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ON CONTACT CR-SUBMANIFOLD OF A KENMOTSU MANIFOLD WITH KILLING TENSOR FIELD

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Abstract. The object of this paper is to study the Contact CR-submanifold of a Kenmotsu manifold with the help of a killing tensor field and deduce some results.

1. Introduction

K. Kenmotsu [5] introduced the notion of Kenmotsu manifold and later several authors studied this manifold [2, 14, 15]. M. Kobayashi and N. Papaghuic [10, 11] investigated the geometry of semi-invariant submanifolds of a Kenmotsu manifold. The geometry of Contact CR-submanifolds, invariant and anti-invariant submanifolds of an almost contact metric structure are studied by A. Bejancu [1].

Gupta et al. [13] studied the intrinsic characterization of a slant submanifold of a Kenmotsu manifold in case of induced metric and obtained some examples of the slant submanifold of a Kenmotsu manifold. Avik De [2] studied and obtained few examples of a 3-dimensional Kenmotsu manifold with parallel Ricci tensor and obtained killing condition for a vector field in Kenmotsu manifold.

Moreover, the Contact CR-submanifolds of Kenmotsu manifolds are studied by some other authors [8, 9]. The notion of a killing tensor field was introduced by Professor D. E. Blair [4]. In [12], we have investigated and characterized a slant submanifold of a Kenmotsu manifold using killing tensor fields. In this paper, we have studied Contact CR-submanifold of a Kenmotsu manifold using the notion of a killing tensor field and obtained some results.

Key words and phrases. Contact manifold, CR-submanifold, Kenmotsu manifold, killing tensor field.

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2. Preliminaries

A $(2m+1)$-dimensional manifold M is said to admit an almost contact metric structure if there exist a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$\varphi\xi = 0, \quad \varphi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \eta(\varphi U) = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad g(U, \xi) = \eta(U), \tag{2.2}$$

where U and V are vector fields on M [3,7].

Moreover, if

$$\left(\overline{\nabla}_U \varphi\right) V = -g(U, \varphi V)\xi - \eta(V)\varphi U, \quad \overline{\nabla}_U \xi = U - \eta(U)\xi, \tag{2.3}$$

where $\overline{\nabla}$ be a Levi-Civita connection on $\overline{M}$, then the structure $(M, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold [5].

Suppose M is an isometrically immersed submanifold in $\overline{M}$ and ∇, $\overline{\nabla}$ be the Riemannian connections on M, $\overline{M}$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.4}$$

and

$$\overline{\nabla}_U W = -\mathrm{A}_W U + \nabla^{\perp}_U W, \tag{2.5}$$

for any vector fields $U, V \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ be the normal connection on $T^{\perp}M$, A and h be the shape operator and second fundamental form of M in $\overline{M}$.

Both h and A are related as

$$g(A_W U, V) = g(h(U, V), W). \tag{2.6}$$

In Kenmotsu manifold, M is isometrically immersed submanifold. For any vector field U tangent to M, we put

$$\varphi U = pU + fU, \tag{2.7}$$

where pU and fU denote the tangent and normal component of φU, respectively.

The covariant derivative of p, f are given by

$$(\nabla_U p)V = \nabla_U pV - p\nabla_U V,$$
$$(\nabla_U f)V = \nabla^{\perp}_U fV - f\nabla_U V.$$

Similarly, for any vector field W normal to M, we have

$$\varphi W = bW + cW, \tag{2.8}$$

where bW and cW are the tangent and normal component of φW.

The covariant derivative of b, c are given by

$$(\nabla_U b)W = \nabla_U bW - b\nabla_U^{\perp} W,$$
$$(\nabla_U c)W = \nabla_U^{\perp} cW - c\nabla_U^{\perp} W.$$

Let p be the endomorphism defined by (2.7), then we have

$$g(pU, V) + g(U, pV) = 0. \tag{2.9}$$

Definition 2.1 ([9]). Let M be a submanifold of a Kenmotsu manifold $\overline{M}$. Then M is said to be a contact CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $D : x \to D_x \subseteq T_x(M)$ on M satisfying the following conditions:
(i) $TM = D \oplus D^{\perp}$, $\xi \in D$;
(ii) D is invariant with respect to φ, that is, $\varphi D_x \subset T_x(M)$;
(iii) the orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subseteq T_x(M)$ satisfies $\varphi D_x^{\perp} \subseteq T_x^{\perp}(M)$ for each $x \in M$.

A contact CR-submanifold is said to be proper if neither $D_x = \{0\}$ nor $D_x^{\perp} = \{0\}$. If $D_x = \{0\}$, then M is anti-invariant submanifold and if $D_x^{\perp} = \{0\}$, then M becomes invariant submanifold.

Now, let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. For any $U, V \in \Gamma(TM)$, by (2.3), (2.7), (2.8) together with the Gauss and Weingarten formulae [9], we have

$$\left(\overline{\nabla}_U \varphi\right) V = \overline{\nabla}_U \varphi V - \varphi \overline{\nabla}_U V \tag{2.10}$$

or

$$-g(U, \varphi V) - \eta(V)\varphi U = \overline{\nabla}_U pV + \overline{\nabla}_U fV - \varphi \nabla_U V - \varphi h(U, V).$$

By comparing the tangent and normal component of the above equation, we have

$$(\nabla_U p)V = A_{fV}U + bh(U, V) + g(pU, V)\xi - \eta(V)pU \tag{2.11}$$

and

$$(\nabla_U f)V = ch(U, V) - h(U, pV) - \eta(V)fU. \tag{2.12}$$

If ξ be the structure vector field tangent to submanifold M, then by (2.3) and (2.6), we have

$$A_W \xi = h(U, \xi) = 0, \tag{2.13}$$

for all $U \in \Gamma(TM)$ and $W \in \Gamma(T^{\perp}M)$. Thus, (2.11) reduces to

$$(\nabla_U p)V = g(pU, V)\xi - \eta(V)pU, \tag{2.14}$$

for any $U, V \in \Gamma(D)$. This shows that, the induced structure p is a Kenmotsu structure on M [9].

Let M is a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$, then equation (2.11) reduces to

$$(\nabla_U p)V = bh(U, V) + g(pU, V)\xi - \eta(V)pU, \tag{2.15}$$

for any $U, V \in \Gamma(D)$ [8].

If the second fundamental form h is zero, then submanifold M is totally geodesic. A submanifold M is totally umbilical if

$$h\left(U,V\right)=g\left(U,V\right)H,$$

where H is the mean curvature vector. In addition, if $H=0$, then the submanifold M is minimal.

A tensor field φ is called killing [4], if it satisfies the following condition

$$\left(\overline{\nabla}_U\varphi\right)V+\left(\overline{\nabla}_V\varphi\right)U=0. \tag{2.16}$$

3. Contact CR-Submanifold of a Kenmotsu Manifold $\overline{M}$ with Killing Tensor Field

In this section, we discuss some results on contact CR-submanifold of a Kenmotsu manifold with killing tensor field.

Theorem 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\overline{\nabla}_U pV+\overline{\nabla}_V pU)+\left(\overline{\nabla}_U fV+\overline{\nabla}_V fU\right)=p(\overline{\nabla}_U V+\overline{\nabla}_V U)+f(\overline{\nabla}_U V+\overline{\nabla}_V U). \tag{3.1}$$

Proof. From the equation (2.10), we have

$$(\overline{\nabla}_U\varphi)V=\overline{\nabla}_U\varphi V-\varphi\overline{\nabla}_U V.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V\varphi\right)U=\overline{\nabla}_V\varphi U-\varphi\overline{\nabla}_V U.$$

On clubbing above equations, we get

$$\left(\overline{\nabla}_U\varphi\right)V+\left(\overline{\nabla}_V\varphi\right)U=\overline{\nabla}_U\varphi V-\varphi\overline{\nabla}_U V+\overline{\nabla}_V\varphi U-\varphi\overline{\nabla}_V U.$$

Using (2.16), we get

$$0=\overline{\nabla}_U\varphi V-\varphi\overline{\nabla}_U V+\overline{\nabla}_V\varphi U-\varphi\overline{\nabla}_V U. \tag{3.2}$$

Using (2.7), above equation yields

$$(\overline{\nabla}_U pV+\overline{\nabla}_V pU)+(\overline{\nabla}_U fV+\overline{\nabla}_V fU)=p(\overline{\nabla}_U V+\overline{\nabla}_V U)+f(\overline{\nabla}_U V+\overline{\nabla}_V U). \quad \square$$

Theorem 3.2. *Suppose M denotes a contact CR-submanifold with killing tensor field φ of a Kenmotsu manifold $\overline{M}$, then*

$$\eta\left(V\right)pU+\eta\left(U\right)pV=0 \tag{3.3}$$

and

$$\eta\left(V\right)fU+\eta\left(U\right)fV=0. \tag{3.4}$$

Proof. From equation (2.3), we have

$$\left(\overline{\nabla}_U \varphi\right) V = g\left(\varphi U, V\right) \xi - \eta\left(V\right) \varphi U.$$

By swapping U and V, above equation becomes

$$\left(\overline{\nabla}_V \varphi\right) U = -g\left(\varphi U, V\right) \xi - \eta\left(U\right) \varphi V.$$

Clubbing above two equations, we get

$$\left(\overline{\nabla}_U \varphi\right) V + \left(\overline{\nabla}_V \varphi\right) U = -\eta\left(V\right) \varphi U - \eta\left(U\right) \varphi V.$$

By using (2.16), we get

$$-\eta\left(V\right) \varphi U - \eta\left(U\right) \varphi V = 0. \tag{3.5}$$

By using (2.7) in above equation, then comparing the tangential and normal components, we get the result. □

Theorem 3.3. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then the induced structure p satisfies*

$$(\nabla_U p)V + (\nabla_V p)U = 0. \tag{3.6}$$

Proof. From (2.14), we have

$$(\nabla_U p)V = -g\left(U, pV\right) \xi - \eta\left(V\right) pU.$$

By swapping U and V in above equation, we get

$$(\nabla_V p)U = g\left(U, pV\right) \xi - \eta\left(U\right) pV.$$

On clubbing above two equations, we have

$$(\nabla_U p)V + (\nabla_V p)U = -\eta\left(V\right) pU - \eta\left(U\right) pV.$$

By using (3.3) in above equation, we get the result. □

Theorem 3.4. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. If second fundamental form h is parallel then contact CR-submanifold M is a totally geodesic.*

Proof. By swapping U and V in (2.15), we have

$$(\nabla_V p)U = bh\left(U, V\right) - g\left(V, pU\right) \xi - \eta\left(U\right) pV. \tag{3.7}$$

Combining (2.15) and (3.7), we have

$$(\nabla_U p)V + (\nabla_V p)U = 2bh\left(U, V\right) - \eta\left(V\right) pU - \eta\left(U\right) pV.$$

Now, using (3.3) and (3.6), yields $h\left(U, V\right) = 0$ for any $U, V \in \Gamma(TM)$. □

Lemma 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$A_{fV}U + A_{fU}V + 2bh\left(U, V\right) = 0. \tag{3.8}$$

Proof. By swapping U and V in (2.11), we have

$$(\nabla_V p)U = A_{fU}V + bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV. \tag{3.9}$$

On clubbing (2.11) and (3.9), we get

$$\begin{aligned}(\nabla_U p)V + (\nabla_V p)U =& A_{fV}U + A_{fU}V + 2bh\,(U,V) + g\,(pU,V)\,\xi \\ &+ g\,(pV,U)\,\xi - \eta\,(U)\,pV - \eta\,(V)\,pU.\end{aligned}$$

By using (2.9), it follows that

$$(\nabla_U p)V + (\nabla_V p)U = A_{fV}U + A_{fU}V + 2bh\,(U,V) - \eta\,(U)\,pV - \eta\,(V)\,pU.$$

Since p satisfies (3.3) and (3.6), we get the desired result. □

Proposition 3.1. *Suppose M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ. Then M is anti-invariant submanifold in $\overline{M}$ if the endomorphism p is parallel.*

Proof. By interchanging U and V in (2.15), we get

$$(\nabla_V p)\,U = bh\,(U,V) + g\,(pV,U)\,\xi - \eta\,(U)\,pV,$$

for any $U, V \in \Gamma(D)$.

Clubbing above equation with (2.15), we get

$$(\nabla_U p)\,V + (\nabla_V p)\,U = 2bh\,(U,V) + g\,(pU,V)\,\xi + g\,(pV,U)\,\xi - \eta\,(V)\,pU - \eta\,(U)\,pV.$$

By using (2.9) and (3.6), above equation yields

$$2bh\,(U,V) - \eta\,(V)\,pU - \eta\,(U)\,pV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), we get $pU = 0$, which establishes our assertion. □

Proposition 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then M is invariant (submanifold) in $\overline{M}$ if the endomorphism f is parallel.*

Proof. By swapping U and V in (2.12), we get

$$(\nabla_V f)\,U = ch\,(U,V) - h\,(V,pU) - \eta\,(U)\,fV, \tag{3.10}$$

for any $U, V \in \Gamma(TM)$.

Clubbing (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

If f is parallel, then above equation becomes

$$2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV = 0.$$

Setting $V = \xi$ and taking into account (2.1) and (2.13), it follows that $fU = 0$. □

Lemma 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\overline{M}$ with killing tensor field φ, then*

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 0 \tag{3.11}$$

if and only if

$$2ch\,(U,V) = h\,(U,pV) + h\,(V,pU)\,. \tag{3.12}$$

Proof. Taking into consideration (2.12) and (3.10), we get

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU) - \eta\,(V)\,fU - \eta\,(U)\,fV.$$

By using (3.4), above equation yields

$$(\nabla_U f)\,V + (\nabla_V f)\,U = 2ch\,(U,V) - h\,(U,pV) - h\,(V,pU)\,.$$

Hence, the result. □

4. Examples

In this section, we give a few examples of Kenmotsu manifolds with killing φ.

Example 4.1. Let us consider the three dimensional manifold $\overline{M} = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Suppose metric g on $\overline{M}$ is given by

$$g = \eta \otimes \eta + e^{2z}(dx \otimes dx + dy \otimes dy).$$

Now, we choose

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} = \xi.$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i,e_j) = 0$ for $i \neq j$ and $g(e_i,e_j) = 1$ for $i = j$, for $1 \leq i,j \leq 3$. The 1-form η is given by $\eta(U) = g(U,e_3)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1,1)$, defined by $\varphi(e_1) = 0$, $\varphi(e_2) = 0$, $\varphi(e_3) = 0$. Now, using the linearity property of φ and g, we get

$$\varphi^2 U = -U + \eta(U)\xi, \quad \eta(e_3) = 1, \quad g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields,

$$\begin{aligned}
\overline{\nabla}_{e_1}e_1 &= -\,e_3, \quad \overline{\nabla}_{e_1}e_2 = 0, \quad \overline{\nabla}_{e_1}e_3 = e_1,\\
\overline{\nabla}_{e_2}e_1 &= 0, \quad \overline{\nabla}_{e_2}e_2 = -e_3, \quad \overline{\nabla}_{e_2}e_3 = e_2,\\
\overline{\nabla}_{e_3}e_1 &= e_1, \quad \overline{\nabla}_{e_3}e_2 = e_2, \quad \overline{\nabla}_{e_3}e_3 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U\xi = U - \eta\,(U)\,\xi$ for $\xi = e_3$. Hence, the manifold is a Kenmotsu manifold. From the above relations, we obtain the following equations

$$(4.1)\quad \begin{cases} (\overline{\nabla}_{e_1}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_1 = 0, & (\overline{\nabla}_{e_1}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_1 = 0, \\ (\overline{\nabla}_{e_1}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_1 = 0, & (\overline{\nabla}_{e_2}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_2 = 0, \\ (\overline{\nabla}_{e_2}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_2 = 0, & (\overline{\nabla}_{e_2}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_2 = 0, \\ (\overline{\nabla}_{e_3}\varphi)e_1 + (\overline{\nabla}_{e_1}\varphi)e_3 = 0, & (\overline{\nabla}_{e_3}\varphi)e_2 + (\overline{\nabla}_{e_2}\varphi)e_3 = 0, \\ (\overline{\nabla}_{e_3}\varphi)e_3 + (\overline{\nabla}_{e_3}\varphi)e_3 = 0. \end{cases}$$

From the equations (4.1), it follows that φ is the killing tensor field. Hence, the manifold $\overline{M}$ is a Kenmotsu manifold with the killing tensor field φ. Moreover, we have

$$(4.2)\quad \begin{cases} \overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 + \overline{\nabla}_{e_1}\varphi e_1 - \varphi\overline{\nabla}_{e_1}e_1 = 0, \\ \overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 + \overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 = 0, \\ \overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 + \overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 = 0, \\ \overline{\nabla}_{e_2}\varphi e_1 - \varphi\overline{\nabla}_{e_2}e_1 + \overline{\nabla}_{e_1}\varphi e_2 - \varphi\overline{\nabla}_{e_1}e_2 = 0, \\ \overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 + \overline{\nabla}_{e_2}\varphi e_2 - \varphi\overline{\nabla}_{e_2}e_2 = 0, \\ \overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 + \overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 = 0, \\ \overline{\nabla}_{e_3}\varphi e_1 - \varphi\overline{\nabla}_{e_3}e_1 + \overline{\nabla}_{e_1}\varphi e_3 - \varphi\overline{\nabla}_{e_1}e_3 = 0, \\ \overline{\nabla}_{e_3}\varphi e_2 - \varphi\overline{\nabla}_{e_3}e_2 + \overline{\nabla}_{e_2}\varphi e_3 - \varphi\overline{\nabla}_{e_2}e_3 = 0, \\ \overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 + \overline{\nabla}_{e_3}\varphi e_3 - \varphi\overline{\nabla}_{e_3}e_3 = 0, \end{cases}$$

and

$$(4.3)\quad \begin{cases} \eta(e_1)\varphi(e_1) + \eta(e_1)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_1) + \eta(e_1)\varphi(e_2) = 0, \\ \eta(e_3)\varphi(e_1) + \eta(e_1)\varphi(e_3) = 0, & \eta(e_1)\varphi(e_2) + \eta(e_2)\varphi(e_1) = 0, \\ \eta(e_2)\varphi(e_2) + \eta(e_2)\varphi(e_2) = 0, & \eta(e_3)\varphi(e_2) + \eta(e_2)\varphi(e_3) = 0, \\ \eta(e_1)\varphi(e_3) + \eta(e_3)\varphi(e_1) = 0, & \eta(e_2)\varphi(e_3) + \eta(e_3)\varphi(e_2) = 0, \\ \eta(e_3)\varphi(e_3) + \eta(e_3)\varphi(e_3) = 0. \end{cases}$$

The equations (4.1) and (4.2) satisfy the equation (3.2) and the equations (4.1) and (4.3) satisfy the equation (3.5).

Analogous to [14], we have the following example of five-dimensional Kenmotsu manifold with the killing tensor field.

Example 4.2. Let us consider the five dimensional manifold $\overline{M} = \{(x_1, x_2, x_3, x_4, v) \in \mathbb{R}^5, v \neq 0\}$, where (x_1, x_2, x_3, x_4, v) are the standard coordinates in $\mathbb{R}^5$. Suppose metric g on $\overline{M}$ is given by

$$g = \eta \otimes \eta + e^{2v}\sum_{i=1}^{4} dx_i \otimes dx_i.$$

Now, we choose

$$e_1 = e^{-v}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-v}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-v}\frac{\partial}{\partial x_3}, \quad e_4 = e^{-v}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial v} = \xi.$$

The above vector fields are linearly independent at the each point of $\overline{M}$ such that $g(e_i, e_j) = 0$ for $i \neq j$ and $g(e_i, e_j) = 1$ for $i = j$, where $i, j = 1, 2, 3, 4, 5$. The 1-form η is given by $\eta(U) = g(U, e_5)$ for chosen U on $\overline{M}$. Let φ be a tensor field of type $(1, 1)$, defined by $\varphi(e_1) = 0$, $\varphi(e_2) = 0$, $\varphi(e_3) = 0$, $\varphi(e_4) = 0$, $\varphi(e_5) = 0$.

Now, using the linearity property of φ and g, we have

$$\varphi^2 U = -U + \eta(U)\xi, \quad \eta(e_5) = 1, \quad g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V),$$

for chosen vector fields U and V on $\overline{M}$.

A simple computation yields

$$\begin{aligned}
&\overline{\nabla}_{e_1} e_1 = -\, e_5, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = 0, \quad \overline{\nabla}_{e_1} e_4 = 0, \quad \overline{\nabla}_{e_1} e_5 = e_1,\\
&\overline{\nabla}_{e_2} e_1 = 0, \quad \overline{\nabla}_{e_2} e_2 = -e_5, \quad \overline{\nabla}_{e_2} e_3 = 0, \quad \overline{\nabla}_{e_2} e_4 = 0, \quad \overline{\nabla}_{e_2} e_5 = e_2,\\
&\overline{\nabla}_{e_3} e_1 = 0, \quad \overline{\nabla}_{e_3} e_2 = 0, \quad \overline{\nabla}_{e_3} e_3 = -e_5, \quad \overline{\nabla}_{e_3} e_4 = 0, \quad \overline{\nabla}_{e_3} e_5 = e_3,\\
&\overline{\nabla}_{e_4} e_1 = 0, \quad \overline{\nabla}_{e_4} e_2 = 0, \quad \overline{\nabla}_{e_4} e_3 = 0, \quad \overline{\nabla}_{e_4} e_4 = -e_5, \quad \overline{\nabla}_{e_4} e_5 = e_4,\\
&\overline{\nabla}_{e_5} e_1 = e_1, \quad \overline{\nabla}_{e_5} e_2 = e_2, \quad \overline{\nabla}_{e_5} e_3 = e_3, \quad \overline{\nabla}_{e_5} e_4 = e_4, \quad \overline{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_U \xi = U - \eta(U)\xi$ for $\xi = e_5$. Moreover, on the similar pattern of Example 4.1, it follows that φ is a killing tensor field. Hence $\overline{M}$ is a five-dimensional Kenmotsu manifold with the killing tensor field. Also, analogous to Example 4.1, it can be seen that the equations (3.2) and (3.5) are satisfied.

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References

[1] A. Bejancu, *Geometry of CR-Submanifolds*, D. Reidel Publ. Co., Dordrecht, Holland, 1986.

[2] A. De, *On Kenmotsu manifold*, Bull. Math. Anal. Appl. **2** (2010), 1–6.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer Verlag, Berlin, New York, 1976.

[4] D. E. Blair, *Almost contact manifolds with Killing structure tensors*, Pacific J. Math. **39**(2) (1971), 285–292.

[5] K. Kenmotsu, *A class of almost contact Riemannian manifold*, Tohoku Math. J. **24** (1972), 93–103.

[6] K. Yano and M. Kon, *On contact CR-submanifolds*, J. Korean Math. Soc. **26**(2) (1989), 231–262.

[7] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific Publishing Co., Singapore, 1984.

[8] M. Atceken, *Contact CR-submanifolds of Kenmotsu manifolds*, Serdica Math. J. **37** (2011), 67–78.

[9] M. Atceken and S. Dirik, *On contact CR-submanifolds of Kenmotsu manifolds*, Acta Univ. Sapientiae Math. **4**(2) (2012), 182–198.

[10] M. Kobayashi, *Semi-invariant submanifolds of a certain class of almost contact manifolds*, Tensor (N.S) **43** (1986), 28–36.

[11] N. Papaghuic, *Semi-invariant submanifolds in a Kenmotsu manifold*, Rend. Mat. Appl. **7**(3) (1983), 607–622.

[12] P. K. Pandey and R. S. Gupta, *Characterization of a slant submanifold of a Kenmotsu manifold*, Novi Sad J. Math. **38**(1) (2008), 97–102.

[13] R. S. Gupta and P. K. Pandey, *Structure on a slant submanifold of a Kenmotsu manifold*, Differential Geometry-Dynamical Systems **10** (2008), 139–147.

[14] U. C. De and P. Majhi, *On Invariant submanifolds of Kenmotsu manifold*, J. Geom. **106** (2015), 109–122.

[15] V. A. Khan, K. A. Khan and Sirajuddin, *Contact CR-warped product submanifolds of Kenmotsu manifolds*, Thai J. Math. **6**(2) (2008), 307–314.

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