Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 143–151.

CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS ASSOCIATED WITH LINEAR OPERATOR

OSAMAH N. KASSAR[1] AND ABDUL RAHMAN S. JUMA[1]

Abstract. In this paper, we introduce and study certain subclass of meromorphic univalent functions by using a linear operator by means of a Hadamard product involving some suitably normalized meromorphically q-Hypergeometric functions, in the punctured open unit disk. Some properties like, coefficients inequalities, growth and distortion theorems, closure theorems, Extreme Points and Radii of meromorphic starlikeness and meromorphic convexity are obtained.

1. Introduction

Let Σ denote the class of meromorphic functions in the punctured open unit disk $\mathbb{D}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{D} - \{0\}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1.1}$$

we denote by $\Sigma_S(\gamma)$, $\Sigma_{\mathbb{k}}(\gamma)$ and $\Sigma^*_S(\gamma)$, $0 \leq \gamma < 1$, the subclasses of Σ that are meromorphic univalent, meromorphically convex functions of order γ and meromorphically starlike functions of order γ, respectively.

A function $f \in \Sigma_{\mathbb{k}}(\gamma)$ if and only if $-\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma$, $z \in \mathbb{D}$. Similarly, a function $f \in \Sigma^*_S(\gamma)$ if and only if $-\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma$, $z \in \mathbb{D}$, where f given by (1.1).

There are many other classes of meromorphically univalent functions that has been extensively studied (see [2, 3, 7, 9] and [11]).

Key words and phrases. Meromorphic functions, Hadamard product (or convolution), convex functions, starlike functions, hypergeometric function.

2010 *Mathematics Subject Classification.* Primary: 30C80. Secondary: 30C45.

DOI 10.46793/KgJMat2301.143K

Received: February 24, 2020.

Accepted: July 23, 2020.

For functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$, we define the Hadamard product or convolution of f and g by

$$(f * g) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Cho et al. [6] and Ghanim and Darus [8] studied the following function

$$q_{\lambda,\mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} z^n, \quad \lambda > 0,\ \mu \geq 0. \tag{1.2}$$

For complex parameters $a_1, \dots, a_l$ and $b_1, \dots, b_m$, $b_j \in \mathbb{C}$, and $b_j \neq 0, -1, \dots$, and $j = 1, 2, \dots, m$, the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} &{}_l\Psi_m\left(a_1, \dots, a_l; b_1, \dots, b_m; q, z\right) \\ &= \sum_{n=0}^{\infty} \frac{\left(a_1, q\right)_n \cdots \left(a_l, q\right)_n}{(q,q)_n \left(b_1, q\right)_n \cdots \left(b_m, q\right)_n} \times \left[(-1)^n q^{q\binom{n}{2}} \right]^{1+m-l} z^n, \end{aligned} \tag{1.3}$$

where $\binom{n}{2} = n(n-1)/2$, $q \neq 0$ and $l > m+1$, $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $z \in \mathbb{D}$. The q -shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.4}$$

and in terms of basic analogue of the gamma function

$$\left(q^a;q\right)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{1.5}$$

It is of interest to note that $\lim_{q\to -1}\left(\left(q^a;q\right)_n/(1-q)^n\right) = (a)_n = a(a+1)\cdots(a+n-1)$ is the familiar Pochhammer symbol and

$${}_l\Psi_m\left(a_1, \dots, a_l; b_1, \dots, b_m; z_2\right) := \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \cdots \left(a_l\right)_n}{\left(b_1\right)_n \cdots \left(b_m\right)_n} \cdot \frac{z^n}{n!}.$$

Now for $z \in \mathbb{D}$, $0 < |q| < 1$ and $l = m+1$ the basic hypergeometric function defined in (1.3) takes the form

$${}_l\Psi_m\left(a_1, \dots, a_l; b_1, \dots, b_m; q, z\right) := \sum_{n=0}^{\infty} \frac{\left(a_1, q\right)_n \cdots \left(a_l, q\right)_n}{(q,q)_n \left(b_1, q\right)_n \cdots \left(b_m, q\right)_n} z^n, \tag{1.6}$$

which converges absolutely in the open unit disk $\mathbb{D}$ (see[1]).

Corresponding to the function ${}_l\Psi_m\left(a_1, \dots, a_l; b_1, \dots, b_m; q, z\right)$ recently for meromorphic functions $f \in \Sigma$ consisting functions of the form (1.1), Al-dweby and Darus [1] introduce q -analogue of Liu-Srivastava operator as below

(1.7)

$$\begin{aligned} {}_l\Upsilon_m\left(a_1, \dots, a_l; b_1, \dots, b_m; q, z\right) * f(z) &= \frac{1}{z} {}_l\Psi_m\left(a_1, \dots, a_l; b_1, \dots, b_m; q, z\right) * f(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l}\left(a_i, q\right)_{n+1}}{(q,q)_{n+1} \prod_{i=1}^{m}\left(b_i, q\right)_{n+1}} a_n z^n, \end{aligned}$$

where $\prod_{k=1}^{s}(a_k,q)_{n+1} = (a_1,q)_{n+1}(a_2,q)_{n+1}\cdots(a_s,q)_{n+1}$, $z \in \mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and

$$\begin{aligned} {}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) &= \frac{1}{z}{}_l\Psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}}z^n. \end{aligned}$$

Corresponding to the functions ${}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z)$, and $q_{\lambda,\mu}(z)$ given in (1.2) and using the Hadamard product for $f(z) \in \Sigma$, we will present a generalization to the linear operator on $\sum$ as follows

$$\mathcal{G}_{\mu}^{\lambda}(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q) : \Sigma \to \Sigma$$

and

$$\begin{aligned} &\mathcal{G}_{\mu}^{\lambda}(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q)\,f(z) \\ =&f(z) * {}_l\Upsilon_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) * q_{\lambda,\mu}(z) \\ =&\frac{1}{z} + \sum_{n=1}^{\infty}\frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}|a_n|\,z^n. \end{aligned} \tag{1.8}$$

For convenience, we shall henceforth denote

$$\mathcal{G}_{\mu}^{\lambda}(a_1,a_2,\ldots a_l;b_1,b_2,\ldots b_m;q)\,f(z) = \mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z). \tag{1.9}$$

Notice that, the linear operator (1.8) in above was introduced and studied by Challab et al. [5]. For convenience, we let

$$\Lambda_n^{\lambda,\mu} = \frac{\prod_{i=1}^{l}(a_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(b_i,q)_{n+1}}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu}. \tag{1.10}$$

Definition 1.1. For $0 \le \gamma < 1$, $k \ge 0$ and $0 \le \eta < \frac{1}{2}$, we let $\Sigma(\gamma,k,\eta)$ be the subclass of $\Sigma_{S(\gamma)}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned} &-\operatorname{Re}\left(\frac{z(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))' + \eta z^2(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))''}{(1-\eta)\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z) + \eta z(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))'} + \gamma\right) \\ >&k\left|\frac{z(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))' + \eta z^2(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))''}{(1-\eta)\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z) + \eta z(\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z))'} + 1\right|, \end{aligned} \tag{1.11}$$

where $\mathcal{G}_{\mu}^{\lambda}(a_l,b_m,q)\,f(z)$ is given by (1.8).

Remark 1.1. For suitable choice of parameters involved in the Definition 1.1, the class reduces to various new subclasses in the following examples, we illustrate two important subclasses.

Example 1.1. For $\eta = 0$, we let $\Sigma(\gamma, k, 0) = \Sigma(\gamma, k)$ denote a subclass of $\Sigma(\gamma, k, \eta)$ consisting functions of the form (1.1) satisfying the condition that

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)} + \gamma\right) > k\left|\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)} + 1\right|,$$

where $\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)$ is given by (1.8).

Example 1.2. For $\eta = 0$, $k = 0$ we let $\Sigma(\gamma, 0, 0) = \Sigma(\gamma)$ denote a subclass of $\Sigma(\gamma, k, \eta)$ consisting functions of the form (1.1) satisfying the condition that

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))'}{\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)} + \gamma\right) > 0,$$

where $\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)$ is given by (1.8).

For more details about class in the [10, Definition 1.1].

2. Set of Lemmas

We now give the preliminary lemmas that we shall employ in the proof of the main results.

Lemma 2.1 ([4]). *If γ is a real number and $\omega = -u - iv$ is a complex number, then*

$$|\omega + (1-\gamma)| - |\omega - (1+\gamma)| \geq 0 \Leftrightarrow \operatorname{Re}(\omega) \geq \gamma.$$

Lemma 2.2 ([4]). *If $\omega = u + iv$ is a complex number and γ, k are real numbers, then*

$$-\operatorname{Re}(\omega) \geq |\omega + 1|k + \gamma \Leftrightarrow -\operatorname{Re}\left(\omega\left(1 + ke^{i\theta}\right) + ke^{i\theta}\right) \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

3. Main Results

Theorem 3.1. *Let $f \in \Sigma$ be given by* (1.1)*. Then $f \in \Sigma(\gamma, k, \eta)$ if and only if*

$$\sum_{n=1}^{\infty}(1 + (n-1)\eta)[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}|a_n| \leq (1-\gamma)(1-2\eta), \tag{3.1}$$

where $\Lambda_n^{\lambda,\mu}$ is given by (1.10)*.*

Proof. Let $f \in \Sigma(\gamma, k, \eta)$. Then by definition and using Lemma 2.2, we get

$$-\operatorname{Re}\left(\frac{z(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))' + \eta z^2(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))''}{(1-\eta)\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z) + \eta z(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z))'}\left(1 + ke^{i\theta}\right) + ke^{i\theta}\right) > \gamma, \tag{3.2}$$

where $\pi \leq \theta \leq \pi$. For easiness, we let

$$\begin{aligned} A(z) :=& -\left[z\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)\right)' + \eta z^2\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)\right)''\right](1 + ke^{i\theta}) \\ &- ke^{i\theta}\left[(1-\eta)\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z) + \eta z\left(\mathcal{G}_\mu^\lambda(a_l, b_m, q) f(z)\right)'\right] \end{aligned}$$

and

$$B(z) := (1-\eta)\mathcal{G}_{\mu}^{\lambda}(a_l, b_m, q)\, f(z) + \eta z \left(\mathcal{G}_{\mu}^{\lambda}(a_l, b_m, q)\, f(z)\right)'.$$

Hence, the equation (3.2) is equivalent to $-\mathrm{Re}\left(\frac{A(z)}{B(z)}\right) \geq \gamma$ and from Lemma 2.1, we only want to prove that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0.$$

Therefore,

$$\begin{aligned}&|A(z) + (1-\gamma)B(z)|\\ \geq&(1-2\eta)(2-\gamma)\frac{1}{|z|} - \sum_{n=1}^{\infty}[(n-1+\gamma) + k(n+1)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n\end{aligned}$$

and

$$\begin{aligned}&|A(z) - (1+\gamma)B(z)|\\ \leq&\gamma(1-2\eta)\frac{1}{|z|} + \sum_{n=1}^{\infty}[(n+1+\gamma) + k(n+1)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n.\end{aligned}$$

Thus,

$$\begin{aligned}&|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|\\ \geq&2(1-2\eta)(1-\gamma)\frac{1}{|z|} - 2\sum_{n=1}^{\infty}[n(1+k) + (\gamma+k)](1+\eta(n-1))\Lambda_n^{\lambda,\mu}|a_n||z|^n \geq 0,\end{aligned}$$

by the provided condition (3.1). On the other hand, let $f \in \Sigma(\gamma, k, \eta)$. Then by Lemma 2.2, we get (3.2).

Choosing the values of z on the positive real axis the inequality (3.2) reduce to

$$\begin{aligned}&\mathrm{Re}\left(\frac{(1-\gamma)(1-2\eta)\frac{1}{z^2} + \sum_{n=1}^{\infty}(1+(n-1)\eta)\left[n\left(1+ke^{i\theta}\right) + \left(\gamma + ke^{i\theta}\right)\right]\Lambda_n^{\lambda,\mu}|a_n|z^{n-1}}{(1-2\eta)\frac{1}{z^2} - \sum_{n=1}^{\infty}(1+(n-1)\eta)\Lambda_n^{\lambda,\mu}|a_n|z^{n-1}}\right)\\ \geq&0.\end{aligned}$$

Since $\mathrm{Re}\left(-e^{i\theta}\right) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$\mathrm{Re}\left(\frac{(1-\gamma)(1-2\eta)\frac{1}{r^2} + \sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)]\Lambda_n^{\lambda,\mu}|a_n|r^{n-1}}{(1-2\eta)\frac{1}{r^2} - \sum_{n=1}^{\infty}(1+(n-1)\eta)\Lambda_n^{\lambda,\mu}|a_n|r^{n-1}}\right) \geq 0.$$

Letting $r \to 1^-$ and by mean value theorem we get desired inequality (3.1). □

Corollary 3.1. *If $f \in \Sigma(\gamma, k, \eta)$, then*

$$|a_n| \leq \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}}.$$

Corollary 3.2. *Let $f(z) \in \Sigma$ be given by* (1.1). *Then $f \in \Sigma(\gamma, k)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1) + (k+\gamma)]\Lambda_n^{\lambda,\mu}(\alpha_1)\,|a_n| \leq (1-\gamma),$$

where $\eta = 0$, *in Theorem* 3.1.

4. Growth and Distortion Theorem

Theorem 4.1. *Let* $f \in \Sigma(\gamma, k, \eta)$ *given by* (1.1). *Then for* $0 < |z| = r < 1$ *we get*

$$\frac{1}{r} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r$$

and

$$\frac{1}{r^2} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} z. \tag{4.1}$$

Proof. Since $f \in \Sigma(\gamma, k, \eta)$ and $0 < |z| = r < 1$, then

$$|f(z)| \leq \frac{1}{|z|} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} |z| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r$$

and

$$|f(z)| \geq \frac{1}{|z|} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} |z| \geq \frac{1}{r} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} r.$$

On the other hand

$$|f'(z)| \leq \left|\frac{-1}{z^2}\right| + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}$$

and

$$|f'(z)| \geq \left|\frac{-1}{z^2}\right| - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}} \geq \frac{1}{r^2} - \frac{(1-\gamma)(1-2\eta)}{(2k+\gamma+1)\Lambda_2^{\lambda,\mu}}.$$

This completes the proof of Theorem 4.1. □

5. Closure Theorems

Let $f_j(z)$, $j = 1, 2, \ldots, I$, be the function given by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,j}| z^n. \tag{5.1}$$

Theorem 5.1. *Let the function* $f_j(z)$ *defined by* (5.1) *be in the class* $\Sigma(\gamma, k, \eta)$ *for every* $j = 1, 2, \ldots, I$. *Then the function* $f(z)$ *defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{m} q_n z^n,$$

belongs to the class $\Sigma(\gamma, k, \eta)$, *where* $q_n = \frac{1}{I}\sum_{n=1}^{I} |a_{n,j}|$, $n = 1, 2, \ldots$

Proof. Since $f_j(z) \in \Sigma(\gamma, k, \eta)$, it follows from Theorem 3.1 that

$$\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu} \quad |a_{n,j}| \leq (1-\gamma)(1-2\eta),$$

for every $j = 1, 2, \ldots, I$. Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu} q_n \\
=&\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}\left\{\frac{1}{I}\sum_{j=1}^{I}|a_{n,j}|\right\} \\
=&\frac{1}{I}\sum_{j=1}^{I}\left(\sum_{n=1}^{\infty}(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}|a_{n,j}|\right) \\
\leq&\frac{1}{I}\sum_{j=1}^{I}(1-\gamma)(1-2\eta) = (1-\gamma)(1-2\eta),
\end{aligned}$$

which implies that f is in $\Sigma(\gamma, k, \eta)$. □

6. Extreme Points

Theorem 6.1. *Let*

$$f_0(z) = \frac{1}{z} \tag{6.1}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}} z^n, \quad n \geq 1. \tag{6.2}$$

Then $f \in \Sigma(\gamma, k, \eta)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty}\omega_n f_n(z), \quad \sum_{n=0}^{\infty}\omega_n = 1,\ \omega_n \geq 0. \tag{6.3}$$

Proof. From (6.1), (6.2) and (6.3), we have

$$f(z) = \frac{1}{z} + \sum_{n=2}^{\infty}\frac{(1-\gamma)(1-2\eta)\omega_n}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}} z^n. \tag{6.4}$$

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(1-\gamma)(1-2\eta)\omega_n}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}} \cdot \frac{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}{(1-\gamma)(1-2\eta)} \\
(6.5)& \\
=&\sum_{n=2}^{\infty}\omega_n = 1-\omega_1 \leq 1,
\end{aligned}$$

it follows from Theorem 3.1 that the function $f \in \Sigma(\gamma, k, \eta)$.

Conversely, suppose that f is in $\Sigma(\gamma,k,\eta)$, since

$$a_n \leq \frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Lambda_n^{\lambda,\mu}}, \quad n \geq 1. \tag{6.6}$$

Setting

$$\omega_n = \frac{(1+(n-1)\eta)[n(k+1)+(k+\gamma)]\Gamma_n}{(1-\gamma)(1-2\eta)} a_n, \quad \omega_1 = 1-\sum_{n=2}^{\infty}\omega_n, \tag{6.7}$$

it follows that

$$f(z) = \sum_{n=1}^{\infty}\omega_n f_n(z). \tag{6.8}$$

This completes the proof of the theorem. □

7. Radii of Meromorphic Starlikeness and Meromorphic Convexity

Theorem 7.1. *Let $f \in \Sigma(\gamma,k,\eta)$. Then f is meromorphically starlike of order δ, $0 \leq \delta < 1$, in the unit disc $|z| < r_3$, where*

$$r_3 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta}\right)\frac{(\eta(n-1)+1)[n(1-k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-2\eta)(1-\gamma)}\right]^{\frac{1}{n+1}}, \quad n \geq 1.$$

The result is sharp for the extremal function $f(z)$ given by (4.1).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\delta, \quad |z| < r_3.$$

Since

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq \frac{(n+1)\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}|z|^{n+1}}{1-\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}|z|^{n+1}}. \tag{7.1}$$

Hence, (7.1) holds true if

$$\begin{aligned}(n+1)&\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}|z|^{n+1}\\ \leq&(1-\delta)\left[1-\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}|z|^{n+1}\right]\end{aligned}$$

or

$$(n+2-\delta)\frac{(1-\gamma)(1-2\eta)}{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}|z|^{n+1} \leq (1-\delta).$$

Thus, for

$$|z|^{n+1} \leq \frac{(1-\delta)}{(n+2-\delta)}\cdot\frac{(1+(n-1)\eta)[n(1+k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-\gamma)(1-2\eta)}.$$

Hence, $f(z)$ is starlike of order δ. □

Corollary 7.1. *Let $f \in \Sigma(\gamma, k, \eta)$. Then f is meromorphically convex of order δ, $0 \le \delta < 1$, in the unit disc $|z| < r_4$, where*

$$r_4 = \inf_n \left[\left(\frac{1-\delta}{n(n+2-\delta)} \right) \frac{(\eta(n-1)+1)[n(1-k)+(\gamma+k)]\Lambda_n^{\lambda,\mu}}{(1-2\eta)(1-\gamma)} \right]^{\frac{1}{n+1}}, \quad n \ge 1.$$

The result is sharp for the extremal function $f(z)$ given by (4.1).

References

[1] H. Aldweby and M. Darus, *Integral operator defined by q-analogue of Liu-Srivastava operator*, Stud. Univ. Babeş-Bolyai Math. **58**(4) (2013), 529–537.

[2] O. Altintas, H. Irmark and H. M. Srivastava, *A family of meromorphically univalent functions with positive coefficients*, PanAmer. Math. J. **5**(1) (1995), 75–81.

[3] M. K. Aouf, *On a certain class of meromorphic univalent functions with positive coefficients*, Rend. Mat. Appl. **7**(11) (1999), 209–219.

[4] E. S. Aqlan, *Some Problems Connected with Geometric Function Theory*, Pune University, Pune, 2004.

[5] K. A. Challab, M. Darus and F. Ghanim, *A linear operator and associated families of meromorphically q-hypergeometric functions*, AIP Conference proceedings - AIP Publishing LLC. **1830**(1) (2017), `https://doi.org/10.1063/1.4980962`

[6] N. E. Cho, O. S. Kwon and H. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl. **300**(2) (2004), `/https://doi.org/10.1016/j.jmaa.2003.12.026`

[7] M. Darus, *Meromorphic functions with positive coefficients*, Int. J. Math. Math. Sci. **4** (2004), `https://doi.org/10.1155/S0161171204306198`

[8] F. Ghanim and M. Darus, *Some properties on a certain class of meromorphic functions related to Cho-Kwon-Srivastava operator*, Asian-Eur. J. Math. **5**(4) (2012), `https://doi.org/10.1142/S1793557112500520`

[9] A. R. S. Juma, H. H. Ebrahim and S. I. Ahmed, *On subclass of meromorphic univalent function defined by derivative operator*, Gen. Math. **26**(2) (2015), 79–96.

[10] S. Mayilvaganan, N. Magesh and N. B. Gatti, *On certain subclasses of meromorphic functions with positive coefficients associated with Liu- Srivastava linear operator*, International Journal of Pure and Applied Mathematics **113**(13) (2017), `https:/doi.org/10.5402/2012/698307`

[11] S. Owa and N. N. Pascu, *Coefficient inequalities for certain classes of meromorphically starlike and meromorphically convex functions*, Journal of Inequalities in Pure and Applied Mathematics **4**(1) (2003), 1–6.

[1]Department of Mathematics,
RAMADI, IRAQ,
UNIVERSITY OF ANBAR
Email address: `os1989ama@uoanbar.edu.iq`

Email address: `dr_juma@hotmail.com`