17]. Moreover, Yin and Zhan researched new types of fuzzy filters in these algebras [19]. Indeed, Haveshki and Eslami investigated n-fold filters of BL-algebras [7] and Motamed et al. studied on n-fold obstinate filters [12] and radicals of filters in BL-algebras [13]. Besides, H. M. Sheffer introduced Sheffer operation [16], and then McCune et al. showed that every Boolean function or axiom may be restated by this operation [11]. Since the Sheffer operation is a commutative, it satisfies that many algebraic structures have more useful axiom system. Recently, Sheffer stroke Hilbert algebras and filters a strong Sheffer stroke non-associative MV-algebras are studied (for details [14] and [15], respectively).

We first give basic definitions and notions related to a Sheffer stroke BL-algebra, and present new properties. Then some kind of (fuzzy) filters are defined and exemplified. Besides, we prove that f is a fuzzy (ultra) filter of a Sheffer stroke BL-algebra if and only if $f_p = \{c_1 \in C : p \leq f(c_1)\} \neq \emptyset$ is its (ultra) filter for any $p \in (0, 1]$, and it is satisfied for $p = f(1)$ and for the characteristic function χ_P of P in which P is the nonempty subset of a Sheffer stroke BL-algebra.

2. Preliminaries

In this section, we give fundamental definitions and notions about Sheffer stroke BL-algebras, BL-algebras, filters and fuzzy filters of BL-algebras.

Definition 2.1 ([4])**.** Let $\mathcal{C} = \langle C, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke* if it satisfies the following conditions:

$(S1)$ $c_1|c_2 = c_2|c_1$;
$(S2)$ $(c_1|c_1)|(c_1|c_2) = c_1$;
$(S3)$ $c_1|((c_2|c_3)|(c_2|c_3)) = ((c_1|c_2)|(c_1|c_2))|c_3$;
$(S4)$ $(c_1|((c_1|c_1)|(c_2|c_2)))|(c_1|((c_1|c_1)|(c_2|c_2))) = c_1$.

Definition 2.2. A Sheffer stroke BL-algebra is an algebra $(C, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions:

$(sBL-1)$ $(C, \vee, \wedge, 0, 1)$ is a bounded lattice;
$(sBL-2)$ $(C, |)$ is a groupoid with the Sheffer stroke;
$(sBL-3)$ $c_1 \wedge c_2 = (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))$;
$(sBL-4)$ $(c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)) = 1$,

for all $c_1, c_2 \in C$.

$1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of C.

Example 2.1. Consider a structure $(C, \vee, \wedge, |, 0, 1)$ with the following Hasse diagram (see Figure 1), where $C = \{0, u, v, 1\}$.

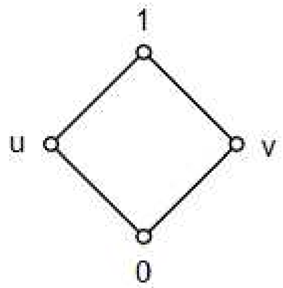

FIGURE 1.

The binary operations $|$, $\vee$ and $\wedge$ on C have Cayley tables as follow in Table 1, 2 and 3.

TABLE 1. The table of a Sheffer stroke $|$

$\|$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

TABLE 2. The table of $\vee$

$\vee$	0	u	v	1
0	0	u	v	1
u	u	u	1	1
v	v	1	v	1
1	1	1	1	1

TABLE 3. The table of $\wedge$

$\wedge$	0	u	v	1
0	0	0	0	0
u	0	u	0	u
v	0	0	v	v
1	0	u	v	1

Then this structure is a Sheffer stroke BL-algebra.

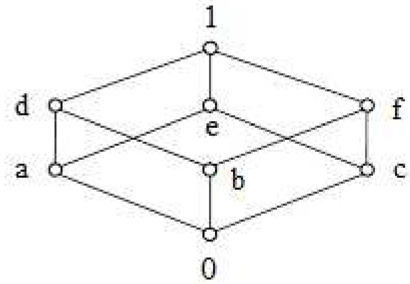

FIGURE 2.

Example 2.2. Consider a structure $(C, \vee, \wedge, |, 0, 1)$ with the following Hasse diagram (see Figure 2), where $C = \{0, a, b, c, d, e, f, 1\}$.

The binary operations $|$, $\vee$ and $\wedge$ on C have Cayley tables as follow in Table 4, 5 and 6. Then this structure is a Sheffer stroke BL-algebra.

TABLE 4. The table of a Sheffer stroke $|$

$\vert$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	1

TABLE 5. The table of $\vee$

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

Proposition 2.1. *In any Sheffer stroke BL-algebra C, the following features hold, for all $c_1, c_2, c_3 \in C$:*

TABLE 6. The table of $\wedge$

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

(1) $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$;
(2) $c_1|(c_1|c_1) = 1$;
(3) $1|(c_1|c_1) = c_1$;
(4) $c_1|(1|1) = 1$;
(5) $(c_1|1)|(c_1|1) = c_1$;
(6) $(c_1|c_2)|(c_1|c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2|(c_3|c_3)$;
(7) $c_1 \leq c_2$ *if and only if* $c_1|(c_2|c_2) = 1$;
(8) $c_1 \leq c_2|(c_1|c_1)$;
(9) $c_1 \leq (c_1|c_2)|c_2$;
(10) (*a*) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1$;
(*b*) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$;
(11) *if* $c_1 \leq c_2$, *then*
(*i*) $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$;
(*ii*) $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$;
(*iii*) $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$;
(12) $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$;
(13) $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$;
(14) $((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3) = ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$;
(15) $c_1 \vee c_2 = ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))$.

Proof. (1) It follows from $(S1)$ and $(S3)$.

(2) We get $c_1|(c_1|c_1) = (c_1|(c_1|c_1)) \vee (c_1|(c_1|c_1)) = 1$ from $(sBL-1)$ and $(sBL-4)$.

(3) We have $1|(c_1|c_1) = (c_1|(c_1|c_1))|(c_1|c_1) = c_1$ from (2), $(S1)$ and $(S2)$.

(4) It is obtained from (3), $(S1)$ and $(S2)$ that $c_1|(1|1) = (1|(c_1|c_1))|(1|1) = 1$.

(5) It follows from $(S1)$, $(S2)$ and (3) that

$$(c_1|1)|(c_1|1) = (1|((c_1|c_1)|(c_1|c_1)))|(1|((c_1|c_1)|(c_1|c_1))) = (c_1|c_1)|(c_1|c_1) = c_1.$$

(6) ($\Rightarrow$) Let $(c_1|c_2)|(c_1|c_2) \leq c_3$. Then it follows from $(sBL-1)$, $(sBL-3)$, $(S1)$ and $(S3)$ that

$$\begin{aligned}(c_1|c_2)|(c_1|c_2) &= ((c_1|c_2)|(c_1|c_2)) \wedge c_3 \\ &= (((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\end{aligned}$$

$$
\begin{aligned}
&|(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\
&= (c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&\quad |(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))|\\
&\quad (c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&\quad |(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad (*_1),
\end{aligned}
$$

for all $c_1, c_2, c_3 \in C$. Thus, we have

$$
\begin{aligned}
c_1 \wedge (c_2|(c_3|c_3)) =&(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \qquad (sBL-3)\\
=&(c_1|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\
&|(c_1|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3))) \qquad (S3)\\
=&(c_1|(((c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))))\\
&|(c_3|c_3)))|(c_1|(((c_2|((c_1|(c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3)))))|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))|\\
&(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))))|(c_3|c_3))) \qquad (*_1)\\
=&(c_1|(((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))\\
&|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \quad ((S1) \text{ and } (S3))\\
=&(c_1|((c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))|((c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3))))|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_1|((c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))|((c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3))))|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad ((S1) \text{ and } (S3))\\
=&(c_1|1)|(c_1|1) \qquad (2)\\
=&c_1 \qquad (5),
\end{aligned}
$$

i.e., $c_1 \leq c_2|(c_3|c_3)$ from $(sBL-1)$.

($\Leftarrow$) Let $c_1 \leq c_2|(c_3|c_3)$. Then we obtain from $(sBL-1)$ and $(sBL-3)$ that

$$
\begin{aligned}
c_1 &= c_1 \wedge (c_2|(c_3|c_3))\\
&= (c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \qquad (*_2),
\end{aligned}
$$

for all $c_1, c_2, c_3 \in C$. Thus, it follows

$$\begin{aligned}((c_1|c_2)|(c_1|c_2)) \wedge c_3 =&(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\ &|(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3))) \qquad (sBL-3)\\ =&(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\ &|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\ &|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\ &|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad ((S1) \text{ and } (S3))\\ =&(c_1|c_2)|(c_1|c_2) \qquad ((*_2) \text{ and } (S1))\end{aligned}$$

i.e., $(c_1|c_2)|(c_1|c_2) \leq C_3$ from $(sBL-1)$.

(7) Let $c_1 \leq c_2$. Then we obtain from (5) and $(S1)$ that $c_1 = (1|c_1)|(1|c_1) \leq c_2$. So, it follows from (6) that $1 \leq c_1|(c_2|c_2)$. Since it is known $c_1|(c_2|c_2) \leq 1$ for all $c_1, c_2 \in A$, we have $c_1|(c_2|c_2) = 1$.

Conversely, let $c_1|(c_2|c_2) = 1$. Because it is known $c_1 \leq 1$ for all $c_1 \in C$, we get $c_1 \leq 1 = c_1|(c_2|c_2)$ by the hypothesis. Thus, it follows from (6) and $(S2)$ that $c_1 = (c_1|c_1)|(c_1|c_1) \leq c_2$.

(8) Since it is known that $c_2 \leq 1$ for all $c_2 \in C$, we have

$$\begin{aligned}c_2 \leq 1 &\Leftrightarrow c_2 \leq c_1|(c_1|c_1) \qquad (2)\\ &\Leftrightarrow (c_1|c_2)|(c_1|c_2) \leq c_1 \qquad ((6) \text{ and } (S1))\\ &\Leftrightarrow c_1 \leq c_2|(c_1|c_1) \qquad (6).\end{aligned}$$

(9) For all $c_1, c_2 \in C$, it follows from (6), $(S2)$ and $(S1)$, respectively, that $(c_1|c_2)|(c_1|c_2) \leq (c_1|c_2)|(c_1|c_2) \Leftrightarrow c_1 \leq (c_1|c_2)|c_2$.

(10)

(a) Because $c_1 \leq c_1$ for all $c_1 \in C$, we get from $(S2)$, $(S1)$ and (6), respectively, that

$$c_1 \leq c_1 \Leftrightarrow c_1 \leq (c_1|(c_2|c_2))|(c_1|c_1) \Leftrightarrow (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1.$$

(b) Since $c_1|(c_2|c_2) \leq c_1|(c_2|c_2)$ for all $c_1, c_2 \in C$, it is obtained from (6) and $(S1)$ that

$$c_1|(c_2|c_2) \leq c_1|(c_2|c_2) \Leftrightarrow (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2.$$

(11) Let $c_1 \leq c_2$.

(i) We have $(c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))) \leq c_1 \leq c_2$ from 10 (b) and the hypothesis. So, we get from $(S1)$ and (6) that $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$.

(ii) We know $c_1 \leq c_2 \leq (c_2|c_3)|c_3$ by (9) and the hypothesis. Therefore, it follows from $(S1)$, $(S2)$ and (6) that $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$.

(iii) It is obtained from (ii) and (10) (b) that

$$c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \leq c_2|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \leq c_3.$$

Then we get from $(S1)$ and (6) that $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$.

(12) Because we know $(c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))) \leq c_1$ from (10) (b), we have from (11) (i) that $c_1|(c_2|c_2) \leq ((c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))))|(y|y)$. Then it is obtained from $(S1)$ and $(S3)$ that $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$.

(13) Since it is known $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$ from (10) (b), it is obtained from (11); (i) that $c_2|(c_3|c_3) \leq ((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))))|(c_3|c_3)$. Then we get $c_2|(c_3|c_3) \leq (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$ from $(S1)$ and $(S3)$. Thus, it follows from (6) and $(S1)$ that $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$.

(14) Because $c_1, c_2 \leq c_1 \vee c_2$, we obtain from (11) (ii) that $(c_1|c_3)|(c_1|c_3), (c_2|c_3)|(c_2|c_3) \leq ((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3)$. Then it follows

$$((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)) \leq ((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3).$$

Since $(c_1|c_3)|(c_1|c_3), (c_2|c_3)|(c_2|c_3) \leq ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$, we have from (6) that

$$c_1, c_2 \leq c_3|((((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))|(((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))).$$

Then

$$c_1 \vee c_2 \leq c_3|((((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))|(((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))).$$

So, it follows from (6) that $((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3) \leq ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$.

(15) We have $c_1, c_2 \leq (c_1|(c_2|c_2))|(c_2|c_2)$ and $c_1, c_2 \leq (c_2|(c_1|c_1))|(c_1|c_1)$. Then $c_1, c_2 \leq ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))$, and so

$$c_1 \vee c_2 \leq ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1)).$$

Also, we obtain

$$\begin{aligned}
&((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1)) \\
=&((((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))) \\
&|((c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1))))|((((c_1|(c_2|c_2))|(c_2|c_2)) \\
&\wedge ((c_2|(c_1|c_1))|(c_1|c_1)))|((c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)))) \quad ((5) \text{ and } (sBL-4)) \\
=&(((c_1|(c_2|c_2))|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1)) \\
&|(c_1|c_1))))|((c_1|(c_2|c_2))|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))) \\
&|(((c_2|(c_1|c_1))|(((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))|((c_2|(c_1|c_1)) \\
&|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))) \qquad ((S1) \text{ and } (14)) \\
\leq&(((c_1|(c_2|c_2))|((c_1|(c_2|c_2))||(c_2|c_2)))|((c_1|(c_2|c_2)) \\
&|((c_1|(c_2|c_2))||(c_2|c_2)))) \vee (((c_2|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1)))| \\
&((c_2|(c_1|c_1))|((c_2|(c_1|c_1))||(c_1|c_1)))) \qquad ((S1) \text{ and } (11)\ (ii)) \\
=&((c_1|(c_2|c_2)) \wedge c_2) \vee ((c_2|(c_1|c_1)) \wedge c_1) \quad (sBL-3) \\
=&c_2 \vee c_1 \quad (8) \\
=&c_1 \vee c_2.
\end{aligned}$$

□

Lemma 2.1. *Let C be a Sheffer stroke BL-algebra. Then* $(c_1|(c_2|c_2))|(c_2|c_2) = (c_2|(c_1|c_1))|(c_1|c_1)$ *for all* $c_1, c_2 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Then it is obtained from Proposition 2.1 (13), $(S1)$ and $(S2)$ that

$$\begin{aligned}(c_1|(c_2|c_2))|(c_2|c_2) &\leq (c_2|(c_1|c_1))|(((c_1|(c_2|c_2))|(c_1|c_1))|((c_1|(c_2|c_2))|(c_1|c_1)))\\ &= (c_2|(c_1|c_1))|(c_1|c_1),\end{aligned}$$

and similarly, $(c_2|(c_1|c_1))|(c_1|c_1) \leq (c_1|(c_2|c_2))|(c_2|c_2)$. Therefore, $(c_1|(c_2|c_2))|(c_2|c_2) = (c_2|(c_1|c_1))|(c_1|c_1)$ for all $c_1, c_2 \in C$. □

Corollary 2.1. *Let C be a Sheffer stroke BL-algebra. Then* $c_1 \vee c_2 = (c_1|(c_2|c_2))|(c_2|c_2)$ *for all* $c_1, c_2 \in C$.

Lemma 2.2. *Let C be a Sheffer stroke BL-algebra. Then* $((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) = c_1|(c_2|c_2)$ *for all* $c_1, c_2 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Then it is known from Proposition 2.1 (9) that $c_1|(c_2|c_2) \leq ((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2)$. Also, it follows from Proposition 2.1 (12) and (1)–(3), respectively, that

$$\begin{aligned}((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) \leq &(c_1|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2))\\ &|(c_2|c_2))))|((c_1|(c_2|c_2))|(c_1|(c_2|c_2)))\\ =&((c_1|(c_2|c_2))|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))))\\ &|((c_1|(c_2|c_2))|(c_1|(c_2|c_2)))\\ =&(c_1|(c_2|c_2)).\end{aligned}$$

Thus, $((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) = c_1|(c_2|c_2)$ for all $c_1, c_2 \in C$. □

Lemma 2.3. *Let C be a Sheffer stroke BL-algebra. Then* $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$ *for all* $c_1, c_2, c_3 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Since $c_2 \leq c_1|(c_2|c_2)$ from Proposition 2.1 (8), it is obtained from Proposition 2.1 (11) (iii) and (1), respectively, that

$$\begin{aligned}(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) \leq &c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))\\ =&c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))).\end{aligned}$$

Besides, it follows from Proposition 2.1 (1), (12), $(S3)$ and $(S2)$, respectively, that

$$\begin{aligned}c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) =&c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))\\ \leq&(c_1|(c_2|c_2))|((c_1|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))))\\ &|(c_1|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))))\\ =&(c_1|(c_2|c_2))|((((c_1|c_1)|(c_1|c_1))|(c_3|c_3))\\ &|(((c_1|c_1)|(c_1|c_1))|(c_3|c_3)))\\ =&(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))).\end{aligned}$$

Therefore, $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$. □

3. Various Filters of Sheffer Stroke BL-Algebras

In this section, we give some types of filters on a Sheffer stroke BL-algebra. Unless orherwise specified, C represents a Sheffer stroke BL-algebra.

Definition 3.1. A filter of C is a nonempty subset $P \subseteq C$ satisfying
$(SF-1)$ if $c_1, c_2 \in P$, then $(c_1|c_2)|(c_1|c_2) \in P$;
$(SF-2)$ if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.

Example 3.1. For the Sheffer stroke BL-algebra in Example 2.2, C, $\{1\}$, $\{a, d, e, 1\}$ and $\{c, e, f, 1\}$ are filters of C.

Proposition 3.1. *Let P be a nonempty subset of C. Then P is a filter of C if and only if the following hold:*
$(SF-3)$ $1 \in P$;
$(SF-4)$ $c_1 \in P$ *and* $c_1|(c_2|c_2) \in P$ *imply* $c_2 \in P$.

Lemma 3.1. *Let P be a filter of C. Then* $c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1))) \in P$ *and* $c_3 \in P$ *imply* $(c_1|(c_2|c_2))|(c_2|c_2) \in P$ *for any* $c_1, c_2, c_3 \in C$.

Proof. Let P be a filter of C. Since $c_3|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2))|(c_2|c_2))) = c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1))) \in P$, from Lemma 2.1 and $c_3 \in P$, it follows from $(SF-4)$ that $(c_1|(c_2|c_2))|(c_2|c_2) \in P$. □

Lemma 3.2. *Let P be a filter of C. Then*
(a) $c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1))) \in P$ *and* $c_3 \in P$ *imply* $((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1) \in P$;
(b) $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ *and* $c_1|(c_2|c_2) \in P$ *imply* $c_1|(c_3|c_3) \in P$;
(c) $c_1|(((c_2|(c_3|c_3))|(c_2|c_2))|((c_2|(c_3|c_3))|(c_2|c_2))) \in P$ *and* $c_1 \in P$ *imply* $c_2 \in P$,
for any $c_1, c_2, c_3 \in C$.

Proof. Let P be a filter of C.
(a) Because $c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1))) \in P$ and $c_3 \in P$, we get from Lemma 2.1, Lemma 2.2 and $(SF-4)$ that

$$((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1) = ((c_2|(c_1|c_1))|(c_1|c_1))|(c_1|c_1) = c_2|(c_1|c_1) \in P.$$

(b) Since $(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) = c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ from Lemma 2.3 and $c_1|(c_2|c_2) \in P$, it follows from $(SF-4)$ that $c_1|(c_3|c_3) \in P$.

(c) Because

$$\begin{aligned} & c_1|(((c_2|(c_3|c_3))|(c_2|c_2))|((c_2|(c_3|c_3))|(c_2|c_2))) \\ =& c_1|(((c_2|c_2)|(c_2|(c_3|c_3)))|((c_2|c_2)|(c_2|(c_3|c_3)))) \\ =& c_1|(c_2|c_2) \in P, \end{aligned}$$

from $(S1)$-$(S2)$ and $c_1 \in P$, we have from $(SF-4)$ that $c_2 \in P$. □

Lemma 3.3. *Let P be a filter of C. Then $c \vee (c|c) \in P$ for any $c \in C$.*

Proof. Let P be a filter of C, and c be any element of C. Since

$$\begin{aligned} c \vee (c|c) &= (c|((c|c)|(c|c)))|((c|c)|(c|c)) \qquad \text{(Corollary 2.1)} \\ &= (c|(c|c)) \qquad (S1)\text{-}(S2) \\ &= 1 \qquad \text{(Proposition 2.1 (2))} \end{aligned}$$

and $1 \in P$, it is obtained $c \vee (c|c) \in P$. □

Definition 3.2. Let P be a filter of C. Then P is called an ultra filter of C if it satisfies $c \in P$ or $c|c \in P$ for all $c \in C$.

Example 3.2. Consider the Sheffer stroke BL-algebra in Example 2.2. Then the filter $\{a, d, e, 1\}$ of C is ultra while the filter $\{1\}$ of C is not an ultra filter of C.

Lemma 3.4. *A filter P of C is an ultra filter of C if and only if $c_1 \notin P$ and $c_2 \notin P$ imply $c_1|(c_2|c_2) \in P$ for all $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let P be an ultra filter of C. Assume that $c_1 \notin P$ and $c_2 \notin P$. Because P is an ultra filter of C, $c_1|c_1 \in P$ and $c_2|c_2 \in P$. Then $c_1|c_1 \leq (c_2|c_2)|((c_1|c_1)|(c_1|c_1)) = c_1|(c_2|c_2)$ from Proposition 2.1 (8) and $(S1)$-$(S2)$. So, $c_1|(c_2|c_2) \in P$.

($\Leftarrow$) Let $c_1 \notin P$ and $c_2 \notin P$. Then $c_1|(c_2|c_2) \in P$ for $c_1, c_2 \in C$. Suppose that $c|c \notin P$ and $c \notin P$ for any $c \in C$. Then $(c|c)|(c|c) = c \in P$ by the hypothesis and $(S2)$, which is a contradiction. Hence, $c|c \in P$ and $c \in P$ for any $c \in C$, i.e., P is an ultra filter of C. □

Lemma 3.5. *A filter P of C is an ultra filter of C if and only if $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$ for all $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let P be an ultra filter of C and $c_1 \vee c_2 \in P$. Suppose that $c_1 \notin P$ or $c_2 \notin P$. Then we have $c_1|(c_2|c_2) \in P$ from Lemma 3.4. Since $(c_1|(c_2|c_2))|(c_2|c_2) \in P$, from Corollary 2.1 and $c_1|(c_2|c_2) \in P$, we get $c_2 \in P$ which is a contrradiction. Thus, $c_1 \in P$ or $c_2 \in P$.

($\Leftarrow$) Let c_1 and c_2 be any elements in C such that $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$. Because $c \vee (c|c) \in P$ for all $c \in C$ from Lemma 3.3, it follows $c \in P$ or $c|c \in P$, i.e., P is an ultra filter of C. □

4. Some Fuzzy Filters of Sheffer Stroke BL-Algebras

In this section, we introduce some fuzzy filters in Sheffer stroke BL-algebras. Unless orherwise specified, C represents a Sheffer stroke BL-algebra.

Definition 4.1. A fuzzy filter of C is a fuzzy subset f of C such that for all $c_1, c_2 \in C$

(1) $f(c_1) \leq f(1)$;
(2) $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$.

Example 4.1. Consider the Sheffer stroke BL-algebra C in Example 2.1. Let $f(0) = f(u) = f(v) = 0,5$ and $f(1) = 1$. Then f is a fuzzy filter of C.

Proposition 4.1. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if for all $c_1, c_2, c_3 \in C$ $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$ implies $f(c_1) \wedge f(c_2) \leq f(c_3)$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C and $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$. Since

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1) \wedge f(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))) \\ &\leq f(c_2|(c_3|c_3)) \qquad \text{(Definition 4.1 (2))}, \end{aligned}$$

it follows from Definition 4.1 (2) that

$$f(c_1) \wedge f(c_2) \leq f(c_2|(c_3|c_3)) \wedge f(c_2) = f(c_2) \wedge f(c_2|(c_3|c_3)) \leq f(c_3).$$

($\Leftarrow$) Let $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$ imply $f(c_1) \wedge f(c_2) \leq f(c_3)$. By substituting $[c_2 := c_1]$ and $[c_3 := 1]$, it is obtained from Proposition 2.1 (4) that $c_1|((c_1|(1|1))|(c_1|(1|1))) = 1$ implies $f(c_1) = f(c_1) \wedge f(c_1) \leq f(1)$. Besides, substituting $[c_2 := c_1|(c_2|c_2)]$ and $[c_3 := c_2]$, simultaneously, it is concluded from $(S1)$, $(S3)$ and Proposition 2.1 (2) that $c_1|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2))|(c_2|c_2))) = 1$, which implies $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$. Thus, f is a fuzzy filter of C. □

Corollary 4.1. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if for all $c_1, c_2, c_3 \in C$ $(c_1|c_2)|(c_1|c_2) \leq c_3$ implies $f(c_1) \wedge f(c_2) \leq f(c_3)$.*

Proposition 4.2. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if*

(1) *f is order-preserving;*

(2) *$f(c_1) \wedge f(c_2) \leq f((c_1|c_2)|(c_1|c_2))$ for any $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C.

(1) Assume that $c_1 \leq c_2$, i.e., $c_1|(c_2|c_2) = 1$ from Proposition 2.1 (7). Then

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1) \wedge f(c_1|(c_2|c_2)) \\ &\leq f(c_2) \qquad \text{(Definition 4.1 (2))}. \end{aligned}$$

(2) Since

$$\begin{aligned} & c_1|((c_2|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|c_2))))|(c_2|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|c_2))))) \\ =& c_1|((c_2|(c_1|c_2))|(c_2|(c_1|c_2))) \qquad (S2) \\ =& ((c_1|c_2)|(c_1|c_2))|(c_1|c_2) \qquad (S3) \\ =& 1, \qquad ((S1) \text{ and Proposition 2.1 (2)}) \end{aligned}$$

it follows from Proposition 4.1 that $f(c_1) \wedge f(c_2) \leq f((c_1|c_2)|(c_1|c_2))$.

($\Leftarrow$) Let f be a fuzzy subset of C satisfying (1) and (2) for all $c_1, c_2, c_3 \in C$. By (1) and the fact that $c_1 \leq 1$ for all $c_1 \in C$, $f(c_1) \leq f(1)$. It is known from Proposition 2.1 (9) that $c_1 \leq (c_1|(c_2|c_2))|(c_2|c_2)$, and so $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$ by Proposition 2.1 (6). Then it is obtained from (1) $-$ (2) that $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))) \leq f(c_2)$. Therefore, f is a fuzzy filter of C. □

Corollary 4.2. *Let f be an order-preserving fuzzy subset of C. f is a fuzzy filter of C if and only if $f((c_1|c_2)|(c_1|c_2)) = f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C. By Proposition 4.2 (2), it is sufficient to show that $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$. Since $c_1 \leq 1$ for all $c_1 \in C$, it follows from $(S1)$, Proposition 2.1 (2) and (6) that $(c_1|c_2)|(c_1|c_2) \leq c_1, c_2$ for all $c_1, c_2 \in C$. Because f is order-preserving, $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1), f(c_2)$, so $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1) \wedge f(c_2)$.

($\Leftarrow$) It is clear by Proposition 4.2. □

Corollary 4.3. *Let f be a fuzzy filter of C. Then $f(c_1 \wedge c_2) = f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$.*

Proof. Let f be a fuzzy filter of C. Since $c_1 \wedge c_2 \leq c_1$, $c_1 \wedge c_2 \leq c_2$ and f is an order-preserving, it is obtained $f(c_1 \wedge c_2) \leq f(c_1)$ and $f(c_1 \wedge c_2) \leq f(c_2)$. Then $f(c_1 \wedge c_2) \leq f(c_1) \wedge f(c_2)$. Because we know $c_2 \leq c_1|(c_2|c_2)$ from Proposition 2.1 (8), it follows from Proposition 2.1 (11) (ii), $(S1)$, and $(sBL-3)$ that $(c_1|c_2)|(c_1|c_2) \leq (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) = c_1 \wedge c_2$. Thus, $f(c_1) \wedge f(c_2) \leq f(c_1 \wedge c_2)$ from Corollary 4.1. □

Theorem 4.1. *Let f be a fuzzy filter of C.*
(a) *If $f(c_1|(c_2|c_2)) = f(1)$, then $f(c_1) \leq f(c_2)$.*
(b) $f(c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1)))) \wedge f(c_3) \leq f((c_1|(c_2|c_2))|(c_2|c_2))$.
(c) $f(c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1)))) \wedge f(c_3) \leq f(((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1))$.
(d) $f(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))) \wedge f(c_1|(c_2|c_2)) \leq f(c_1|(c_3|c_3))$,
for any $c_1, c_2, c_3 \in C$.

Proof. (a) Since

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1)} \\ &= f(c_1) \wedge f(c_1|(c_2|c_2)) \\ &= f((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))) \qquad \text{(Corollary 4.2)} \\ &= f(c_1 \wedge c_2) \qquad (sBL-3) \\ &= f(c_1) \wedge f(c_2), \qquad \text{(Corollary 4.3)}, \end{aligned}$$

we get $f(c_1) \leq f(c_2)$.

(b) It is proved from Definition 4.1 (2) and Lemma 2.1.

(c) We have from Definition 4.1 (2), Lemma 2.2 and Lemma 2.1 that

$$\begin{aligned} f(c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1)))) \wedge f(c_3) &\leq f(c_2|(c_1|c_1)) \\ &= f(((c_2|(c_1|c_1))|(c_1|c_1))|(c_1|c_1)) \\ &= f(((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1)). \end{aligned}$$

(d) It is proved from Lemma 2.3 and Definition 4.1 (2). □

Definition 4.2. A fuzzy subset f of C is called a fuzzy ultra filter of C if it is a fuzzy filter of C that satisfies the following conditions $f(c_1) = f(1)$ or $f(c_1|c_1) = f(1)$, for all $c_1 \in C$.

Example 4.2. Consider the Sheffer stroke BL-algebra C in Example 2.2. Let the fuzzy filter f of C be defined by $f(a) = f(d) = f(e) = f(1) = 1$ and $f(b) = f(c) = f(f) = f(0) = 0$. Since $f(a) = f(d) = f(e) = f(1) = 1$ and $1 = f(a) = f(d) = f(e) = f(1) = f(f|f) = f(c|c) = f(b|b) = f(0|0)$, f is a fuzzy ultra filter of C.

Theorem 4.2. *A fuzzy subset f of C is a fuzzy ultra filter of C if and only if $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$ imply $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1)) = f(1)$ for all $c_1, c_2 \in C$.*

Proof. Let $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$ imply $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1)) = f(1)$. Supoose that $f(c_1) \neq f(1)$ and $f(1|1) \neq f(1)$ for any $c_1 \in C$. Then we have from Proposition 2.1 (4)-(5) and $(S1)$-$(S2)$ that $f(c_1|c_1) = f(c_1|1) = f(c_1|((1|1)|(1|1))) = f(1)$ and $f(1) = f((c_1|c_1)|(1|1)) = f((1|1)|(c_1|c_1)) = f(1)$. Similarly, $f(c_1) = f(1)$ whenever $f(c_1|c_1) \neq f(1)$ and $f(1|1) \neq f(1)$. Thus, f is a fuzzy ultra filter of C.

Conversely, let f be a fuzzy ultra filter of C. Assume that c_1 and c_2 are any elements in C such that $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$. So, $f(c_1|c_1) = f(1)$ and $f(c_2|c_2) = f(1)$. Because

$$(c_1|c_1)|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) = (c_2|c_2)|((c_1|(c_1|c_1))|(c_1|(c_1|c_1))) = 1,$$

from $(S1)$, $(S3)$, Proposition 2.1 (2) and (4), it is obtained

$$\begin{aligned} f(1) &= f(1) \wedge f(1) \\ &= f(c_1|c_1) \wedge f((c_1|c_1)|((c_1|(c_2|c_2))|(c_1|(c_2|c_2)))) \\ &\leq f(c_1|(c_2|c_2)) \qquad \text{(Definition 4.1 (2))}, \end{aligned}$$

which gives $f(c_1|(c_2|c_2)) = f(1)$. Similarly, $f(c_2|(c_1|c_1)) = f(1)$. □

Theorem 4.3. *A fuzzy subset f of C is a fuzzy ultra filter of C if and only if $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$ for all $c_1, c_2 \in C$.*

Proof. Let f be a fuzzy ultra filter of C. When $f(c_1) = f(1)$ or $f(c_2) = f(1)$, the proof is completed from Definition 4.1 (1). So, let $f(c_1) \neq f(1)$ or $f(c_2) \neq f(1)$. Then $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1)) = f(1)$ by Theorem 4.2. Since

$$\begin{aligned} f(c_1 \vee c_2) &= f(1) \wedge f(c_1 \vee c_2) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1|(c_2|c_2)) \wedge f((c_1|(c_2|c_2))|(c_2|c_2)) \qquad \text{(Corollary 2.1)} \\ &\leq f(c_2) \qquad \text{(Definition 4.1 (2))} \end{aligned}$$

and

$$\begin{aligned} f(c_1 \vee c_2) &= f(c_2 \vee c_1) \\ &= f(1) \wedge f(c_2 \vee c_1) \qquad \text{(Definition 4.1 (1))} \end{aligned}$$

$$= f(c_2|(c_1|c_1)) \wedge f((c_2|(c_1|c_1))|(c_1|c_1)) \quad \text{(Corollary 2.1)}$$
$$\leq f(c_1) \quad \text{(Definition 4.1 (2))},$$

$f(c_1 \vee c_2) \leq f(c_1), f(c_2)$ and so, $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$.

Conversely, let c_1 and c_2 be any elements in C such that $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$. Then

$$\begin{aligned} f(1) &= f(c|(c|c)) \quad \text{(Proposition 2.1 (2))} \\ &= f((c|((c|c)|(c|c)))|((c|c)|(c|c))) \quad (S2) \\ &= f(c \vee (c|c)) \quad \text{(Corollary 2.1)} \\ &\leq f(c) \vee f(c|c), \end{aligned}$$

i.e., $f(c) \vee f(c|c) = f(1)$. Thus, $f(c) = f(1)$ or $f(c|c) = f(1)$, which implies that f is a fuzzy ultra filter of C. □

Proposition 4.3. *f is a fuzzy filter of C if and only if $f_p = \{c_1 \in C : p \leq f(c_1)\} \neq \emptyset$ is a filter of C for any $p \in (0, 1]$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C.

- Since $f_p \neq \emptyset$, there exists some $c \in C$ such that $p \leq f(c)$. Then we obtain from Definition 4.1 (1) that $p \leq f(c) \leq f(1)$, i.e., $1 \in f_p$.
- Let $c_1, c_1|(c_2|c_2) \in f_p$, i.e., $p \leq f(c_1), f(c_1|(c_2|c_2))$. It is concluded from Definition 4.1 (2) that $p \leq f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$, that is, $c_2 \in f_p$. Therefore, f_p is a filter of C.

($\Leftarrow$) Let $f_p \neq \emptyset$ is a filter of C.

- Let $c \in C$ such that $f(c) > f(1)$. If $p = \frac{f(c)+f(1)}{2}$, then $f(1) < p < f(c)$. So, $1 \notin f_p$ which contradicts with $(SF-3)$. Hence, $f(c) \leq f(1)$.
- Suppose that $c_1, c_2 \in C$ such that $f(c_2) < f(c_1) \wedge f(c_1|(c_2|c_2))$. If $f(c_1) = \gamma$, $f(c_2) = \theta$ and $f(c_1|(c_2|c_2)) = \lambda$, then $\theta < \min(\gamma, \lambda)$. Consider $\lambda_1 = \frac{1}{2}(\theta + \min(\gamma, \lambda))$. Then $\theta < \lambda_1 < \gamma$ and $\theta < \lambda_1 < \lambda$. For $p = \lambda_1 \in (0, 1]$, $c_1 \in f_p$ and $c_1|(c_2|c_2) \in f_p$ but $c_2 \notin f_p$ which contradicts with $(SF-4)$. Thus, $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$. □

Theorem 4.4. *Let f be a fuzzy filter of C. Then f is a fuzzy ultra filter of C if and only if f_p is either empty or an ultra filter of C for each $p \in [0, 1]$.*

Proof. Assume that f is a fuzzy ultra filter of C, and $f_p \neq \emptyset$. Let $c_1 \vee c_2 \in f_p$, i.e., $p \leq f(c_1 \vee c_2)$. Then $p \leq f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$ from Theorem 4.3. So, $p \leq f(c_1)$ or $p \leq f(c_2)$, i.e., $c_1 \in f_p$ or $c_2 \in f_p$. Hence, f_p is an ultra filter of C.

Conversely, suppose that f_p is an ultra filter of C. Let $p = f(c_1 \vee c_2)$, i. e., $c_1 \vee c_2 \in f_p$. Then $c_1 \in f_p$ or $c_2 \in f_p$ from Lemma 3.5. Thus, $f(c_1 \vee c_2) = p \leq f(c_1)$ or $f(c_1 \vee c_2) \leq f(c_2)$, and so, $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$. Therefore, f is a fuzzy ultra filter of C. □

Corollary 4.4. *Let f be a fuzzy filter of C. Then f is a fuzzy ultra filter of C if and only if $f_{f(1)}$ is an ultra filter of C.*

Corollary 4.5. *Let P be a nonempty subset of C. Then P is an ultra filter of C if and only if χ_P is a fuzzy ultra filter of C in which χ_P is the characterictic function of P.*

5. Conclusion

In the present work, we have studied on (fuzzy) filters of Sheffer stroke BL-algebras, and the relationships between them. After giving basic definitions and notions about Sheffer stroke BL-algebra, we introduce some types of (fuzzy) filters of a Sheffer stroke BL-algebra, and present their some properties. Then we show that f is a fuzzy filter of a Sheffer stroke BL-algebra if and only if f_p is empty or is its filter for any $p \in (0, 1]$, and it holds in the case of (fuzzy) ultra filter. Indeed, it is concluded that above property holds for $p = f(1)$ and for the characteristic function of a nonempty subset of a Sheffer stroke BL-algebra. In a similar way, it can be examined relationships between them by defining some kinds of (fuzzy) ideals of Sheffer stroke BL-algebras.

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GENERALIZED EXTENDED RIEMANN-LIOUVILLE TYPE FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT. In this paper, we present new extensions of incomplete gamma, beta, Gauss hypergeometric, confluent hypergeometric function and Appell-Lauricella hypergeometric functions, by using the extended Bessel function due to Boudjelkha [4]. Some recurrence relations, transformation formulas, Mellin transform and integral representations are obtained for these generalizations. Further, an extension of the Riemann-Liouville fractional derivative operator is established.

1. INTRODUCTION

In recent years, incomplete gamma functions have been used in many problems in applied mathematics, statistics, engineering and many other fields including physics and biology. Most generally, special functions became powerful tools to treat all these areas. Classical gamma and Euler's beta functions are defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad \text{Re}(\alpha) > 0, \tag{1.1}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt, \tag{1.2}$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \text{Re}(x) > 0, \text{Re}(y) > 0. \tag{1.3}$$

Key words and phrases. Generalized extended incomplete gamma function, generalized extended beta function, extended Riemann-Liouville fractional derivative, Mellin transform, extended Gauss hypergeometric function, integral representation.

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Using an exponential regulazing term, Chaudhry et al. [9] extended the incomplete gamma function as follows

$$\gamma(\alpha, x; p) = \int_0^x t^{\alpha-1} e^{-t-\frac{p}{t}} dt, \quad \mathrm{Re}(p) > 0; \, p = 0, \, \mathrm{Re}(\alpha) > 0, \tag{1.4}$$

$$\Gamma(\alpha, x; p) = \int_x^\infty t^{\alpha-1} e^{-t-\frac{p}{t}} dt. \tag{1.5}$$

They proved the following recurrence formula

$$\gamma(\alpha, x; p) + \Gamma(\alpha, x; p) = 2p^{\alpha/2} K_\alpha(2\sqrt{p}), \quad \mathrm{Re}(p) > 0,$$

where $K_\alpha(z)$ is the Macdonald function, known also as modified Bessel function of the third kind, defined for any $\mathrm{Re}(z) > 0$ by

$$K_\alpha(z) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1} e^{-t-z^2/4t} dt.$$

A first extension of Euler's beta function is given by Chaudhry et al. [8] as follows

$$B(x, y, p) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt, \quad \mathrm{Re}(p) > 0; \, p = 0, \, \mathrm{Re}(x) > 0, \, \mathrm{Re}(y) > 0. \tag{1.6}$$

These extensions are useful and provide new connections with error and Whittaker functions. For $p = 0$, (1.4), (1.5) and (1.6) will be reduced to known incomplete gamma and beta functions (1.1), (1.2) and (1.3), respectively. Instead of using the exponential function, Chaudhry and Zubair [11] proposed a generalized extension of (1.4), (1.5) in the following form

$$\gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \tag{1.7}$$

$$\Gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \tag{1.8}$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(p) > 0$, $-\infty < \alpha < \infty$.

Nowadays, many authors are developing new extensions of Euler's gamma, beta and hypergeometric functions based on the paper of Chaudhry and Zubair [11] by considering an exponential kernel and some modified special functions (see for more details [13,14,20,22,23,25–27]). Very recently, Agarwal et al. [1] developed an extension of the Euler's beta function as follows

$$B_\mu(x, y; p; m) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) dt, \tag{1.9}$$

where $x, y \in \mathbb{C}$, $m > 0$ and $\mathrm{Re}(p) > 0$.

In the present paper, we introduce a new generalized incomplete gamma and Euler's beta functions by substituting in (1.7), (1.8) and (1.9) the Macdonald function $K_\alpha(z)$

by it's extended one developed by Boudjelkha [4], namely

$$R_K(z,\alpha,q,\lambda) = \frac{(z/2)^\alpha}{2}\int_0^\infty t^{-\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}}dt, \tag{1.10}$$

where $|\arg z^2| < \pi/2$, $0 < q \le 1$ and $-1 \le \lambda \le 1$.

Clearly, when $\lambda = 0$ and $q = 1$, $R_K(z,\alpha,q,\lambda)$ is reduced to $K_\alpha(z)$. Moreover, Boudjelkha proved that the $R_K(z,-\alpha,q,\lambda)$ function can be expanded in terms of $K_\alpha(z)$ as follows

$$R_K(z,-\alpha,q,\lambda) = \sum_{n=0}^\infty \lambda^n \frac{K_\alpha(z\sqrt{q+n})}{(q+n)^{\alpha/2}}, \quad \mathrm{Re}(z^2) > 0,\ 0<q\le 1,\ -1\le\lambda\le 1,$$

and showed that the behavior of the function $R_K(z,-\alpha,q,\lambda)$ for small values of z is described by the asymptotic formulas:

$$R_K(z,-\alpha,q,\lambda) \sim \begin{cases} \frac{1}{2}\frac{\Gamma(-z)}{(z/2)^{-\alpha}}(1-\lambda)^{-1},\ z\to 0,\ -1<\lambda<1, & \mathrm{Re}(\alpha)<0, \\ \frac{1}{2}\frac{\Gamma(z)}{(z/2)^{\alpha}}\Phi(\lambda,\alpha,q),\ z\to 0,\ -1\le\lambda\le 1, & \mathrm{Re}(\alpha)>1, \end{cases}$$

where $\Phi(\lambda,\alpha,q)$ stands for the Lerch function. As for the asymptotic behavior of this function, when $z\to\infty$, it is given by

$$R_K(z,-\alpha,q,\lambda) \sim \sqrt{\frac{\pi}{2z}}\cdot\frac{e^{-z\sqrt{q}}}{q^{\alpha/2+1/4}}, \quad \text{as } z\to\infty,\ |\arg z| < \frac{\pi}{4},\ -1\le\lambda\le 1.$$

In particular, when $q = 1$, we have

$$R_K(z,-\alpha,1,\lambda) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad \text{as } z\to\infty,\ |\arg z| < \frac{\pi}{4},$$

which is the same asymptotic formula as that of K_α.

Further, by using the generalized extended beta function we get other extensions of Gauss hypergeometric, confluent hypergeometric, Appell and Lauricella hypergeometric functions and we investigate some of their properties.

Recently, fractional derivative operators become significant research topics due to their wide applications in various areas including mathematical, physical, life sciences and engineering problems. To cite only a few of this operator's applications, we refer to [5–7, 16, 29] and the references therein. The use of fractional derivative operators in obtaining generating relations for some special functions can be found in [22, 28]. There are two important fractional derivatives operators: Riemann-Liouville and Caputo operators. Undoubtedly, the difference between them is very important for applications to differential equations because of required initial conditions which are of different types (see e.g [19] and [31]). It is worth being pointed out that nowadays a great attention is devoted to develop extensions of fractional differential operators, readers may refer to [1–3, 5–7, 17, 18, 21–23, 30]. Making use of the R_K function and inspired by the work of Agarwal et al. [1], we introduce new generalized incomplete Riemann-Liouville fractional derivative operators, and we obtain some generating relations involving generalized extended Gauss hypergeometric function.

The paper is organized as follows. In Section 2, we introduce the generalized extended incomplete Gamma and Euler's beta functions, some of their properties are investigated. Section 3 is devoted to introduce extended hypergeometric and confulent hypergeometric functions by the extended Euler's beta function given in Section 2, their related properties are established. The extended Appell and Lauricella hypergeometric functions are given in Section 4. In Section 5, we give another result which consits to introduce the generalized extended Riemann Liouville fractional derivative operator and establish most important properties such Mellin transform among others. Finally, in the last section, we obtain linear and bilinear generating relations for the generalized extended hypergeometric functions.

2. The Generalized Extended Incomplete Gamma and Euler's Beta Functions

In this section, we define new extended incomplete Gamma and Euler's beta functions based on the extension of Bessel function (1.10) and we give some properties.

2.1. The generalized extended incomplete Gamma function.

Definition 2.1. The generalized extended incomplete gamma functions are given by

$$\gamma_\mu(\alpha,x;q;\lambda;p)=\sqrt{\frac{2p}{\pi}}\int_0^x t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{2.1}$$

$$\Gamma_\mu(\alpha,x;q;\lambda;p)=\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{2.2}$$

where $\mathrm{Re}(x)>0$, $0<q\leq 1$, $-1\leq\lambda\leq 1$ and $\mathrm{Re}(p)>0$.

Remark 2.1. When $\lambda=0$ and $q=1$, (2.1) and (2.2) are respectively reduced to the extended incomplete gamma functions (1.7) and (1.8) defined by Chaudhry and Zubair [10, 11].

Proposition 2.1 (Decomposition theorem)**.**

$$\begin{aligned}\Gamma_\mu(\alpha,x;q;\lambda;p)+\gamma_\mu(\alpha,x;q;\lambda;p)=&\frac{\Gamma(\alpha+\mu)}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu}\Phi_{1-\frac{\alpha+\mu}{2},\frac{1}{2}-\frac{\alpha+\mu}{2}}\left(\lambda,\mu+\frac{1}{2},q,\frac{p^2}{16}\right)\\&+\frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2\sqrt{\pi}}\left(\frac{p}{2}\right)^{\alpha}\Phi_{\frac{1}{2},\frac{\alpha+\mu+2}{2}}\left(\lambda,\frac{\mu-\alpha+1}{2},q,\frac{p^2}{16}\right)\\&-\frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2\sqrt{\pi}}\left(\frac{p}{2}\right)^{\alpha+1}\Phi_{\frac{3}{2},\frac{\alpha+\mu+3}{2}}\left(\lambda,\frac{\mu-\alpha}{2},q,\frac{p^2}{16}\right),\end{aligned}$$

with $\mathrm{Re}(p)>0$, $-\infty<\alpha<\infty$ *and*

$$\Phi_{b_1,b_2}(\lambda,s,q,\xi)=\int_0^\infty\frac{t^{s-1}e^{-qt}}{1-\lambda e^{-t}}\,{}_0F_2\left(\begin{matrix}-\\ \\ b_1,b_2\end{matrix};-\frac{\xi}{t}\right)dt$$

$$= \int_0^\infty \frac{t^{s-1}e^{-(q-1)t}}{e^t-\lambda}\,{}_0F_2\left(\begin{matrix} - \\ b_1, b_2 \end{matrix}; -\frac{\xi}{t}\right)dt, \tag{2.3}$$

$s \in \mathbb{C}$, $\mathrm{Re}(\xi) > 0$ *and* $b_1, b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Proof. We have

$$\begin{aligned}
&\Gamma_\mu(\alpha, x; q; \lambda; p) + \gamma_\mu(\alpha, x; q; \lambda; p) \\
=&\sqrt{\frac{2p}{\pi}} \int_0^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) dt \\
=&\frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu} \int_0^\infty t^{\alpha+\mu-1} e^{-t} \left(\int_0^\infty \tau^{\mu-\frac{1}{2}} \frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}} d\tau\right) dt \\
=&\frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu} \int_0^\infty \frac{\tau^{\mu-\frac{1}{2}} e^{-q\tau}}{1-\lambda e^{-\tau}} \left(\int_0^\infty t^{\alpha+\mu-1} e^{-t} e^{-\frac{p^2}{4t^2\tau}} dt\right) d\tau. \qquad (2.4)
\end{aligned}$$

Using the integral [24, page 31, (6)], we obtain

(2.5)

$$\begin{aligned}
\int_0^\infty t^{\alpha+\mu-1} e^{-t} e^{-\frac{p^2}{4t^2\tau}} dt =&\Gamma(\alpha+\mu)\,{}_0F_2\left(\begin{matrix} - \\ 1-\frac{\alpha+\mu}{2}, \frac{1}{2}-\frac{\alpha+\mu}{2} \end{matrix}; -\frac{p^2}{16\tau}\right) \\
&+ \frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu}{2}} {}_0F_2\left(\begin{matrix} - \\ \frac{1}{2}, \frac{\alpha+\mu+2}{2} \end{matrix}; -\frac{p^2}{16\tau}\right) \\
&- \frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu+1}{2}} {}_0F_2\left(\begin{matrix} - \\ \frac{3}{2}, \frac{\alpha+\mu+3}{2} \end{matrix}; -\frac{p^2}{16\tau}\right).
\end{aligned}$$

Finally, substituting (2.5) in (2.4) and by using the notation (2.3) we get the desired result. □

Proposition 2.2 (Recurrence relation)**.**

$$\begin{aligned}
\Gamma_\mu(\alpha+1, x; q; \lambda; p) =&(\alpha+\mu)\Gamma_\mu(\alpha, x; q; \lambda; p) + p\Gamma_{\mu-1}(\alpha-1, x; q; \lambda; p) \\
&+ \sqrt{\frac{2p}{\pi}} x^{\alpha-\frac{1}{2}} e^{-x} R_K\left(\frac{p}{x}, -\mu-\frac{1}{2}, q, \lambda\right),
\end{aligned}$$

where $\mathrm{Re}(p) > 0$, $-\infty < \alpha < \infty$.

Proof. We have

$$\frac{d}{dt} R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) = \frac{d}{dt}\left[\frac{\left(\frac{p}{2t}\right)^{-\mu-\frac{1}{2}}}{2} \int_0^\infty \tau^{\mu-\frac{1}{2}} \frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}} d\tau\right]$$

$$= \frac{\mu+\frac{1}{2}}{t} R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) + \frac{p}{t^2} R_K\left(\frac{p}{t}, -\mu+\frac{1}{2}, q, \lambda\right). \tag{2.6}$$

Differentiating $t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right)$ with respect to t and by using (2.6), we get

$$\frac{d}{dt}\left[t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right)\right] \tag{2.7}$$

$$\begin{aligned} =&(\alpha+\mu)t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) + p\, t^{\alpha-\frac{5}{2}}e^{-t}R_K\left(\frac{p}{t}, -\mu+\frac{1}{2}, q, \lambda\right) \\ &- t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right). \end{aligned} \tag{2.8}$$

Multiplying both sides of (2.7) by $\sqrt{\frac{2p}{\pi}}$ and integrating from x to ∞ and using (2.2), we find

$$\begin{aligned} &0 - \sqrt{\frac{2p}{\pi}} x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x}, -\mu-\frac{1}{2}, q, \lambda\right) \\ =&(\alpha+\mu)\Gamma_\mu(\alpha, x; q; \lambda; p) + p\Gamma_{\mu-1}(\alpha-1, x; q; \lambda; p) - \Gamma_\mu(\alpha+1, x; q; \lambda; p), \end{aligned}$$

which can be also written as

$$\begin{aligned} \Gamma_\mu(\alpha+1, x; q; \lambda; p) =&(\alpha+\mu)\Gamma_\mu(\alpha, x; q; \lambda; p) + p\Gamma_{\mu-1}(\alpha-1, x; q; \lambda; p) \\ &+ \sqrt{\frac{2p}{\pi}} x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x}, -\mu-\frac{1}{2}, q, \lambda\right). \end{aligned}$$

□

Proposition 2.3. *The following formula holds*

$$\begin{aligned} &\Gamma_{\mu-1}(\alpha, x; 1; \lambda; p) - \Gamma_{\mu+1}(\alpha, x; 1; \lambda; p) + \frac{2\mu+1}{p}\Gamma_\mu(\alpha+1, x; 1; \lambda; p) \\ =&\lambda\frac{\partial}{\partial\lambda}\Gamma_{\mu+1}(\alpha, x; 1; \lambda; p), \end{aligned}$$

where $\operatorname{Re}(p) > 0$, $-\infty < \alpha < \infty$.

Proof. By using (2.2), for $q = 1$ and the following relation [4, (22)], we get

$$R_K(z, -\alpha+1, 1, \lambda) - R_K(z, -\alpha-1, 1, \lambda) + \frac{2\alpha}{z}R_K(z, -\alpha, 1, \lambda) = \lambda\frac{\partial}{\partial\lambda}R_K(z, -\alpha-1, 1, \lambda).$$

□

Proposition 2.4 (Laplace transform)**.** *Let*

$$H(\tau) = \begin{cases} 1, & \tau > 0, \\ 0, & \tau < 0, \end{cases}$$

be the Heaviside unit step function and $\mathcal{L}$ *be the Laplace transform operator. Then*

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right)H(t-x); s\right\} = \sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha, sx; q; \lambda; sp), \tag{2.9}$$

(2.10)
$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)H(t-x)H(t);s\right\}=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\gamma_\mu(\alpha,sx;q;\lambda;sp),$$

where $x>0$, $\mathrm{Re}(p)>0$, $-\infty<\alpha<\infty$.

Proof. We have

$$\begin{aligned}&\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)H(t-x);s\right\}\\ =&\int_0^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}H(t-x)dt\\ =&\int_x^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}dt.\end{aligned}$$

Substituting $t=\frac{\tau}{s}$, $dt=\frac{d\tau}{s}$, we get

$$\begin{aligned}&\int_x^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}dt\\ =&s^{-\alpha+\frac{1}{2}}\int_{sx}^\infty \tau^{\alpha-\frac{3}{2}}e^{-\tau}R_K\left(\frac{sp}{\tau},-\mu-\frac{1}{2},q,\lambda\right)dt=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha,sx;q;\lambda;sp).\end{aligned}$$

The proof of (2.10) is omitted since it is quite similar as that of (2.9). □

Proposition 2.5 (Parametric differentiation)**.**

$$\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right)=-\frac{1}{p}\left[\mu\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\right].$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right)=&\frac{1}{2p}\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt\\ &+\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right)dt.\end{aligned}\tag{2.11}$$

We have

$$\begin{aligned}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right)=&-\frac{\mu+\frac{1}{2}}{p}\frac{(p/2t)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\ &-\frac{1}{t}\frac{(p/2t)^{-\mu+\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{3}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\ =&-\frac{\mu+\frac{1}{2}}{p}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\\ &-\frac{1}{t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right),\end{aligned}\tag{2.12}$$

Finally, by substituting (2.12) into (2.11) we get the desired result. □

2.2. **The generalized extended beta function.**

Definition 2.2. The generalized extended beta function is given by

$$B_\mu(x,y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{2.13}$$

where $x,y\in\mathbb{C}$, $0<q\leq 1$, $-1\leq\lambda\leq 1$, $m>0$ and $\operatorname{Re}(p)>0$.

Remark 2.2. Taking $\lambda=0$ and $q=1$, (2.13) is reduced to the extended Euler's beta function (1.9) defined by Agarwal et al. [1].

Proposition 2.6 (Functional relations)**.** 1. *The following formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=B_\mu(x+1,y;q;\lambda;p;m)+B_\mu(x,y+1;q;\lambda;p;m). \tag{2.14}$$

2. *Let $n\in\mathbb{N}$. Then the following summation formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{k=0}^{n}B_\mu(x+k,y+n-k;q;\lambda;p;m). \tag{2.15}$$

Proof. 1. The right-hand side of (2.14) yields to

$$\sqrt{\frac{2p}{\pi}}\int_0^1\left\{t^{x-\frac{1}{2}}(1-t)^{y-\frac{3}{2}}+t^{x-\frac{3}{2}}(1-t)^{y-\frac{1}{2}}\right\}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which, after simplification, implies

$$\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which is equal to the left-hand side of (2.14).

2. The case $n=0$ of (2.15) holds easily. The case $n=1$ of (2.15) is just (2.14). For the other cases we can easily proceed by induction on n. ☐

Proposition 2.7. *The following formula holds*

$$B_\mu(x,1-y;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_\mu(x+n,1;q;\lambda;p;m). \tag{2.16}$$

Proof. We have

$$B_\mu(x,1-y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{-y-\frac{1}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt. \tag{2.17}$$

By substituting the formula

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!},\quad |t|<1,\ y\in\mathbb{C},$$

in the right-hand of (2.17) and after interchanging the order of integral and summation, we get (2.16). ☐

Proposition 2.8. *The following formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{n=0}^{\infty}B_\mu(x+n,y+1;q;\lambda;p;m).$$

Proof. By substituting again the formula

$$(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n,\quad |t|<1,$$

in the right-hand of (2.13) and similarly as in the proof of Proposition 2.7 we get the desired result. □

Lemma 2.1. *Let* $\mathcal{M}$ *be the Mellin transform operator. Then*

$$\mathcal{M}\{R_K(z,-\alpha,q,\lambda),z\to s\}=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s+\alpha}{2}\right)\Phi\left(\lambda,\frac{s+\alpha}{2},q\right),$$

where $0<q\le 1$, *or* $-1\le\lambda<1$, $\mathrm{Re}(s)>|\mathrm{Re}(\alpha)|$ *or* $\lambda=1$, $\mathrm{Re}(s)>\max\{\mathrm{Re}(\alpha),2-\mathrm{Re}(\alpha)\}$ *and* $\Phi\left(\lambda,\frac{s+\alpha}{2},q\right)$ *stands for the Lerch function* (*see* [12, 15]).

Proof.

$$\begin{aligned}\mathcal{M}\{R_K(z,-\alpha,q,\lambda),z\to s\}&=\int_0^\infty z^{s-1}R_K(z,-\alpha,q,\lambda)dz\\&=2^{\alpha-1}\int_0^\infty z^{s-\alpha-1}\left(\int_0^\infty t^{\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}}dt\right)dz\\&=2^{\alpha-1}\int_0^\infty t^{\alpha-1}\frac{e^{-qt}}{1-\lambda e^{-t}}\left(\int_0^\infty z^{s-\alpha-1}e^{-z^2/4t}dz\right)dt\\&=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\int_0^\infty t^{\frac{s+\alpha}{2}-1}\frac{e^{-qt}}{1-\lambda e^{-t}}dt\\&=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s+\alpha}{2}\right)\Phi\left(\lambda,\frac{s+\alpha}{2},q\right).\end{aligned}$$ □

Proposition 2.9 (Mellin transform)**.** *The following expression holds true*

$$\begin{aligned}\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m),p\to s\}=&\frac{2^{s-1}}{\sqrt{\pi}}B\left(x+ms+\frac{m-1}{2},y+ms+\frac{m-1}{2}\right)\\&\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right),\end{aligned}$$

where $x,y\in\mathbb{C}$, $m>0$ *and* $0<q\le 1$ *or* $1\le\lambda<1$,

$$\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),-1-\mathrm{Re}(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(y)}{m}\right\},$$

or $\lambda=1$,

$$\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),1-\mathrm{Re}(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(y)}{m}\right\}.$$

Proof.

$$
\begin{aligned}
&\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m), p\to s\}\\
=&\int_0^\infty p^{s-1}B_\mu(x,y;q;\lambda;p;m)dp\\
=&\int_0^\infty p^{s-1}\sqrt{\frac{2p}{\pi}}\left(\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\right)dp\\
=&\sqrt{\frac{2}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\left(\int_0^\infty p^{s+\frac{1}{2}-1}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dp\right)dt\\
=&\sqrt{\frac{2}{\pi}}\int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}}(1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}}dt\int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u,-\mu-\frac{1}{2},q,\lambda\right)du\\
=&\sqrt{\frac{2}{\pi}}B\left(x+ms+\frac{m-1}{2},y+ms+\frac{m-1}{2}\right)\int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u,-\mu-\frac{1}{2},q,\lambda\right)du.
\end{aligned}
$$

Finally, by using Lemma 2.1 we get the desired result. □

3. Extended Gauss Hypergeometric and Confluent Hypergeometric Functions

We use the generalized extended beta function (2.13) to extend hypergeometric and confluent hypergeometric functions, respectively, as follows.

Definition 3.1. The extended Gauss hypergeometric function $F_\mu(a,b;c;z;q;\lambda;p;m)$ and the confluent hypergeometric function $\Phi_\mu(b;c;z;q;\lambda;p;m)$ are respectively defined by

$$F_\mu(a,b;c;z;q;\lambda;p;m)=\sum_{n=0}^\infty (a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^n}{n!}, \tag{3.1}$$

$|z|<1$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $0<q\leq 1$, $-1\leq\lambda\leq 1$, $m>0$, $\mathrm{Re}(p)>0$,

$$\Phi_\mu(b;c;z;q;\lambda;p;m)=\sum_{n=0}^\infty \frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^n}{n!},$$

$z\in\mathbb{C}$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $-1\leq\lambda\leq 1$, $m>0$, $\mathrm{Re}(p)>0$.

Remark 3.1. Taking $\lambda=0$ and $q=1$, (3.1) reduces to the extended Gauss hypergeometric function defined by Agarwal et al. [1, Definition 2.8].

Proposition 3.1 (Integral representation)**.** 1. *The following integral representation for the extended Gauss hypergeometric function $F_\mu(a,b;c;z;q;\lambda;p;m)$ is valid*

$$
\begin{aligned}
F_\mu(a,b;c;z;q;\lambda;p;m)=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}(1-zt)^{-a}\\
&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,
\end{aligned}\tag{3.2}
$$

where $\arg(1-z) < \pi$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

2. *The following integral representation for the extended confluent hypergeometric function* $\Phi_\mu(b; c; z; q; \lambda; p; m)$ *is valid*

$$\Phi_\mu(b; c; z; q; \lambda; p; m) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b, c-b)} \int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}} e^{zt} \times R_K\left(\frac{p}{t^m(1-t)^m}, -\mu - \frac{1}{2}, q, \lambda\right) dt, \tag{3.3}$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. 1. By using (2.13) and the generalized binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!}, \quad |zt| < 1,$$

we get the required result.

2. Similarly as in the proof of 1. □

Proposition 3.2 (Differentiation formula)**.** (a) *For* $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{F_\mu(a, b; c; z; q; \lambda; p; m)\} = \frac{(a)_n (b)_n}{(c)_n} F_\mu(a+n, b+n; c+n; z; q; \lambda; p; m), \tag{3.4}$$

where $|z| < 1$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

(b) *For* $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{\Phi_\mu(b; c; z; q; \lambda; p; m)\} = \frac{(b)_n}{(c)_n} \Phi_\mu(b+n; c+n; z; q; \lambda; p; m),$$

where $z \in \mathbb{C}$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. (a) For $n = 1$, we have

$$\begin{aligned} \frac{d}{dz}\{F_\mu(a, b; c; z; q; \lambda; p; m)\} &= \sum_{n=1}^{\infty} (a)_n \frac{B_\mu(b+n, c-b; q; \lambda; p; m)}{B(b, c-b)} \cdot \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} (a)_{n+1} \frac{B_\mu(b+n+1, c-b; q; \lambda; p; m)}{B(b, c-b)} \cdot \frac{z^n}{n!}. \end{aligned} \tag{3.5}$$

Using identities $B(b, c-b) = \frac{c}{b} B(b+1, c-b)$ and $(a)_{n+1} = a(a+1)_n$ in (3.5), we get

$$\begin{aligned} \frac{d}{dz}\{F_\mu(a, b; c; z; q; \lambda; p; m)\} &= \frac{ab}{c} \sum_{n=0}^{\infty} (a+1)_n \frac{B_\mu(b+n+1, c-b; q; \lambda; p; m)}{B(b+1, c-b)} \cdot \frac{z^n}{n!} \\ &= \frac{ab}{c} F_\mu(a+1, b+1; c+1; z; q; \lambda; p; m), \end{aligned}$$

and hence

$$\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} = \frac{ab}{c}F_\mu(a+1,b+1;c+1;z;q;\lambda;p;m). \tag{3.6}$$

Then, by using (3.6) repeatedly, we get (3.4).

The proof of part (b) is similar as that of part (a). □

Proposition 3.3 (Transformation formulas)**.**

1. *For* $\arg(1-z) < \pi$ *we have*

$$F_\mu(a,b;c;z;q;\lambda;p;m) = (1-z)^{-a}F_\mu\left(a, c-b; c; \frac{z}{z-1}; q;\lambda;p;m\right),$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

2. $\Phi_\mu(b;c;z;q;\lambda;p;m) = e^z\Phi_\mu(c-b;c;-z;q;\lambda;p;m)$, *where* $z \in \mathbb{C}$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. Replacing t by $1-t$ in the integral representations (3.2) and (3.3). □

4. Extended Appell and Lauricella Hypergeometric Functions

Definition 4.1. Extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, defined by

(4.1)

$$F_{1,\mu}(a,b,c;d;x,y;q;\lambda;p;m) = \sum_{n,k=0}^{\infty}(b)_n(c)_k\frac{B_\mu(a+n+k,d-a;q;\lambda;p;m)}{B(a,d-a)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!},$$

where $|x| < 1$, $|y| < 1$, $\mathrm{Re}(d) > \mathrm{Re}(a) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$,

$$\begin{aligned} F_{2,\mu}(a,b,c;d,e;x,y;q;\lambda;p;m) = &\sum_{n,k=0}^{\infty}(a)_{n+k}\frac{B_\mu(b+n,d-b;q;\lambda;p;m)}{B(b,d-b)} \\ &\times\frac{B_\mu(c+k,e-c;q;\lambda;p;m)}{B(c,e-c)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}, \end{aligned} \tag{4.2}$$

where $|x| + |y| < 1$, $\mathrm{Re}(d) > \mathrm{Re}(b) > 0$, $\mathrm{Re}(e) > \mathrm{Re}(c) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$,

$$\begin{aligned} &F^3_{D,\mu}(a,b,c,d;e;x,y,z;q;\lambda;p;m) \\ &= \sum_{n,k,r=0}^{\infty}(b)_n(c)_k(d)_r\frac{B_\mu(a+n+k+r,e-a;q;\lambda;p;m)}{B(a,e-a)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\cdot\frac{z^r}{r!}, \end{aligned} \tag{4.3}$$

where $|x| < 1$, $|y| < 1$, $|z| < 1$, $\mathrm{Re}(e) > \mathrm{Re}(a) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Remark 4.1. Taking $\lambda = 0$ and $q = 1$, (4.1), (4.2) and (4.3) are reduced to extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$, defined by Agarwal et al. [1, Definitions 2.9, 2.10, 2.11].

Proposition 4.1 (Integral representation)**.** *The following integral representations for the extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, valid*

$$\begin{aligned}&F_{1,\mu}(a,b,c;d;x,y;q;\lambda;p;m)\\=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,d-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{d-a-\frac{3}{2}}(1-xt)^{-b}\times(1-yt)^{-c}\\&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,\\&F_{2,\mu}(a,b,c;d;x,y;q;\lambda;p;m)\\=&\frac{2p}{\pi}\cdot\frac{1}{B(b,d-b)B(c,e-c)}\\&\times\int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}\times w^{b-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}(1-xt-yw)^{-a}\\&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)R_K\left(\frac{p}{w^m(1-w)^m},-\mu-\frac{1}{2},q,\lambda\right)dtdw,\\&F^3_{D,\mu}(a,b,c,d;e;x,y,z;q;\lambda;p;m)\\=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,e-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}(1-xt)^{-b}\times(1-yt)^{-c}(1-zt)^{-d}\\&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt.\end{aligned}$$

Proof. The proofs are very similar to those of Theorems 2.13, 2.15 and 2.16 in [1]. □

5. The Generalized Extended Riemann-Liouville Fractional Derivative Operator

The classical Riemann-Liouville fractional derivative operator is defined by

$$D_z^{\delta}f(z):=\frac{1}{\Gamma(-\delta)}\int_0^z(z-t)^{-\delta-1}f(t)dt, \tag{5.1}$$

where $\mathrm{Re}(\delta)<0$. It coincides with the fractional integral of order $-\delta$. In the case $n-1<\mathrm{Re}(\delta)<n$, $n\in\mathbb{N}$, we write

$$D_z^{\delta}f(z):=\frac{d^n}{dz^n}D_z^{\delta-n}f(z)=\frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\delta)}\int_0^z(z-t)^{n-\delta-1}f(t)dt\right\}.$$

Definition 5.1. The generalized extended Riemann-Liouville fractional derivative is defined as follows

$$D_z^{\delta,\mu;p;q;\lambda;m}f(z):=\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z(z-t)^{-\delta-1}f(t)R_K\left(\frac{pz^{2m}}{t^m(z-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{5.2}$$

where $\operatorname{Re}(\delta) < 0$, $\operatorname{Re}(p) > 0$, $\operatorname{Re}(m) > 0$, $\operatorname{Re}(\mu) \geq 0$ and $0 < q \leq 1$, $-1 \leq \lambda \leq 1$.

For $n-1 < \operatorname{Re}(\delta) < n$, $n \in \mathbb{N}$, we have

$$D_z^{\delta,\mu;p;q;\lambda;m} f(z) := \frac{d^n}{dz^n} D_z^{\delta-n,\mu;p;q;\lambda;m} f(z) = \frac{d^n}{dz^n} \left\{ \frac{1}{\Gamma(n-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{n-\delta-1} f(t) \times R_K \left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu - \frac{1}{2}, q, \lambda \right) dt \right\}.$$

Remark 5.1. 1. Taking $\lambda = 0$ and $q = 1$, the generalized extended Riemann-Liouville fractional derivative operator (5.2) is reduced to the extended Riemann-Liouville fractional derivative operator given by Agarwal et al. [1]

$$D_z^{\delta,\mu;p;m} f(z) := \frac{1}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-\delta-1} f(t) K_{\mu+\frac{1}{2}} \left(\frac{pz^{2m}}{t^m(z-t)^m} \right) dt,$$

where $\operatorname{Re}(\delta) < 0$, $\operatorname{Re}(p) > 0$, $\operatorname{Re}(m) > 0$, $\operatorname{Re}(\mu) > 0$.

2. If $\lambda = 0$, $q = 1$, $m = 0$, $\mu = 0$ and $p \to 0$, then the generalized extended Riemann-Liouville fractional derivative operator (5.2) reduces to the classical Riemann-Liouville fractional derivative operator (5.1).

In order to calculate generalized extended fractional derivatives for some functions, we give two results concerning the generalized extended Riemann-Liouville fractional derivative operator of some elementary functions which will be useful in the sequel.

Lemma 5.1. *Let* $\operatorname{Re}(\delta) < 0$. *Then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} = \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu \left(\beta + \frac{3}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m \right).$$

Proof. Using Definition 5.1 and a local setting $t = zu$, we obtain

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} &= \frac{1}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-\delta-1} t^\beta R_K \left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu - \frac{1}{2}, q, \lambda \right) dt \\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^1 (1-u)^{(-\delta+\frac{1}{2})-\frac{3}{2}} u^{(\beta+\frac{3}{2})-\frac{3}{2}} \\ &\quad \times R_K \left(\frac{p}{u^m(1-u)^m}, -\mu - \frac{1}{2}, q, \lambda \right) du \\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu (\beta + \frac{3}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m). \end{aligned}$$

□

More generally, we give the generalized extended Riemann-Liouville fractional derivative of an analytic function $f(z)$ at the origin.

Lemma 5.2. *Let* $\operatorname{Re}(\delta) < 0$. *If a function* $f(z)$ *is analytic at the origin, then*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}.$$

Proof. Since f is analytic at the origin, its Maclaurin expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (for $|z| < \rho$ with $\rho \in \mathbb{R}^+$ is the convergence radius). By substituting entire power series in Definition 5.1, we obtain

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} =& \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} \\ & \times R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right)\sum_{n=0}^{\infty} a_n t^n dt. \end{aligned}$$

By virtue of the uniform continuity on the convergence disk, we can do integration term by term in the equation above. Thus

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} =& \sum_{n=0}^{\infty} a_n \left\{\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}\right. \\ & \left. \times R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right) t^n dt\right\} \\ =& \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}. \end{aligned}$$

□

Corollary 5.1.

$$D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} = \frac{z^{-\delta}}{\Gamma(-\delta)} B\left(\frac{3}{2}, -\delta+\frac{1}{2}\right) F_\mu\left(\alpha, \frac{3}{2}, -\delta+2; z; q; \lambda; p; m\right),$$

where $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\delta) < 0$.

Proof. Using binomial theorem for $(1-z)^{-\alpha}$ and Lemma 5.1, we obtain:

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} &= D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n \frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!} D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\} \\ &= \frac{z^{-\delta}}{\Gamma(-\delta)}\sum_{n=0}^{\infty}(\alpha)_n B_\mu\left(n+\frac{3}{2}, -\delta+\frac{1}{2}; p, q; \lambda; m\right)\frac{z^n}{n!}. \end{aligned}$$

Hence, the result. □

Combining previous lemmas, we obtain the generalized extended derivative of the product of analytic function with a power function.

Theorem 5.1. *Let* $\mathrm{Re}(\delta) < 0$. *Suppose that a function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < \rho$, *for some* $\rho \in \mathbb{R}^+$. *Then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\}$$

$$= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)} \sum_{n=0}^{\infty} a_n B_\mu \left(\beta + n + \frac{1}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m\right) z^n.$$

A subsequent result can be given as follows.

Theorem 5.2. *For* $\mathrm{Re}(\delta) > \mathrm{Re}(\beta) > -\frac{1}{2}$, *we have*

$$\begin{aligned} &D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} \\ =&\frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) F_\mu\left(\alpha, \beta+\frac{1}{2}; \delta+1; z; q; \lambda; p; m\right), \end{aligned}$$

where $|z| < 1$, $\alpha \in \mathbb{C}$.

Proof. The result is easily established by taking $f(z) = (1-z)^{-\alpha}$, so we have

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} &= D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}\sum_{k=0}^{\infty}(\alpha)_k \frac{z^k}{k!}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k-1}\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{B_\mu(\beta+k+\frac{1}{2}, \delta-\beta+\frac{1}{2}; p; q; \lambda; m)}{\Gamma(\delta-\beta)} z^{\delta+k-1}. \end{aligned}$$

By the expression (3.1), we get

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} =&\frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) \\ &\times F_\mu\left(\alpha, \beta+\frac{1}{2}; \delta+1; z; q; \lambda; p; m\right). \end{aligned}$$ ☐

Theorem 5.3. *For* $\mathrm{Re}(\delta) > \mathrm{Re}(\beta) > -\frac{1}{2}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\gamma) > 0$, $|az| < 1$ *and* $|bz| < 1$. *Then, the following generating relation holds true*

$$\begin{aligned} &D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} \\ =&\frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}\right) F_{1,\mu}\left(\beta+\frac{1}{2}, \alpha, \gamma; \delta+1; az, bz; q; \lambda; p; m\right). \end{aligned}$$

Proof. By applying the binomial Theorem to $(1-az)^{-\alpha}$ and $(1-bz)^{-\gamma}$ and making use of Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned} &D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} \\ =&D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(\alpha)_k(\gamma)_r \frac{(az)^k}{k!} \cdot \frac{(bz)^r}{r!}\right\} \\ =& \sum_{k,r=0}^{\infty} (\alpha)_k(\gamma)_r D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k+r-1}\} \frac{a^k}{k!} \cdot \frac{b^r}{r!} \end{aligned}$$

$$=z^{\delta-1}\sum_{k,r=0}^{\infty}(\alpha)_k(\gamma)_r\frac{B_\mu(\beta+k+r+\frac{1}{2},\delta-\beta+\frac{1}{2};p;q;\lambda;m)}{\Gamma(\delta-\beta)}\cdot\frac{(az)^k}{k!}\cdot\frac{(bz)^r}{r!}.$$

By using (4.1), we can get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{1,\mu}\left(\beta+\frac{1}{2},\alpha,\gamma;\delta+1;az,bz;q;\lambda;p;m\right). \quad \square$$

Theorem 5.4. *For* $\mathrm{Re}(\delta)>\mathrm{Re}(\beta)>-\frac{1}{2}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\tau)>0$, $|az|<1$, $|bz|<1$ *and* $|cz|<1$, *we have*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}(1-cz)^{-\tau}\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{D,\mu}^3\left(\beta+\frac{1}{2},\alpha,\gamma,\tau;\delta+1;az,bz;q;\lambda;p;m\right).$$

Proof. The proof is similar to that of Theorem 5.3, it is sufficient to use the binomial Theorem for $(1-az)^{-\alpha}$, $(1-bz)^{-\gamma}$, $(1-cz)^{-\tau}$, then applying Lemmas 5.1 and 5.2. $\square$

Theorem 5.5. *For* $\mathrm{Re}(\delta)>\mathrm{Re}(\beta)>-\frac{1}{2}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\tau)>\mathrm{Re}(\gamma)>0$, $\left|\frac{x}{1-z}\right|<1$ *and* $|x|+|z|<1$, *we have*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=z^{\delta-1}\frac{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}{\Gamma(\delta-\beta)}F_{2,\mu}\left(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m\right).$$

Proof. By the binomial formula and according to Definition 3.1, we expand $z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m)$ to get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\cdot\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}\left(\frac{x}{1-z}\right)^n\right\}$$
$$=\sum_{n=0}^{\infty}(\alpha)_n\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}\frac{x^n}{n!}.$$

In order to exhibit $F_{2,\mu}$, we apply Theorem 5.2 for $D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}$ and substitute the extended hypergeometric function F_μ by its series representation, we obtain

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)\sum_{n,k=0}^{\infty}(\alpha)_{n+k}\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}$$

$$\times \frac{B_\mu(\beta+k+\frac{1}{2},\delta-\beta+\frac{1}{2};q;\lambda;p;m)}{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}\cdot\frac{x^n z^k}{n!z!}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{2,\mu}\left(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m\right).$$

This completes the proof. □

Proposition 5.1 (Mellin transform)**.** *The following expression holds true*

$$\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta,p\to s\}=2^{s-1}z^{\beta-\delta}\frac{1}{\sqrt{\pi}}B\left(\beta+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)$$
$$\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right),$$

for $\operatorname{Re}(\mu)\geq 0$*,* $m>0$ *and* $\operatorname{Re}(s)>\max\left\{\operatorname{Re}(\mu),-\frac{1}{2}-\frac{1}{m}-\frac{\operatorname{Re}(\beta)}{m},\frac{\operatorname{Re}(\delta)}{m}-\frac{1}{2}\right\}$.

Proof. We can prove this result by applying Mellin transform and using Lemma 5.1.

$$\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta,p\to s\}=\frac{1}{\Gamma(-\delta)}\int_0^\infty p^{s-1}z^{\beta-\delta}B_\mu\left(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m\right)dp$$
$$=\frac{z^{\beta-\delta}}{\Gamma(-\delta)}\int_0^\infty p^{s-1}B_\mu\left(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m\right)dp.$$

As the last integral is the Mellin transform of $B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m)$, the result immediately follows via Proposition 2.9. □

Proposition 5.2. *The following expression holds true*

$$\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}(1-z)^{-\beta},p\to s\}$$
$$=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)$$
$$\times\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right){}_2F_1\left(\beta,m\left(s+\frac{1}{2}\right)+1;-\delta+m(2s+1)+1;z\right),$$

where $\operatorname{Re}(\mu)\geq 0$*,* $\operatorname{Re}(\delta)<0$*,* $m>0$*,* $|z|<1$*,* $\operatorname{Re}(s)>\max\left\{\operatorname{Re}(\mu),-\frac{1}{2}+\frac{1}{m},\frac{\delta}{m}-\frac{1}{2}\right\}$ *and* ${}_2F_1$ *is the well-known Gauss hypergeometric function.*

Proof. The result can be proved using the Binomial theorem for $(1-z)^{-\alpha}$ and the Mellin transform of the general term. Indeed,

$$\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\},p\to s\}$$
$$=\mathcal{M}\left\{D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\right\},p\to s\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^n,p\to s\}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}2^{s-1}z^{n-\delta}\frac{1}{\sqrt{\pi}}B\left(n+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)$$
$$\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right).$$
$$=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)$$
$$\times\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}B\left(n+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)z^n$$
$$=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)$$
$$\times\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right){}_2F_1\left(\beta,m\left(s+\frac{1}{2}\right)+1;-\delta+m(2s+1)+1;z\right).\qquad\square$$

6. Generating Function Involving the Extended Generalized Gauss Hypergeometric Function

In this section, we establish some generating functions for the generalized Gauss hypergeometric functions.

Theorem 6.1. *Let* $\mathrm{Re}(\beta)>0$ *and* $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;p;\lambda;m\right)t^n \tag{6.1}$$
$$=(1-t)^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-t};q;p;\lambda;m\right),$$

where $|z|<\min\{1,|1-t|\}$.

Proof. By considering the following elementary identity

$$(1-z)^{-\beta}\left(1-\frac{t}{1-z}\right)^{-\beta}=(1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta}$$

and expanding its left-hand side to give

$$(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta},\quad\text{for }|t|<|1-z|. \tag{6.2}$$

Multiplying both sides of (6.2) by $z^{\alpha-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D^{\alpha-\gamma;\mu;q;p;\lambda;m}$, we find

$$D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n t^n}{n!}z^{\alpha-1}(1-z)^{-\beta-n}\right\}$$

$$=D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

Uniform convergence of the involved series allows us to permute the summation and fractional derivative operator to get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\beta-n}\}t^n$$
$$=(1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

The result easily follows using Theorem 5.2. □

Theorem 6.2. *Let* $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\tau)>0$ *and* $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta-n,\alpha+\frac{1}{2};\gamma+1;z;q;p;\lambda;m\right)t^n$$
$$=(1-t)^{-\beta}F_{1,\mu}\left(\alpha+\frac{1}{2},\tau,\beta;\gamma+1;z;\frac{-zt}{1-t};q;p;\lambda;m\right),$$

where $|z|<1$, $|t|<|1-z|$ *and* $|z||t|<|1-t|$.

Proof. By considering the following identity

$$[1-(1-z)t]^{-\beta}=(1-t)^{-\beta}\left(1+\frac{zt}{1-t}\right)^{-\beta},$$

and expanding its left-hand side as power series, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(1-z)^nt^n=(1-t)^{-\beta}\left(1-\frac{-zt}{1-t}\right)^{-\beta},\quad \text{for } |t|<|1-z|.$$

Multiplying both sides by $z^{\alpha-1}(1-z)^{-\tau}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}$ on both sides, we find

$$D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}z^{\alpha-1}(1-z)^{-\tau}(1-z)^nt^n\right\}$$
$$=D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}(1-z)^{-\tau}\left(1-\frac{-zt}{1-t}\right)^{-\beta}\right\}.$$

Interchanging the order of the summation and fractional derivative under the given conditions, we obtain

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\tau+n}\}t^n$$

$$=(1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}(1-z)^{-\tau}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

Finally, the desired result follows by Theorems 5.2 and 5.3. ☐

Theorem 6.3. *Let* $\mathrm{Re}(\xi) > \mathrm{Re}(\upsilon) > -\frac{1}{2}$, $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > -\frac{1}{2}$ *and* $\mathrm{Re}(\beta) > 0$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)F_{\mu}\left(-n,\upsilon+\frac{1}{2};\xi+1;u;q;\lambda;p;m\right)t^n$$
$$=(1-t)^{-\beta}F_{2,\mu}\left(\beta,\alpha+\frac{1}{2},\upsilon+\frac{1}{2};\gamma+1,\xi+1;\frac{z}{1-t},\frac{-ut}{1-t};q;\lambda;p;m\right),$$

where $|z|<1$, $|\frac{1-u}{1-z}t|<1$ *and* $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$.

Proof. By replacing t by $(1-u)t$ in (6.1) and multiplying both sides of the resulting identity by $u^{\upsilon-1}$, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)u^{\upsilon-1}(1-u)^n t^n$$
$$=u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_{\mu}\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right), \tag{6.3}$$

where $\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$.

Next, applying the fractional derivative $D^{\upsilon-\xi,\mu;q;\lambda;p;m}$ to both sides of (6.3) and changing the order of the summation and the fractional derivative under conditions $|z|<1$, $|\frac{1-u}{1-z}t|<1$ and $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$, yields

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n$$
$$=D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_{\mu}\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right)\right\},$$

The last identity can be written as follows:

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n$$
$$=(1-t)^{-\beta}D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}\left[1-\frac{-ut}{1-t}\right]^{-\beta}\right.$$
$$\left.\times F_{\mu}\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;\frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}};q;\lambda;p;m\right)\right\}.$$

Thus, by using Theorems 5.2 and 5.5 in the resulting identity, we obtain the desired result. ☐

7. Concluding remarks

In this paper, by using an extension of macdonald given by Boudjekha function we developed a generalized extension of some special functions namely: incomplete gamma, beta, hypergeomtric and confluent functions and we obtained a new extended Riemann-Liouville fractional derivative operator. We conclude first, for $\lambda = 0$ and $q = 1$, that extended incomplete gamma functions are respectively reduced to incomplete gamma functions (see [9]) and all the results established here will coincide with those obtained in [1]. Finally, if we letting $\lambda = m = \mu = 0$, $q = 1$ and $p \to 0$ then all the results established in this paper will reduce to the results associated with classical Riemann-Liouville fractional derivative operator (see [16]).

We intend to investigate aslo some other extensions based on Lerch and Hurwitz functions and Pochhammer Symbol, recently initiated in [25, 27].

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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS TO A SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. In this work, we obtain necessary and sufficient conditions for oscillation of solutions of second-order neutral impulsive differential system

$$\begin{cases} \Big(r(t)\big(z'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, \quad t \geq t_0,\, t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\big(z'(\lambda_k)\big)^{\gamma}\Big) + \sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)) = 0, \quad k = 1,2,3,\dots, \end{cases}$$

where $z(t) = x(t) + p(t)x(\tau(t))$. Under the assumption $\int_0^\infty \big(r(\eta)\big)^{-1/\gamma}d\eta = \infty$, we consider two cases when $\gamma > \alpha_i$ and $\gamma < \alpha_i$. Our main tool is Lebesgue's Dominated Convergence theorem. Examples are given to illustrate our main results and we state an open problem.

1. Introduction

In this article we consider the neutral impulsive differential system

$$\begin{cases} \Big(r(t)\big(z'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, \quad t \geq t_0,\, t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\big(z'(\lambda_k)\big)^{\gamma}\Big) + \sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)) = 0, \quad k = 1,2,3,\dots, \end{cases} \tag{1.1}$$

where

$$z(t) = x(t) + p(t)x(\tau(t)), \quad \Delta x(a) = \lim_{s\to a^+} x(s) - \lim_{s\to a^-} x(s),$$

the functions $p, q_i, h_i, r, \sigma_i, \tau$ are continuous that satisfy the conditions stated below and assume that the sequence $\{\lambda_k\}$ satisfies $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k < \cdots$ as $k \to \infty$

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and γ and α_i are the quotient of two odd positive integers and λ_k's are fixed moment of impulsive effects..

(A1) $\sigma_i \in C([0,\infty),\mathbb{R}_+)$, $\tau \in C^2([0,\infty),\mathbb{R}_+)$, $\sigma_i(t) < t$, $\tau(t) < t$, $\lim_{t\to\infty}\sigma_i(t) = \infty$, $\lim_{t\to\infty}\tau(t) = \infty$.

(A2) $r \in C^1([0,\infty),\mathbb{R}_+)$, $q_i, h_i \in C([0,\infty),\mathbb{R}_+)$, $0 < r(t)$, $0 \le q_i(t)$, $0 \le h_i(t)$ for all $t \ge 0$ and $i = 1,2,\dots,m$, $\sum q_i(t)$ is not identically zero in any interval $[b,\infty)$.

(A3) $\int_0^\infty r^{-1/\gamma}(s)\,ds = \infty$ and let $\Pi(t) = \int_0^t r^{-1/\gamma}(\eta)\,d\eta$.

(A4) $-1 < -p_0 \le p(t) \le 0$ for $t \ge t_0$.

(A5) There exists a differentiable function $\sigma_0(t)$ such that $0 < \sigma_0(t) = \min\{\sigma_i(t) : t \ge t^*\}$ and $\sigma_0'(t) \ge \alpha$ for $t \ge t^*$, $\alpha > 0$, $i = 1,2,\dots,m$.

The main feature of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (1.1). Sufficient conditions for the oscillation and nonoscillation of all solutions to the first and second order neutral impulsive differential systems are provided in [12–15, 18–22]. The necessary and sufficient conditions for oscillation of all solutions to the first order neutral impulsive differential systems are discussed in [20, 21]. In this work, our main aim is to present the necessary and sufficient conditions for oscillation of all solutions of (1.1).

In 2011, Dimitrova and Donev [13–15] have considered the first order impulsive differential system of the form

$$\begin{cases} \Big(x(t) + p(t)x(\tau(t))\Big)' + q(t)x(\sigma(t)) = 0, \quad t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\tau(\lambda_k))\Big) + q(\lambda_k)x(\sigma(\lambda_k)) = 0, \quad k \in \mathbb{N}, \end{cases} \tag{1.2}$$

and established several sufficient conditions for oscillation of the solutions of (1.2).

In 2014, Tripathy [19] have established sufficient conditions for oscillation of all solutions of

$$\begin{cases} \Big(x(t) + p(t)x(t-\tau)\Big)' + q(t)f\Big(x(t-\sigma)\Big) = 0, \quad t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\tau(\lambda_k - \tau))\Big) + q(\lambda_k)f\Big(x(\sigma(\lambda_k - \sigma))\Big) = 0, \quad k \in \mathbb{N}. \end{cases} \tag{1.3}$$

In 2015, Tripathy and Santra [20] obtained the necessary and sufficient conditions for oscillatory and asymptotic behavior of solutions of

$$\begin{cases} \Big(x(t) + p(t)x(t-\tau)\Big)' + q(t)f\Big(x(t-\sigma)\Big) = g(t), \quad t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\lambda_k - \tau)\Big) + q(\lambda_k)f\Big(x(\lambda_k - \sigma)\Big) = h(\lambda_k), \quad k \in \mathbb{N}. \end{cases}$$

In 2016, Tripathy, Santra and Pinelas [21] obtained necessary and sufficient conditions of (1.3). In the subsequent year, Tripathy and Santra [22] established sufficient conditions for oscillation and existence of positive solutions of

$$\begin{cases} \Big(r(t)\Big(x(t) + p(t)x(t-\tau)\Big)'\Big)' + q(t)f\Big(x(t-\sigma)\Big) = 0, \quad t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(r(\lambda_k)\Big(x(\lambda_k) + p(\lambda_k)x(\lambda_k - \tau)\Big)'\Big) + q(\lambda_k)f\Big(x(\lambda_k - \sigma)\Big) = 0, \quad k \in \mathbb{N}. \end{cases}$$

In 2018, Santra [18] established sufficient conditions for oscillations of solutions of

$$\begin{cases} \Big(r(t)\Big(x(t)+p(t)x(\tau(t))\Big)'\Big)' + q(t)f\Big(x(\sigma(t))\Big) = 0, \quad t \neq \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(r(\lambda_k)\Big(x(\lambda_k)+p(\lambda_k)x(\tau(\lambda_k))\Big)'\Big) + q(\lambda_k)f\Big(x(\sigma(\lambda_k))\Big) = 0, \quad k \in \mathbb{N}. \end{cases}$$

By a solution x we mean a function differentiable on $[t_0, \infty)$ such that $z(t)$ and $z'(t)$ are differentiable for $t \neq t_k$, and $z(t)$ is left continuous at λ_k and has right limit at λ_k, and x satisfies (1.1). We restrict our attention to solutions for which $\sup_{t \geq b} |x(t)| > 0$ for every $b \geq 0$. A solution is called oscillatory it has arbitrarily large zeros; otherwise is non-oscillatory.

To define a particular solution, we need an initial function $\phi(t)$ which is twice differentiable for t in the interval

$$\min\Big\{\inf\{\tau(t) : t_0 \leq t\},\ \inf\{\sigma_i(t) : t_0 \leq t, i = 1, 2, \dots, m\}\Big\} \leq t.$$

Then a solution is obtained using the method of steps: When replacing $x(\tau(t))$ by $\phi(\tau(t))$, and $x(\sigma_i(t))$ by $\phi(\sigma_i(t))$ in (1.1), we obtain a second-order differential equation. We solve this equation taking into account discrete equation of (1.1), say on an interval $[t_0, t_1]$. Then repeat the process starting at $t = t_1$.

2. Necessary and Sufficient Conditions

Lemma 2.1. *Assume that* (A1)-(A4) *hold for $t \geq t_0$. If x is an eventually positive solution of* (1.1)*, then z satisfies any one of the following two cases:*

(i) $z(t) < 0,\ z'(t) > 0,\ \Big(r(z')^\gamma\Big)'(t) \leq 0$;

(ii) $z(t) > 0,\ z'(t) > 0,\ \Big(r(z')^\gamma\Big)'(t) \leq 0$,

for all sufficiently large t.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t^*$ and $i = 1, 2, \dots, m$. From (1.1) it follows that

$$\begin{aligned} \Big(r(t)\Big(z'(t)\Big)^\gamma\Big)' &= -\sum_{i=1}^m q_i(t) x^{\alpha_i}(\sigma_i(t)) \leq 0, \quad \text{for } t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\Big(z'(\lambda_k)\Big)^\gamma\Big) &= -\sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)) \leq 0, \quad \text{for } k = 1, 2, \dots \end{aligned} \tag{2.1}$$

Therefore, $r(t)\Big(z'(t)\Big)^\gamma$ is non-increasing for $t \geq t^*$, including jumps of discontinuity. Next we show the $r(t)\Big(z'(t)\Big)^\gamma$ is positive. By contradiction assume that $r(t)\Big(z'(t)\Big)^\gamma \leq 0$ at a certain time $t \geq t^*$. Using that $\sum q_i$ is not identically zero on any interval $[b, \infty)$, and by (2.1), there exists $t_2 \geq t^*$ such that

$$r(t)\Big(z'(t)\Big)^\gamma \leq r(t_2)\Big(z'(t_2)\Big)^\gamma < 0, \quad \text{for all } t \geq t_2.$$

Recall that γ is the quotient of two positive odd integers. Then

$$z'(t) \le \Big(\frac{r(t_2)}{r(t)}\Big)^{1/\gamma} z'(t_2), \quad \text{for } t \ge t_2.$$

Since $r(\lambda_k)\Big(z'(\lambda_k)\Big)^{\gamma} \le r(t_2)\Big(z'(t_2)\Big)^{\gamma} < 0$ for all $\lambda_k \ge t_2$. Integrating from t_2 to t, we have

$$\begin{aligned} z(t) &\le z(t_2) + \sum_{t_2 \le \lambda_k < \infty} z'(\lambda_k) + \Big(r(t_2)\Big)^{1/\gamma} z'(t_2)\Big(\Pi(t) - \Pi(t_2)\Big) \\ &\le z(t_2) + \Big(r(t_2)\Big)^{1/\gamma} z'(t_2)\Big(\Pi(t) - \Pi(t_2)\Big) \to -\infty, \end{aligned}$$

as $t \to \infty$ due to (A3). Now, we consider the following two possibilities.

If x is unbounded, then there exists a sequence $\{\eta_k\} \to \infty$ such that $x(\eta_k) = \sup\{x(\eta) : \eta \le \eta_k\}$. By $\tau(\eta_k) \le \eta_k$, we have $x(\tau(\eta_k)) \le x(\eta_k)$ and hence

$$z(\eta_k) = x(\eta_k) + p(\eta_k)x(\tau(\eta_k)) \ge (1 + p(\eta_k))x(\eta_k) \ge (1 - p_0))x(\eta_k) \ge 0,$$

which contradicts $\lim_{k\to\infty} z(t) = -\infty$. Recall that $\{\lambda_k\}$ are the sequence of points for $t \ge \lambda_k$, then by similar argument we can show that $z(\lambda_k) \ge 0$ to get a contradiction to $\lim_{k\to\infty} z(t) = -\infty$. Therefore, $r(t)\Big(z'(t)\Big)^{\gamma} > 0$ for all $t \ge t^*$.

If x is bounded, then z is also bounded, which is a contradiction to $\lim_{k\to\infty} z(t) = -\infty$.

From $r(t)\Big(z'(t)\Big)^{\gamma} > 0$ and $r(t) > 0$, it follows that $z'(t) > 0$. Then there is $t_1 \ge t^*$ such that z satisfies only one of two cases (i) and (ii). This completes the proof. □

Lemma 2.2. *Assume that* (A1)-(A4) *hold. If x is an eventually positive solution of* (1.1)*, then any one of following two cases exists:*

(1) *if z satisfies* (i), $\lim_{t\to\infty} x(t) = 0$;
(2) *if z satisfies* (ii)*, there exist $t_1 \ge t_0$ and $\delta > 0$ such that*

$$0 < z(t) \le \delta\Pi(t), \tag{2.2}$$

$$\begin{aligned} &\Big(\Pi(t) - \Pi(t_1)\Big)\Big[\int_t^{\infty} \sum_{i=1}^{m} q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta + \sum_{\lambda_k \ge t}\sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]^{1/\gamma} \\ &\le z(t) \le x(t), \end{aligned} \tag{2.3}$$

for all $t \ge t_1$.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \ge t^*$ and $i = 1, 2, \dots, m$. Then Lemma 2.1 holds and we have following two possible cases.

Case 1. Let z satisfies (i) for all $t \ge t_1$. Note that $\lim_{t\to\infty} z(t)$ exists and by (A1), $\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} x(\tau(t))$. Then $0 > z(t) \ge x(t) - p_0x(\tau(t))$ implies

$$0 \ge \lim_{t\to\infty} z(t) \ge \lim_{t\to\infty} \Big[x(t) - p_0x(\tau(t))\Big] \ge (1 - p_0)\limsup_{t\to\infty} x(t).$$

Since $(1-p_0)>0$, it follows that $\limsup_{t\to\infty} x(t)=0$, hence $\lim_{t\to\infty} x(t)=0$ for $t\neq\lambda_k$, $k\in\mathbb{N}$. We may note that $\{x(\lambda_k-0)\}_{k\in\mathbb{N}}$ and $\{x(\lambda_k+0)\}_{k\in\mathbb{N}}$ are sequences of real numbers and because of continuity of x

$$\lim_{k\to\infty} x(\lambda_k-0)=0=\lim_{k\to\infty} x(\lambda_k+0)$$

due to $\liminf_{t\to\infty} x(t)=0=\limsup_{t\to\infty} x(t)$. Hence, $\lim_{t\to\infty} x(t)=0$ for all t and λ_k, $k\in\mathbb{N}$.

Case 2. Let z satisfies (ii) for all $t\geq t_1$. Note that $x(t)\geq z(t)$ and z is positive and increasing so x cannot converge to zero. From $r(t)\big(z'(t)\big)^\gamma$ being non-increasing, there exists a constant $\delta>0$ and $t\geq t_1$ such that $\big(r(t)\big)^{1/\gamma}z'(t)\leq\delta$ and hence $z(t)\leq\delta\Pi(t)$ for $t\geq t_1$.

Since $r(t)\big(z'(t)\big)^\gamma$ is positive and non-increasing, $\lim_{t\to\infty} r(t)\big(z'(t)\big)^\gamma$ exists and is non-negative. Integrating (1.1) from t to a, we have

$$r(a)\big(z'(a)\big)^\gamma-r(t)\big(z'(t)\big)^\gamma=-\int_t^a\sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\,d\eta+\sum_{t\leq\lambda_k<a}\Delta\Big(r(\lambda_k)z'(\lambda_k)\Big)^\gamma.$$

Computing the limit as $a\to\infty$

$$r(t)\big(z'(t)\big)^\gamma\geq\int_t^\infty\sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\,d\eta+\sum_{\lambda_k\geq t}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)). \tag{2.4}$$

Then

$$z'(t)\geq\bigg[\frac{1}{r(t)}\Big[\int_t^\infty\sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\,d\eta+\sum_{t\leq\lambda_k}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\bigg]^{1/\gamma}.$$

Since $z(t_1)>0$, integrating the above inequality yields

$$z(t)\geq\int_{t_1}^t\bigg[\frac{1}{r(\eta)}\Big[\int_\eta^\infty\sum_{i=1}^m q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta+\sum_{\eta\leq\lambda_k}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\bigg]^{1/\gamma}d\eta.$$

Since the integrand is positive, we can increase the lower limit of integration from η to t, and then use the definition of $\Pi(t)$, to obtain

$$z(t)\geq\Big(\Pi(t)-\Pi(t_1)\Big)\Big[\int_t^\infty\sum_{i=1}^m q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta+\sum_{t\leq\lambda_k}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]^{1/\gamma},$$

which yields (2.3). ☐

2.1. The Case $\alpha_i<\gamma$.

In this subsection, we assume that there exists a constant β_1, the quotient of two positive odd integers such that $0<\alpha_i<\beta_1<\gamma$.

Theorem 2.1. *Under assumptions* (A1)-(A4)*, each solution of* (1.1) *is either oscillatory or converge to zero if and only if*

$$\int_0^\infty\sum_{i=1}^m q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta))\,d\eta+\sum_{k=1}^\infty\sum_{i=1}^m h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k))=\infty. \tag{2.5}$$

Proof. We prove the sufficiency by contradiction. Initially, we assume that a solution x is eventually positive which does not converge to zero. So, Lemma 2.1 holds and z satisfies any one of two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim_{t\to\infty} x(t) = 0$ which is a contradiction.

For Case 2, we can find a $t_1 > 0$ such that

$$x(t) \geq z(t) \geq \Big(\Pi(t) - \Pi(t_1)\Big) w^{1/\gamma}(t) \geq 0, \quad \text{for } t \geq t_1,$$

where

$$w(t) = \int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta + \sum_{\lambda_k \geq t} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)) \geq 0.$$

As $\lim_{t\to\infty} \Pi(t) = \infty$, there exists $t_2 \geq t_1$, such that $\Pi(t) - \Pi(t_1) \geq \frac{1}{2}\Pi(t)$ for $t \geq t_2$ and hence

$$z(t) \geq \frac{1}{2}\Pi(t) w^{1/\gamma}(t). \tag{2.6}$$

Note that w is left continuous at λ_k,

$$w'(t) = -\sum_{i=1}^m q_i(t) x^{\alpha_i}(\sigma_i(t)), \quad \text{for } t \neq \lambda_k,$$
$$\Delta w(\lambda_k) = -\sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)) \leq 0.$$

Thus w is non-negative and non-increasing for $t \geq t_2$. Using (2.2), $\alpha_i - \beta_1 < 0$ and (2.6), we have

$$\begin{aligned} x^{\alpha_i}(t) &\geq z^{\alpha_i - \beta_1}(t) z^{\beta_1}(t) \geq (\delta \Pi(t))^{\alpha_i - \beta_1} z^{\beta_1}(t) \\ &\geq \Big(\delta\Pi(t)\Big)^{\alpha_i-\beta_1} \Big(\frac{\Pi(t) w^{1/\gamma}(t)}{2}\Big)^{\beta_1} = \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(t) w^{\beta_1/\gamma}(t), \quad \text{for } t \geq t_2. \end{aligned}$$

Since w is non-increasing, $\frac{\beta_1}{\gamma} > 0$, and $\sigma_i(\eta) < \eta$, it follows that

$$x^{\alpha_i}(\sigma_i(\eta)) \geq \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(\sigma_i(\eta)) w^{\beta_1/\gamma}(\sigma_i(\eta)) \geq \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(\sigma_i(\eta)) w^{\beta_1/\gamma}(\eta). \tag{2.7}$$

Now, we have

$$\Big(w^{1-\beta_1/\gamma}(t)\Big)' = \Big(1 - \frac{\beta_1}{\gamma}\Big) w^{-\beta_1/\gamma}(t) \Big(-\sum_{i=1}^m q_i(t) x^{\alpha_i}(\sigma_i(t))\Big), \quad \text{for } t \neq \lambda_k. \tag{2.8}$$

To estimate the discontinuities of $w^{1-\beta_1/\gamma}$ we use a Taylor polynomial of order 1 for the function $h(x) = x^{1-\beta_1/\gamma}$, with $0 < \beta_1 < \gamma$ about $x = a$

$$b^{1-\beta_1/\gamma} - a^{1-\beta_1/\gamma} \leq \Big(1 - \frac{\beta_1}{\gamma}\Big) a^{-\beta_1/\gamma}(b - a).$$

Then $\Delta w^{1-\beta_1/\gamma}(\lambda_k) \le \big(1-\frac{\beta_1}{\gamma}\big) w^{-\beta_1/\gamma}(\lambda_k)\Delta w(\lambda_k)$. Integrating (2.8) from t_2 to t, we have

(2.9)

$$\begin{aligned}
w^{1-\beta_1/\gamma}(t_2) \ge& \Big(1-\frac{\beta_1}{\gamma}\Big)\Big[-\int_{t_2}^{t} w^{-\beta_1/\gamma}(\eta) w'(\eta)\, d\eta - \sum_{t_2\le\lambda_k<t} w^{-\beta_1/\gamma}(\lambda_k)\Delta w(\lambda_k)\Big]\\
=& \Big(1-\frac{\beta_1}{\gamma}\Big)\Big[\int_{t_2}^{t} w^{-\beta_1/\gamma}(\eta)\Big(\sum_{i=1}^{m} q_i(\eta) x^{\alpha_i}(\sigma_i(\eta))\Big)\, d\eta\\
&+ \sum_{t_2\le\lambda_k<t} w^{-\beta_1/\gamma}(\lambda_k)\sum_{i=1}^{m} h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\\
\ge& \frac{1-\frac{\beta_1}{\gamma}}{2^{\beta_1}\delta^{(\beta_1-\alpha_i)}}\Big[\int_{t_2}^{t}\sum_{i=1}^{m} q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta)))\, d\eta + \sum_{t_2\le\lambda_k<t}\sum_{i=1}^{m} h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)))\Big],
\end{aligned}$$

which contradicts (2.5) as $t \to \infty$ and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution x, we introduce the variables $y = -x$ so that we can apply the above process for the solution y.

Next we show the necessity part by a contrapositive argument. Let (2.5) do not hold. Then it is possible to find $t_1 > 0$ such that

$$\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Pi^{\alpha_i}(\sigma_i(\zeta))\, d\zeta + \sum_{\lambda_k\ge\eta}\sum_{i=1}^{m} h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)) \le \frac{\epsilon}{\delta^{\alpha_i}}, \tag{2.10}$$

for all $\eta \ge t_1$ and $\delta, \epsilon > 0$ satisfying the relation

$$(2\epsilon)^{1/\gamma} = (1-p_0)\delta, \tag{2.11}$$

so that $0 < \epsilon^{1/\gamma} \le (1-p_0)\delta/2^{1/\gamma} < \delta$. Define the set of continuous functions

$$M = \{x \in C([0,\infty)) : \epsilon^{1/\gamma}\big(\Pi(t)-\Pi(t_1)\big) \le x(t) \le \delta\big(\Pi(t)-\Pi(t_1)\big),\ t \ge t_1\},$$

and define an operator Φ on M by

$$(\Phi x)(t) = \begin{cases} 0, & \text{if } t \le t_1,\\ -p(t)x(\tau(t)) + \displaystyle\int_{t_1}^{t}\Big[\frac{1}{r(\eta)}\Big[\epsilon + \int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta & \\ + \displaystyle\sum_{\lambda_k\ge\eta}\sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\Big]^{1/\gamma} d\eta, & \text{if } t > t_1. \end{cases}$$

We need to show that if x is a fixed point of Φ, i.e., $\Phi x = x$, then x is a solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. For $x \in M$, we have $0 \le \epsilon^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big) \le x(t)$ and by (A2) and (A3) we have

$$(\Phi x)(t) \ge 0 + \int_{t_1}^{t} \left[\frac{1}{r(\eta)}[\epsilon + 0 + 0]\right]^{1/\gamma} d\eta = \epsilon^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big).$$

Now we estimate $(\Phi x)(t)$ from above. For x in M, by definition of the set M we have $x^{\alpha_i}(\sigma_i(\eta)) \le (\delta\Pi(\sigma_i(\eta)))^{\alpha_i}$. Therefore, by (2.10),

$$\begin{aligned}(\Phi x)(t) \le& p_0\delta(\Pi(t) - \Pi(t_1)) + \int_{t_1}^{t} \left[\frac{1}{r(\eta)}\Big[\epsilon + \delta^{\alpha_i}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Pi^{\alpha_i}(\sigma_i(\zeta))\, d\zeta\right.\\ &\left.+ \delta^{\alpha_i}\sum_{\lambda_k \ge \eta}\sum_{i=1}^{m} h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k))\Big]\right]^{1/\gamma} d\eta\\ \le& p_0\delta\Big(\Pi(t) - \Pi(t_1)\Big) + (2\epsilon)^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big) = \delta\Big(\Pi(t) - \Pi(t_1)\Big).\end{aligned}$$

Therefore, Φ maps M to M.

To find a fixed point for Φ in M, let us define a sequence of functions in M by the recurrence relation

$$\begin{aligned}u_0(t) =& 0, \quad \text{for } t = 0,\\ u_1(t) =& (\Phi u_0)(t) = \begin{cases} 0, & \text{if } t < t_1,\\ \epsilon^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big), & \text{if } t \ge t_1,\end{cases}\\ u_{n+1}(t) =& (\Phi u_n)(t), \quad \text{for } n \ge 1,\ t \ge t_1.\end{aligned}$$

Note that for each fixed t, we have $u_1(t) \ge u_0(t)$. Using mathematical induction, we can show that $u_{n+1}(t) \ge u_n(t)$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using the Lebesgue Dominated Convergence Theorem, we can show that u is a fixed point of Φ in M. This shows under assumption (2.10), there a non-oscillatory solution that does not converge to zero. □

Corollary 2.1. Under the assumptions of Theorem 2.1, every unbounded solution of (1.1) is oscillatory if and only if (2.5) holds.

Proof. The proof of the corollary is an immediate consequence of Theorem 2.1. □

2.2. **The Case $\alpha_i > \gamma$.** In this subsection, we assume that there exists a constant β_2, the quotient of two positive odd integers such that $\gamma < \beta_2 < \alpha_i$.

Theorem 2.2. *Under assumptions* (A1)-(A5) *and $r(t)$ is non-decreasing, every solution of* (1.1) *is either oscillatory or converges to zero if and only if*

$$\int_{0}^{\infty} \left[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\, d\zeta + \sum_{k=1}^{\infty}\sum_{i=1}^{m} h_i(\lambda_k)\Big]\right]^{1/\gamma} d\eta = \infty. \tag{2.12}$$

Proof. We prove the sufficiency by contradiction. Initially, we assume that x is an eventually positive solution not converging to zero. So, Lemma 2.1 holds and z satisfies any one of two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim_{t\to\infty} x(t) = 0$, which is a contradiction.

For Case 2, $z(t) > 0$ is non-decreasing for $t \geq t_1$ and

$$x^{\alpha_i}(t) \geq z^{\alpha_i}(t) \geq z^{\alpha_i-\beta_2}(t) z^{\beta_2}(t) \geq z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(t)$$

implies that

$$x^{\alpha_i}(\sigma_i(t)) \geq z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(\sigma_i(t)), \quad \text{for } t \geq t_2 > t_1. \tag{2.13}$$

Using (2.4), (2.13) and $\sigma_i(t) \geq \sigma_0(t)$, we have

$$r(t)\Big(z'(t)\Big)^{\gamma} \geq z^{\alpha_i-\beta_2}(t_1)\left[\int_t^\infty \sum_{i=1}^m q_i(\eta)\, d\eta + \sum_{\lambda_k\geq t}\sum_{i=1}^m h_i(\lambda_k)\right] z^{\beta_2}(\sigma_0(t)), \tag{2.14}$$

for $t \geq t_2$. Being $r(t)\Big(z'(t)\Big)^{\gamma}$ non-increasing and $\sigma_0(t) \leq t$, we have

$$r(\sigma_0(t))\Big(z'(\sigma_0(t))\Big)^{\gamma} \geq r(t)\Big(z'(t)\Big)^{\gamma}.$$

Using the last inequality in (2.14) and then dividing by $z^{\beta_2/\gamma}(\sigma_0(t)) > 0$, we get

$$\frac{z'(\sigma_0((t))}{z^{\beta_2/\gamma}(\sigma_0(t))} \geq \left[\frac{z^{\alpha_i-\beta_2}(t_1)}{r(\sigma_0(t))}\left[\int_t^\infty \sum_{i=1}^m q_i(\eta)\, d\eta + \sum_{\lambda_k\geq t}\sum_{i=1}^m h_i(\lambda_k)\right]\right]^{1/\gamma},$$

for $t \geq t_2$. Multiplying the left-hand side by $\sigma_0'(t)/\alpha \geq 1$ and integrating from t_2 to t, we find

$$\begin{aligned}\frac{1}{\alpha}\int_{t_2}^t \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta_2/\gamma}(\sigma_0(\eta))}\, d\eta \geq & z^{(\alpha_i-\beta_2)/\gamma}(t_1)\int_{t_2}^t \left[\frac{1}{r(\sigma_0(\eta))}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta\right.\right.\\ & \left.\left. + \sum_{\eta\leq\lambda_k}\sum_{i=1}^m h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta, \quad \text{for } t \geq t_2.\end{aligned} \tag{2.15}$$

Since $\gamma < \beta_2$, $r(\sigma_0(\eta)) \leq r(\eta)$ and

$$\frac{1}{\alpha(1-\beta_2/\gamma)}\Big[z^{1-\beta_2/\gamma}(\sigma_0(\eta))\Big]_{\eta=t_2}^t \leq \frac{1}{\alpha(\beta_2/\gamma - 1)} z^{1-\beta_2/\gamma}(\sigma_0(t_2)),$$

then (2.15) becomes

$$\int_{t_2}^t \left[\frac{1}{r(\eta)}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta + \sum_{\eta\leq\lambda_k}\sum_{i=1}^m h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta < \infty,$$

which is a contradiction to (2.12). This contradiction implies that the solution x cannot be eventually positive. The case with an eventually negative solution is proved.

To prove the necessity part, we assume that (2.12) does not hold. For given $\epsilon = \Big(2/(1-p_0)\Big)^{-\alpha_i/\gamma} > 0$, we can find a $t_1 > 0$ such that

$$\int_{t_1}^{\infty} \left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta)\, d\zeta + \sum_{\lambda_k \geq s} \sum_{i=1}^{m} h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta < \epsilon. \tag{2.16}$$

Consider

$$M = \left\{x \in C([0,\infty)) : 1 \leq x(t) \leq \frac{2}{1-p_0} \text{ for } t \geq t_1\right\}.$$

Define the operator

$$(\Phi x)(t) = \begin{cases} 0, & \text{if } t < t_1, \\ 1 - p(t)x(\tau(t)) & \\ + \int_{t_1}^{t} \Big[\frac{1}{r(\eta)} \Big[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta & \\ + \sum_{\lambda_k \geq \eta} \sum_{i=1}^{m} h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\Big]^{1/\gamma} d\eta, & \text{if } t \geq t_1. \end{cases}$$

Indeed, $\Phi x = x$ implies that x is a solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. Let $x \in M$. Then $1 \leq x$ implies that $(\Phi x)(t) \geq 1$, on $[t_1, \infty)$. Estimating $(\Phi x)(t)$ from above. Let $x \in M$. Then $x \leq 2/(1-p_0)$ and thus

$$\begin{aligned}(\Phi x)(t) \leq & 1 - p(t)\frac{2}{1-p_0} + \int_{t_1}^{t} \left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta)\Big(\frac{2}{1-p_0}\Big)^{\alpha_i} d\zeta\right.\right. \\ & \left.\left. + \sum_{\lambda_k \geq \eta} \sum_{i=1}^{m} h_i(\lambda_k)\Big(\frac{2}{1-p_0}\Big)^{\alpha_i}\right]\right]^{1/\gamma} d\eta.\end{aligned}$$

Since $\sigma_0(\eta) \leq \eta$ and $r(\cdot)$ is non-decreasing, we can replace $r(\eta)$ by $r(\sigma_0(\eta))$ and the above inequality is still valid. By (2.16) and the definition of ϵ, we have

$$(\Phi x)(t) \leq 1 + \frac{2p_0}{1-p_0} + \Big(2/(1-p_0)\Big)^{\alpha_i/\gamma}\epsilon = 1 + \frac{2p_0}{1-p_0} + 1 = \frac{2}{1-p_0}.$$

Therefore, Φ maps M to M.

To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{aligned} u_0(t) =& 0, \quad \text{for } t = 0, \\ u_1(t) =& (\Phi u_0)(t) = 1, \quad \text{for } t \geq t_1, \\ u_{n+1}(t) =& (\Phi u_n)(t), \quad \text{for } n \geq 1,\ t \geq t_1. \end{aligned}$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can prove that $u_{n+1}(t) \geq u_n(t)$. Therefore, $\{u_n\}$

converges pointwise to a function u in M. Then u is a fixed point of Φ and a positive solution to (1.1) that does not converge to zero. □

Corollary 2.2. Under the assumptions of Theorem 2.2, every unbounded solution of (1.1) is oscillatory if and only if (2.12) hold.

Example 2.1. Consider the neutral differential equation

(2.17)
$$\begin{cases} \Big(e^{-t}\Big(\big(x(t)-e^{-t}x(\tau(t))\big)'\Big)^{11/3}\Big)' + \frac{1}{t+1}(x(t-2))^{1/3} + \frac{1}{t+2}(x(t-1))^{5/3} = 0, \\ \Big(e^{-k}\Big(\big(x(k)-e^{-k}x(\tau(k))\big)'\Big)^{11/3}\Big)' + \frac{1}{t+4}(x(k-2))^{1/3} + \frac{1}{t+5}(x(k-1))^{5/3} = 0. \end{cases}$$

Here $\gamma = 11/3$, $r(t) = e^{-t}$, $-1 < p(t) = -e^{-t} \le 0$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\lambda_k = k$ for $k \in \mathbb{N}$, $\Pi(t) = \int_0^t e^{11s/3}\,ds = \frac{3}{11}\Big(e^{11t/3}-1\Big)$, $\alpha_1 = 1/3$ and $\alpha_2 = 5/3$. For $\beta_1 = 7/3$, we have $0 < \max\{\alpha_1,\alpha_2\} < \beta_1 < \gamma$, and $u^{\alpha_i-\beta_1} = u^{-2}$ and $u^{\alpha_2-\beta_1} = u^{-2/3}$ which both are decreasing functions. To check (2.5) we have

$$\begin{aligned} &\int_0^\infty \sum_{i=1}^m q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta))\,d\eta + \sum_{k=1}^\infty\sum_{i=1}^m h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)) \\ \ge &\int_0^\infty \sum_{i=1}^m q_i(s)\Pi^{\alpha_i}(\sigma_i(\eta)))\,d\eta \\ \ge &\int_0^\infty q_1(\eta)\Pi^{\alpha_i}(\sigma_1(\eta)))\,d\eta \\ = &\int_0^\infty \frac{1}{\eta+1}\Big(\frac{3}{11}\Big(e^{5(\eta-2)/3}-1\Big)\Big)^{1/3}\,d\eta = \infty, \end{aligned}$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 2.1 hold, and therefore, each solution of (2.17) is oscillatory or converges to zero.

Example 2.2. Consider the neutral differential equation

(2.18)
$$\begin{cases} \Big(\Big(\big(x(t)-e^{-t}x(\tau(t))\big)'\Big)^{1/3}\Big)' + t(x(t-2))^{7/3} + (t+1)(x(t-1))^{11/3} = 0, \\ \Big(\Big(\big(x(2^k)-e^{-2^k}x(\tau(2^k))\big)'\Big)^{1/3}\Big)' + \frac{t}{2}(x(2^k-2))^{7/3} + \frac{t}{3}(x(2^k-1))^{11/3} = 0. \end{cases}$$

Here $\gamma = 1/3$, $r(t) = 1$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\alpha_1 = 7/3$ and $\alpha_2 = 11/3$. For $\beta_2 = 5/3$, we have $\min\{\alpha_1,\alpha_2\} > \beta_2 > \gamma$ and $u^{\alpha_1-\beta_2} = u^{2/3}$ and $u^{\alpha_2-\beta_2} = u^2$, which both are increasing functions. To check (2.12) we have

$$\begin{aligned}
&\int_{t_1}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta+\sum_{\lambda_k\geq\eta}\sum_{i=1}^{m}h_i(\lambda_k)\Big]\Big]^{1/\gamma}d\eta\\
&\geq\int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta\Big]\Big]^{1/\gamma}d\eta\\
&\geq\int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}q_1(\zeta)\,d\zeta\Big]\Big]^{1/\gamma}d\eta\geq\int_{2}^{\infty}\Big[\int_{\eta}^{\infty}\zeta\,d\zeta\Big]^{3}d\eta=\infty.
\end{aligned}$$

So, all the conditions of of Theorem 2.2 hold. Thus, all solution of (2.18) is oscillatory or converges to zero.

Remark 2.1. Based on this work and [13–15, 18–22] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation (1.1) for $p > 0$ and $-\infty < p \leq -1$.

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ON CONTACT CR-SUBMANIFOLD OF A KENMOTSU MANIFOLD WITH KILLING TENSOR FIELD

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Abstract. The object of this paper is to study the Contact CR-submanifold of a Kenmotsu manifold with the help of a killing tensor field and deduce some results.

1. Introduction

K. Kenmotsu [5] introduced the notion of Kenmotsu manifold and later several authors studied this manifold [2, 14, 15]. M. Kobayashi and N. Papaghuic [10, 11] investigated the geometry of semi-invariant submanifolds of a Kenmotsu manifold. The geometry of Contact CR-submanifolds, invariant and anti-invariant submanifolds of an almost contact metric structure are studied by A. Bejancu [1].

Gupta et al. [13] studied the intrinsic characterization of a slant submanifold of a Kenmotsu manifold in case of induced metric and obtained some examples of the slant submanifold of a Kenmotsu manifold. Avik De [2] studied and obtained few examples of a 3-dimensional Kenmotsu manifold with parallel Ricci tensor and obtained killing condition for a vector field in Kenmotsu manifold.

Moreover, the Contact CR-submanifolds of Kenmotsu manifolds are studied by some other authors [8, 9]. The notion of a killing tensor field was introduced by Professor D. E. Blair [4]. In [12], we have investigated and characterized a slant submanifold of a Kenmotsu manifold using killing tensor fields. In this paper, we have studied Contact CR-submanifold of a Kenmotsu manifold using the notion of a killing tensor field and obtained some results.

Key words and phrases. Contact manifold, CR-submanifold, Kenmotsu manifold, killing tensor field.

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2. Preliminaries

A $(2m+1)$-dimensional manifold M is said to admit an almost contact metric structure if there exist a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$\varphi\xi = 0, \quad \varphi^2 U = -U + \eta(U)\,\xi, \quad \eta(\xi) = 1, \quad \eta(\varphi U) = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U,V) - \eta(U)\,\eta(V), \quad g(U,\xi) = \eta(U), \tag{2.2}$$

where U and V are vector fields on M [3,7].

Moreover, if

$$\left(\overline{\nabla}_U \varphi\right) V = -g(U, \varphi V)\,\xi - \eta(V)\,\varphi U, \quad \overline{\nabla}_U \xi = U - \eta(U)\,\xi, \tag{2.3}$$

where $\overline{\nabla}$ be a Levi-Civita connection on $\overline{M}$, then the structure $(M, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold [5].

Suppose M is an isometrically immersed submanifold in $\overline{M}$ and ∇, $\overline{\nabla}$ be the Riemannian connections on M, $\overline{M}$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_U V = \nabla_U V + h(U,V) \tag{2.4}$$

and

$$\overline{\nabla}_U W = -\mathrm{A}_W U + \nabla^{\perp}_U W, \tag{2.5}$$

for any vector fields $U, V \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ be the normal connection on $T^{\perp}M$, A and h be the shape operator and second fundamental form of M in $\overline{M}$.

Both h and A are related as

$$g(A_W U, V) = g(h(U,V), W). \tag{2.6}$$

In Kenmotsu manifold, M is isometrically immersed submanifold. For any vector field U tangent to M, we put

$$\varphi U = pU + fU, \tag{2.7}$$

where pU and fU denote the tangent and normal component of φU, respectively.

The covariant derivative of p, f are given by

$$(\nabla_U p)V = \nabla_U pV - p\nabla_U V,$$
$$(\nabla_U f)V = \nabla^{\perp}_U fV - f\nabla_U V.$$

Similarly, for any vector field W normal to M, we have

$$\varphi W = bW + cW, \tag{2.8}$$

where bW and cW are the tangent and normal component of φW.

The covariant derivative of b, c are given by

$$(\nabla_U b)W = \nabla_U bW - b\nabla_U^{\perp}W,$$
$$(\nabla_U c)W = \nabla_U^{\perp} cW - c\nabla_U^{\perp}W.$$

Let p be the endomorphism defined by (2.7), then we have

$$g(pU, V) + g(U, pV) = 0. \tag{2.9}$$

Definition 2.1 ([9]). Let M be a submanifold of a Kenmotsu manifold $\overline{M}$. Then M is said to be a contact CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $D : x \to D_x \subseteq T_x(M)$ on M satisfying the following conditions:

(i) $TM = D \oplus D^{\perp}$, $\xi \in D$;

(ii) D is invariant with respect to φ, that is, $\varphi D_x \subset T_x(M)$;

(iii) the orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subseteq T_x(M)$ satisfies $\varphi D_x^{\perp} \subseteq T_x^{\perp}(M)$ for each $x \in M$.

A contact CR-submanifold is said to be proper if neither $D_x = \{0\}$ nor $D_x^{\perp} = \{0\}$. If $D_x = \{0\}$, then M is anti-invariant submanifold and if $D_x^{\perp} = \{0\}$, then M becomes invariant submanifold.

Now, let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. For any $U, V \in \Gamma(TM)$, by (2.3), (2.7), (2.8) together with the Gauss and Weingarten formulae [9], we have

$$\left(\overline{\nabla}_U \varphi\right) V = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V \tag{2.10}$$

or

$$-g(U, \varphi V) - \eta(V)\varphi U = \overline{\nabla}_U pV + \overline{\nabla}_U fV - \varphi \nabla_U V - \varphi h(U, V).$$

By comparing the tangent and normal component of the above equation, we have

$$(\nabla_U p)V = A_{fV}U + bh(U, V) + g(pU, V)\xi - \eta(V)pU \tag{2.11}$$

and

$$(\nabla_U f)V = ch(U, V) - h(U, pV) - \eta(V)fU. \tag{2.12}$$

If ξ be the structure vector field tangent to submanifold M, then by (2.3) and (2.6), we have

$$A_W \xi = h(U, \xi) = 0, \tag{2.13}$$

for all $U \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$. Thus, (2.11) reduces to

$$(\nabla_U p)V = g(pU, V)\xi - \eta(V)pU, \tag{2.14}$$

for any $U, V \in \Gamma(D)$. This shows that, the induced structure p is a Kenmotsu structure on M [9].

Let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$, then equation (2.11) reduces to

$$(\nabla_U p)V = bh(U, V) + g(pU, V)\xi - \eta(V)pU, \tag{2.15}$$

for any $U, V \in \Gamma(D)$ [8].

If the second fundamental form h is zero, then submanifold M is totally geodesic. A submanifold M is totally umbilical if

$$h(U, V) = g(U, V) H,$$

where H is the mean curvature vector. In addition, if $H = 0$, then the submanifold M is minimal.

A tensor field φ is called killing [4], if it satisfies the following condition

$$\left(\overline{\nabla}_U \varphi\right) V + \left(\overline{\nabla}_V \varphi\right) U = 0. \tag{2.16}$$

3. Contact CR-Submanifold of a Kenmotsu Manifold $\overline{M}$ with Killing Tensor Field

In this section, we discuss some results on contact CR-submanifold of a Kenmotsu manifold with killing tensor field.

Theorem 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\overline{\nabla}_U pV + \overline{\nabla}_V pU) + \left(\overline{\nabla}_U fV + \overline{\nabla}_V fU\right) = p(\overline{\nabla}_U V + \overline{\nabla}_V U) + f(\overline{\nabla}_U V + \overline{\nabla}_V U). \tag{3.1}$$

Proof. From the equation (2.10), we have

$$(\overline{\nabla}_U \varphi)V = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V \varphi\right) U = \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U.$$

On clubbing above equations, we get

$$\left(\overline{\nabla}_U \varphi\right) V + \left(\overline{\nabla}_V \varphi\right) U = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V + \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U.$$

Using (2.16), we get

$$0 = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V + \overline{\nabla}_V \varphi U - \varphi \overline{\nabla}_V U. \tag{3.2}$$

Using (2.7), above equation yields

$$(\overline{\nabla}_U pV + \overline{\nabla}_V pU) + (\overline{\nabla}_U fV + \overline{\nabla}_V fU) = p(\overline{\nabla}_U V + \overline{\nabla}_V U) + f(\overline{\nabla}_U V + \overline{\nabla}_V U). \quad \square$$

Theorem 3.2. *Suppose M denotes a contact CR-submanifold with killing tensor field φ of a Kenmotsu manifold $\overline{M}$, then*

$$\eta(V) pU + \eta(U) pV = 0 \tag{3.3}$$

and

$$\eta(V) fU + \eta(U) fV = 0. \tag{3.4}$$

Proof. From equation (2.3), we have

$$\left(\overline{\nabla}_U\varphi\right)V = g\left(\varphi U, V\right)\xi - \eta\left(V\right)\varphi U.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V\varphi\right)U = -g\left(\varphi U, V\right)\xi - \eta\left(U\right)\varphi V.$$

Clubbing above two equations, we get

$$\left(\overline{\nabla}_U\varphi\right)V + \left(\overline{\nabla}_V\varphi\right)U = -\eta\left(V\right)\varphi U - \eta\left(U\right)\varphi V.$$

By using (2.16), we get

$$-\eta\left(V\right)\varphi U - \eta\left(U\right)\varphi V = 0. \tag{3.5}$$

By using (2.7) in above equation, then comparing the tangential and normal components, we get the result. □

Theorem 3.3. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then the induced structure p satisfies*

$$(\nabla_U p)V + (\nabla_V p)U = 0. \tag{3.6}$$

Proof. From (2.14), we have

$$(\nabla_U p)V = -g\left(U, pV\right)\xi - \eta\left(V\right)pU.$$

By swapping U and V in above equation, we get

$$(\nabla_V p)U = g\left(U, pV\right)\xi - \eta\left(U\right)pV.$$

On clubbing above two equations, we have

$$(\nabla_U p)V + (\nabla_V p)U = -\eta\left(V\right)pU - \eta\left(U\right)pV.$$

By using (3.3) in above equation, we get the result. □

Theorem 3.4. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. If second fundamental form h is parallel then contact CR-submanifold M is a totally geodesic.*

Proof. By swapping U and V in (2.15), we have

$$(\nabla_V p)U = bh\left(U, V\right) - g\left(V, pU\right)\xi - \eta\left(U\right)pV. \tag{3.7}$$

Combining (2.15) and (3.7), we have

$$(\nabla_U p)V + (\nabla_V p)U = 2bh\left(U, V\right) - \eta\left(V\right)pU - \eta\left(U\right)pV.$$

Now, using (3.3) and (3.6), yields $h\left(U, V\right) = 0$ for any $U, V \in \Gamma(TM)$. □

Lemma 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$A_{fV}U + A_{fU}V + 2bh\left(U, V\right) = 0. \tag{3.8}$$

Proof. By swapping U and V in (2.11), we have

$$(\nabla_V p)U = A_{fU}V + bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV. \tag{3.9}$$

On clubbing (2.11) and (3.9), we get

$$\begin{aligned}(\nabla_U p)V + (\nabla_V p)U =& A_{fV}U + A_{fU}V + 2bh\,(U,V) + g\,(pU,V)\,\xi \\ &+ g\,(pV,U)\,\xi - \eta\,(U)\,pV - \eta\,(V)\,pU.\end{aligned}$$

By using (2.9), it follows that

$$(\nabla_U p)V + (\nabla_V p)U = A_{fV}U + A_{fU}V + 2bh\,(U,V) - \eta\,(U)\,pV - \eta\,(V)\,pU.$$

Since p satisfies (3.3) and (3.6), we get the desired result. □

Proposition 3.1. *Suppose M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. Then M is anti-invariant submanifold in $\overline{M}$ if the endomorphism p is parallel.*

Proof. By interchanging U and V in (2.15), we get

$$(\nabla_V p)\,U = bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV,$$

for any $U, V \in \Gamma(D)$.

Clubbing above equation with (2.15), we get

$$(\nabla_U p)\,V + (\nabla_V p)\,U = 2bh\,(U,V) + g\,(pU,V)\,\xi + g\,(pV,U)\,\xi - \eta\,(V)\,pU - \eta\,(U)\,pV.$$

By using (2.9) and (3.6), above equation yields

$$2bh\,(U,V) - \eta\,(V)\,pU - \eta\,(U)\,pV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), we get $pU = 0$, which establishes our assertion. □

Proposition 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then M is invariant (submanifold) in $\overline{M}$ if the endomorphism f is parallel.*

Proof. By swapping U and V in (2.12), we get

$$(\nabla_V f)\,U = ch\,(U,V) - h\,(V,pU) - \eta\,(U)\,fV, \tag{3.10}$$

for any $U, V \in \Gamma(TM)$.

Clubbing (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

If f is parallel, then above equation becomes

$$2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), it follows that $fU = 0$. □

Lemma 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 0 \tag{3.11}$$

if and only if

$$2ch\,(U,V) = h\,(U,pV) + h\,(V,pU)\,. \tag{3.12}$$

Proof. Taking into consideration (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

By using (3.4), above equation yields

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU)\,.$$

Hence, the result. □

4. Examples

In this section, we give a few examples of Kenmotsu manifolds with killing φ.

Example 4.1. Let us consider the three dimensional manifold $\overline{M} = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Suppose metric g on $\overline{M}$ is given by

$$g = \eta \otimes \eta + e^{2z}(dx \otimes dx + dy \otimes dy).$$

Now, we choose

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} = \xi.$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i,e_j) = 0$ for $i \neq j$ and $g(e_i,e_j) = 1$ for $i = j$, for $1 \leq i,j \leq 3$. The 1-form η is given by $\eta(U) = g(U,e_3)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1,1)$, defined by $\varphi(e_1) = 0$, $\varphi(e_2) = 0$, $\varphi(e_3) = 0$. Now, using the linearity property of φ and g, we get

$$\varphi^2 U = -U + \eta(U)\xi, \quad \eta(e_3) = 1, \quad g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields,

$$\begin{aligned}
\overline{\nabla}_{e_1} e_1 &= -\,e_3, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = e_1,\\
\overline{\nabla}_{e_2} e_1 &= 0, \quad \overline{\nabla}_{e_2} e_2 = -e_3, \quad \overline{\nabla}_{e_2} e_3 = e_2,\\
\overline{\nabla}_{e_3} e_1 &= e_1, \quad \overline{\nabla}_{e_3} e_2 = e_2, \quad \overline{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U \xi = U - \eta\,(U)\,\xi$ for $\xi = e_3$. Hence, the manifold is a Kenmotsu manifold. From the above relations, we obtain the following equations

$$(4.1)\quad \begin{cases} (\overline{\nabla}_{e_1}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_1 = 0, & (\overline{\nabla}_{e_1}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_1 = 0,\\ (\overline{\nabla}_{e_1}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_1 = 0, & (\overline{\nabla}_{e_2}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_2 = 0,\\ (\overline{\nabla}_{e_2}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_2 = 0, & (\overline{\nabla}_{e_2}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_2 = 0,\\ (\overline{\nabla}_{e_3}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_3 = 0, & (\overline{\nabla}_{e_3}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_3 = 0,\\ (\overline{\nabla}_{e_3}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_3 = 0. \end{cases}$$

From the equations (4.1), it follows that φ is the killing tensor field. Hence, the manifold $\overline{M}$ is a Kenmotsu manifold with the killing tensor field φ. Moreover, we have

$$(4.2)\quad \begin{cases} \overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 + \overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 = 0,\\ \overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 + \overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 = 0,\\ \overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 + \overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 = 0,\\ \overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 + \overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 = 0,\\ \overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 + \overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 = 0,\\ \overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 + \overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 = 0,\\ \overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 + \overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 = 0,\\ \overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 + \overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 = 0,\\ \overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 + \overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 = 0, \end{cases}$$

and

$$(4.3)\quad \begin{cases} \eta(e_1)\varphi(e_1) + \eta(e_1)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_1) + \eta(e_1)\varphi(e_2) = 0,\\ \eta(e_3)\varphi(e_1) + \eta(e_1)\varphi(e_3) = 0, & \eta(e_1)\varphi(e_2) + \eta(e_2)\varphi(e_1) = 0,\\ \eta(e_2)\varphi(e_2) + \eta(e_2)\varphi(e_2) = 0, & \eta(e_3)\varphi(e_2) + \eta(e_2)\varphi(e_3) = 0,\\ \eta(e_1)\varphi(e_3) + \eta(e_3)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_3) + \eta(e_3)\varphi(e_2) = 0,\\ \eta(e_3)\varphi(e_3) + \eta(e_3)\varphi(e_3) = 0. \end{cases}$$

The equations (4.1) and (4.2) satisfy the equation (3.2) and the equations (4.1) and (4.3) satisfy the equation (3.5).

Analogous to [14], we have the following example of five-dimensional Kenmotsu manifold with the killing tensor field.

Example 4.2. Let us consider the five dimensional manifold $\overline{M} = \{(x_1, x_2, x_3, x_4, v) \in \mathbb{R}^5, v \neq 0\}$, where (x_1, x_2, x_3, x_4, v) are the standard coordinates in $\mathbb{R}^5$. Suppose metric g on $\overline{M}$ is given by

$$g = \eta \otimes \eta + e^{2v} \sum_{i=1}^{4} dx_i \otimes dx_i.$$

Now, we choose

$$e_1 = e^{-v}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-v}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-v}\frac{\partial}{\partial x_3}, \quad e_4 = e^{-v}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial v} = \xi.$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i,e_j)=0$ for $i\neq j$ and $g(e_i,e_j)=1$ for $i=j$, where $i,j=1,2,3,4,5$. The 1-form η is given by $\eta(U)=g(U,e_5)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1,1)$, defined by $\varphi(e_1)=0$, $\varphi(e_2)=0$, $\varphi(e_3)=0$, $\varphi(e_4)=0$, $\varphi(e_5)=0$.

Now, using the linearity property of φ and g, we have

$$\varphi^2U=-U+\eta(U)\xi,\quad \eta(e_5)=1,\quad g(\varphi U,\varphi V)=g(U,V)-\eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields

$$\begin{aligned}
&\overline{\nabla}_{e_1}e_1=-\,e_5,\quad \overline{\nabla}_{e_1}e_2=0,\quad \overline{\nabla}_{e_1}e_3=0,\quad \overline{\nabla}_{e_1}e_4=0,\quad \overline{\nabla}_{e_1}e_5=e_1,\\
&\overline{\nabla}_{e_2}e_1=0,\quad \overline{\nabla}_{e_2}e_2=-e_5,\quad \overline{\nabla}_{e_2}e_3=0,\quad \overline{\nabla}_{e_2}e_4=0,\quad \overline{\nabla}_{e_2}e_5=e_2,\\
&\overline{\nabla}_{e_3}e_1=0,\quad \overline{\nabla}_{e_3}e_2=0,\quad \overline{\nabla}_{e_3}e_3=-e_5,\quad \overline{\nabla}_{e_3}e_4=0,\quad \overline{\nabla}_{e_3}e_5=e_3,\\
&\overline{\nabla}_{e_4}e_1=0,\quad \overline{\nabla}_{e_4}e_2=0,\quad \overline{\nabla}_{e_4}e_3=0,\quad \overline{\nabla}_{e_4}e_4=-e_5,\quad \overline{\nabla}_{e_4}e_5=e_4,\\
&\overline{\nabla}_{e_5}e_1=e_1,\quad \overline{\nabla}_{e_5}e_2=e_2,\quad \overline{\nabla}_{e_5}e_3=e_3,\quad \overline{\nabla}_{e_5}e_4=e_4,\quad \overline{\nabla}_{e_5}e_5=0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U\xi=U-\eta\,(U)\,\xi$ for $\xi=e_5$. Moreover, on the similar pattern of Example 4.1, it follows that φ is a killing tensor field. Hence $\overline{M}$ is a five-dimensional Kenmotsu manifold with the killing tensor field. Also, analogous to Example 4.1, it can be seen that the equations (3.2) and (3.5) are satisfied.

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SOME MATHEMATICAL PROPERTIES FOR MARGINAL MODEL OF POISSON-GAMMA DISTRIBUTION

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Abstract. Recently, Casadei [4] provided an explicit formula for statistical marginal model in terms of Poisson-Gamma mixture. This model involving certain polynomials which play the key role in reference analysis of the signal and background model in counting experiments. The principal object of this paper is to present a natural further step toward the mathematical properties concerning this polynomials. We first obtain explicit representations for these polynomials in form of the Laguerre polynomials and the confluent hyper-geometric function and then based on these representations we derive a number of useful properties including generating functions, recurrence relations, differential equation, Rodrigueś formula, finite sums and integral transforms.

1. Introduction

In statistics, marginal models [7] are a technique for obtaining regression estimates in multilevel modeling, also called hierarchical linear models. People often want to know the effect of a predictor/explanatory variable X, on a response variable Y. One way to get an estimate for such effects is through regression analysis. Marginal model is generally compared to conditional model (random-effects model). Casadei [4], (see also [5]) investigated the model representing two independent Poisson processes, labeled as signal and background and both contributing additively to the total number of counted events, is considered from a Bayesian point of view (see [2] and [3]). This is a widely used model for the searches of rare or exotic events in presence of a background source, as for example in the searches performed by high energy physics experiments. The starting point in [4] is the marginal model $p(k|s)$, specifying the

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probability of counting $k \geq 0$ events in the hypothesis that the signal yield is $s \geq 0$ with the assumed knowledge about the background contribution:

$$p(k|s) = \int_0^\infty \mathrm{Poi}(k|s+b)\mathrm{Ga}(b|\alpha,\beta)db, \tag{1.1}$$

where $\mathrm{Ga}(b|\alpha,\beta)$ is a Gamma density of the form

$$p(b) = \mathrm{Ga}(b|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1}e^{-\alpha\beta}, \quad \alpha > 0,\, \beta \neq -1, \tag{1.2}$$

and $\mathrm{Poi}(k|s+b)$ is Poisson probability given by the formula (see [4])

$$\mathrm{Poi}(k|s+b) = e^{-s-b}\sum_{n=0}^{k} \frac{s^{k-n}b^n}{n!(k-n)!}. \tag{1.3}$$

Now, in view of (1.2) and (1.3) we find from (1.1) that

$$p(k|s) = \sum_{n=0}^{k} \frac{s^{k-n}\beta^\alpha e^{-s}}{n!(k-n)!\Gamma(\alpha)} \int_0^\infty e^{-(1+\beta)b} b^{\alpha+n-1} db,$$

which on using the Euler's integral [6]

$$\int_0^\infty e^{-at}t^{\nu-1}dt = a^{-\nu}\Gamma(\nu), \quad \nu > 0,$$

yields the marginal model (see [4])

$$p(k|x) = \left(\frac{\beta}{1+\beta}\right)^\alpha e^{-x} f_n(x;\alpha,\beta), \tag{1.4}$$

where

$$f_n(x;\alpha,\beta) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k} \frac{x^{n-k}}{(n-k)!(1+\beta)^k}. \tag{1.5}$$

The model $\mathrm{Poi}(k|s+b)$ is used to compute the Fisher's information (see [4, page 5, (1.4)])

$$I(s) = \mathrm{E}\left[\left(\frac{\partial}{\partial s}\log p(k|s)\right)^2\right] = -\mathrm{E}\left[\left(\frac{\partial^2}{\partial s^2}\mathrm{log} p(k|s)\right)\right], \tag{1.6}$$

and the reference prior [16]

$$\pi(s) \propto |I(s)|^{1/2}.$$

Starting from equation (1.6) and after certain mathematical computations, Casadei [4] derived the following expression for the Fisher's information:

$$I(s) = \left(\frac{\beta}{1+\beta}\right)^\alpha e^{-s} \sum_{n=0}^{\infty} \frac{[f_n(s;\alpha,\beta)]^2}{f_{n+1}(s;\alpha,\beta)} - 1. \tag{1.7}$$

From equation (1.7) one obtains

$$|I(s)|^{1/2} = \left|\left(\frac{\beta}{1+\beta}\right)^\alpha e^{-s} \sum_{n=0}^{\infty} \frac{[f_n(s;\alpha,\beta)]^2}{f_{n+1}(s;\alpha,\beta)} - 1\right|^{1/2}. \tag{1.8}$$

The function $|I(s)|^{1/2}$ has its single maximum at zero, hence a possible definition of the reference prior $\pi(s)$ for the signal is (see [4, page 9, (4.3)])

$$\pi(s) = \frac{|I(s)|^{1/2}}{|I(0)|^{1/2}}. \tag{1.9}$$

Because $\pi(s)$ does not explicitly depend on b (see (1.1)), the marginal posterior is proportional to the product of the reference prior (1.9) and the marginal likelihood (1.4)

$$p(k|x) \propto \left(\frac{\beta}{1+\beta}\right)^{\alpha} e^{-x} f_n(x;\alpha,\beta)\pi(s). \tag{1.10}$$

Casadei [4] derived a number of interesting properties for the polynomials $f_n(x;\alpha,\beta)$ which were useful in his investigation and following the prescription by [16], the reference prior for the signal parameter s is computed from the conditional model (1.1). Clearly, from (1.6), the polynomials $f_n(x;\alpha,\beta)$ play the key role in implementing all results in the work of Casadei [4]. For the evaluation of $f_n(x;\alpha,\beta)$ the author suggested some methods based on the logarithms, because this avoids rounding problems related to expressions featuring very big and very small values. Motivated by the important role of the marginal model $p(k|s)$ in several diverse fields of physics, analysis and statistical methods and the contributions in [4, 5] toward the the marginal model-polynomials $f_n(x;\alpha,\beta)$, this work aims at introducing several representations and properties for the polynomials $f_n(x;\alpha,\beta)$ in terms of known hyper-geometric functions and polynomials, for example, confluent hypergeometric function ${}_1F_1$ and Laguerre polynomials, which will be useful for the evaluation of the marginal model $p(k|s)$.

2. Explicit and Integral Representations

Based on the formulas

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}$$

and

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n,$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ denotes the Pochhammer symbol, the assertion (1.5) can be written in the form

$$f_n(x;\alpha,\beta) = \sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}. \tag{2.1}$$

By exploiting the result [1]

$$(-n)_k = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{2.2}$$

and the definition of the hyper-geometric function ${}_2F_0$ (see [1])

$$ {}_2F_0[a,b;-;x] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{n!}, $$

we find from (2.1) that

$$ f_n(x;\alpha,\beta) = \frac{x^n}{n!} {}_2F_0 \left[-n, \alpha; -; \frac{-1}{x(1+\beta)} \right]. $$

Now, with help of the representations of the hyper-geometric function ${}_2F_0$ (see [12, page 614])

$$ {}_2F_0[-n,a;-;z] = (a)_n(-z)^n {}_1F_1[-n;1-a-n;-z^{-1}] = n! z^n L_n^{-a-n}(-z^{-1}), $$

we can easily establish the explicit representations

$$ f_n(x;\alpha,\beta) = \frac{(\alpha)_n}{n!(1+\beta)^n} {}_1F_1[-n;1-\alpha-n;x(1+\beta)], \tag{2.3} $$

or equivalently

$$ f_n(x;\alpha,\beta) = \left(\frac{-1}{(1+\beta)} \right)^n L_n^{(-a-n)}(x(1+\beta)), \tag{2.4} $$

where ${}_1F_1$ is the confluent hyper-geometric function [6]

$$ {}_1F_1[a;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}, \tag{2.5} $$

and $L_n^{(\alpha)}$ is the associated Laguerre polynomials (see [1] or [13])

$$ L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+n+1) x^k}{k!(n-k)!\Gamma(\alpha+k+1)}. $$

Since the polynomials $f_n(x;\alpha,\beta)$ can be expressed in terms of representation involving the confluent hypergeometric function ${}_1F_1$ and the Laguerre polynomials $L_n^{(\alpha)}$, the properties of these function and polynomials assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties for the polynomials $f_n(x;\alpha,\beta)$. In this work formula (2.3) will play the key role in obtaining a number of main results for the polynomials $f_n(x;\alpha,\beta)$. Next, according to the relation between Laguerre polynomials $L_n^{(\alpha)}$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [1, page 294, (35)]

$$ L_n^{(\alpha)} = \lim_{\lambda\to\infty} P_n^{(\alpha,\lambda)} \left(1 - \frac{2x}{\lambda} \right) $$

and the assertion (2.4), we can obtain the explicit relation

$$ f_n(x;\alpha,\beta) = \left(\frac{-1}{(1+\beta)} \right)^n \lim_{\lambda\to\infty} P_n^{(-\alpha-n,\lambda)} \left(1 - \frac{2x(1+\beta)}{\lambda} \right). $$

Further, the Laguerre polynomials have the following asymptotic representation which describe their behavior for large value of the degree n [8, page 87, (4.22.18)]; see also [9]:

$$L_n^{(\alpha)}(x) \approx \frac{\Gamma(\alpha+n+1)}{n!} e^{\frac{x}{2}} (Nx)^{\frac{-\alpha}{2}} J_\alpha\left(2\sqrt{(Nx)}\right), \quad n \to \infty, N = n + \frac{\alpha+1}{2}. \tag{2.6}$$

In view of the explicit representation (2.4) it follows from (2.6) that

$$f_n(x;\alpha,\beta) \approx \left(\frac{-1}{(1+\beta)}\right)^n \frac{\Gamma(1-\alpha)}{n!} e^{\frac{x(1+\beta)}{2}} (Nx(1+\beta))^{\frac{\alpha+n}{2}} J_{-\alpha-n}\left(2\sqrt{(Nx(1+\beta))}\right),$$

$n \to \infty$, $N = \frac{n-\alpha+1}{2}$. From (2.3), we can easily seen that

$$f_n(0;\alpha,\beta) = \frac{(\alpha)_n}{n!(1+\beta)^n} \tag{2.7}$$

and

$$f_{n+1}(x;\alpha,\beta) = \frac{(\alpha)_{n+1}}{(n+1)!(1+\beta)^{n+1}}. \tag{2.8}$$

Formulas (2.7) and (2.8) are useful in computing the reference prior $\pi(s)$ in (1.9). It is often convenient to identify the various special functions and polynomials with contour integrals along certain paths in the complex plane. These integrals provide recursion formulas, asymptotic forms, and analytic continuations of the special functions. Also, they are sometimes used as definitions of special functions and polynomials . Now, we consider some integral representations for the polynomials $f_n(x;\alpha,\beta)$. To obtain integral representations, we first recall the results (see [1, page 300, (9.13) and (9.17)], [6, (6.11.1)(3)])

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt, \tag{2.9}$$

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^x x^{\left(\frac{1-c}{2}\right)} \int_0^1 e^{-t} t^{\frac{1}{2}(c-1)-a} J_{c-1}(2\sqrt{(xt)}) dt \tag{2.10}$$

and

$${}_1F_1[a;c;x] = \frac{\Gamma(\gamma)\Gamma(1-a)}{2\pi i \Gamma(c-a)} \oint_\gamma e^{xs} \left(\frac{s}{s-1}\right)^a (1-s)^{c-1} \frac{ds}{s}, \tag{2.11}$$

where c is a positive integer and the contour γ starts and ends at the point $s = 1$ on the $s-axis$ and encircles the origin in a positive sense and that Re(c) > Re(a). Also, a fourth representation can be obtained from equation (6.11.1) (7) of [6], for Re(c) > 0, $\gamma > 1$ and $a \neq 1, 2, 3, \ldots, c-1$. This is achieved with $b = c = n+1, n = 0, 1, \ldots,$ in (6.11.2) (6) of [6], where the integrand is a one-valued function of the parameter s and the path of integration may be replaced by a contour, for instance a circle $|s| = \rho > 1$. This representation is given by (see [6])

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{2\pi i x^{c-1}} \oint_\gamma e^{xs} \left(\frac{s}{s-1}\right)^a \frac{ds}{s^c}. \tag{2.12}$$

Directly from the results (2.9), (2.10), (2.11) and (2.12) and based on the definition (2.3), we can establish the following integral representations:

$$f_n(x;\alpha,\beta) = \frac{(-1)^n}{n!(1+\beta)^n\Gamma(-n)} \int_0^1 e^{x(\beta+1)t} t^{-(n+1)}(1-t)^{-\alpha} dt,$$

$$f_n(x;\alpha,\beta) = \frac{(-1)^n e^{x(1+\beta)}[x(1+\beta)]^{\frac{\alpha+n}{2}}}{n!(1+\beta)^n\Gamma(-n)} \int_0^1 e^{-t} t^{\frac{n-\alpha}{2}} J_{-(\alpha+n)}(2\sqrt{(x(1+\beta)t)})dt,$$

$$f_n(x;\alpha,\beta) = \frac{(\alpha)_n\Gamma(\gamma)\Gamma(1+n)}{2\pi i\ n!(1+\beta)^n\Gamma(1-\alpha)} \oint_\gamma e^{x(1+\beta)s}(s-1)^{-\alpha}\frac{ds}{s^{n+1}}$$

and

$$f_n(x;\alpha,\beta) = \frac{(\alpha)_n\Gamma(1-\alpha-n)}{2\pi i\ n!(1+\beta)^{-\alpha}x^{-\alpha-n}} \oint_\gamma e^{x(1+\beta)s}(s-1)^n\frac{ds}{s^{1-\alpha}},$$

respectively. By using the previous explicit and integral representations the computing of the conditional model will be more easily. In this regard by virtue of the results (2.7) and (2.8), in conjunction with (1.7), we find that

$$\begin{aligned} I(0) =& \frac{(1+\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)}\left(\frac{\beta}{1+\beta}\right)^\alpha \sum_{n=0}^\infty \frac{(\alpha)_n(\alpha)_n(2)_n}{n!(\alpha+1)_n(1)_n}\left(\frac{1}{1+\beta}\right)^n - 1 \\ =& \frac{(1+\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)}\left(\frac{\beta}{1+\beta}\right)^\alpha {}_3F_2\left[\alpha,\alpha,2;\alpha+1,1;\frac{1}{(1+\beta)}\right] - 1, \end{aligned} \tag{2.13}$$

where ${}_3F_2$ is special case of the generalized hypergeometric series ${}_pF_q$ (see [1]). Hence, from the assertions (2.13) and (2.3), we find the following elegant explicit representation for the marginal posterior defined by (1.10):

$$\begin{aligned} p(k|x) \propto& \left(\frac{\beta}{1+\beta}\right)^\alpha \frac{(\alpha)_n e^{-x}}{n!(1+\beta)^n} {}_1F_1[-n;1-\alpha-n;x(1+\beta)] \\ &\times \frac{\left|\left(\frac{\beta}{1+\beta}\right)^\alpha e^{-x}\sum_{n=0}^\infty \frac{(n+1)(\alpha)_n({}_1F_1[-n;1-\alpha-n;x(1+\beta)])^2}{n!(\alpha+n)_1(\beta+1)^{n-1}{}_1F_1[-n-1;-\alpha-n;x(1+\beta)]} - 1\right|^{1/2}}{\left|\frac{\beta^\alpha}{\alpha(1+\beta)^{\alpha-1}}{}_3F_2[\alpha,\alpha,2;\alpha+1,1;1/(1+\beta)] - 1\right|^{1/2}}. \end{aligned}$$

3. Generating Functions

A generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a power series. The sum of this infinite series is the generating function. Generating functions are often expressed in closed form (rather than as a series), by some expression involving operations defined for formal series. These expressions in terms of the indeterminate x may involve arithmetic operations, differentiation with respect to x and composition with (i.e., substitution into) other generating functions; since these operations are also defined for functions, the result looks like a function of x. Indeed, the closed form expression can often be interpreted as a function that can be evaluated at (sufficiently small) concrete values of x, and which has the formal series as its series expansion. Also, the generating functions offer a direct way to investigate the properties of the polynomials they define. Directly from (2.1) of the prececding section we obtain

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k} t^n}{k!(n-k)!(1+\beta)^k}.$$

On replacing n by $n+k$, we obtain

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{n} \frac{(\alpha)_k t^k}{k!(1+\beta)^k}.$$

Hence the polynomials $f_n(x;\alpha,\beta)$ have the following generating relation:

$$e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n. \tag{3.1}$$

A set of other generating functions for these polynomials is easily obtained. Let λ be arbitrary and proceed as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda)_n f_n(x;\alpha,\beta)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_k(\lambda)_n x^{n-k}t^n}{k!(n-k)!(1+\beta)^k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha)_k(\lambda)_{n+k} x^n t^{n+k}}{k!n!(1+\beta)^k} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k(\lambda)_k t^k}{k!(1+\beta)^k} \sum_{n=0}^{\infty} \frac{(\lambda+k)_n (xt)^n}{n!}.\end{aligned}$$

We thus arrive at the generating function

$$\sum_{n=0}^{\infty}(\lambda)_n f_n(x;\alpha,\beta)t^n = (1-xt)^{-\lambda}{}_2F_0\left[\alpha,\lambda;-;\frac{t}{(1+\beta)(1-xt)}\right].$$

The following two formulas are well-known consequences of the derivative operator $\hat{D}_x = \frac{\partial}{\partial x}$ and the integral operator $\hat{D}_x^{-1}$ (see [10]):

$$\hat{D}_x^n x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}x^{\lambda-n}, \quad \hat{D}_x^{-n} x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n+1)}x^{\lambda+n},$$

$m \in \mathbb{N}\cup\{0\}, \lambda \in \mathbb{C}-\{-1,-2,\dots\}$. Since

$$\hat{D}_x^k x^n = \frac{n!x^{n-k}}{(n-k)!},$$

formula (2.1) yields the operational relation

$$f_n(x;\alpha,\beta) = \frac{1}{n!}\sum_{k=0}^{\infty}\frac{(\alpha)_k \hat{D}_x^k x^n}{k!(1+\beta)^k} = \frac{1}{n!}\left(1-\frac{\hat{D}_x}{1+\beta}\right)^{-\alpha} x^n. \tag{3.2}$$

On multiplying both sides of (3.2) by t^n and taking the sum, we then get the generating relation

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \left(1-\frac{\hat{D}_x}{1+\beta}\right)^{-\alpha} e^{xt}.$$

On the other hand, since

$$\hat{D}_x^k x^{\alpha+k-1} = (\alpha)_k x^{\alpha-1},$$

we get from (2.1) the operational relation

$$f_n(x;\alpha,\beta) = \frac{x^{n-\alpha+1}}{n!}\left(1-\frac{\hat{D}_x x}{x(1+\beta)}\right)^n x^{\alpha-1}.$$

Now, we can easily derive the following generating relation:

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = x^{1-\alpha} exp\left[x\left(1-\frac{t\hat{D}_x x}{x(1+\beta)}\right)\right] x^{\alpha-1}.$$

From (2.1), we can easily derive the $m-th$ partial derivative of $f_n(x;\alpha,\beta)$ with respect to x as follows:

$$\hat{D}_x^m f_n(x;\alpha,\beta) = f_n^m(x;\alpha,\beta) = \sum_{k=0}^{n-m} \frac{(\alpha)_k x^{n-k-m}}{k!(n-k-m)!(1+\beta)^k} = f_{n-m}(x;\alpha,\beta). \tag{3.3}$$

Hence, from assertions (3.1) and (3.3), we get the result

$$e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=0}^{\infty} f_{n-m}(x;\alpha,\beta)t^{n-m}. \tag{3.4}$$

Remark 3.1. If in the generating functions (3.1) and (3.4), we let $t=1$, we find that

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta) = \sum_{n=0}^{\infty} f_{n-m}(x;\alpha,\beta) = e^x\left(\frac{1+\beta}{\beta}\right)^{\alpha},$$

which are the properties 1 and 3 of the polynomials in (2.1) derived by Casadei (see [4, pages 6–7]).

Remark 3.2. From (3.3), we have

$$f_n^m(x;\alpha,\beta) = \sum_{k=0}^{n-m} \frac{(-1)^{k+m}(-n)_{k+m}(\alpha)_k x^{n-k-m}}{n!k!(1+\beta)^k}. \tag{3.5}$$

We know that [13]

$$(-n)_{k+m} = \begin{cases} 0, & \text{if } k>n \text{ or } m>n, \\ \dfrac{(-1)^{k+m}n!}{(n-k-m)!}, & \text{if } 0\le k+m\le n. \end{cases} \tag{3.6}$$

Hence, equation (3.5), in conjunction with (3.6), gives

$$f_n(x;\alpha,\beta) = \begin{cases} 0, & \text{if } m>n, \\ f_n^m(x;\alpha,\beta) = f_{n-m}(x;\alpha,\beta), & \text{if } 0\le m\le n, \\ 1, & \text{if } n=m, \end{cases}$$

which is the tired property for the polynomials in (2.1) proved by Casadei [4, page 6, (2.7)]. Note that the proofs of the properties 1, 2 and 3 above are very short in compare with their proofs in [4]. In fact these three properties have their origin in the properties of Laguerre polynomials and the confluent hyper-geometric series ${}_1F_1$ (see [13, pages 200–213]).

4. Recurrence Relations and Differential Equation

First, in view of definition (2.1), we find that

$$\hat{D}_x f_n^m(x;\alpha,\beta) = f_{n-1}(x;\alpha,\beta) \tag{4.1}$$

and

$$\hat{D}_x^{-1} f_n^m(x;\alpha,\beta) = f_{n+1}(x;\alpha,\beta).$$

Secondly, differentiating both the sides of (3.1) with respect to t, we get

$$xe^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} + \left(\frac{\alpha}{1+\beta-t}\right)e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n-1},$$

or

$$\begin{aligned}&x(1+\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n - x\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^{n+1} + \alpha\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n\\ =&(1+\beta)\sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n-1} - \sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n+1},\end{aligned}$$

or

$$\begin{aligned}&x(1+\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n - x\sum_{n=0}^{\infty} f_{n-1}(x;\alpha,\beta)\ t^{n} + \alpha\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n\\ =&(1+\beta)\sum_{n=0}^{\infty} f_{n+1}(x;\alpha,\beta)\ (n+1)t^{n-1} - \sum_{n=0}^{\infty} f_{n-1}(x;\alpha,\beta)\ (n-1)t^{n}.\end{aligned}$$

Equating the coefficients of t^n from both sides, we find

$$[x(1+\beta)+\alpha]f_n(x;\alpha,\beta)-(x-n+1)f_{n-1}(x;\alpha,\beta)-(1+\beta)(n+1)f_{n+1}(x;\alpha,\beta) = 0. \tag{4.2}$$

From [13, Section 48, (15), (18) and (20)], using $p=q=1$, $\alpha_1=-n$, $\beta_1=1-\alpha-n$, $x\mapsto x(1+\beta)$, we obtain

$$\begin{aligned}\alpha_1 F_1[-n;1-\alpha-n;x(1+\beta)] =& -n_1F_1[-n+1;1-\alpha-n;x(1+\beta)]\\ &+(\alpha+1)_1F_1[-n;1-\alpha-n;x(1+\beta)],\end{aligned} \tag{4.3}$$

(4.4)

$$\begin{aligned}&[x(1+\beta)-n]_1F_1[-n;1-\alpha-n;x(1+\beta)]\\ =&-n_1F_1[-n;1-\alpha-n;x(1+\beta)] - \frac{x(1+\beta)(\alpha-1)}{(1-\alpha-n)}{}_1F_1[-n;2-\alpha-n;x(1+\beta)],\end{aligned}$$

$$ {}_1F_1[-n;1-\alpha-n;x(1+\beta)] = {}_1F_1[-n-1;1-\alpha-n;x(1+\beta)] + \frac{x}{(1-\alpha-n)}{}_1F_1[-n;2-\alpha-n;x(1+\beta)]. \tag{4.5}$$

Since (see (2.3))

$$ {}_1F_1[-n;1-\alpha-n;x(1+\beta)] = \frac{n!(1+\beta)^n}{(\alpha)_n} f_n(x;\alpha,\beta),$$

equations (4.3), (4.4) and (4.5) may be converted into the mixed recurrence formulas

$$ f_n(x;\alpha,\beta) = \left(1+\frac{n}{\alpha}\right) f_n(x;\alpha+1,\beta) - \frac{n}{\alpha} f_n(x;\alpha,\beta),$$
$$ f_n(x;\alpha,\beta) = \frac{n}{(n-x(1+\beta))} f_{n-1}(x;\alpha,\beta) + \frac{x(1+\beta)(\alpha-1)}{(n-x(1+\beta))(1-\alpha-n)} f_n(x;\alpha-1,\beta),$$
$$ f_n(x;\alpha,\beta) = f_{n+1}(x;\alpha,\beta) + \frac{x}{(1-\alpha-n)} f_n(x;\alpha-1,\beta).$$

Next, we derive the differential equation of the polynomials $f_n(x;\alpha,\beta)$. Our starting point is the recurrence formula (4.2). We have from (4.2)

$$ x(1+\beta)f_n(x;\alpha,\beta) + \alpha f_n(x;\alpha,\beta) - x f_{n-1}(x;\alpha,\beta) + (n-1) f_{n-1}(x;\alpha,\beta) - (1+\beta)(n+1) f_{n+1}(x;\alpha,\beta) = 0. \tag{4.6}$$

Using (4.1), equation (4.6) yields

$$ x(1+\beta)f_n(x;\alpha,\beta) + \alpha f_n(x;\alpha,\beta) - x\frac{\partial}{\partial x} f_n(x;\alpha,\beta) + (n-1)\frac{\partial}{\partial x} f_n(x;\alpha,\beta) - (1+\beta)(n+1) f_{n+1}(x;\alpha,\beta) = 0. \tag{4.7}$$

Next, on differentiating equation (4.7) with respect to x and simplify we obtain the following second order differential equation for the polynomials $f_n(x;\alpha,\beta)$:

$$ (n-x-1)\frac{\partial^2}{\partial x^2} f_n(x;\alpha,\beta) + (x(1+\beta)+\alpha-1)\frac{\partial}{\partial x} f_n(x;\alpha,\beta) - n(1+\beta) f_n(x;\alpha,\beta) = 0.$$

5. Rodrigueś-Type Formula

From (2.4), we have

$$ f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \sum_{k=0}^{n} \frac{(1-\alpha-n)_n[-x(1+\beta)]^k}{k!(n-k)!(1-\alpha-n)_k}. \tag{5.1}$$

Since

$$ \hat{D}_x^{n-k} x^{-\alpha} = \frac{(1-\alpha-n)_n}{(1-\alpha-n)_k} x^{(-\alpha-n+k)},$$

equation (5.1) can be written in the form

$$f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!} \sum_{k=0}^{n} \frac{(-1)^k(1+\beta)^k}{k!(n-k)!} \hat{D}_x^{n-k} x^{-\alpha}$$
$$= \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k(1+\beta)^k \hat{D}_x^{n-k} x^{-\alpha}.$$

Again, since

$$\hat{D}_x^k e^{-(1+\beta)x} = (-1)^k(1+\beta)^k e^{-(1+\beta)x},$$

we may conclude that

$$f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!} e^{(1+\beta)x} \sum_{k=0}^{n} \binom{n}{k} [\hat{D}_x^{n-k} x^{-\alpha}][\hat{D}_x^k e^{-(1+\beta)x}]. \tag{5.2}$$

Therefore, by Leibnitz theorem, equation (5.2) can be written in the following interesting Rodrigueś-type formula:

$$f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!} e^{(1+\beta)x} \hat{D}_x^n [x^{-\alpha} e^{-(1+\beta)x}].$$

6. Finite Sums and Integral Transforms

Using the result

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n+k,k),$$

we can write

$$\sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(\alpha-\lambda)_{n-s}}{(n-s)!}(1+\beta)^{n-s} f_s(x;\lambda,\beta) t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(\alpha-\lambda)_{n}}{n!}(1+\beta)^{n} f_s(x;\lambda,\beta) t^{n+s}$$
$$= \left(1-\frac{t}{1+\beta}\right)^{\lambda-\alpha} \sum_{s=0}^{\infty} f_s(x;\lambda,\beta) t^s.$$

Now, employing the generating relation (3.1) and comparing the coefficients of t^n in the resulting expression, we get the following finite sum:

$$\sum_{s=0}^{n} \frac{(\alpha-\lambda)_{n-s}}{(n-s)!}(1+\beta)^{n-s} f_s(x;\lambda,\beta) = f_n(x;\alpha,\beta).$$

Similarly, one can derive the following result:

$$\sum_{s=0}^{n} \frac{(\alpha+s)_{n-s}(1+y)^n}{(n-s)!}(1+\beta)^{n-s} f_s(x;\alpha,\beta)\left(\frac{y}{1-y}\right)^s = f_n\left(\frac{xy(1+\beta)}{\beta-y};\alpha,\beta\right).$$

Since

$$\begin{aligned}\sum_{p=0}^{\infty} f_p(x+y;\alpha+\gamma,\beta)t^n &= e^{(x+y)t}\left(1-\frac{t}{1+\beta}\right)^{-\alpha-\gamma}\\ &= e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} e^{yt}\left(1-\frac{t}{1+\beta}\right)^{-\gamma}\\ &= \sum_{p=0}^{\infty} f_p(y;\gamma,\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^{p+n}.\end{aligned}$$

Now, on letting $p \mapsto p-n$ and comparing the coefficients of t^p, we get the desired result

$$f_p(x+y;\alpha+\gamma,\beta) = \sum_{n=0}^{p} f_n(x;\alpha,\beta)f_{p-n}(y;\gamma,\beta)t^{p+n}.$$

Similarly, from the equation

$$\sum_{p=0}^{\infty} f_p(x;\alpha,\beta)t^n = e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\gamma}\left(1-\frac{t}{1+\beta}\right)^{\gamma-\alpha},$$

we can show that

$$f_p(x;\alpha,\beta) = \sum_{p=0}^{n} \frac{(\alpha-\gamma)_{n-p} f_p(x;\lambda,\beta)}{(p-n)!(1+\beta)^{p-n}}.$$

Next, we turn to some integral transforms for the polynomials $f_p(x;\alpha,\beta)$. In view of the definition (3.1), we obtain

$$\int_0^{\infty} e^{-t}t^{\lambda-1} f_n(xt;\alpha,\beta)dt = \sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}\int_0^{\infty} e^{-t}t^{\lambda+n-k-1}dt.$$

Now, on using the formulas (1.5) and (2.2) and the result $(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}$, and considering the definition of the Gaussian hyper-geometric function ${}_2F_1$ (see [15])

$$ {}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{6.1}$$

we get the integral transform

$$\int_0^{\infty} e^{-t}t^{\lambda-1} f_n(xt;\alpha,\beta)dt = \frac{\Gamma(\lambda+n)}{n!}\,{}_2F_1\left[-n,\alpha;1-\lambda-n;\frac{1}{x(1+\beta)}\right].$$

In view of (2.1), we get

$$\begin{aligned}\int_0^{t} (t-z)^{\lambda-1} z^{\alpha-\lambda+n-1} f_n(x/z;\alpha,\beta)dt =& \sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}\\ &\times \int_0^{t} (t-z)^{\lambda-1} z^{\alpha-\lambda+n+k-1}dt.\end{aligned}$$

Putting $(t-z)=t(1-p)$, we get

$$\int_0^t (t-z)^{\lambda-1} z^{\alpha-\lambda+n-1} f_n(x/z;\alpha,\beta)dt$$
$$=\sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k} t^{\alpha+\lambda+k} \int_0^1 p^{\alpha-\lambda+k-1}(1-p)^{\lambda-1}dp.$$

Now, projection integral transform would occur if we use the definition of Beta function

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$

and considering (2.5) and this asserts

$$\int_0^t (t-z)^{\lambda-1} z^{\alpha-\lambda+n-1} f_n(x/z;\alpha,\beta)dt = f_n(x/t;\alpha-\lambda,\beta).$$

7. Conclusions

In the previous sections we etablished a number of properties and representations for the polynomials $f_n(x;\alpha,\beta)$, which are useful tools for computing the marginal model $p(k|x)$, the Fisher's information $I(x)$ and the reference prior $\pi(x)$. In this regard, if we make use of the series representation (2.5), the assertion (1.6) gives us

$$p(k|x) = \left(\frac{\beta}{1+\beta}\right)^{\alpha} e^{-x} \frac{(\alpha)_n}{n!(1+\beta)^n} {}_1F_1[-n;1-\alpha-n;x(1+\beta)] \tag{7.1}$$
$$=\frac{\beta^{\alpha}(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^k(-n)_s(1+\beta)^s x^{s+k}}{k!s!(1-\alpha-n)_s}.$$

On letting $s \mapsto s-k$ in (7.1) and using the following relations [13]:

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n} \quad \text{and} \quad (-n)_k = \frac{(-1)^k n!}{(n-k)!},$$

we obtain

$$p(k|x) = \frac{\beta^{\alpha}(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{s=0}^{\infty} \frac{(-n)_s[x(1+\beta)]^s}{s!(1-\alpha-n)_s} \sum_{k=0}^{\infty} \frac{(-s)_k(\alpha+n-s)_k}{k!(1+n-s)_k} \left(\frac{1}{1+\beta}\right)^k.$$

According to the definition of Gaussian hyper-geometric series in (6.1), we get

$$p(k|x) = \frac{\beta^{\alpha}(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{s=0}^{\infty} \frac{(-n)_s[x(1+\beta)]^s}{s!(1-\alpha-n)_s} {}_2F_1\left[-s,\alpha+n-s;1+n-s;\frac{1}{(1+\beta)}\right].$$

On other hand, by letting $k \mapsto k-s$ in (7.1) and proceeding in the manner described above, it is not difficult to obtain from the series expansion

$$p(k|x) = \frac{\beta^{\alpha}(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{x^k}{k!} {}_2F_1\left[-n,-k;1-\alpha-n;1+\beta\right].$$

We conclude this investigation by remarking that the schema suggested in the derivation of the results in this work can be applied to find other needed properties for the polynomials defined in (2.1). Therefore, the properties of the polynomials $f_n(x;\alpha,\beta)$ assume noticeable importance.

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CHARACTERIZATION OF GRAPHS OF CONNECTED DETOUR NUMBER 2

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Abstract. Let $G = (V, E)$ be a connected graph of order $P(G) \geq 2$. The connected detour number of G, denoted $cdn(G)$, is introduced and studied by A. P. Santhakumaran and S. Athisayanathan [7]. In this paper, we characterize connected graph G of $cdn(G) = 2$ and of detour diameter $D(G) = 5, 6$.

1. Introduction

Let $G = (V, E)$ be a connected simple graph of p vertices and q edges. We assume that $p \geq 2$ and it is finite. For $u, v \in V(G)$, the length of a maximum $u - v$ path is called ***detour distance*** between u and v, and denoted by $D(u, v)$. A $u - v$ path of length $D(u, v)$ is called **u-v *detour***. For a vertex $v \in V$, the ***detour eccentricity*** $e_D(v)$ is defined by:

$$e_D(v) = \max\{D(u, v) : u \in V\},$$
$$\operatorname{diam}_D(G) = \max\{e_D(v) : v \in V(G)\}.$$

A vertex $w \in V(G)$ is said ***to lie*** on a $u - v$ detour Q, if w is a vertex of $V(Q)$ including u and v. A ***detour set*** (denoted d.s.) of G is a subset S of $V(G)$ such that every vertex v of G lies on $x - y$ detour for some $x, y \in S$. The ***detour number*** of G, denoted $dn(G)$, is defined by:

$$dn(G) = \min\{|S| : S \text{ is a detour set of } G\}.$$

A ***detour basis*** of G is a detour set of order $dn(G)$. If S is a detour set of G and the induced subgraph $G[S]$ is connected, then S is called ***connected detour set***

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$$cdn\,(G) = \min\{|S| : S \text{ is a connected detour set of } G\}.$$

A connected detour basis of G is a connected detour set of G of order $cdn(G)$. For the definitions of the concepts not given here, we refer to [1,3–7]. There are many research on connected detour number and edge detour graphs (see [8–10]). Ahmed and Ali [2], determined detour number for three special classes of graphs G, namely, unicyclic graphs, bicyclic graphs, and cog-graphs for C_p, K_p and $K_{m,n}$. In [7], the authors A. P. Santhakumaran and S. Athisayanathan characterized connected graphs G of $cdn\,(G) = 2$ and $D\,(G) \leq 4$. In this paper, we characterize graphs G of $D\,(G) = 5$ and 6 for which $cdn\,(G) = 2$.

2. Characterizations of Graphs G with $D(G) = 5$ and $cdn(G) = 2$

We start with the following proposition for graphs G having $cdn(G) = 2$.

Proposition 2.1. *Let G be a connected graph of order $P(G) \geq 3$. If $cdn\,(G) = 2$, then G contains neither end-vertices nor cut-vertices.*

Proof. (1) If v is an end-vertex of G and u is the vertex adjacent to v, then v is a cut-vertex, and $G - \{u, v\}$ contains at least one vertex, say w. Since u and v are in every c.d.s. of G; and uv is the only $u - v$ detour, then $\{u, v\}$ is not a c.d.s. of G [7]. Thus, $cdn(G) \geq 3$, contradicting the hypothesis. Therefore, G does not contain end-vertices.

(2) Now, assume that G contains a cut-vertex x and $\{x, y\}$ is a connected detour basis of G. By the proof of part (1), G contains no end-vertices, so y is not end-vertex. Let H_1 and H_2 be components of $G - \{x\}$, and let $y \in V(H_1)$. Since $P(G) \geq 3$, then H_2 contains at least one vertex. Clearly, every $x - y$ detour does not contain vertices from H_2, contradicting the definition of d.s. Thus, G does not contain cut-vertices. □

Now we proceed to find graphs G with detour diameter $D\,(G) = 5$ for which $cdn\,(G) = 2$.

Theorem 2.1. *Let G be a connected graph of $P(G) \geq 6$ and with $D\,(G) = 5$. Then, $cdn\,(G) = 2$ if and only if G is a cycle graph C_6, with or without any number of chords, or like the graph G_i $(i = 1, 2)$ depicted in Figure* 1.

Proof. It is easy to verify that for C_6 and for each G_i (i=1,2) $D(C_6) = D(G_i)$=5 and $cdn\,(C_6) = cdn\,(G_i) = 2$, in which a, b is a detour basis of G_i.

To prove the converse, let G be a connected graph of $P(G) \geq 6$ and with $D\,(G) = 5$, $cdn\,(G) = 2$. Then, by Proposition 2.1, G does not contain end-vertices and cut-vertices. Since $D\,(G) = 5$ and G is connected, then the circumference of G (denoted by $cir(G)$) is $3 \leq cir(G) \leq 6$. Therefore, we shall consider four cases for $cir(G)$.

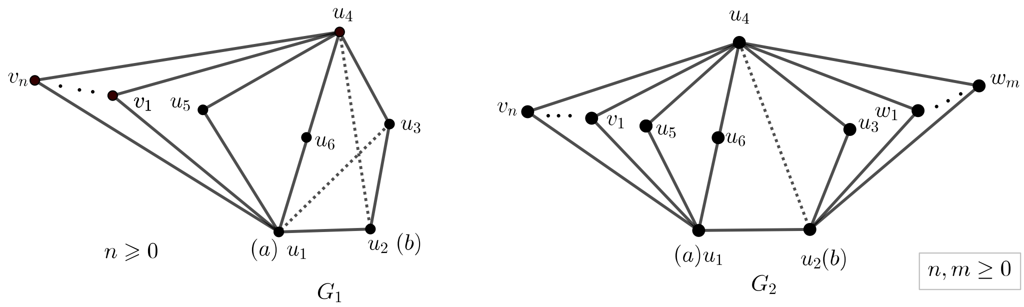


Figure 1.

Case (1). Let $cir\,(G) = 3$ and $P = (v_1, v_2, \ldots, v_6)$ is a $v_1 - v_6$ detour diameter in G (see Figure 2).

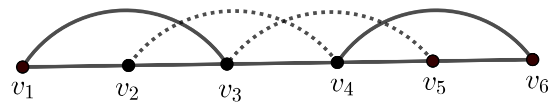


Figure 2. P for $cir\,(G) = 3$.

Then v_1 is not adjacent to v_4, v_5, v_6; and v_6 is not adjacent to v_2 and v_3. Moreover, v_1 and v_6 are not adjacent to any vertex other than $V(P)$. Since $\deg v_i = 2$, $(i = 1, \ldots, 6)$, then v_1 must be adjacent to v_3, and v_6 must be adjacent to v_4. By Proposition 2.1, G contains no cut-vertices, therefore there is either a $v_2 - v_5$ path in G, or $v_2 - v_4$ path and $v_3 - v_5$ path. Each of the two possibilities implies the existence of a cycle of length ≥ 6 in G, contradicting our assumption. Thus, in this case there is no graph that fulfills the required conditions.

Case (2). Let $cir\,(G) = 4$, and $P = (v_1, v_2, \ldots, v_6)$ be a $v_1 - v_6$ detour diameter of G (see Figure 3).

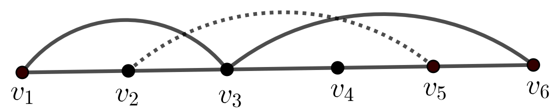


Figure 3.

Then v_1 is not adjacent to v_5 and v_6; and v_6 is not adjacent to v_2. Thus, v_1 is adjacent to v_3 or v_4, and v_6 is adjacent to v_3 or v_4. Therefore, we consider four subcases.

(a) If $v_1v_3, v_6v_4 \in E\,(G)$, then, as explained in case (1), $cir(G) \geq 6$, a contradiction.

(b) If $v_1v_3, v_6v_3 \in E(G)$, then either there is in G a $v_2 - v_4$ path or $v_2 - v_5$ path. Each of the two possibilities produces a graph G having $cir(G) \geq 5$; a contradiction.

(c) If $v_1v_4, v_6v_4 \in E(G)$, then, as in subcase (b), we arrive to a contradiction.

(d) If $v_1v_4, v_6v_3 \in E(G)$, then G contains the 6-cycle $(v_1, v_2, v_3, v_6, v_5, v_4, v_1)$ and so $cir(G) \geq 6$, a contradiction.

Therefore, in case (2) there is no graph that satisfies the required conditions of the theorem.

Case (3). Let $cir\,(G) = 5$ and $P = (v_1, v_2, \ldots, v_6)$ is a $v_1 - v_6$ detour diameter, then v_1 is not adjacent to v_6, and each of v_1, v_6 is not adjacent to any vertex not in $V(P)$. By Proposition 2.1, $\deg v_i \geq 2$ $(i = 1, 6)$. Therefore, we have consider the following nine subcases.

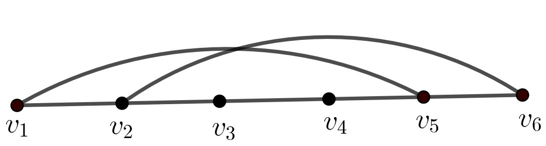


FIGURE 4.

(a) If $v_1v_5, v_2v_6 \in E(G)$, then such graph is like G_1 with $n = 0$ and without the edges u_1u_3, u_2u_4, in Figure 1.

(b) If $v_1v_5, v_3v_6 \in E(G)$, then such graph is like G_1 with $n = 0$ and without the edges u_2u_4.

(c) If $v_1v_5, v_4v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_5, v_6, v_4, v_3, v_2, v_1)$, contradicting our assumption.

(d) If $v_1v_4, v_2v_6 \in E(G)$, then G is like G_2 with $m = n = 0$ and without the edge u_1u_3 and u_2u_4.

(e) If $v_1v_4, v_3v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_2, v_3, v_6, v_5, v_4, v_1)$, contradicting our assumption.

(f) If $v_1v_4, v_4v_6 \in E(G)$, then by Proposition 2.1, there must be a $v_3 - v_5$ path or $v_2 - v_5$ path. If G contains $v_3 - v_5$ path, then G contains a cycle of length $\geq$6, a contradiction. Now, assume that G contains a $v_2 - v_5$ path, of length $\geq$ 2 then G contains $v_2v_5 \in E(G)$, then G is like G_2 in Figure 1 with $m = n = 0$.

(g) If $v_1v_3,\ v_2v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_3, v_4, v_5, v_6, v_2, v_1)$, contradicting the assumption.

(h) If $v_1v_3, v_3v_6 \in E(G)$, then as in subcase (f) either G is like G_2 with $m = n = 0$, or $cir(G) \geq 6$.

(i) If $v_1v_3, v_4v_6 \in E(G)$, then by Proposition 2.1, either G contains $v_2 - v_5$ path, or $v_2 - v_4$ path and $v_3 - v_5$ path, see Figure 5.

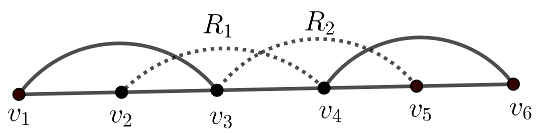


FIGURE 5.

If G contains a $v_2 - v_5$ path Q, then G contains a cycle $(v_1, v_3, v_4, v_6, v_5, Q, v_2, v_1)$, of length ≥ 6, a contradiction. If G contains a $v_2 - v_4$ path R_1 and $v_3 - v_5$ path R_2, then G contains a cycle $(v_1, v_3, R_2, v_5, v_6, v_4, R_1, v_2, v_1)$, of length ≥ 6 contradicting our assumption.

In view of the explanations in the subcases (a)-(i) we deduce that G_1 and G_2 in Figure 1 are of the general forms that satisfy the requirements of the theorem in this case.

Case (4). Let $cir\,(G) = 6$, and C be a 6-cycle in G. Because $D\,(G) = 5$, then there is no vertex in G, other than the vertices of C, adjacent to a vertex of C. Therefore, $P\,(G) = 6$ and so G is C_6 with, or without some chords. Hence, the proof of the theorem is completed. ☐

3. Characterization of Graphs G with $D(G) = 6$ and $cdn(G) = 2$

In the following proposition we establish that if G is a block of $D\,(G) = 6$, then the circumference of G is more than four.

Proposition 3.1. *Let G be a block of order $p \geq 7$ and with $D\,(G) = 6$, then $cir\,(G) = 5, 6$ or 7.*

Proof. Let $P = (u_1, u_2, \ldots, u_6, u_7)$ be a detour diameter of G, shown in Figure 6.

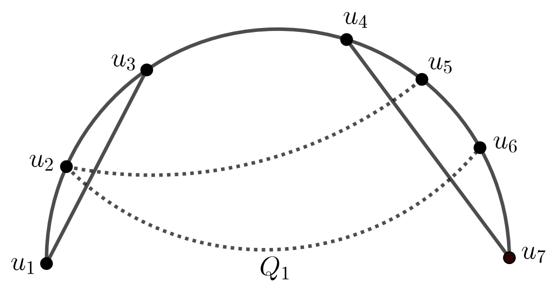


Figure 6.

Since G is a block, then it does not contain cut-vertices and end-vertices. Because $D\,(G) = 6$, then u_1 and u_7 each is not adjacent to any vertex other than $u_2, u_3, \ldots, u_6$. It is clear that $cir(G) \leq 7$. If u_1 is adjacent to u_5, u_6 or u_7, and/or u_7 is adjacent to u_1, u_2 or u_3, then G contains a cycle of length more than four (see Figure 6). To compute the proof we shall show that G contains a cycle of length 5, 6 or 7 if u_1 is adjacent to u_3 or u_4, and u_7 is adjacent to u_4 or u_5. So, we consider the following four cases.

Case (1). If $u_1u_3, u_7u_4 \in E(G)$, then we have the following four subcases.

(a) G contains a $u_2 - u_6$ path Q_1 which is edge-disjoint from P, this implies that G contains $l-$cycle $(u_3, u_1, u_2, (Q_1), u_6, u_7, u_4, u_3)$ of length $l \geq 6$.

(b) G contains the edge u_2u_5 which implies that G contains the $7-$cycle $(u_3, u_1, u_2, u_5, u_6, u_7, u_4, u_3)$.

(c) G contains edges u_2u_4 and u_3u_5, this implies that G contains the $7-$cycle $(u_2, u_1, u_3, u_5, u_6, u_7, u_4, u_2)$.

(d) G contains a $u_2 - u_4$ path Q_2 and a $u_3 - u_6$ path Q_3, which are edge-disjoint from P; this implies that G contains the cycle $(u_3, u_1, u_2, (Q_2), u_4, u_5, u_6, (Q_3), u_3)$ of length $l \geq 6$ (see Figure 7).

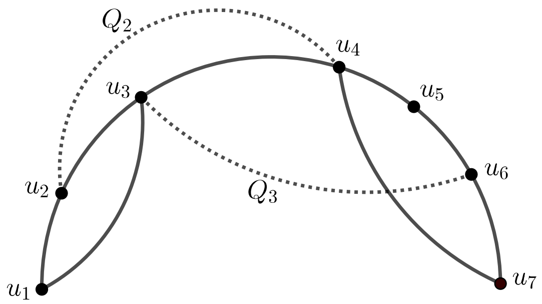


FIGURE 7.

Case (2). If $u_1u_3, u_7u_5 \in E(G)$, then we have two subcases.

(i) G contains the edge u_2u_6, which implies that G contains the 7-cycle $(u_3, u_1, u_2, u_6, u_7, u_5, u_4, u_3)$.

(ii) G contains a $u_2 - u_5$ path R_1 and a $u_3 - u_6$ path R_2 which are edge disjoint from $E(P)$, which implies that G contains cycle $(u_3, u_1, u_2, (R_1), u_5, u_7, u_6, (R_2), u_3)$ of length $l \geq 6$ (see Figure 8).

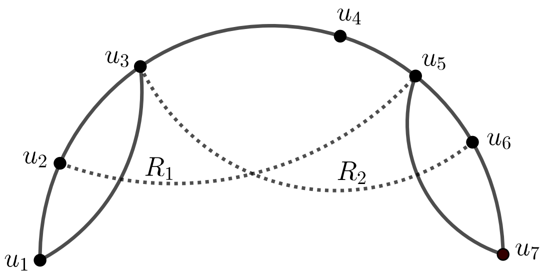


FIGURE 8.

Case (3). If $u_1u_4, u_5u_7 \in E(G)$, then, as in case (2), G contains a cycle of length 6 or 7.

Case (4). If $u_1u_4, u_4u_7 \in E(G)$, then we have four subcases for the cycles in G.

(α) G contains a $u_2 - u_5$ path F_1 other than (u_2, u_3, u_4, u_5), this implies that G contains a cycle $(u_2, u_1, u_4, u_7, u_6, u_5, (F_1), u_2)$ of length ≥ 6 (see Figure 9).

(β) G contains a $u_2 - u_6$ path F_2, this produces that G contains a cycle $(u_2, u_3, u_4, u_5, (F_2), u_2)$ of length $l \geq 5$.

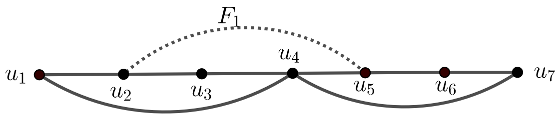


FIGURE 9.

(γ) G contains the edge u_3u_5 implying that G contains the 7-cycle $(u_3, u_2, u_1, u_4, u_7, u_6, u_5, u_3)$.

(δ) G contains a $u_3 - u_6$ path F_3, this produces that G contains a cycle $(u_3, u_2, u_1, u_4, u_7, u_6, (F_3), u_3)$ of length $l \geq 6$.

Hence, the proof of the proposition is completed. □

Theorem 3.1. *Let G be a connected graph of order $p \geq 7$ and with detour diameter $D(G) = 6$. Then, $cdn(G) = 2$ if and only if G is a cycle graph C_7^*, with or without any number of chords, or G belongs to the family F shown in Figure* 10.

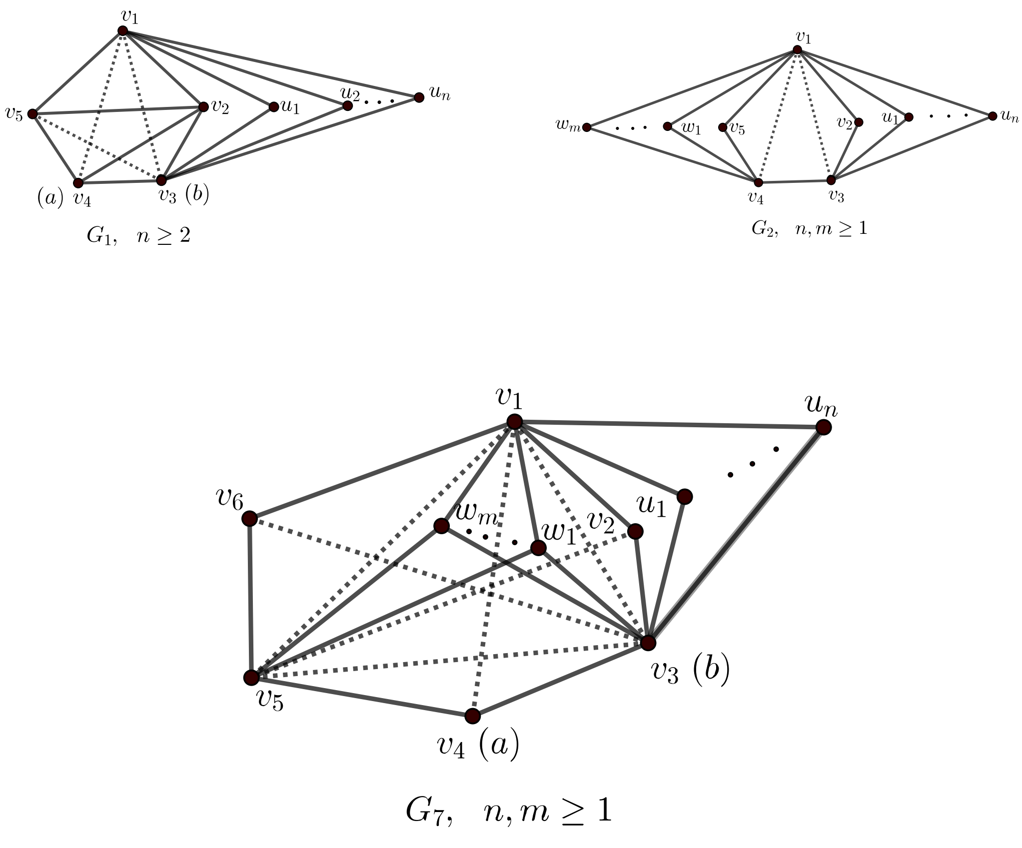
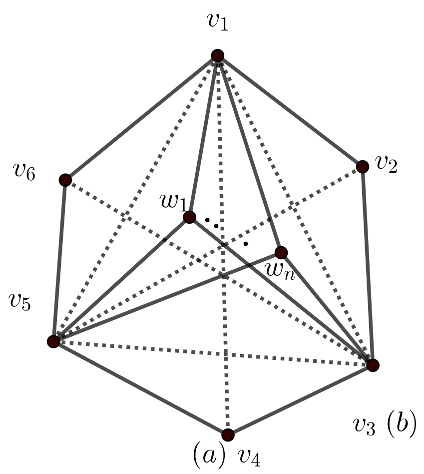
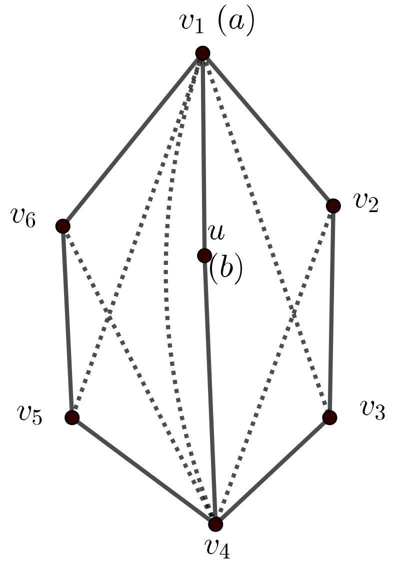


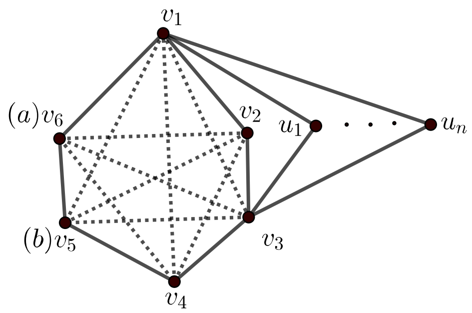
FIGURE 10. The family F



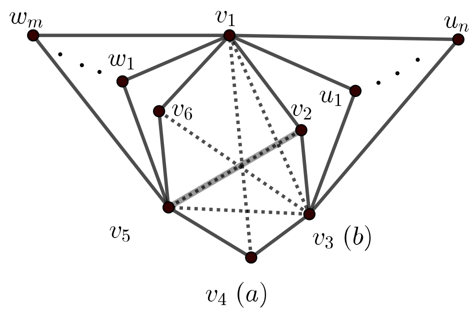
G_3, $n \geq 1$



G_4



G_5, $n \geq 1$



G_6, $n, m \geq 1$

Proof. It is straightforward to verify that $D(C_7^*) = D(G_i) = 6$, and $cdn(C_7^*) = cdn(G_i) = 2$, in which $\{a, b\}$ is a connected detour basis of G_i $(1 \leq i \leq 7)$.

To prove the converse, let G be a connected graph of order $p \geq 7$, $D(G) = 6$ and $cdn(G) = 2$. Then, by the Proposition 2.1, G is a block, and by Proposition 3.1, $cir(G) = 5$, 6 or 7. Thus, we shall consider three cases depending on the circumference of G.

Case (1). Let $cir(G) = 5$ and $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Since G is connected and $P(G) \geq 7$, then there is a vertex $u_1 \neq v_i$ $(1 \leq i \leq 5)$ adjacent to a vertex, say v_1, of C. Because $\deg u_1 \geq 2$, then either u_1 is adjacent to another vertex of C not adjacent to v_1, or it is adjacent to a vertex $x \neq v_i$ $(1 \leq i \leq 5)$. If $u_1x \in E(G)$, then x is not adjacent to any other vertex $x \notin V(C)$, and, also, it is not adjacent to any vertex of C, because, otherwise $D(G) \geq 7$ or $cir(G) \geq 6$. Therefore u_1 must be adjacent to non-adjacent vertices of C, say v_1 and v_3 and it is not adjacent to any other vertex of G, that is $\deg u_1 = 2$. It is clear that every vertex $y \notin V(C)$ is of degree 2 and adjacent to two non-adjacent vertices of C.

Let $w_1 \in V(G)$, $w_1 \notin V(C)$ and $w_1 \neq u_1$, then the following hold.

(a) If $w_1v_1, w_1v_3 \in E(G)$, then G is like the graph G_1, in Figure 10, with $n = 2$ (taking $u_2 = w_1$) and with edge v_2v_4 or v_2v_5, and G may contain edge $\{v_1v_3, v_1v_4, v_3v_5\}$. Therefore, G_1 is of a general form of this subcase, because $P(G) \geq 7$.

(b) If $w_1v_2, w_1v_5 \in E(G)$, then $cir(G) \geq 7$, a contradiction.

(c) If $w_1v_2, w_1v_4 \in E(G)$, then $cir(G) \geq 6$, a contradiction.

(d) If $w_1v_1, w_1v_4 \in E(G)$, (or $w_1v_3, w_1v_5 \in E(G)$), then G is like the graph G_2, in Figure 10, with $m = n = 1$ and G may contain some of the edges v_1v_4 or v_1v_3. Therefore, G_2 is of a general form of this subcase, because $P(G) \geq 7$.

Case (2). Let $cir(G) = 6$, $C = (v_1, v_2, \ldots, v_6, v_1)$ and let $W = V(G) - V(C)$. If $w \in W$, then w S is adjacent to at least two vertices of C, for otherwise $D(G) \geq 7$. Since $cir(G) = 6$, then w is not adjacent to any two adjacent vertices of C. Therefore, every vertex of W is of degree 3 or 2, and it is not adjacent to any vertex other than the vertices of C. Thus, we shall consider G in the following three subcases.

(a) Let every vertex of W is of degree 3. If $w \in W$, and w is adjacent to v_1 then it is adjacent to v_3 and v_5. If in addition to w, there is $w' \in W$ adjacent to v_2, v_4 and v_6, then G contains the 8-cycle $(v_1, w, v_3, v_2, w', v_4, v_5, v_6, v_1)$ (see Figure 11) contradicting the assumption. Thus, without loss of generality every vertex of W is adjacent to v_1, v_3 and v_5. Therefore, G is like the graph G_3 in Figure 10 with $n \geq 1$ and a number of dotted chords of C.

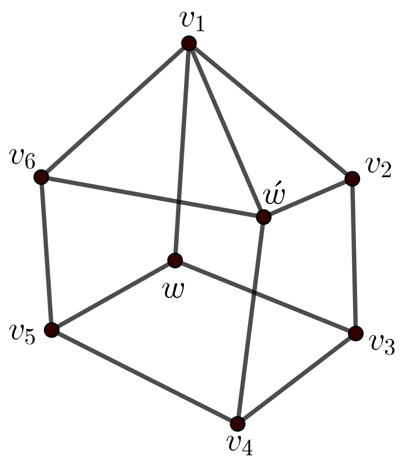


FIGURE 11.

(b) Let every vertex of W is of degree 2. Let u be any vertex in W and assume that u is adjacent to v_1. Then u is adjacent to v_3, v_4 or v_5. Therefore, we have two general possibilities, namely:

(i) $uv_1, uv_4 \in E(G)$;

(ii) uv_1, uv_3 (or uv_1, uv_5) $\in E(G)$.

For subcase (i), if u' is another vertex of W, then, for all connections of u' with a pair of non-adjacent vertices of C, the graph G will not satisfy the requirements $D = 6$ and $cdn = 2$. Therefore W consists of exactly one vertex u, and so $P(G) = 7$. Hence, G is like the graph G_4 shown in Figure 10.

(ii) Let $uv_1, uv_3 \in E(G)$. If each $u \in W$ is adjacent to the some non-adjacent pair of $V(C)$ like v_1, v_3, then G is like G_5 shown in Figure 10. If there is a vertex $u_1 \in V(G)$ adjacent to, say v_1, v_3, and there is at least one vertex $w_1 \in V(G)$ adjacent to v_1, v_5 (or v_3, v_5), then G is like G_6 with $n, m \geq 1$. For other connections of the vertices of W to pairs of non-adjacent vertices of $V(C)$, we have the following.

(a) If $uv_1, uv_3; wv_1, wv_5; xv_3, xv_5 \in V(G)$, where $x \in W$, then we have a graph like H_1 shown in Figure 12. Clearly, $cdn(H_1) = 3$, so H_1 does not fulfill the requirements.

(b) If $uv_1, uv_3; wv_4, wv_6 \in V(G)$, then we have a graph like H_2 shown in Figure 12. Clearly, $D(H_2) = 7$, so H_2 does not fulfil the required conditions.

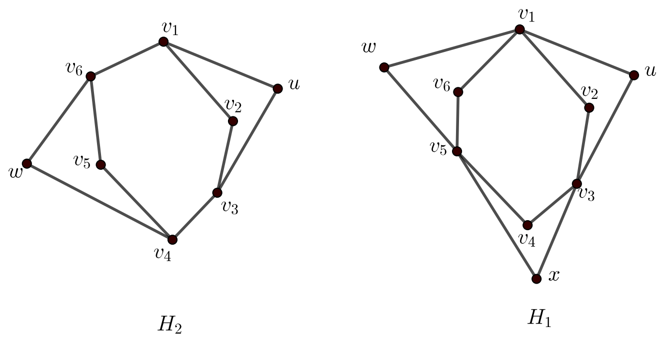


Figure 12.

(c) Now, assume that W consists of vertices of degree 2 and of degree 3. Let w be a vertex in W of degree 3. Then, without loss of generality, assume that w is adjacent to v_1, v_3 and v_5. Let $u \in W$ of degree 2, then we have the following possibilities.

(1) If u is adjacent to v_1 and v_3, then G is like the graph G_7, with $n, m \geq 1$, shown in Figure 10.

(2) If u is adjacent to v_2 and v_4, then G contains a 7-cycle $(v_1, v_6, v_5, w, v_3, v_4, u, v_2, v_1)$, a contradiction.

(3) If u is adjacent to v_1 and v_4, then $cdn(G) \geq 2$, a contradiction.

(4) If u is adjacent to v_3 and v_5, then G is like the graph G_7 in Figure 10.

Hence, the graph G in Case (2), for which $cir(G) = 6$, is in general construction, is like G_i $(i = 3, 4, 5, 6, 7)$.

Case (3). Let $cir(G) = 7$ and $C = (v_1, v_2, \ldots, v_7, v_1)$. If there is a vertex u in G other than the vertices of C, then u is adjacent to a vertex of C, say v_1. This implies that G contains a 7-path, namely $u, v_1, v_2, \ldots, v_7$, contradicting the hypothesis of the theorem. Therefore, $P(G) = 7$, and so G is the 7-cycle graph C_7^* with some chords of C.

Hence, the proof of the theorem is completed. □

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ON DEGREE OF APPROXIMATION OF SIGNALS IN THE GENERALIZED ZYGMUND CLASS BY USING $(E, r)(N, q_n)$ MEAN

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Abstract. In the present article, we have established a result on degree of approximation of function (or signal) in the generalized Zygmund class $Z_l^{(m)}$,$(l \geq 1)$ by using $(E, r)(N, q_n)$- mean of Trigonometric Fourier series.

1. Introduction

Signal Analysis describes the field of study whose objective is to collect, understand and deduce information and intelligence from various signals. Now-a-days the analysis of signals is a fundamental problem for many engineers and scientists. In the recent past, we have seen the applications of mathematical methods such as Probability theory, Mathematical statistics etc. in the analysis of signals. Very recently, approximation theory has got a large popularity as it has given a new dimension in approximating the signals (or functions). The estimation of error functions in Lipschitz class and Zygmund space using different summability techniques of Fourier series and conjugate Fourier series have been of great interest among the researchers in the last decades (for details see [2, 3, 9, 10, 13–18]). Later, the generalized Zygmund class $Z_l^{(m)}$, $l \geq 1$, is investigated by Leindler [8], Moricz [4], Moricz and Nemeth [5]. Very recently Nigam [7] and Singh et al. [11] proved approximation of functions in the generalized Zygmund class by using Hausdroff means. Lal and Shireen [12] proved a result on approximation of functions of generalized Zygmund class by Matrix Euler summability mean of Fourier series. In the present paper, we investigate on the degree

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of approximation of a signal (or function) in the generalized Zygmund class $Z_l^{(m)}$, $l \geq 1$, by $(E,r)(N,q_n)$ product mean of the trigonometric Fourier series.

2. Definitions and Notations

Let h be a function, which is periodic in $[0,2\pi]$ such that $\int_0^{2\pi} |h(x)|^l dx < \infty$. We denote

$$L_l[0,2\pi] = \Big\{ h : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |h(x)|^l dx < \infty \Big\}, \quad l \geq 1.$$

The Fourier series of $h(x)$ is given by

$$\sum_{n=0}^{\infty} u_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big). \tag{2.1}$$

Let $S_p(h;x)$ denotes the p-th partial sum of $h(x)$ and is given by

$$S_p(h;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x+v) \frac{\sin \left(p + \frac{1}{2}\right) v}{2 \sin \frac{v}{2}} dv.$$

We define

$$\|h\|_l = \left(\frac{1}{2\pi} \int_0^{2\pi} |h(x)|^l dx \right)^{\frac{1}{l}}, \quad 1 \leq l < \infty,$$

and

$$\|h\|_l = \operatorname{esssup}_{0 \leq x \leq 2\pi} |h(x)|, \quad l = \infty.$$

Let the Zygmund modulus of continuity of $h(x)$ be

$$m(h;r) = \sup_{0 \leq r, x \in \mathbb{R}} |h(x+v) + h(x-v) - 2h(x)| \quad \text{(see [1])}.$$

Suppose $\mathbf{B}$ represents the Banach space of all 2π periodic functions which are continuous and defined over $[0,2\pi]$ under the supremum norm. Clearly,

$$Z_{(\alpha)} = \Big\{ h \in \mathbf{B} : |h(x+v) + h(x-v) - 2h(x)| = O\left(|v|^{\alpha}\right), 0 < \alpha \leq 1 \Big\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha)}$ defined by

$$\|h\|_{(\alpha)} = \sup_{0 \leq x \leq 2\pi} |h(x)| + \sup_{x, v \neq 0} \frac{|h(x+v) + h(x-v) - 2h(x)|}{|v|^{\alpha}}.$$

For $h \in L_l[0,2\pi]$, $l \geq 1$, the integral Zygmund modulus of continuity is defined by

$$m_l(h;r) = \sup_{0 < v \leq r} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h(x+v) + h(x-v) - 2h(x)|^l dx \right\}^{\frac{1}{l}},$$

and for $h \in \mathbf{B}$, $l = \infty$,

$$m_\infty(h;r) = \sup_{0 < v \leq r} \max_x |h(x+v) + h(x-v) - 2h(x)|.$$

Clearly, $m_l(h;r) \to 0$ as $l \to 0$. Again,

$$Z_{(\alpha),\ l} = \left\{ h \in L_l[0,2\pi] : \left(\int_0^{2\pi} |h(x+v) + h(x-v) - 2h(x)|^l dx \right)^{\frac{1}{l}} = O(|v|^\alpha) \right\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha),l}$ for $0 < \alpha \le 1$ and $l \ge 1$. Clearly,

$$\|h\|_{(\alpha),l} = \|h\|_l + \sup_{v \ne 0} \frac{\|h(\cdot+v) + h(\cdot-v) - 2h(\cdot)\|_l}{|v|^\alpha}.$$

Let

$$Z^{(m)} = \Big\{ h \in \mathbf{B} : |h(x+v) + h(x-v) - 2h(x)| = O\left(m(v)\right) \Big\},$$

where m is a Zygmund modulus of continuity satisfying

(a) $m(0) = 0$;

(b) $m(v_1 + v_2) \le m(v_1) + m(v_2)$.

Define

$$Z_l^{(m)} = \left\{ h \in L_l : 1 \le l < \infty, \sup_{v \ne 0} \frac{\|h(\cdot+v) + h(\cdot-v) - 2h(\cdot)\|_l}{m(v)} < \infty \right\},$$

where

$$\|h\|_l^{(m)} = \|h\|_l + \sup_{v \ne 0} \frac{\|h(\cdot+v) + h(\cdot-v) - 2h(\cdot)\|_l}{m(v)}, \quad l \ge 1.$$

Clearly, $\|\cdot\|_l^{(m)}$ is a norm $Z_l^{(m)}$. Also, $Z_l^{(m)}$ is complete since L_l, $l \ge 1$, is complete. So, $Z_l^{(m)}$ is a Banach space under $\|\cdot\|_l^{(m)}$. Agian, suppose $m(v)$ and $\mu(v)$ represents the Zygmund moduli of continuity such that $\frac{m(v)}{\mu(v)}$ is positive and non-decreasing then

$$\|h\|_l^{(\mu)} \le \max\left\{1, \frac{m(2\pi)}{\mu(2\pi)}\right\} \|h\|_l^{(m)} \le \infty. \tag{2.2}$$

Clearly,

$$Z_l^{(m)} \subseteq Z_l^{(\mu)} \subseteq L_l, \quad l \ge 1.$$

Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Suppose $\{q_k\}$ represents the sequence of non-negative integers such that

$$Q_n = \sum_{k=0}^{n} q_k \to \infty \quad \text{as} \quad n \to \infty. \tag{2.3}$$

If

$$\tau_n^N = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} s_k, \quad n = 0, 1, 2, \ldots \tag{2.4}$$

represents the (N, q_n) mean of $\{s_n\}$ generated by the sequence $\{q_n\}$, then the series $\sum u_n$ is said to be summable to 's' whenever

$$\lim_{n \to \infty} \tau_n^N \to s.$$

We know, (N, q_n) method is regular [6]. The (E, r) transform of $\{s_n\}$ is given by

$$E_n^r = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) r^{n-k} s_k. \tag{2.5}$$

If $E_n^r \to s$ as $n \to \infty$, then $\sum u_n$ is summable to 's' by (E, r) summability. Also, (E, r) method is regular [6].

The $(E, r)(N, q_n)$ transform of $\{s_n\}$ is given by

$$\tau_n^{E_r,N} = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) \Big\{ \frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} s_\nu \Big\}. \tag{2.6}$$

The series $\sum u_n$ is summable to s by the $(E, r)(N, q_n)$ transform if $\tau_n^{E_r,N} \to s$ as $n \to \infty$.

The following notations are used in the rest part of our paper:

$$\begin{aligned} \varphi(x, v) =& h(x+v) + h(x-v) - 2h(x), \\ \kappa_n^{E_r,N}(v) =& \frac{1}{2\pi(1+r)^n} \sum_{k=0}^{n} C(n,k) \left\{ \frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} \frac{\sin\left(\nu + \frac{1}{2}\right)v}{\sin\left(\frac{v}{2}\right)} \right\}. \end{aligned}$$

3. Known Results

Using Matrix Euler summability means, Lal and Shireen [12] proved the following theorems.

Theorem 3.1. *Let the lower triangular matrix $A = (a_{n,k})$ satisfy the following conditions:*

$$\sum_{k=0}^{n} a_{n,k} = 1, \quad a_{n,k} \geq 0, \quad n = 0, 1, 2, \ldots, \; k = 0, 1, 2, \ldots, \tag{3.1}$$

$$\sum_{k=0}^{n} |\Delta a_{n,k}| = O\Big(\frac{1}{n+1}\Big) \quad \textit{and} \quad (n+1) a_{n,n} = O(1). \tag{3.2}$$

The best approximation of the Fourier series (2.1) *by Matrix-Euler mean is given by*

$$E_n(h) = \inf_{t_n{}^{\Delta,E}} \| t_n{}^{\Delta,E} - h \|_l^\mu = O\left(\frac{1}{n+1} \int_{\frac{1}{(n+1)}}^{\pi} \frac{m(v)}{v^2 \mu(v)} dv \right), \tag{3.3}$$

where

$$t_n{}^{\Delta,E} = \sum_{k=0}^{n} a_{n,k} \frac{1}{2^k} \sum_{\nu=0}^{k} C(k,\nu) s_\nu,$$

represents the Matrix-Euler mean of a 2π periodic and Lebesgue integrable function $h : [0, 2\pi] \to \mathbb{R}$, that belongs to $Z_l^{(m)}$, $l \geq 1$. Here, m and μ are the Zygmund moduli of continuity and $\frac{m(v)}{\mu(v)}$ is positive and non-decreasing.

Theorem 3.2. *Let $A = (a_{n,k})$ be a lower triangular matrix satisfying* (3.1) *and* (3.2) *in Theorem* 3.1 *along with the condition that* $\frac{m(v)}{\mu(v)}$ *is non-increasing. Then, for* $h \in Z_l^{(m)}$, $l \geq 1$, *the best approximation by Matrix-Euler mean* $\left(t_n{}^{\Delta,E}\right)$ *is given by*

$$E_n(h) = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)} \log(n+1)\pi\right). \tag{3.4}$$

4. Main Theorems

Theorem 4.1. *Let* $h : [0, 2\pi] \to \mathbb{R}$ *be a periodic function* (*with period* 2π) *belonging to* $Z_l^{(m)}$, $l \geq 1$, *which is integrable in the sense of Lebesgue. Then the degree of approximation of h by using* $(E, r)(N, q_n)$ *mean of* (2.1) *is given by*

$$E_n(h) = \inf_{\tau_n{}^{E_r,N}} \|\tau_n{}^{E_r,N} - h\|_l^{\mu} = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v\mu(v)} dv\right), \tag{4.1}$$

where $m(v)$ *and* $\mu(v)$ *are the Zygmund moduli of continuity and* $\frac{m(v)}{v\mu(v)}$ *is positive and non-decreasing.*

Theorem 4.2. *The degree of approximation of a* 2π *periodic and Lebesgue integrable function h,* $h : [0, 2\pi] \to \mathbb{R}$, *using* $(E, r)(N, q_n)$ *mean of* (2.1) *is given by*

$$E_n(h) = \inf_{\tau_n{}^{E_r,N}} \|\tau_n{}^{E_r,N} - h\|_l^{\mu} = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)} \left(\frac{\pi}{n+1} - \frac{1}{(n+1)^2}\right)\right), \tag{4.2}$$

where $h \in Z_l^{(m)}$, $l \geq 1$, $m(v)$ *and* $\mu(v)$ *are the Zygmund moduli of continuity and* $\frac{m(v)}{v\ \mu(v)}$ *is positive and non-increasing.*

We require the below mentioned lemmas to prove our main theorems.

5. Lemmas

Lemma 5.1.

$$|\kappa_n^{E_r,N}| = O(n), \quad \text{for } 0 \leq v \leq \frac{1}{n+1}.$$

Lemma 5.2.

$$|\kappa_n^{E_r,N}| = O\left(\frac{1}{v}\right), \quad \text{for } \frac{1}{n+1} \leq v \leq \pi.$$

Lemma 5.3. *Let* $h \in Z_l^{(m)}$. *Then, for* $0 < v \leq \pi$,

(*i*) $\|\varphi(\cdot, v)\|_l = O(m(v))$;

(*ii*) $\|\varphi(\cdot + y, v) + \varphi(\cdot - y, v) - 2\varphi(\cdot, v)\|_l = O(m(v))$ *or* $O(m(y))$;

(iii) If $m(v)$ and $\mu(v)$ are as defined in Theorem 4.1*, then*

$$\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l = O\left(\mu(y)\frac{m(v)}{\mu(v)}\right),$$

where $\varphi(x,v)=h(x+v)+h(x-v)-2h(x)$.

6. Proof of the Lemmas

Proof of Lemma 5.1. For $0\le v\le \frac{1}{n+1}$ and $\sin nv \le n\sin v$, we have

$$|\kappa_n^{E_r,N}(v)| = \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}\right|$$
$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{(2\nu+1)\sin\frac{v}{2}}{\sin\frac{v}{2}}\right\}\right|$$
$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}(2k+1)\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\right\}\right|$$
$$\le \frac{(2n+1)}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\right|$$
$$= O(n).$$

□

Proof of Lemma 5.2. By Jordan's lemma

$$\sin\left(\frac{v}{2}\right)\ge\frac{v}{\pi},\quad \sin nv\le 1,\quad \frac{1}{n+1}\le v\le\pi.$$

Now,

$$|\kappa_n^{E_r,N}(v)| = \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}\right|$$
$$\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k}\frac{\pi}{v}\ q_{k-\nu}\right\}\right|$$
$$= \frac{1}{2v(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k}\ q_{k-\nu}\right\}\right|$$
$$= \frac{1}{2v(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\right|$$
$$= O\left(\frac{1}{v}\right).$$

□

Proof of Lemma 5.3. See [12]. □

7. Proof of Main Theorems

Proof of Theorem 4.1. Let $S_k(h;x)$ denotes the k-th partial sum of the series (2.1). We have

$$S_k(h;x)-h(x)=\frac{1}{2\pi}\int_0^{\pi}\varphi(x,v)\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}dv$$

and the (N,q_n) transform of it is given by

$$\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\{S_k(h;x)-h(x)\}=\frac{1}{2\pi}\int_0^{\pi}\varphi(x,v)\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}dv.$$

Let the $(E,r)(N,q_n)$ transform of $S_k(h;x)$ by $\tau_n^{E_r,N}$. Then

$$\begin{aligned}\tau_n^{E_r,N}-h(x)=&\frac{1}{2\pi(1+r)^n}\int_0^{\pi}\varphi(x,v)\sum_{k=0}^{n}C(n,k)\ r^{n-k}\left\{\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}dv\\ =&\int_0^{\pi}\varphi(x;v)\kappa_n^{E_r,N}(v)dv\\ =&\chi_n(x).\end{aligned}$$

Then

$$\chi_n(x+y)+\chi_n(x-y)-2\chi_n(x)=\int_0^{\pi}\Big\{\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)\Big\}\kappa_n^{E_r,N}(v)dv.$$

Using Minkowski's inequality, we have

$$\begin{aligned}&\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l\\ =&\left\{\frac{1}{2\pi}\int_0^{2\pi}|\chi_n(x+y)+\chi_n(x-y)-2\chi_n(x)|^l dx\right\}^{\frac{1}{l}}\\ =&\frac{1}{2\pi}\left[\int_0^{2\pi}\left|\int_0^{\pi}\Big\{\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)\Big\}\kappa_n^{E_r,N}(v)dv\right|^l dx\right]^{\frac{1}{l}}\\ \le&\int_0^{\pi}|\kappa_n^{E_r,N}(v)|\left\{\frac{1}{2\pi}\int_0^{2\pi}|\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)|^l dx\right\}^{\frac{1}{l}}dv\\ =&\int_0^{\pi}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ =&\int_0^{\frac{1}{n+1}}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ &+\int_{\frac{1}{n+1}}^{\pi}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ =&\Gamma_1+\Gamma_2.\end{aligned}\tag{7.1}$$

Using Lemma 5.1, Lemma 5.3 and monotonicity of $\frac{m(v)}{\mu(v)}$, with respect to 'v', we have

$$\begin{aligned}\Gamma_1 &= \int_0^{\frac{1}{n+1}} \|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l |\kappa_n^{E_r,N}(v)|dv \\ &= \int_0^{\frac{1}{n+1}} O\Big(\mu(y)\frac{m(v)}{\mu(v)}\Big)O(n)dv.\end{aligned}$$

By using the second mean value theorem of integral, we have

$$\Gamma_1 \le O\left(n\mu(y)\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\int_0^{\frac{1}{n+1}} dv\right)$$

$$= O\left(\frac{n}{n+1}\ \mu(y)\ \frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) = O\left(\mu(y)\ \frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right). \tag{7.2}$$

Again, by using Lemma 5.2 and Lemma 5.3, we get

$$\begin{aligned}\Gamma_2 &= \int_{\frac{1}{n+1}}^{\pi} \|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l |\kappa_n^{E_r,N}(v)|dv \\ &\le \int_{\frac{1}{n+1}}^{\pi} O\Big(\mu(y)\frac{m(v)}{\mu(v)}\Big)\frac{1}{v}dv\end{aligned}$$

$$= O\Big(\mu(y)\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv\Big). \tag{7.3}$$

By (7.1), (7.2) and (7.3), we have

$$\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l = O\left(\mu(y)\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) + O\Big(\mu(y)\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}dv\Big).$$

Therefore, we have

$$\sup_{y\neq 0}\frac{\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l}{\mu(y)} = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) + O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}dv\Big). \tag{7.4}$$

As

$$\varphi(x,v) = |h(x+v)+h(x-v)-2h(x)|,$$

by applying Minkowski's inequality, we get

$$\|\varphi(x,v)\|_l = \|h(x+v)+h(x-v)-2h(x)\|_l = O(m(v)). \tag{7.5}$$

Now, using Lemma 5.1, Lemma 5.2 and (7.5),

$$\|\chi_n(.)\|_l \le \Big(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\Big)\|\varphi(\cdot,v)\|_l |\kappa_n^{E_r,N}(v)|dv$$

$$=O\Big(n\int_0^{\frac{1}{n+1}} m(v)\ dv\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big)$$

$$=O\Big(m\Big(\frac{1}{n+1}\Big)\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big). \tag{7.6}$$

From (7.4) and (7.6), we have

$$\begin{aligned}
&\|\chi_n(\cdot)\|_l^{\mu}\\
=&\|\chi_n(\cdot)\|_l+\sup_{y\neq 0}\frac{\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l}{\mu(y)}\\
=&O\Big(m\Big(\frac{1}{n+1}\Big)\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big)+O\left(\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\right)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}\ dv\Big)\\
=&\sum_{j=1}^{4}G_j.
\end{aligned}$$

In view of monotonicity of $\mu(v)$ for $0<v\leq\pi$, we have

$$m(v)=\frac{m(v)}{\mu(v)}\mu(v)\leq\mu(\pi)\frac{m(v)}{\mu(v)}=O\Big(\frac{m(v)}{\mu(v)}\Big).$$

Therefore,

$$G_1=O(G_3).$$

Again, by using monotonicity of $\mu(v)$,

$$G_2=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}\mu(v)dv\leq\mu(\pi)\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv=O(G_4).$$

Since $\frac{m(v)}{\mu(v)}$ is positive and increasing

$$G_4=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}dv=\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\int_{\frac{1}{n+1}}^{\pi}\frac{dv}{v}\geq\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}.$$

Therefore,

$$G_3=O(G_4).$$

Thus,

$$\|\chi_n(\cdot)\|_l^{\mu}=O(G_4)=O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv\Big).$$

Hence,

$$E_n(h)=\inf_n\|\chi_n(\cdot)\|_l^{\mu}=O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv\Big).$$

This completes the proof of the Theorem 4.1. □

Proof of Theorem 4.2. In this theorem, since we have assumed $\frac{m(v)}{v\mu(v)}$ is positive and decreasing, proceeding as in Theorem 4.1. We have

$$E_n(h) = \inf_n \|\chi_n(.)\|_l^\mu = O\left(\frac{m\left(\frac{1}{n+1}\right)}{(n+1)\ \mu\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} dv\right),$$

i.e.,

$$E_n(h) = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\left(\frac{\pi}{(n+1)} - \frac{1}{(n+1)^2}\right)\right).$$

This is what we need to prove in Theorem 4.2. □

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CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS ASSOCIATED WITH LINEAR OPERATOR

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Abstract. In this paper, we introduce and study certain subclass of meromorphic univalent functions by using a linear operator by means of a Hadamard product involving some suitably normalized meromorphically q-Hypergeometric functions, in the punctured open unit disk. Some properties like, coefficients inequalities, growth and distortion theorems, closure theorems, Extreme Points and Radii of meromorphic starlikeness and meromorphic convexity are obtained.

1. Introduction

Let Σ denote the class of meromorphic functions in the punctured open unit disk $\mathbb{D}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{D} - \{0\}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1.1}$$

we denote by $\Sigma_S(\gamma)$, $\Sigma_{\Bbbk}(\gamma)$ and $\Sigma_S^*(\gamma)$, $0 \leq \gamma < 1$, the subclasses of Σ that are meromorphic univalent, meromorphically convex functions of order γ and meromorphically starlike functions of order γ, respectively.

A function $f \in \Sigma_{\Bbbk}(\gamma)$ if and only if $-\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma$, $z \in \mathbb{D}$. Similarly, a function $f \in \Sigma_S^*(\gamma)$ if and only if $-\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma$, $z \in \mathbb{D}$, where f given by (1.1).

There are many other classes of meromorphically univalent functions that has been extensively studied (see [2,3,7,9] and [11]).

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For functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$, we define the Hadamard product or convolution of f and g by

$$(f * g) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Cho et al. [6] and Ghanim and Darus [8] studied the following function

$$q_{\lambda,\mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} z^n, \quad \lambda > 0,\ \mu \geq 0. \tag{1.2}$$

For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$, $b_j \in \mathbb{C}$, and $b_j \neq 0, -1, \ldots$, and $j = 1, 2, \ldots, m$, the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} &{}_l\Psi_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} \times \left[(-1)^n q^{q\binom{n}{2}}\right]^{1+m-l} z^n, \end{aligned} \tag{1.3}$$

where $\binom{n}{2} = n(n-1)/2$, $q \neq 0$ and $l > m+1$, $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $z \in \mathbb{D}$. The q -shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a; q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.4}$$

and in terms of basic analogue of the gamma function

$$(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{1.5}$$

It is of interest to note that $\lim_{q \to -1} \left((q^a; q)_n / (1-q)^n\right) = (a)_n = a(a+1)\cdots(a+n-1)$ is the familiar Pochhammer symbol and

$${}_l\Psi_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; z_2\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \cdot \frac{z^n}{n!}.$$

Now for $z \in \mathbb{D}$, $0 < |q| < 1$ and $l = m+1$ the basic hypergeometric function defined in (1.3) takes the form

$${}_l\Psi_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} z^n, \tag{1.6}$$

which converges absolutely in the open unit disk $\mathbb{D}$ (see[1]).

Corresponding to the function ${}_l\Psi_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z\right)$ recently for meromorphic functions $f \in \Sigma$ consisting functions of the form (1.1), Al-dweby and Darus [1] introduce q -analogue of Liu-Srivastava operator as below

(1.7)

$$\begin{aligned} {}_l\Upsilon_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z\right) * f(z) &= \frac{1}{z} {}_l\Psi_m\left(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z\right) * f(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l} (a_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^{m} (b_i, q)_{n+1}} a_n z^n, \end{aligned}$$

where $\prod_{k=1}^{s}(a_k,q)_{n+1} = (a_1,q)_{n+1}(a_2,q)_{n+1}\cdots(a_s,q)_{n+1}$, $z \in \mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and

$$\begin{aligned} {}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) &= \frac{1}{z}{}_l\Psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}} z^n. \end{aligned}$$

Corresponding to the functions ${}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z)$, and $q_{\lambda,\mu}(z)$ given in (1.2) and using the Hadamard product for $f(z) \in \Sigma$, we will present a generalization to the linear operator on $\sum$ as follows

$$\mathcal{G}_\mu^\lambda(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q) : \Sigma \to \Sigma$$

and

$$\begin{aligned} &\mathcal{G}_\mu^\lambda(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q)f(z) \\ =&f(z) * {}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) * q_{\lambda,\mu}(z) \\ =&\frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} |a_n| z^n. \end{aligned} \tag{1.8}$$

For convenience, we shall henceforth denote

$$\mathcal{G}_\mu^\lambda(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q)f(z) = \mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z). \tag{1.9}$$

Notice that, the linear operator (1.8) in above was introduced and studied by Challab et al. [5]. For convenience, we let

$$\Lambda_n^{\lambda,\mu} = \frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}. \tag{1.10}$$

Definition 1.1. For $0 \le \gamma < 1$, $k \ge 0$ and $0 \le \eta < \frac{1}{2}$, we let $\Sigma(\gamma,k,\eta)$ be the subclass of $\Sigma_{S(\gamma)}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned} &-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))' + \eta z^2(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))''}{(1-\eta)\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z) + \eta z(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))'} + \gamma\right) \\ &>k\left|\frac{z(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))' + \eta z^2(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))''}{(1-\eta)\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z) + \eta z(\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z))'} + 1\right|, \end{aligned} \tag{1.11}$$

where $\mathcal{G}_\mu^\lambda(a_l,b_m,q)f(z)$ is given by (1.8).

Remark 1.1. For suitable choice of parameters involved in the Definition 1.1, the class reduces to various new subclasses in the following examples, we illustrate two important subclasses.

Example 1.1. For $\eta = 0$, we let $\Sigma(\gamma, k, 0) = \Sigma(\gamma, k)$ denote a subclass of $\Sigma(\gamma, k, \eta)$ consisting functions of the form (1.1) satisfying the condition that

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)} + \gamma\right) > k\left|\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)} + 1\right|,$$

where $\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)$ is given by (1.8).

Example 1.2. For $\eta = 0$, $k = 0$ we let $\Sigma(\gamma, 0, 0) = \Sigma(\gamma)$ denote a subclass of $\Sigma(\gamma, k, \eta)$ consisting functions of the form (1.1) satisfying the condition that

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)} + \gamma\right) > 0,$$

where $\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)$ is given by (1.8).

For more details about class in the [10, Definition 1.1].

2. Set of Lemmas

We now give the preliminary lemmas that we shall employ in the proof of the main results.

Lemma 2.1 ([4])**.** *If γ is a real number and $\omega = -u - iv$ is a complex number, then*

$$|\omega + (1-\gamma)| - |\omega - (1+\gamma)| \geq 0 \Leftrightarrow \operatorname{Re}(\omega) \geq \gamma.$$

Lemma 2.2 ([4])**.** *If $\omega = u + iv$ is a complex number and γ, k are real numbers, then*

$$-\operatorname{Re}(\omega) \geq |\omega + 1|k + \gamma \Leftrightarrow -\operatorname{Re}\left(\omega\left(1 + ke^{i\theta}\right) + ke^{i\theta}\right) \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

3. Main Results

Theorem 3.1. *Let $f \in \Sigma$ be given by* (1.1)*. Then $f \in \Sigma(\gamma, k, \eta)$ if and only if*

$$\sum_{n=1}^{\infty}(1 + (n-1)\eta)[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}|a_n| \leq (1-\gamma)(1-2\eta), \tag{3.1}$$

where $\Lambda_n^{\lambda,\mu}$ is given by (1.10)*.*

Proof. Let $f \in \Sigma(\gamma, k, \eta)$. Then by definition and using Lemma 2.2, we get

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))' + \eta z^2(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))''}{(1-\eta)\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z) + \eta z(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z))'}\left(1 + ke^{i\theta}\right) + ke^{i\theta}\right) > \gamma, \tag{3.2}$$

where $\pi \leq \theta \leq \pi$. For easiness, we let

$$\begin{aligned} A(z) := &-\left[z\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)\right)' + \eta z^2\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)\right)''\right](1 + ke^{i\theta}) \\ &- ke^{i\theta}\left[(1-\eta)\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z) + \eta z\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q)\, f(z)\right)'\right] \end{aligned}$$

and

$$B(z) := (1-\eta)\mathcal{G}_{\mu}^{\lambda}(a_l, b_m, q)\, f(z) + \eta z\left(\mathcal{G}_{\mu}^{\lambda}(a_l, b_m, q)\, f(z)\right)'.$$

Hence, the equation (3.2) is equivalent to $-\mathrm{Re}\left(\frac{A(z)}{B(z)}\right) \geq \gamma$ and from Lemma 2.1, we only want to prove that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0.$$

Therefore,

$$\begin{aligned}&|A(z) + (1-\gamma)B(z)|\\ \geq&(1-2\eta)(2-\gamma)\frac{1}{|z|} - \sum_{n=1}^{\infty}[(n-1+\gamma) + k(n+1)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n\end{aligned}$$

and

$$\begin{aligned}&|A(z) - (1+\gamma)B(z)|\\ \leq&\gamma(1-2\eta)\frac{1}{|z|} + \sum_{n=1}^{\infty}[(n+1+\gamma) + k(n+1)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n.\end{aligned}$$

Thus,

$$\begin{aligned}&|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|\\ \geq&2(1-2\eta)(1-\gamma)\frac{1}{|z|} - 2\sum_{n=1}^{\infty}[n(1+k) + (\gamma+k)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n \geq 0,\end{aligned}$$

by the provided condition (3.1). On the other hand, let $f \in \Sigma(\gamma, k, \eta)$. Then by Lemma 2.2, we get (3.2).

Choosing the values of z on the positive real axis the inequality (3.2) reduce to

$$\begin{aligned}&\mathrm{Re}\left(\frac{(1-\gamma)(1-2\eta)\frac{1}{z^2} + \sum_{n=1}^{\infty}(1+(n-1)\eta)\left[n\left(1+ke^{i\theta}\right) + \left(\gamma+ke^{i\theta}\right)\right]\Lambda_n^{\lambda,\mu}|a_n|z^{n-1}}{(1-2\eta)\frac{1}{z^2} - \sum_{n=1}^{\infty}(1+(n-1)\eta)\Lambda_n^{\lambda,\mu}|a_n|z^{n-1}}\right)\\ \geq&0.\end{aligned}$$

Since $\mathrm{Re}\left(-e^{i\theta}\right) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$\mathrm{Re}\left(\frac{(1-\gamma)(1-2\eta)\frac{1}{r^2} + \sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)]\Lambda_n^{\lambda,\mu}|a_n|r^{n-1}}{(1-2\eta)\frac{1}{r^2} - \sum_{n=1}^{\infty}(1+(n-1)\eta)\Lambda_n^{\lambda,\mu}|a_n|r^{n-1}}\right) \geq 0.$$

Letting $r \to 1^-$ and by mean value theorem we get desired inequality (3.1). □

Corollary 3.1. *If $f \in \Sigma(\gamma, k, \eta)$, then*

$$|a_n| \leq \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}}.$$

Corollary 3.2. *Let $f(z) \in \Sigma$ be given by* (1.1). *Then $f \in \Sigma(\gamma, k)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}(\alpha_1)\,|a_n| \leq (1-\gamma),$$

where $\eta = 0$, in Theorem 3.1.

4. Growth and Distortion Theorem

Theorem 4.1. *Let $f \in \Sigma(\gamma, k, \eta)$ given by (1.1). Then for $0 < |z| = r < 1$ we get*

$$\frac{1}{r} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r$$

and

$$\frac{1}{r^2} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} z. \tag{4.1}$$

Proof. Since $f \in \Sigma(\gamma, k, \eta)$ and $0 < |z| = r < 1$, then

$$|f(z)| \leq \frac{1}{|z|} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} |z| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r$$

and

$$|f(z)| \geq \frac{1}{|z|} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} |z| \geq \frac{1}{r} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r.$$

On the other hand

$$|f'(z)| \leq \left|\frac{-1}{z^2}\right| + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}$$

and

$$|f'(z)| \geq \left|\frac{-1}{z^2}\right| - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \geq \frac{1}{r^2} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}.$$

This completes the proof of Theorem 4.1. □

5. Closure Theorems

Let $f_j(z)$, $j = 1, 2, \ldots, I$, be the function given by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,j}| z^n. \tag{5.1}$$

Theorem 5.1. *Let the function $f_j(z)$ defined by (5.1) be in the class $\Sigma(\gamma, k, \eta)$ for every $j = 1, 2, \ldots, I$. Then the function $f(z)$ defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{m} q_n z^n,$$

belongs to the class $\Sigma(\gamma, k, \eta)$, where $q_n = \frac{1}{I}\sum_{n=1}^{I} |a_{n,j}|$, $n = 1, 2, \ldots$

Proof. Since $f_j(z) \in \Sigma(\gamma, k, \eta)$, it follows from Theorem 3.1 that

$$\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu} \ |a_{n,j}| \le (1-\gamma)(1-2\eta),$$

for every $j = 1, 2, \dots, I$. Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu} q_n \\
=&\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}\left\{\frac{1}{I}\sum_{j=1}^{I}|a_{n,j}|\right\} \\
=&\frac{1}{I}\sum_{j=1}^{I}\left(\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}|a_{n,j}|\right) \\
\le&\frac{1}{I}\sum_{j=1}^{I}(1-\gamma)(1-2\eta) = (1-\gamma)(1-2\eta),
\end{aligned}$$

which implies that f is in $\Sigma(\gamma, k, \eta)$. □

6. Extreme Points

Theorem 6.1. *Let*

$$f_0(z) = \frac{1}{z} \tag{6.1}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}z^n, \quad n \ge 1. \tag{6.2}$$

Then $f \in \Sigma(\gamma, k, \eta)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty}\omega_n f_n(z), \quad \sum_{n=0}^{\infty}\omega_n = 1,\ \omega_n \ge 0. \tag{6.3}$$

Proof. From (6.1), (6.2) and (6.3), we have

$$f(z) = \frac{1}{z} + \sum_{n=2}^{\infty}\frac{(1-\gamma)(1-2\eta)\omega_n}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}z^n. \tag{6.4}$$

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(1-\gamma)(1-2\eta)\omega_n}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}\cdot\frac{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}{(1-\gamma)(1-2\eta)} \\
&=\sum_{n=2}^{\infty}\omega_n = 1-\omega_1 \le 1,
\end{aligned} \tag{6.5}$$

it follows from Theorem 3.1 that the function $f \in \Sigma(\gamma, k, \eta)$.

Conversely, suppose that f is in $\Sigma(\gamma, k, \eta)$, since

$$a_n \leq \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}, \quad n \geq 1. \tag{6.6}$$

Setting

$$\omega_n = \frac{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Gamma_n}{(1-\gamma)(1-2\eta)} a_n, \quad \omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n, \tag{6.7}$$

it follows that

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z). \tag{6.8}$$

This completes the proof of the theorem. □

7. Radii of Meromorphic Starlikeness and Meromorphic Convexity

Theorem 7.1. *Let $f \in \Sigma(\gamma, k, \eta)$. Then f is meromorphically starlike of order δ, $0 \leq \delta < 1$, in the unit disc $|z| < r_3$, where*

$$r_3 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{(\eta(n-1)+1)[n(1-k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-2\eta)(1-\gamma)} \right]^{\frac{1}{n+1}}, \quad n \geq 1.$$

The result is sharp for the extremal function $f(z)$ given by (4.1).

Proof. We must show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \delta, \quad |z| < r_3.$$

Since

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq \frac{(n+1)\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} |z|^{n+1}}{1 - \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} |z|^{n+1}}. \tag{7.1}$$

Hence, (7.1) holds true if

$$\begin{aligned} &(n+1)\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} |z|^{n+1} \\ \leq &(1-\delta)\left[1 - \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} |z|^{n+1}\right] \end{aligned}$$

or

$$(n+2-\delta)\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} |z|^{n+1} \leq (1-\delta).$$

Thus, for

$$|z|^{n+1} \leq \frac{(1-\delta)}{(n+2-\delta)} \cdot \frac{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-\gamma)(1-2\eta)}.$$

Hence, $f(z)$ is starlike of order δ. □

Corollary 7.1. *Let* $f \in \Sigma(\gamma, k, \eta)$*. Then* f *is meromorphically convex of order* δ*,* $0 \leq \delta < 1$*, in the unit disc* $|z| < r_4$*, where*

$$r_4 = \inf_n \left[\left(\frac{1-\delta}{n(n+2-\delta)} \right) \frac{(\eta(n-1)+1)[n(1-k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-2\eta)(1-\gamma)} \right]^{\frac{1}{n+1}}, \quad n \geq 1.$$

The result is sharp for the extremal function $f(z)$ *given by* (4.1).

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$\overline{q}$-LAPLACE TRANSFORM ON QUANTUM INTEGRAL

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Abstract. In this paper, we present $\overline{q}$-Laplace transform by $\overline{q}$-integral definition on quantum analogue. We present some properties and obtain formulaes of $\overline{q}$-Laplace transform with its aplications.

1. Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. The quantum calculus or q-calculus began with FH Jackson in the early twentieth century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it arose interest due to high demand of mathematics that models quantum computing. q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences quantum theory, mechanics and the theory of relativity.

There are many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains.

In 2017, Alp and Sarikaya [1] gave a new defination of q-integral which is showed $\overline{q}$-integral.

The aim of this paper present Laplace transform on $\overline{q}$-integral. In second section we give notations and preliminaries for q-analogue. In third section we give definition of Laplace transform on $\overline{q}$-integral and obtain some auxiliary results. In fourth section we calculate $\overline{q}$-Laplace transforms of functions and some properties of $\overline{q}$-Laplace transform.

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Now remember following Laplace transform on classical analysis.

For $t > 0$ Laplace transform of $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt. \tag{1.1}$$

We say that transform converges if the limit exists, and diverges if not.

2. Notations and Preliminaries

In this section, first we give definition and notations of q-analogue with q-derivates then definition and properties of $\overline{q}$-integral. For $0 < q < 1$ here and further we use the following notations [3, 4]:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \tag{2.1}$$

(2.2)

$$(x-a)_q^n = \prod_{i=0}^{n-1}\left(x - q^i a\right) = (x-a)(x-qa)\left(x-q^2a\right)\cdots\left(x-q^{n-1}a\right), \quad n \in \mathbb{Z}^+,$$

$$(a:q)_0 = 1,$$

$$(1-a)_q^n = (a:q)_n = \prod_{i=0}^{n}\left(1-q^i a\right),$$

(2.3)

$$(1-a)_q^\infty = (a:q)_\infty = \prod_{i=0}^{\infty}\left(1-q^i a\right),$$

(2.4)

$$(1-a)_q^n = \frac{(1-a)_q^\infty}{(1-q^n a)_q^\infty} = \frac{(a:q)_\infty}{(q^n a:q)_\infty}, \quad n \in \mathbb{C}.$$

Notice that, under our assumptions on q, the infinite product (2.3) is convergent. Moreover, the definitions (2.2) and (2.4) are consistent.

Definition 2.1. In [2], for f has $D_q^n f(a)$, Jackson introduced the following q-counterpart of Taylor series:

$$f(x) = \sum_{n=0}^\infty \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)(x-a)_q^n = \sum_{n=0}^\infty \frac{D_q^n f(a)(x-a)_q^n}{[n]_q!}, \tag{2.5}$$

D_q is the q-difference operator.

Here E_q^x and e_q^x are two q-analogues of the exponential functions and their q-Taylor series ([4]):

$$E_q^x = \sum_{n=0}^\infty q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^\infty = ((q-1)x:q)_\infty, \tag{2.6}$$

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}} = \frac{1}{((1-q)x:q)_{\infty}}. \tag{2.7}$$

Lemma 2.1 ([4]). *The q-exponential functions satisfy the following properties:*

$$e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1, \quad E_q^x = e_{1/q}^x.$$

For $E_q^{-x} = \frac{1}{e_q^x}$ we have

$$\lim_{x\to\infty} E_q^{-x} = \lim_{x\to\infty} \frac{1}{e_q^x} = 0.$$

Let $J := [a,b] \subset \mathbb{R}$, $J^\circ := (a,b)$ be interval and $0 < q < 1$ be a constant. Definiton of q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ as follows.

Definition 2.2 ([5]). Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$\begin{aligned} {}_aD_q f(x) &= \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \\ {}_aD_q f(a) &= \lim_{x\to a} {}_aD_q f(x), \end{aligned} \tag{2.8}$$

is called the q-derivative on J of function f at x.

We say that f is q-differentiable on J provided ${}_aD_q f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.8), then ${}_aD_q f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$${}_aD_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [4].

Lemma 2.2 ([5]). *Let $\alpha \in \mathbb{R}$, then we have*

$${}_aD_q (x-a)^{\alpha} = [\alpha]_q (x-a)^{\alpha-1}. \tag{2.9}$$

The following definitions and theorems with respect to $\overline{q}$-integral were referred in [1, page 148].

Definition 2.3. Let $f : J \to \mathbb{R}$ is continuous function. For $0 < q < 1$

$$\int_a^b f(s)\, {}_ad_{\overline{q}}s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) - f(b)\right], \tag{2.10}$$

which second sense quantum integral definition that call $\overline{q}$-integral for $x \in J$.

Moreover, if $c \in (a,x)$ then the definite $\overline{q}$-integral on J is defined by

$$\int_c^x f(s)\, {}_ad_{\overline{q}}s \tag{2.11}$$

$$=\int_a^x f(s)\,{}_ad_{\overline{q}}s-\int_a^c f(s)\,{}_ad_{\overline{q}}s$$
$$=\frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)a)-f(x)\right]$$
$$-\frac{(1-q)(c-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^nf(q^nc+(1-q^n)a)-f(c)\right].$$

Theorem 2.1 ([1]). *Let $f:J\to\mathbb{R}$ be a continuous function. Then we have the following properties of $\overline{q}$-integral*

i)
$${}_aD_q\int_a^x f(s)\,{}_ad_{\overline{q}}s=\frac{f(x)+f(qx+(1-q)a)}{2};$$

ii)
$$\int_0^1 f(sb+(1-s)a)\,{}_0d_{\overline{q}}s=\frac{1}{b-a}\int_a^b f(t)\,{}_ad_{\overline{q}}t;$$

iii)
$$\int_c^x {}_aD_qf(s)\,{}_ad_{\overline{q}}s$$
$$=\frac{qf(x)+f(qx+(1-q)a)-qf(c)-f(qc+(1-q)a)}{2q},\quad \text{for } c\in(a,x);$$

iv)
$$\int_a^x [f(s)+g(s)]\,{}_ad_{\overline{q}}s=\int_a^x f(s)\,{}_ad_{\overline{q}}s+\int_a^x g(s)\,{}_ad_{\overline{q}}s;$$

v)
$$\int_a^x (\alpha f)(s)\,{}_ad_{\overline{q}}s=\alpha\int_a^x f(s)\,{}_ad_{\overline{q}}s,\quad \alpha\in\mathbb{R};$$

vi) partial integration property:

$$\int_c^x f(s)\,{}_aD_qg(s)\,{}_ad_{\overline{q}}s \tag{2.12}$$
$$=\left.\frac{qf(s)g(s)+f(qs+(1-q)a)g(qs+(1-q)a)}{2q}\right|_c^x$$
$$-\int_c^x g(qs+(1-q)a)\,{}_aD_qf(s)\,{}_ad_{\overline{q}}s;$$

vii)

$$\int_a^x (s-a)^{\alpha}\, {}_a d_{\overline{q}} s = \frac{1}{[\alpha+1]_q}\left(\frac{1+q^{\alpha}}{2}\right)(x-a)^{\alpha+1}.$$

3. Auxiliary Results

For using in further theorems lets give an example on q-derivative.

Example 3.1. For $s>0,\ t\in\mathbb{R}$, we have

$$D_q E_q^{-st} = -\, sE_q^{-qst}, \qquad (3.1)$$
$$D_q E_q^{-qst} = -\, qsE_q^{-q^2st}.$$

Proof. By using q-derivative and (2.6), we obtain that:

$$\begin{aligned} D_q E_q^{-st} =& D_q\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(st)^n\right)\\ =& \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}} s^n}{[n]_q!} D_q t^n\\ =& \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}} s^n}{[n-1]_q!} t^{n-1}\\ =& \sum_{n=0}^{\infty}\frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}} s^{n+1}}{[n]_q!} t^{n}\\ =& -\, s\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(qst)^n\\ =& -\, sE_q^{-qst} \end{aligned}$$

and in the same way we have

$$D_q E_q^{-qst} = -qsE_q^{-q^2st}$$

and the proof is completed. □

Now we present $\overline{q}$-Laplace transform on $\overline{q}$-integral below.

Definition 3.1. Let $s>0$ and $f:[0,\infty)\to\mathbb{R}$ be a function. Then the $\overline{q}$-Laplace transform is defined by

$$L_{\overline{q}}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, E_q^{-qst} d_{\overline{q}} t. \qquad (3.2)$$

Assume f, g are two functions and $\alpha,\beta\in\mathbb{C}$ by using (3.2) linearity property of $\overline{q}$-Laplace transform is written as follow:

$$L_{\overline{q}}\{\alpha f(t)+\beta g(t)\} = \alpha L_{\overline{q}}\{f(t)\} + \beta L_{\overline{q}}\{g(t)\}.$$

4. $\overline{q}$-Laplace Transform of Functions

In this section, we proved $\overline{q}$-Laplace transform of functions and n degrees of quantum derivative function. Let's first calculate the $\overline{q}$-Laplace transformation of the constant function as below.

Theorem 4.1. *The $\overline{q}$-Laplace transform of function $f(t) = 1$ is*

$$L_{\overline{q}}\{1\} = F(s) = \frac{1+q}{2q} \cdot \frac{1}{s}.$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$F(s) = L_{\overline{q}}\{1\} = \lim_{\alpha\to\infty} \int_0^\alpha E_q^{-qst} d_{\overline{q}}t = \int_0^\infty E_q^{-qst} d_{\overline{q}}t.$$

Then calculate above integral by using the $\overline{q}$-integral, we have

$$\begin{aligned}
&\int_0^\alpha E_q^{-qst} d_{\overline{q}}t \\
&= \int_0^\alpha \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} (qst)^n d_{\overline{q}}t \\
&= \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}} (qs)^n}{[n]_q!} \int_0^\alpha t^n d_{\overline{q}}t \\
&= \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}} (qs)^n}{[n]_q!} \cdot \frac{1+q^n}{2[n+1]_q} \alpha^{n+1} \\
&= \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}} (qs)^n}{2[n+1]_q!} \alpha^{n+1} + \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}} (qs)^n q^n}{2[n+1]_q!} \alpha^{n+1} \\
&= -\frac{1}{2s} \sum_{n=0}^\infty \frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}}{[n+1]_q!} (s\alpha)^{n+1} - \frac{1}{2qs} \sum_{n=0}^\infty \frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}}{[n+1]_q!} (qs\alpha)^{n+1} \\
&= -\frac{1}{2s} \sum_{n=1}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} (s\alpha)^n - \frac{1}{2qs} \sum_{n=1}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} (qs\alpha)^n \\
&= -\frac{1}{2s} \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} (s\alpha)^n + \frac{1}{2s} - \frac{1}{2qs} \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} (qs\alpha)^n + \frac{1}{2qs} \\
&= -\frac{1}{2s} E_q^{-s\alpha} + \frac{1}{2s} - \frac{1}{2qs} E_q^{-qs\alpha} + \frac{1}{2qs}
\end{aligned}$$

and by taking the limit the proof is obtained as follows

$$L_{\overline{q}}\{1\} = F(s) = \lim_{\alpha\to\infty} \left(-\frac{1}{2s} E_q^{-s\alpha} + \frac{1}{2s} - \frac{1}{2qs} E_q^{-qs\alpha} + \frac{1}{2qs} \right) = \frac{1+q}{2q} \cdot \frac{1}{s},$$

where

$$\lim_{\alpha\to\infty} E_q^{-qs\alpha} = \lim_{\alpha\to\infty} E_q^{-s\alpha} = 0. \qquad \square$$

Theorem 4.2. *For $n \in \mathbb{R}$ with $n > -1$, the $\overline{q}$-Laplace transform of function $f(t) = t^n$ is*

$$L_{\overline{q}}\{t^n\} = \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\}. \tag{4.1}$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$L_{\overline{q}}\{t^n\} = F(s) = \lim_{\alpha\to\infty} \int_0^\alpha t^n E_q^{-qst} d_{\overline{q}}t = \int_0^\infty t^n E_q^{-qst} d_{\overline{q}}t.$$

Then, calculate above integral by using (2.12) and (3.1) with the $\overline{q}$-integral, we have

$$\begin{aligned}
\int_0^\alpha t^n E_q^{-qst} d_{\overline{q}}t &= -\frac{1}{s}\int_0^\alpha t^n D_q E_q^{-st} d_{\overline{q}}t \\
&= -\frac{1}{s}\left[\left.\frac{qt^n E_q^{-st} + (qt)^n E_q^{-qst}}{2q}\right|_0^\alpha - [n]_q \int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t\right] \\
&= \frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs} \\
&= \frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs}
\end{aligned}$$

and by taking the limit

$$\begin{aligned}
L_{\overline{q}}\{t^n\} &= F(s) = \lim_{\alpha\to\infty}\int_0^\alpha t^n E_q(-qst) d_{\overline{q}}t \\
&= \lim_{\alpha\to\infty}\left[\frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs}\right] \\
&= \frac{[n]_q}{s}\int_0^\infty t^{n-1} E_q^{-qst} d_{\overline{q}}t = \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\}
\end{aligned}$$

and the proof is completed. $\square$

Theorem 4.3. *Let $n \in \mathbb{N}$, then the $\overline{q}$-Laplace transform of function $f(t) = t^n$ is*

$$L_{\overline{q}}\{t^n\} = \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}}.$$

Proof. By using (4.1), it follows that

$$L_{\overline{q}}\{t^n\} = \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\}$$

$$
\begin{aligned}
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} L_{\overline{q}} \left\{ t^{n-2} \right\} \\
&\vdots \\
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} \cdots L_{\overline{q}} \{1\} \\
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} \cdots \frac{1+q}{2q} \cdot \frac{1}{s} \\
&= \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}}.
\end{aligned}
$$

□

Theorem 4.4. *The $\overline{q}$-Laplace transform of function $f(t) = e_q^{at}$ is*

$$
L_{\overline{q}} \left\{ e_q^{at} \right\} = \frac{1+q}{2q} \cdot \frac{1}{s-a}, \quad s > a.
$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$
\begin{aligned}
L_{\overline{q}} \left\{ e_q^{at} \right\} &= \lim_{\alpha \to \infty} \int_0^\alpha e_q^{at} E_q^{-qst} d_{\overline{q}} t = \lim_{\alpha \to \infty} \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \int_0^\alpha t^n E_q^{-qst} d_{\overline{q}} t \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \lim_{\alpha \to \infty} \int_0^\alpha t^n E_q^{-qst} d_{\overline{q}} t = \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} L_{\overline{q}} \{t^n\} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \cdot \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} \left(\frac{a}{s} \right)^n = \frac{1+q}{2q} \cdot \frac{1}{s-a},
\end{aligned}
$$

and the proof is completed. □

Theorem 4.5. *The $\overline{q}$-Laplace transform of function $f(t) = E_q^{at}$ is*

$$
L_{\overline{q}} \left\{ E_q^{at} \right\} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{a}{s} \right)^n, \quad s > 0.
$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$
\begin{aligned}
L_{\overline{q}} \left\{ E_q^{at} \right\} &= \int_0^\infty E_q^{at} E_q^{-qst} d_{\overline{q}} t \\
&= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{a^n}{[n]_q!} \int_0^\infty t^n E_q^{-qst} d_{\overline{q}} t \\
&= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{a^n}{[n]_q!} \cdot \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}} \\
&= \frac{1+q}{2qs} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{a}{s} \right)^n.
\end{aligned}
$$

□

Theorem 4.6. *The $\overline{q}$-Laplace transform of q-cosine, q-sine, q-Cosine, q-Sine functions are that*

$$\begin{aligned}
L_{\overline{q}}\{\cos_q at\} =&\frac{1+q}{2q}\cdot\frac{s}{s^2+a^2},\\
L_{\overline{q}}\{\sin_q at\} =&\frac{1+q}{2q}\cdot\frac{a}{s^2+a^2},\\
L_{\overline{q}}\{\mathrm{Cos}_q at\} =&\frac{1+q}{2qs}\sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)}\left(\frac{a}{s}\right)^{2n},\\
L_{\overline{q}}\{\mathrm{Sin}_q at\} =&\frac{1+q}{2qs}\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}\left(\frac{a}{s}\right)^{2n+1}.
\end{aligned}$$

Proof. Consider the following definition of q-cosine, q-sine, q-Cosine and q-Sine functions:

$$\cos_q at = \frac{e_q^{iat}+e_q^{-iat}}{2} \quad\text{and}\quad \sin_q at = \frac{e_q^{iat}-e_q^{-iat}}{2i},$$
$$\mathrm{Cos}_q at = \frac{E_q^{iat}+E_q^{-iat}}{2} \quad\text{and}\quad \mathrm{Sin}_q at = \frac{E_q^{iat}-E_q^{-iat}}{2i}.$$

Then, by using linearity of $\overline{q}$-Laplace transform,

$$\begin{aligned}
L_{\overline{q}}\{\cos_q at\} =&L_{\overline{q}}\left\{\frac{e_q^{iat}+e_q^{-iat}}{2}\right\} = \frac{1}{2}\left(L_{\overline{q}}\left\{e_q^{iat}\right\}+L_{\overline{q}}\left\{e_q^{-iat}\right\}\right)\\
=&\frac{1}{2}\left(\frac{1+q}{2q}\cdot\frac{1}{s-ia}+\frac{1+q}{2q}\cdot\frac{1}{s+ia}\right)\\
=&\frac{1+q}{2q}\cdot\frac{s}{s^2+a^2}
\end{aligned}$$

and in the same way we have

$$L_{\overline{q}}\{\sin_q at\} = \frac{1+q}{2q}\cdot\frac{a}{s^2+a^2}.$$

Now, we obtain $\overline{q}$-Laplace transform of q-Cosine and q-Sine functions

$$\begin{aligned}
L_{\overline{q}}\{\mathrm{Cos}_q at\} =&L_{\overline{q}}\left\{\frac{E_q^{iat}+E_q^{-iat}}{2}\right\}\\
=&\frac{1}{2}\left(L_{\overline{q}}\left\{E_q^{iat}\right\}+L_{\overline{q}}\left\{E_q^{-iat}\right\}\right)\\
=&\frac{1+q}{4qs}\left(\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\left(\frac{ia}{s}\right)^n+\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\left(\frac{-ia}{s}\right)^n\right)\\
=&\frac{1+q}{4qs}\sum_{n=0}^{\infty}\left[1+(-1)^n\right]q^{\frac{n(n-1)}{2}}\left(\frac{ia}{s}\right)^n\\
=&\frac{1+q}{2qs}\sum_{n=0}^{\infty}(-1)^n q^{n(2n-1)}\left(\frac{a}{s}\right)^{2n}
\end{aligned}$$

and

$$
\begin{aligned}
L_{\overline{q}}\left\{\operatorname{Sin}_q at\right\} =& L_{\overline{q}}\left\{\frac{E_q^{iat}-E_q^{-iat}}{2i}\right\} \\
=& \frac{1}{2i}\left(L_{\overline{q}}\left\{E_q^{iat}\right\}-L_{\overline{q}}\left\{E_q^{-iat}\right\}\right) \\
=& \frac{1+q}{4qsi}\left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\left(\frac{ia}{s}\right)^n-\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\left(\frac{-ia}{s}\right)^n\right) \\
=& \frac{1+q}{4qsi}\sum_{n=0}^{\infty}\left[1-(-1)^n\right] q^{\frac{n(n-1)}{2}}\left(\frac{ia}{s}\right)^n \\
=& \frac{1+q}{2qsi}\sum_{n=0}^{\infty} q^{n(2n+1)}\left(\frac{ia}{s}\right)^{2n+1} \\
=& \frac{1+q}{2qs}\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}\left(\frac{a}{s}\right)^{2n+1}.
\end{aligned}
$$

So, the proof is completed. □

Theorem 4.7. *The $\overline{q}$-Laplace transform of hyperbolic q-cosine, hyperbolic q-sine functions are*

$$
\begin{aligned}
L_{\overline{q}}\left\{\cosh_q at\right\} &= \frac{1+q}{2q}\cdot\frac{s}{s^2-a^2}, \\
L_{\overline{q}}\left\{\sinh_q at\right\} &= \frac{1+q}{2q}\cdot\frac{a}{s^2-a^2}.
\end{aligned}
$$

Proof. Hyperbolic q-cosine, hyperbolic q-sine are defined by

$$
\cosh_q at = \frac{e_q^{at}+e_q^{-at}}{2} \quad \text{and} \quad \sinh_q at = \frac{e_q^{at}-e_q^{-at}}{2}.
$$

Then, by using linearity of $\overline{q}$-Laplace transform,

$$
\begin{aligned}
L_{\overline{q}}\left\{\cosh_q at\right\} &= L_{\overline{q}}\left\{\frac{e_q^{at}+e_q^{-at}}{2}\right\} = \frac{1}{2}\left(L_{\overline{q}}\left\{e_q^{at}\right\}+L_{\overline{q}}\left\{e_q^{-at}\right\}\right) \\
&= \frac{1}{2}\left(\frac{1+q}{2q}\cdot\frac{1}{s-a}+\frac{1+q}{2q}\cdot\frac{1}{s+a}\right) \\
&= \frac{1+q}{2q}\cdot\frac{s}{s^2-a^2}
\end{aligned}
$$

and in the same way we have

$$
L_{\overline{q}}\left\{\sinh_q at\right\} = \frac{1+q}{2q}\cdot\frac{a}{s^2-a^2}.
$$

□

If $f(t)$ is piecewise continuous on the interval $(0,\infty)$ and of exponential order c, then $L_{\overline{q}}\left\{f(t)\right\}$ exists for $s>c$. Therefore, we obtain the following theorem.

Theorem 4.8. *If $f, D_q f, D_q^2 f, \ldots, D_q^{n-1} f$ are continuous and $D_q^n f$ is piecewise continuous on $(0,\infty)$ and are of exponential order then we have*

$$L_{\overline{q}}\left\{D_q^n f(t)\right\} = s^n L_{\overline{q}}\left\{f(t)\right\} - \frac{(1+q)}{2q}\sum_{i=0}^{n-1} s^{n-1-i} D_q^i f(0).$$

Proof. A function f is said to be of exponential order c if there exist c, $K > 0$ and $T > 0$ such that

$$|f(t)| \leq Ke^{ct}, \quad \text{for all } t < T.$$

Therefore, we have

$$\lim_{t\to\infty} E_q^{-qst} f(t) = 0. \tag{4.2}$$

Then, by using (4.2) we write

$$\begin{aligned}
L_{\overline{q}}\left\{D_q f(t)\right\} &= \int_0^\infty E_q^{-qst} D_q f(t)\, d_{\overline{q}}t \\
&= \left.\frac{qE_q^{-qst} f(t) + E_q^{-q^2 st} f(qt)}{2q}\right|_0^\infty - \int_0^\infty f(qt)\, D_q E_q^{-qst} d_{\overline{q}}t \\
&= -\frac{(1+q)}{2q} f(0) + qs\int_0^\infty f(qt)\, E_q^{-q^2 st} d_{\overline{q}}t \\
&= -\frac{(1+q)}{2q} f(0) + s\int_0^\infty f(u)\, E_q^{-qsu} d_{\overline{q}}u \\
&= sL_{\overline{q}}\left\{f(t)\right\} - \frac{(1+q)}{2q} f(0).
\end{aligned}$$

If we replace $f(t)$ by $D_q f(t)$ we have

$$\begin{aligned}
L_{\overline{q}}\left\{D_q^2 f(t)\right\} &= \int_0^\infty E_q^{-qst} D_q^2 f(t)\, d_{\overline{q}}t \\
&= \left.\frac{qE_q^{-qst} D_q f(t) + E_q^{-q^2 st} D_q f(qt)}{2q}\right|_0^\infty - \int_0^\infty D_q f(qt)\, D_q E_q^{-qst} d_{\overline{q}}t \\
&= -\frac{(1+q)}{2q} D_q f(0) + qs\int_0^\infty f(qt)\, E_q^{-q^2 st} d_{\overline{q}}t \\
&= -\frac{(1+q)}{2q} D_q f(0) + s\int_0^\infty D_q f(t)\, E_q^{-qst} d_{\overline{q}}t \\
&= sL_{\overline{q}}\left\{D_q f(t)\right\} - \frac{(1+q)}{2q} D_q f(0)
\end{aligned}$$

$$=s\left[sL_{\overline{q}}\left\{f(t)\right\}-\frac{(1+q)}{2q}f\left(0\right)\right]-\frac{(1+q)}{2q}D_{q}f\left(0\right)$$
$$=s^{2}L_{\overline{q}}\left\{f(t)\right\}-\frac{(1+q)}{2q}\left(D_{q}f\left(0\right)+sf\left(0\right)\right).$$

If we continue with this process, we get

$$L_{\overline{q}}\left\{D_{q}^{n}f(t)\right\}=s^{n}L_{\overline{q}}\left\{f(t)\right\}-\frac{(1+q)}{2q}\sum_{i=0}^{n-1}s^{n-1-i}D_{q}^{i}f\left(0\right),$$

and the proof is completed. □

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