Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 21–38.

b-GENERALIZED SKEW DERIVATIONS ON MULTILINEAR POLYNOMIALS

BALCHAND PRAJAPATI[1]

Abstract. Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and F, G be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1,\dots,x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $P \notin Z(R)$ such that

$$\big[P,[F(f(r)),f(r)]\big]=[G(f(r)),f(r)],$$

for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:

(1) there exist $\lambda,\mu\in C$ such that $F(x)=\lambda x$, $G(x)=\mu x$ for all $x\in R$;
(2) there exist $a,b\in U$, $\lambda,\mu\in C$ such that $F(x)=ax+\lambda x+xa$, $G(x)=bx+\mu x+xb$ for all $x\in R$ and $f(x_1,\dots,x_n)^2$ is central valued on R.

1. Introduction

Throughout the article R always denotes an associative ring with the center $Z(R)$, U denotes the Utumi quotient ring of ring R. The definition and axiomatic formulation of Utumi quotient ring U can be found in [5] and [10]. We notice that U is a prime ring with unity and $Z(U)=C$ is called the extended centroid of ring R. The extended centroid C is a field. For $x,y\in R$, the commutator of x and y is $xy-yx$ and it is denoted by $[x,y]$. Sometimes commutator of x and y is called Lie product of x and y. Let $S\subseteq R$, a function f on R is called centralizing (or commuting) function on S if $[f(s),s]\in Z(R)$ (or $[f(s),s]=0$) for all $s\in S$. In this direction, Divinsky [17] studied the commuting automorphism on rings. More precisely, it is proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Mayne [9] generalized this result and proved that if R is

Key words and phrases. b-Generalized skew derivations, multilinear polynomials, prime rings, the extended centroid, Utumi quotient ring.
2010 *Mathematics Subject Classification.* Primary: 16N60. Secondary: 16W25.
DOI 10.46793/KgJMat2301.021P
Received: November 21, 2019.
Accepted: June 14, 2020.

a prime ring with a nontrivial centralizing automorphism then R is a commutative integral domain. Further, Posner [8] studied the centralizing derivations on prime rings. More precisely, he proved that there does not exist any non zero centralizing derivation on non commutative prime ring. This was the starting point for the research by several authors. By derivation, we mean an additive mapping d on R such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$, define a mapping f on R such that $f(x) = [a, x]$ for all $x \in R$. Here, we notice that f is a derivation on R. This kind of derivation is called an inner derivation induced by an element a. Derivation is called outer if it not an inner.

Brešar [13] extended the Posner's [8] result by taking two derivations and proved that if d and δ are two derivations of R with atleast one derivation is non zero, such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then R is commutative. Notable work has been done by several mathematicians to generalize these results on some appropriate subsets of prime ring R. Natural question will arise that what will happen if we replace x with multilinear polynomial in Posner's theorem [8] as well as Brešar's theorem [13] and in this direction many results have been done. One of these results in this direction is given by De Filippis and Wei [27] for skew derivation on multilinear polynomial. Note that an additive mapping d on R is said to be skew derivation associated with an automorphism α if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. It is natural to ask that what will happen if derivation replaced by generalized derivation. The notion of generalized derivation introduced by Brešar in [12] which is a generalization of derivation. An additive mapping F is said to be a generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Note that if R is a prime or a semiprime ring then the derivation d is uniquely determined by F and d is called the associated derivation of F.

Argaç and De Filippis [16] have given the partial generalization of Posner's theorem [8]. More precisely, they describe the structure of additive mapping satisfying the identity $F(f(r))f(r) - f(r)G(f(r)) = 0$ for all $r \in R^n$, where f is a multilinear polynomial and F, G are two generalized derivations on prime ring R. In 2018, Tiwari [19] studied the commuting generalized derivations on prime ring, which is generalization of the work of Argaç and De Filippis [16]. The generalization of Posner's theorem for generalized derivation on multilinear polynomial in [26] (where further generalization can be found in [1, 2, 20, 21]) is given below.

Let K be a commutative ring with unity, R be a prime algebra over K and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K, not central valued on R. Suppose that d is a non zero derivation and F is a non zero generalized derivation of R such that $d([F(f(r)), f(r)]) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;
(2) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.

2. b-Generalized Skew Derivation

Generalizations of derivations and generalized derivations are b-generalized derivations and b-generalized skew derivations. The definition of b-generalized derivation is given below which is from [14]. Let R be a prime ring and U be its Utumi ring of quotient. Let $b \in U$.

Definition 2.1. An additive mapping $F : R \to U$ is called b-generalized derivation of R if $F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$, where $d : R \to U$ is an additive map.

In [14] Košan and Lee proved that if R is a prime ring and $b \neq 0$ then the associated map d must be a derivation of R. Here, we see that a 1-generalized derivation is a generalized derivation. For some $a, b, c \in U$, define a map $F : R \to U$ as $F(x) = ax + bxc$ for all $x \in R$. This is a b-generalized derivation which is called b-generalized inner derivation.

Let α be an automorphism on R. This α is said to be an inner automorphism of R if there exists an invertible element $p \in U$ such that $\alpha(x) = pxp^{-1}$ for all $x \in R$ otherwise it is called outer automorphism. An additive mapping F on R is called generalized skew derivation associated with an automorphism α if there exists a skew derivation d on R such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$. Note that a skew derivation on R associated with an automorphism α is an additive mapping such that $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. A skew derivation associated with the identity automorphism is a derivation and generalized skew derivation associated with identity automorphism is a generalized derivation.

Let α be an inner automorphism on R, that is, $\alpha(x) = pxp^{-1}$ for some $p \in U$ and for all $x \in R$. Now by definition of generalized skew derivation associated with this α, we have $F(xy) = F(x)y + pxp^{-1}d(y)$ for all $x, y \in R$. If d is a skew inner derivation associated with same α, then we know that $d(x) = ax - \alpha(x)a = ax - pxp^{-1}a$. Thus we have $F(xy) = F(x)y + pxp^{-1}(ay - pyp^{-1}a)$, which implies that $F(xy) = F(x)y + pxp^{-1}ay - pxp^{-1}pyp^{-1}a = F(x)y + px\{p^{-1}ay - yp^{-1}a\}$. This gives that $F(xy) = F(x)y + pxd(y)$, where $d(y) = [p^{-1}a, y]$ for all $y \in R$, is an inner derivation induced by $p^{-1}a$. This implies that it is a p-generalized derivation on R. Thus, if α is an inner automorphism on R, then every generalized skew derivation on R is a b-generalized derivation.

The following definition given by De Filippis and Wei [28] is a generalization of above.

Definition 2.2. Let R be an associative ring, $b \in U$, $d : R \to R$ a linear mapping and α be an automorphism of R. A linear mapping $F : R \to R$ is said to be b-generalized skew derivation of R associated with an automorphism α if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$.

As par the above definition b-generalized skew derivations cover the concepts of derivations, generalized derivations, skew derivations, generalized skew derivations and b-generalized derivations. In the same article it is proved that if $b \neq 0$ and R

is a prime ring then the associated additive mapping d becomes a skew derivation associated with the same automorphism α. Further, it is proved that F can be extended to U and it assumes the form $F(x) = ax + bd(x)$, where $a \in U$.

Recently, Liu [6] generalized the result of Posner [8] by taking b-generalized derivation with Engel conditions on prime ring R.

More recently, Sharma et al. [18] studied an identity related to generalized derivations on prime ring with multilinear polynomial over C. More precisely, they proved the following.

Suppose that R is a prime ring of characteristic different from 2 with Utumi quotient ring U and $f(x_1, \dots, x_n)$ is a non central multilinear polynomial over C. Let F and G be two generalized derivations of R and d a non zero derivation of R such that

$$d([F(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)]) = [G(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)],$$

for all $r_1, \dots, r_n \in R$, then one of the following holds:

(a) there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;

(b) there exist $a, b \in U$ and $\lambda, \mu \in C$ such that $F(x) = ax + \lambda x + xa$, $G(x) = bx + \mu x + xb$ for all $x \in R$ and $f(x_1, \dots, x_n)^2$ is central valued on R.

Motivated by above result we prove our main theorem. In this case we take d to be an inner derivation and F, G are b-generalized skew derivations. More precisely, the statement of our main theorem is the following.

Theorem 2.1 (Main Theorem)**.** *Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and F, G be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $P \notin Z(R)$ such that*

$$\Big[P, [F(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)]\Big] = [G(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)],$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

(1) *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*

(2) *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax+\lambda x+xa$, $G(x) = bx+\mu x+xb$ for all $x \in R$ and $f(x_1, \dots, x_n)^2$ is central valued on R.*

The following corollaries are immediate consequence of our main result.

Corollary 2.1. *Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and G be a b-generalized skew derivation on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C which is not central valued on R. If*

$$[G(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)] = 0,$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

(1) *there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in R$;*

(2) *there exist* $a \in U$, $\lambda \in C$ *such that* $G(x) = ax + \lambda x + xa$ *for all* $x \in R$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

Corollary 2.2. *Let R be a prime ring of characteristic different from 2 with the center $Z(R)$ and F be a b-generalized skew derivations on R. Let U be Utumi quotient ring of R with the extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. If*

$$[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in Z(R),$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *there exist $a \in U$, $\lambda \in C$ such that $F(x) = ax + \lambda x + xa$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

If we take $F = d$, a skew derivation, then we get the following.

Corollary 2.3. *Let R be a prime ring of characteristic different from 2 and d be a skew derivation on R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is a commutative ring.*

Let α be any automorphism, then $\alpha - 1$ is a skew derivation. From above corollary we get $[(\alpha-1)(x), x] \in Z(R)$ which implies either R is commutative or α is an identity automorphism. Therefore we state the result of Mayne [9].

Corollary 2.4. *Let R be a prime ring of characteristic different from 2 and α be an automorphism on R such that $[\alpha(x), x] \in Z(R)$ for all $x \in R$, then either α is an identity automorphism or R is a commutative ring.*

3. Preliminaries and Notations

Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C. Then $f(x_1, \ldots, x_n)$ has the following form:

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},$$

where $\gamma_\sigma \in C$ and S_n be the symmetric group of n symbols.

If d is a skew derivation associated with an automorphism α then

$$\begin{aligned} d(\gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}) =& d(\gamma_\sigma) x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} \\ &+ \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)} \end{aligned}$$

and therefore

$$\begin{aligned} d(f(x_1, \ldots, x_n)) =& f^d(x_1, \ldots, x_n) \\ &+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)}, \end{aligned} \tag{3.1}$$

where $f^d(x_1, \ldots, x_n)$ is a multilinear polynomial originated from $f(x_1, \ldots, x_n)$ after replacing each coefficients γ_σ with $d(\gamma_\sigma)$. Similarly, we use $f^\alpha(x_1, \ldots, x_n)$ to denote a multilinear polynomial originated from $f(x_1, \ldots, x_n)$ after replacing each coefficients γ_σ with $\alpha(\gamma_\sigma)$. Let SD denotes the set of all skew derivations and SD_{in} denotes the set of all skew inner derivations of R.

Further, we will frequently use some important theory of generalized polynomial identities and differential identities. We recall some of the remarks.

Remark 3.1. If I is a two-sided ideal of R then R, I and U satisfy the same differential identities [23].

Remark 3.2. If I is a two-sided ideal of R then R, I and U satisfy the same generalized polynomial identities with coefficients in U [5].

Remark 3.3. Let R be a prime ring and $\alpha \in \mathrm{Aut}(R)$ be an outer automorphism of R. If $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates [29].

Remark 3.4. If $f(x_i, d(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, d is an outer skew derivation and α is an outer automorphism of R then R also satisfies the generalized polynomial identity $f(x_i, y_i, z_i)$, where x_i, y_i, z_i are distinct indeterminates ([4, Theorem 1], also see [29]).

Remark 3.5. If d is a non zero skew derivation of R, then the associated automorphism α is unique [11].

Remark 3.6. From [4] we can state the following result. Let R be a prime ring, d a non zero skew derivation on R and I a non zero ideal of R. If I satisfies the skew differential polynomial identity

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0,$$

for any $r_1, \ldots, r_n \in I$ then either

(i) I satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, x_1, \ldots, x_n) = 0$ or
(ii) d is skew U-inner.

Remark 3.7. Let R be a prime ring. Suppose $\sum_{i=1}^n a_i x b_i + \sum_{j=1}^m c_j x q_j = 0$ for all $x \in R$, where $a_i, b_i, c_j, q_j \in U$, $1 \le i \le n$ and $1 \le j \le m$. If $a_1, \ldots, a_n$ are C-independent, then each b_i is C-dependent on $q_1, \ldots, q_m$. Similarly, if $b_1, \ldots, b_n$ are C-independent, then each a_i is C-dependent on $c_1, \ldots, c_m$ (see [24, Lemma 1]).

4. F and G be b-Generalized Skew Inner Derivations

In this section, we study the situation when F and G are b-generalized skew inner derivations of R. Let $F(x) = ax + b\alpha(x)u$ and $G(x) = cx + b\alpha(x)v$ for all $x \in R$ and for some $a, b, c, u, v \in U$. Then we prove the following proposition.

Proposition 4.1. *Let R be a prime ring of characteristic different from 2, U be Utumi ring of quotient of R with the extended centroid C. Suppose F and G are b-generalized skew derivations defined as $F(x) = ax + bpxp^{-1}u$ and $G(x) = cx + bpxp^{-1}v$ for all $x \in R$ and for some $a, b, c, u, v, p \in U$ with invertible p. Let $f(x_1, \ldots, x_n)$ be a non central multilinear polynomial over C. If $P \in R$ be non central such that*

$$\Big[P, [F(f(r)), f(r)]\Big] = [G(f(r)), f(r)],$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following conditions holds:

(1) *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*
(2) *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax+\lambda x+xa$, $G(x) = bx+\mu x+xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

For proof of this proposition, we need the following.

Lemma 4.1. ([26, Lemma 1]). *Let C be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are non scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that each matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all non zero entries.*

Proposition 4.2. *Let $R = M_k(C)$, $k \geq 2$, be the ring of all $k \times k$ matrices over the infinite field C with characteristic different from 2. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2 + b'xq'x - Pxax - Pxqxq' - ax^2P - qxq'xP + xaxP + xqxc' - qxq''x - xcx - xqxq'' = 0$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then either q or q' or P is central.*

Proof. By our assumption

$$\begin{aligned} &a'f(r)^2 + b'f(r)q'f(r) - Pf(r)af(r) - Pf(r)qf(r)q' - af(r)^2P - qf(r)q'f(r)P \\ &+f(r)af(r)P + f(r)qf(r)c' - qf(r)q''f(r) - f(r)cf(r) - f(r)qf(r)q'' = 0, \end{aligned} \tag{4.1}$$

for all $r = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \in R$. We shall prove this result by contradiction. Suppose that $q \notin C$, $q' \notin C$ and $P \notin C$. Then by Lemma 4.1 there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(q)$, $\phi(q')$ and $\phi(P)$ have all non zero entries. Clearly $\phi(q)$, $\phi(q')$, $\phi(P)$, $\phi(a)$, $\phi(a')$, $\phi(b')$, $\phi(c)$, $\phi(c')$ and $\phi(q'')$ must satisfy the condition (4.1).

Let e_{ij} be the matrix whose (i, j)-entry is 1 and rest entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [23] (see also [25]), there exist $s_1, \ldots, s_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(s_1, \ldots, s_n) = \gamma e_{ij}$, with $i \neq j$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Since ϕ is an automorphism, without loss of generality we write (4.1) after replacing $f(r_1, \ldots, r_n) = e_{ij}$

$$\begin{aligned} &a'e_{ij}^2 + b'e_{ij}q'e_{ij} - Pe_{ij}ae_{ij} - Pe_{ij}qe_{ij}q' - ae_{ij}^2P - qe_{ij}q'e_{ij}P \\ &+e_{ij}ae_{ij}P + e_{ij}qe_{ij}c' - qe_{ij}q''e_{ij} - e_{ij}ce_{ij} - e_{ij}qe_{ij}q'' = 0. \end{aligned}$$

It implies that

$$\begin{aligned} &b'e_{ij}q'e_{ij} - Pe_{ij}ae_{ij} - Pe_{ij}qe_{ij}q' - qe_{ij}q'e_{ij}P \\ &+e_{ij}ae_{ij}P + e_{ij}qe_{ij}c' - qe_{ij}q''e_{ij} - e_{ij}ce_{ij} - e_{ij}qe_{ij}q'' = 0. \end{aligned} \tag{4.2}$$

Left and right multiplying by e_{ij} in (4.2), we obtain

$$-e_{ij}Pe_{ij}qe_{ij}q'e_{ij} - e_{ij}qe_{ij}q'e_{ij}Pe_{ij} = 0.$$

From this we have $2(P)_{ji}(q)_{ji}(q')_{ji}e_{ij} = 0$ or get $(P)_{ji}(q)_{ji}(q')_{ji}e_{ij} = 0$, since $\mathrm{char}(R) \neq 2$. It gives that either $(P)_{ji} = 0$ or $(q)_{ji} = 0$ or $(q')_{ji} = 0$, a contradiction, since P, q and q' have all non zero entries. Thus, we conclude that either q or q' or P is central. $\square$

Proposition 4.3. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all matrices over the field C with characteristic different from 2 and $f(x_1, \ldots, x_n)$ a non central multilinear polynomial over C. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2 + b'xq'x - Pxax - Pxqxq' - ax^2P - qxq'xP + xaxP + xqxc' - qxq''x - xcx - xqxq'' = 0$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then either q or q' or P is central.*

Proof. The conclusions follow from Proposition 4.2 in the case of infinite field C. Now we assume that C is a finite field. Suppose that K is an infinite extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_n)$ such that

$$\begin{aligned} Q(r_1, \ldots, r_n) =& a'f(r_1, \ldots, r_n)^2 + b'f(r_1, \ldots, r_n)q'f(r_1, \ldots, r_n) \\ &- Pf(r_1, \ldots, r_n)af(r_1, \ldots, r_n) - Pf(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)q' \\ &- af(r_1, \ldots, r_n)^2P - qf(r_1, \ldots, r_n)q'f(r_1, \ldots, r_n)P \\ &+ f(r_1, \ldots, r_n)af(r_1, \ldots, r_n)P + f(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)c' \\ &- qf(r_1, \ldots, r_n)q''f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)cf(r_1, \ldots, r_n) \\ &- f(r_1, \ldots, r_n)qf(r_1, \ldots, r_n)q'' \end{aligned} \tag{4.3}$$

is a generalized polynomial identity for R. It is a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Hence the complete linearization of $Q(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ in $2n$ indeterminates, moreover $\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^nQ(r_1, \ldots, r_n)$. It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for both R and $\overline{R}$. By assumption $\mathrm{char}(R) \neq 2$ we obtain $Q(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 4.3. $\square$

Lemma 4.2. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, which is not central valued on R. Let $a, a', b', c, c', P, q, q', q'' \in R$ such that $a'x^2 + b'xq'x - Pxax - Pxqxq' - ax^2P - qxq'xP + xaxP + xqxc' - qxq''x - xcx - xqxq'' = 0$*

for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then either q or q' or P is central.

Proof. We shall prove this by contradiction. Suppose that none of q, q' and P is in C. By hypothesis, we have

$$\begin{aligned} h(x_1, \ldots, x_n) =& a' f(x_1, \ldots, x_n)^2 + b' f(x_1, \ldots, x_n) q' f(x_1, \ldots, x_n) \\ & - P f(x_1, \ldots, x_n) a f(x_1, \ldots, x_n) - P f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) q' \\ & - a f(x_1, \ldots, x_n)^2 P - q f(x_1, \ldots, x_n) q' f(x_1, \ldots, x_n) P \\ & + f(x_1, \ldots, x_n) a f(x_1, \ldots, x_n) P + f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) c' \\ & - q f(x_1, \ldots, x_n) q'' f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n) c f(x_1, \ldots, x_n) \\ & - f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) q'', \end{aligned} \tag{4.4}$$

for all $x_1, \ldots, x_n \in R$. Since R and U satisfy same generalized polynomial identity (GPI) (see [5]), U satisfies $h(x_1, \ldots, x_n) = 0_T$. Suppose that $h(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in non commuting indeterminates $x_1, \ldots, x_n$. Then $h(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. Since neither q nor q' nor P is central, hence the term

$$-P f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) q' - q f(x_1, \ldots, x_n) q' f(x_1, \ldots, x_n) P$$

appears nontrivially in $h(x_1, \ldots, x_n)$. Thus, U satisfies

$$-P f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) q' - q f(x_1, \ldots, x_n) q' f(x_1, \ldots, x_n) P = 0_T.$$

Since $P \notin C$, hence it implies that $P f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) q' = 0$. This gives a contradiction, i.e., we have either $P \in C$ or $q' \in C$ or $q \in C$.

Next, suppose that $h(x_1, \ldots, x_n)$ is a non trivial GPI for U. In case C is infinite, we have $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [22, Theorem 2.5 and Theorem 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$. By Martindale's theorem [30], R is then a primitive ring with non zero socle $\mathrm{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [15, page 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be non commutative and so $m \geq 2$. In this case, by Proposition 4.3, we get that either $P \in C$ or $q' \in C$ or $q \in C$, a contradiction.

Next we suppose that V is infinite dimensional over C. By Martindale's theorem [30, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since we have assumed that neither P nor q nor q' is in the center. Then there exist $h_1, h_2, h_3 \in \mathrm{soc}(R)$ such that $[P, h_1] \neq 0$, $[q, h_2] \neq 0$ and

$[q', h_3] \neq 0$. By Litoff's Theorem [7], there exists an idempotent $e \in \mathrm{soc}(R)$ such that $Ph_1, h_1P, qh_2, h_2q, q'h_3, h_3q', h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned}
&e\Big\{a'f(ex_1e,\dots,ex_ne)^2 + b'f(ex_1e,\dots,ex_ne)q'f(ex_1e,\dots,ex_ne)\\
&- Pf(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne) - Pf(ex_1e,\dots,ex_ne)qf(ex_1e,\dots,ex_ne)q'\\
&- af(ex_1e,\dots,ex_ne)^2P - qf(ex_1e,\dots,ex_ne)q'f(ex_1e,\dots,ex_ne)P\\
&+ f(ex_1e,\dots,ex_ne)af(ex_1e,\dots,ex_ne)P + f(ex_1e,\dots,ex_ne)qf(ex_1e,\dots,ex_ne)c'\\
&- qf(ex_1e,\dots,ex_ne)q''f(ex_1e,\dots,ex_ne) - f(ex_1e,\dots,ex_ne)cf(ex_1e,\dots,ex_ne)\\
&- f(ex_1e,\dots,ex_ne)qf(ex_1e,\dots,ex_ne)q''\Big\}e,
\end{aligned}$$

the subring eRe satisfies

$$\begin{aligned}
&\Big\{ea'ef(x_1,\dots,x_n)^2 + eb'ef(x_1,\dots,x_n)eq'ef(x_1,\dots,x_n)\\
&- ePef(x_1,\dots,x_n)eaef(x_1,\dots,x_n) - ePef(x_1,\dots,x_n)eqef(x_1,\dots,x_n)eq'e\\
&- eaef(x_1,\dots,x_n)^2ePe - eqef(x_1,\dots,x_n)eq'ef(x_1,\dots,x_n)ePe\\
&+ f(x_1,\dots,x_n)eaef(x_1,\dots,x_n)ePe + f(x_1,\dots,x_n)eqef(x_1,\dots,x_n)ec'e\\
&- eqef(x_1,\dots,x_n)eq''ef(x_1,\dots,x_n) - f(x_1,\dots,x_n)ecef(x_1,\dots,x_n)\\
&- f(x_1,\dots,x_n)eqef(x_1,\dots,x_n)eq''e\Big\}.
\end{aligned}$$

Then by the above finite dimensional case, either ePe or eqe or $eq'e$ is central element of eRe. Thus either $Ph_1 = (ePe)h_1 = h_1ePe = h_1P$ or $qh_2 = (eqe)h_2 = h_2(eqe) = h_2q$ or $q'h_3 = (eq'e)h_3 = h_3(eq'e) = h_3q'$, a contradiction. □

Lemma 4.3. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let F and G are mappings defined as $F(x) = ax + bxu$, $G(x) = cx + bxv$ for some $a, b, c, u, v \in R$. Let $P \in R$ be non central such that $\Big[P, [F(x), x]\Big] = [G(x), x]$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in R, then either b is central or u is central.*

Proof. By applying similar argument as we have used in Lemma 4.2, we get our desired result. □

Remark 4.1. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let $P, p, q \in R$ and P non central be such that $\Big[P, [p, x]\Big] = [q, x]$ for all $x \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in R, then p and q are central.

Proof. Similar as proof of Lemma 4.2. □

Remark 4.2. Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. Let $a \in R$ be such that $af(r_1,\dots,r_n) \in C$ for all $r_1,\dots,r_n \in R$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1,\dots,x_n)$ in R, then $a = 0$.

Proof. By applying similar argument as we have used in Lemma 4.2 we get $a \in C$. If $a \neq 0$ then from $af(x_1,\dots,x_n) \in C$, we get $f(x_1,\dots,x_n) \in C$ for all $x_1,\dots,x_n \in R$, a contradiction. Therefore, we must have $a = 0$. $\square$

Now we are in position to prove Proposition 4.1.

Proof of Proposition 4.1. From Lemma 4.2, we get either $P \in C$ or $bp \in C$ or $p^{-1}u \in C$. Since $P \notin C$, we shall study following cases.

Case-I. If $bp \in C$ then $F(x) = ax + xbu$ and $G(x) = cx + xbv$ are generalized inner derivations. By [18, Lemma 3.6] we get our conclusions.

Case-II. If $p^{-1}u \in C$ then $F(x) = (a+bu)x = u'x$, $G(x) = cx+bpxp^{-1}v = cx+qxq''$, where $u' = a+bu, q = bp, q'' = p^{-1}v$. Then from $\Big[P, [F(x), x]\Big] = [G(x), x]$, R satisfies the generalized polynomial identity $\theta(x_1,\dots,x_n)$ which can be written as

$$\begin{aligned}\theta(x_1,\dots,x_n) =&Pu'f(x_1,\dots,x_n)^2 - Pf(x_1,\dots,x_n)u'f(x_1,\dots,x_n)\\&- u'f(x_1,\dots,x_n)^2P + f(x_1,\dots,x_n)u'f(x_1,\dots,x_n)P\\&- cf(x_1,\dots,x_n)^2 - qf(x_1,\dots,x_n)q''f(x_1,\dots,x_n)\\&+ f(x_1,\dots,x_n)cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q''.\end{aligned} \tag{4.5}$$

If $\theta(x_1,\dots,x_n)$ is a trivial generalized polynomial identity for R then each of the following is a trivial generalized polynomial identity for R:

- $Pu'f(x_1,\dots,x_n)^2 - Pf(x_1,\dots,x_n)u'f(x_1,\dots,x_n) - u'f(x_1,\dots,x_n)^2P$ $+ f(x_1,\dots,x_n)u'f(x_1,\dots,x_n)P$;
- $-cf(x_1,\dots,x_n)^2 + f(x_1,\dots,x_n)cf(x_1,\dots,x_n)$;
- $-qf(x_1,\dots,x_n)q''f(x_1,\dots,x_n) + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q''$.

Therefore, we must have $u' \in C$, $c \in C$ and $q, q'' \in C$. In this case we get our conclusion.

If $\theta(x_1,\dots,x_n)$ is a non trivial generalized polynomial identity for R then by Matindale's theorem [30] U is a primitive ring having non zero socle with the field C as its associated division ring. By [15, page 35] U is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing non zero linear transformations of finite rank. Assume first that $\dim_C V = k \geq 2$ is a finite positive integer, then $U \cong M_k(C)$ and the conclusion follows from Proposition 4.3.

Now suppose that $\dim_C V = \infty$. Then the set $f(U) = \{f(r_1,\dots,r_n) : r_i \in U\}$ is dense on U, see [31, Lemma 2]. By the fact that $\theta(x_1,\dots,x_n) = 0$ is a generalized polynomial identity for U, therefore U satisfies the generalized polynomial identity

$$Pu'x^2 - Pxu'x - u'x^2P + xu'xP = cx^2 + qxq''x - xcx - xqxq''. \tag{4.6}$$

In (4.6) replace x by $x+1$ we get

$$Pu'x - Pxu' - u'xP + xu'P - cx - qq''x + xc + xqq'' = 0, \tag{4.7}$$

for all $x \in U$. Replace x by xy in the expression (4.7) we get

$$Pu'xy - Pxyu' - u'xyP + xyu'P - cxy - qq''xy + xyc + xyqq'' = 0, \tag{4.8}$$

for all $x, y \in U$. Now multiply from right side by y in expression (4.7) we get

$$Pu'xy - Pxu'y - u'xPy + xu'Py - cxy - qq''xy + xcy + xqq''y = 0. \tag{4.9}$$

Comparing (4.8) and (4.9) we get

$$Px[u', y] + u'x[P, y] + x[-u'P - c - qq'', y] = 0, \tag{4.10}$$

for all $x, y \in U$. By Remark 3.7 either $u' \in C$ or there exist λ_y, μ_y depending on y such that $[P, y] = \lambda_y[u', y]$ and $[-u'P - c - qq'', y] = \mu_y[u', y]$. If $u' \in C$ then by [3, Main theorem] we get our conclusions. If $u' \notin C$ then there is some $y_0 \in U$ such that $[u', y_0] \neq 0$. Therefore, we have $[P, y_0] = \lambda_{y_0}[u', y_0]$ and $[-u'P - c - qq'', y_0] = \mu_{y_0}[u', y_0]$. Substituting these values in (4.10) we get

$$Px[u', y_0] + u'x\lambda_{y_0}[u', y_0] + x\mu_{y_0}[u', y_0] = (P + u'\lambda_{y_0} + \mu_{y_0})x[u', y_0] = 0,$$

by primeness of U we get $P + u'\lambda_{y_0} + \mu_{y_0} = 0$. We note that $\lambda_{y_0} \neq 0$ otherwise $P \in C$, a contradiction. Substituting the value of P in (4.10) we get

$$2\lambda_{y_0}u'x[u', y] + x[-\lambda_{y_0}u'^2 - c - qq'', y] = 0.$$

Again by Remark 3.7 there exists η_y depending on y such that $[-\lambda_{y_0}u'^2 - c - qq'', y] = \eta_y[u', y]$. Since $u' \notin C$ there is some y_0' such that $[u', y_0'] \neq 0$. For fixed $\eta_{y_0'}$ we have $[-\lambda_{y_0}u'^2 - c - qq'', y_0'] = \eta_{y_0'}[u', y_0']$. Thus, we get $(2\lambda_{y_0}u' + \eta_{y_0'})x[u', y_0'] = 0$ for all $x \in U$. The primeness of U gives $2\lambda_{y_0}u' + \eta_{y_0'} = 0$. Since char$R \neq 2$ and $\lambda_{y_0} \neq 0$ we get $u' \in C$, a contradiction. □

Proposition 4.4. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and the extended centroid C, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R. Suppose that F and G are b-generalized skew derivations associated with an outer automorphism α defined as $F(x) = ax + b\alpha(x)u$, $G(x) = cx + b\alpha(x)v$ for all $x \in f(R)$ and for some $a, b, c, u, v \in R$. Let $P \in R$ be non central element of R such that $\Big[P, [F(f(r)), f(r)]\Big] = [G(f(r)), f(r)]$ for all $f(r) \in f(R)$, where $f(R)$ denotes the set of all evaluations of the polynomial $f(x_1, \ldots, x_n)$ in R, then one of the following holds:*

1. *there exist $\lambda, \mu \in C$ such that $F(x) = \lambda x$, $G(x) = \mu x$ for all $x \in R$;*
2. *there exist $a, b \in U$, $\lambda, \mu \in C$ such that $F(x) = ax + \lambda x + xa$, $G(x) = bx + \mu x + xb$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Proof. From the given hypothesis we get

$$\begin{aligned}&\Big[P, [af(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))u, f(x_1, \ldots, x_n)]\Big]\\ =&[cf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))v, f(x_1, \ldots, x_n)],\end{aligned}$$

for all $x_1,\dots,x_n \in R$. Since R and U satisfy the same polynomial identity we get

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))u,f(x_1,\dots,x_n)]\Big]\\ =&[cf(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))v,f(x_1,\dots,x_n)],\end{aligned}$$

for all $x_1,\dots,x_n \in U$. By Remark 3.3 above expression becomes

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+bf^{\alpha}(y_1,\dots,y_n)u,f(x_1,\dots,x_n)]\Big]\\ =&[cf(x_1,\dots,x_n)+bf^{\alpha}(y_1,\dots,y_n)v,f(x_1,\dots,x_n)],\end{aligned}$$

for all $x_1,\dots,x_n,y_1,\dots,y_n \in U$. In particular, U satisfies the blended component

$$\Big[P,[af(x_1,\dots,x_n),f(x_1,\dots,x_n)]\Big]=[cf(x_1,\dots,x_n),f(x_1,\dots,x_n)].$$

Now result follows from Proposition 4.1 by taking $F(x)=ax$ and $G(x)=cx$. □

5. Proof of the Main Theorem

We can write $F(x)=ax+bd(x)$, $G(x)=cx+b\delta(x)$ for all $x\in R$ and for some $a,b,c\in U$, where d,δ are skew derivations on R. If d and δ both are skew inner derivations on R then by Proposition 4.1 and Proposition 4.4, we get our conclusions. If $b=0$ then also we get our conclusions from Proposition 4.1. So assume $b\neq 0$. Now we assume that both are not skew inner derivations. We shall study the following cases.

Case-I. Let d be skew inner and δ be outer. In this case we write $F(x)=ax+b\alpha(x)u$ and $G(x)=cx+b\delta(x)$. From given hypothesis we get

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))u,f(x_1,\dots,x_n)]\Big]\\ =&[cf(x_1,\dots,x_n)+b\delta(f(x_1,\dots,x_n)),f(x_1,\dots,x_n)].\end{aligned}\tag{5.1}$$

Substituting the value of $\delta(f(x_1,\dots,x_n))$ from (3.1) in equation (5.1), we get

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))u,f(x_1,\dots,x_n)]\Big]\\ =&\Big[cf(x_1,\dots,x_n)+bf^{\delta}(x_1,\dots,x_n)\\ &+b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\Big].\end{aligned}\tag{5.2}$$

Since δ is outer, by using Remark 3.6 in above expression, we get

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))u,f(x_1,\dots,x_n)]\Big]\\ =&\Bigg[cf(x_1,\dots,x_n)+bf^{\delta}(x_1,\dots,x_n)\\ &+b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\Bigg].\end{aligned}\tag{5.3}$$

In particular U satisfies the blended component

$$\left[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\right]. \tag{5.4}$$

Suppose α is an inner automorphism. In (5.4) replace $y_{\sigma(1)}=x_{\sigma(1)}$ and $y_{\sigma(i)}=0$ for all $i>1$, we get

$$\Big[bf(x_1,\dots,x_n),f(x_1,\dots,x_n)\Big]=0.$$

Conclusions follow from the inner case.

Suppose α is an outer automorphism, then for $y_{\sigma(n)}=\alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)})=t_{\sigma(i)}$ for all i and $y_{\sigma(i)}=0$ for $i<n$ in (5.4) we get

$$\Big[bf^{\alpha}(t_1,\dots,t_n),f(x_1,\dots,x_n)\Big]=0,$$

for all $t_1,\dots,t_n,x_1,\dots,x_n\in U$. By Remark 4.1 we get $bf^{\alpha}(t_1,\dots,t_n)\in C$ for all $t_1,\dots,t_n\in U$ and by Remark 4.2 we get $b=0$, a contradiction.

Case-II. Now we assume that d is an outer derivation and δ is a skew inner derivation then we write $F(x)=ax+bd(x)$ and $G(x)=cx+b\alpha(x)v$. Then our hypothesis becomes

$$\begin{aligned}&\Big[P,[af(x_1,\dots,x_n)+bd(f(x_1,\dots,x_n)),f(x_1,\dots,x_n)]\Big]\\ &=[cf(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))v,f(x_1,\dots,x_n)].\end{aligned} \tag{5.5}$$

We substitute the value of $d(f(x_1,\dots,x_n))$ from (3.1) in above equation, we get that U satisfies

$$\begin{aligned}&\Bigg[P,\Big[af(x_1,\dots,x_n)+bf^{d}(x_1,\dots,x_n)\\ &+b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\Big]\Bigg]\\ &=[cf(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))v,f(x_1,\dots,x_n)].\end{aligned}$$

Since d is outer derivation, by using Remark 3.6 in above expression, we get

$$\begin{aligned}&\Bigg[P,\Big[af(x_1,\dots,x_n)+bf^{d}(x_1,\dots,x_n)\\ &+b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\Big]\Bigg]\\ &=[cf(x_1,\dots,x_n)+b\alpha(f(x_1,\dots,x_n))v,f(x_1,\dots,x_n)],\end{aligned}$$

where $d(x_i)=y_i$. In particular, U satisfies the blended component

$$\left[P,\left[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)},f(x_1,\dots,x_n)\right]\right]. \tag{5.6}$$

Suppose α is an inner automorphism. Replace $y_{\sigma(1)} = x_{\sigma(1)}$ and $y_{\sigma(i)} = 0$ for all $i > 1$ in (5.6) we get $\Big[P, [bf(x_1, \dots, x_n), f(x_1, \dots, x_n)]\Big] = 0$ for all $x_1, \dots, x_n \in U$. Conclusions follow from inner case.

Now suppose α is an outer automorphism, then for $y_{\sigma(n)} = \alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)}) = t_{\sigma(i)}$ for all i and $y_{\sigma(i)} = 0$ for $i < n$ in (5.6) we get

$$\Big[P, [bf^{\alpha}(t_1, \dots, t_n), f(x_1, \dots, x_n)]\Big] = 0,$$

for all $t_1, \dots, t_n, x_1, \dots, x_n \in U$. From Remark 4.1 we get $bf^{\alpha}(t_1, \dots, t_n) \in C$ for all $t_1, \dots, t_n \in U$ and by Remark 4.2 we get $b = 0$, a contradiction.

Case-III. Now we suppose that none of d and δ are skew inner derivations. In this case we write $F(x) = ax + bd(x)$, $G(x) = cx + b\delta(x)$, where d and δ both are outer derivations. Now we have the following two subcases.

d and δ be C-Linearly Independent Modulo SD_{in}

In this case from our hypothesis, U satisfies

$$\begin{aligned}&\Big[P, [af(x_1, \dots, x_n) + bd(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)]\Big]\\ &=[cx + b\delta(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)].\end{aligned} \tag{5.7}$$

We substitute the value of $d(f(x_1, \dots, x_n))$ and $\delta(f(x_1, \dots, x_n))$ from (3.1) and use Remark 3.6 to (5.7) then U satisfies

$$\begin{aligned}&\Bigg[P, \Big[af(x_1, \dots, x_n) + bf^{d}(x_1, \dots, x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1, \dots, x_n)\Big]\Bigg]\\ =&\Bigg[cf(x_1, \dots, x_n) + bf^{\delta}(x_1, \dots, x_n)\\ &+ b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1, \dots, x_n)\Bigg],\end{aligned}$$

where $y_{\sigma(j+1)} = d(x_{\sigma(j+1)})$ and $z_{\sigma(j+1)} = \delta(x_{\sigma(j+1)})$. In particular, U satisfies the blended component

$$\Bigg[b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}, f(x_1, \dots, x_n)\Bigg]. \tag{5.8}$$

Suppose α is an inner automorphism. Replace $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for all $i > 1$ in expression (5.8) we get $[bf(x_1, \dots, x_n), f(x_1, \dots, x_n)] = 0$ for all $x_1, \dots, x_n \in U$. Conclusions follow from inner case.

Now suppose α is an outer automorphism. Then for $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for $i > 1$ in (5.8) we get

$$[bf^{\alpha}(t_1, \ldots, t_n), f(x_1, \ldots, x_n)] = 0, \tag{5.9}$$

for all $t_1, \ldots, t_n, x_1, \ldots, x_n \in U$. By Remark 4.1 we get $bf^{\alpha}(t_1, \ldots, t_n) \in C$ for all $t_1, \ldots, t_n \in U$. By Remark 4.2 we get $b = 0$, a contradiction.

d and δ be C-Linearly Dependent Modulo SD_{in}

Since d and δ be C-linearly dependent modulo SD_{in} there are some $\lambda, \mu \in C$, $q' \in U$ such that $\lambda d(x) + \mu\delta(x) = q'x - \alpha(x)q'$ for all $x \in R$.

If $\lambda = 0$ and $\mu \neq 0$ then $\delta(x) = qx - \alpha(x)q$, where $q = \mu^{-1}q'$ is a skew inner derivation, a contradiction.

If $\lambda \neq 0$ and $\mu = 0$ then $d(x) = qx - \alpha(x)q$, where $q = \lambda^{-1}q'$ is a skew inner derivation, a contradiction.

Suppose $\lambda \neq 0$ and $\mu \neq 0$ and we write $d(x) = \beta\delta(x) + qx - \alpha(x)q$, where $\beta = -\lambda^{-1}\mu$, $q = \lambda^{-1}q'$. Now from our hypothesis we have

$$\Big[P, [af(x_1, \ldots, x_n) + b\beta\delta(f(x_1, \ldots, x_n)) + bqf(x_1, \ldots, x_n) - b\alpha(f(x_1, \ldots, x_n))q, f(x_1, \ldots, x_n)]\Big] = [cf(x_1, \ldots, x_n) + b\delta(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)].$$

Substituting the value of $\delta(f(x_1, \ldots, x_n))$ from (3.1) in above expression we get

$$\begin{aligned}
&\Bigg[P, \Bigg[af(x_1, \ldots, x_n) + b\beta f^{\delta}(x_1, \ldots, x_n) \\
&\quad + b\beta \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)} + bqf(x_1, \ldots, x_n) \\
&\quad - b\alpha(f(x_1, \ldots, x_n))q, f(x_1, \ldots, x_n)\Bigg]\Bigg] \\
&= \Bigg[cf(x_1, \ldots, x_n) + bf^{\delta}(x_1, \ldots, x_n) \\
&\quad + b \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg],
\end{aligned} \tag{5.10}$$

where $z_{\sigma(j+1)} = \delta(x_{\sigma(j+1)})$. In particular, U satisfies the blended component

$$\begin{aligned}
&\Bigg[P, \Big[b\beta \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Big]\Bigg] \\
&= \Bigg[b \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \ldots x_{\sigma(j)}) z_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Bigg].
\end{aligned} \tag{5.11}$$

Suppose α is an inner automorphism. Replacing $z_{\sigma(1)} = x_{\sigma(1)}$ and $z_{\sigma(i)} = 0$ for $i > 1$ in (5.11) we get

$$\Big[P, [b\beta f(x_1, \ldots, x_n), f(x_1, \ldots, x_n)]\Big] = [bf(x_1, \ldots, x_n), f(x_1, \ldots, x_n)],$$

for all $x_1, \ldots, x_n \in U$. Conclusions follow from inner case.

Suppose α is an outer automorphism. Replace $z_{\sigma(n)} = \alpha(x_{\sigma(n)})$, $\alpha(x_{\sigma(i)}) = t_{\sigma(i)}$ for all i and $z_{\sigma(i)} = 0$ for $i < n$ in (5.11) we get

$$\Big[P, [b\beta f^{\alpha}(t_1, \ldots, t_n), f(x_1, \ldots, x_n)]\Big] = [bf^{\alpha}(t_1, \ldots, t_n), f(x_1, \ldots, x_n)],$$

for all $t_1, \ldots, t_n, x_1, \ldots, x_n \in U$. By Remark 4.1 we get $b\beta f^{\alpha}(t_1, \ldots, t_n) \in C$ and $bf^{\alpha}(t_1, \ldots, t_n) \in C$ for all $t_1, \ldots, t_n \in U$. In both cases $bf^{\alpha}(t_1, \ldots, t_n) \in C$ for all $t_1, \ldots, t_n$, since $0 \neq \beta \in C$. By Remark 4.2 we get $b = 0$, a contradiction.

Similarly, if we consider $\delta(x) = \beta d(x) + qx - \alpha(x)q$ for all $x \in R$ then we get a contradiction.

Acknowledgements. Author thanks to the referees for their valuable comments and suggestions. Author is partially supported by Department of Science and Technology Research Grant-SERB EMR/2016/001550.

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[1]School of Liberal Studies,
Ambedkar University Delhi,
Kashmere Gate, Delhi 110006
Email address: `balchand@aud.ac.in`