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SOME MATHEMATICAL PROPERTIES FOR MARGINAL MODEL OF POISSON-GAMMA DISTRIBUTION

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Abstract. Recently, Casadei [4] provided an explicit formula for statistical marginal model in terms of Poisson-Gamma mixture. This model involving certain polynomials which play the key role in reference analysis of the signal and background model in counting experiments. The principal object of this paper is to present a natural further step toward the mathematical properties concerning this polynomials. We first obtain explicit representations for these polynomials in form of the Laguerre polynomials and the confluent hyper-geometric function and then based on these representations we derive a number of useful properties including generating functions, recurrence relations, differential equation, Rodrigueś formula, finite sums and integral transforms.

1. Introduction

In statistics, marginal models [7] are a technique for obtaining regression estimates in multilevel modeling, also called hierarchical linear models. People often want to know the effect of a predictor/explanatory variable X, on a response variable Y. One way to get an estimate for such effects is through regression analysis. Marginal model is generally compared to conditional model (random-effects model). Casadei [4], (see also [5]) investigated the model representing two independent Poisson processes, labeled as signal and background and both contributing additively to the total number of counted events, is considered from a Bayesian point of view (see [2] and [3]). This is a widely used model for the searches of rare or exotic events in presence of a background source, as for example in the searches performed by high energy physics experiments. The starting point in [4] is the marginal model $p(k|s)$, specifying the

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probability of counting $k \geq 0$ events in the hypothesis that the signal yield is $s \geq 0$ with the assumed knowledge about the background contribution:

$$p(k|s) = \int_0^\infty \mathrm{Poi}(k|s+b)\mathrm{Ga}(b|\alpha,\beta)db, \tag{1.1}$$

where $\mathrm{Ga}(b|\alpha,\beta)$ is a Gamma density of the form

$$p(b) = \mathrm{Ga}(b|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} e^{-\alpha\beta}, \quad \alpha > 0,\, \beta \neq -1, \tag{1.2}$$

and $\mathrm{Poi}(k|s+b)$ is Poisson probability given by the formula (see [4])

$$\mathrm{Poi}(k|s+b) = e^{-s-b} \sum_{n=0}^{k} \frac{s^{k-n} b^n}{n!(k-n)!}. \tag{1.3}$$

Now, in view of (1.2) and (1.3) we find from (1.1) that

$$p(k|s) = \sum_{n=0}^{k} \frac{s^{k-n}\beta^\alpha e^{-s}}{n!(k-n)!\Gamma(\alpha)} \int_0^\infty e^{-(1+\beta)b} b^{\alpha+n-1} db,$$

which on using the Euler's integral [6]

$$\int_0^\infty e^{-at} t^{\nu-1} dt = a^{-\nu}\Gamma(\nu), \quad \nu > 0,$$

yields the marginal model (see [4])

$$p(k|x) = \left(\frac{\beta}{1+\beta}\right)^\alpha e^{-x} f_n(x;\alpha,\beta), \tag{1.4}$$

where

$$f_n(x;\alpha,\beta) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k} \frac{x^{n-k}}{(n-k)!(1+\beta)^k}. \tag{1.5}$$

The model $\mathrm{Poi}(k|s+b)$ is used to compute the Fisher's information (see [4, page 5, (1.4)])

$$I(s) = \mathrm{E}\left[\left(\frac{\partial}{\partial s}\log p(k|s)\right)^2\right] = -\mathrm{E}\left[\left(\frac{\partial^2}{\partial s^2}\mathrm{log} p(k|s)\right)\right], \tag{1.6}$$

and the reference prior [16]

$$\pi(s) \propto |I(s)|^{1/2}.$$

Starting from equation (1.6) and after certain mathematical computations, Casadei [4] derived the following expression for the Fisher's information:

$$I(s) = \left(\frac{\beta}{1+\beta}\right)^\alpha e^{-s} \sum_{n=0}^{\infty} \frac{[f_n(s;\alpha,\beta)]^2}{f_{n+1}(s;\alpha,\beta)} - 1. \tag{1.7}$$

From equation (1.7) one obtains

$$|I(s)|^{1/2} = \left|\left(\frac{\beta}{1+\beta}\right)^\alpha e^{-s} \sum_{n=0}^{\infty} \frac{[f_n(s;\alpha,\beta)]^2}{f_{n+1}(s;\alpha,\beta)} - 1\right|^{1/2}. \tag{1.8}$$

The function $|I(s)|^{1/2}$ has its single maximum at zero, hence a possible definition of the reference prior $\pi(s)$ for the signal is (see [4, page 9, (4.3)])

$$\pi(s) = \frac{|I(s)|^{1/2}}{|I(0)|^{1/2}}. \tag{1.9}$$

Because $\pi(s)$ does not explicitly depend on b (see (1.1)), the marginal posterior is proportional to the product of the reference prior (1.9) and the marginal likelihood (1.4)

$$p(k|x) \propto \left(\frac{\beta}{1+\beta}\right)^{\alpha} e^{-x} f_n(x;\alpha,\beta)\pi(s). \tag{1.10}$$

Casadei [4] derived a number of interesting properties for the polynomials $f_n(x;\alpha,\beta)$ which were useful in his investigation and following the prescription by [16], the reference prior for the signal parameter s is computed from the conditional model (1.1). Clearly, from (1.6), the polynomials $f_n(x;\alpha,\beta)$ play the key role in implementing all results in the work of Casadei [4]. For the evaluation of $f_n(x;\alpha,\beta)$ the author suggested some methods based on the logarithms, because this avoids rounding problems related to expressions featuring very big and very small values. Motivated by the important role of the marginal model $p(k|s)$ in several diverse fields of physics, analysis and statistical methods and the contributions in [4, 5] toward the the marginal model-polynomials $f_n(x;\alpha,\beta)$, this work aims at introducing several representations and properties for the polynomials $f_n(x;\alpha,\beta)$ in terms of known hyper-geometric functions and polynomials, for example, confluent hypergeometric function ${}_1F_1$ and Laguerre polynomials, which will be useful for the evaluation of the marginal model $p(k|s)$.

2. Explicit and Integral Representations

Based on the formulas

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}$$

and

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n,$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ denotes the Pochhammer symbol, the assertion (1.5) can be written in the form

$$f_n(x;\alpha,\beta) = \sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}. \tag{2.1}$$

By exploiting the result [1]

$$(-n)_k = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{2.2}$$

and the definition of the hyper-geometric function ${}_2F_0$ (see [1])

$$ {}_2F_0[a,b;-;x]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n x^n}{n!}, $$

we find from (2.1) that

$$ f_n(x;\alpha,\beta)=\frac{x^n}{n!}{}_2F_0\left[-n,\alpha;-;\frac{-1}{x(1+\beta)}\right]. $$

Now, with help of the representations of the hyper-geometric function ${}_2F_0$ (see [12, page 614])

$$ {}_2F_0[-n,a;-;z]=(a)_n(-z)^n{}_1F_1[-n;1-a-n;-z^{-1}]=n!z^nL_n^{-a-n}(-z^{-1}), $$

we can easily establish the explicit representations

$$ f_n(x;\alpha,\beta)=\frac{(\alpha)_n}{n!(1+\beta)^n}{}_1F_1[-n;1-\alpha-n;x(1+\beta)], \tag{2.3} $$

or equivalently

$$ f_n(x;\alpha,\beta)=\left(\frac{-1}{(1+\beta)}\right)^n L_n^{(-a-n)}(x(1+\beta)), \tag{2.4} $$

where ${}_1F_1$ is the confluent hyper-geometric function [6]

$$ {}_1F_1[a;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(c)_n n!}, \tag{2.5} $$

and $L_n^{(\alpha)}$ is the associated Laguerre polynomials (see [1] or [13])

$$ L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k\Gamma(\alpha+n+1)x^k}{k!(n-k)!\Gamma(\alpha+k+1)}. $$

Since the polynomials $f_n(x;\alpha,\beta)$ can be expressed in terms of representation involving the confluent hypergeometric function ${}_1F_1$ and the Laguerre polynomials $L_n^{(\alpha)}$, the properties of these function and polynomials assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties for the polynomials $f_n(x;\alpha,\beta)$. In this work formula (2.3) will play the key role in obtaining a number of main results for the polynomials $f_n(x;\alpha,\beta)$. Next, according to the relation between Laguerre polynomials $L_n^{(\alpha)}$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [1, page 294, (35)]

$$ L_n^{(\alpha)}=\lim_{\lambda\to\infty}P_n^{(\alpha,\lambda)}\left(1-\frac{2x}{\lambda}\right) $$

and the assertion (2.4), we can obtain the explicit relation

$$ f_n(x;\alpha,\beta)=\left(\frac{-1}{(1+\beta)}\right)^n\lim_{\lambda\to\infty}P_n^{(-\alpha-n,\lambda)}\left(1-\frac{2x(1+\beta)}{\lambda}\right). $$

Further, the Laguerre polynomials have the following asymptotic representation which describe their behavior for large value of the degree n [8, page 87, (4.22.18)]; see also [9]:

$$L_n^{(\alpha)}(x) \approx \frac{\Gamma(\alpha+n+1)}{n!} e^{\frac{x}{2}} (Nx)^{\frac{-\alpha}{2}} J_\alpha \left(2\sqrt{(Nx)}\right), \quad n \to \infty, N = n + \frac{\alpha+1}{2}. \tag{2.6}$$

In view of the explicit representation (2.4) it follows from (2.6) that

$$f_n(x;\alpha,\beta) \approx \left(\frac{-1}{(1+\beta)}\right)^n \frac{\Gamma(1-\alpha)}{n!} e^{\frac{x(1+\beta)}{2}} (Nx(1+\beta))^{\frac{\alpha+n}{2}} J_{-\alpha-n}\left(2\sqrt{(Nx(1+\beta))}\right),$$

$n \to \infty$, $N = \frac{n-\alpha+1}{2}$. From (2.3), we can easily seen that

$$f_n(0;\alpha,\beta) = \frac{(\alpha)_n}{n!(1+\beta)^n} \tag{2.7}$$

and

$$f_{n+1}(x;\alpha,\beta) = \frac{(\alpha)_{n+1}}{(n+1)!(1+\beta)^{n+1}}. \tag{2.8}$$

Formulas (2.7) and (2.8) are useful in computing the reference prior $\pi(s)$ in (1.9). It is often convenient to identify the various special functions and polynomials with contour integrals along certain paths in the complex plane. These integrals provide recursion formulas, asymptotic forms, and analytic continuations of the special functions. Also, they are sometimes used as definitions of special functions and polynomials . Now, we consider some integral representations for the polynomials $f_n(x;\alpha,\beta)$. To obtain integral representations, we first recall the results (see [1, page 300, (9.13) and (9.17)], [6, (6.11.1)(3)])

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt, \tag{2.9}$$

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^x x^{\left(\frac{1-c}{2}\right)} \int_0^1 e^{-t} t^{\frac{1}{2}(c-1)-a} J_{c-1}(2\sqrt{(xt)}) dt \tag{2.10}$$

and

$${}_1F_1[a;c;x] = \frac{\Gamma(\gamma)\Gamma(1-a)}{2\pi i \Gamma(c-a)} \oint_\gamma e^{xs} \left(\frac{s}{s-1}\right)^a (1-s)^{c-1} \frac{ds}{s}, \tag{2.11}$$

where c is a positive integer and the contour γ starts and ends at the point $s = 1$ on the $s - axis$ and encircles the origin in a positive sense and that Re(c) > Re(a). Also, a fourth representation can be obtained from equation (6.11.1) (7) of [6], for Re(c) > 0, $\gamma > 1$ and $a \neq 1, 2, 3, \ldots, c-1$. This is achieved with $b = c = n+1, n = 0, 1, \ldots,$ in (6.11.2) (6) of [6], where the integrand is a one-valued function of the parameter s and the path of integration may be replaced by a contour, for instance a circle $|s| = \rho > 1$. This representation is given by (see [6])

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{2\pi i x^{c-1}} \oint_\gamma e^{xs} \left(\frac{s}{s-1}\right)^a \frac{ds}{s^c}. \tag{2.12}$$

Directly from the results (2.9), (2.10), (2.11) and (2.12) and based on the definition (2.3), we can establish the following integral representations:

$$
\begin{aligned}
f_n(x;\alpha,\beta) =& \frac{(-1)^n}{n!(1+\beta)^n\Gamma(-n)}\int_0^1 e^{x(\beta+1)t}t^{-(n+1)}(1-t)^{-\alpha}dt,\\
f_n(x;\alpha,\beta) =& \frac{(-1)^n e^{x(1+\beta)}[x(1+\beta)]^{\frac{\alpha+n}{2}}}{n!(1+\beta)^n\Gamma(-n)}\int_0^1 e^{-t}t^{\frac{n-\alpha}{2}}J_{-(\alpha+n)}(2\sqrt{(x(1+\beta)t)})dt,\\
f_n(x;\alpha,\beta) =& \frac{(\alpha)_n\Gamma(\gamma)\Gamma(1+n)}{2\pi i\ n!(1+\beta)^n\Gamma(1-\alpha)}\oint_\gamma e^{x(1+\beta)s}(s-1)^{-\alpha}\frac{ds}{s^{n+1}}
\end{aligned}
$$

and

$$
f_n(x;\alpha,\beta) = \frac{(\alpha)_n\Gamma(1-\alpha-n)}{2\pi i\ n!(1+\beta)^{-\alpha}x^{-\alpha-n}}\oint_\gamma e^{x(1+\beta)s}(s-1)^n\frac{ds}{s^{1-\alpha}},
$$

respectively. By using the previous explicit and integral representations the computing of the conditional model will be more easily. In this regard by virtue of the results (2.7) and (2.8), in conjunction with (1.7), we find that

$$
\begin{aligned}
I(0) =& \frac{(1+\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)}\left(\frac{\beta}{1+\beta}\right)^\alpha\sum_{n=0}^\infty\frac{(\alpha)_n(\alpha)_n(2)_n}{n!(\alpha+1)_n(1)_n}\left(\frac{1}{1+\beta}\right)^n-1\\
=& \frac{(1+\beta)\Gamma(\alpha)}{\Gamma(\alpha+1)}\left(\frac{\beta}{1+\beta}\right)^\alpha {}_3F_2\left[\alpha,\alpha,2;\alpha+1,1;\frac{1}{(1+\beta)}\right]-1,
\end{aligned}
\tag{2.13}
$$

where ${}_3F_2$ is special case of the generalized hypergeometric series ${}_pF_q$ (see [1]). Hence, from the assertions (2.13) and (2.3), we find the following elegant explicit representation for the marginal posterior defined by (1.10):

$$
\begin{aligned}
p(k|x) \propto& \left(\frac{\beta}{1+\beta}\right)^\alpha\frac{(\alpha)_n e^{-x}}{n!(1+\beta)^n}{}_1F_1[-n;1-\alpha-n;x(1+\beta)]\\
&\times\frac{\left|\left(\frac{\beta}{1+\beta}\right)^\alpha e^{-x}\sum_{n=0}^\infty\frac{(n+1)(\alpha)_n({}_1F_1[-n;1-\alpha-n;x(1+\beta)])^2}{n!(\alpha+n)_1(\beta+1)^{n-1}{}_1F_1[-n-1;-\alpha-n;x(1+\beta)]}-1\right|^{1/2}}{\left|\frac{\beta^\alpha}{\alpha(1+\beta)^{\alpha-1}}{}_3F_2[\alpha,\alpha,2;\alpha+1,1;1/(1+\beta)]-1\right|^{1/2}}.
\end{aligned}
$$

3. Generating Functions

A generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a power series. The sum of this infinite series is the generating function. Generating functions are often expressed in closed form (rather than as a series), by some expression involving operations defined for formal series. These expressions in terms of the indeterminate x may involve arithmetic operations, differentiation with respect to x and composition with (i.e., substitution into) other generating functions; since these operations are also defined for functions, the result looks like a function of x. Indeed, the closed form expression can often be interpreted as a function that can be evaluated at (sufficiently small) concrete values of x, and which has the formal series as its series expansion. Also, the generating functions offer a direct way to investigate the properties of the polynomials they define. Directly from (2.1) of the prececding section we obtain

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k} t^n}{k!(n-k)!(1+\beta)^k}.$$

On replacing n by $n+k$, we obtain

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{n} \frac{(\alpha)_k t^k}{k!(1+\beta)^k}.$$

Hence the polynomials $f_n(x;\alpha,\beta)$ have the following generating relation:

$$e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n. \tag{3.1}$$

A set of other generating functions for these polynomials is easily obtained. Let λ be arbitrary and proceed as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda)_n f_n(x;\alpha,\beta)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_k(\lambda)_n x^{n-k}t^n}{k!(n-k)!(1+\beta)^k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha)_k(\lambda)_{n+k} x^n t^{n+k}}{k!n!(1+\beta)^k}\\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k(\lambda)_k t^k}{k!(1+\beta)^k} \sum_{n=0}^{\infty} \frac{(\lambda+k)_n (xt)^n}{n!}.\end{aligned}$$

We thus arrive at the generating function

$$\sum_{n=0}^{\infty}(\lambda)_n f_n(x;\alpha,\beta)t^n = (1-xt)^{-\lambda}{}_2F_0\left[\alpha,\lambda;-;\frac{t}{(1+\beta)(1-xt)}\right].$$

The following two formulas are well-known consequences of the derivative operator $\hat{D}_x = \frac{\partial}{\partial x}$ and the integral operator $\hat{D}_x^{-1}$ (see [10]):

$$\hat{D}_x^n x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}x^{\lambda-n}, \quad \hat{D}_x^{-n} x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n+1)}x^{\lambda+n},$$

$m \in \mathbb{N}\cup\{0\}, \lambda \in \mathbb{C}-\{-1,-2,\dots\}$. Since

$$\hat{D}_x^k x^n = \frac{n! x^{n-k}}{(n-k)!},$$

formula (2.1) yields the operational relation

$$f_n(x;\alpha,\beta) = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(\alpha)_k \hat{D}_x^k x^n}{k!(1+\beta)^k} = \frac{1}{n!}\left(1-\frac{\hat{D}_x}{1+\beta}\right)^{-\alpha} x^n. \tag{3.2}$$

On multiplying both sides of (3.2) by t^n and taking the sum, we then get the generating relation

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = \left(1 - \frac{\hat{D}_x}{1+\beta}\right)^{-\alpha} e^{xt}.$$

On the other hand, since

$$\hat{D}_x^k x^{\alpha+k-1} = (\alpha)_k x^{\alpha-1},$$

we get from (2.1) the operational relation

$$f_n(x;\alpha,\beta) = \frac{x^{n-\alpha+1}}{n!}\left(1 - \frac{\hat{D}_x x}{x(1+\beta)}\right)^n x^{\alpha-1}.$$

Now, we can easily derive the following generating relation:

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^n = x^{1-\alpha} exp\left[x\left(1 - \frac{t\hat{D}_x x}{x(1+\beta)}\right)\right] x^{\alpha-1}.$$

From (2.1), we can easily derive the $m-th$ partial derivative of $f_n(x;\alpha,\beta)$ with respect to x as follows:

$$\hat{D}_x^m f_n(x;\alpha,\beta) = f_n^m(x;\alpha,\beta) = \sum_{k=0}^{n-m} \frac{(\alpha)_k x^{n-k-m}}{k!(n-k-m)!(1+\beta)^k} = f_{n-m}(x;\alpha,\beta). \tag{3.3}$$

Hence, from assertions (3.1) and (3.3), we get the result

$$e^{xt}\left(1 - \frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=0}^{\infty} f_{n-m}(x;\alpha,\beta)t^{n-m}. \tag{3.4}$$

Remark 3.1. If in the generating functions (3.1) and (3.4), we let $t = 1$, we find that

$$\sum_{n=0}^{\infty} f_n(x;\alpha,\beta) = \sum_{n=0}^{\infty} f_{n-m}(x;\alpha,\beta) = e^x \left(\frac{1+\beta}{\beta}\right)^{\alpha},$$

which are the properties 1 and 3 of the polynomials in (2.1) derived by Casadei (see [4, pages 6–7]).

Remark 3.2. From (3.3), we have

$$f_n^m(x;\alpha,\beta) = \sum_{k=0}^{n-m} \frac{(-1)^{k+m}(-n)_{k+m}(\alpha)_k x^{n-k-m}}{n!k!(1+\beta)^k}. \tag{3.5}$$

We know that [13]

$$(-n)_{k+m} = \begin{cases} 0, & \text{if } k > n \text{ or } m > n, \\ \dfrac{(-1)^{k+m} n!}{(n-k-m)!}, & \text{if } 0 \le k+m \le n. \end{cases} \tag{3.6}$$

Hence, equation (3.5), in conjunction with (3.6), gives

$$f_n(x;\alpha,\beta) = \begin{cases} 0, & \text{if } m > n, \\ f_n^m(x;\alpha,\beta) = f_{n-m}(x;\alpha,\beta), & \text{if } 0 \le m \le n, \\ 1, & \text{if } n = m, \end{cases}$$

which is the tired property for the polynomials in (2.1) proved by Casadei [4, page 6, (2.7)]. Note that the proofs of the properties 1, 2 and 3 above are very short in compare with their proofs in [4]. In fact these three properties have their origin in the properties of Laguerre polynomials and the confluent hyper-geometric series ${}_1F_1$ (see [13, pages 200–213]).

4. Recurrence Relations and Differential Equation

First, in view of definition (2.1), we find that

$$\hat{D}_x f_n^m(x;\alpha,\beta) = f_{n-1}(x;\alpha,\beta) \tag{4.1}$$

and

$$\hat{D}_x^{-1} f_n^m(x;\alpha,\beta) = f_{n+1}(x;\alpha,\beta).$$

Secondly, differentiating both the sides of (3.1) with respect to t, we get

$$xe^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} + \left(\frac{\alpha}{1+\beta-t}\right)e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} = \sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n-1},$$

or

$$\begin{aligned} &x(1+\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n - x\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^{n+1} + \alpha\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n \\ =&(1+\beta)\sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n-1} - \sum_{n=1}^{\infty} f_n(x;\alpha,\beta)\ nt^{n+1}, \end{aligned}$$

or

$$\begin{aligned} &x(1+\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n - x\sum_{n=0}^{\infty} f_{n-1}(x;\alpha,\beta)\ t^{n} + \alpha\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)\ t^n \\ =&(1+\beta)\sum_{n=0}^{\infty} f_{n+1}(x;\alpha,\beta)\ (n+1)t^{n-1} - \sum_{n=0}^{\infty} f_{n-1}(x;\alpha,\beta)\ (n-1)t^{n}. \end{aligned}$$

Equating the coefficients of t^n from both sides, we find

$$[x(1+\beta)+\alpha]f_n(x;\alpha,\beta)-(x-n+1)f_{n-1}(x;\alpha,\beta)-(1+\beta)(n+1)f_{n+1}(x;\alpha,\beta) = 0. \tag{4.2}$$

From [13, Section 48, (15), (18) and (20)], using $p = q = 1$, $\alpha_1 = -n$, $\beta_1 = 1-\alpha-n$, $x \mapsto x(1+\beta)$, we obtain

$$\begin{aligned} \alpha_1 F_1[-n;1-\alpha-n;x(1+\beta)] =& - n_1F_1[-n+1;1-\alpha-n;x(1+\beta)] \\ &+ (\alpha+1)_1F_1[-n;1-\alpha-n;x(1+\beta)], \end{aligned} \tag{4.3}$$

(4.4)

$$\begin{aligned} &[x(1+\beta)-n]_1F_1[-n;1-\alpha-n;x(1+\beta)] \\ =& - n_1F_1[-n;1-\alpha-n;x(1+\beta)] - \frac{x(1+\beta)(\alpha-1)}{(1-\alpha-n)}{}_1F_1[-n;2-\alpha-n;x(1+\beta)], \end{aligned}$$

$$
\begin{aligned}
(4.5)\qquad {}_1F_1[-n;1-\alpha-n;x(1+\beta)] =& {}_1F_1[-n-1;1-\alpha-n;x(1+\beta)] \\
&+\frac{x}{(1-\alpha-n)}{}_1F_1[-n;2-\alpha-n;x(1+\beta)].
\end{aligned}
$$

Since (see (2.3))

$$
{}_1F_1[-n;1-\alpha-n;x(1+\beta)] = \frac{n!(1+\beta)^n}{(\alpha)_n} f_n(x;\alpha,\beta),
$$

equations (4.3), (4.4) and (4.5) may be converted into the mixed recurrence formulas

$$
\begin{aligned}
f_n(x;\alpha,\beta) =& \left(1+\frac{n}{\alpha}\right) f_n(x;\alpha+1,\beta) - \frac{n}{\alpha} f_n(x;\alpha,\beta),\\
f_n(x;\alpha,\beta) =& \frac{n}{(n-x(1+\beta))} f_{n-1}(x;\alpha,\beta) + \frac{x(1+\beta)(\alpha-1)}{(n-x(1+\beta))(1-\alpha-n)} f_n(x;\alpha-1,\beta),\\
f_n(x;\alpha,\beta) =& f_{n+1}(x;\alpha,\beta) + \frac{x}{(1-\alpha-n)} f_n(x;\alpha-1,\beta).
\end{aligned}
$$

Next, we derive the differential equation of the polynomials $f_n(x;\alpha,\beta)$. Our starting point is the recurrence formula (4.2). We have from (4.2)

$$
\begin{aligned}
(4.6)\qquad & x(1+\beta)f_n(x;\alpha,\beta) + \alpha f_n(x;\alpha,\beta) - x f_{n-1}(x;\alpha,\beta) + (n-1) f_{n-1}(x;\alpha,\beta) \\
& - (1+\beta)(n+1) f_{n+1}(x;\alpha,\beta) = 0.
\end{aligned}
$$

Using (4.1), equation (4.6) yields

$$
\begin{aligned}
(4.7)\qquad & x(1+\beta)f_n(x;\alpha,\beta) + \alpha f_n(x;\alpha,\beta) - x\frac{\partial}{\partial x} f_n(x;\alpha,\beta) + (n-1)\frac{\partial}{\partial x} f_n(x;\alpha,\beta) \\
& - (1+\beta)(n+1) f_{n+1}(x;\alpha,\beta) = 0.
\end{aligned}
$$

Next, on differentiating equation (4.7) with respect to x and simplify we obtain the following second order differential equation for the polynomials $f_n(x;\alpha,\beta)$:

$$
(n-x-1)\frac{\partial^2}{\partial x^2} f_n(x;\alpha,\beta) + (x(1+\beta)+\alpha-1)\frac{\partial}{\partial x} f_n(x;\alpha,\beta) - n(1+\beta) f_n(x;\alpha,\beta) = 0.
$$

5. Rodrigueś-Type Formula

From (2.4), we have

$$
(5.1)\qquad f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \sum_{k=0}^{n} \frac{(1-\alpha-n)_n[-x(1+\beta)]^k}{k!(n-k)!(1-\alpha-n)_k}.
$$

Since

$$
\hat{D}_x^{n-k} x^{-\alpha} = \frac{(1-\alpha-n)_n}{(1-\alpha-n)_k} x^{(-\alpha-n+k)},
$$

equation (5.1) can be written in the form

$$\begin{aligned} f_n(x;\alpha,\beta) &= \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!}\sum_{k=0}^{n}\frac{(-1)^k(1+\beta)^k}{k!(n-k)!}\hat{D}_x^{n-k}x^{-\alpha} \\ &= \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1+\beta)^k\hat{D}_x^{n-k}x^{-\alpha}. \end{aligned}$$

Again, since

$$\hat{D}_x^k e^{-(1+\beta)x} = (-1)^k(1+\beta)^k e^{-(1+\beta)x},$$

we may conclude that

$$f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!}e^{(1+\beta)x}\sum_{k=0}^{n}\binom{n}{k}[\hat{D}_x^{n-k}x^{-\alpha}][\hat{D}_x^k e^{-(1+\beta)x}]. \tag{5.2}$$

Therefore, by Leibnitz theorem, equation (5.2) can be written in the following interesting Rodrigueś-type formula:

$$f_n(x;\alpha,\beta) = \left(\frac{-1}{1+\beta}\right)^n \frac{x^{\alpha+n}}{n!}e^{(1+\beta)x}\hat{D}_x^n[x^{-\alpha}e^{-(1+\beta)x}].$$

6. Finite Sums and Integral Transforms

Using the result

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n+k,k),$$

we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(\alpha-\lambda)_{n-s}}{(n-s)!}(1+\beta)^{n-s}f_s(x;\lambda,\beta)t^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(\alpha-\lambda)_n}{n!}(1+\beta)^n f_s(x;\lambda,\beta)t^{n+s} \\ &= \left(1-\frac{t}{1+\beta}\right)^{\lambda-\alpha}\sum_{s=0}^{\infty}f_s(x;\lambda,\beta)t^s. \end{aligned}$$

Now, employing the generating relation (3.1) and comparing the coefficients of t^n in the resulting expression, we get the following finite sum:

$$\sum_{s=0}^{n}\frac{(\alpha-\lambda)_{n-s}}{(n-s)!}(1+\beta)^{n-s}f_s(x;\lambda,\beta) = f_n(x;\alpha,\beta).$$

Similarly, one can derive the following result:

$$\sum_{s=0}^{n}\frac{(\alpha+s)_{n-s}(1+y)^n}{(n-s)!}(1+\beta)^{n-s}f_s(x;\alpha,\beta)\left(\frac{y}{1-y}\right)^s = f_n\left(\frac{xy(1+\beta)}{\beta-y};\alpha,\beta\right).$$

Since

$$\begin{aligned}\sum_{p=0}^{\infty} f_p(x+y;\alpha+\gamma,\beta)t^n &= e^{(x+y)t}\left(1-\frac{t}{1+\beta}\right)^{-\alpha-\gamma}\\ &= e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\alpha} e^{yt}\left(1-\frac{t}{1+\beta}\right)^{-\gamma}\\ &= \sum_{p=0}^{\infty} f_p(y;\gamma,\beta)\sum_{n=0}^{\infty} f_n(x;\alpha,\beta)t^{p+n}.\end{aligned}$$

Now, on letting $p \mapsto p-n$ and comparing the coefficients of t^p, we get the desired result

$$f_p(x+y;\alpha+\gamma,\beta) = \sum_{n=0}^{p} f_n(x;\alpha,\beta)f_{p-n}(y;\gamma,\beta)t^{p+n}.$$

Similarly, from the equation

$$\sum_{p=0}^{\infty} f_p(x;\alpha,\beta)t^n = e^{xt}\left(1-\frac{t}{1+\beta}\right)^{-\gamma}\left(1-\frac{t}{1+\beta}\right)^{\gamma-\alpha},$$

we can show that

$$f_p(x;\alpha,\beta) = \sum_{p=0}^{n}\frac{(\alpha-\gamma)_{n-p}f_p(x;\lambda,\beta)}{(p-n)!(1+\beta)^{p-n}}.$$

Next, we turn to some integral transforms for the polynomials $f_p(x;\alpha,\beta)$. In view of the definition (3.1), we obtain

$$\int_0^{\infty} e^{-t}t^{\lambda-1}f_n(xt;\alpha,\beta)dt = \sum_{k=0}^{n}\frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}\int_0^{\infty} e^{-t}t^{\lambda+n-k-1}dt.$$

Now, on using the formulas (1.5) and (2.2) and the result $(a)_{n-k} = \frac{(-1)^k(a)_n}{(1-a-n)_k}$, and considering the definition of the Gaussian hyper-geometric function ${}_2F_1$ (see [15])

$$(6.1)\qquad {}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,$$

we get the integral transform

$$\int_0^{\infty} e^{-t}t^{\lambda-1}f_n(xt;\alpha,\beta)dt = \frac{\Gamma(\lambda+n)}{n!}{}_2F_1\left[-n,\alpha;1-\lambda-n;\frac{1}{x(1+\beta)}\right].$$

In view of (2.1), we get

$$\begin{aligned}\int_0^{t}(t-z)^{\lambda-1}z^{\alpha-\lambda+n-1}f_n(x/z;\alpha,\beta)dt =& \sum_{k=0}^{n}\frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k}\\ &\times\int_0^{t}(t-z)^{\lambda-1}z^{\alpha-\lambda+n+k-1}dt.\end{aligned}$$

Putting $(t-z)=t(1-p)$, we get

$$\int_0^t (t-z)^{\lambda-1} z^{\alpha-\lambda+n-1} f_n(x/z;\alpha,\beta)dt$$
$$=\sum_{k=0}^{n} \frac{(\alpha)_k x^{n-k}}{k!(n-k)!(1+\beta)^k} t^{\alpha+\lambda+k} \int_0^1 p^{\alpha-\lambda+k-1}(1-p)^{\lambda-1}dp.$$

Now, projection integral transform would occur if we use the definition of Beta function

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$

and considering (2.5) and this asserts

$$\int_0^t (t-z)^{\lambda-1} z^{\alpha-\lambda+n-1} f_n(x/z;\alpha,\beta)dt = f_n(x/t;\alpha-\lambda,\beta).$$

7. Conclusions

In the previous sections we etablished a number of properties and representations for the polynomials $f_n(x;\alpha,\beta)$, which are useful tools for computing the marginal model $p(k|x)$, the Fisher's information $I(x)$ and the reference prior $\pi(x)$. In this regard, if we make use of the series representation (2.5), the assertion (1.6) gives us

$$p(k|x) = \left(\frac{\beta}{1+\beta}\right)^{\alpha} e^{-x} \frac{(\alpha)_n}{n!(1+\beta)^n} {}_1F_1[-n;1-\alpha-n;x(1+\beta)] \tag{7.1}$$
$$=\frac{\beta^\alpha(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^k(-n)_s(1+\beta)^s x^{s+k}}{k!s!(1-\alpha-n)_s}.$$

On letting $s \mapsto s-k$ in (7.1) and using the following relations [13]:

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n} \quad \text{and} \quad (-n)_k = \frac{(-1)^k n!}{(n-k)!},$$

we obtain

$$p(k|x) = \frac{\beta^\alpha(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{s=0}^{\infty} \frac{(-n)_s[x(1+\beta)]^s}{s!(1-\alpha-n)_s} \sum_{k=0}^{\infty} \frac{(-s)_k(\alpha+n-s)_k}{k!(1+n-s)_k}\left(\frac{1}{1+\beta}\right)^k.$$

According to the definition of Gaussian hyper-geometric series in (6.1), we get

$$p(k|x) = \frac{\beta^\alpha(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{s=0}^{\infty} \frac{(-n)_s[x(1+\beta)]^s}{s!(1-\alpha-n)_s} {}_2F_1\left[-s,\alpha+n-s;1+n-s;\frac{1}{(1+\beta)}\right].$$

On other hand, by letting $k \mapsto k-s$ in (7.1) and proceeding in the manner described above, it is not difficult to obtain from the series expansion

$$p(k|x) = \frac{\beta^\alpha(\alpha)_n}{n!(1+\beta)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{x^k}{k!} {}_2F_1\left[-n,-k;1-\alpha-n;1+\beta\right].$$

We conclude this investigation by remarking that the schema suggested in the derivation of the results in this work can be applied to find other needed properties for the polynomials defined in (2.1). Therefore, the properties of the polynomials $f_n(x;\alpha,\beta)$ assume noticeable importance.

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