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ON DEGREE OF APPROXIMATION OF SIGNALS IN THE GENERALIZED ZYGMUND CLASS BY USING $(E, r)(N, q_n)$ MEAN

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Abstract. In the present article, we have established a result on degree of approximation of function (or signal) in the generalized Zygmund class $Z_l^{(m)}$,$(l \geq 1)$ by using $(E, r)(N, q_n)$- mean of Trigonometric Fourier series.

1. Introduction

Signal Analysis describes the field of study whose objective is to collect, understand and deduce information and intelligence from various signals. Now-a-days the analysis of signals is a fundamental problem for many engineers and scientists. In the recent past, we have seen the applications of mathematical methods such as Probability theory, Mathematical statistics etc. in the analysis of signals. Very recently, approximation theory has got a large popularity as it has given a new dimension in approximating the signals (or functions). The estimation of error functions in Lipschitz class and Zygmund space using different summability techniques of Fourier series and conjugate Fourier series have been of great interest among the researchers in the last decades (for details see [2, 3, 9, 10, 13–18]). Later, the generalized Zygmund class $Z_l^{(m)}$, $l \geq 1$, is investigated by Leindler [8], Moricz [4], Moricz and Nemeth [5]. Very recently Nigam [7] and Singh et al. [11] proved approximation of functions in the generalized Zygmund class by using Hausdroff means. Lal and Shireen [12] proved a result on approximation of functions of generalized Zygmund class by Matrix Euler summability mean of Fourier series. In the present paper, we investigate on the degree

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of approximation of a signal (or function) in the generalized Zygmund class $Z_l^{(m)}$, $l \geq 1$, by $(E,r)(N,q_n)$ product mean of the trigonometric Fourier series.

2. Definitions and Notations

Let h be a function, which is periodic in $[0,2\pi]$ such that $\int_0^{2\pi} |h(x)|^l dx < \infty$. We denote

$$L_l[0,2\pi] = \Big\{h : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |h(x)|^l dx < \infty\Big\}, \quad l \geq 1.$$

The Fourier series of $h(x)$ is given by

$$\sum_{n=0}^{\infty} u_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx\Big). \tag{2.1}$$

Let $S_p(h;x)$ denotes the p-th partial sum of $h(x)$ and is given by

$$S_p(h;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x+v) \frac{\sin\Big(p+\frac{1}{2}\Big)v}{2\sin\frac{v}{2}} dv.$$

We define

$$\|h\|_l = \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(x)|^l dx\Big)^{\frac{1}{l}}, \quad 1 \leq l < \infty,$$

and

$$\|h\|_l = \text{esssup}_{0\leq x\leq 2\pi} |h(x)|, \quad l = \infty.$$

Let the Zygmund modulus of continuity of $h(x)$ be

$$m(h;r) = \sup_{0\leq r, x\in\mathbb{R}} |h(x+v) + h(x-v) - 2h(x)| \quad \text{(see [1])}.$$

Suppose $\mathbf{B}$ represents the Banach space of all 2π periodic functions which are continuous and defined over $[0,2\pi]$ under the supremum norm. Clearly,

$$Z_{(\alpha)} = \Big\{h \in \mathbf{B} : |h(x+v) + h(x-v) - 2h(x)| = O\left(|v|^\alpha\right), 0 < \alpha \leq 1\Big\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha)}$ defined by

$$\|h\|_{(\alpha)} = \sup_{0\leq x\leq 2\pi} |h(x)| + \sup_{x,v\neq 0} \frac{|h(x+v) + h(x-v) - 2h(x)|}{|v|^\alpha}.$$

For $h \in L_l[0,2\pi]$, $l \geq 1$, the integral Zygmund modulus of continuity is defined by

$$m_l(h;r) = \sup_{0<v\leq r} \left\{\frac{1}{2\pi}\int_0^{2\pi} |h(x+v) + h(x-v) - 2h(x)|^l dx\right\}^{\frac{1}{l}},$$

and for $h \in \mathbf{B}$, $l = \infty$,

$$m_\infty(h;r) = \sup_{0<v\leq r} \max_x |h(x+v) + h(x-v) - 2h(x)|.$$

Clearly, $m_l(h;r) \to 0$ as $l \to 0$. Again,

$$Z_{(\alpha),\ l} = \left\{ h \in L_l[0,2\pi] : \left(\int_0^{2\pi} |h(x+v) + h(x-v) - 2h(x)|^l dx \right)^{\frac{1}{l}} = O(|v|^\alpha) \right\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha),l}$ for $0 < \alpha \leq 1$ and $l \geq 1$. Clearly,

$$\|h\|_{(\alpha),l} = \|h\|_l + \sup_{v \neq 0} \frac{\|h(\cdot + v) + h(\cdot - v) - 2h(\cdot)\|_l}{|v|^\alpha}.$$

Let

$$Z^{(m)} = \Big\{ h \in \mathbf{B} : |h(x+v) + h(x-v) - 2h(x)| = O\left(m(v)\right) \Big\},$$

where m is a Zygmund modulus of continuity satisfying

(a) $m(0) = 0$;

(b) $m(v_1 + v_2) \leq m(v_1) + m(v_2)$.

Define

$$Z_l^{(m)} = \left\{ h \in L_l : 1 \leq l < \infty, \sup_{v \neq 0} \frac{\|h(\cdot + v) + h(\cdot - v) - 2h(\cdot)\|_l}{m(v)} < \infty \right\},$$

where

$$\|h\|_l^{(m)} = \|h\|_l + \sup_{v \neq 0} \frac{\|h(\cdot + v) + h(\cdot - v) - 2h(\cdot)\|_l}{m(v)}, \quad l \geq 1.$$

Clearly, $\|\cdot\|_l^{(m)}$ is a norm $Z_l^{(m)}$. Also, $Z_l^{(m)}$ is complete since L_l, $l \geq 1$, is complete. So, $Z_l^{(m)}$ is a Banach space under $\|\cdot\|_l^{(m)}$. Agian, suppose $m(v)$ and $\mu(v)$ represents the Zygmund moduli of continuity such that $\frac{m(v)}{\mu(v)}$ is positive and non-decreasing then

$$\|h\|_l^{(\mu)} \leq \max \left\{ 1, \frac{m(2\pi)}{\mu(2\pi)} \right\} \|h\|_l^{(m)} \leq \infty. \tag{2.2}$$

Clearly,

$$Z_l^{(m)} \subseteq Z_l^{(\mu)} \subseteq L_l, \quad l \geq 1.$$

Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Suppose $\{q_k\}$ represents the sequence of non-negative integers such that

$$Q_n = \sum_{k=0}^{n} q_k \to \infty \quad \text{as} \quad n \to \infty. \tag{2.3}$$

If

$$\tau_n^N = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} s_k, \quad n = 0, 1, 2, \ldots \tag{2.4}$$

represents the (N, q_n) mean of $\{s_n\}$ generated by the sequence $\{q_n\}$, then the series $\sum u_n$ is said to be summable to 's' whenever

$$\lim_{n \to \infty} \tau_n^N \to s.$$

We know, (N, q_n) method is regular [6]. The (E, r) transform of $\{s_n\}$ is given by

$$E_n^r = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) r^{n-k} s_k. \tag{2.5}$$

If $E_n^r \to s$ as $n \to \infty$, then $\sum u_n$ is summable to 's' by (E, r) summability. Also, (E, r) method is regular [6].

The $(E, r)(N, q_n)$ transform of $\{s_n\}$ is given by

$$\tau_n^{E_r,N} = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) \Big\{ \frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} s_\nu \Big\}. \tag{2.6}$$

The series $\sum u_n$ is summable to s by the $(E, r)(N, q_n)$ transform if $\tau_n^{E_r,N} \to s$ as $n \to \infty$.

The following notations are used in the rest part of our paper:

$$\begin{aligned}
\varphi(x, v) =& h(x+v) + h(x-v) - 2h(x), \\
\kappa_n^{E_r,N}(v) =& \frac{1}{2\pi(1+r)^n} \sum_{k=0}^{n} C(n,k) \left\{ \frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} \frac{\sin\left(\nu + \frac{1}{2}\right)v}{\sin\left(\frac{v}{2}\right)} \right\}.
\end{aligned}$$

3. Known Results

Using Matrix Euler summability means, Lal and Shireen [12] proved the following theorems.

Theorem 3.1. *Let the lower triangular matrix $A = (a_{n,k})$ satisfy the following conditions:*

$$\sum_{k=0}^{n} a_{n,k} = 1, \quad a_{n,k} \geq 0, \quad n = 0, 1, 2, \dots, \; k = 0, 1, 2, \dots, \tag{3.1}$$

$$\sum_{k=0}^{n} |\Delta a_{n,k}| = O\Big(\frac{1}{n+1}\Big) \quad \text{and} \quad (n+1) a_{n,n} = O(1). \tag{3.2}$$

The best approximation of the Fourier series (2.1) *by Matrix-Euler mean is given by*

$$E_n(h) = \inf_{t_n{}^{\Delta,E}} \| t_n{}^{\Delta,E} - h \|_l^\mu = O\left(\frac{1}{n+1} \int_{\frac{1}{(n+1)}}^{\pi} \frac{m(v)}{v^2 \mu(v)} dv \right), \tag{3.3}$$

where

$$t_n{}^{\Delta,E} = \sum_{k=0}^{n} a_{n,k} \frac{1}{2^k} \sum_{\nu=0}^{k} C(k,\nu) s_\nu,$$

represents the Matrix-Euler mean of a 2π periodic and Lebesgue integrable function $h : [0, 2\pi] \to \mathbb{R}$, that belongs to $Z_l^{(m)}$, $l \geq 1$. Here, m and μ are the Zygmund moduli of continuity and $\frac{m(v)}{\mu(v)}$ is positive and non-decreasing.

Theorem 3.2. *Let $A = (a_{n,k})$ be a lower triangular matrix satisfying* (3.1) *and* (3.2) *in Theorem* 3.1 *along with the condition that* $\frac{m(v)}{\mu(v)}$ *is non-increasing. Then, for* $h \in Z_l^{(m)}$, $l \geq 1$, *the best approximation by Matrix-Euler mean* $\left({t_n}^{\Delta,E}\right)$ *is given by*

$$E_n(h) = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\log(n+1)\pi\right). \tag{3.4}$$

4. Main Theorems

Theorem 4.1. *Let* $h : [0, 2\pi] \to \mathbb{R}$ *be a periodic function* (*with period* 2π) *belonging to* $Z_l^{(m)}$, $l \geq 1$, *which is integrable in the sense of Lebesgue. Then the degree of approximation of* h *by using* $(E, r)(N, q_n)$ *mean of* (2.1) *is given by*

$$E_n(h) = \inf_{{\tau_n}^{E_r,N}} \|{\tau_n}^{E_r,N} - h\|_l^{\mu} = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v\mu(v)}dv\right), \tag{4.1}$$

where $m(v)$ *and* $\mu(v)$ *are the Zygmund moduli of continuity and* $\frac{m(v)}{v\mu(v)}$ *is positive and non-decreasing.*

Theorem 4.2. *The degree of approximation of a* 2π *periodic and Lebesgue integrable function* h, $h : [0, 2\pi] \to \mathbb{R}$, *using* $(E, r)(N, q_n)$ *mean of* (2.1) *is given by*

$$E_n(h) = \inf_{{\tau_n}^{E_r,N}} \|{\tau_n}^{E_r,N} - h\|_l^{\mu} = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\left(\frac{\pi}{n+1} - \frac{1}{(n+1)^2}\right)\right), \tag{4.2}$$

where $h \in Z_l^{(m)}$, $l \geq 1$, $m(v)$ *and* $\mu(v)$ *are the Zygmund moduli of continuity and* $\frac{m(v)}{v\ \mu(v)}$ *is positive and non-increasing.*

We require the below mentioned lemmas to prove our main theorems.

5. Lemmas

Lemma 5.1.

$$|\kappa_n^{E_r,N}| = O(n), \quad \text{for } 0 \leq v \leq \frac{1}{n+1}.$$

Lemma 5.2.

$$|\kappa_n^{E_r,N}| = O\Big(\frac{1}{v}\Big), \quad \text{for } \frac{1}{n+1} \leq v \leq \pi.$$

Lemma 5.3. *Let* $h \in Z_l^{(m)}$. *Then, for* $0 < v \leq \pi$,

(*i*) $\|\varphi(\cdot, v)\|_l = O(m(v))$;

(*ii*) $\|\varphi(\cdot + y, v) + \varphi(\cdot - y, v) - 2\varphi(\cdot, v)\|_l = O(m(v))$ *or* $O(m(y))$;

(iii) If $m(v)$ and $\mu(v)$ are as defined in Theorem 4.1, then

$$\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l = O\left(\mu(y)\frac{m(v)}{\mu(v)}\right),$$

where $\varphi(x,v)=h(x+v)+h(x-v)-2h(x)$.

6. Proof of the Lemmas

Proof of Lemma 5.1. For $0\le v\le \frac{1}{n+1}$ and $\sin nv\le n\sin v$, we have

$$\begin{aligned}
|\kappa_n^{E_r,N}(v)| &= \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}\right|\\
&\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{(2\nu+1)\sin\frac{v}{2}}{\sin\frac{v}{2}}\right\}\right|\\
&\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}(2k+1)\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\right\}\right|\\
&\le \frac{(2n+1)}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\right|\\
&= O(n).
\end{aligned}$$

□

Proof of Lemma 5.2. By Jordan's lemma

$$\sin\left(\frac{v}{2}\right)\ge\frac{v}{\pi},\quad \sin nv\le 1,\quad \frac{1}{n+1}\le v\le\pi.$$

Now,

$$\begin{aligned}
|\kappa_n^{E_r,N}(v)| &= \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}\right|\\
&\le \frac{1}{2\pi(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k}\frac{\pi}{v}\ q_{k-\nu}\right\}\right|\\
&= \frac{1}{2v(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\left\{\frac{1}{Q_k}\sum_{\nu=0}^{k}\ q_{k-\nu}\right\}\right|\\
&= \frac{1}{2v(1+r)^n}\left|\sum_{k=0}^{n} C(n,k)\ r^{n-k}\right|\\
&= O\left(\frac{1}{v}\right).
\end{aligned}$$

□

Proof of Lemma 5.3. See [12]. □

7. Proof of Main Theorems

Proof of Theorem 4.1. Let $S_k(h;x)$ denotes the k-th partial sum of the series (2.1). We have

$$S_k(h;x)-h(x)=\frac{1}{2\pi}\int_0^{\pi}\varphi(x,v)\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}dv$$

and the (N,q_n) transform of it is given by

$$\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\{S_k(h;x)-h(x)\}=\frac{1}{2\pi}\int_0^{\pi}\varphi(x,v)\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}dv.$$

Let the $(E,r)(N,q_n)$ transform of $S_k(h;x)$ by $\tau_n^{E_r,N}$. Then

$$\begin{aligned}\tau_n^{E_r,N}-h(x)=&\frac{1}{2\pi(1+r)^n}\int_0^{\pi}\varphi(x,v)\sum_{k=0}^{n}C(n,k)\ r^{n-k}\left\{\frac{1}{Q_n}\sum_{k=0}^{n}q_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}\right\}dv\\ =&\int_0^{\pi}\varphi(x;v)\kappa_n^{E_r,N}(v)dv\\ =&\chi_n(x).\end{aligned}$$

Then

$$\chi_n(x+y)+\chi_n(x-y)-2\chi_n(x)=\int_0^{\pi}\Big\{\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)\Big\}\kappa_n^{E_r,N}(v)dv.$$

Using Minkowski's inequality, we have

$$\begin{aligned}&\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l\\ =&\left\{\frac{1}{2\pi}\int_0^{2\pi}|\chi_n(x+y)+\chi_n(x-y)-2\chi_n(x)|^l dx\right\}^{\frac{1}{l}}\\ =&\frac{1}{2\pi}\left[\int_0^{2\pi}\left|\int_0^{\pi}\Big\{\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)\Big\}\kappa_n^{E_r,N}(v)dv\right|^l dx\right]^{\frac{1}{l}}\\ \le&\int_0^{\pi}|\kappa_n^{E_r,N}(v)|\left\{\frac{1}{2\pi}\int_0^{2\pi}|\varphi(x+y,v)+\varphi(x-y,v)-2\varphi(x,v)|^l dx\right\}^{\frac{1}{l}}dv\\ =&\int_0^{\pi}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ =&\int_0^{\frac{1}{n+1}}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ &+\int_{\frac{1}{n+1}}^{\pi}\|\varphi(\cdot+y,v)+\varphi(\cdot-y,v)-2\varphi(\cdot,v)\|_l|\kappa_n^{E_r,N}(v)|dv\\ =&\Gamma_1+\Gamma_2.\end{aligned}\tag{7.1}$$

Using Lemma 5.1, Lemma 5.3 and monotonicity of $\frac{m(v)}{\mu(v)}$, with respect to 'v', we have

$$\begin{aligned}\Gamma_1 &= \int_0^{\frac{1}{n+1}} \|\varphi(\cdot + y, v) + \varphi(\cdot - y, v) - 2\varphi(\cdot, v)\|_l |\kappa_n^{E_r,N}(v)| dv \\ &= \int_0^{\frac{1}{n+1}} O\Big(\mu(y)\frac{m(v)}{\mu(v)}\Big) O(n) dv.\end{aligned}$$

By using the second mean value theorem of integral, we have

$$\begin{aligned}\Gamma_1 \le& O\left(n\mu(y)\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\int_0^{\frac{1}{n+1}} dv\right) \\ =& O\left(\frac{n}{n+1}\ \mu(y)\ \frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) = O\left(\mu(y)\ \frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right). \qquad (7.2)\end{aligned}$$

Again, by using Lemma 5.2 and Lemma 5.3, we get

$$\begin{aligned}\Gamma_2 &= \int_{\frac{1}{n+1}}^{\pi} \|\varphi(\cdot + y, v) + \varphi(\cdot - y, v) - 2\varphi(\cdot, v)\|_l |\kappa_n^{E_r,N}(v)| dv \\ &\le \int_{\frac{1}{n+1}}^{\pi} O\Big(\mu(y)\frac{m(v)}{\mu(v)}\Big)\frac{1}{v} dv \\ &= O\Big(\mu(y)\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v\mu(v)} dv\Big). \qquad (7.3)\end{aligned}$$

By (7.1), (7.2) and (7.3), we have

$$\|\chi_n(\cdot + y) + \chi_n(\cdot - y) - 2\chi_n(\cdot)\|_l = O\left(\mu(y)\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) + O\Big(\mu(y)\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v\ \mu(v)} dv\Big).$$

Therefore, we have

$$\sup_{y\neq 0} \frac{\|\chi_n(\cdot + y) + \chi_n(\cdot - y) - 2\chi_n(\cdot)\|_l}{\mu(y)} = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)}\right) + O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v\ \mu(v)} dv\Big). \qquad (7.4)$$

As

$$\varphi(x, v) = |h(x + v) + h(x - v) - 2h(x)|,$$

by applying Minkowski's inequality, we get

$$\|\varphi(x, v)\|_l = \|h(x + v) + h(x - v) - 2h(x)\|_l = O(m(v)). \qquad (7.5)$$

Now, using Lemma 5.1, Lemma 5.2 and (7.5),

$$\|\chi_n(.)\|_l \le \Big(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\Big) \|\varphi(\cdot, v)\|_l |\kappa_n^{E_r,N}(v)| dv$$

$$=O\Big(n\int_0^{\frac{1}{n+1}} m(v)\ dv\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big)$$

$$=O\Big(m\Big(\frac{1}{n+1}\Big)\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big). \tag{7.6}$$

From (7.4) and (7.6), we have

$$\begin{aligned}&\|\chi_n(\cdot)\|_l^{\mu}\\ =&\|\chi_n(\cdot)\|_l+\sup_{y\neq 0}\frac{\|\chi_n(\cdot+y)+\chi_n(\cdot-y)-2\chi_n(\cdot)\|_l}{\mu(y)}\\ =&O\Big(m\Big(\frac{1}{n+1}\Big)\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv\Big)+O\left(\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\right)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}\ dv\Big)\\ =&\sum_{j=1}^{4}G_j.\end{aligned}$$

In view of monotonicity of $\mu(v)$ for $0<v\leq\pi$, we have

$$m(v)=\frac{m(v)}{\mu(v)}\mu(v)\leq\mu(\pi)\frac{m(v)}{\mu(v)}=O\Big(\frac{m(v)}{\mu(v)}\Big).$$

Therefore,

$$G_1=O(G_3).$$

Again, by using monotonicity of $\mu(v)$,

$$G_2=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v}dv=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}\mu(v)dv\leq\mu(\pi)\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv=O(G_4).$$

Since $\frac{m(v)}{\mu(v)}$ is positive and increasing

$$G_4=\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\ \mu(v)}dv=\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\int_{\frac{1}{n+1}}^{\pi}\frac{dv}{v}\geq\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}.$$

Therefore,

$$G_3=O(G_4).$$

Thus,

$$\|\chi_n(\cdot)\|_l^{\mu}=O(G_4)=O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv\Big).$$

Hence,

$$E_n(h)=\inf_n\|\chi_n(\cdot)\|_l^{\mu}=O\Big(\int_{\frac{1}{n+1}}^{\pi}\frac{m(v)}{v\mu(v)}dv\Big).$$

This completes the proof of the Theorem 4.1. □

Proof of Theorem 4.2. In this theorem, since we have assumed $\frac{m(v)}{v\mu(v)}$ is positive and decreasing, proceeding as in Theorem 4.1. We have

$$E_n(h) = \inf_n \|\chi_n(.)\|_l^\mu = O\left(\frac{m\left(\frac{1}{n+1}\right)}{(n+1)\ \mu\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} dv\right),$$

i.e.,

$$E_n(h) = O\left(\frac{m\left(\frac{1}{n+1}\right)}{\mu\left(\frac{1}{n+1}\right)} \left(\frac{\pi}{(n+1)} - \frac{1}{(n+1)^2}\right)\right).$$

This is what we need to prove in Theorem 4.2. □

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