Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 119–129.

CHARACTERIZATION OF GRAPHS OF CONNECTED DETOUR NUMBER 2

GASHAW A. MOHAMMEDSALEH[1]

Abstract. Let $G = (V, E)$ be a connected graph of order $P(G) \geq 2$. The connected detour number of G, denoted $cdn(G)$, is introduced and studied by A. P. Santhakumaran and S. Athisayanathan [7]. In this paper, we characterize connected graph G of $cdn(G) = 2$ and of detour diameter $D(G) = 5, 6$.

1. Introduction

Let $G = (V, E)$ be a connected simple graph of p vertices and q edges. We assume that $p \geq 2$ and it is finite. For $u, v \in V(G)$, the length of a maximum $u - v$ path is called ***detour distance*** between u and v, and denoted by $D(u, v)$. A $u - v$ path of length $D(u, v)$ is called **u-v *detour***. For a vertex $v \in V$, the ***detour eccentricity*** $e_D(v)$ is defined by:

$$e_D(v) = \max \{D(u, v) : u \in V\},$$
$$\operatorname{diam}_D(G) = \max \{e_D(v) : \ v \in V(G)\}.$$

A vertex $w \in V(G)$ is said ***to lie*** on a $u - v$ detour Q, if w is a vertex of $V(Q)$ including u and v. A ***detour set*** (denoted d.s.) of G is a subset S of $V(G)$ such that every vertex v of G lies on $x - y$ detour for some $x, y \in S$. The ***detour number*** of G, denoted $dn(G)$, is defined by:

$$dn(G) = \min \{|S| : S \text{ is a detour set of } G\}.$$

A ***detour basis*** of G is a detour set of order $dn(G)$. If S is a detour set of G and the induced subgraph $G[S]$ is connected, then S is called ***connected detour set***

Key words and phrases. Detour distance, detour number, connected detour number.

2010 *Mathematics Subject Classification.* Primary: 05C30. Secondary: 05C31, 05C38.
DOI 10.46793/KgJMat2301.119M
Received: January 26, 2020.
Accepted: July 21, 2020.

(denoted c.d.s.) of G. The ***connected detour number*** of G, denoted $cdn(G)$, is defined as:

$$cdn\,(G) = \min\{|S| : S \text{ is a connected detour set of } G\}.$$

A connected detour basis of G is a connected detour set of G of order $cdn(G)$. For the definitions of the concepts not given here, we refer to [1,3–7]. There are many research on connected detour number and edge detour graphs (see [8–10]). Ahmed and Ali [2], determined detour number for three special classes of graphs G, namely, unicyclic graphs, bicyclic graphs, and cog-graphs for C_p, K_p and $K_{m,n}$. In [7], the authors A. P. Santhakumaran and S. Athisayanathan characterized connected graphs G of $cdn\,(G) = 2$ and $D\,(G) \leq 4$. In this paper, we characterize graphs G of $D\,(G) = 5$ and 6 for which $cdn\,(G) = 2$.

2. Characterizations of Graphs G with $D(G) = 5$ and $cdn(G) = 2$

We start with the following proposition for graphs G having $cdn(G) = 2$.

Proposition 2.1. *Let G be a connected graph of order $P(G) \geq 3$. If $cdn\,(G) = 2$, then G contains neither end-vertices nor cut-vertices.*

Proof. (1) If v is an end-vertex of G and u is the vertex adjacent to v, then v is a cut-vertex, and $G - \{u, v\}$ contains at least one vertex, say w. Since u and v are in every c.d.s. of G; and uv is the only $u - v$ detour, then $\{u, v\}$ is not a c.d.s. of G [7]. Thus, $cdn(G) \geq 3$, contradicting the hypothesis. Therefore, G does not contain end-vertices.

(2) Now, assume that G contains a cut-vertex x and $\{x, y\}$ is a connected detour basis of G. By the proof of part (1), G contains no end-vertices, so y is not end-vertex. Let H_1 and H_2 be components of $G - \{x\}$, and let $y \in V(H_1)$. Since $P(G) \geq 3$, then H_2 contains at least one vertex. Clearly, every $x - y$ detour does not contain vertices from H_2, contradicting the definition of d.s. Thus, G does not contain cut-vertices. □

Now we proceed to find graphs G with detour diameter $D\,(G) = 5$ for which $cdn\,(G) = 2$.

Theorem 2.1. *Let G be a connected graph of $P(G) \geq 6$ and with $D\,(G) = 5$. Then, $cdn\,(G) = 2$ if and only if G is a cycle graph C_6, with or without any number of chords, or like the graph G_i $(i = 1, 2)$ depicted in Figure 1.*

Proof. It is easy to verify that for C_6 and for each G_i (i=1,2) $D(C_6) = D(G_i)$=5 and $cdn\,(C_6) = cdn\,(G_i) = 2$, in which a, b is a detour basis of G_i.

To prove the converse, let G be a connected graph of $P(G) \geq 6$ and with $D\,(G) = 5$, $cdn\,(G) = 2$. Then, by Proposition 2.1, G does not contain end-vertices and cut-vertices. Since $D\,(G) = 5$ and G is connected, then the circumference of G (denoted by $cir(G)$) is $3 \leq cir(G) \leq 6$. Therefore, we shall consider four cases for $cir(G)$.

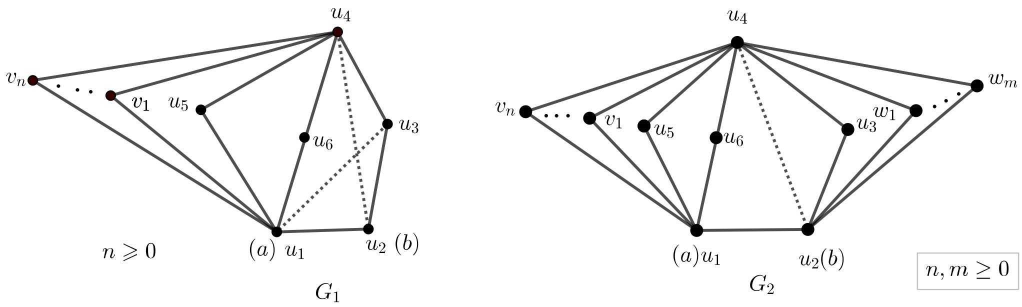


Figure 1.

Case (1). Let $cir\,(G) = 3$ and $P = (v_1, v_2, \ldots, v_6)$ is a $v_1 - v_6$ detour diameter in G (see Figure 2).

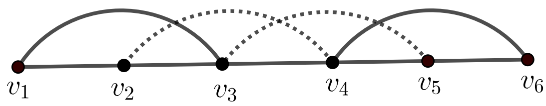


Figure 2. P for $cir\,(G) = 3$.

Then v_1 is not adjacent to v_4, v_5, v_6; and v_6 is not adjacent to v_2 and v_3. Moreover, v_1 and v_6 are not adjacent to any vertex other than $V(P)$. Since $\deg v_i = 2$, $(i = 1, \ldots, 6)$, then v_1 must be adjacent to v_3, and v_6 must be adjacent to v_4. By Proposition 2.1, G contains no cut-vertices, therefore there is either a $v_2 - v_5$ path in G, or $v_2 - v_4$ path and $v_3 - v_5$ path. Each of the two possibilities implies the existence of a cycle of length ≥ 6 in G, contradicting our assumption. Thus, in this case there is no graph that fulfills the required conditions.

Case (2). Let $cir\,(G) = 4$, and $P = (v_1, v_2, \ldots, v_6)$ be a $v_1 - v_6$ detour diameter of G (see Figure 3).

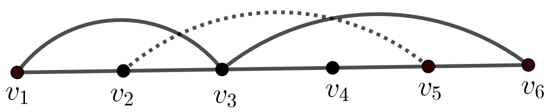


Figure 3.

Then v_1 is not adjacent to v_5 and v_6; and v_6 is not adjacent to v_2. Thus, v_1 is adjacent to v_3 or v_4, and v_6 is adjacent to v_3 or v_4. Therefore, we consider four subcases.

(a) If $v_1v_3, v_6v_4 \in E\,(G)$, then, as explained in case (1), $cir(G) \geq 6$, a contradiction.

(b) If $v_1v_3, v_6v_3 \in E(G)$, then either there is in G a $v_2 - v_4$ path or $v_2 - v_5$ path. Each of the two possibilities produces a graph G having $cir(G) \geq 5$; a contradiction.

(c) If $v_1v_4, v_6v_4 \in E(G)$, then, as in subcase (b), we arrive to a contradiction.

(d) If $v_1v_4, v_6v_3 \in E(G)$, then G contains the 6-cycle $(v_1, v_2, v_3, v_6, v_5, v_4, v_1)$ and so $cir(G) \geq 6$, a contradiction.

Therefore, in case (2) there is no graph that satisfies the required conditions of the theorem.

Case (3). Let $cir\,(G) = 5$ and $P = (v_1, v_2, \ldots, v_6)$ is a $v_1 - v_6$ detour diameter, then v_1 is not adjacent to v_6, and each of v_1, v_6 is not adjacent to any vertex not in $V(P)$. By Proposition 2.1, $\deg v_i \geq 2$ $(i = 1, 6)$. Therefore, we have consider the following nine subcases.

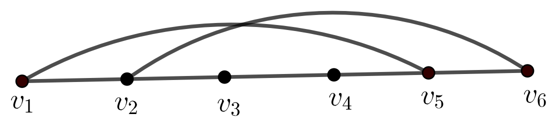


Figure 4.

(a) If $v_1v_5, v_2v_6 \in E(G)$, then such graph is like G_1 with $n = 0$ and without the edges u_1u_3, u_2u_4, in Figure 1.

(b) If $v_1v_5, v_3v_6 \in E(G)$, then such graph is like G_1 with $n = 0$ and without the edges u_2u_4.

(c) If $v_1v_5, v_4v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_5, v_6, v_4, v_3, v_2, v_1)$, contradicting our assumption.

(d) If $v_1v_4, v_2v_6 \in E(G)$, then G is like G_2 with $m = n = 0$ and without the edge u_1u_3 and u_2u_4.

(e) If $v_1v_4, v_3v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_2, v_3, v_6, v_5, v_4, v_1)$, contradicting our assumption.

(f) If $v_1v_4, v_4v_6 \in E(G)$, then by Proposition 2.1, there must be a $v_3 - v_5$ path or $v_2 - v_5$ path. If G contains $v_3 - v_5$ path, then G contains a cycle of length $\geq$6, a contradiction. Now, assume that G contains a $v_2 - v_5$ path, of length $\geq$ 2 then G contains $v_2v_5 \in E(G)$, then G is like G_2 in Figure 1 with $m = n = 0$.

(g) If $v_1v_3,\ v_2v_6 \in E(G)$, then G contains the 6-cycle $(v_1, v_3, v_4, v_5, v_6, v_2, v_1)$, contradicting the assumption.

(h) If $v_1v_3, v_3v_6 \in E(G)$, then as in subcase (f) either G is like G_2 with $m = n = 0$, or $cir(G) \geq 6$.

(i) If $v_1v_3, v_4v_6 \in E(G)$, then by Proposition 2.1, either G contains $v_2 - v_5$ path, or $v_2 - v_4$ path and $v_3 - v_5$ path, see Figure 5.

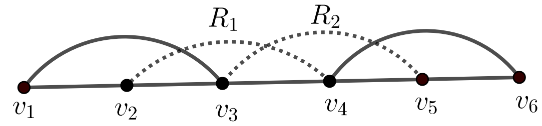


Figure 5.

If G contains a $v_2 - v_5$ path Q, then G contains a cycle $(v_1, v_3, v_4, v_6, v_5, Q, v_2, v_1)$, of length ≥ 6, a contradiction. If G contains a $v_2 - v_4$ path R_1 and $v_3 - v_5$ path R_2, then G contains a cycle $(v_1, v_3, R_2, v_5, v_6, v_4, R_1, v_2, v_1)$, of length ≥ 6 contradicting our assumption.

In view of the explanations in the subcases (a)-(i) we deduce that G_1 and G_2 in Figure 1 are of the general forms that satisfy the requirements of the theorem in this case.

Case (4). Let $cir\,(G) = 6$, and C be a 6-cycle in G. Because $D\,(G) = 5$, then there is no vertex in G, other than the vertices of C, adjacent to a vertex of C. Therefore, $P\,(G) = 6$ and so G is C_6 with, or without some chords. Hence, the proof of the theorem is completed. □

3. Characterization of Graphs G with $D(G) = 6$ and $cdn(G) = 2$

In the following proposition we establish that if G is a block of $D\,(G) = 6$, then the circumference of G is more than four.

Proposition 3.1. *Let G be a block of order $p \geq 7$ and with $D\,(G) = 6$, then $cir\,(G) = 5, 6$ or 7.*

Proof. Let $P = (u_1, u_2, \ldots, u_6, u_7)$ be a detour diameter of G, shown in Figure 6.

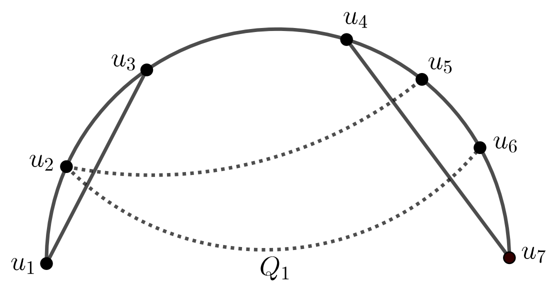


Figure 6.

Since G is a block, then it does not contain cut-vertices and end-vertices. Because $D\,(G) = 6$, then u_1 and u_7 each is not adjacent to any vertex other than $u_2, u_3, \ldots, u_6$. It is clear that $cir(G) \leq 7$. If u_1 is adjacent to u_5, u_6 or u_7, and/or u_7 is adjacent to u_1, u_2 or u_3, then G contains a cycle of length more than four (see Figure 6). To compute the proof we shall show that G contains a cycle of length 5, 6 or 7 if u_1 is adjacent to u_3 or u_4, and u_7 is adjacent to u_4 or u_5. So, we consider the following four cases.

Case (1). If $u_1u_3, u_7u_4 \in E(G)$, then we have the following four subcases.

(a) G contains a $u_2 - u_6$ path Q_1 which is edge-disjoint from P, this implies that G contains $l-$cycle $(u_3, u_1, u_2, (Q_1), u_6, u_7, u_4, u_3)$ of length $l \geq 6$.

(b) G contains the edge u_2u_5 which implies that G contains the $7-$cycle $(u_3, u_1, u_2, u_5, u_6, u_7, u_4, u_3)$.

(c) G contains edges u_2u_4 and u_3u_5, this implies that G contains the 7−cycle $(u_2, u_1, u_3, u_5, u_6, u_7, u_4, u_2)$.

(d) G contains a $u_2 - u_4$ path Q_2 and a $u_3 - u_6$ path Q_3, which are edge-disjoint from P; this implies that G contains the cycle $(u_3, u_1, u_2, (Q_2), u_4, u_5, u_6, (Q_3), u_3)$ of length $l \geq 6$ (see Figure 7).

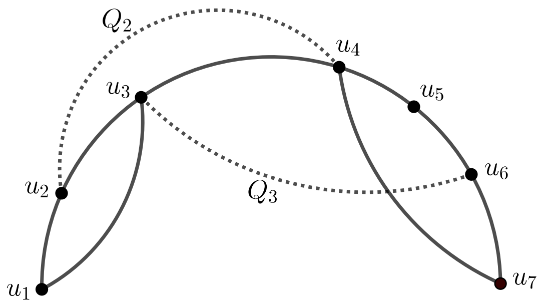


FIGURE 7.

Case (2). If $u_1u_3, u_7u_5 \in E(G)$, then we have two subcases.

(i) G contains the edge u_2u_6, which implies that G contains the 7-cycle $(u_3, u_1, u_2, u_6, u_7, u_5, u_4, u_3)$.

(ii) G contains a $u_2 - u_5$ path R_1 and a $u_3 - u_6$ path R_2 which are edge disjoint from $E(P)$, which implies that G contains cycle $(u_3, u_1, u_2, (R_1), u_5, u_7, u_6, (R_2), u_3)$ of length $l \geq 6$ (see Figure 8).

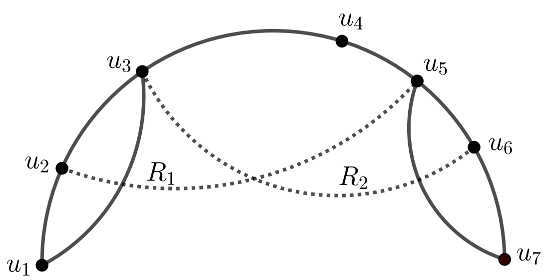


FIGURE 8.

Case (3). If $u_1u_4, u_5u_7 \in E(G)$, then, as in case (2), G contains a cycle of length 6 or 7.

Case (4). If $u_1u_4, u_4u_7 \in E(G)$, then we have four subcases for the cycles in G.

(α) G contains a $u_2 - u_5$ path F_1 other than (u_2, u_3, u_4, u_5), this implies that G contains a cycle $(u_2, u_1, u_4, u_7, u_6, u_5, (F_1), u_2)$ of length ≥ 6 (see Figure 9).

(β) G contains a $u_2 - u_6$ path F_2, this produces that G contains a cycle $(u_2, u_3, u_4, u_5, (F_2), u_2)$ of length $l \geq 5$.

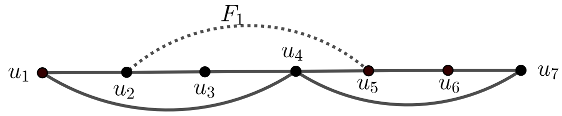


FIGURE 9.

(γ) G contains the edge u_3u_5 implying that G contains the 7-cycle $(u_3, u_2, u_1, u_4, u_7, u_6, u_5, u_3)$.

(δ) G contains a $u_3 - u_6$ path F_3, this produces that G contains a cycle $(u_3, u_2, u_1, u_4, u_7, u_6, (F_3), u_3)$ of length $l \geq 6$.

Hence, the proof of the proposition is completed. $\square$

Theorem 3.1. *Let G be a connected graph of order $p \geq 7$ and with detour diameter $D(G) = 6$. Then, $cdn(G) = 2$ if and only if G is a cycle graph C_7^*, with or without any number of chords, or G belongs to the family F shown in Figure* 10.

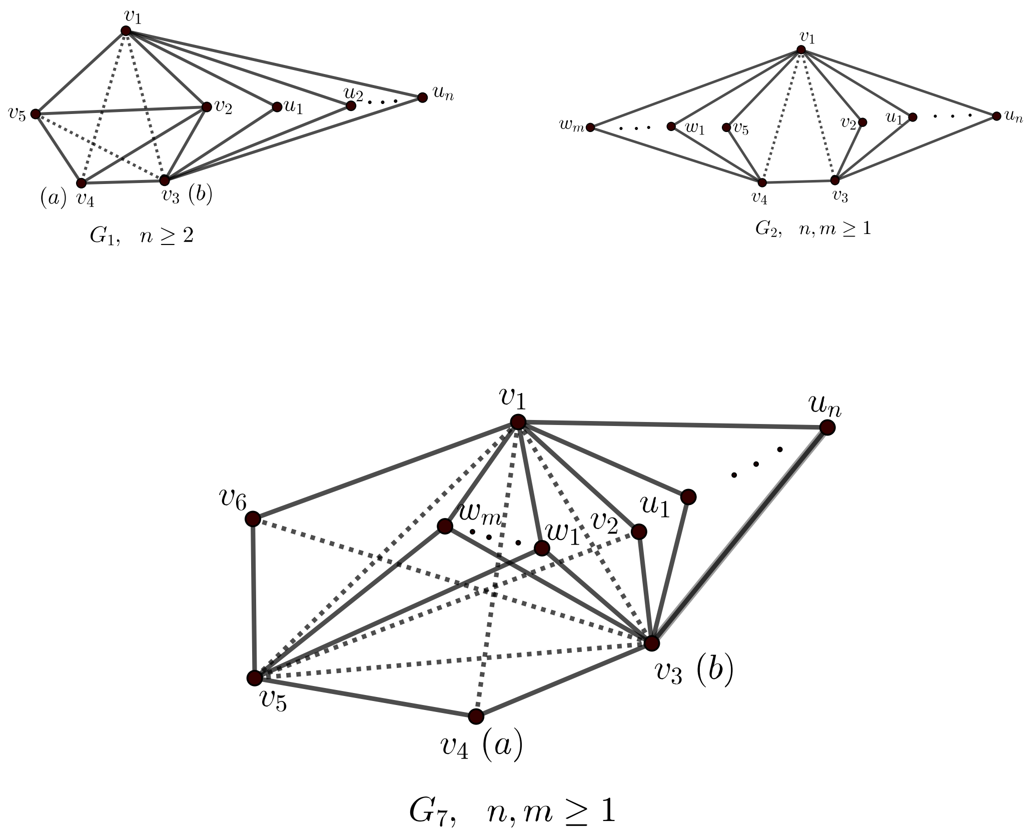
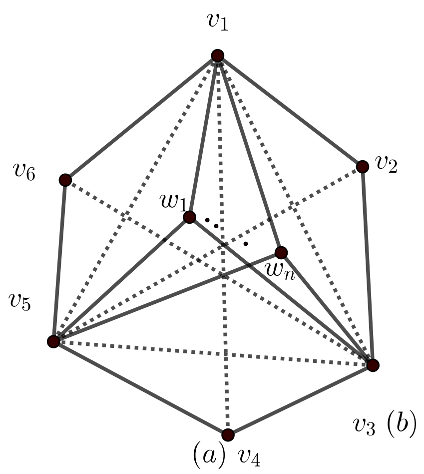
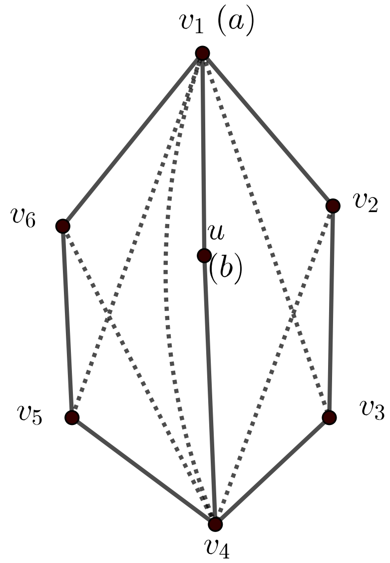


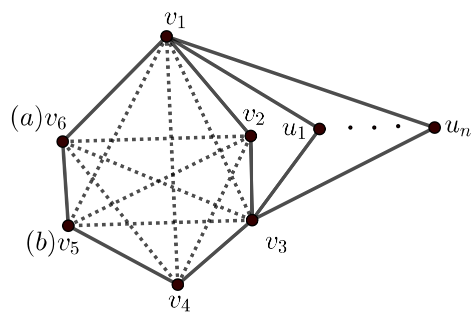
FIGURE 10. The family F



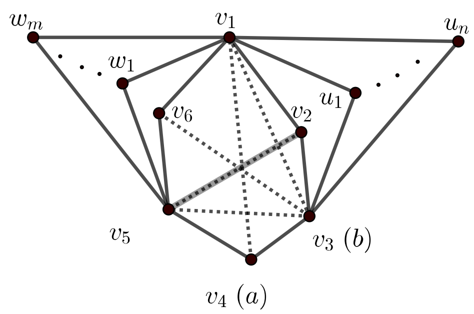
G_3, $n \geq 1$



G_4



G_5, $n \geq 1$



G_6, $n, m \geq 1$

Proof. It is straightforward to verify that $D\left(C_7^*\right) = D\left(G_i\right) = 6$, and $cdn(C_7^*) = cdn\left(G_i\right) = 2$, in which $\{a, b\}$ is a connected detour basis of G_i $(1 \leq i \leq 7)$.

To prove the converse, let G be a connected graph of order $p \geq 7$, $D\left(G\right) = 6$ and $cdn\left(G\right) = 2$. Then, by the Proposition 2.1, G is a block, and by Proposition 3.1, $cir\left(G\right) = 5$, 6 or 7. Thus, we shall consider three cases depending on the circumference of G.

Case (1). Let $cir\left(G\right) = 5$ and $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Since G is connected and $P(G) \geq 7$, then there is a vertex $u_1 \neq v_i$ $(1 \leq i \leq 5)$ adjacent to a vertex, say v_1, of C. Because $\deg u_1 \geq 2$, then either u_1 is adjacent to another vertex of C not adjacent to v_1, or it is adjacent to a vertex $x \neq v_i$ $(1 \leq i \leq 5)$. If $u_1x \in E(G)$, then x is not adjacent to any other vertex $x \notin V(C)$, and, also, it is not adjacent to any vertex of C, because, otherwise $D(G) \geq 7$ or $cir(G) \geq 6$. Therefore u_1 must be adjacent to non-adjacent vertices of C, say v_1 and v_3 and it is not adjacent to any other vertex of G, that is $\deg u_1 = 2$. It is clear that every vertex $y \notin V(C)$ is of degree 2 and adjacent to two non-adjacent vertices of C.

Let $w_1 \in V(G)$, $w_1 \notin V(C)$ and $w_1 \neq u_1$, then the following hold.

(a) If $w_1v_1, w_1v_3 \in E(G)$, then G is like the graph G_1, in Figure 10, with $n = 2$ (taking $u_2 = w_1$) and with edge v_2v_4 or v_2v_5, and G may contain edge $\{v_1v_3, v_1v_4, v_3v_5\}$. Therefore, G_1 is of a general form of this subcase, because $P(G) \geq 7$.

(b) If $w_1v_2, w_1v_5 \in E(G)$, then $cir(G) \geq 7$, a contradiction.

(c) If $w_1v_2, w_1v_4 \in E(G)$, then $cir(G) \geq 6$, a contradiction.

(d) If $w_1v_1, w_1v_4 \in E(G)$, (or $w_1v_3, w_1v_5 \in E(G)$), then G is like the graph G_2, in Figure 10, with $m = n = 1$ and G may contain some of the edges v_1v_4 or v_1v_3. Therefore, G_2 is of a general form of this subcase, because $P(G) \geq 7$.

Case (2). Let $cir(G) = 6$, $C = (v_1, v_2, \ldots, v_6, v_1)$ and let $W = V(G) - V(C)$. If $w \in W$, then w S is adjacent to at least two vertices of C, for otherwise $D(G) \geq 7$. Since $cir(G) = 6$, then w is not adjacent to any two adjacent vertices of C. Therefore, every vertex of W is of degree 3 or 2, and it is not adjacent to any vertex other than the vertices of C. Thus, we shall consider G in the following three subcases.

(a) Let every vertex of W is of degree 3. If $w \in W$, and w is adjacent to v_1 then it is adjacent to v_3 and v_5. If in addition to w, there is $w' \in W$ adjacent to v_2, v_4 and v_6, then G contains the 8-cycle $(v_1, w, v_3, v_2, w', v_4, v_5, v_6, v_1)$ (see Figure 11) contradicting the assumption. Thus, without loss of generality every vertex of W is adjacent to v_1, v_3 and v_5. Therefore, G is like the graph G_3 in Figure 10 with $n \geq 1$ and a number of dotted chords of C.

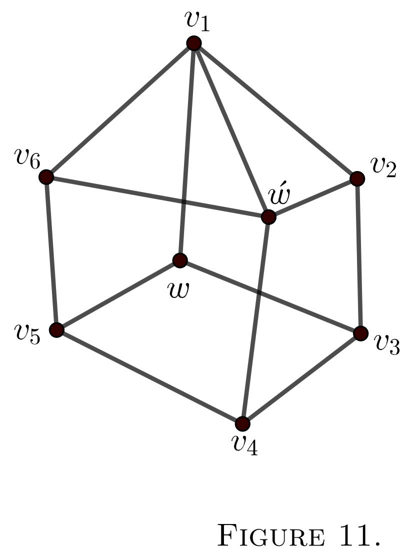


FIGURE 11.

(b) Let every vertex of W is of degree 2. Let u be any vertex in W and assume that u is adjacent to v_1. Then u is adjacent to v_3, v_4 or v_5. Therefore, we have two general possibilities, namely:

(i) $uv_1, uv_4 \in E(G)$;

(ii) uv_1, uv_3 (or uv_1, uv_5) $\in E(G)$.

For subcase (i), if u' is another vertex of W, then, for all connections of u' with a pair of non-adjacent vertices of C, the graph G will not satisfy the requirements $D = 6$ and $cdn = 2$. Therefore W consists of exactly one vertex u, and so $P(G) = 7$. Hence, G is like the graph G_4 shown in Figure 10.

(ii) Let $uv_1, uv_3 \in E(G)$. If each $u \in W$ is adjacent to the some non-adjacent pair of $V(C)$ like v_1, v_3, then G is like G_5 shown in Figure 10. If there is a vertex $u_1 \in V(G)$ adjacent to, say v_1, v_3, and there is at least one vertex $w_1 \in V(G)$ adjacent to v_1, v_5 (or v_3, v_5), then G is like G_6 with $n, m \geq 1$. For other connections of the vertices of W to pairs of non-adjacent vertices of $V(C)$, we have the following.

(a) If $uv_1, uv_3; wv_1, wv_5; xv_3, xv_5 \in V(G)$, where $x \in W$, then we have a graph like H_1 shown in Figure 12. Clearly, $cdn\,(H_1) = 3$, so H_1 does not fulfill the requirements.

(b) If $uv_1, uv_3; wv_4, wv_6 \in V(G)$, then we have a graph like H_2 shown in Figure 12. Clearly, $D\,(H_2) = 7$, so H_2 does not fulfil the required conditions.

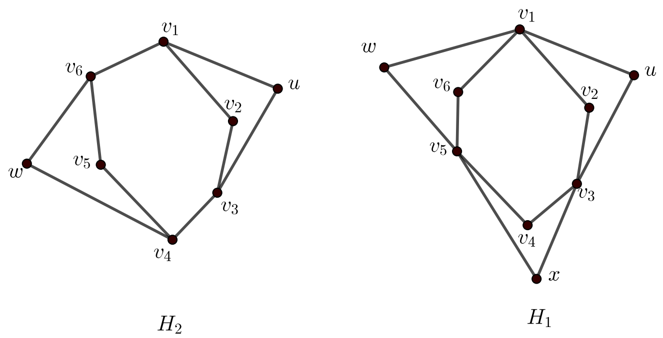


FIGURE 12.

(c) Now, assume that W consists of vertices of degree 2 and of degree 3. Let w be a vertex in W of degree 3. Then, without loss of generality, assume that w is adjacent to v_1, v_3 and v_5. Let $u \in W$ of degree 2, then we have the following possibilities.

(1) If u is adjacent to v_1 and v_3, then G is like the graph G_7, with $n, m \geq 1$, shown in Figure 10.

(2) If u is adjacent to v_2 and v_4, then G contains a 7-cycle $(v_1, v_6, v_5, w, v_3, v_4, u, v_2, v_1)$, a contradiction.

(3) If u is adjacent to v_1 and v_4, then $cdn(G) \geq 2$, a contradiction.

(4) If u is adjacent to v_3 and v_5, then G is like the graph G_7 in Figure 10.

Hence, the graph G in Case (2), for which $cir\,(G) = 6$, is in general construction, is like G_i $(i = 3, 4, 5, 6, 7)$.

Case (3). Let $cir\,(G) = 7$ and $C = (v_1, v_2, \ldots, v_7, v_1)$. If there is a vertex u in G other than the vertices of C, then u is adjacent to a vertex of C, say v_1. This implies that G contains a 7-path, namely $u, v_1, v_2, \ldots, v_7$, contradicting the hypothesis of the theorem. Therefore, $P(G) = 7$, and so G is the 7-cycle graph C_7^* with some chords of C.

Hence, the proof of the theorem is completed. □

References

[1] K. Abhishek and A. Ganesan, *Detour distance pattern of a graph*, International Journal of Pure and Applied Mathematics **87**(5) (2013), 719–728.

[2] A. M. Ahmed and A. A. Ali, *The connected detour numbers of special classes of connected graphs*, J. Math. **20** (2019), 1–9.

[3] G. Chartrand, H. Escuadro and P. Zang, *Detour distance in graph*, J. Combin. Math. Combin. Copmut. **53** (2005), 75–94.

[4] G. Chartrand, L. Johns and P. Zang, *Detour numbers of a graph*, Util. Math. Combin. **64** (2003), 97–113.

[5] G. Chartrand and L. Lesnik, *Graphs and Digraphs*, 6th Edition, Taylor and Francis Group, CRC Press, New York, 2016.

[6] G. Chartrand and P. Zang, *Distance in graphs-taking the long view*, AKCE Int. J. Graphs Comb. **1**(1) (2004), 1–13.

[7] A. P. Santhakumaran and S. Athisayanathan, *On the connected detour number of a graph*, J. Prime Res. Math. **5** (2009), 149–170.

[8] A. P. Santhakumaran and S. Athisayanathan, *The connected detour number of a graph*, J. Combin. Math. Combin. Copmut. **69** (2009), 205–218.

[9] A. P. Santhakumaran and P. Titus, *The vertex detour number of a graph*, AKCE Int. J. Graphs Comb. **4**(1) 2007), 99–112.

[10] A. P. Santhakumaran and P. Titus, *The connected vertex detour number of a graph*, Acta Univ. Sapientiae Math. **2**(2) (2010), 146–159.

[1]Department of Mathematics,
College of Science,
Salahaddin University/Erbil.
Email address: gashaw.mohammed@su.edu.krd