ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 47, Number 2, 2023

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić
. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Niš : Grafika Galeb). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2302

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Email: krag_j_math@kg.ac.rs
Website: http://kjm.pmf.kg.ac.rs

Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: Grafika Galeb, Niš, Serbia
From 2021 the journal appears in one volume and six issues per annum.

Contents

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(2) (2023), PAGES 173–186.

GRAPHS WITH AT MOST FOUR SEIDEL EIGENVALUES

MODJTABA GHORBANI[1], MARDJAN HAKIMI-NEZHAAD[1], AND BO ZHOU[2]

ABSTRACT. Let G be a graph of order n with adjacency matrix $A(G)$. The eigenvalues of matrix $S(G) = J_n - I_n - 2A(G)$, where J_n is the n by n matrix with all entries 1, are called the Seidel eigenvalues of G. Let $\mathcal{G}(n,r)$ be the set of all graphs of order n with a single Seidel eigenvalue with multiplicity r. In the present work, we will characterize all graphs in the class $\mathcal{G}(n,n-i)$ for $i=1,2$ and for the case $i=3$ our characterization is done by this condition that the nullity of $S(G)$ is zero. If the nullity of $S(G)$ is not zero the problem is solved in special cases.

1. INTRODUCTION

Let G be a simple graph on n vertices with adjacency matrix $A(G)$. The roots of the characteristic polynomial $P_G(\lambda) = \det(\lambda I_n - A(G))$ of G, where I_n is the identity matrix of order n, are called the eigenvalues of G. The spectrum of an adjacency matrix $A(G)$ of G is the multiset of its eigenvalues and forms the spectrum of G denoted by $\mathrm{Spec}(G)$.

Lint and Seidel in [13] introduced a symmetric $(0,-1,1)$-adjacency matrix for a graph G called the Seidel matrix of G as $S(G) = J_n - I_n - 2A(G)$, where J_n is the n by n matrix with entries 1 in every position.

The rank of the matrix $S(G)$ denoted by $\mathrm{rank}(S(G))$ is equal to the maximum number of linearly independent columns of $S(G)$. The multiplicity of the eigenvalue zero of $A(G)$ is called the nullity of G denoted by $\eta(G)$.

Let $\mu_1(G),\dots,\mu_n(G)$ be the Seidel eigenvalues of G, namely the roots of $\det(\mu I - S(G))$, arranged in non-increasing order. The multiset of distinct Seidel eigenvalues of G composes the Seidel spectrum of G and we denote it by $\mathrm{Spec}_S(G)$. If G has exactly s distinct Seidel eigenvalues $\mu_1(G),\dots,\mu_s(G)$ with multiplicities $t_1,\dots,t_s$,

Key words and phrases. Interlacing theorem, Seidel eigenvalue, Seidel switching, nullity.
2010 *Mathematics Subject Classification.* Primary: 05C50. Secondary: 05C35.
DOI 10.46793/KgJMat2302.173G
Received: March 23, 2020.
Accepted: August 03, 2020.

respectively, then we write $\mathrm{Spec}_S(G) = \{[\mu_1(G)]^{t_1}, \ldots, [\mu_s(G)]^{t_s}\}$. We encourage the interested readers to consult papers [7,9] for more information about the mathematical properties of this matrix.

A Seidel switching of graph G can be constructed as follows. Let $V(G) = U_1 \cup U_2$ be a partition of vertices of G and G' be a graph obtained from G by removing all edges between U_1 and U_2 and adding all edges between them not presented in G. We say that G' is a Seidel switching of G with respect to U_1 and in this case G' and G are Seidel co-spectral, see [8]. Two graphs G and G' are called switching equivalent, if G' is constructed by a sequence of Seidel switching from G.

The Figure 1 contains the class of graphs of order n, $2 \leq n \leq 6$, and their Seidel switching together with their Seidel spectra, see [13]. For example, in Figure 1 three switching equivalent classes of all graphs of order 4 are presented.

We proceed as follows. In the rest of this section, further definition are given and known results needed are stated. In Section 2, we provide some preparatory results. Section 3 contains the main results of this paper. In other words, in this section, we give the characterization of some graphs in $\mathcal{G}(n, n-i)$ for $i = 1, 2, 3$ in terms of their Seidel eigenvalues.

The complement of graph G is denoted by $\overline{G}$. Also, the complete graph, cycle graph and path graph on n vertices are denoted by K_n, C_n and P_n, respectively. A complete bipartite graph with a bipartition of sizes a and b is denoted by $K_{a,b}$, where $a + b = n$.

A graph obtained by removing a perfect matching from $K_{a,b}$ is denoted by $K^-_{a,b}$.

The union of two disjoint graphs G and H is denoted by $G \cup H$. The join $G + H$ is the graph obtained from $G \cup H$ by connecting all vertices from $V(G)$ with all vertices from $V(H)$.

The graph $G + e$ is a new graph obtained from G by adding an edge e.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ be two arbitrary matrices. A new $mp \times nq$ product matrix constructed from A by replacing each element a_{ij} with the block $a_{ij}B$ is called as Kronecker product or Tensor product of them and we denote it by $A \otimes B$.

2. Auxiliary Results

Lemma 2.1 ([3]). *For any graph G with n vertices, where $n \geq 2$, we have*

i) $\sum_{i=1}^{n} \mu_i(G) = 0$;

ii) $\sum_{i=1}^{n} \mu_i^2(G) = n(n-1)$.

Lemma 2.2 ([3]). *If G is a graph on n vertices, then* $\mathrm{rank}(S(G)) = n - 1$ *or* n.

Theorem 2.1 (Interlacing Theorem, [3]). *Let G be a graph of order n with induced subgraph H of order m. Let $\mu_1(G) \geq \cdots \geq \mu_n(G)$ and $\mu_1(H) \geq \cdots \geq \mu_m(H)$ be eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, we have $\mu_i(G) \geq \mu_i(H) \geq \mu_{n-m+i}(G)$.*

Let $\mathcal{G}(n, r)$ be the set of a graphs on n vertices which has a single Seidel eigenvalue with multiplicity r. Here, we give the characterization of some graphs in $\mathcal{G}(n, n-i)$ for $i = 1, 2, 3$ in terms of their Seidel eigenvalues.

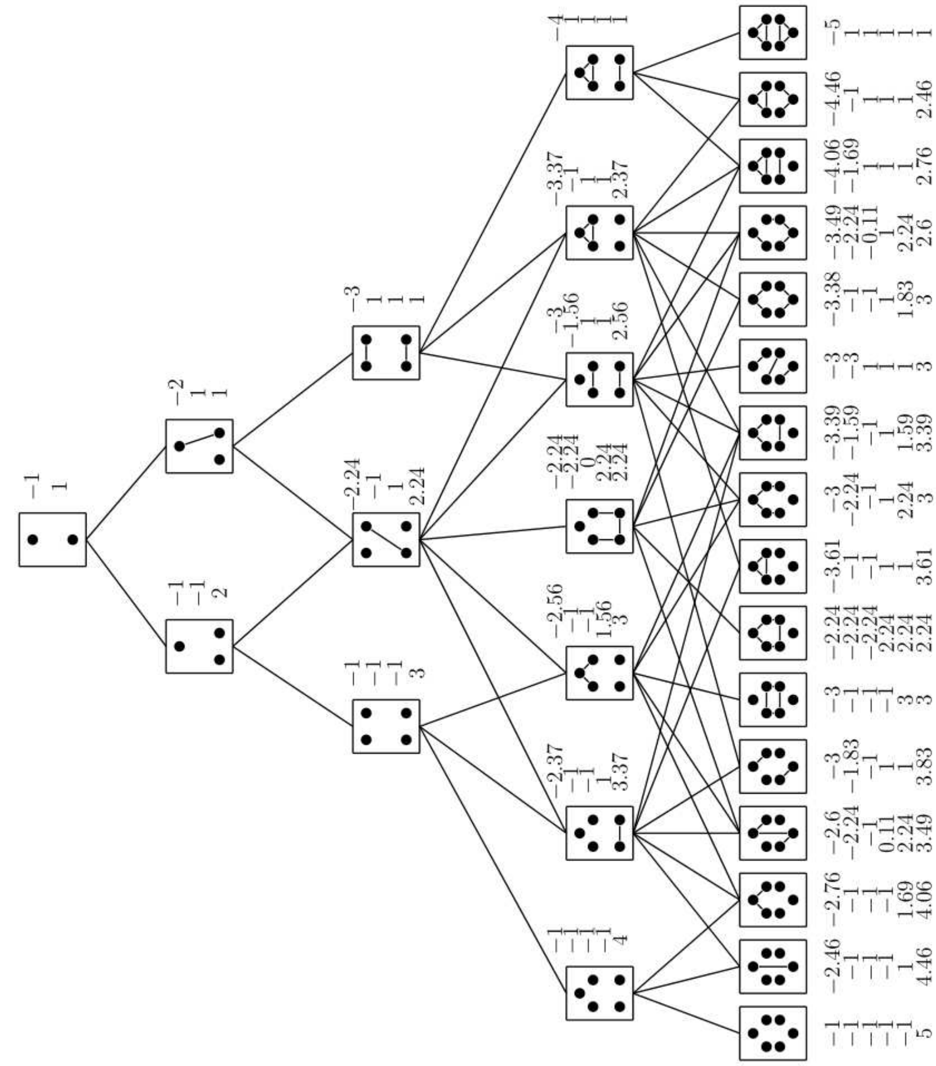

FIGURE 1. Each graph in above diagram is a representative of a class of graphs of order n $(2 \leq n \leq 6)$ together with their switching equivalent graphs which have the same Seidel spectra. Also, all Seidel eigenvalues are written in the right hand side of each graph.

Theorem 2.2. *A graph of order $n \geq 2$ has exactly one positive Seidel eigenvalue if and only if it is a complete bipartite graph or an empty graph.*

Proof. Let G be a graph of order n. If $G \neq K_{n_1,n_2}$, where $n_1 + n_2 = n$ and $n \geq 2$, then we have

$$\mathrm{Spec}_S(K_{n_1,n_2}) = \{[-1]^{n-1}, [n-1]^1\},$$

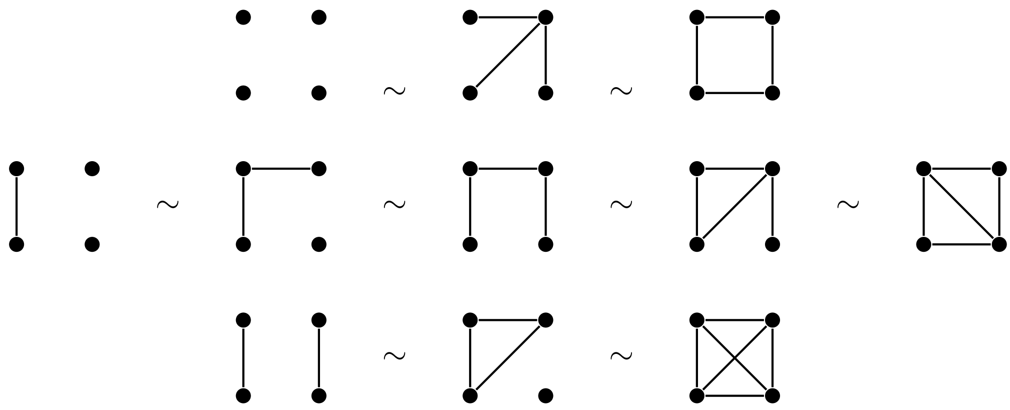

Figure 2. Three switching equivalent classes of graphs of order 4

thus $\mu_2(G) = -1$. Now suppose that G is connected graph, where $\mu_2(G) < 0$ and $G \neq K_{n_1,n_2}$. Hence, either K_3 or P_4 are as an induced subgraph of G. Since

$$\mathrm{Spec}_S(K_3) = \{[-2]^1, [1]^2\}$$

and

$$\mathrm{Spec}_S(P_4) = \{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\},$$

the interlacing theorem yields that $\mu_2(G) \geq \mu_2(K_3) = 1$ or $\mu_2(G) \geq \mu_2(P_4) = 1$, a contradiction. If G is a disconnected graph with exactly one positive Seidel eigenvalue, then G is Seidel equivalent to a connected graph (e.g., by letting U_1 be the vertex set of a component), thus G is Seidel equivalent to a complete bipartite graph (by the first part of the proof) and consequently to an empty graph. $\square$

Corollary 2.1. *If $G \neq K_{n_1,n_2}$, $n_1 + n_2 = n$, is a connected graph with at least two vertices, then $\mu_2 \geq 1$.*

Corollary 2.2. *A connected graph G has exactly two positive Seidel eigenvalues if and only if it has K_3 or P_4 as an induced subgraph.*

Theorem 2.3. *A graph of order $n \geq 3$ has exactly one negative Seidel eigenvalue if and only if it is a complete graph or it is isomorphic with $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$.*

Proof. By regarding $S(\overline{G}) = -S(G)$, one can see that if G has exactly one negative Seidel eigenvalue then $\overline{G}$ has exactly one positive Seidel eigenvalue. By Theorem 2.2 the proof is complete. $\square$

Corollary 2.3. *If $G \neq K_n$ is a connected graph with at least three vertices, then $\mu_{n-1}(G) \leq -1$.*

Corollary 2.4. *The connected graph G has exactly two negative Seidel eigenvalues if and only if it has graph P_3 as an induced subgraph.*

3. Main Results

The main goal of this paper is to classify some classes of graphs $G \in \mathcal{G}(n, n-i)$ for $i = 1, 2, 3$. For $i = 1, 2$ the problem is completely solved but for $i = 3$, in the case that $\eta(G) = 0$, we are done. But if $\eta > 0$, we characterized the graphs in special cases. At first, suppose G is a graph with a single eigenvalue with multiplicity $n-1$ or $n-2$. The following result can be obtained.

Theorem 3.1.

(i) *For $n \geq 2$, $\mathcal{G}(n, n-1) = \{K_n, \overline{K}_n, K_{n_1,n_2}, K_{n_1} \cup K_{n_2}\}$, where $n_1 + n_2 = n$.*

(ii) *For $n \geq 3$, $\mathcal{G}(n, n-2) = \{K_3, \overline{K}_3, P_3, K_2 \cup K_1\}$.*

Proof. **(i)** If $G \in \mathcal{G}(n, n-1)$, then G has exactly two distinct Seidel eigenvalues and so G has one single positive or one single negative Seidel eigenvalue. By Theorems 2.2 and 2.3, G is Seidel equivalent to one of graphs $K_n, \overline{K}_n, K_{n_1,n_2}$ or $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$. This completes the proof of the first claim.

(ii) If $G \in \mathcal{G}(n, n-2)$, then G has at most three distinct Seidel eigenvalues and thus we can consider the following cases.

Case 1. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-2}, [\beta]^2\}$, where $\alpha \neq \beta$ are two real numbers.

Subcase 1. If $\beta < 0 < \alpha$, then by Lemma 2.1, we obtain

$$\alpha = \frac{1}{n-2}\sqrt{2(n-1)(n-2)} \quad \text{and} \quad \beta = -\frac{1}{2}\sqrt{2(n-1)(n-2)}. \tag{3.1}$$

Suppose G is a graph of order greater than 2. If K_3 or $K_2 \cup K_1$ is an induced subgraph of G, then by interlacing theorem we have $\alpha = 1$ and $\beta \leq -2$. Hence (3.1) implies that $n = 0$, a contradiction. If $\overline{K}_3$ or P_3 is an induced subgraph of G, then interlacing theorem yields that $\alpha = 2$ and $\beta \leq -1$. Thus, (3.1) implies that $n = 3$ and so G is Seidel equivalent to one of graphs $\overline{K}_3$ or P_3.

Subcase 2. If $\alpha < 0 < \beta$, then a similar argument shows that G is isomorphic to one of graphs K_3 or $K_2 \cup K_1$.

Case 2. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-2}, [\beta]^1, [\gamma]^1\}$ and α, β, γ are distinct Seidel eigenvalues. Lemma 2.2 implies that the multiplicity of the Seidel eigenvalue zero is at most 1. If $[0]^1 \in \mathrm{Spec}_S(G)$, then G has a single positive or a single negative Seidel eigenvalue and by Theorem 2.2 and 2.3 we conclude that G is Seidel equivalent to one of graphs K_n, $\overline{K}_n$, K_{n_1,n_2} or $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$, both of them are contradictions. By a similar argument, the cases $\beta < 0 < \alpha < \gamma$ and $\gamma < \alpha < 0 < \beta$ and $\beta < 0 < \gamma < \alpha$ and $\alpha < \gamma < 0 < \beta$ are impossible. Also, if either $\alpha < 0 < \beta < \gamma$ or $\gamma < \beta < 0 < \alpha$, then G is Seidel equivalent to one of graphs K_3, $K_2 \cup K_1$, $\overline{K}_3$ or P_3, all of which are impossible, and we are done. $\square$

For the graph G in $\mathcal{G}(n, n-3)$, we know that G has at most four distinct Seidel eigenvalues. In terms of the number and multiplicity of Seidel eigenvalues, we can divide all graphs in $\mathcal{G}(n, n-3)$ into three classes:

$$\mathcal{G}_1(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^3\}\Big\},$$

$$\mathcal{G}_2(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^2, [\gamma]^1\}\Big\},$$
$$\mathcal{G}_3(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^1, [\gamma]^1, [\rho]^1\}\Big\}.$$

Theorem 3.2 ([6])**.** *Let G be a graph of order n. Let $d \geq 1$ and $S(G)$ be a Seidel matrix of order $n \geq 2$ with smallest eigenvalue $\mu_n(G)$ of multiplicity $n-d \geq 1$ and suppose $\mu_n^2(G) \geq d+2$. Then*

$$n \leq \frac{d(\mu_n^2(G) - 1)}{\mu_n^2(G) - d},$$

with equality holds if and only if the spectrum of $S(G)$ is $\Big\{[\mu_n(G)]^{n-d}, [\frac{\mu_n(G)}{d}(n-d)]^d\Big\}$.

Example 3.1. Suppose $G \in \mathcal{G}(5, 2)$ and $\mathrm{rank}(S(G)) = 4$. Then by Figure 1, $\mathcal{G}(5, 2) = \{G_1, G_2, C_5, P_4 \cup K_1\}$, where G_1 and G_2 are as depicted in Figure 3. Furthermore, their Seidel spectra are $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^2, [0]^1, [\sqrt{5}]^2\}$.

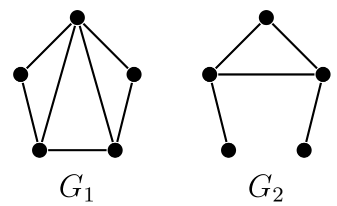

FIGURE 3. Two graphs G_1 and G_2 in Theorem 3.3

Theorem 3.3. *Let $G \in \mathcal{G}(n, n-3)$ be a graph of order $n \geq 6$ and* $\mathrm{rank}(S(G)) = n-1$. *Then $\mathcal{G}(n, n-3)$ is empty.*

Proof. If $\mathrm{rank}(S(G)) = n-1$, then $[0]^1 \in \mathrm{Spec}_S(G)$. Hence, we have the following cases.

Case 1. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [0]^1, [\beta]^2\}$, $\alpha \neq \beta \neq 0$. If $\alpha < \beta$, then by Lemma 2.1, obtain

$$\alpha = -\frac{1}{n-3}\sqrt{2n(n-3)} \quad \text{and} \quad \beta = \frac{1}{2}\sqrt{2n(n-3)}. \tag{3.2}$$

Suppose G is a graph of order at least 6 and contains one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ as an induced subgraph. First, notice that

$$\mathrm{Spec}_S(K_4) = \mathrm{Spec}_S(K_2 \cup K_2) = \mathrm{Spec}_S(K_1 \cup K_3) = \{[-3]^1, [1]^3\}.$$

Hence, interlacing theorem, yields that $\alpha = -3$, $\beta \geq 1$ and $1 \leq 0$, a contradiction. Suppose $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Since

$$\mathrm{Spec}_S(\overline{K}_4) = \mathrm{Spec}_S(C_4) = \mathrm{Spec}_S(K_{1,3}) = \{[-1]^3, [3]^1\},$$

the interlacing theorem implies that $\alpha = -1$ and $\beta \geq 3$. Hence, by (3.2), we find $n = -3$ which contradicts this fact that $n \geq 6$. If there is no graph with either Seidel eigenvalues $\{[-1]^3, [3]^1\}$ or $\{[-3]^1, [1]^3\}$ as an induced subgraph of G, then, by Figure 1, every induced subgraph on 4 vertices has the Seidel spectrum $\{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\}$. Hence, the interlacing theorem implies that $\alpha = -\sqrt{5}$, $\beta \geq \sqrt{5}$ and so by (3.2) we obtain $n = 5$ and $\beta = \sqrt{5}$, a contradiction. Now, suppose $\alpha > \beta$. By a similar argument, we can show that this case is also impossible.

Case 2. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [0]^1, [\beta]^1, [\gamma]^1\}$, where α, β, γ are three distinct non-zero real numbers. Suppose that G is not a graph with a single negative (positive) Seidel eigenvalue. Then we yield $n \geq 6$ and the following cases hold.

Subcase 2.1. If $\alpha < 0 < \beta < \gamma$, then we can suppose either K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G. The interlacing theorem yields that $\alpha = -3$ and $\gamma > \beta \geq 1$, a contradiction. Also, we may assume one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Again, interlacing theorem implies that $\alpha = -1$, $\beta > 0$ and $\gamma \geq 3$. Thus by Lemma 2.1 we find $\beta = \gamma = \frac{1}{2}(n - 3 + \sqrt{n^2 + 2n - 3})$, a contradiction. This means that P_4, $C_4 + e$, $K_3 + e$, $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G. Then interlacing theorem implies that $\alpha = -\sqrt{5}$, $\beta \geq 1$ and $\gamma \geq \sqrt{5}$. Hence, by Lemma 2.1, we obtain $\beta = \gamma = \frac{1}{2}(\sqrt{5}(n-3) + \sqrt{-3n^2 + 18n - 15})$ which contradicts this fact that $\beta < \gamma$.

Subcase 2.2. Let $\beta < \gamma < 0 < \alpha$. Since $S(\overline{G}) = -S(G)$ a similar argument with Subcase 2.1 shows that this case is also impossible. This completes the proof. $\square$

Theorem 3.4. *Let $G \in \mathcal{G}_1(n, n-3)$ be a graph of order $n \geq 4$. Then*

$$\mathcal{G}_1(n, n-3) = \{K_4, \overline{K}_4, C_4, K_{1,3}, K_2 \cup K_2, K_3 \cup K_1, C_5 \cup K_1, H_1, H_2, H_3\},$$

where H_i, $1 \leq i \leq 3$, are as depicted in Figure 4.

Proof. Let $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^3\}$. If $\alpha < \beta$, then by Lemma 2.1, we get

$$\alpha = \frac{-1}{n-3}\sqrt{3(n-1)(n-3)} \quad \text{and} \quad \beta = \frac{1}{3}\sqrt{3(n-1)(n-3)}. \tag{3.3}$$

Similar to the Theorem 3.3, we can show that one of graphs K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G and thus $\alpha = -3$ and $\beta \geq 1$. Hence, (3.3) implies that $n = 4$, $\beta = 1$ and G has either K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ as an induced subgraph of G. If $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then we have $\alpha = -1$, $\beta \geq 3$ and so by (3.3), we find $n = 0$ or $n = 3$, a contradiction with $n \geq 4$. If G has one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ as an induced subgraph, by interlacing theorem, we conclude that $\alpha = -\sqrt{5}$, $\beta \geq \sqrt{5}$ and (3.3) yields $n = 6$. Hence, $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$. By Figure 1, G is Seidel equivalent to one of graphs $C_5 \cup K_1$, H_1, H_2 or H_3. Next suppose that $\beta < \alpha$. It is not difficult to see that G is Seidel equivalent to one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ or $C_5 \cup K_1$ or H_1 or H_2 or H_3. This completes the proof. $\square$

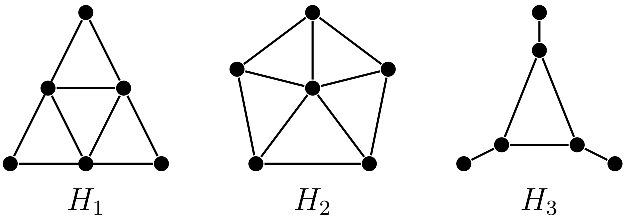

FIGURE 4. Three graphs H_1, H_2 and H_3 in Theorem 3.4

Theorem 3.5. *There is no graph in $\mathcal{G}_2(n, n-3)$ of order $n \geq 4$ with Seidel spectrum $\{[\alpha]^{n-3}, [\beta]^2, [\gamma]^1\}$, where α, β and γ satisfy in the following conditions:*

(i) $\gamma < 0 < \alpha < \beta$ *or* $\gamma < 0 < \beta < \alpha$ *or* $\beta < \alpha < 0 < \gamma$ *or* $\alpha < \beta < 0 < \gamma$*;*
(ii) $\alpha < 0 < \beta < \gamma$ *or* $\gamma < \beta < 0 < \alpha$*;*
(iii) $\beta < 0 < \gamma < \alpha$ *or* $\alpha < \gamma < 0 < \beta$*;*
(iv) $\beta < \gamma < 0 < \alpha$ *or* $\alpha < 0 < \gamma < \beta$*.*

Proof. **(i)** If G has a single positive or a single negative Seidel eigenvalue with multiplicity 1, then Theorems 2.2 and 2.3 yield that $\mathcal{G}_2(n, n-3)$ is empty.

(ii) Suppose that $\alpha < 0 < \beta < \gamma$ and $n \geq 5$ (if $n = 4$ then G has only one negative Seidel eigenvalue and $\mathcal{G}_2(n, n-3)$ is empty). If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then by interlacing theorem, we get $\alpha = -3$ and $\gamma > \beta \geq 1$. Thus Lemma 2.1 implies that

$$\begin{cases} -3(n-3) + 2\beta + \gamma = 0, \\ 9(n-3) + 2\beta^2 + \gamma^2 = n(n-1). \end{cases}$$

Consequently, $\gamma = 3(n-3) - 2\beta$ and so $\beta = \frac{1}{3}(3n - 3 \pm \sqrt{-3n(n-4)})$. Thus, $-3n(n-4) \geq 0$ if and only if $n = 4$. This means that $\beta = 1$ and $\gamma = 1$, a contradiction. Now, suppose one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Thus, $\alpha = -1$, $\beta > 0$ and $\gamma \geq 3$. Thus, by Lemma 2.1, we find $\gamma = n - 1$ and $\beta = -1$, a contradiction. If one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G, again one can prove that $\alpha = -\sqrt{5}$, $\beta \geq 1$ and $\gamma \geq \sqrt{5}$. Hence, Theorem 3.2 implies that $n \leq 6$. There is no graph with these conditions and thus in this case $\mathcal{G}_2(n, n-3)$ is empty.

By a similar argument, we can show that in all cases **(ii)**-**(iv)**, $\mathcal{G}_2(n, n-3)$ is empty and the proof is complete. $\square$

Theorem 3.6 ([3])**.** *Let G be a k-regular graph of order n. Then the Seidel spectrum of G is $\{[n-1-2k]^1, [-1-2\lambda_{n-1}]^1, \ldots, [-1-2\lambda_1]^1\}$, where λ_i $(1 \leq i \leq n)$ are eigenvalues of adjacency matrix $A(G)$.*

Theorem 3.7 ([4])**.** *Suppose that G is a graph of order n without isolated vertices. Then $\eta(G) = n - 3$ if and only if G is isomorphic to the complete tripartite graph K_{n_1,n_2,n_3}, where $n_1 + n_2 + n_3 = n$, $n_1, n_2, n_3 > 0$.*

In continuing by $\mathcal{Q}_n(4,2)$ we mean the collection of all connected regular graphs of order n with spectrum $\{[\lambda_1]^1, [\lambda_2]^1, [\lambda_3]^{t_1}, [\lambda_4]^{t_2}\}$, $t_1+t_2=n-2$. Also, $\mathcal{Q}_n(4,2,-1)$ (resp. $\mathcal{Q}_n(4,2,0)$) denotes the set of all graphs in $\mathcal{Q}_n(4,2)$, in which -1 (resp. 0) is an eigenvalue.

Let G be a graph of order n and adjacency matrix A. By $G\circledast J_m$ we mean a new graph obtained from G by replacing every vertex of G with a clique K_m and two such cliques are adjacent (namely for two cliques Q_1 and Q_2 all vertices of Q_1 are adjacent with all vertices of Q_2) if and only if their corresponding vertices are joined in G, see [11]. One can see that the adjacency matrix of $G\circledast J_m$ is $A\circledast J_m=(A+I_n)\otimes J_m-I_{nm}$.

Theorem 3.8 ([11])**.** *The connected regular graph G is in $\mathcal{Q}_n(4,2,0)$ if and only if $G=\overline{K_{s,s}^{-}\circledast J_t}$, where $n=2st$, $s\geq 3$ and $t\geq 1$.*

Theorem 3.9 ([11])**.** *The connected regular graph G is in $\mathcal{Q}_n(4,2,-1)$ if and only if $G=K_{s,s}\circledast J_t$, where $s,t\geq 2$, or $G=K_{s,s}^{-}\circledast J_t$, where $n=2st$, $s\geq 3$ and $t\geq 1$.*

Theorem 3.10 ([11])**.** *There is no connected k-regular graph of order $n\geq 4$ with adjacency spectrum $\{[k]^1, [\lambda_2]^1, [\lambda_3]^1, [\lambda_4]^{n-3}\}$.*

Theorem 3.11. *Let $G\in\mathcal{G}_2(n,n-3)$ be a connected regular graph of order $n\geq 4$. Then the following cases hold.*

- **(i)** *If $\gamma<\alpha<\beta$, then G is isomorphic to the one of graphs $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$, $n\equiv 0 \pmod 3$ or $\overline{K_{3,3}^{-}\circledast J_{\frac{n}{6}}}$, $n\equiv 0 \pmod 6$.*
- **(ii)** *If $\beta<\alpha<\gamma$, then G is isomorphic to $K_{3,3}^{-}\circledast J_{\frac{n}{6}}$, $n\equiv 0 \pmod 6$.*

Proof. **(i)** Let G be a graph of order n. By Theorem 3.5, we can assume that $\gamma<\alpha<0<\beta$. If one of graphs K_4 or $K_2\cup K_2$ or $K_1\cup K_3$ is an induced subgraph of G, then by interlacing theorem we get $\alpha=1$, a contradiction. If G has one of graphs P_4 or C_4+e or K_3+e or $K_2\cup\overline{K}_2$ or $P_3\cup K_1$ as an induced subgraph of G, then we obtain $\gamma\leq-\sqrt{5}$, $\alpha=-1$, $\beta\geq\sqrt{5}$ and so by Lemma 2.1, we get $\beta=\frac{2n}{3}-1$ and $\gamma=-\frac{n}{3}-1$. As well as, if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, Lemma 2.1 implies that $\beta=\frac{2n}{3}-1$ and $\gamma=-\frac{n}{3}-1$. Therefore,

$$\mathrm{Spec}_S(G)=\left\{\left[\frac{-n}{3}-1\right]^1, [-1]^{n-3}, \left[\frac{2n}{3}-1\right]^2\right\},$$

where $n\equiv 0 \pmod 3$. By Theorem 3.6, the adjacency spectrum of G is

$$\mathrm{Spec}(G)=\left\{\left[\frac{-n}{3}\right]^2, [0]^{n-3}, \left[\frac{2n}{3}\right]^1\right\}$$

or

$$\mathrm{Spec}(G)=\left\{\left[\frac{-n}{3}\right]^2, [0]^{n-4}, \left[\frac{n}{6}\right]^1, \left[\frac{n}{2}\right]^1\right\}.$$

Suppose $\mathrm{Spec}(G)=\{[\frac{-n}{3}]^2, [0]^{n-3}, [\frac{2n}{3}]^1\}$, since $\eta(G)=n-3$, by Theorem 3.7, G is isomorphic to $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$. If $\mathrm{Spec}(G)=\{[\frac{-n}{3}]^2, [0]^{n-4}, [\frac{n}{6}]^1, [\frac{n}{2}]^1\}$, then Theorem 3.8 implies that G is isomorphic to $\overline{K_{3,3}^{-}\circledast J_{\frac{n}{6}}}$, where $n\equiv 0 \pmod 6$.

(ii) Assume that $\beta < 0 < \alpha < \gamma$. It is not difficult to see that $\alpha = 1$ and Lemma 2.1 yields that $\gamma = \frac{n}{3} + 1$ and $\beta = 1 - \frac{2n}{3}$. By Theorem 3.6, we obtain

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{6} - 1\right]^1, [-1]^{n-3}, \left[\frac{n}{3} - 1\right]^1, \left[\frac{5n}{6} - 1\right]^1 \right\} \tag{3.4}$$

or

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{6} - 1\right]^1, [-1]^{n-4}, \left[\frac{n}{3} - 1\right]^2, \left[\frac{n}{2} - 1\right]^1 \right\}. \tag{3.5}$$

Theorem 3.10 implies that (3.4) is impossible. If (3.5) holds, then Theorem 3.9 yields that G is isomorphic to the graph $K_{3,3}^{-} \circledast J_{\frac{n}{6}}$, $n \equiv 0 \pmod 6$ and this completes the proof. $\square$

Example 3.2. Suppose $n \equiv 0 \pmod 3$. For two graphs $G_1 = K_{\frac{n}{3},\frac{n}{3}} \cup \overline{K}_{\frac{n}{3}}$ and $G_2 = \overline{K_{\frac{n}{3},\frac{n}{3}} \cup \overline{K}_{\frac{n}{3}}}$, we obtain

$$\begin{aligned} \mathrm{Spec}_S(G_1) &= \left\{ \left[\frac{-n}{3} - 1\right]^1, [-1]^{n-3}, \left[\frac{2n}{3} - 1\right]^2 \right\}, \\ \mathrm{Spec}_S(G_2) &= \left\{ \left[\frac{-2n}{3} + 1\right]^2, [1]^{n-3}, \left[\frac{n}{3} + 1\right]^1 \right\}. \end{aligned}$$

This implies that both graphs G_1 and G_2 are in $\mathcal{G}_2(n, n-3)$.

Example 3.3. Suppose $n = 6$. By using a program in SageMath software [12], we conclude that all graphs in $\mathcal{G}_2(6, 3)$ are as depicted in Figures 5 and 6.

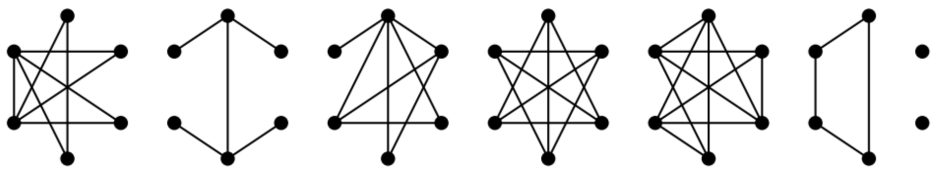

FIGURE 5. All graphs in $\mathcal{G}_2(6, 3)$ with Seidel spectrum $\{[3]^2, [-1]^3, [-3]^1\}$

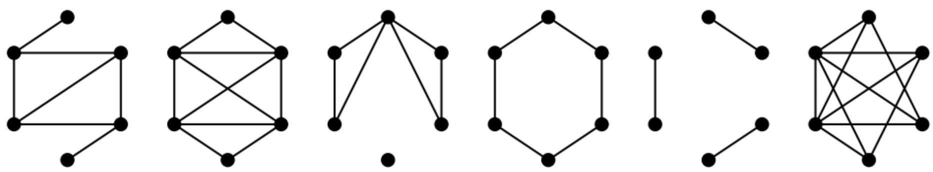

FIGURE 6. All graphs in $\mathcal{G}_2(6, 3)$ with Seidel spectrum $\{[3]^1, [1]^3, [-3]^2\}$

In what follows, by mG we mean the disjoint union of m copies of G, namely $\underbrace{G \cup \cdots \cup G}_{m \text{ times}}$.

Theorem 3.12 ([3]). **(i)** *A graph G with the smallest Seidel eigenvalue larger than -3 is switching equivalent to graphs $\overline{K}_n$ or $K_2 \cup \overline{K}_{n-2}$ or one of graphs depicted in Figure* 7.

(ii) *A graph G with smallest Seidel eigenvalue greater than or equal with -3 is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$ (namely the complement of the line graph of K_8).*

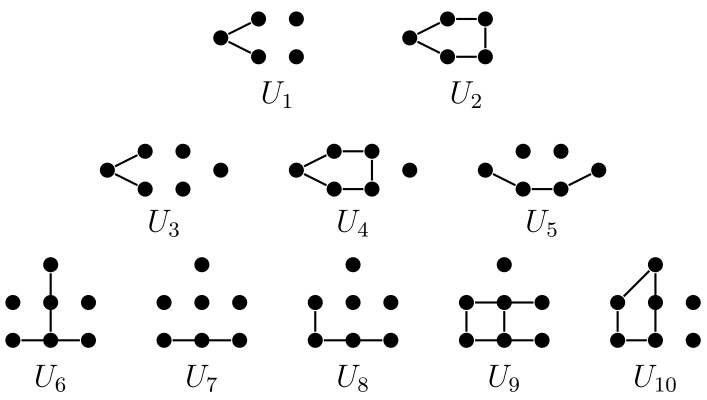

FIGURE 7. Ten graphs with the smallest Seidel eigenvalue larger than -3.

TABLE 1. Graphs together with the Seidel spectra in Theorem 3.12.

Graphs	Seidel spectrum
$\overline{K}_n$	$\{[-1]^{n-1}, [n-1]^1\}$
$K_2 \cup \overline{K}_{n-2}$	$\{[\frac{n}{2} - 2 - \frac{1}{2}\sqrt{(n+6)(n-2)}]^1, [-1]^{n-3}, [1]^1, [\frac{n}{2} - 2 + \frac{1}{2}\sqrt{(n+6)(n-2)}]^1\}$
U_1	$\{[-2.56]^1, [-1]^2, [1.56]^1, [3]^1\}$
U_2	$\{[-\sqrt{5}]^2, [0]^1, [\sqrt{5}]^2\}$
U_3	$\{[-2.75]^1, [-1]^3, [1.69]^1, [4.06]^1\}$
U_3	$\{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$
U_5	$\{[-2.6]^1, [-2.24]^1, [-1], [0.11]^1, [2.24]^1, [3.49]^1\}$
U_6	$\{[-2.78]^1, [-2.46]^1, [-1]^2, [0.29]^1, [2.49]^1, [4.46]^1\}$
U_7	$\{[-2.9]^1, [-1]^4, [1.74]^1, [5.15]^1\}$
U_8	$\{[-2.83]^1, [-2.24]^1, [-1]^2, [0.15]^1, [2.24]^1, [4.68]^1\}$
U_9	$\{[-2.6]^2, [-2]^1, [0.11]^2, [3.49]^2\}$
U_{10}	$\{[-2.7]^1, [-2.24]^1, [-1]^1, [2.24]^2, [3.7]^1\}$
mK_2 ($m \geq 2$)	$\{[-3]^{m-1}, [1]^m, [n-3]^1\}$
$\overline{T(8)}$	$\{[-3]^{21}, [9]^7\}$

Example 3.4. Suppose $G \in \mathcal{G}_3(4,1)$, then we have $\mathcal{G}_3(4,1) = \{K_2 \cup \overline{K}_2, P_3 \cup K_1, K_3 + e, P_4, C_4 + e\}$ and $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\}$.

Theorem 3.13. *Let $G \in \mathcal{G}_3(n, n-3)$ be a graph of order $n \geq 5$. Then the following cases hold.*

- **(i)** *There is no graph in $\mathcal{G}_3(n, n-3)$ which satisfies in the following conditions:*

  $\alpha < \beta < \gamma < 0 < \rho$ *or* $\beta < \alpha < \gamma < 0 < \rho$ *or* $\beta < \gamma < \alpha < 0 < \rho$ *or*
  $\rho < 0 < \gamma < \beta < \alpha$ *or* $\rho < 0 < \gamma < \alpha < \beta$ *or* $\rho < 0 < \alpha < \gamma < \beta$ *or*
  $\alpha < \beta < 0 < \gamma < \rho$ *or* $\rho < \gamma < 0 < \beta < \alpha$.

- **(ii)** *If $\alpha < 0 < \beta < \gamma < \rho$ or $\rho < \gamma < \beta < 0 < \alpha$, then G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$.*

Proof. **(i)** If G has a single positive or a single negative Seidel eigenvalue with multiplicity 1, then by Theorems 2.2 and 2.3, $\mathcal{G}_3(n, n-3)$ is empty. Now, suppose $\alpha < \beta < 0 < \gamma < \rho$ and $n \geq 5$. If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then by interlacing theorem, we get $\alpha = -3$ and $\beta \geq 1$, a contradiction. If G has one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ as an induced subgraph, then we yield $\alpha = -\sqrt{5}$, $-1 \leq \beta < 0$, $\gamma \geq 1$ and $\rho \geq \sqrt{5}$. As well as, if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then we obtain $\alpha = -1$, $-1 \leq \beta < 0$, $\gamma > 0$ and $\rho \geq 3$. Since, $\alpha > -3$, applying Theorem 3.12 (i) and Table 1, we achieve a contradiction. By a similar argument the case $\rho < \gamma < 0 < \beta < \alpha$ is impossible.

(ii) Suppose $\alpha < 0 < \beta < \gamma < \rho$ and $n \geq 5$. If one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G, then $\alpha = -\sqrt{5}$, $\beta > 0$, $\gamma \geq 1$ and $\rho \geq \sqrt{5}$ and if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then $\alpha = -1$, $\beta, \gamma > 0$ and $\rho \geq 3$, a contradiction with $\alpha > -3$. If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then interlacing theorem, yields $\alpha = -3$ and $\beta, \gamma, \rho \geq 1$. Theorem 3.12 (ii) implies that G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$. Let $\rho < \gamma < \beta < 0 < \alpha$. Since $S(G) = -S(\overline{G})$, a similar argument shows that in this case G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$. □

Example 3.5. Suppose $n = 5$. By Figure using a method described in 1, we conclude that all graphs in $\mathcal{G}_3(5, 2)$, where $\beta < \alpha < 0 < \gamma < \rho$ and $\rho < \gamma < 0 < \alpha < \beta$, are as depicted in Figures 8 and 9, respectively.

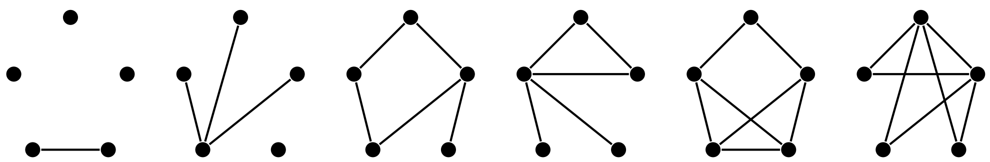

FIGURE 8. All graphs in $\mathcal{G}_3(5, 2)$ with Seidel spectrum $\{[-2.37]^1, [-1]^2, [1]^1, .37]^1\}$

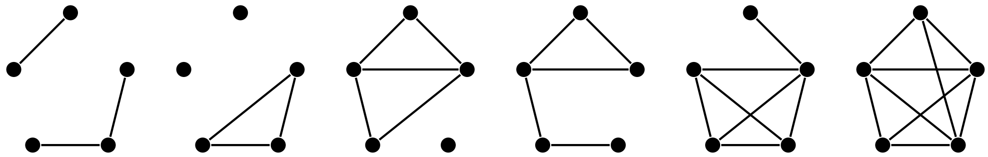

FIGURE 9. All graphs in $\mathcal{G}_3(5,2)$ with Seidel spectrum $\{[-3.37]^1, [-1]^1, [1]^2, [2.37]^1\}$

Conjecture 3.1. Let $G \in \mathcal{G}_3(n, n-3)$ be a graph of order $n \geq 6$. Then the following cases hold:

i) if $\beta < \alpha < 0 < \gamma < \rho$, then G is Seidel equivalent to $K_{i,j} \cup \overline{K}_p$;
ii) if $\rho < \gamma < 0 < \alpha < \beta$, then G is Seidel equivalent to $\overline{K_{i,j} \cup \overline{K}_p}$,

where $1 \leq i \leq [\frac{n}{3}]$, $i \leq j \leq n-3$ and $3 \leq p \leq n-(i+j)$ unless $n \equiv 0 \pmod 3$ and $i = j = p = \frac{n}{3}$.

Remark 3.1. Suppose $G \in \mathcal{G}_3(n, n-3)$ is a graph of order $n \geq 6$. If the Seidel eigenvalues of G are ordered as $\beta < \alpha < 0 < \gamma < \rho$, then it is not difficult to see that one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ or $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G and by interlacing theorem, we have $\alpha = -1$. Also, if the Seidel eigenvalues of G satisfy in $\rho < \gamma < 0 < \alpha < \beta$, by a similar argument we can show that $\alpha = 1$.

REFERENCES

[1] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1993.
[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* Elseyier Science Publishing Co., New York, 1976.
[3] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs,* Universitext, Springer, New York, 2012.
[4] M. N. Ellingham, *Basic subgraphs and graph spectra,* Austras. J. Combin. **8** (1993), 247–265.
[5] E. Ghorbani, *On eigenvalues of Seidel matrices and Haemers' conjecture*, Des. Codes Cryptogr. **84**(1-2) (2017), 189–195. `https://doi.org/10.1007/s10623-016-0248-x`
[6] G. R. W. Greaves, *Equiangular line systems and switching classes containing regular graphs,* Linear Algebra Appl. **536** (2018), 31–51. `https://doi.org/10.1016/j.laa.2017.09.008`
[7] G. Greaves, J. H. Koolen, A. Munemasa and F. Szöllősi, *Equiangular lines in Euclidean spaces,* J. Combin. Theory Ser. A **138** (2016), 208–235. `https://doi.org/10.1016/j.jcta.2015.09.008`
[8] W. H. Haemers, *Seidel switching and graph energy,* MATCH Commun. Math. Comput. Chem. **68** (2012), 653–659. `https://doi.org/10.2139/ssrn.2026916`
[9] M. Hakimi-Nezhaad and M. Ghorbani, *On the Estrada index of Seidel matrix*, Math. Interdisc. Res. **5**(1)(2020), 43–54. `https://doi.org/10.22052/MIR.2019.179267.1128`
[10] M. Hakimi-Nezhaad and M. Ghorbani, *Seidel borderenergetic graphs*, TWMS J. App. Eng. Math. **10**(2) (2020), 389–399.
[11] X. Huang and Q. Huang, *On regular graphs with four distinct eigenvalues,* Linear Algebra Appl. **512** (2017), 219–233. `https://doi.org/10.1016/j.laa.2016.09.043`
[12] Sage Mathematics Software (Version 8.6), 2019, `www.sagemath.org`.

[13] J. H. van Lint and J. J. Seidel, *Equilateral point sets in elliptic geometry,* Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series A: Mathematical Sciences **28** (1966), 335–348. https://doi.org/10.1016/S1385-7258(66)50038-5

[1]Department of Mathematics, Faculty of Science,
Shahid Rajaee Teacher Training University,
Tehran, 16785-136, I. R. Iran
Email address: mghorbani@sru.ac.ir
Email address: m.hakiminezhaad@sru.ac.ir

[2]School of Mathematical Sciences,
South China Normal University,
Guangzhou 510631, P. R. China
Email address: zhoubo@scnu.edu.cn

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 187–201.

GROWTH OF SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

SAMIR CHERIEF[1] AND SAADA HAMOUDA[1]

Abstract. In this paper, we investigate the growth of solutions of the differential equation

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0,$$

where $A(z)$, $B(z)$ are analytic functions in the closed complex plane except at z_0 and a, b are complex constants such that $ab \neq 0$ and $a = cb$, $c > 1$. Another case has been studied for higher order linear differential equations with analytic coefficients having the same order near a finite singular point.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [11, 15, 20]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [2, 12–14, 16]. Recently in [6, 10], Fettouch and Hamouda investigated the growth of solutions of certain linear differential equations near a finite singular point. In this paper, we continue this investigation near a finite singular point to study other types of linear differential equations.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function near z_0

Key words and phrases. Linear differential equations, growth of solutions, finite singular point.
2010 *Mathematics Subject Classification.* Primary: 34M10. Secondary: 30D35.
DOI 10.46793/KgJMat2302.187C
Received: March 01, 2019.
Accepted: August 03, 2020.

by

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t} dt - n(\infty,f)\log r, \tag{1.1}$$

where $n(t,f)$ counts the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ each pole according to its multiplicity and the proximity function by

$$m_{z_0}(r,f) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^{+} \left| f\left(z_0 - re^{i\varphi}\right) \right| d\varphi. \tag{1.2}$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f). \tag{1.3}$$

In addition, the order of meromorphic function $f(z)$ near z_0 is defined by

$$\sigma_T(f,z_0) = \limsup_{r \to 0} \frac{\log^{+} T_{z_0}(r,f)}{-\log r}. \tag{1.4}$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$\sigma_M(f,z_0) = \limsup_{r \to 0} \frac{\log^{+}\log^{+} M_{z_0}(r,f)}{-\log r}, \tag{1.5}$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$.

If $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $0 < \sigma_T(f,z_0) = \sigma < \infty$, then we can define the type of f as the following:

$$\tau_T(f,z_0) = \limsup_{r \to 0} r^{\sigma} T_{z_0}(r,f).$$

If $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $0 < \sigma_M(f,z_0) = \sigma < \infty$, we have also another definition of the type of f as the following:

$$\tau_M(f,z_0) = \limsup_{r \to 0} r^{\sigma} \log^{+} M_{z_0}(r,f).$$

In the usual manner, we define the hyper order near z_0 as follows:

$$\sigma_{2,T}(f,z_0) = \limsup_{r \to 0} \frac{\log^{+}\log^{+} T_{z_0}(r,f)}{-\log r}, \tag{1.6}$$

$$\sigma_{2,M}(f,z_0) = \limsup_{r \to 0} \frac{\log^{+}\log^{+}\log^{+} M_{z_0}(r,f)}{-\log r}. \tag{1.7}$$

Remark 1.1. It is shown in [6] that if f is a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ and $g(w) = f\left(z_0 - \frac{1}{w}\right)$ then $g(w)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R,g) = T_{z_0}\left(\frac{1}{R}, f\right),$$

and so $\sigma(f,z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then $g(w)$ is entire and thus $\sigma_T(f,z_0) = \sigma_M(f,z_0)$ and $\sigma_{2,T}(f,z_0) = \sigma_{2,M}(f,z_0)$.

So, we can use the notation $\sigma(f, z_0)$ without any ambiguity. But concerning the type, as in the complex plane, $\tau_T(f, z_0)$ does not equal to $\tau_M(f, z_0)$. For example, for the function $f(z) = \exp\left\{\frac{1}{z_0 - z}\right\}$, we have $M_{z_0}(r, f) = \exp\left\{\frac{1}{r}\right\}$, then $\sigma_M(f, z_0) = 1$ and $\tau_M(f, z_0) = 1$. On the other side, we have

$$T_{z_0}(r, f) = m_{z_0}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left| f\left(z_0 - re^{i\varphi}\right)\right| d\varphi = \frac{1}{\pi r},$$

so $\sigma_T(f, z_0) = 1$ and $\tau_T(f, z_0) = \frac{1}{\pi}$.

Definition 1.1. The linear measure of a set $E \subset (0, \infty)$ is defined as $\int_0^\infty \chi_E(t)\, dt$ and the logarithmic measure of E is defined by $\int_0^\infty \frac{\chi_E(t)}{t} dt$, where $\chi_E(t)$ is the characteristic function of the set E.

The linear differential equation

$$f'' + A(z)\, e^{az} f' + B(z)\, e^{bz} f = 0, \tag{1.8}$$

where $A(z)$ and $B(z)$ are entire functions, is investigated by many authors; see for example [1, 3, 4, 7]. In [3], Chen proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$ or $c > 1$, then every solution $f(z) \not\equiv 0$ of (1.8) is of infinite order. In 2012, Hamouda proved results similar to (1.8) in the unit disc concerning the differential equation

$$f'' + A(z)\, e^{\frac{a}{(z_0 - z)^\mu}} f' + B(z)\, e^{\frac{b}{(z_0 - z)^\mu}} f = 0, \tag{1.9}$$

where $\mu > 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$, see [8]. Recently, Fettouch and Hamouda proved the following two results.

Theorem 1.1 ([6])**.** *Let z_0, a, b be complex constants such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$ and n be a positive integer. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z) \exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0$$

satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = n$.

Theorem 1.2 ([6])**.** *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.10}$$

satisfies $\sigma(f, z_0) = \infty$, with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In this paper, we will investigate the case $c > 1$ to complete the remaining case in Theorem 1.1, in the following two results.

Theorem 1.3. *Let* $n \in \mathbb{N} \setminus \{0\}$, $A(z) \not\equiv 0$, $B(z) \not\equiv 0$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *such that* $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. *Let* a, b *be complex constants such that* $ab \neq 0$ *and* $a = cb$, $c > 1$. *Then every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f'' + A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z) \exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \tag{1.11}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfies* $\sigma(f, z_0) = \infty$.

Theorem 1.4. *Let* $n \in \mathbb{N} \setminus \{0\}$, $A(z) \not\equiv 0$, $B(z) \not\equiv 0$ *be polynomials. Let* a, b *be complex constants such that* $ab \neq 0$ *and* $a = cb$, $c > 1$. *Then every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f'' + A\left(\frac{1}{z_0 - z}\right) \exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B\left(\frac{1}{z_0 - z}\right) \exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \tag{1.12}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfies* $\sigma(f, z_0) = \infty$, *with* $\sigma_2(f, z_0) = n$.

In the following result, we will improve Theorem 1.2 by studying the case when $\max\{\sigma(A_j, z_0) : j \neq 0\} \leq \sigma(A_0, z_0)$.

Theorem 1.5. *Let* $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying the following conditions*

i) $0 < \sigma(A_j, z_0) \leq \sigma(A_0, z_0) < \infty$, $j = 1, \ldots, k-1$;

ii) $\max\{\tau_M(A_j, z_0) : \sigma(A_j, z_0) = \sigma(A_0, z_0)\} < \tau_M(A_0, z_0)$.

Then every solution $f(z) \not\equiv 0$ *of* (1.10) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfies* $\sigma(f, z_0) = \infty$, *with* $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

Remark 1.2. If we replace τ_M by τ_T in the condition *ii)* in Theorem 1.5 we get the same result.

We can find the analogs of Theorem 1.5 in the complex plane and in the unit disc in ([18, Theorem 1], [9, Theorem 3]).

We signal here that when the coefficients $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$, it may happen that the solution f of (1.10) is not analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$. For example, $f(z) = z$ is a solution of the differential equation

$$f'' - \exp\left\{\frac{1}{z}\right\} f' + \frac{1}{z} \exp\left\{\frac{1}{z}\right\} f = 0, \tag{1.13}$$

where the coefficients of (1.13) are analytic in $\overline{\mathbb{C}} \setminus \{0\}$, but the solution $f(z) = z$ is not analytic in $\overline{\mathbb{C}} \setminus \{0\}$. That's why we wrote in our results (every solution $f(z) \not\equiv 0$ of (1.10), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}, \ldots$) So, it is a priori assumed that f is analytic in Theorem 1.1 and Theorem 1.2. It is similar to the case when the coefficients are meromorphic in $\mathbb{C}$, it is well known that the solutions of (1.10) may be non meromorphic in $\mathbb{C}$.

2. Preliminary Lemmas

To prove these results we need the following lemmas.

Lemma 2.1 ([6]). *Let $A(z) \not\equiv 0$ be analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$, with $\sigma(A, z_0) < n$, n is a positive integer. Set $g(z) = A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\}$, where $a = \alpha + i\beta \neq 0$ is complex number, $z_0 - z = re^{i\varphi}$, $\delta_a(\varphi) = \alpha \cos(n\varphi) + \beta \sin(n\varphi)$ and $H = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$ (obviously, H is of linear measure zero). Then for any given $\varepsilon > 0$ and for any $\varphi \in [0, 2\pi) \setminus H$, there exists $r_0 > 0$ such that for $0 < r < r_0$, we have*

(i) if $\delta_a(\varphi) > 0$, then

$$\exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\}; \tag{2.1}$$

(ii) if $\delta_a(\varphi) < 0$, then

$$\exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\}. \tag{2.2}$$

Lemma 2.2 ([6]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Let $\alpha > 0$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$. Then*

i) there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $A > 0$ that depends on α and j such that for all $r = |z - z_0|$ satisfying $r \in (0,1) \setminus E_1$ we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq A\left[\frac{1}{r^2}T_{z_0}(\alpha r, f)\log T_{z_0}(\alpha r, f)\right]^j; \tag{2.3}$$

ii) there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero and a constant $A > 0$ that depends on α and j such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that (2.3) holds for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$.*

Lemma 2.3 ([10]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma(f, z_0) < \infty$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0,1) \setminus E_1$ we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{r^{k(\sigma+1)+\varepsilon}}, \qquad k \in \mathbb{N}. \tag{2.4}$$

Lemma 2.4. *Let $f(z)$ be a non-constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Then $\sigma(f', z_0) = \sigma(f, z_0)$.*

Proof. By Remark 1.1, $g(w) = f\left(z_0 - \frac{1}{w}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma(g) = \sigma(f, z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g') = \sigma(g)$ (see [17, 19]). We have $f'(z) = \frac{1}{w^2}g'(w)$. Set $h(w) = \frac{1}{w^2}g'(w)$. Obviously, we have $\sigma(h) = \sigma(g')$. In the other hand, by Remark 1.1, we have $\sigma(h) = \sigma(f', z_0)$. So, we conclude that $\sigma(f', z_0) = \sigma(f, z_0)$. □

Lemma 2.5. *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ and suppose that $\left|f^{(k)}(z)\right|$ is unbounded on some ray $\arg(z_0-z)=\theta$. Then there exists an infinite sequence of points $z_m=z_0-r_me^{i\theta}$, $m=1,2,\dots$, where $r_m\to 0$, such that $f^{(k)}(z_m)\to\infty$ and*

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right|\leq M,\quad M>0,\ j=0,1,\dots,k-1. \tag{2.5}$$

Proof. Let $M\left(r,\theta,f^{(k)}\right)$ denotes the maximum modulus of $f^{(k)}$ on the line segment $\left[z_0-r_1e^{i\theta},z_0-re^{i\theta}\right]$. Clearly, we may construct a sequence of points $z_m=z_0-r_me^{i\theta}$, $m\geq 1$, $r_m\to 0$, such that $M\left(r,\theta,f^{(k)}\right)=\ f^{(k)}(z_m)\to\infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1,z_m]$ we have

$$\begin{aligned} f^{(j)}(z_m)=&f^{(j)}(z_1)+f^{(j+1)}(z_1)(z_m-z_1)\\ &+\cdots+\frac{1}{(k-j-1)}f^{(k-1)}(z_1)(z_m-z_1)^{k-j-1}+\int_{z_1}^{z_m}\cdots\int_{z_1}^{y}f^{(k)}(x)\,dxdy\cdots dt, \end{aligned}$$

and by an elementary triangle inequality estimate we obtain

$$\begin{aligned} \left|f^{(j)}(z_m)\right|\leq&\left|f^{(j)}(z_1)\right|+\left|f^{(j+1)}(z_1)\right||z_m-z_1|\\ &+\cdots+\frac{1}{(k-j-1)}\left|f^{(k-1)}(z_1)\right||z_m-z_1|^{k-j-1}\\ &+\frac{1}{(k-j)}\left|f^{(k)}(z_m)\right||z_m-z_1|^{k-j}. \end{aligned} \tag{2.6}$$

From (2.6) and by taking into account that when $m\to\infty$, $f^{(k)}(z_m)\to\infty$, $z_m\to z_0$, we obtain

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right|\leq M,\quad M>0. \qquad\square$$

Lemma 2.6. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite order $\sigma(f,z_0)=\sigma>0$ and finite type $\tau_M(f,z_0)=\tau>0$. Then for any given $0<\beta<\tau$ there exists a set $F\subset(0,1)$ of infinite logarithmic measure such that for all $r\in F$ we have*

$$\log M_{z_0}(r,f)>\frac{\beta}{r^\sigma}.$$

Proof. By the definition of $\tau_M(f,z_0)$, there exists a decreasing sequence $\{r_m\}\to 0$ satisfying $\frac{m}{m+1}r_m>r_{m+1}$ and

$$\lim_{m\to\infty}r_m^\sigma\log M_{z_0}(r_m,f)=\tau.$$

Then there exists m_0 such that for all $m>m_0$ and for a given $\varepsilon>0$ we have

$$\log M_{z_0}(r_m,f)>\frac{\tau-\varepsilon}{r_m^\sigma}. \tag{2.7}$$

There exists m_1 such that for all $m > m_1$ and for a given $0 < \varepsilon < \tau - \beta$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{\tau - \varepsilon}. \tag{2.8}$$

By (2.7) and (2.8), for all $m > m_2 = \max\{m_0, m_1\}$ and for any $r \in \left[\frac{m}{m+1} r_m, r_m\right]$, we have

$$\log M_{z_0}(r, f) > \log M_{z_0}(r_m, f) > \frac{\tau - \varepsilon}{r_m^{\sigma}} > \frac{\tau - \varepsilon}{r^{\sigma}} \left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{r^{\sigma}}.$$

Set $F = \bigcup_{m=m_2}^{\infty} \left[\frac{m}{m+1} r_m, r_m\right]$. Then we have

$$\sum_{m=m_2}^{\infty} \int_{\frac{m}{m+1} r_m}^{r_m} \frac{dt}{t} = \sum_{m>m_2} \log \frac{m+1}{m} = \infty. \qquad \square$$

Lemma 2.7. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with hyper-order $\sigma_2(f, z_0) = \sigma$ and let $V_{z_0}(r)$ be the central index of f (see* [10]*). Then*

$$\limsup_{r \to 0} \frac{\log^+ \log^+ V_{z_0}(r)}{-\log r} = \sigma. \tag{2.9}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f, z_0) = \sigma$ and if $V(R)$ denotes the central index of g, then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [5, Lemma 2], we have

$$\limsup_{R \to +\infty} \frac{\log^+ \log^+ V(R)}{\log R} = \sigma. \tag{2.10}$$

Substituting R by $\frac{1}{r}$ in (2.10), we get (2.9). $\square$

Lemma 2.8. *Let $A_j(z)$, $j = 0, \ldots, k-1$, be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that $\sigma(A_j, z_0) \leq \alpha < \infty$. If f is a solution of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.11}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then $\sigma_2(f, z_0) \leq \alpha$.

Proof. For any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $0 < r = |z_0 - z| < r_0$, we have

$$|A_j(z)| \leq \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.12}$$

By the Wiman-Valiron near a finite singular point (see [10]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^{j}, \quad j = 0, \ldots, k-1, \tag{2.13}$$

where $V_{z_0}(r)$ is the central index of f and $|f(z_r)| = M(r,f) = \max_{|z_0-z|=r} |f(z)|$. From (2.11), we can write

$$-\frac{f^{(k)}}{f} = A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z). \tag{2.14}$$

Substituting (2.12) and (2.13) into (2.14), we obtain

$$(1+o(1))\frac{(V_{z_0}(r))^k}{r^k} \leq k\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}\frac{(V_{z_0}(r))^{k-1}}{r^{k-1}}(1+o(1)),$$

and so

$$V_{z_0}(r) \leq kr\exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}(1+o(1)). \tag{2.15}$$

By (2.15), we get

$$\sigma_2(f,z_0) \leq \alpha.$$

□

It is easy to prove the following lemma.

Lemma 2.9. *Let $P(z) = a_n z^n + \cdots + a_0$, with $a_n \neq 0$ be a polynomial and $A(z) = P\left(\frac{1}{z_0-z}\right)$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $0 < r = |z_0 - z| \leq r_0$, the inequalities*

$$(1-\varepsilon)\frac{|a_n|}{r^n} \leq |P(z)| \leq (1+\varepsilon)\frac{|a_n|}{r^n}$$

hold.

Lemma 2.10. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with the hyper-order $\sigma_2(f,z_0) = \alpha$, and let $V_{z_0}(r)$ be the central index of f. Let $E \subset (0,1]$ be a set of finite logarithmic measure. Then, there exists a sequence of points $\left\{z_m = z_0 - r_m e^{i\theta_m}\right\}$, $m \geq 1$, such that $|f(z_m)| = M_{z_0}(r_m, f)$, $\lim_{m\to\infty}\theta_m = \theta^* \in [0,2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have*

$$\limsup_{r\to 0}\frac{\log^+ V_{z_0}(r)}{-\log r} = \infty, \tag{2.16}$$

$$\exp\left\{\frac{1}{r^{\alpha-\varepsilon}}\right\} \leq V_{z_0}(r) \leq \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.17}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f,z_0) = \alpha$ and if $V(R)$ denotes the central index of g then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [3, Remark 1] we have

$$\limsup_{R\to\infty}\frac{\log^+ V_{z_0}(R)}{\log R} = \infty, \tag{2.18}$$

$$\exp\left\{R^{\alpha-\varepsilon}\right\} \leq V(R) \leq \exp\left\{R^{\alpha+\varepsilon}\right\}. \tag{2.19}$$

Substituting R by $\frac{1}{r}$ in (2.18) and (2.19), we get (2.16) and (2.17). □

3. Proof of Theorems

Proof of Theorem 1.3. We assume that $\sigma(f, z_0) = \sigma < \infty$, and we prove that is failing. By Lemma 2.3, for any given $\varepsilon > 0$ there exists a set $E \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \backslash E$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \backslash E$ and $r = |z - z_0| < r_0$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{r^{\sigma+1+\varepsilon}}. \tag{3.1}$$

Set $a = \alpha + i\beta$, $z_0 - z = re^{i\theta}$, $\delta = \delta_a(\theta) = \alpha\cos(n\theta) + \beta\sin(n\theta)$,

$$H = \{\theta \in [0, 2\pi) : \delta_a(\theta) = 0\}, \tag{3.2}$$

(obviously, H is of linear measure zero). By Lemma 2.1, *for any given* $0 < \varepsilon < 1$ and *for any* $\theta \in [0, 2\pi) \backslash E \cup H$, *there exists* $r_0 > 0$ *such that for* $0 < r < r_0$, (2.1) and (2.2) hold.

Now we take $\theta \in [0, 2\pi) \backslash E \cup H$ (obviously, $E \cup H$ is of linear measure zero). Then we have two cases: $\delta_a(\theta) < 0$ or $\delta_a(\theta) > 0$.

Case (i). $\delta_a = \delta < 0$. By $a = cb$, $c > 1$, $\delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta$. By (1.11), we get

$$1 \leq \left|A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\}\right| \cdot \left|\frac{f'}{f''}\right| + \left|B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\}\right| \cdot \left|\frac{f}{f''}\right|. \tag{3.3}$$

If $|f''(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.5 there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$, $m \geq 1$, where $r_m \to 0$ such that $f''(z_m) \to \infty$ and

$$\left|\frac{f(z_m)}{f''(z_m)}\right| \leq M_1, \quad \left|\frac{f'(z_m)}{f''(z_m)}\right| \leq M_2. \tag{3.4}$$

Using Lemma 2.1 and (3.4) into (3.3), we get as $m \to \infty$

$$1 \leq M_1 \exp\left\{(1 - \varepsilon)\frac{\delta}{r_m^n}\right\} + M_2 \exp\left\{(1 - \varepsilon)\frac{1}{c}\frac{\delta}{r_m^n}\right\} \to 0,$$

a contradiction. Hence,

$$|f''(z)| \leq C_1, \tag{3.5}$$

holds on $\arg(z_0 - z) = \theta$, where C_1 is a constant. By integration along the line segment $\left[z_0 - r_1 e^{i\theta}, z_0 - re^{i\theta}\right]$, from (3.5) and the equality

$$f'(z) = f'(z_1) + \int_{z_1}^{z} f''(t)\,dt,$$

we obtain

$$|f'(z)| \leq C_2 + C_1|z - z_1| \leq C_3, \tag{3.6}$$

as $z \to z_0$. Analogously, by (3.6), we can obtain

$$|f(z)| \le C_4, \tag{3.7}$$

holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Case (ii). $\delta > 0$. We have $\delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta > 0$. By (1.11), we have

$$\left|A(z)\exp\left\{\frac{a}{(z_0 - z_k)^n}\right\}\right| \le \left|\frac{f''(z)}{f'(z)}\right| + \left|B(z)\exp\left\{\frac{b}{(z_0 - z_k)^n}\right\}\right| \cdot \left|\frac{f(z)}{f'(z)}\right|. \tag{3.8}$$

If $|f'(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.5, there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$, $m \ge 1$, where $r_m \to 0$ such that $f'(z_m) \to \infty$ and

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \le M_3. \tag{3.9}$$

Substituting (3.1) and (3.9) into (3.8) and by Lemma 2.1, we obtain

$$\begin{aligned} \exp\left\{(1-\varepsilon)\frac{\delta}{r_m^n}\right\} &\le \frac{1}{r_m^{\sigma+1+\varepsilon}} + M_3 \exp\left\{(1+\varepsilon)\frac{1}{c}\cdot\frac{\delta}{r_m^n}\right\} \\ &\le \frac{M_3}{r_m^{\sigma+1+\varepsilon}} \exp\left\{(1+\varepsilon)\frac{1}{c}\cdot\frac{\delta}{r_m^n}\right\}, \end{aligned}$$

which implies that

$$1 \le \frac{M_3}{r_m^{\sigma+1+\varepsilon}} \exp\left\{\left[(1+\varepsilon)\frac{1}{c} - (1-\varepsilon)\right]\frac{\delta}{r_m^n}\right\}. \tag{3.10}$$

By taking $0 < \varepsilon < \frac{c-1}{1+c}$, a contradiction follows in (3.10) as $m \to \infty$. So, $|f'(z)| \le C_5$. As above, we obtain that $|f(z)| \le C_6$, holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Now, we proved that $|f(z)| \le C$ on any ray $\arg(z_0 - z) = \theta \in [0, 2\pi) \setminus E \cup H$. Set $g(w) = f(z)$ such that $w = \frac{1}{z_0 - z}$. $g(w)$ is entire function in $\mathbb{C}$ and $|g(w)| \le C'$ $(C' > 0)$ on any ray $\arg(w) = -\theta$ such that $\theta \in [0, 2\pi) \setminus E \cup H$. By Phragmen-Lindelof theorem in sectors, we get that $|g(w)| \le C'$ in $\mathbb{C}$ and By Liouville theorem we conclude that $g(w)$ is a constant. So, $f(z)$ is constant. We know that the only constant solution of (1.11) is $f \equiv 0$. Hence, every solution $f(z) \not\equiv 0$ of (1.11) is of infinite order. □

Proof of Theorem 1.4. Assume that $f \not\equiv 0$ is an analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (1.12). By Theorem 1.3 and Lemma 2.8, we have $\sigma(f, z_0) = \infty$ and $\sigma_2(f, z_0) = \alpha \le n$. We assume that $\sigma_2(f, z_0) = \alpha < n$, and we prove that is failing. Since the Wiman-Valiron near a finite singular point (see [10]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j, \quad j = 1, 2, \tag{3.11}$$

where $|f(z_r)| = M_{z_0}(r,f) = \max_{|z_0-z|=r}|f(z)|$. By Lemma 2.10, there is a sequence $\left\{z_m = z_0 - r_m e^{i\theta_m}\right\}$, $m \geq 1$, such that $|f(z_m)| = M_{z_0}(r_m,f)$, $\lim_{m\to\infty}\theta_m = \theta^* \in [0,2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have

$$\limsup_{m\to\infty}\frac{\log V_{z_0}(r_m)}{-\log r_m} = \infty,$$

$$\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\} \leq V_{z_0}(r_m) \leq \exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}. \tag{3.12}$$

Set $a = \alpha + i\beta$, $z_0 - z = re^{i\theta_0}$, $\delta = \delta_a(\theta^*) = \alpha\cos(n\theta^*) + \beta\sin(n\theta^*)$. Since $a = cb$, $c > 1$, we have $\delta_b(\theta^*) = \frac{1}{c}\delta_a(\theta^*) = \frac{1}{c}\delta$. There is three cases: (i) $\delta < 0$; (ii) $\delta > 0$; (iii) $\delta = 0$.

Case (i). $\delta < 0$. By $\lim_{m\to\infty}\theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m < 0$, $\delta_a(\theta_m) = c\delta_m < 0$. From (1.12), we can write

$$\left|\exp\left\{\frac{-b}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f''}{f}\right| \leq \left|A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a-b}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f'}{f}\right| + \left|B\left(\frac{1}{z_0-z_m}\right)\right|. \tag{3.13}$$

Substituting (3.11)–(3.12) into (3.13) and by Lemma 2.1 and Lemma 2.9, for any given ε $(0 < \varepsilon < n-\alpha)$ as m is sufficiently large, we have

$$\begin{aligned}
&\exp\left\{(1-\varepsilon)\frac{-\delta_m}{r_m^n}\right\}\exp\left\{\frac{2}{r_m^{\alpha-\varepsilon}}\right\}\frac{1}{r_m^2}(1+o(1))\\
\leq &\left|\exp\left\{\frac{-b}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f''}{f}\right|\\
\leq &\exp\left\{(1-\varepsilon)(c-1)\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1+o(1)}{r_m} + \frac{1}{r_m^{d+1}}\\
\leq &\exp\left\{(1-\varepsilon)(c-1)\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1}{r_m^{d+2}},
\end{aligned}$$

where $d = \deg B$, which implies

$$\exp\left\{\frac{2}{r_m^{\alpha-\varepsilon}}\right\}(1+o(1)) \leq \exp\left\{(1-\varepsilon)c\frac{\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}\frac{1}{r_m^d}. \tag{3.14}$$

By taking $0 < \varepsilon < \max\{1, n-\alpha\}$, the right side of inequality (3.14) tends to zero as $m \to \infty$. This is a contradiction.

Case (ii). $\delta > 0$. By $\lim_{m\to\infty}\theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m > 0$, $\delta_a(\theta_m) = c\delta_m > 0$. By (1.12), we can write

$$\left|A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f'}{f}\right| \leq \left|\frac{f''}{f}\right| + \left|B\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}\right|. \tag{3.15}$$

Substituting (3.11)–(3.12) into (3.15) and by Lemma 2.1, as m is sufficiently large,we have

$$\begin{aligned}
&\exp\left\{(1-\varepsilon)\frac{c\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}\frac{(1+o(1))}{r_m}\\
\leq&\left|A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\cdot\left|\frac{f'}{f}\right|\\
\leq&\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\frac{(1+o(1))}{r_m^2}+\exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\}\\
\leq&\frac{1}{r_m^2}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\},
\end{aligned}$$

which implies the following inequality

$$\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}(1+o(1))\leq\frac{1}{r_m}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{[(1+\varepsilon)-(1-\varepsilon)c]\frac{\delta_m}{r_m^n}\right\}. \tag{3.16}$$

By taking $0<\varepsilon<\max\left\{\frac{c-1}{c+1},n-\alpha\right\}$, the right side of inequality (3.16) tends to zero as $m\to\infty$ and so a contradiction follows.

Case (iii) $\delta=0$. Since $\arg(z_0-z)=\theta^*$ is an asymptotic line of $\frac{a}{(z_0-z_m)^n}$, there is $m_0>0$ such that as $m>m_0$ we have

$$e^{-1}\leq\left|\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\leq e, \tag{3.17}$$

$$e^{\frac{-1}{c}}\leq\left|\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}\right|\leq e^{\frac{1}{c}}. \tag{3.18}$$

By (1.12), (3.11) and (3.17)–(3.18), we obtain

$$\begin{aligned}
-\left(\frac{V_{z_0}(r_m)}{z_0-z_m}\right)^2(1+o(1))=&A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\left(\frac{V_{z_0}(r)}{z_0-z_m}\right)(1+o(1))\\
&+B\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}.
\end{aligned}\tag{3.19}$$

By (3.17)–(3.19) and Lemma 2.1, for m large enough, we have

$$\left(\frac{V_{z_0}(r_m)}{r_m}\right)^2(1+o(1))\leq\frac{1}{r_m^{d+1}}\left(\frac{V_{z_0}(r_m)}{r_m}\right)(1+o(1)),$$

and so

$$V_{z_0}(r_m)\leq\frac{1}{r_m^d}(1+o(1)), \tag{3.20}$$

where $d=\max\{\deg A,\deg B\}$. (3.20) contradicts (3.12). Thus, $\sigma_2(f,z_0)\geq n$ and by Lemma 2.8, we obtain $\sigma_2(f,z_0)=n$. □

Proof of Theorem 1.5. Assume that $f \not\equiv 0$ is an analytic solution of (1.10) in $\overline{\mathbb{C}} \setminus \{z_0\}$. From (1.10), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \cdot \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \cdot \left|\frac{f'}{f}\right|. \tag{3.21}$$

By Lemma 2.2, for any given $\alpha > 0$ there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $\lambda > 0$ that depends only on α such that for all $r = |z - z_0|$ satisfying $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j}, \quad j = 1, \ldots, k. \tag{3.22}$$

There exist β_1, β_2 such that $\max\{\tau_M(A_j, z_0) : \sigma(A_j, z_0) = \sigma(A_0, z_0)\} < \beta_1 < \beta_2 < \tau_M(A_0, z_0)$. There exists a set $E_2 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0|$ satisfying $r \notin E_2$, we have

$$|A_j(z)| \leq \exp\left\{\frac{\beta_1}{r^\sigma}\right\}, \quad j = 1, \ldots, k. \tag{3.23}$$

By Lemma 2.6, there exists a set $F \subset (0,1)$ of infinite logarirhmic measure such that for all $r \in F$ we have

$$M_{z_0}(r, A_0) > \exp\left\{\frac{\beta_2}{r^\sigma}\right\}. \tag{3.24}$$

From (3.21)–(3.24), for all z satisfying $r = |z - z_0| \in F \backslash E_1 \cup E_2$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we obtain

$$\exp\left\{\frac{\beta_2}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k} \exp\left\{\frac{\beta_1}{r^\sigma}\right\},$$

and thus

$$\exp\left\{\frac{\beta_2 - \beta_1}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k}. \tag{3.25}$$

From (3.25), it is easy to obtain that $\sigma_2(f, z_0) \geq \sigma$ and combining this with Lemma 2.8, we get the equality $\sigma_2(f, z_0) = \sigma = \sigma(A_0, z_0)$. ☐

Acknowledgements. The author would like to thank the anonymous referees for their careful reading and necessary comments which have improved the presentation of the paper. This work has been supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120180003).

References

[1] I. Amemiya and M. Ozawa, *Non-existence of finite order solutions of* $w'' + e^{-z}w' + Q(z)w = 0$, Hokkaido Math. J. **10** (1981), 1–17.
[2] L. Bieberbach, *Theorie der Gewöhnlichen Differentialgleichungen,* Springer-Verlag, Berlin, Heidelberg, New York, 1965.
[3] Z. X. Chen, *The growth of solutions of* $f'' + e^{-z}f' + Q(z)f = 0$*, where the order* $(Q) = 1$, Sci. China Math. **45** (2002), 290–300.
[4] Z. X. Chen and K. H. Shon, *On the growth of solutions of a class of higher order linear differential equations*, Acta Math. Sci. Ser. A **24**B(1) (2004), 52–60. https://doi.org/10.1016/S0252-9602(17)30359-4
[5] Z. X. Chen and C. C. Yang, *Some further results on zeros and growths of entire solutions of second order linear differential equations,* Kodai Math. J. **22** (1999), 273–285. https://doi.org/10.2996/kmj/1138044047
[6] H. Fettouch and S. Hamouda, *Growth of local solutions to linear differential equations around an isolated essential singularity,* Electron. J. Differential Equations **2016** (2016), 10 pages.
[7] G. G. Gundersen, *On the question of whether* $f'' + e^{-z}f' + B(z)f = 0$ *can admit a solution* $f \not\equiv 0$ *of finite order*, Proc. Roy. Soc. Edinburgh Sect. A **102** (1986), 9–17. https://doi.org/10.1017/S0308210500014451
[8] S. Hamouda, *Properties of solutions to linear differential equations with analytic coefficients in the unit disc*, Electron. J. Differential Equations **2012** (2012), 9 pages.
[9] S. Hamouda, *Iterated order of solutions of linear differential equations in the unit disc*, Comput. Methods Funct. Theory **13**(4) (2013), 545–555.https://doi.org/10.1007/s40315-013-0034-y
[10] S. Hamouda, *The possible orders of growth of solutions to certain linear differential equations near a singular point,* J. Math. Anal. Appl. **458** (2018), 992–1008. https://doi.org/10.1016/j.jmaa.2017.10.005
[11] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[12] A. Ya. Khrystiyanyn and A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli*, Mat. Stud. **23**(1) (2005), 19–30.
[13] A. A. Kondratyuk and I. Laine, *Meromorphic functions in multiply connected domains,* in: *Fourier Series Methods in Complex Analysis*, Univ. Joensuu Dept. Math. Rep. Ser. **10**, Univ. Joensuu, Joensuu, 2006, 9–111.
[14] R. Korhonen, *Nevanlinna theory in an annulus,* in: *Value Distribution Theory and Related Topics*, Adv. Complex Anal. Appl. **3**, Kluwer Acad. Publ., Boston, MA, 2004, 167–179. https://doi.org/10.1007/1-4020-7951-6_7
[15] I. Laine, *Nevanlinna Theory and Complex Differential Equations,* W. de Gruyter, Berlin, 1993. https://doi.org/10.1515/9783110863147
[16] E. L. Mark and Y. Zhuan, *Logarithmic derivatives in annulus,* J. Math. Anal. Appl. **356** (2009), 441–452. https://doi.org/10.1016/j.jmaa.2009.03.025
[17] M. Tsuji, *Potential Theory in Modern Function Theory,* Chelsea, New York, 1975 (reprint of the 1959 edition).
[18] J. Tu and C-F. Yi, *On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order,* J. Math. Anal. Appl. **340** (2008), 487–497. https://doi.org/10.1016/j.jmaa.2007.08.041
[19] J. M. Whittaker, *The order of the derivative of a meromorphic function,* J. Lond. Math. Soc. **s1-11** (1936), 82–87. https://doi.org/10.1112/jlms/s1-11.2.82
[20] L. Yang, *Value Distribution Theory,* Springer-Verlag Science Press, Berlin, Beijing, 1993. https://doi.org/10.1007/978-3-662-02915-2

[1]Laboratory of pure and applied mathematics,
Department of mathematics, University of Mostaganem,
27000, Mostaganem, Algeria

Email address: samir.cherief@univ-mostadz
Email address: saada.hamouda@univ-mostadz

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 203–211.

BELL GRAPHS ARE DETERMINED BY THEIR LAPLACIAN SPECTRA

ALI ZEYDI ABDIAN[1]

Abstract. A graph G is said to be determined by the spectrum of its Laplacian spectrum (DLS, for short) if every graph with the same spectrum is isomorphic to G. An ∞-graph is a graph consisting of two cycles with just a vertex in common. Consider the coalescence of an ∞-graph and the star graph $K_{1,s}$, with respect to their unique maximum degree. We call this a bell graph. In this paper, we aim to prove that all bell graphs are DLS.

1. Introduction

As usual $G = (V(G), E(G))$ is a simple graph having n vertices and m edges, with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. The complement of G is denoted by $\overline{G}$.

The degree sequence of G, denoted by $\deg(G)$, is the sequence of vertex degrees; in fact $\deg(G) = (d_1, d_2, \dots, d_n)$ in which $d_i = d_i(G) = d_G(v_i)$ for $i = 1, \dots, n$, is the degree of the vertex v_i so that $d_1 \geq d_2 \geq \cdots \geq d_n$.

Let $A(G)$ and $D(G) = \mathrm{Diag}(d_1, d_2, \dots, d_n)$ denote the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian matrix* of G is defined as $L(G) = A(G) - D(G)$. The polynomial $\varphi_{L(G)}(x) = \det(x\mathbb{I}_n - L(G))$, where $\mathbb{I}_n$ is the identity matrix of order n, is called the Laplacian characteristic polynomial of G. Any root of $\varphi_{L(G)}(x)$ is called a *Laplacian eigenvalue* of G. The multi-set of Laplacian eigenvalue of G is called the *Laplacian spectrum* or *L-spectrum* of G. Note that $L(G)$ is a symmetric, positive semidefinite matrix, and thus its eigenvalues are all real non-negative numbers. We denote its eigenvalues in the non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

Key words and phrases. Bell graph, Laplacian spectrum, L-cospectral, cospectral graphs, spectral characterization.

2010 *Mathematics Subject Classification.* Primary: 05C50.

DOI 10.46793/KgJMat2302.203A

Received: September 22, 2019.

Accepted: August 04, 2020.

Although, the spectral graph theory originated with the eigenvalues of the adjacency matrices, but Laplacian matrices have come to have comparable importance.

The coalescence of two graphs G_1 and G_2, with respect to $u_1 \in V(G_1)$ and $u_2 \in V(G_1)$, is the graph obtained by identifying u_1 and u_2 in the disjoint union of G_1 and G_2. We denote it by $(G_1 \circ G_2)(u_1, u_2)$. In the case when it does not make deference which vertex in G_1 and G_2 is identified to obtain a coalescence, we denote this graph by $G_1 \circ G_2$. This operation is extended, inductively, to any arbitrary number of graphs. For example, the coalescence of k arbitrary cycles is called a k-rose graph; in fact, this is a graph with $k \geq 1$ cycles meeting in one vertex. For $i, j \geq 3$, $C_i \circ C_j$ is a 2-rose graph called an ∞-graph.

Van Dam and Haemers [12] conjectured that almost all graphs are determined by their Laplacian spectrum, that is, they are the only graph (up to isomorphism) with that spectrum. However, very few graphs are known to have that property, and so discovering new classes of such graphs is an interesting problem. Formally, we define two graphs G and H to be *L-cospectral* if they have the same L-spectrum, and a graph G is *determined by its Laplacian spectrum*, abbreviated by DLS, if no other graphs are L-cospectral with G. Let us mention some known DLS graphs obtained by coalescence of other DLS graphs:

- Liu et al. [10] proved that any rose graph, each cycle of which is a triangle, is DLS;
- Wang et al. [14] showed that triangle-free 2-rose graphs are almost DLS (notice that not all 2-rose graphs are DLS (see [9]);
- Wang et al. [15] proved that all 3-rose graphs, having at least one triangle, are DLS.

It is known that the Laplacian eigenvalues of a graph give the Laplacian eigenvalues of its complement. Therefore, complement of a DLS graph, is also DLS. Hence, all the complements of the above graphs are DLS.

In the current article, we consider a new graph being coalescence of a 2-rose graph and a star graph with respect to their vertices of maximum degree. In fact, this graph is the coalescence of $C_i \circ C_j$, with the vertex v_1 of maximum degree 4 and the star graph $K_{1,s}$ with the vertex v_2 of maximum degree s. Let us call this graph a *bell graph* and denote it by $BG(C_i, C_j, s)$, $i \leq j$, see Figure 1.

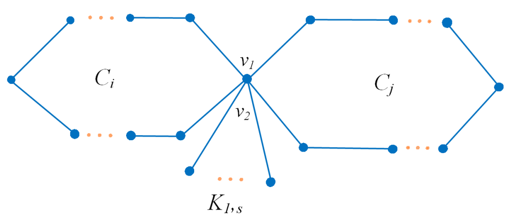

FIGURE 1. The bell graph $BG(C_i, C_j, s)$

In this paper, it is proved that bell graphs and their complements are DLS. The rest of this article is organized as follows: In Section 2, we recall some previously established results playing a crucial role throughout this paper. In Section 3, we fisrt prove that no two non-isomorphic bell graphs are L-cospectral, and then we determine the degree sequence of graphs L-cospectral with the bell graphs. Finally, we obtain all bell graphs are DLS.

2. Preliminaries

In this section, we recall some previously established results playing a crucial role throughout this paper.

Theorem 2.1 ([12, 13]). *The following can be obtained from the Laplacian spectrum of a graph:*

(i) *the number of vertices;*
(ii) *the number of edges;*
(iii) *the number of spanning trees;*
(iv) *the number of components;*
(v) *the sum of the squares of the degrees of the vertices.*

Lemma 2.1 ([3]). *For a graph G, we have $\mu_{n-1} \geq 0$ with equality if and only if G is connected.*

Theorem 2.2 ([7]). *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of G and $\overline{G}$, respectively. Then $\overline{\mu}_i = n - \mu_{n-i}$ for $i = 1, 2, \ldots, n-1$.*

For any two graphs G and H, we denote by $\mathcal{N}_G(H)$ and $\mathcal{W}_G(i)$, the number of subgraphs of G being isomorphic to H, and the number of closed walks of length i in G, respectively. Note that the trace of a matrix M is denoted by $\mathrm{tr}(M)$.

Theorem 2.3 ([1, 13]). *Suppose G is a graph with m edges. The number of closed walks of lengths 2, 3, and 4 in G can be computed by the following formulas:*

(a) $\mathcal{W}_G(2) = 2m$;
(b) $\mathcal{W}_G(3) = \mathrm{tr}(A^3(G)) = 6\mathcal{N}_G(C_3)$;
(c) $\mathcal{W}_G(4) = 2m + 4\mathcal{N}_G(P_3) + 8\mathcal{N}_G(C_4)$.

Theorem 2.4 ([6]). *If G is a non-empty graph with n vertices, then*

$$\mu_1(G) \geq d_1(G) + 1. \tag{2.1}$$

Furthermore, if G is connected, then the equality in (2.1) holds if and only if $d_1(G) = n - 1$.

A graph G is called *regular* if $d_1(G) = \cdots = d_n(G)$. A bipartite graph is called *semi-regular* if the degrees of vertices in each part, are constant.

The next result uses the quantity $\theta_G(u) = \sum_{v \in N_G(u)} \frac{d_G(v)}{d_G(u)}$, where $N_G(u)$ denotes the set of neighbors of the vertex u in G.

Theorem 2.5 ([13]). *For a connected graph G, we have*

$$\mu_1(G) \leq \max\{d_G(u) + \theta_G(u) \mid u \in V(G)\}. \tag{2.2}$$

Besides, the equality in (2.2) *holds if and only if G is either regular or semi-regular, bipartite graph.*

Theorem 2.6 ([1,8]). *Let G be a non-empty graph. Then $\mu_1(G) \leq d_1(G) + d_2(G)$. Moreover, G is connected only if $\mu_2(G) \geq d_2(G)$.*

Cevetcovic et al. in [2] obtained the first three coefficients of the Laplacian characteristic polynomials, while the forth one, was obtained by Oliveira et al. in [11].

Theorem 2.7 ([2,11]). *Let G be a graph with n vertices and m edges with the degree set* $\deg(G) = (d_1, d_2, \ldots, d_n)$. *Then we have the following: $\varphi_{L(G)}(x) = \sum_{i=0}^{n} l_i(G)x^i$, are obtained as follows:*

$$\begin{aligned} l_0(G) =& 1, \quad l_1(G) = -2m, \quad l_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2, \\ l_3(G) =& \frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6\mathcal{N}_G(C_3)\right). \end{aligned}$$

As an immediate consequence of Theorem 2.7, we have following result.

Corollary 2.1. *If G and H are L-cospectral graphs such that* $\deg(H) = \deg(G)$, *then they have the same number of triangles, i.e., $\mathcal{N}_G(C_3) = \mathcal{N}_H(C_3)$.*

Let G and H be two L-cospectral graphs. It follows from Theorem 2.1 (i), (ii), (iv), (v) and Theorem 2.7 and Corollary 2.1 that

$$\operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} d_i^3(G) = \operatorname{tr}(A^3(H)) - \sum_{i=1}^{n} d_i^3(H).$$

Based on this, Liu and Huang [9] defined the following invariant for a graph G:

$$\varepsilon(G) = \operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} (d_i(G) - 2)^3.$$

Theorem 2.8 ([13]). *If G and H are L-cospectral, then $\varepsilon(G) = \varepsilon(H)$.*

Theorem 2.9 ([3]). *If u is a vertex of G and $G - u$ is the subgraph obtained from G by deleting u, then $\mu_i(G) \geq \mu_i(G - u) \geq \mu_{i+1}(G) - 1$, $i = 1, 2, \ldots, n - 1$.*

3. Main Results

In this section, we establish bound on the first and the second largest Laplacian eigenvalues of bell graphs.

Lemma 3.1. *For a bell graph G with s pendent vertices, we have*

(i) $5 + s \leq \mu_1(G) < 6 + s$;

(ii) $\mu_2(G) < 5$.

Proof. (i) It follows from Theorems 2.4 and 2.5 that

$$5+s \leq \mu_1(G) \leq 4+s+\frac{4+s+4}{4+s} = 5+s+\frac{4}{4+s} < 6+s.$$

(ii) This is a direct consequence of Theorem 2.9 and this fact that the greatest eigenvalue of a path is less than 4. $\square$

Let G be a connected graph with n vertices and m edges. Then G is called *k-cyclic* if $m = n+k-1$. For a bell graph $G = BG(C_i, C_j, s)$, we have $n = n(G) = (i+j)-1+s$ and $m = m(G) = (i+j)+s$ and so $m = m(G) = n+1 = n+2-1$, implying that G is a 2-cyclic graph.

Lemma 3.2. *If H is L-cospectral with $G = BG(C_i, C_j, s)$, then H is connected, and*

$$\deg(H) = \deg(G) = (s+4, \underbrace{2, \ldots, 2}_{i+j-2 \text{ times}}, \underbrace{1, \ldots, 1}_{s \text{ times}}).$$

Proof. Connectedness of H is clear by Theorem 2.1 (iv) and Lemma 3.1 (iii). Let us determine its degree sequence. By Lemma 3.1, $\mu_2(H) < 5$, and thus, it follows from Theorem 2.6 that $d_2(H) \leq 4$. Since H and G are L-cospectral, by Theorem 2.1, H is also connected, and has the same order, size, and sum of the squares of its degrees as G. Let n_i denote the number of vertices of degree i in H for $i = 1, 2, \ldots, d_1(H)$. Then

$$\sum_{i=1}^{d_1(H)} n_i = n(G) = (i+j)-1+s, \tag{3.1}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(G) = 2((i+j)+s), \tag{3.2}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + d_1^2(G), \tag{3.3}$$

where n_i' is the number of vertices of G of degree i for $i = 1, 2$. By adding up (3.1), (3.2) and (3.3) with coefficients 2, -3, 1, respectively, we get:

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = (s+2)(s+3). \tag{3.4}$$

By Lemma 3.1, $5+s \leq \mu_1(G) < 6+s$. From Theorem 2.4 it follows that

$$d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) < 6+s,$$

which leads to $d_1(H) \leq 4+s$. On the other hand, by Lemma 3.1 and Theorem 2.6, one can conclude that

$$5+s \leq \mu_1(G) = \mu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 4,$$

from which we have $d_1(H) \geq s+1$. Therefore, we have $s+1 \leq d_1(H) \leq s+4$. From Theorem 2.8 it follows that

$$6\mathcal{N}_H(C_3) - \sum_{i=1}^{n} (d_i(H)-2)^3 = 6\mathcal{N}_G(C_3) - ((s+2)^3 - s). \tag{3.5}$$

Therefore,

$$\mathcal{N}_H(C_3) = \frac{1}{6}\left(\sum_{i=1}^{n} (d_i(H)-2)^3 + 6\mathcal{N}_G(C_3) - ((s+2)^3 - s)\right). \tag{3.6}$$

We consider the following three main cases.

Case A. $d_1(H) = s+4$. By (3.4) one can deduce that

$$((s+4)^2 - 3(s+4) + 2) + 2n_3 + 6n_4 = (s+2)(s+3), \tag{3.7}$$

from which it follows that $n_3 = 0$. Combining (3.2) and (3.3), we find that $n_1 = s$ and $n_2 = n-(s+1)$. Therefore, $\deg(H) = \deg(G)$. In this case, it follows from (3.6) that $\mathcal{N}_H(C_3) = \mathcal{N}_G(C_3)$. Obviously, $d_1(H) \geq 5 > 4 \geq d_2(H)$ or $n_{s+4} = 1$.

Case B. $s+3 = d_1(H)$. Then $n_{s+3} = 1$. By an argument similar to that of (3.6), we have the following:

$$((s+3)^2 - 3(s+3) + 2) + 2n_3 = (s+2)(s+3), \tag{3.8}$$

By (3.1), (3.2) and (3.4) we get

$$\begin{cases} n_1 = 2s+11, \\ n_2 = -3s+n-20, \\ n_3 = s+8. \end{cases}$$

It follows from (3.6) that $\mathcal{N}_H(C_3) = \frac{6\mathcal{N}_G(C_3)-s^3-6s^2-12s-11}{6}$. Therefore, $\mathcal{N}_H(C_3) < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$. We assume that $n_{s+3} \geq 2$. Then

$$s+3 = d_1(H) = d_2(H) \leq 3,$$

which is a contradiction, since $s \geq 1$.

Case C. $d_1(H) = s+2$. We first assume that $n_{s+2} = 1$. In this case, $s+2 = d_1(H) > 3 \geq d_2(H)$ and as a result $s \geq 2$. From (3.4) and by a straightforward calculation, we get:

$$((s+2)^2 - 3(s+2) + 2) + 2n_3 = (s+2)(s+3). \tag{3.9}$$

By (3.1), (3.2) and (3.4) we get:

$$\begin{cases} n_1 = 3s+1, \\ n_2 = -5s+n-5, \\ n_3 = 2s+3. \end{cases}$$

It follows from (3.6) that

$$\mathcal{N}_H(C_3) = \frac{6\mathcal{N}_G(C_3) - s^3 - 6s^2 - 12s - 10}{6}.$$

Therefore, $\mathcal{N}_H(C_3) < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$. Next we assume that $n_{s+2} \geq 2$. Then $s+2 = d_1(H) = d_2(H) \leq 3$ implying that $s = 1$. By (3.1), (3.2) and (3.4) we get

$$\begin{cases} n_1 = 4, \\ n_2 = n - 10, \\ n_3 = 6. \end{cases}$$

It follows from (3.6) that $\mathcal{N}_H(C_3) = \mathcal{N}_G(C_3) - \frac{14}{3} < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$, which is a contradiction.

Case D. $d_1(H) = s + 1$. By a similar argument, we will have a contradiction. $\square$

In the following, we show that any graph L-cospectral with a bell graph G, is also a bell graph with the same degree sequence as G.

Corollary 3.1. *Let H be a graph L-cospectral with a bell graph $G = BG(C_i, C_j, s)$. Then H is a bell graph with the same degree sequence as G.*

Proof. By Lemma 3.2, $\deg(H) = \deg(G) = (s+4, \underbrace{2, \ldots, 2}_{i+j-2 \text{ times}}, \underbrace{1, \ldots, 1}_{s \text{ times}})$. So, H has a unique vertex of degree greater than 2, say $d_H(v) = s + 4 > 2$. It is clear that the maximum degree of $H - v$ is most 2, i.e., $d_1(H - v) \leq 2$. Moreover, $H - v$ contains no cycles, otherwise, since it is connected, there would be another vertex of degree greater than 2. Consequently, $H - v$ must be a forest each component of which is a path. Therefore, H consists of exactly 2 cycles intersecting in a single vertex. Hence, H must be a bell graph. $\square$

Before proving our main result, we state some essential lemmas and notations.

Lemma 3.3 ([4]). *Let G be a graph with a set of vertices $X = \{u_1, u_2, \ldots, u_k\}$ such that*

$$N_G(u_1) = N_G(u_2) = \cdots = N_G(u_k) = \{w_1, w_2, \ldots, w_p\}.$$

If G^* is the graph obtained from G by adding any q, $1 \leq q \leq \frac{k(k-1)}{2}$, edges among $\{u_1, u_2, \ldots, u_k\}$, then the eigenvalues of $L(G^*)$ are as follows: those eigenvalues of $L(G)$ which are equal to p are incremented by $\lambda_i(G^*[X])$, $i = 1, 2, \ldots, k-1$, and the remaining eigenvalues are the same.

Lemma 3.4 ([5]). *No two non-isomorphic starlike trees are L-cospectral.*

Suppose that $H = BG(C_i, C_j, s)$ is a bell graph, and let v be the vertex of H such that $d_H(v) = s + 4$. Now, we remove an arbitrary edge not being adjacent to v, from cycle C_i and C_j. Then we obtain a starlike tree, say, $S(H)$. Hereafter, $S(H) = (s, l_1, l_2)$ means $S(H) - v = P_{l_1} \cup P_{l_2} \cup \overline{K_s}$ such that $l_1 + l_2 = (i + j) - 1$.

Note that in the proof of Lemma 3.4, it was shown that if $S_1 = S(l_1, \ldots, l_t)$ and $S_2 = S(j_1, \ldots, j_t)$ are two non-isomorphic starlike trees, then $\mu_1(S_1) \neq \mu_1(S_2)$, where $l_1 \geq l_2 \geq \cdots \geq l_t \geq 1$ and $j_1 \geq j_2 \geq \cdots \geq j_t \geq 1$.

Corollary 3.2. *If $S(G) = (s, l_1, l_2)$ and $S(H) = (s, j_1, j_2)$ are two non-isomorphic starlike trees, then $\mu_1(S(G)) \neq \mu_1(S(H))$.*

Now we express our main result.

Theorem 3.1. *Bell graphs are determined by their Laplacian spectrum.*

Proof. Let H be a graph L-cospectral with a bell graph $G = BG(C_{t_1}, C_{t_2}, s)$. It follows from Corollary 3.1 that H is also a bell graph with the same degree sequence as G. Assuming that $H = BG(C_{k_1}, C_{k_2}, s)$ we need to prove that $\{t_1, t_2\} = \{k_1, k_2\}$. To do so, consider the corresponding starlike trees $S(G) = (s, l_1, l_2)$ and $S(H) = (s, j_1, j_2)$. We claim that H and G are isomorphic, otherwise, $\mu_1(S(G)) \neq \mu_1(S(H))$ and so $\mu_1(G) \neq \mu_1(H)$, contradicting Lemma 3.2. $\square$

From Theorem 2.2, it follows that the Laplacian eigenvalues of a graph give the Laplacian eigenvalues of its complement. Therefore, the complement of a DLS graph, is also DLS. Hence, the following fact is immediately follows from Theorem 3.1.

Corollary 3.3. *The complements of bell graphs are also DLS.*

Acknowledgements. The authors are deeply indebted to the anonymous referee for valuable comments and suggestions, which led to an improvement of the original manuscript.

References

[1] W. N. Anderson and T. D. Morley, *Eigenvalues of the Laplacian of a graph*, Linear Multilinear Algebra **18** (1985), 141–145. https://doi.org/10.1080/03081088508817681

[2] D. M. Cvetković, P. Rowlinson and S. K. Simić, *Signless Laplacian of finite graphs*, Linear Algebra Appl. **423** (2007), 155–171. https://doi.org/10.1016/j.laa.2007.01.009

[3] D. M. Cvetković, P. Rowlinson and S. K. Simić, *An Introduction to the Theory of Graph Spectra*, London Mathematical Society Student Texts 75, Cambridge Univ. Press, Cambridge, 2010.

[4] K. Ch. Das, *On conjectures involving second largest signless Laplacian eigenvalue of graphs*, Linear Algebra Appl. **432** (2010), 3018–3029. https://doi.org/10.1016/j.laa.2010.01.005

[5] L. Feng and G. Yu, *No starlike trees are Laplacian cospectral*, Publikacije Elektrotehničkog fakulteta. Serija Matematika **18** (2007), 46–51. https://doi.org/10.2298/PETF0718046F

[6] R. Grone and R. Merris, *The Laplacian spectrum of graph II*, SIAM J. Discrete Math. **7** (1994), 221–229.

[7] A. K. Kelmans, *The number of trees in a graph I, II*, Automat, Remote Control **26** (1965), 2118–2129 and **27** (1966), 233–241 (Translated from Avtomat. i Telemekh. **26** (1965), 2118–2129 and **27** (1966), 233–241 [in Russian]).

[8] J. S. Li and Y. L. Pan, *A note on the second largest eigenvalue of the Laplacian matrix of a graph*, Linear and Multilinear Algebra **48** (2000), 117–121. https://doi.org/10.1080/03081080008818663

[9] F. J. Liu and Q. X. Huang, *Laplacian spectral characterization of* 3*-rose graphs*, Linear Algebra Appl. **439** (2013), 2914–2920. https://doi.org/10.1016/j.laa.2013.07.029

[10] X. Liu, Y. Zhang and X. Gui, *The multi-fan graphs are determined by their Laplacian spectra*, Discrete Math. **308** (2008), 4267–4271. https://doi.org/10.1016/j.disc.2007.08.002

[11] C. S. Oliveira, N. M. M. de. Abreu and S. Jurkiewilz, *The characteristic polynomial of the Laplacian of graphs in* (a, b)*-linear cases*, Linear Algebra Appl. **356** (2002), 113–121. `https://doi.org/10.1016/S0024-3795(02)00357-9`

[12] E. R. van Dam and W. H. Haemers, *Which graphs are determined by their spectrum?*, Linear Algebra Appl. **373** (2003), 241–272. `https://doi.org/10.1016/S0024-3795(03)00483-X`

[13] F. Wen, Q. Huang, X. Huang and F. Liu, *The spectral characterization of wind-wheel graphs*, Indian J. Pure Appl. Math. **46** (2015), 613–631. `https://doi.org/10.1007/s13226-015-0141-8`

[14] J. Wang, Q. Huang, F. Belardo and E. M. Li Marzi, *On the spectral characterizations of* ∞*-graphs*, Discrete Math. **310** (2010), 1845–1855. `https://doi.org/10.1016/j.disc.2010.01.021`

[15] J. Wang, Q. Huang and F. Belardo, *On the spectral characterizations of* 3*-rose graphs*, Util. Math. **91** (2013), 33–46.

[1]Department of the mathematical Science, College of Science,
Lorestan University,
Lorestan, Khoramabad 41566, Iran
Email address: `abdian.al@fs.lu.ac.ir, aabdian67@gmail.com, azeydiabdi@gmail.com`

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 213–228.

KONTSEVICH GRAPHONS

ALI SHOJAEI-FARD[1]

Abstract. The article applies graph functions to extend the Kontsevich differential graded Lie algebraic formalism (in Deformation Quantization) to infinite Kontsevich graphs on the basis of the Connes-Kreimer Hopf algebraic renormalization and the theory of noncommutative differential geometry.

1. Introduction

The motivation of this work has been inspired from the recent progresses about the mathematical foundations of the Connes-Kreimer renormalization theory of gauge field theories under two different settings. The one setting concerns finding a new interpretation of the BPHZ Hopf algebraic perturbative renormalization in the context of the Kontsevich Deformation Quantization theory. In this direction, the Hopf-Birkhoff factorization of Feynman rules characters has been described in terms of the Baker-Campbell-Hausdorff formula and the Kontsevich's bi-differential symplectic operator for quantum deformations [5, 12, 16]. The other setting concerns finding some new applications of the theory of graphons in dealing with large Feynman diagrams (namely, infinite Feynman graphs) as sparse graphs generated by sequences of expansions of Feynman diagrams. In this direction, solutions of combinatorial Dyson-Schwinger equations in Quantum Field Theory have been described in terms of graph limits of sequences of random graphs derived from graphon models [17–19]. In addition, in arXiv:1811.05333: *A mathematical perspective on the phenomenology of non-perturbative Quantum Field Theory*, 2020, The MPIM Preprint Series 2018 (65),

Key words and phrases. Graphons, Kontsevich's admissible graphs, renormalization Hopf algebra, noncommutative differential calculus, Maurer-Cartan equations.

2010 *Mathematics Subject Classification.* Primary: 53D55, 05C63. Secondary: 81R60, 16T05, 81Q70, 53Z05.

DOI 10.46793/KgJMat2302.213S

Received: January 08, 2020.

Accepted: August 09, 2020.

the author has addressed some recent applications of graphon models in Quantum Field Theory.

Thanks to the combination of these topics, in this work we aim to show the existence of a new class of infinite Kontsevich graphs generated by sequences of finite Kontsevich's admissible graphs. These infinite graphs allow us to extend the Hochschild-Kontsevich products to a non-perturbative setting. One immediate consequence of this investigation is the formulation of a new class of non-commutative differential calculi which can encode some geometric information (such as quantized motion integral equations) about the evolution of sequences of Kontsevich's admissible graphs. Our main task in this work is to formulate a new non-perturbative modification of the Kontsevich deformation theory via infinite combinatorial tools and the Connes-Kreimer renormalization Hopf algebra. We first apply the theory of graphons for sparse graphs [1–3, 9, 14, 15] to determine a new compact Hausdorff sub-space of graph functions namely, the space of Kontsevich graphons equipped with the cut-distance topology. This topological space can encode the convergent limits of sequences of finite Kontsevich's admissible graphs. Thanks to the Kreimer's renormalization coproduct and Kontsevich graphons, we explain the structure of a new topological Hopf algebra $H^{\mathrm{cut}}_{\mathrm{Kont}}$ on Kontsevich's admissible graphs which is closely related to the structure of a new topological Hopf algebra $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ on Kontsevich graphons. Then we apply this Hopf algebraic setting together with the BPHZ perturbative renormalization to build a new noncommutative differential calculus machinery on Kontsevich's admissible graphs on the basis of the Nijenhuis property of the minimal subtraction map as the renormalization scheme. This study enables us to formulate a new class of quantized motion integrals associated to Kontsevich's admissible graphs. This formalism can be modified for Kontsevich graphons which leads us to obtain a new non-perturbative version of Kontsevich $\star$-products. Finally, we lift the Maurer-Cartan equations onto the level of Kontsevich graphons and their corresponding infinite Kontsevich graphs.

The Connes-Kreimer renormalization Hopf algebra of Feynman diagrams in Quantum Field Theory is derived from the Bogoliubov-Zimmermann forest formula in perturbative renormalization [4, 10, 11]. This Hopf algebra has been applied by Ionescu in arXiv:hep-th/0307062: *Perturbative Quantum Field Theory and configuration space integrals*, 2003 and [8] to build a differential graded Hopf algebra of the Kontsevich's graph complex. In this work, we determine a new class of graphon models for Kontsevich's admissible graphs namely, Kontsevich graphons and then we apply these graphon representations to build a new topological Hopf algebra on the space of Kontsevich's admissible graphs. We equip also the space of Kontsevich graphons with a new compact Hausdorff topological Hopf algebra structure where objects in the boundary region enable us to determine a new collection of infinite Kontsevich graphs. These infinite graphs can be studied in terms of graphon models. The resulting topological Hopf algebra might be useful to search for a completion of the differential graded Hopf algebra of the Kontsevich's graph complex with respect to the cut-distance topology.

Deformation Quantization focuses on the construction of a mathematical model for the description of quantum systems under Dirac's correspondence principle. The model is actually based on quantizing the space of observables on a Poisson manifold in terms of defining a new associative multiplication as a deformation of pointwise multiplication in the direction of the Poisson bracket. The Kontsevich approach has provided a universal deformation quantization for any open domain in $\mathbb{R}^d$ via a graphical representation for bi-differential operators [12, 13]. In this work we apply our new topological Hopf algebraic setting to formulate a new non-perturbative generalization for Deformation Quantization. For this purpose we explain the construction of a new class of noncommutative differential calculi on Kontsevich's admissible graphs originated from the Connes-Kreimer renormalization theory of gauge field theories [4, 21] and the theory of noncommutative differential geometry [6]. We show that the Connes-Kreimer Renormalization Group can provide a new class of quantized integrable systems which can encode the evolution of sequences of Kontsevich's admissible graphs. We then extend this study to the level of Kontsevich graphons which enable us to formulate a new non-perturbative generalization for the Kontsevich $\star$-products in Deformation Quantization. These quantized star type of products are actually the results of the quantization of Poisson structures which are generated by the minimal subtraction map in the BPHZ renormalization theory. Furthermore, we formulate a new version of the Maurer-Cartan equations on infinite Kontsevich graphs in terms of their graphon models.

2. Graphons

We can study a dense or sparse graph in terms of the ratio between the number of its edges and the maximal number of possible edges. Passing from discrete graphs to dense graphs requires to apply sequences of edge weighted graphs such that their vertex sets tend to a continuum set of vertices. The notion of convergence for an arbitrary sequence of graphs with the growing number of vertices can be formulated via graph functions or graphons. At first, the theory of graphons has been initiated in infinite combinatorics for the study of dense graphs derived from sequences of finite weighted graphs with growing density values. The basic idea was to build a convergent limit for any sequence of this type in terms of the behavior of subgraph densities. Homomorphism densities play the fundamental rules for the construction of graph limits in this setting. However this theory has been developed immediately for the study of graph limits of sequences of finite sparse graphs in the context of random graphs and measure theoretic tools. The basic idea in this setting was to generate non-zero graph limits from sequences of graphs with almost zero densities. [1–3, 9, 14, 15]

The convergence of a sequence of pixel pictures can provide the most fundamental example for graphons. It is possible to generate different pixel picture presentations (as labeled graphons) for a graph in terms of the rescaling of the ground measure space or relabeling procedures. However we can encapsulate all these pixel picture presentations into a suitable isomorphic class to achieve the notion of uniqueness for this class of graph limits. Graphons, as analytic objects in infinite combinatorics, can be redefined in terms of a class of graph functions.

Definition 2.1. For a given measure space or a probability space (J, μ_J), a graphon is a symmetric bounded measurable function such as $W : J \times J \to [a, b] \subset \mathbb{R}$. It is called a bigraphon if we remove the symmetric property.

In the standard graphon models, we can work on the closed interval $J = [0, 1]$ equipped with the Lebesgue measure as the ground measure space to build graphons. In this setting, invertible Lebesgue measure preserving transformations on $[0, 1]$ such as ρ can generate relabeled versions of a given graphon. In other words, a relabeled graphon W^ρ is defined by $W^\rho(x, y) := W(\rho(x), \rho(y))$.

In general, graphons W_1, W_2 are called weakly isomorphic (or weakly equivalent), if there exist μ_J-measure preserving transformations σ_1, σ_2 on J such that $W_1^{\sigma_1}$ and $W_2^{\sigma_2}$ are the same almost everywhere. We can define an equivalence class $[W]$, known as unlabeled graphon class, which contains all relabeled graphons and weakly isomorphic versions with respect to a fixed graphon W.

We can define the cut-norm (as a semi-norm) on the space of labeled graphons. It is given by

$$\|W^\rho\|_{\text{cut}} := \sup_{A,B \subsetneq J} \left| \int_{A \times B} W^\rho(x, y) d\mu_J(x) d\mu_J(y) \right|. \tag{2.1}$$

This semi-norm is the key tool to define graph limits where we need to work on the space of unlabeled graphon classes to define the notion of unique convergence for the space of finite graphs. The cut-norm (2.1) gives us a metric structure on the space of unlabeled graphon classes. It is defined by

$$d_{\text{cut}}([W_1], [W_2]) := \inf_{\rho_1, \rho_2} \|W_1^{\rho_1} - W_2^{\rho_2}\|_{\text{cut}}, \tag{2.2}$$

such that the resulting topological space is compact and Hausdorff [9, 14].

Lemma 2.1. *Each finite simple weighted graph can determine a unique unlabeled graphon class.*

Proof. We consider labeled graphons on the closed interval $[0, 1]$ equipped with the Lebesgue measure. Each finite simple weighted graph $G = (V, E)$ can determine a class of labeled graph functions generated via its corresponding adjacency matrix A_G. They are pixel picture presentations. The set of vertices V can be seen as a finite probability space with the uniform measure and the set of edges E as the indicator of adjacency. Then we define the labeled graph function W_G^σ by fixing a partition σ on the closed interval such as dividing $[0, 1]$ into $|V|$ equal sub-intervals I_is. Now define $W_G^\sigma(x, y) := a_{ij} \in A_G$ for $x \in I_i$ and $y \in I_j$. Up to the weakly isomorphic relation, now we can associate an unlabeled graphon class $[W_G^\sigma]$ to the graph G which contains all possible labeled graph functions W_G^σ which are equivalent in terms of relabeling via invertible measure preserving transformations or they are weakly isomorphic. $\square$

The metric (2.2) is the key tool for the study of the behavior of extremely large graphs or complex networks whenever the vertex set of these graphs goes to infinity. In this setting, we can check that two graphons are weakly isomorphic if they have zero cut-distance from each other. Graphons generated by relabeling are weakly isomorphic. The graphon corresponding to the empty graph (i.e., 0-graphon) is identified by the class $[W_{\mathbb{I}}^{\sigma}]$ of graph functions such that $\int_{[0,1]\times[0,1]} W_{\mathbb{I}}^{\sigma}(x,y)dxdy = 0$. Graph limits can be interpreted as objects of the boundary region of the topological space of all finite graphs with respect to the cut-distance topology [9, 14, 15].

The theory of graphons has also been developed for the study of sparse graphs where we need to renormalize graph functions or rescale the base measure of the ground measure space to build non-zero graphons via the convergent limits of sequences of sparse graphs with weak densities [1–3, 15]. We recently applied this class of graphon models to formulate an analytic generalization for Feynman diagrams in Quantum Field Theory. These graphon models have led us to find some new combinatorial tools in dealing with Dyson-Schwinger equations as fixed point equations of Green's functions. It is then shown that non-perturbative solutions of quantum motions in gauge field theories can be described in terms of cut-distance convergent limits of sequences of random graphs generated by graphon representations of Feynman diagrams and their formal expansions [17–19].

3. Topological Hopf Algebra Structures on Kontsevich's Admissible Graphs and their Graphon Models

In this part, we study the fundamental elements of Deformation Quantization namely, Kontsevich's admissible graphs, Hochschild-Kontsevich products and their connection to the Connes-Kreimer insertion operator on Feynman diagrams. We then define Kontsevich graphons which are useful to study graph limits of Kontsevich's admissible graphs. We then equip the space of finite Kontsevich's admissible graphs and the space of their corresponding graphon models with the cut-distance topology together with some new Hopf algebra structures derived from the Connes-Kreimer renormalization Hopf algebra of Feynman diagrams. The Hopf algebra of Kontsevich's admissible graphs is topologically completed via the topology of graphons which can lead us to formulate the concept of convergence for sequences of these graphs. Our study provides a new class of infinite Kontsevich graphs which can be described in terms of convergent limits of sequences of random graphs derived from graphon models.

Definition 3.1. A Kontsevich's admissible graph is a simple oriented graph which contains two classes of totally ordered disjoint sets of vertices called internal and boundary vertices. Boundary vertices are leaves while there are no multiple edges or self-loops in the graph. There is also a total order on the set of all edges.

Remark 3.1. A Kontsevich's admissible graph can be presented via a disk such that internal vertices live inside the disk and boundary vertices live on the boundary region of the disk.

Definition 3.2. Let $\mathcal{G}^{p,q}$ be the set of isomorphism classes of all Kontsevich's admissible graphs such as K with q internal vertices such that $v(K) - e(K) - 1 = p$. Set $\mathfrak{g}^{\bullet,\bullet}$ as the bigraded vector space generated by $\bigcup_{p,q=0}^{\infty} \mathcal{G}^{p,q}$.

A subgraph G of K is called a normal subgraph if the quotient graph $H = K/G$ as the result of collapsing the subgraph G to a vertex v_G is itself a graph in $\mathfrak{g}^{\bullet,\bullet}$. Each normal subgraph G should be a full subgraph which means that every edge of K connecting two vertices of G is an edge of G [7, 12].

Remark 3.2. We can describe K as an extension of H by G in terms of inserting the graph G into a vertex of H. This process can be summarized by the notation $G \hookrightarrow K \twoheadrightarrow H$ such that the extension is called internal or boundary with respect to the type of that vertex which G is inserted into.

Definition 3.3. We can define two different Hochschild-Kontsevich products on $\mathfrak{g}^{\bullet,\bullet}$ in terms of types of vertices. They are given by

$$H \bullet G := \sum_{G \hookrightarrow K \twoheadrightarrow H,\ \text{internal}} \pm K, \qquad H \circ G := \sum_{G \hookrightarrow L \twoheadrightarrow H,\ \text{boundary}} \pm L \tag{3.1}$$

such that $\bullet$ is a $(0, -1)$ degree product and $\circ$ is a bigraded product.

Feynman diagrams in Quantum Field Theory are finite oriented labeled graphs which contains two classes of edges namely, internal and external edges. Each internal edge has begining and ending points while each external edge has only begining or ending point. Decorations in each Feynman diagram can encode fundamental data of physical systems such as conservation of momenta while vertices encode interactions among elementary particles (i.e., edges). Each Feynman diagram is a simplified model for a complicated iterated ill-defined integral which exists in the Green's functions of the physical theory. In Connes-Kreimer theory, we can describe the perturbative renormalization machinery in terms of a factorization algorithm on Feynman diagrams originated from the insertion operator. Rebuilding Feynman diagrams from the components of this factorization might not be unique in gauge field theories where we need to apply some new shuffle type products on Feynman diagrams or some identities among Feynman diagrams to generate a uniqueness [10, 11, 20].

Lemma 3.1. *The Hochschild-Kontsevich products $\bullet$ and $\circ$ can determine a pre-Lie operator on the set of Feynman diagrams.*

Proof. In terms of types of vertices and types of edges, we can glue Feynman diagrams to obtain a new diagram or decompose a complicated Feynman diagram into its primitive components. For any given Feynman diagrams Γ_1, Γ_2, suppose there exists a vertex $v_i \in \Gamma_1$ such that $f_{v_i} \sim \Gamma_2^{[1],\mathrm{ext}}$. Then we can define the insertion of Γ_2 inside Γ_1 via v_i in terms of the formula

$$\Gamma_1 *_{v_i} \Gamma_2 := \Gamma_1/\{v_i\} \cup \Gamma_2/\Gamma_2^{[1],\mathrm{ext}}, \tag{3.2}$$

which is a new graph such that for each edge $e_j \in f_i$, $\{v_{e_j}\}$ contains only one vertex of Γ_2. The sum over all possible vertices which have equivalent type with $\Gamma_2^{[1],\mathrm{ext}}$ gives us the insertion of Γ_2 inside Γ_1. We have

$$\Gamma_1 *_{\mathrm{ins}} \Gamma_2 := \sum_{v \in \Gamma_1, f_v \sim \Gamma_2^{[1],\mathrm{ext}}} \Gamma_1 *_v \Gamma_2, \tag{3.3}$$

which is known as the Connes-Kreimer insertion operator and it provides a pre-Lie algebra structure on Feynman diagrams. The commutator with respect to the insertion operator defines a Lie algebra structure on Feynman diagrams which leads us to build the Connes-Kreimer renormalization Hopf algebra [4, 10, 21]. The insertion operator $*_{\mathrm{ins}}$ is a non-homogeneous product which can be described as a combination of the Hochschild-Kontsevich products $\bullet$ and $\circ$ (3.1). $\square$

We can formulate an analytic generalization for Kontsevich's admissible graphs in the context of the theory of graphons.

Lemma 3.2. *Any Kontsevich's admissible graph K can determine a unique unlabeled (bi)graphon class $[W_K]$.*

Proof. We need to update Lemma 2.1. We choose the closed interval $[0, 1]$ equipped with the Lebesgue measure as the ground measure space. Thanks to Definition 3.1, we can build the adjacency matrix A_K corresponding to the graph K. This matrix can be presented by a pixel picture P_K presentation built by the scaling of $[0, 1]^2$ where 1's in A_K turn into black squares and 0's in A_K turn into white squares. This class of presentations can be encoded by choosing partitions σ on $[0, 1]$ together with symmetric bounded Lebesgue measurable maps W_K^σ defined on $[0, 1]^2$.

We call $[W_K]$ the unlabeled Kontsevich graphon class corresponding to the graph K. This class contains all relabeled Kontsevich graphons corresponding to K and all other Kontsevich graphons which are weakly isomorphic to W_K. $\square$

Definition 3.4. A sequence $\{K_n\}_{n\geq 0}$ of finite Kontsevich's admissible graphs is called convergent when n tends to infinity, if the corresponding sequence $\{[W_{K_n}]\}_{n\geq 0}$ of unlabeled Kontsevich graphon classes converges to a non-zero unlabeled Kontsevich graphon class $[W_\infty]$ with respect to the cut-distance topology.

The non-zero graph limit W_∞ can be built by rescaling methods explained in [1–3, 15] which enable us to renormalize the canonical graphons.

Definition 3.5. The Kontsevich's admissible graph generated by the information of the Kontsevich graphon W_∞ is an infinite graph K_{W_∞}. It contains infinite number of internal or boundary vertices or (infinite) number of edges. We call K_{W_∞} an infinite Kontsevich graph.

Kontsevich graphons are useful to study the asymptotic behavior of growing sequences of Kontsevich normal subgraphs with respect to the cut distance topology.

Proposition 3.1. *We can lift products $\circ$ and $\bullet$ onto the level of Kontsevich graphons.*

Proof. Let $\{K_n\}_{n\geq 0}$ be a sequence of Kontsevich's admissible graphs which is cut-distance convergent to the unlabeled Kontsevich graphon class $[W_{K_\infty}]$ with the corresponding infinite Kontsevich graph K_∞. Let $\{G_n\}_{n\geq 0}$ be another sequence of Kontsevich's admissible graphs such that for each n, G_n is a normal subgraph of K_n. Let the sequence $\{G_n\}_{n\geq 0}$ is cut-distance convergent to the unlabeled Kontsevich graphon class $[W_{G_\infty}]$ with the corresponding infinite Kontsevich graph G_∞.

We can build a new sequence $\{H_n\}_{n\geq 0} := \{K_n/G_n\}_{n\geq 0}$ of quotient graphs which is cut-distance convergent to the infinite Kontsevich graph H_∞. Thanks to Kontsevich graphon representations W_{K_∞}, W_{G_∞} and W_{K_∞/G_∞}, we can show that $W_{H_\infty} \in [W_{K_\infty/G_\infty}]$. Therefore, $H_\infty = K_\infty/G_\infty$. Now for each n, we can define

$$H_n \bullet G_n = \sum_{G_n \hookrightarrow K_n \twoheadrightarrow H_n, \text{ internal}} \pm K_n, \quad H_n \circ G_n = \sum_{G_n \hookrightarrow K_n \twoheadrightarrow H_n, \text{ boundary}} \pm K_n. \tag{3.4}$$

As the result, we can define $H_\infty \bullet G_\infty$ as the infinite Kontsevich graph corresponding to the cut-distance convergent limit of the sequence $\{H_n \bullet G_n\}_{n\geq 0}$ and define $H_\infty \circ G_\infty$ as the infinite Kontsevich graph corresponding to the cut-distance convergent limit of the sequence $\{H_n \circ G_n\}_{n\geq 0}$. □

Definition 3.6. The bigraded vector space $\mathfrak{g}^{\bullet,\bullet}$ (i.e., Definition 3.2) together with the cut-distance topology give us a topological vector space. We present this new space with $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$ such that its objects have graphon representations determined by Lemma 3.2, Definition 3.4 and Definition 3.5.

Remark 3.3. $H_\infty \bullet G_\infty$ or $H_\infty \circ G_\infty$ could have infinite terms in their series. The compactness of the topology of graphons enables us to describe these infinite series in terms of objects in the boundary of the space $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$.

Ionescu in arXiv:hep-th/0307062: *Perturbative Quantum Field Theory and configuration space integrals*, 2003 and [8] has applied the Kreimer's renormalization coproduct to build a differential graded Hopf algebra structure on Kontsevich's graph complex. Thanks to our explained graphon models, now we can formulate a new topological Hopf algebra structure on Kontsevich's admissible graphs which can be completed in terms of the cut-distance topology.

Proposition 3.2. *The completion map with respect to normal subgraphs together with the graphon representations of Kontsevich's admissible graphs can determine a topological Hopf algebra structure on* $\mathfrak{g}^{\bullet,\bullet}_{\text{cut}}$.

Proof. Thanks to the Connes-Kreimer renormalization Hopf algebra of Feynman diagrams, the structure of a differential graded Hopf algebra on Kontsevich's admissible graphs has been explained in [8]. We work on the free commutative algebra generated by Kontsevich's admissible graphs over the field $\mathbb{Q}$ or $\mathbb{R}$ such that the empty graph is its unit. For any given Kontsevich's admissible graph K, define

$$\Delta(K) = \mathbb{I} \otimes K + K \otimes \mathbb{I} + \sum_{G} G \otimes K/G, \tag{3.5}$$

as a coproduct such that the sum is over all non-trivial normal subgraphs of K and $\mathbb{I}$ is the empty graph. Terms in this expansion are in an one to one correspondence with all possible internal or boundary extensions of normal subgraphs of the original graph.

The counit is defined by $\varepsilon(\mathbb{I}) = 1$ and $\varepsilon(K) = 0$ for $K \neq \mathbb{I}$. If we apply the graduation parameter on Kontsevich's admissible graphs given by Definition 3.1 and Definition 3.2, then we can define an antipode recursively. This completes the construction of the renormalization Hopf algebra of Kontsevich's admissible graphs.

Now we plan to topologically complete this Hopf algebra in terms of graphon representations of Kontsevich's admissible graphs (i.e., Lemma 3.2 and Definition 3.4). It is enough to show the continuity of the coproduct and antipode with respect to the topology of graphons.

We work on the free commutative algebra generated by unlabeled Kontsevich graphon classes over the field $\mathbb{Q}$ or $\mathbb{R}$ such that $[W_{\mathbb{I}}]$ corresponding to the empty graph is its unit. Thanks to the coproduct (3.5), for any unlabeled Kontsevich graphon class $[W_K]$ corresponding to a finite graph K, its coproduct is given by

$$\Delta([W_K]) = [W_{\mathbb{I}}] \otimes [W_K] + [W_K] \otimes [W_{\mathbb{I}}] + \sum [W_G] \otimes [W_{K/G}], \tag{3.6}$$

such that the sum is controlled by Kontsevich graphons associated to non-trivial normal subgraphs of K. This coproduct is a bounded and linear map which makes it a continuous map with respect to the cut-distance topology.

In addition, let K_∞ be an infinite Kontsevich graph as the graph limit of the sequence $\{K_n\}_{n\geq 0}$ of finite Kontsevich's admissible graphs. Let $[W_\infty]$ be the unique unlabeled Kontsevich graphon class corresponding to K_∞. This means that the sequence $\{[W_{K_n}]\}_{n\geq 1}$ is cut-distance convergent to $[W_\infty]$. Thanks to the continuity of the coproduct (3.6), $\Delta([W_\infty])$ can be defined as the cut-distance convergent limit of the sequence $\{\Delta([K_n])\}_{n\geq 0}$.

The counit is defined by $\varepsilon([W_{\mathbb{I}}]) = 1$ and $\varepsilon([W_K]) = 0$, for $K \neq \mathbb{I}$. We can also define the antipode map on unlabeled Kontsevich graphon classes recursively in terms of the cut-distance convergent limit of a sequence of antipodes of finite Kontsevich graphs. The compactness of the cut-distance topology is enough to observe that the defined coproduct and antipode are bounded. The linearity and boundary condition guarantee the continuity of the coproduct and antipode.

We use the notation $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ for the resulting topological Hopf algebra of unlabeled Kontsevich graphon classes. We also use the notation $H^{\mathrm{cut}}_{\mathrm{Kont}}$ for the resulting topological Hopf algebra of Kontsevich's admissible graphs which is generated by $\mathfrak{g}^{\bullet,\bullet}_{\mathrm{cut}}$ as a vector space.

Any linear combination $\alpha_1 K_1 + \cdots + \alpha_n K_n$ of Kontsevich's admissible graphs can generate a Kontsevich graphon class in $\mathcal{S}^{\text{Kont}}_{\text{graphon}}$. The corresponding labeled Kontsevich graphon $W_{\alpha_1 K_1 + \cdots + \alpha_n K_n}$ can be determined in terms of the normalizing or rescaling methods used on each $W_{\alpha_i K_i}$. In other words, for each $1 \leq i \leq n$, we first project the labeled Kontsevich graphon $W_{\alpha_i K_i}$ into the subinterval I_i of $[0,1]$ where $\{I_i\}_i$ is a partition for $[0,1]$. We present the resulting labeled graphons with $W_{\tilde{\alpha}_i \tilde{K}_i}$. Then we can define

$$W_{\tilde{\alpha}_1 \tilde{K}_1 + \cdots + \tilde{\alpha}_n \tilde{K}_n} := \frac{W_{\tilde{\alpha}_1 \tilde{K}_1} + \cdots + W_{\tilde{\alpha}_n \tilde{K}_n}}{\| W_{\tilde{\alpha}_1 \tilde{K}_1} + \cdots + W_{\tilde{\alpha}_n \tilde{K}_n} \|_{\text{cut}}}. \tag{3.7}$$

Thanks to the correspondences $K \mapsto [W_K]$ and $\{K_n\}_{n \geq 0} \mapsto K_{[W_\infty]}$, we can complete the Hopf algebra of Kontsevich's admissible graphs and formulate a surjective topological Hopf algebra homomorphism

$$\Psi_{\text{Kont}} : \mathcal{S}^{\text{Kont}}_{\text{graphon}} \to H^{\text{cut}}_{\text{Kont}}. \tag{3.8}$$ □

Thanks to this study, now it is possible to define the notion of distance between Kontsevich's admissible graphs via their graphon representations.

Definition 3.7. The distance between Kontsevich's admissible graphs K_1 and K_2 is defined in terms of the cut-distance between their corresponding unlabeled Kontsevich graphon classes. In other words, thanks to the metric (2.2), we have

$$d(K_1, K_2) := d_{\text{cut}}([W_{K_1}], [W_{K_2}]). \tag{3.9}$$

Corollary 3.1. *A sequence of Kontsevich's admissible graphs is convergent if and only if it is a cut-distance Cauchy sequence.*

Corollary 3.2. *For a given Kontsevich graphon W_∞, there exists a sequence of finite random graphs which is cut-distance convergent to W_∞.*

Proof. For each n, we can define a finite random graph $G(W_\infty, n)$ which contains n points $x_1, \ldots, x_n$ from the Kontsevich graphon W_∞ such that the existence of an edge between x_i and x_j is determined by the probability $W_\infty(x_i, x_j)$. Thanks to [9, 19], we can show that the sequence $\{G(W_\infty, n)\}_{n \geq 0}$ is cut-distance convergent to W_∞. □

Infinite polydifferential operators can be described in terms of multiplication of functions and infinite vector fields which act as polyderivations on infinite functions. We can define these operators as the cut-distance convergent limit of sequences of finite operators. In this setting, the multiplication of infinite functions is represented by the Kontsevich graphon $b_{0,\infty}$ with no internal vertices and infinite (countable) boundary vertices. The resulting Kontsevich graphon is actually the cut-distance convergent limit of Kontsevich's admissible graphs which belong to $\mathcal{G}^{m-1,0}$ when m tends to infinity. In addition, ∞-vector field with infinite polyderivations is represented by the Kontsevich graphon $b_{1,\infty}$ with one internal vertex and infinite countable boundary vertices with infinite countable edges.

4. Noncommutative Differential Calculi on Kontsevich's Admissible Graphs and their Graphon Models via the Renormalization Map

In [12] it is shown that the Hopf-Birkhoff factorization of Feynman rules characters in the Connes-Kreimer perturbative renormalization process can be interpreted as a deformation of the pointwise multiplication of some exponential functions under the Kontsevich product. In this part we plan to work on the space of linear functionals on the topological Hopf algebra of Kontsevich's admissible graphs or Kontsevich graphons with values in the algebra A_{dr} of Laurent series with finite pole parts equipped with the minimal subtraction map to build a new class of differential graded Lie algebras and Poisson structures with respect to deformed versions of the convolution product.

The Rota-Baxter algebra $(A_{\mathrm{dr}}, R_{\mathrm{ms}})$ determines a class of deformed convolution products on the space $L(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ of linear maps given by

$$\phi_1 \circ_\lambda \phi_2 := \mathcal{R}_\lambda(\phi_1) * \phi_2 + \phi_1 * \mathcal{R}_\lambda(\phi_2) - \mathcal{R}_\lambda(\phi_1 * \phi_2), \tag{4.1}$$

such that $\mathcal{R}_\lambda := \mathcal{R} - \lambda(\mathrm{Id} - \mathcal{R})$, where $\mathcal{R}$ is the extension of R_{ms} on $L(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ and λ is a real number.

The convolution product $*$ is defined in terms of the coproduct (3.5) on Kontsevich's admissible graphs. In other words, for any $\phi_1, \phi_2 \in L(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ and any Kontsevich's admissible graph K, we have

$$\phi_1 * \phi_2(K) := \phi_1(\mathbb{I})\phi_2(K) + \phi_1(K)\phi_2(\mathbb{I}) + \sum_G \phi_1(G)\phi_2(K/G), \tag{4.2}$$

such that G are non-trivial normal subgraphs of K.

Let an infinite Kontsevich graph K_∞ is the result of the cut-distance convergent limit of a sequence $\{K_n\}_{n\geq 1}$ of finite Kontsevich's admissible graphs. Thanks to the continuity of the coproduct (3.5) with respect to the cut-distance topology and Proposition 3.1, we can show that the sequence $\{\sum_{G_n} \phi_1(G_n)\phi_2(K_n/G_n)\}_{n\geq 1}$ is cut-distance convergent to $\sum_{G_\infty} \phi_1(G_\infty)\phi_2(K_\infty/G_\infty)$. This means that we can extend the convolution product $*$ on infinite Kontsevich graphs where $\phi_1 * \phi_2(K_\infty)$ can be defined as the convergent limit of the sequence $\{\phi_1 * \phi_2(K_n)\}_{n\geq 1}$.

The associative products $\circ_\lambda$ on $L(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ are actually the direct consequence of the Nijenhuis property of the map $\mathcal{R}_\lambda$. The non-cocommutativity of $H^{\mathrm{cut}}_{\mathrm{Kont}}$ ensures that each product $\circ_\lambda$ is noncommutative. Therefore we can define a new Lie bracket $[\cdot,\cdot]_\lambda$ via the commutator with respect to $\circ_\lambda$.

Proposition 4.1. *There exists a noncommutative differential calculus on* $H^{\wedge\lambda}_{\mathrm{Kont}} := (L(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}}), \circ_\lambda)$.

Proof. Set $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$ as the center of the algebra and $\mathrm{Der}^{\lambda}_{\mathrm{Kont}}$ as the space of all linear maps $\theta : H^{\wedge\lambda}_{\mathrm{Kont}} \to H^{\wedge\lambda}_{\mathrm{Kont}}$ which obey the Leibniz rule. The Lie bracket $[\cdot,\cdot]_\lambda$, which satisfies the Jacobi identity, can determine the corresponding Poisson bracket $\{\cdot,\cdot\}_\lambda$. For each $\phi \in H^{\wedge\lambda}_{\mathrm{Kont}}$, define $\psi \mapsto \{\phi,\psi\}_\lambda$ as the corresponding Hamiltonian derivation. Set $\mathrm{Ham}^{\lambda}_{\mathrm{Kont}}$ as the $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$-module generated by all Hamiltonian derivations.

Thanks to the theory of noncommutative differential geometry on the basis of the space of derivations [6], for $n \geq 1$, define $\Omega^n_{\mathrm{Kont},\lambda}$ as the space of all $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$-multilinear anti-symmetric maps from $\mathrm{Ham}^{\lambda}_{\mathrm{Kont}} \times \cdots^{n} \times \mathrm{Ham}^{\lambda}_{\mathrm{Kont}}$ to $H^{\wedge\lambda}_{\mathrm{Kont}}$. We have the differential graded algebra $(\Omega^{\bullet}_{\mathrm{Kont},\lambda}, d_\lambda)$ such that the degree one anti-derivative differential operator d_λ is given by

$$\begin{aligned} d_\lambda \omega(\theta_0, \dots, \theta_n) := &\sum_{k=0}^{n} (-1)^k \theta_k \omega(\theta_0, \dots, \hat{\theta}_k, \dots, \theta_n) \\ &+ \sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, \dots, \hat{\theta}_r, \dots, \hat{\theta}_s, \dots, \theta_n). \qquad \square \end{aligned}$$

Corollary 4.1. *There exists a new class of integrable systems which can geometrically evaluate Kontsevich's admissible graphs.*

Proof. We apply the renormalization map $R_{\mathrm{ms}} : A_{\mathrm{dr}} \to A_{\mathrm{dr}}$ and work on the noncommutative deRham complex derived from Proposition 4.1. We have

$$\mathrm{DR}^{\bullet}_{\mathrm{Kont},\lambda} := \frac{\Omega^{\bullet}_{\mathrm{Kont},\lambda}}{[\Omega^{\bullet}_{\mathrm{Kont},\lambda}, \Omega^{\bullet}_{\mathrm{Kont},\lambda}]_\lambda}. \tag{4.3}$$

The deformed Lie bracket $[\cdot,\cdot]_\lambda$ allows us to define a class of $Z(H^{\wedge\lambda}_{\mathrm{Kont}})$-bilinear anti-symmetric non-degenerate closed 2-forms for the presentation of the Poisson bracket $\{\cdot,\cdot\}_\lambda$. For any derivations $\theta_1 = \sum u_i \circ_\lambda \mathrm{ham}(f_i)$, $\theta_2 = \sum v_j \circ_\lambda \mathrm{ham}(h_j)$, define the symplectic form

$$\omega_\lambda(\theta_1, \theta_2) = \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [f_i, h_j]_\lambda, \tag{4.4}$$

such that $\{f_1, \dots, f_n, h_1, \dots, h_m\} \subsetneq H^{\wedge\lambda}_{\mathrm{Kont}}$, $\{u_1, \dots, u_n, v_1, \dots, v_m\} \subsetneq Z(H^{\wedge\lambda}_{\mathrm{Kont}})$.

If θ^λ_f is the symplectic vector field associated to the symplectic form ω_λ, then we have

$$\{f, g\}_\lambda = \omega_\lambda(\theta^\lambda_f, \theta^\lambda_g), \tag{4.5}$$

as the quantization of the Poisson structure on Kontsevich's admissible graphs and Kontsevich graphons in the direction of the minimal subtraction scheme.

Thanks to [4,21], we can build the Connes-Kreimer Renormalization Group $\{F_t\}_t$ of the topological Hopf algebra $H^{\mathrm{cut}}_{\mathrm{Kont}}$ of Kontsevich's admissible graphs. This is a 1-parameter subgroup of the Lie group $\mathrm{Hom}(H^{\mathrm{cut}}_{\mathrm{Kont}}, A_{\mathrm{dr}})$ of characters. Then we can check that $\{F_t, F_s\}_0 = 0$. $\square$

Remark 4.1. Thanks to the surjective homomorphism Ψ_{Kont} (3.8), Proposition 4.1 and Corollary 4.1, we can build a noncommutative differential calculus on $\mathcal{S}^{\mathrm{Kont},\wedge\lambda}_{\mathrm{graphon}}$ and then we can show that the Connes-Kreimer Renormalization Group of $\mathcal{S}^{\mathrm{Kont}}_{\mathrm{graphon}}$ can determine a new class of integrable systems.

Corollary 4.2. *The Kontsevich's Deformation Quantization* [7,12] *can be lifted onto the level of Kontsevich graphons.*

Proof. For the algebra

$$S_{\text{Kont}}^{\wedge_\lambda} := (L(\mathcal{S}_{\text{graphon}}^{\text{Kont}}, A_{\text{dr}}), \circ_\lambda), \tag{4.6}$$

we work on the Lie algebra $\text{der}_{\text{Kont}}^{\lambda}$ of all derivations $\rho : \mathcal{S}_{\text{graphon}}^{\text{Kont}} \to A_{\text{dr}}$. This space is generated by infinitesimal characters such as $\rho_{[W]}$ corresponding to each Kontsevich graphon $[W]$. Let $\mathcal{A}^d$ as the space of functions with the domain $\text{der}_{\text{Kont}}^{\lambda} \times \cdots^{d} \times \text{der}_{\text{Kont}}^{\lambda}$ and with the images in $\mathcal{S}_{\text{graphon}}^{\text{Kont}}$.

For any Kontsevich graphon $[W_K]$ corresponding to the graph $K \in H_{\text{Kont}}^{\text{cut}}$, we can define the bi-differential operator $B_{[W_K],\lambda} : \mathcal{A}^d \times \mathcal{A}^d \longrightarrow \mathcal{A}^d$ in terms of the differential operator d_λ (determined by the Poisson structure $\{\cdot,\cdot\}_\lambda$) and derivations ρ_K.

Set G_n, $n \geq 0$, as the collection of all Kontsevich graphs with $n+2$ vertices $\{1,\ldots,n\} \cup \{X,Y\}$ and $2n$ edges such that for each vertex k, there exist two edges staring at k. We can now define a new $\star$-product on $\mathcal{A}^d$ as the Kontsevich's quantization of $\circ_\lambda$. For any functions $F, G \in \mathcal{A}^d$, $F \star_\lambda G$ is defined as the convergent limit of the sequence

$$\left\{ \sum_{j=0}^{n} \epsilon^j \sum_{L \in G_j} \omega_K(L) B_{[W_K],\lambda}(F,G) \right\}_{n \geq 0}, \tag{4.7}$$

with respect to the cut-distance topology defined on Kontsevich graphons when n tends to infinity. $\square$

The quantization $F \star_\lambda G$ can contain an infinite formal expansion of growing Kontsevich's admissible graphs which can not be handled by the perturbative setting. Therefore we name it a non-perturbative generalization of the standard Kontsevich's Deformation Quantization. Thanks to the compactness of the topology of graphons [9, 14], we can search for cut-distance graph limits for these infinite expansions.

5. Maurer-Cartan Equations on Kontsevich Graphons

In this section, we aim to formulate a new generalization of the Maurer-Cartan equations for infinite Kontsevich graphs (i.e., Definition 3.5) generated as the graph limits of sequences of finite Kontsevich's admissible graphs.

The commutator with respect to the operation $\circ$ gives a Lie algebraic structure on $\mathfrak{g}_{\text{cut}}^{\bullet;\bullet}$. This Lie bracket is actually obtained as an extension of the Hochschild-Kontsevich Lie bracket with respect to the cut-distance topology. It determines the differential operator d_1 of degree $(1,0)$. In addition, we can also extend the Kontsevich's vertical differential operator on $\mathfrak{g}_{\text{cut}}^{\bullet;\bullet}$ to define the differential operator d_2 on infinite Kontsevich's admissible graphs. For a given infinte Kontsevich graph $K_{[U_\infty]}$ corresponding to the unlabeled Kontsevich graphon class $[U_\infty]$, $d_2(K_{[U_\infty]})$ is the result of the cut-distance convergent limit of the sequence $\{d_2(K_n)\}_{n\geq 0}$, where for each n

$$d_2(K_n) := \sum_{e \hookrightarrow G \twoheadrightarrow K_n,\ \text{internal}} \pm G = K_n \bullet e, \tag{5.1}$$

which is expanding the internal vertices of K_n by the insertion of an additional edge. d_2 is a differential operator of degree $(0,1)$.

Proposition 5.1. *There exists a Hochschild-Kontsevich differential graded Lie algebra on Kontsevich graphons.*

Proof. Set $\mathfrak{g}^n_{\text{cut}} := \oplus_{p+q=n}\mathfrak{g}^{p,q}$ as the graded vector space equipped with the cut-distance topology. We can show that differential operators d_1, d_2 commute on the total complex $\mathfrak{g}^\bullet_{\text{cut}}$ and therefore $d := d_1 \pm d_2$ is a total differential operator which is compatible with the graded Lie bracket $[\cdot,\cdot]_\circ$ induced by $\circ$. $\square$

Now we can formulate the Maurer-Cartan equations on an infinite generalization of Kontsevich's admissible graphs.

Corollary 5.1. *There exists a modified version of the Maurer-Cartan equation on infinite Kontsevich graphs.*

Proof. The topological Hopf algebra $H^{\text{cut}}_{\text{Kont}}$ of Kontsevich's admissible graphs (built by Proposition 3.2) and the noncommutative differential calculus (i.e., Proposition 4.1) can be applied to associate the 1-form

$$\alpha_{MC}(K) = \sum_G S(G) d_\lambda \theta_{K/G}, \tag{5.2}$$

such that S is the antipode of $H^{\text{cut}}_{\text{Kont}}$, the sum is taken over all normal subgraphs G of K and $\theta_{K/G}$ is the infinitesimal characters with respect to Kontsevich's admissible quotient graphs K/G. We can check that

$$\alpha_{MC}(KL) = \alpha_{MC}(K)\varepsilon(L) + \varepsilon(K)\alpha_{MC}(L). \tag{5.3}$$

Therefore, a general presentation of the Maurer-Cartan equation has the form

$$d_\lambda \alpha_{MC}(K) = -\sum_G \alpha_{MC}(G)\alpha_{MC}(K/G). \tag{5.4}$$

Now suppose $\{K_n\}_{n\geq 0}$ be a sequence of finite Kontsevich's admissible graphs which satisfy the equation (5.4) for each $n \geq 0$ and the sequence is cut-distance convergent to the Kontsevich graphon W_∞. Then it can be seen that the infinite Kontsevich graph $K_{[W_\infty]}$ is also a solution for (5.4). $\square$

It is possible to define a new morphism $\bar{\mathcal{U}}$ of differential graded Lie algebras (as a generalization of the map $\mathcal{U}$ given in [7]) between $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}$ and the Chevalley-Eilenberg complex $\text{CE}^{\bullet;\bullet}_{\text{cut}}(T^\bullet_{\text{poly}}, D^\bullet_{\text{poly}})$ equipped with the cut-distance topology. This enables us to formulate the Maurer-Cartan equations on the complex of Kontsevich graphons in the language of morphisms between $T^\bullet_{\text{poly}}$ and $D^\bullet_{\text{poly}}$.

6. Conclusion

The main achievement of this work is to provide some new mathematical tools for the study of the Kontsevich Deformation Quantization under a non-perturbative setting. We applied graphon models for the study of the space of Kontsevich's admissible graphs to formulate some new topological Hopf algebra structures $H_{\mathrm{Kont}}^{\mathrm{cut}}$ on these graphs and $\mathcal{S}_{\mathrm{graphon}}^{\mathrm{Kont}}$ on their corresponding graphon models. Then we worked on the basis of this Hopf algebraic setting to build a new class of noncommutative differential calculi on Kontsevich's admissible graphs originated from the BPHZ perturbative renormalization. This study has led us to determine a new class of quantized integrable systems which can geometrically describe the evolution of sequences of Kontsevich's admissible graphs. In addition, thanks to the topologically completion of our Hopf algebra model, we formulated the Kontsevich's Deformation Quatization for Kontsevich graphons which has led us to obtain a non-perturbative generalization for deformation quantization procedure. Furthermore, we have obtained a new modified version of the Maurer-Cartan equations on infinite Kontsevich graphs.

As the final note, the topological Hopf algebras $H_{\mathrm{Kont}}^{\mathrm{cut}}$ and $\mathcal{S}_{\mathrm{graphon}}^{\mathrm{Kont}}$ are also useful to work on combinatorial Dyson-Schwinger equations on Kontsevich's admissible graphs in the context of Hochschild type of equations. Solutions of these equations can be described in terms of random graphs generated by Kontsevich graphons. This study can be useful to find some new interconnections between combinatorial Dyson-Schwinger equations and the non-perturbative generalization of the Kontsevich Deformation Quantization.

Acknowledgements. The author would like to thank the Reviewer because of the important comments and advices which were useful to clarify the results of this work.

References

[1] B. Bollobas and O. Riordan, *Metrics for Sparse Graphs*, Surveys in Combinatorics 2009, LMS Lecture Notes Series **365**, Cambridge Univ. Press, Cambridge, 2009, 211–288. https://doi.org/10.1017/CBO9781107325975.009

[2] C. Borgs, J. T. Chayes, H. Cohn and N. Holden, *Sparse exchangeable graphs and their limits via graphon processes*, J. Mach. Learn. Res. **18**(210), 1–71, 2018. https://www.jmlr.org/papers/v18/16-421.html

[3] C. Borgs, J. T. Chayes, H. Cohn and Y. Zhao, *An L^p theory of sparse graph convergence I: Limits, sparse random graph models, and power law distributions*, Trans. Amer. Math. Soc. **372**(5) (2019), 3019–3062. https://doi.org/10.1090/tran/7543

[4] A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Commun. Math. Phys. **199**(1) (1998), 203–242. https://doi.org/10.1007/s002200050499

[5] C. Chryssomalakos, H. Quevedo, M. Rosenbaum and J. D. Vergara, *Normal coordinates and primitive elements in the Hopf algebra of renormalization*, Commun. Math. Phys. **225**(3) (2002), 465–485. https://doi.org/10.1007/s002200100588

[6] M. Dubois-Violette, *Lectures on graded differential algebras and noncommutative geometry*, in: *Noncommutative Differential Geometry and Its Applications to Physics*, Mathematical Physics Studies **23**, Springer, Dordrecht, 2001, 245–306. https://doi.org/10.1007/978-94-010-0704-7_15

[7] D. Fiorenza and L. M. Ionescu, *Graph complexes in deformation quantization*, Lett. Math. Phys. **73**(3) (2005), Article ID 193. https://doi.org/10.1007/s11005-005-0017-7

[8] L. M. Ionescu, *Cohomology of Feynman graphs and perturbative quantum field theory*, in: O. Kovras (Ed.), *Focus on Quantum Field Theory*, Nova Publishers Inc., 2004.

[9] S. Janson, *Graphons, cut norm and distance, couplings and rearrangements*, NYJM Monographs **4** (2013), 76 pages.

[10] D. Kreimer, *Factorization in quantum field theory: an exercise in Hopf algebras and local singularities*, in: Cartier P., Moussa P., Julia B., Vanhove P. (eds) *Frontiers in Number Theory, Physics, and Geometry II*, Springer, Berlin, 2007, 715–736. https://doi.org/10.1007/978-3-540-30308-4_14

[11] D. Kreimer, *Unique factorization in pertubative QFT*, Nuclear Physics B - Proceedings Supplements **116** (2003), 392–396. https://doi.org/10.1016/S0920-5632(03)80206-2

[12] V. Kathotia, *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, Internat. J. Math. **11**(4) (2000), 523–551. https://doi.org/10.1142/S0129167X0000026X

[13] M. Kontsevich, *Formal* (*non*)*commutative symplectic geometry*, in: *The Gelfand Mathematical Seminars* **1990**–**1992**, Birkhauser, Boston, Basel, Berlin, 1993, 173–187. https://doi.org/10.1007/978-1-4612-0345-2_11

[14] L. Lovasz, *Large Networks and Graph Limits*, American Mathematical Society Colloquium Publications **60**, Amer. Math. Soc., Providence, RI, 2012. https://doi.org/10.1090/coll/060

[15] J. Nesetril and P. Ossona de Mendez, *Sparsity-Graphs, Structures, and Algorithms*, Algorithms and Combinatorics **28**, Springer, Heidelberg, 2012. https://doi.org/10.1007/978-3-642-27875-4

[16] M. Rosenbaum and J. D. Vergara, *The Hopf algebra of renormalization, normal coordinates and Kontsevich deformation quantization*, J. Phys. A **37** (2004), 7939–7954. https://doi.org/10.1088/0305-4470/37/32/008

[17] A. Shojaei-Fard, *Formal aspects of non-perturbative quantum field theory via an operator theoretic setting*, Int. J. Geom. Methods Mod. Phys. **16**(12) (2019), Paper ID 1950192, 23 pages. https://doi.org/10.1142/S0219887819501925

[18] A. Shojaei-Fard, *Non-perturbative β-functions via Feynman graphons*, Modern Phys. Lett. A **34**(14) (2019), Paper ID 1950109, 10 pages. https://doi.org/10.1142/S0217732319501098

[19] A. Shojaei-Fard, *Graphons and renormalization of large Feynman diagrams*, Opuscula Math. **38**(3) (2018), 427–455. https://doi.org/10.7494/OpMath.2018.38.3.427

[20] A . Shojaei-Fard, *From Dyson-Schwinger equations to the Riemann-Hilbert correspondence*, Int. J. Geom. Methods Mod. Phys. **7**(4) (2010), 519–538. https://doi.org/10.1142/S0219887810004427

[21] W. D. van Suijlekom, *Renormalization of gauge fields: a Hopf algebra approach*, Commun. Math. Phys. **276**(3) (2007), 773–798. https://doi.org/10.1007/s00220-007-0353-9

[1] Ph.D. Mathematician and Independent Scholar,
1461863596 Marzdaran Blvd., Tehran, Iran
ORCID iD: 0000-0002-6418-3227
Email address: shojaeifa@yahoo.com

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(2) (2023), PAGES 229–243.

SOME L_1-BICONSERVATIVE LORENTZIAN HYPERSURFACES IN THE LORENTZ-MINKOWSKI SPACES

FIROOZ PASHAIE[1]

ABSTRACT. The biconservative hypersurfaces of Euclidean spaces have conservative stress-energy with respect to the bienergy functional. We study Lorentzian hypersurfaces of Minkowski spaces, satisfying an extended condition (namely, L_1-biconservativity condition), where L_1 (as an extension of the Laplace operator $\Delta = L_0$) is the *linearized operator* arisen from the first normal variation of 2nd mean curvature vector field. A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ is said to be L_1-biconservative if the tangent component of vector field $L_1^2 x$ is identically zero. The geometric motivation of this subject is a well-known conjecture of Bang-Yen Chen saying that the only biharmonic submanifolds (i.e., satisfying condition $L_0^2 x = 0$) of Euclidean spaces are the minimal ones. We discuss on L_1-biconservative Lorentzian hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$. After illustrating some examples, we prove that these hypersurfaces, with at most two distinct principal curvatures and constant ordinary mean curvature, have constant 2nd mean curvature.

1. INTRODUCTION

The main geometric motivation of the subject of biconservative hypersurfaces is a well-known conjecture of Bang-Yen Chen (in 1987) which states that every biharmonic submanifold of a Euclidean space is harmonic. Further, Chen proved that his conjecture is true for biharmonic surfaces in $\mathbb{E}^3$. In 1992, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([10]). Let $\mathbf{x} : M^n \to \mathbb{E}^{n+1}$ denotes an isometric immersion of a hypersurface M^n into the $(n+1)$-dimensional Euclidean space with the Laplace operator Δ, the shape operator A associated to a unit normal vector field $\mathbf{n}$ and the ordinary mean curvature H on M^n. The hypersurface M^n is said to be harmonic if $\mathbf{x}$ satisfies condition $\Delta\mathbf{x} = 0$.

Key words and phrases. Lorentzian hypersurface, L_1-biconservative, Lorentz-Minkowski space.
2010 *Mathematics Subject Classification.* Primary: 53C43, 53C40, 53C42. Secondary: 58B25.
DOI 10.46793/KgJMat2302.229P
Received: March 31, 2020.
Accepted: August 12, 2020.

It is said to be biharmonic if $\mathbf{x}$ satisfies condition $\Delta^2\mathbf{x} = 0$. Also, M^n is said to be biconservative if the tangential part of $\Delta^2\mathbf{x}$ vanishes identically. A famous law due to Beltrami says that $\Delta\mathbf{x} = -nH\mathbf{n}$, so the condition $\Delta\mathbf{x} = 0$ is equivalent to $H \equiv 0$ and the condition $\Delta^2\mathbf{x} = 0$ is equivalent to $\Delta(H\mathbf{n}) = 0$. In 1995, Hasanis and Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-space ([11]). As an extended case, a hypersurface $\mathbf{x} : M_p^3 \to \mathbb{E}_s^4$, whose mean curvature vector field is an eigenvector of the Laplace operator Δ, has been studied, for instance, in [8, 9] for the Euclidean case (where $p = s = 0$), and for the Lorentz case in [4, 5] (for $s = 1$ and $p = 0, 1$). On the other hand, Chen himself had found a nice relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject initiated by Chen (for instance, in [6, 7]) and also studied by L. J. Alias, S. M. B. Kashani and others. In [12], Kashani has studied the notion of L_1-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([6]).

The map L_1 is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized operator of the first variation of the 2th mean curvature of the hypersurface (see, for instance, [1, 17, 20]). This operator is defined by $L_1(f) = tr(P_1 \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_1 = nHI - A$ denotes the first Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. It is interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_1. Recently, in [15], we have studied the L_1-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space $\mathbb{L}^4$. In this paper, we show that every L_1-biconservative Lorentzian hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$, with constant mean curvature and at most two distinct principal curvatures, has constant 2nd mean curvature.

We present the organization of paper. In Section 2, we remember some preliminaries which will be needed in paper. In Section 3, we present some examples of L_1-biconservative Lorentzian hypersurfaces in $\mathbb{L}^{n+1}$. Section 4 is dedicated to L_1-biconservative Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$. First, in Theorem 4.1, 4.2 and 4.3 we discuss on L_1-biconservative Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$ with diagonalizable shape operator. The other cases that the shape operator of hypersurface is non-diagonalizable will be seen in Theorem 4.4, 4.5 and 4.6.

2. Preliminaries

In this section, we recall preliminaries from [1, 13, 14] and [16–19]. The m-dimensional Lorentz-Minkowski space $\mathbb{L}^m$ means the pseudo-Euclidean space with index 1, $\mathbb{E}_1^m$, which is the real vector space $\mathbb{R}^m$ endowed with the scalar product defined by $\langle x, y\rangle := -x_1y_1 + \Sigma_{i=2}^m x_iy_i$ for every $x, y \in \mathbb{R}^m$. Throughout the paper, we study on every Lorentzian hypersurface of $\mathbb{L}^{n+1}$, defined by an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^n and $\mathbb{L}^{n+1}$, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by $\bar{\nabla}_X Y = \tilde{\nabla}_X Y + \langle AX, Y\rangle\mathbf{n}$ for every $X, Y \in \chi(M)$, where $\mathbf{n}$ is a (locally) unit normal vector field on M and A is the shape operator of M relative to $\mathbf{n}$. For each non-zero vector $X \in \mathbb{L}^{n+1}$, the real value $\langle X, X\rangle$ may be a negative, zero or positive number and then, the vector X is said to be time-like, light-like or space-like, respectively.

Definition 2.1. For a n-dimensional Lorentzian vector space V_1^n, a basis $\mathcal{B} := \{e_1, \dots, e_n\}$ is said to be *orthonormal* if it satisfies $\langle e_i, e_j\rangle = \epsilon_i \delta_i^j$ for $i, j = 1, \dots, n$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, \dots, n$. As usual, δ_i^j stands for the Kronecker delta. $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_j\rangle = \delta_i^j$ for $i = 1, \dots, n$ and $j = 3, \dots, n$.

As well-known, the shape operator A of the Lorentzian hypersurface M_1^n in $\mathbb{L}^{n+1}$, as a self-adjoint linear map on the tangent bundle of M_1^n, locally can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^n, the matrix representation of the induced metric on M_1^n is $G_1 = \mathrm{diag}_n[-1, 1, \dots, 1]$ and the shape operator of M_1^n can be put into matrix forms $B_1 = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$ and

$$B_4 = \mathrm{diag}\left[\begin{bmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{bmatrix}, \eta_1, \dots, \eta_{n-2}\right],$$

where $\lambda \neq 0$, respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^n, the induced metric on which has matrix form $G_2 = \mathrm{diag}_n[\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], 1, \dots, 1]$ and the shape operator of M_1^n can be put into matrix forms

$$B_2 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}, \lambda_1, \dots, \lambda_{n-2}\right]$$

and

$$B_3 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{bmatrix}, \lambda_1, \dots, \lambda_{n-3}\right],$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.1. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, \dots, e_n\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, \dots, e_n\}$, where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrices $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \mathrm{diag}_n\left[\begin{bmatrix} \kappa + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa - \frac{1}{2} \end{bmatrix}, \lambda_1, \dots, \lambda_{n-2}\right]$$

and

$$\tilde{B_3} = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}, \lambda_1, \dots, \lambda_{n-3}\right].$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

Notation. According to four possible matrix representations of the shape operator of M_1^n, we define its principal curvatures, denoted by unified notations κ_i for $i = 1, \ldots, n$, as follow. In case I, we put $\kappa_i := \lambda_i$ for $i = 1, \ldots, n$, where λ_i's are the eigenvalues of B_1. In cases II, where the matrix representation of A is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_i := \lambda_{i-2}$ for $i = 3, \ldots, n$. In case III, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$ and $\kappa_i := \lambda_{i-3}$ for $i = 4, \ldots, n$. Finally, in the case IV, where the shape operator has matrix representation $\tilde{B_4}$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$ and $\kappa_i := \eta_{i-2}$ for $i = 3, \ldots, n$.

The characteristic polynomial of A on M_1^n is of the form $Q(t) = \prod_{i=1}^n (t - \kappa_i) = \sum_{j=0}^n (-1)^j s_j t^{n-j}$, where $s_0 := 1$, $s_i := \sum_{1 \le j_1 < \cdots < j_i \le n} \kappa_{j_1} \cdots \kappa_{j_i}$ for $i = 1, 2, \ldots, n$.

For $j = 1, \ldots, n$, the *jth mean curvature* H_j of M_1^n is defined by $H_j = \frac{1}{\binom{n}{j}} s_j$. When H_j is identically null, M_1^n is said to be $(j-1)$-*minimal*.

Definition 2.2. (i) A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with diagonalizable shape operator, is said to be *isoparametric* if all of it's principal curvatures are constant.

(ii) A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of it's shape operator is constant.

Remark 2.2. Here we remember Theorem 4.10 from [14], which assures us that there is no isoparametric Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with complex principal curvatures.

The well-known Newton transformations $P_j : \chi(M) \to \chi(M)$ on M_1^n, is defined by

$$P_0 = I, \quad P_j = s_j I - A \circ P_{j-1}, \quad j = 1, 2, \ldots, n,$$

where I is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^j (-1)^i s_{j-i} A^i$, where $A^0 = I$, which gives, by the Cayley-Hamilton theorem (stating that any operator is annihilated by its characteristic polynomial), that $P_n = 0$. It can be seen that, P_j is self-adjoint and commutative with A (see [1, 17]).

Now, we define a notation as

$$\mu_{i_1, i_2, \cdots i_t; k} = \sum_{1 \le j_1 < \cdots < j_k \le n; j_l \notin \{i_1, i_2, \cdots i_t\}} \kappa_{j_1} \cdots \kappa_{j_k}, \quad i = 1, \ldots, n, \ 1 \le k \le n-1,$$

$\mu_{i_1, i_2, \cdots i_t; 0} := 1$ and $\mu_{i_1, i_2, \cdots i_t; s} := 0$ for $s < 0$. Corresponding to four possible forms $\tilde{B_i}$ for $1 \le i \le 4$ of A, the Newton transformation P_j has different representations. In the case I, where $A = \tilde{B_1}$, we have $P_j = \mathrm{diag}[\mu_{1;j}, \ldots, \mu_{n;j}]$ for $j = 1, 2, \ldots, n-1$.

When $A = B_2$ (in the case II), we have

$$P_j = \mathrm{diag}\left[\begin{bmatrix} \mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1} & -\frac{1}{2}\mu_{1,2;j-1} \\ \frac{1}{2}\mu_{1,2;j-1} & \mu_{1,2;j} + (\kappa + \frac{1}{2})\mu_{1,2;j-1} \end{bmatrix}, \mu_{3;j}, \ldots, \mu_{n;j}\right]$$

and $s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + \kappa^2\mu_{1,2;j-2}$ for $j = 1, \ldots, n-1$.

In the case III, we have $A = B_3$ and putting

$$\Lambda_j := \begin{bmatrix} u_j + 2\kappa u_{j-1} + (\kappa^2 - \frac{1}{2})u_{j-2} & -\frac{1}{2}u_{j-2} & -\frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{1}{2}u_{j-2} & u_j + 2\kappa u_{j-1} + (\kappa^2 + \frac{1}{2})u_{j-2} & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & u_j + 2\kappa u_{j-1} + \kappa^2 u_{j-2} \end{bmatrix},$$

we have $P_j = \mathrm{diag}[\Lambda_j, \mu_{4;j}, \dots, \mu_{n;j}]$, where $u_l := \mu_{1,2,3;l}$ and

$$s_j = u_j + 3\kappa u_{j-1} + 3\kappa^2 u_{j-2} + \kappa^3 u_{j-3}, \quad \text{for } j = 1, \dots, n-1.$$

In the case IV, we have $A = B_4$,

$$P_j = \mathrm{diag}\left[\begin{bmatrix} \kappa\mu_{1,2;j-1} + \mu_{1,2;j} & -\lambda\mu_{1,2;j-1} \\ \lambda\mu_{1,2;j-1} & \kappa\mu_{1,2;j-1} + \mu_{1,2;j} \end{bmatrix}, \mu_{3;j}, \dots, \mu_{n;j}\right]$$

and $s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + (\kappa^2 + \lambda^2)\mu_{1,2;j-2}$ for $j = 1, \dots, n-1$.

In all cases, the following important identities occur for $j = 1, \dots, n-1$, similar to those in [1–3, 17, 18]:

$$\begin{aligned} s_{j+1} =& \kappa_i\mu_{i;j} + \mu_{i;j+1}, \quad 1 \le i \le n, \\ \mu_{i;j+1} =& \kappa_l\mu_{i,l;j} + \mu_{i,l;j+1}, \quad 1 \le i, l \le n, \ i \ne l, \\ tr(P_j) =& (n-j)s_j = c_j H_j, \\ tr(P_j \circ A) =& (n - (n-j-1))s_{j+1} = (j+1)s_{j+1} = c_j H_{j+1}, \\ tr(P_j \circ A^2) =& \tbinom{n}{j+1}[nH_1H_{j+1} - (n-j-1)H_{j+2}], \end{aligned}$$

where $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$.

The *linearized operator* of the $(j+1)$th mean curvature of M, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every $X, Y \in \chi(M)$.

Associated to the orthonormal frame $\{e_1, \dots, e_n\}$ of tangent space on a local coordinate system in the hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, $L_1(f)$ has an explicit expression as $L_1(f) = \sum_{i=1}^n \epsilon_i\mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. For a Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}_1^{n+1}$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$, where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [1, 17]:

$$\begin{aligned} &\nabla\langle \mathbf{x}, \mathbf{a}\rangle = \mathbf{a}^T, \quad \nabla\langle \mathbf{n}, \mathbf{a}\rangle = -A\mathbf{a}^T, \\ &\qquad L_1\mathbf{x} = n(n-1)H_2\mathbf{n}, \quad L_1\mathbf{n} = -\frac{n(n-1)}{2}\left(\nabla(H_2) + (nH_1H_2 - (n-2)H_3)\mathbf{n}\right), \end{aligned}$$

and finally, we have

$$\begin{aligned} L_1^2\mathbf{x} =& n(n-1)\left(2P_2\nabla H_2 - \frac{3}{2}n(n-1)H_2\nabla H_2\right) \\ &+ n(n-1)\left(L_1H_2 - \frac{n(n-1)}{2}H_2(nH_1H_2 - (n-2)H_3)\right)\mathbf{n}. \end{aligned}$$

Assume that a hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies the condition $L_1^2\mathbf{x} = 0$, then it is said to be L_1-*biharmonic*. By the last equalities, from the condition $L_1(H_2\mathbf{n}) = 0$

(which is equivalent to L_1-biharmonicity) we obtain simpler conditions on M_1^n to be a L_1-biharmonic hypersurface in $\mathbb{L}^{n+1}$, as:

$$(2.1)\qquad L_1H_2 = \frac{n(n-1)}{2}H_2(nH_1H_2-(n-2)H_3),\quad P_2\nabla H_2 = \frac{3}{4}n(n-1)H_2\nabla H_2.$$

A Lorentzian hypersurface $\mathbf{x}: M_1^n \to \mathbb{L}^{n+1}$ is said to be L_1-*bicoservative*, if its 2th mean curvature satisfies the second condition in (2.1).

The well-known structure equations on $\mathbb{L}^{n+1}$ are given by $d\omega_i = \sum_{j=1}^{n+1}\omega_{ij}\wedge\omega_j$, $\omega_{ij}+\omega_{ji}=0$ and $d\omega_{ij} = \sum_{l=1}^{n+1}\omega_{il}\wedge\omega_{lj}$. Restricted on M, we have $\omega_{n+1}=0$ and then, $0 = d\omega_{n+1} = \sum_{i=1}^{n}\omega_{n+1,i}\wedge\omega_i$. So, by Cartan's lemma, there exist functions h_{ij} such that $\omega_{n+1,i} = \sum_{j=1}^{n}h_{ij}\omega_j$ and $h_{ij}=h_{ji}$, which give the second fundamental form of M, as $B=\sum_{i,j}h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature H is given by $H=\frac{1}{n}\sum_{i=1}^{n}h_{ii}$. Therefore, we obtain the structure equations on M as $d\omega_i = \sum_{j=1}^{n}\omega_{ij}\wedge\omega_j$, $\omega_{ij}+\omega_{ji}=0$ and $d\omega_{ij} = \sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n}R_{ijkl}\omega_k\wedge\omega_l$ for $i,j=1,2,\dots,n-1$, and the Gauss equations $R_{ijkl}=(h_{ik}h_{jl}-h_{il}h_{jk})$, where R_{ijkl} denotes the components of the Riemannian curvature tensor of M. Denoting the covariant derivative of h_{ij} by h_{ijk}, we have $dh_{ij} = \sum_{k=1}^{n}h_{ijk}\omega_k + \sum_{k=1}^{n}h_{kj}\omega_{ik} + \sum_{k=1}^{n}h_{ik}\omega_{jk}$ and by the Codazzi equation we get $h_{ijk}=h_{ikj}$.

Finally, we recall the definition of an L_1-finite type hypersurface from [12], which is the basic notion of the paper.

Definition 2.3. An isometrically immersed hypersurface $\mathbf{x}: M_1^n \to \mathbb{L}^{n+1}$ is said to be of L_1-finite type if $\mathbf{x}$ has a finite decomposition $\mathbf{x} = \sum_{i=0}^{m}\mathbf{x}_i$, for some positive integer m, satisfying the condition $L_1\mathbf{x}_i = \tau_i\mathbf{x}_i$, where $\tau_i\in\mathbb{R}$ and $\mathbf{x}_i: M_1^n\to\mathbb{L}^{n+1}$ is smooth maps, for $i=1,2,\dots,m$, and $\mathbf{x}_0$ is constant. If all τ_i's are mutually different, M_1^n is said to be of L_1-m-type. An L_1-m-type hypersurface is said to be null if for at least one i, $1\le i\le m$, we have $\tau_i=0$.

3. Examples

Now, we provide two families of examples of L_1-biconservative Lorentzian hypersurfaces in $\mathbb{L}^{n+1}$, some of them are not L_1-biharmonic.

Example 3.1. Consider the subset $\{(y_1,\dots,y_{n+1})\in\mathbb{L}^{n+1} \mid -y_1^2+\cdots+y_{l+1}^2 = r^2\}$ representing the cylindrical hypersurface $\mathbb{S}_1^l(r)\times\mathbb{E}^{n-l}\subset\mathbb{L}^{n+1}$ for $r>0$ and $l=1,2,\dots,n-1$, with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(y_1,\dots,y_{n-l+1},0,\dots,0)$. Clearly, it has two distinct constant principal curvatures $\kappa_1=\cdots=\kappa_l=\frac{1}{r}$ and $\kappa_{l+1}=\cdots=\kappa_n=0$ and constant higher order mean curvatures $H_1=\frac{l}{n}r^{-1}$ and $H_2=\frac{l(l-1)}{n(n-1)}r^{-2}$. One can see that $\mathbb{S}_1^1(r)\times\mathbb{E}^{n-1}$ is L_1-biharmonic, but $\mathbb{S}_1^l(r)\times\mathbb{E}^{n-l}$ is not L_1-biharmonic for $l=2,\dots,n-1$.

Example 3.2. Consider the subset $\{(y_1,\dots,y_{n+1})\in\mathbb{L}^{n+1} \mid y_{l+1}^2+\cdots+y_{n+1}^2=r^2\}$ denoting the hypersurface $\mathbb{L}^l\times\mathbb{S}^{n-l}(r)\subset\mathbb{L}^{n+1}$ with $\mathbf{n}(y) = -\frac{1}{r}(0,\dots,,0,y_{l+1},\dots,y_{n+1})$ as the Gauss map for $r>0$ and $l=1,2,\dots,n-1$. It has two distinct principal curvatures $\kappa_1 = \cdots = \kappa_l = 0$ and $\kappa_{l+1} = \cdots = \kappa_n = \frac{1}{r}$ and constant higher order mean curvatures $H_1 = \frac{n-l}{n} r^{-1}$, and $H_2 = \frac{(n-l)(n-l-1)}{n(n-1)} r^{-2}$. One can see that $\mathbb{L}^l \times \mathbb{S}^{n-l}(r)$ is not L_1-biharmonic for $l = 1, 2, \ldots, n-2$, but $\mathbb{L}^{n-1} \times \mathbb{S}^1(r)$ is L_1-biharmonic.

4. L_1-Biconservative Lorentzian Hypersurfeces in $\mathbb{L}^{n+1}$

In this section, we give six theorems on the L_1-biconservative connected orientable timelike hypersurface in $\mathbb{L}^{n+1}$ with constant ordinary mean curvature. Theorem 4.1, 4.2 and 4.3 are appropriated to the case that the shape operator on hypersurface is diagonalizable. Theorem 4.4, 4.5 and 4.6 are related to the cases that the shape operator on hypersurface is of type II, III and IV, respectively.

4.1. **Hypersurfaces with diagonalizable shape operator.**

Theorem 4.1. *Every L_1-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ for any natural number $n \geq 2$, having a diagonalizable shape operator with exactly one eigenvalue function of multiplicity n, has constant 2nd mean curvature.*

Proof. Let $x : M_1^n \to \mathbb{L}^{n+1}$ be a L_1-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with assumed conditions. Defining the open subset $\mathcal{U}$ of M as $\mathcal{U} := \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \emptyset$, we consider $\{e_1, \ldots, e_n\}$ as a local orthonormal frame of principal directions of A on $\mathcal{U}$ such that for $i = 1, \ldots, n$, we have $Ae_i = \lambda e_i$ and

$$\mu_{i,2} = \frac{1}{2}(n-1)(n-2)\lambda^2, \quad H_2 = \lambda^2. \tag{4.1}$$

By assumption, we have $P_2(\nabla H_2) = \frac{3}{4} n(n-1) H_2 \nabla H_2$, which using the polar decomposition $\nabla H_2 = \sum_{i=1}^n \epsilon_i \langle \nabla H_2, e_i \rangle e_i$, gives

$$\epsilon_i \langle \nabla H_2, e_i \rangle \left(\mu_{i,2} - \frac{3}{4} n(n-1) H_2 \right) = 0$$

on $\mathcal{U}$ for $i = 1, \ldots, n$. Hence, if for some i we have $\langle \nabla H_2, e_i \rangle \neq 0$ on $\mathcal{U}$, then we get $\mu_{i,2} = \frac{3}{4} n(n-1) H_2$, which, using (4.1), gives $\lambda^2 = 0$ and then $H_2 = 0$ on $\mathcal{U}$, which is a contradiction. Hence, $\mathcal{U}$ is empty and H_2 is constant on M. □

Theorem 4.2. *Let $\boldsymbol{x} : M_1^n \to \mathbb{L}^{n+1}$ be an L_1-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions λ and η of multiplicities $n-1$ and 1, respectively. Then M_1^n has constant 2nd mean curvature.*

Proof. Taking the open subset $\mathcal{V}$ of M_1^n as $\mathcal{V} := \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V} \neq \emptyset$, we consider $\{e_1, \ldots, e_n\}$ as a local orthonormal frame of principal directions of A on $\mathcal{V}$ such that $Ae_i = \lambda e_i$ for $i = 1, \ldots, n-1$ and $Ae_n = \eta e_n$. Therefore, we obtain

$$\mu_{1,2} = \cdots = \mu_{n-1,2} = \frac{1}{2}(n-2)(n-3)\lambda^2 + (n-2)\lambda\eta, \tag{4.2}$$

$$\begin{aligned}\mu_{n,2} =& \frac{1}{2}(n-1)(n-2)\lambda^2,\\ nH_1 =&(n-1)\lambda+\eta, \quad n(n-1)H_2=(n-1)(n-2)\lambda^2+2(n-1)\lambda\eta,\\ \binom{n}{3}H_3 =&\binom{n-1}{3}\lambda^3+\binom{n-1}{2}\lambda^2\eta.\end{aligned}$$

Using the polar decomposition $\nabla H_2=\sum_{i=1}^n \epsilon_i\langle\nabla H_2,e_i\rangle e_i$, from (2.1) we have

$$\epsilon_i\langle\nabla H_2,e_i\rangle\left(\mu_{i,2}-\frac{3}{4}n(n-1)H_2\right)=0,$$

on $\mathcal{V}$ for $i=1,\dots,n$. Since, by definition of the subset $\mathcal{V}$, we have $\langle\nabla H_2,e_i\rangle\neq 0$ on $\mathcal{V}$ for some i, then we get

$$\mu_{i,2}=\frac{3}{4}n(n-1)H_2, \tag{4.3}$$

for some i which gives one of the following states.

State 1. $\langle\nabla H_2,e_i\rangle\neq 0$, for some $i\in\{1,\dots,n-1\}$. Using (4.2), from (4.3) we obtain $(n-2)(n-9)\lambda^2-4(n+l)\lambda\eta=0$, which gives $\lambda=0$ or $\eta=-\frac{(n-2)(n+3)}{2(n+1)}\lambda$. If $\lambda=0$, then $H_2=0$. Otherwise, we get $\lambda=\frac{2n(n+1)}{n^2-n+4}H_1$ and $H_2=-\frac{8n(n+1)(n-2)}{(n^2-n+4)^2}H_1^2$.

State 2. $\langle\nabla H_2,e_i\rangle=0$ for all $i\in\{1,\dots,n-1\}$ and $\langle\nabla H_2,e_n\rangle\neq 0$. By (4.2) and (4.3), we obtain $\lambda=0$ or $\eta=\frac{2-n}{6}\lambda$. If $\lambda=0$, then $H_2=0$. Otherwise, we get $\lambda=\frac{6n}{5n-4}H_1$ and $H_2=\frac{24n(n-2)}{(5n-4)^2}H_1^2$.

Therefore, H_2 is constant on M_1^n. □

Theorem 4.3. *Let $\boldsymbol{x}: M_1^n\to\mathbb{L}^{n+1}$ be an L_1-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions λ and η of multiplicities $n-k$ and k, respectively, where $2\leq k\leq n-2$. Then, the 2nd mean curvature of M_1^n has to be constant.*

Proof. Defining the open subset $\mathcal{V}$ of M_1^n as $\mathcal{V}:=\{p\in M_1^n \mid \nabla H_2^2(p)\neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V}\neq\emptyset$, we consider $\{e_1,\dots,e_n\}$ as a local orthonormal frame of principal directions of A on $\mathcal{V}$ such that $Ae_i=\lambda e_i$ for $i=1,\dots,n-k$ and $Ae_i=\eta e_i$ for $i=n-k+1,\dots,n$. Therefore, we obtain

$$\begin{aligned}\mu_{1,2} =& \cdots=\mu_{n-k,2}\\ =&\frac{1}{2}(n-k-1)(n-k-2)\lambda^2+\frac{1}{2}k(k-1)\eta^2+(n-k-1)k\lambda\eta,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mu_{n-k+1,2} =& \cdots=\mu_{n,2}\\ =&(n-k)\left(\frac{1}{2}(n-k-1)\lambda^2+(k-1)\lambda\eta\right)+\frac{1}{2}(k-1)(k-2)\eta^2,\end{aligned} \tag{4.5}$$

$$\begin{aligned}nH_1 =&(n-k)\lambda+k\eta,\\ n(n-1)H_2 =&(n-k)((n-k-1)\lambda^2+2k\lambda\eta)+k(k-1)\eta^2,\end{aligned}$$

$$\binom{n}{3}H_3 = \binom{n-k}{3}\lambda^3 + k\binom{n-k}{2}\lambda^2\eta + (n-k)\binom{k}{2}\lambda\eta^2 + \binom{k}{3}\eta^3.$$

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^n \epsilon_i\langle\nabla H_2, e_i\rangle e_i$, from (2.1) we have $\epsilon_i\langle\nabla H_2, e_i\rangle(\mu_{i,2} - \frac{3}{4}n(n-1)H_2) = 0$ on $\mathcal{V}$ for $i = 1, \ldots, n$. Hence, $\langle\nabla H_2, e_i\rangle \neq 0$ on $\mathcal{V}$ for some i and then

$$\mu_{i,2} = \frac{3}{4}n(n-1)H_2. \tag{4.6}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

State 1. $\langle\nabla H_2, e_i\rangle \neq 0$ for some $i \in \{1, \ldots, n-k\}$. Using (4.4), from (4.6) we obtain $(n-k-1)(n-k+4)\lambda^2 + k(k-1)\eta^2 + 2k(n-k+2)\lambda\eta = 0$, which gives $\eta = d_0\lambda$, where

$$d_0 = -\left(\frac{n-k+2}{k-1} \pm \frac{\sqrt{kn(n-k+3) + k(5k-4)}}{k(k-1)}\right).$$

Hence, we get $\lambda = \frac{n}{n-k(1-d_0)}H_1$ and $\eta = \frac{nd_0}{n-k(1-d_0)}H_1$, which give $H_2 = d_1H_1^2$ for a fixed coefficient d_1 (i.e., H_2 is constant on M_1^n).

State 2. $\langle\nabla H_2, e_i\rangle = 0$ for all $i \in \{1, \ldots, n-l\}$ and $\langle\nabla H_2, e_i\rangle \neq 0$ for some $i \in \{n-l+1, \ldots, n\}$. By (4.4) and (4.6), we obtain

$$(n-l)(n-l-1)\lambda^2 + (l+4)(l-1)\eta^2 + 2(n-l)(l+2)\lambda\eta = 0,$$

which gives $(n-1)\lambda(6\eta + (n-2)\lambda) = 0$. If $\lambda = 0$, then $H_2 = 0$. Otherwise, we have $\eta = -\frac{n-2}{6}\lambda$, which gives $\lambda = \frac{6n}{(6-k)n-4k}H_1$ and $\eta = -\frac{n(n-2)}{(6-k)n-4k}H_1$ and then $H_2 = d_2H_1^2$ for a fixed coefficient d_2 (i.e., H_2 is constant on M_1^n). ☐

4.2. **Hypersurfaces with non-diagonalizable shape operator.** This subsection is appropriated to cases that the Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$ have shape operator of type II, III or IV.

Theorem 4.4. *Every L_1-biconservative Lorentzian hypersurface M_1^n in $\mathbb{L}^{n+1}$, where $n \geq 3$ with shape operator of type II, having constant ordinary mean curvature and at most two distinct principal curvatures, has constant 2nd mean curvature.*

Proof. Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies all conditions of the theorem. So, it is L_1-biconservative with shape operator of type II, constant ordinary mean curvature and two distinct principal curvatures. Taking the open subset $\mathcal{U} = \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we show that $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \ldots, e_n\}$ on M_1^n, the shape operator A has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$ and $Ae_i = \lambda e_i$ for $i = 3, \ldots, n$. Then we have the following equalities:

$$\begin{aligned}
&nH_1 = 2\kappa + (n-2)\lambda,\ n(n-1)H_2 = 2\kappa^2 + (n-2)(n-3)\lambda^2 + 4(n-2)\kappa\lambda,\\
&P_2e_1 = \left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)(\kappa - \frac{1}{2})\lambda\right) e_1 + \frac{n-2}{2}\lambda e_2,\\
&P_2e_2 = -\frac{n-2}{2}\lambda e_1 + \left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)(\kappa + \frac{1}{2})\lambda\right) e_2,\\
&P_2e_i = \left(\kappa^2 + 2(n-3)\kappa\lambda + \frac{(n-3)(n-4)}{2}\lambda^2\right) e_i, \quad i = 3, \dots, n.
\end{aligned}$$

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{n} \epsilon_i e_i(H_2) e_i$, from (2.1) we get

$$\begin{aligned}
&\left((n-3)\lambda^2 + (2\kappa - 1)\lambda - \frac{3n(n-1)}{2(n-2)}H_2\right) \epsilon_1 e_1(H_2) = \lambda\epsilon_2 e_2(H_2),\\
&\left((n-3)\lambda^2 + (2\kappa + 1)\lambda - \frac{3n(n-1)}{2(n-2)}H_2\right) \epsilon_2 e_2(H_2) = -\lambda\epsilon_1 e_1(H_2),\\
&\left(\kappa^2 + 2(n-3)\kappa\lambda + \frac{(n-3)(n-4)}{2}\lambda^2 - \frac{3}{4}n(n-1)H_2\right) \epsilon_i e_i(H_2) = 0, \quad i = 3, \dots, n.
\end{aligned} \tag{4.7}$$

Now, we prove the main claim.

Claim. $e_i(H_2) = 0$ for $i = 1, \dots, n$. If $e_1(H_2) \neq 0$, then by dividing both sides of two equalities in (4.7) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa - \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2 = \frac{n-2}{2}\lambda u,\\
&\left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa + \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2\right) u = -\frac{n-2}{2}\lambda,
\end{aligned} \tag{4.8}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. From (4.8) we obtain $\lambda(1+u)^2 = 0$, then $\lambda = 0$ or $u = -1$. If $\lambda = 0$. Then we obtain $H_2 = 0$, which means H_2 is constant. Otherwise, we have $u = -1$, which gives $\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, then we obtain $6\kappa^2 + (n-2)(n-3)\lambda^2 + 8(n-2)\kappa\lambda = 0$. Since $nH_1 = 2\kappa + (n-2)\lambda$ is assumed to be constant on M, by substituting which in the last equality, we get $(4-3n)(n-2)\lambda^2 + 2n(n-2)H_1\lambda + 3n^2H_1^2 = 0$, which means λ, κ and the kth mean curvatures for $k = 2, \dots, n$, are also constant on M_1^n. So, we got a contradiction and therefore, the first part of the claim is proved.

If $e_2(H_2) \neq 0$, then by dividing both sides of two equalities in (4.7) by $\epsilon_2 e_2(H_2)$ we get

$$\begin{aligned}
&\left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa - \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2\right) v = \frac{n-2}{2}\lambda,\\
&\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa + \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2 = -\frac{n-2}{2}\lambda v,
\end{aligned} \tag{4.9}$$

where $v := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$. From (4.9) we obtain $\lambda(1+v)^2 = 0$. If $\lambda = 0$, from (4.9) we obtain $H_2 = 0$, which means H_2 is constant. Otherwise, we have $v = -1$, which gives $\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, then similar to the first part, we obtain that λ, κ and the kth mean curvatures for $k = 2,\dots,n$ are also constant on M_1^n. So, we got a contradiction and therefore, the second part of the claim is proved.

Finally, each of assumptions $e_i(H_2) \neq 0$ for $i = 3,\dots,n$, gives the equality $\kappa^2 + \frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, which gives $\kappa^2 + n(n-3)\lambda^2 + 4(n-1)\kappa\lambda = 0$. Similar to two first cases, Using formula $nH_1 = 2\kappa + (n-2)\lambda$, from the last equation we obtain that λ, κ and the kth mean curvatures for $k = 2,\dots,n$, are also constant on M_1^n. The contradiction that H_2 is constant on M. So, the claim is confirmed. $\square$

Theorem 4.5. *Every L_1-biconservative timelike hypersurface M_1^n in $\mathbb{L}^{n+1}$ with shape operator of type III, having at most two distinct principal curvatures and constant ordinary mean curvature, has constant 2nd mean curvature.*

Proof. Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies all conditions of the theorem. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1,\dots,e_n\}$ on M_1^n, the shape operator A has the matrix form $\tilde{B}_3$, such that $Ae_1 = \kappa e_1 - \frac{\sqrt{2}}{2}e_3$, $Ae_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Ae_3 = \frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and $Ae_i = \lambda e_i$ for $i = 4,\dots,n$. Then we have

$$\begin{aligned}
nH_1 =& 3\kappa + (n-3)\lambda, \ n(n-1)H_2 = 3\kappa^2 + \frac{(n-3)(n-4)}{2}\lambda^2 + 3(n-3)\kappa\lambda,\\
P_2e_1 =& \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2 - \frac{1}{2}\right)e_1 + \frac{1}{2}e_2 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_3,\\
P_2e_2 =& \frac{1}{2}e_1 + \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2 + \frac{1}{2}\right)e_2 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_3,\\
P_2e_3 =& \frac{-\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_1 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_2\\
&+ \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2\right)e_3,\\
P_2e_i =& \left(3\kappa^2 + 3(n-4)\kappa\lambda + \frac{(n-4)(n-5)}{2}\lambda^2\right)e_i, \quad i = 4,\dots,n.
\end{aligned}$$

Similar to proof of Theorem 4.4, we assume that H_2 is non-constant and considering the open subset $\mathcal{U} = \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U} = \emptyset$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{n} \epsilon_i e_i(H_2)e_i$, from (2.1) we get the following system of conditions:

(4.10)
$$\left((n-3)\lambda\left(\frac{n-4}{2}\lambda + 2\kappa\right) + \kappa^2 - \frac{1}{2} - \frac{3}{4}n(n-1)H_2\right)\epsilon_1 e_1(H_2) + \frac{1}{2}\epsilon_2 e_2(H_2)$$

$$
\begin{aligned}
=&\frac{\sqrt{2}}{2}\left((n-3)\lambda+\kappa\right)\epsilon_3 e_3(H_2),\\
&\frac{1}{2}\epsilon_1 e_1(H_2)+\left((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)+\kappa^2+\frac{1}{2}-\frac{3}{4}n(n-1)H_2\right)\epsilon_2 e_2(H_2)\\
=&-\frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)\epsilon_3 e_3(H_2),\\
&\frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)(\epsilon_1 e_1(H_2)+\epsilon_2 e_2(H_2))\\
=&-\left((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)+\kappa^2-\frac{3}{4}n(n-1)H_2\right)\epsilon_3 e_3(H_2),\\
&\left(3\kappa^2+((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)-\frac{3}{4}n(n-1)H_2)\right)\epsilon_i e_i(H_2)=0,\quad i=4,\dots,n.
\end{aligned}
$$

Now, we prove that H_2 is constant.

Claim. $e_i(H_2)=0$ for $i=1,\dots,n$.

If $e_1(H_2)\neq 0$, then by dividing both sides of three first equalities in (4.10) by $\epsilon_1 e_1(H_2)$, and using the notations $u_1:=\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2:=\frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$
\begin{aligned}
\frac{1}{4}(\alpha-2)+\frac{1}{2}u_1-\beta u_2=&0,\\
\frac{1}{2}+\frac{1}{4}(\alpha+2)u_1+\beta u_2=&0,\\
\beta(1+u_1)+\frac{1}{4}\alpha u_2=&0,
\end{aligned}
\tag{4.11}
$$

where $\alpha:=(n-3)\lambda\left(\frac{n-4}{2}\lambda-\kappa\right)-5\kappa^2$ and $\beta:=\frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)$. From (4.11) we obtain

$$
\beta u_2(1+u_1)=\frac{1}{2}(u_1^2-1)-u_1,\quad \frac{1}{4}\alpha(1+u_1)=-u_1. \tag{4.12}
$$

On the other hand, since $nH_1=3\kappa+(n-3)\lambda$ is assumed to be constant, we can restate α and β in terms of κ as:

$$
\begin{aligned}
\alpha=&\frac{1}{2(n-3)}\left((5n-24)\kappa^2-(8n^2-30n)H\kappa+n^2(n-4)H_1^2\right),\\
\beta=&\frac{\sqrt{2}}{2}(nH_1+2\kappa).
\end{aligned}
\tag{4.13}
$$

Now, using (4.12), from (4.11) we get a polynomial equation in terms of κ as $64\beta^2+\alpha^3-8\alpha=0$. This result says that κ and then λ and H_2 have constant values on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_1(H_2)\equiv 0$ is proved.

If $e_2(H_2)\neq 0$, then by dividing both sides of three first equalities in (4.10) by $\epsilon_2 e_2(H_2)$ and using the identities recalled in the first paragraph of the proof and notations $v_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$ and $v_3 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_2 e_2(H_2)}$, we get

$$\begin{aligned} \frac{1}{4}(\alpha-2)v_1 + \frac{1}{2} - \beta v_3 =&0,\\ \frac{1}{2}v_1 + \frac{1}{4}(\alpha+2) + \beta v_3 =&0,\\ \beta(v_1+1) + \frac{1}{4}\alpha v_3 =&0, \end{aligned} \tag{4.14}$$

where α and β are as the first case. From (4.14) we obtain

$$\beta v_3(1+v_1) = \frac{1}{2}(1-v_1^2) - v_1, \quad \frac{1}{4}\alpha(1+v_1) = -1. \tag{4.15}$$

Now, using (4.13) and (4.15), from the third equation in (4.14) we get a polynomial equation in terms of κ as $64\beta^2 + \alpha^2\beta - 8\alpha = 0$. This result says that κ, λ and H_2 have constant values on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_2(H_2) \equiv 0$ is proved.

If $e_3(H_2) \neq 0$, then by dividing both sides of equalities in (4.10) by $\epsilon_3 e_3(H_2)$, and using notations $w_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_3 e_3(H_2)}$ and $w_2 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_3 e_3(H_2)}$, we get

$$\begin{aligned} \frac{1}{4}(\alpha-2)w_1 + \frac{1}{2}w_2 =&\beta,\\ \frac{1}{2}w_1 + \frac{1}{4}(\alpha+2)w_2 =&-\beta,\\ \beta(w_1+w_2) =&-\frac{1}{4}\alpha, \end{aligned} \tag{4.16}$$

where α and β are as the first case. From (4.16) we obtain

$$\beta(w_1+w_2) = -\frac{1}{2}(w_1+w_2)^2, \quad \frac{1}{4}\alpha(w_1+w_2) = -w_2. \tag{4.17}$$

Using (4.13) and (4.17), From (4.16) we get a polynomial equation in terms of κ as $\alpha - 8\beta^2 = 0$. This result says that κ and then λ and H_2 have constant value on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_3(H_2) \equiv 0$ is proved.

The forth stage is assumption $e_i(H_2) \neq 0$ for some $i \geq 4$. By the same manner, from (4.10) we get $\alpha + 8\kappa^2 = 0$, which by using (4.13) gives a polynomial equation in terms of κ. This result says that κ and then λ and H_2 have constant value on $\mathcal{U}$. This is a contradiction and implies that $e_i(H_2) \equiv 0$ for $i = 4, 5, \ldots, n$. □

Theorem 4.6. *Every L_1-biconservative connected orientable Lorentzian hypersurface M_1^n with shape operator of type IV in $\mathbb{L}^{n+1}$, having at most two distinct principal curvatures, has constant 2nd mean curvature.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M \mid \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By assumption, the shape operator A of M_1^4 is of type IV with at most two distinct nonzero eigenvalue functions, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \ldots, e_n\}$ on M_1^n, the shape operator A has the matrix form B_4, such that $Ae_1 = -\lambda e_2$, $Ae_2 = \lambda e_1$, $Ae_i = 0$ for $i = 3, \ldots, n$. Then we have $P_2 e_1 = P_2 e_2 = 0$, $P_2 e_i = \lambda^2 e_i$ for $i = 3, \ldots, n$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^4 \epsilon_i e_i(H_2) e_i$, from (2.1) we get

$$\begin{aligned} \frac{3}{4} n(n-1) H_2 \epsilon_i e_i(H_2) =& 0, \quad i = 1, 2, \\ \left(\lambda^2 - \frac{3}{4} n(n-1) H_2\right) \epsilon_i e_i(H_2) =& 0, \quad i = 3, \ldots, n, \end{aligned}$$

which clearly gives $e_i(H_2) = 0$ for $i = 1, \ldots, n$. Then H_2 is constant on M_1^n. □

Acknowledgements. The author would like to gratefully thank the anonymous referees for their careful reading of the paper and the corrections.

References

[1] L. J. Alias and N. Gürbüz, *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata **121** (2006), 113–127. https://doi.org/10.1007/s10711-006-9093-9

[2] L. J. Alias and S. M. B. Kashani, *Hypersurfaces in space forms satisfying the condition $L_k x = Ax + b$*, Taiwanese J. Math. **14**(5) (2010), 1957–1977. https://doi.org/1011650/twjm/1500406026

[3] M. Aminian and S. M. B. Kashani, *L_k-Biharmonic hypersurfaces inthe Euclidean space*, Taiwanese J. Math. **19** (2015), 861–874. https://www.jstor.org/stable/taiwjmath.19.3.861

[4] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis and B. J. Papantoniou, *Biharmonic Lorentz hypersurfaces in E_1^4*, Pac. J. Math. **229** (2007), 293–306. https://doi.org/10.2140/pjm.2007.229.293

[5] A. Arvanitoyeorgos, F. Defever and G. Kaimakamis, *Hypersurfaces in E_s^4 with proper mean curvature vector*, J. Math. Soc. Japan **59** (2007), 797–809.

[6] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, 2nd Edition, Series in Pure Mathematics **27**, World Scientific Publishing Co, Singapore, 2015.

[7] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. **17** (1991), 169–188.

[8] F. Defever, *Hypersurfaces of E^4 satisfying $\Delta \vec{H} = \lambda \vec{H}$*, Michigan Math. J. **44** (1997), 355–363.

[9] F. Defever, *Hypersurfaces of E^4 with harmonic mean curvature vector*, Math. Nachr. **196** (1998), 61–69.

[10] I. Dimitrić, *Submanifolds of E^n with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sin. **20** (1992), 53–65.

[11] T. Hasanis and T. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr. **172** (1995), 145–169. https://doi.org/10.1002/mana.19951720112

[12] S. M. B. Kashani, *On some L_1-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$*, Bull. Korean Math. Soc. **46**(1) (2009), 35–43. https://doi.org/10.4134/BKMS.2009.46.1.035

[13] P. Lucas and H. F. Ramirez-Ospina, *Hypersurfaces in the Lorentz-Minkowski space satisfying $L_k \psi = A\psi + b$*, Geom. Dedicata **153** (2011), 151–175. https://doi.org/10.1007/s10711-010-9562-z

[14] M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pac. J. Math. **118**(1) (1985), 165–197. https://projecteuclid.org/euclid.pjm/1102706671

[15] F. Pashaie and A. Mohammadpouri, *L_k-Biharmonic spacelike hypersurfaces in Minkowski* 4-*space* $\mathbb{E}_1^4$, Sahand Communications in Mathematical Analysis **5**(1) (2017), 21–30. https://dx.doi.org/10.22130/scma.2017.20589

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Acad. Press Inc., San Diego, 1983.

[17] F. Pashaie and S. M. B. Kashani, *Spacelike hypersurfaces in Riemannian or Lorentzian space forms satisfying $L_k x = Ax + b$*, Bull. Iranian Math. Soc. **39**(1) (2013), 195–213.

[18] F. Pashaie and S. M. B. Kashani, *Timelike hypersurfaces in the Lorentzian standard space forms satisfying $L_k x = Ax + b$*, Mediterr. J. Math. **11**(2) (2014), 755–773. https://doi.org/10.1007/s00009-013-0336-3

[19] A. Z. Petrov, *Einstein Spaces*, Pergamon Press, Hungary, Oxford, New York, 1969.

[20] R. C. Reilly, *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geom. **8**(3) (1973), 465–477. https://doi.org/10.4310/jdg/1214431802

[1]Department of Mathematics, Faculty of Sciences,
University of Maragheh,
P.O. Box 55181-83111, Maragheh, Iran
Email address: f_pashaie@maragheh.ac.ir, f_pashaie@yahoo.com.

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 245–257.

GENERALIZED MIXED TYPE BERNOULLI-GEGENBAUER POLYNOMIALS

YAMILET QUINTANA[1]

Abstract. The generalized mixed type Bernoulli-Gegenbauer polynomials of order $\alpha > -\frac{1}{2}$ are special polynomials obtained by use of the generating function method. These polynomials represent an interesting mixture between two classes of special functions, namely generalized Bernoulli polynomials and Gegenbauer polynomials. The main purpose of this paper is to discuss some of their algebraic and analytic properties.

1. Introduction

Bernoulli and Gegenbauer polynomials are among classical families of algebraic polynomials whose history goes back centuries. Each one of these polynomials, as well as their natural generalizations, have showed their useful in several disciplines [1–3, 6–9, 16, 17, 19–21, 23–25, 27–29]. In this paper we shall be concerned with the some of the main properties of the generalized mixed type Bernoulli-Gegenbauer polynomials $\mathscr{V}_n^{(\alpha)}(x)$ of order $\alpha \in (-1/2, \infty)$, $n \geq 0$ (GBG polynomials, in short). This is a special family of polynomials defined through the generating functions and series expansions as follows:

$$\left[\frac{z}{(e^z - 1)\left(1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}\right)}\right]^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathscr{V}_n^{(\alpha)}(x)\frac{z^n}{n!}, \tag{1.1}$$

Key words and phrases. Generalized Bernoulli polynomials, Gegenbauer polynomials, GBG polynomials, inversion formula, matrix representations, matrix-inversion formula.

2010 *Mathematics Subject Classification.* Primary: 11B83. Secondary: 11C08, 33B99.

DOI 10.46793/KgJMat2302.245Q

Received: April 22, 2020.

Accepted: August 13, 2020.

where $|z| < 2\pi$, $|x| \leq 1$ and $\alpha \in (-1/2, \infty) \setminus \{0\}$,

$$\left[\frac{2\pi - xz}{1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}}\right] e^{xz} = \sum_{n=0}^{\infty} \mathscr{V}_n^{(0)}(x) \frac{z^n}{n!}, \quad |z| < 2\pi,\ |x| \leq 1. \tag{1.2}$$

The polynomials $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ represent an interesting mixture between two classes of special functions, namely generalized Bernoulli polynomials and Gegenbauer polynomials. The separate emergence of these families of polynomials in different fields as such as physical mathematics, information theory, combinatorics, approximation theory, number theory, numerical analysis and partial differential equations and so on, has been a well-known fact and documented [1, 3, 4, 6, 7, 12, 14, 18–20, 27, 28]. However, in recent years new connections between these families of polynomials have been given (see, for instance [2, 9, 29]). The aim of this note is to investigate some properties of the GBG polynomials, focusing our attention on their explicit expressions, derivatives formulas, matrix representations, matrix-inversion formulas, and other relations connecting them with Gegenbauer polynomials.

The paper is organized as follows. In Section 2 some relevant properties of the generalized Bernoulli polynomials and the Gegenbauer polynomials are given. Section 3 contains the main algebraic and analytic properties of the GBG polynomials (see e.g., Proposition 3.1, Lemmas 3.1 and 3.2, and Theorem 3.1), as well as, some illustrative examples.

2. Basic Facts: Generalized Bernoulli Polynomials and Gegenbauer Polynomials

This section is devoted to present some structural properties of the generalized Bernoulli polynomials and Gegenbauer polynomials which will be useful in the sequel. We will begin with the generalized Bernoulli polynomials. As is well known, these polynomials play an important role in the calculus of finite differences since the coefficients in all the usual central-difference formulas for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of them (see e.g., [10] and the references therein).

Recent and interesting works dealing with generalized Bernoulli and Euler polynomials, Appell and Apostol type polynomials, their properties and applications in several areas can be found by reviewing the current literature on this subject. For a broad information on old literature and new research trends about these classes of polynomials we strongly recommend to the interested reader see [8, 10, 13, 14, 16, 17, 20, 21, 24, 25].

From now on, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n.

2.1. **Generalized Bernoulli Polynomials.**

The classical Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined as follows (see, for details, [3, 14, 20, 23]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z|<2\pi,\ 1^{\alpha}:=1, \tag{2.1}$$

and

$$B_n(x) := B_n^{(1)}(x), \quad n\in\mathbb{N}_0, \tag{2.2}$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

The numbers $B_n^{(\alpha)} := B_n^{(\alpha)}(0)$ are called generalized Bernoulli numbers of order α, $n\in\mathbb{N}_0$. Clearly, we have

$$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(x-\alpha),$$

so that

$$B_n^{(\alpha)}(\alpha) = (-1)^n B_n^{(\alpha)}. \tag{2.3}$$

From the generating relation (2.1), it is fairly straightforward to deduce the addition formula:

$$B_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{(\alpha)}(x) B_{n-k}^{(\beta)}(y). \tag{2.4}$$

Making the substitution $\beta=0$ into (2.4) and interchanging x and y, we obtain the well known representation:

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k} B_k^{(\alpha)} x^{n-k}. \tag{2.5}$$

The following theorem summarizes some properties of the generalized Bernoulli polynomials.

Theorem 2.1. (a) ([26, (3)]) *Explicit formula for the generalized Bernoulli polynomials in terms of the Gaussian hypergeometric function:*

$$\begin{aligned} B_n^{(\alpha)}(x) \;=\; & \sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} j^{2k}(x+j)^{n-k} \\ & \times {}_2F_1(k-n, k-\alpha; 2k+1; j/(x+j)), \end{aligned} \tag{2.6}$$

where ${}_2F_1$ *denotes the Gaussian hypergeometric function given by*

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}, \quad c\notin\{0,-1,-2,\dots\},$$

with $(a)_0=1$, $(a)_n = a(a+1)\cdots(a+n-1)$, $n\in\mathbb{N}$, *being the Pochhammer's symbol.*

(b) ([26, (13)]) *The substitution* $x=0$ *into* (2.6) *yields the following representation for the generalized Bernoulli numbers:*

$$B_n^{(\alpha)} = \sum_{k=0}^{n}\binom{\alpha+n}{n-k}\binom{\alpha+k-1}{k}\frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} j^{n+k}. \tag{2.7}$$

The interested reader also may consult [20, 22, 26] for detailed proofs of the above assertions.

In addition to (2.2) classical Bernoulli polynomials $B_n(x)$ admit a variety of different representations. For instance, we recall that the classical Bernoulli polynomials $B_n(x)$ may be inverted in order to give a representation of the monomial basis (cf., [17, Eq. (4)] and the references therein). This resulting representation is commonly called inversion formula:

$$\begin{aligned} x^n &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x) \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B_{n-k}(x), \quad n\geq 0. \end{aligned} \tag{2.8}$$

Consequently, the set $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ is a basis for $\mathbb{P}_n$.

In the next lemma we show an inversion formula for a subfamily of generalized Bernoulli polynomials.

Lemma 2.1. *For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{(m)}(x)\right\}_{n\geq 0}$ be the sequence of generalized Bernoulli polynomials of order m. Then we have*

$$x^n = \frac{1}{(n+1)_m}\sum_{r=0}^{n}\binom{n+m}{r+m}a_r(m)B_{n-r}^{(m)}(x), \quad n\geq 0, \tag{2.9}$$

where the coefficients $a_r(m)$ are given by

$$a_r(m) = \sum_{k_1=0}^{r}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}}\binom{r+m}{k_1+m-1}\binom{k_1+m-1}{k_2+m-2}\cdots\binom{k_{m-2}+2}{k_{m-1}+1}, \quad r=0,\ldots,n.$$

Proof. From (2.1) it follows that

$$z^m e^{xz} = (e^z-1)^m\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!}. \tag{2.10}$$

It is not difficult to show by repeated application of the Cauchy product of series that

$$(e^z-1)^m = \sum_{n=0}^{\infty}a_n(m)\frac{z^{n+m}}{(n+m)!},$$

where

$$a_n(m) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}}\binom{n+m}{k_1+m-1}\binom{k_1+m-1}{k_2+m-2}\cdots\binom{k_{m-2}+2}{k_{m-1}+1}.$$

Thus, the right-hand side of (2.10) becomes

$$\begin{aligned} (e^z-1)^m\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!} &= \left[\sum_{n=0}^{\infty}a_n(m)\frac{z^{n+m}}{(n+m)!}\right]\left[\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!}\right] \\ &= \sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n+m}{r+m}a_r(m)B_{n-r}^{(m)}(x)\right]\frac{z^{n+m}}{(n+m)!}. \end{aligned} \tag{2.11}$$

Likewise, the left-hand side of (2.10) can be expressed by use of the Cauchy product of series as follows

$$z^m e^{xz} = \sum_{n=0}^{\infty} x^n \frac{z^{n+m}}{n!} = \sum_{n=0}^{\infty} (n+1)_m \, x^n \frac{z^{n+m}}{(n+m)!}. \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\sum_{n=0}^{\infty} (n+1)_m \, x^n \frac{z^{n+m}}{(n+m)!} = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n+m}{r+m} a_r(m) B_{n-r}^{(m)}(x) \right] \frac{z^{n+m}}{(n+m)!}, \tag{2.13}$$

and comparing the coefficients on both sides of (2.13), we get the desired inversion formula (2.9). □

As a straightforward consequence of the inversion formula (2.9) we obtain an expected algebraic property.

Corollary 2.1. *For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\left\{ B_0^{(m)}(x), \ldots, B_n^{(m)}(x) \right\}$ is a basis for $\mathbb{P}_n$, i.e.,*

$$\mathbb{P}_n = \operatorname{span} \left\{ B_0^{(m)}(x), B_1^{(m)}(x), \ldots, B_n^{(m)}(x) \right\}.$$

2.2. **Gegenbauer polynomials.** For $\alpha > -\frac{1}{2}$ we denote by $\{\hat{C}_n^{(\alpha)}\}_{n\geq 0}$ the sequence of Gegenbauer polynomials, orthogonal on $[-1, 1]$ with respect to the measure $d\mu(x) = (1-x^2)^{\alpha - \frac{1}{2}} \, dx$ (cf., [27, Chapter IV]), normalized by

$$\hat{C}_n^{(\alpha)}(1) = \frac{\Gamma(n + 2\alpha)}{n!\Gamma(2\alpha)}.$$

More precisely,

$$\int_{-1}^{1} \hat{C}_n^{(\alpha)}(x) \hat{C}_m^{(\alpha)}(x) \, d\mu(x) = \int_{-1}^{1} \hat{C}_n^{(\alpha)}(x) \hat{C}_m^{(\alpha)}(x) (1-x^2)^{\alpha - \frac{1}{2}} \, dx = M_n^{\alpha} \, \delta_{n,m}, \quad n, m \geq 0,$$

where the constant M_n^{α} is positive. It is clear that the normalization above does not allow α to be zero or a negative integer. Nevertheless, the following limits exist for every $x \in [-1, 1]$ (see [27, (4.7.8)])

$$\lim_{\alpha \to 0} \hat{C}_0^{(\alpha)}(x) = T_0(x), \quad \lim_{\alpha \to 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n} T_n(x),$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence $\{\hat{C}_n^{(0)}(x)\}_{n\geq 0}$ as follows

$$\hat{C}_0^{(0)}(1) = 1, \quad \hat{C}_n^{(0)}(1) = \frac{2}{n}, \quad \hat{C}_n^{(0)}(x) = \frac{2}{n} T_n(x), \quad n \geq 1.$$

We denote the nth monic Gegenbauer orthogonal polynomial by

$$C_n^{(\alpha)}(x) = (k_n^{\alpha})^{-1} \hat{C}_n^{(\alpha)}(x),$$

where the constant k_n^α (cf., [27, formula (4.7.31)]) is given by

$$\begin{aligned} k_n^\alpha &= \frac{2^n\Gamma(n+\alpha)}{n!\Gamma(\alpha)}, \quad \alpha \neq 0, \\ k_n^0 &= \lim_{\alpha\to 0}\frac{k_n^\alpha}{\alpha} = \frac{2^n}{n}, \quad n \geq 1. \end{aligned}$$

Then for $n \geq 1$, we have $C_n^{(0)}(x) = \lim_{\alpha\to 0}(k_n^\alpha)^{-1}\hat{C}_n^{(\alpha)}(x) = \frac{1}{2^{n-1}}T_n(x)$.

It is well known that the Gegenbauer polynomials are closely connected with axially symmetric potentials in n dimensions and contain the Legendre and Chebyshev polynomials as special cases [6, 7]. Furthermore, they inherit practically all the formulae known in the classical theory of Legendre polynomials.

Proposition 2.1. ([15, cf., Proposition 2.1]) *Let $\{C_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.*

(*a*) *Three-term recurrence relation.*

$$xC_n^{(\alpha)}(x) = C_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)}C_{n-1}^{(\alpha)}(x), \quad \alpha > -\frac{1}{2}, \alpha \neq 0, \tag{2.14}$$

with initial conditions $C_0^{(\alpha)}(x) = 1$, $C_1^{(\alpha)}(x) = x$ and recurrence coefficient $\gamma_n^{(\alpha)} = \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}$.

(*b*) *For every $n \in \mathbb{N}$ (see* [27, (4.7.15)])

$$h_n^\alpha := \|C_n^{(\alpha)}\|_\mu^2 = \int_{-1}^{1}[C_n^{(\alpha)}(x)]^2 d\mu(x) = \pi 2^{1-2\alpha-2n}\frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}. \tag{2.15}$$

(*c*) *Rodrigues formula.*

$$(1-x^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(x) = \frac{(-1)^n\Gamma(n+2\alpha)}{\Gamma(2n+2\alpha)}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-\frac{1}{2}}\right], \quad x \in (-1,1).$$

(*d*) *Structure relation (see* [27, (4.7.29)]). *For every $n \geq 2$*

$$C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) + \xi_{n-2}^{(\alpha)}C_{n-2}^{(\alpha)}(x),$$

where

$$\xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, \quad n \geq 0.$$

(*e*) *For every $n \in \mathbb{N}$ (see* [27, formula (4.7.14)])

$$\frac{d}{dx}C_n^{(\alpha)}(x) = nC_{n-1}^{(\alpha+1)}(x).$$

As is well known the monic Gegenbauer orthogonal polynomials admit other different definitions [1, 4, 27, 28]. In order to deal with the definitions (1.1) and (1.2) of the

GBG polynomials, we also are interested in the definition of the monic Gegenbauer orthogonal polynomials by means of the following generating functions:
(2.16)
$$\left(1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}\right)^{-\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x)\frac{z^n}{n!},\quad |z|<2\pi,|x|\leq 1,\alpha\in(-1/2,\infty)\setminus\{0\},$$

and
(2.17)
$$\frac{2\pi-xz}{1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}}=\sum_{n=0}^{\infty}\frac{1}{\pi^{n-1}}C_n^{(0)}(x)z^n=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\pi^{n-1}}C_n^{(0)}(x)\frac{z^n}{n!},\quad |z|<2\pi,\ |x|\leq 1.$$

Remark 2.1. Note that (2.16) and (2.17) are suitable modifications of the generating functions for the Gegenbauer polynomials $\hat{C}_n^{(\alpha)}(x)$:

$$\begin{aligned}\left(1-2xz+z^2\right)^{-\alpha} &= \sum_{n=0}^{\infty}\hat{C}_n^{(\alpha)}(x)z^n,\quad |z|<1,\ |x|\leq 1,\ \alpha\in(-1/2,\infty)\setminus\{0\},\\ \frac{1-xz}{1-xz+z^2} &= 1+\sum_{n=1}^{\infty}\frac{n}{2}\hat{C}_n^{(0)}(x)z^n,\quad |z|<1,\ |x|\leq 1.\end{aligned}$$

3. Some Algebraic and Analytic Properties of the GBG Polynomials

Now we are in a position to investigate some properties of the GBG polynomials as follows.

Proposition 3.1. *For $\alpha\in(-1/2,\infty)$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then the following explicit formulas hold.*

$$\mathscr{V}_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(k+\alpha)}{\pi^k\Gamma(\alpha)}C_k^{(\alpha)}(x)B_{n-k}^{(\alpha)}(x),\quad n\geq 0,\ \alpha\neq 0, \tag{3.1}$$

$$\begin{aligned}\mathscr{V}_n^{(0)}(x) &= \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}C_k^{(0)}(x)B_{n-k}^{(0)}(x)\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}C_k^{(0)}(x)x^{n-k},\quad n\geq 0.\end{aligned} \tag{3.2}$$

Proof. On account of the generating functions (1.1) and (2.16), it suffices the appropriate use of Cauchy product of series in order to deduce the expression (3.1).

Similarly, taking into account the generating functions (1.2) and (2.17), we can use an analogous reasoning to the previous one for getting the expression (3.2). □

Thus, the suitable use of (2.3), (2.5), (2.7), (2.14) and (3.1) allow us to check that for $\alpha \in (-1/2, \infty) \setminus \{0\}$ the first four GBG polynomials are:

$$
\begin{aligned}
\mathscr{V}_0^{(\alpha)}(x) =& 1,\\
\mathscr{V}_1^{(\alpha)}(x) =& \left(1+\frac{\alpha}{\pi}\right)x - \frac{\alpha}{2},\\
\mathscr{V}_2^{(\alpha)}(x) =& \left(1+\frac{2\alpha}{\pi}+\frac{(\alpha+1)\alpha}{\pi^2}\right)x^2 - \left(\alpha+\frac{\alpha^2}{\pi}\right)x + \frac{\alpha(3\alpha-1)}{12} - \frac{\alpha}{2\pi^2},\\
\mathscr{V}_3^{(\alpha)}(x) =& \left(\sum_{k=0}^{3}\binom{3}{k}\frac{(\alpha)_k}{\pi^k}\right)x^3 - \frac{3\alpha}{2}\left(\sum_{k=0}^{2}\binom{2}{k}\frac{(\alpha)_k}{\pi^k}\right)x^2\\
&+\left(\frac{(3\alpha-1)\alpha}{4}+\frac{(3\alpha-1)\alpha^2}{4\pi}-\frac{3\alpha}{2\pi^2}-\frac{3(\alpha+1)\alpha}{2\pi^3}\right)x + \frac{\alpha^2(1-\alpha)}{8} + \frac{3\alpha^2}{4\pi^2}.
\end{aligned}
$$

It is worth pointing out that the left hand side of (1.1) can be expressed as $G^{(\alpha)}(z)(1-xg(z))^{-\alpha}e^{xz}$, where

$$
G^{(\alpha)}(z) = \left[\frac{4\pi^2 z}{(e^z-1)(z^2+4\pi^2)}\right]^{\alpha} \quad \text{and} \quad g(z) = \frac{2\pi z}{z^2+4\pi^2},
$$

hence the polynomials $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 1}$ are not generalized Appell polynomials (cf., [5, Chapters I, III]). Also, in contrast to the generalized Bernoulli polynomials and Gegenbauer polynomials, the GBG polynomials neither satisfy a Hanh condition nor an Appell condition. More precisely, we have the following result.

Lemma 3.1. *For $\alpha \in (-1/2, \infty) \setminus \{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then we have*

$$
\frac{d}{dx}\mathscr{V}_{n+1}^{(\alpha)}(x) = (n+1)!\sum_{k=0}^{n}\frac{\mathscr{V}_k^{(\alpha)}(x)}{k!}A_{n-k}^{(\alpha)}(x), \quad n \geq 0, \tag{3.3}
$$

where

$$
A_n^{(\alpha)}(x) = \begin{cases} 1+\dfrac{\alpha}{\pi}, & n=0,\\ \dfrac{\alpha}{\pi^{n+1}}C_n^{(1)}(x), & n\geq 1. \end{cases}
$$

Proof. The identity (3.3) it is a straightforward consequence of (1.1) and (2.16). □

Also, it is possible to obtain some integral relations between the GBG polynomials and monic Gegenbauer polynomials.

Lemma 3.2. *For $\alpha \in (-1/2, \infty) \setminus \{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then the following formula holds.*

$$
\int_{-1}^{1}\mathscr{V}_n^{(\alpha)}(x)C_n^{(\alpha)}(x)d\mu(x) = \frac{n!\Gamma(n+2\alpha)}{\pi^{2\alpha+2n}\Gamma(n+\alpha+1)\Gamma(n+\alpha)}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k}{\pi^{k-1}}, \tag{3.4}
$$

whenever $n \geq 0$.

Proof. In order to obtain (3.4) it suffices to use the orthogonality of the monic Gegenbauer polynomials, (2.5), (2.15) and (3.1). $\square$

Finally, from a matrix framework we can use the expression (3.1) in order to obtain a matrix form of $\mathscr{V}_r^{(\alpha)}(x)$, $r = 0, 1, \ldots, n$, as follows.

The expression (3.1) yields

$$\mathscr{V}_r^{(\alpha)}(x) = \mathbf{C}_r^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x), \tag{3.5}$$

where

$$\mathbf{C}_r^{(\alpha)}(x) = \begin{bmatrix} \binom{r}{r}\frac{\Gamma(r+\alpha)}{\pi^r\Gamma(\alpha)}C_r^{(\alpha)}(x) & \binom{r}{r-1}\frac{\Gamma(r-1+\alpha)}{\pi^{r-1}\Gamma(\alpha)}C_{r-1}^{(\alpha)}(x) & \cdots & C_0^{(\alpha)}(x) & 0 & \cdots & 0 \end{bmatrix},$$

the null entries of the matrix $\mathbf{C}_r^{(\alpha)}(x)$ appear $(n-r)$-times and the matrix $\mathbf{B}^{(\alpha)}(x)$ is given by $\mathbf{B}^{(\alpha)}(x) = \begin{pmatrix} B_0^{(\alpha)}(x) & B_1^{(\alpha)}(x) & \cdots & B_r^{(\alpha)}(x) & \cdots & B_n^{(\alpha)}(x) \end{pmatrix}^T$.

Then, by (3.5) the matrix $\mathbf{V}^{(\alpha)}(x) = \begin{pmatrix} \mathscr{V}_0^{(\alpha)}(x) & \mathscr{V}_1^{(\alpha)}(x) & \cdots & \mathscr{V}_n^{(\alpha)}(x) \end{pmatrix}^T$, can be expressed as follows:

$$\mathbf{V}^{(\alpha)}(x) = \mathbf{C}^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x), \tag{3.6}$$

where $\mathbf{C}^{(\alpha)}(x)$ is the following $(n+1)\times(n+1)$ matrix

$$\mathbf{C}^{(\alpha)}(x) = \begin{bmatrix} C_0^{(\alpha)}(x) & 0 & 0 & \cdots & 0 \\ \binom{1}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & C_0^{(\alpha)}(x) & 0 & \cdots & 0 \\ \binom{2}{2}\frac{\Gamma(2+\alpha)}{\pi^2\Gamma(\alpha)}C_2^{(\alpha)}(x) & \binom{2}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & C_0^{(\alpha)}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x) & \binom{n}{n-1}\frac{\Gamma(n-1+\alpha)}{\pi^{n-1}\Gamma(\alpha)}C_{n-1}^{(\alpha)}(x) & \binom{n}{n-2}\frac{\Gamma(n-2+\alpha)}{\pi^{n-2}\Gamma(\alpha)}C_{n-2}^{(\alpha)}(x) & \cdots & C_0^{(\alpha)}(x) \end{bmatrix}.$$

The following theorem summarizes the ideas described above.

Theorem 3.1. *For $\alpha \in (-1/2, \infty)\setminus\{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then, the matrix $\mathbf{V}^{(\alpha)}(x) = \begin{pmatrix} \mathscr{V}_0^{(\alpha)}(x) & \cdots & \mathscr{V}_n^{(\alpha)}(x) \end{pmatrix}^T$ has the following matrix form:*

$$\mathbf{V}^{(\alpha)}(x) = \mathbf{C}^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x).$$

Remark 3.1. Note that according to (3.5) the rows of the matrix $\mathbf{C}^{(\alpha)}(x)$ are precisely the matrices $\mathbf{C}_r^{(\alpha)}(x)$ for $r = 0, \ldots, n$. Furthermore, the matrix $\mathbf{C}^{(\alpha)}(x)$ is an $(n+1)\times(n+1)$ lower triangular matrix for each $x \in \mathbb{R}$, so that

$$\det\left(\mathbf{C}^{(\alpha)}(x)\right) = \left(C_0^{(\alpha)}(x)\right)^{n+1} = (1)^{n+1} = 1.$$

Therefore, $\mathbf{C}^{(\alpha)}(x)$ is an invertible matrix for each $x \in \mathbb{R}$.

The following example shows how Theorem 3.1 can be used.

Example 3.1. Let us consider $n = 3$ and $\alpha = 1$. From (2.14), (3.1), (3.6) and a standard computation we obtain

$$\mathbf{B}(x) := \mathbf{B}^{(1)}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{\pi} & 1 & 0 & 0 \\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0 \\ \frac{6x^3-3x}{\pi^3} & \frac{3(4x^2-1)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1} \mathbf{V}^{(1)}(x), \tag{3.7}$$

where

$$\mathbf{V}^{(1)}(x) = \begin{bmatrix} 1 \\ \left(1+\frac{1}{\pi}\right)x - \frac{1}{2} \\ \left(1+\frac{2}{\pi}+\frac{2}{\pi^2}\right)x^2 - \left(1+\frac{1}{\pi}\right)x + \frac{1}{6} - \frac{1}{2\pi^2} \\ \left(1+\frac{3}{\pi}+\frac{6}{\pi^2}+\frac{6}{\pi^3}\right)x^3 - \frac{3}{2}\left(1+\frac{2}{\pi}+\frac{2}{\pi^2}\right)x^2 + \frac{1}{2}\left(1+\frac{1}{\pi}-\frac{3}{\pi^2}-\frac{6}{\pi^3}\right)x + \frac{3}{4\pi^2} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{\pi} & 1 & 0 & 0 \\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0 \\ \frac{6x^3-3x}{\pi^3} & \frac{3(4x^2-1)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{x}{\pi} & 1 & 0 & 0 \\ \frac{1}{2\pi^2} & -\frac{2x}{\pi} & 1 & 0 \\ 0 & \frac{3}{2\pi^2} & -\frac{3x}{\pi} & 1 \end{bmatrix},$$

then (3.7) becomes

$$\mathbf{B}(x) = \begin{bmatrix} 1 \\ x - \frac{1}{2} \\ x^2 - x + \frac{1}{6} \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{bmatrix}.$$

That is the entries of the matrix $\mathbf{B}(x)$ are the first four classical Bernoulli polynomials (2.2).

Another interesting algebraic property of the GBG polynomials is related to the inversion formula satisfied by the classical Bernoulli polynomials (2.8). The following example shows the inversion formula for the GBG polynomials $\mathscr{V}_n(x) := \mathscr{V}_n^{(1)}(x)$, $n \geq 0$.

Example 3.2. Making the substitution $\alpha = 1$ into (2.5), we obtain the well known representation:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Then the matrix $\mathbf{B}(x)$ can be expressed as follows (cf., [17, (8)]):

$$\mathbf{B}(x) \quad = \quad \mathbf{M}\mathbf{T}(x),$$

where

$$\mathbf{M} = \begin{bmatrix} B_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{1}B_1 & \binom{1}{0}B_0 & 0 & 0 & \cdots & 0 \\ \binom{2}{2}B_2 & \binom{2}{1}B_1 & \binom{2}{0}B_0 & 0 & \cdots & 0 \\ \binom{3}{3}B_3 & \binom{3}{2}B_2 & \binom{3}{1}B_1 & \binom{3}{0}B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}B_n & \binom{n}{n-1}B_{n-1} & \binom{n}{n-2}B_{n-2} & \binom{n}{n-3}B_{n-3} & \cdots & \binom{n}{0}B_0 \end{bmatrix},$$

and $\mathbf{T}(x) = \begin{pmatrix} 1 & x & \cdots & x^n \end{pmatrix}^T$. It is clear that $\det(\mathbf{M}) = (B_0)^{n+1} = (1)^{n+1} = 1$. So, $\mathbf{M}$ is an invertible matrix.

Making the substitution $\alpha = 1$ into (3.6), we get the matrix representation:

$$\mathbf{V}(x) := \mathbf{V}^{(1)}(x) = \mathbf{C}^{(1)}(x)\mathbf{B}(x) = \mathbf{C}^{(1)}(x)\mathbf{M}\mathbf{T}(x).$$

It follows that

$$\mathbf{T}(x) = \left[\mathbf{C}^{(1)}(x)\mathbf{M}\right]^{-1}\mathbf{V}(x) = \mathbf{M}^{-1}\left(\mathbf{C}^{(1)}(x)\right)^{-1}\mathbf{V}(x).$$

On the account of (2.8), we can deduce the following matrix equation

$$\mathbf{T}(x) = \mathbf{Q}\mathbf{B}(x), \tag{3.8}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{2!}{3!} & \frac{2!}{2!} & 1 & 0 & \cdots & 0 \\ \frac{3!}{4!} & \frac{3!}{3!} & \frac{3}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n!}{(n+1)!} & \frac{n!}{n!} & \frac{n!}{2!(n-1)!} & \frac{n!}{3!(n-2)!} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & 1 & 1 & 0 & \cdots & 0 \\ \frac{1}{4} & 1 & \frac{3}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & 1 & \frac{n}{2} & \frac{n(n-1)}{6} & \cdots & 1 \end{bmatrix}.$$

Notice that $\mathbf{M}^{-1} = \mathbf{Q}$. Consequently, from (3.8) we deduce a matrix-inversion formula for $\mathbf{V}(\mathbf{x})$ as follows

$$\mathbf{T}(x) \quad = \quad \mathbf{Q}\mathbf{B}(x) = \mathbf{Q}\left(\mathbf{C}^{(1)}(x)\right)^{-1}\mathbf{V}(x). \tag{3.9}$$

Also, the matrix identity (3.9) allows us to conclude that the set $\{\mathscr{V}_0(x), \ldots, \mathscr{V}_n(x)\}$ is a basis for $\mathbb{P}_n$, i.e.,

$$\mathbb{P}_n = \operatorname{span}\{\mathscr{V}_0(x), \mathscr{V}_1(x), \ldots, \mathscr{V}_n(x)\}.$$

Remark 3.2. In view of (2.9) it is possible to deduce a matrix-inversion formula for $\mathbf{B}^{(m)}(x)$ as follows

$$\mathbf{T}(x) \;=\; \mathbf{Q}^{(m)}\mathbf{B}^{(m)}(x),$$

where $\mathbf{Q}^{(m)}$ is an $(n+1)\times(n+1)$ lower triangular and invertible matrix, for $m\in\mathbb{N}$ fixed.

Applying Theorem 3.1 (or equivalently, making the substitution $\alpha=m$ into (3.6)) we obtain the following matrix-inversion formula for $\mathbf{V}^{(m)}(x)$

$$\mathbf{T}(x) \;=\; \mathbf{Q}^{(m)}\left[\mathbf{C}^{(m)}(x)\right]^{-1}\mathbf{V}^{(m)}(x).$$

Finally, we leave to the reader the formulation of the analogous identities for the GBG polynomials $\mathscr{V}_n^{(0)}(x)$, $n\geq 0$.

Acknowledgements. The author would like to thank the reviewers for their valuable comments and advice. This research was partially supported by Decanato de Investigación y Desarrollo, Universidad Simón Bolívar, Venezuela, grant DID-USB (S1-IC-CB-004-17).

References

[1] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publising Co., New York, 1984.

[2] S. Araci, M. Acikgoz, A. Bagdasaryan and Ş. Ergoğan, *The Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials*, Turkish Journal of Analysis and Number Theory **1**(1) (2013), 1–3. `https://doi.org/10.12691/tjant-1-1-1`

[3] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer Monographs in Mathematics, Springer, New York, 2014.

[4] R. Askey, *Orthogonal Polynomials and Special Functions*, CBMS-NSF Regional Conf. Ser. Appl. Math. **21** SIAM, Philadelphia, 1975.

[5] R. P. Boas Jr. and R. C. Buck, *Polynomial Expansions of Analytic Functions*, Springer-Verlag, Berlin, 1958.

[6] V. S. Buyarov, P. López-Artés, A. Martínez-Finkelshtein and W. Van Assche, *Information entropy of Gegenbauer polynomials*, J. Phys. A **33**(37) (2000), 6549–6560. `https://doi.org/10.1088/0305-4470/33/37/307`

[7] J. I. de Vicente, S. Gandy and J. Sánchez-Ruiz, *Information entropy of Gegenbauer polynomials of integer parameter*, J. Phys. A **40**(29) (2007), 8345–8361. `https://doi.org/10.1088/1751-8113/40/29/010`

[8] P. Hernández-Llanos, Y. Quintana and A. Urieles, *About extensions of generalized Apostol-type polynomials*, Results Math. **68** (2015), 203–225. `https://doi.org/10.1007/s00025-014-0430-2`

[9] D. S. Kim, S.-H. Rim and T. Kim, *Some identities on Bernoulli and Euler polynomials arising from orthogonality of Legendre polynomials*, J. Inequal. Appl. **2012**(237) (2012), 1–8. `https://doi.org/10.1186/1029-242X-2012-227`

[10] J. L. López and N. M. Temme, *Large degree asymptotics of generalized Bernoulli and Euler polynomials*, J. Math. Anal. Appl. **363** (2010), 197–208. `https://doi.org/10.1016/j.jmaa.2009.08.034`

[11] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl. **308**(1) (2005), 290–302. `https://doi.org/10.1016/j.jmaa.2005.01.020`

[12] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan and Co., Ltd., London, 1933.

[13] L. M. Navas, F. J. Ruiz and J. L. Varona, *Existence and reduction of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Arch. Math. (Brno) **55** (2019), 157–165. `https://doi.org/10.5817/AM2019-3-157`

[14] N. E. Nørlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.

[15] V. G. Paschoa, D. Pérez and Y. Quintana, *On a theorem by Bojanov and Naidenov applied to families of Gegenbauer-Sobolev polynomials*, Commun. Math. Anal. **16** (2014), 9–18.

[16] Y. Quintana, W. Ramírez and A. Urieles, *Generalized Apostol-type polynomial matrix and its algebraic properties*, Math. Rep. (Bucur.) **21**(2) (2019), 249–264.

[17] Y. Quintana, W. Ramírez and A. Urieles, *On an operational matrix method based on generalized Bernoulli polynomials of level m*, Calcolo **55**(3) (2018), 1–29. `https://doi.org/10.1007/s10092-018-0272-5`

[18] S. Roman, *The Umbral Calculus*, Academic Press, Inc., Orlando, 1984.

[19] M. Shah, *Applications of Gegenbauer* (*ultraspherical*) *polynomials in cooling of a heated cylinder*, An. Univ. Timisoara, Ser. Sti. Mat. **8** (1970), 207–212.

[20] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, London, 2012.

[21] H. M. Srivastava, I. Kucukoğlu and Y. Simsek, *Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials*, J. Number Theory **181** (2017), 117–146. `https://doi.org/10.1016/j.jnt.2017.05.008`

[22] H. M. Srivastava, J.-L. Lavoie and R. Tremblay, *A class of addition theorems*, Canad. Math. Bull. **26** (1983), 438–445. `https://doi.org/10.4153/CMB-1983-072-1`

[23] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., West Sussex, England, 1984.

[24] H. M. Srivastava, M. Masjed-Jamei and M. Reza Beyki, *A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Appl. Math. Inf. Sci. **12**(5) (2018), 907–916. `https://doi.org/10.18576/amis/120502`

[25] H. M. Srivastava and Á. Pintér, *Remarks on some relationships between the Bernoulli and Euler polynomials*, Appl. Math. Lett. **17** (2004), 375–380. `https://doi.org/10.1016/S0893-9659(04)90077-8`

[26] H. M. Srivastava, P. G. Todorov, *An explicit formula for the generalized Bernoulli polynomials*, J. Math. Anal. Appl. **130** (1988), 509–513. `https://doi.org/10.1016/0022-247X(88)90326-5`

[27] G. Szegő, *Orthogonal Polynomials*, 4th Ed., Coll. Publ. Amer. Math. Soc. **23**, Amer. Math. Soc., Providence, 1975.

[28] N. M. Temme, *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons Inc., New York, 1996.

[29] Z. Zhang, *Some identities involving generalized Gegenbauer polynomials*, Adv. Differ. Equ. **2017** (391), 1–12. `https://doi.org/10.1186/s13662-017-1445-2`

[1]Departamento de Matemáticas Puras y Aplicadas
Universidad Simón Bolívar
Edificio Matemáticas y Sistemas (MYS)
Apartado Postal: 89000, Caracas 1080 A, Venezuela
Email address: `yquintana@usb.ve`

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 259–269.

CONVERGENCE AND DIFFERENCE ESTIMATES BETWEEN MASTROIANNI AND GUPTA OPERATORS

NEHA[1] AND NAOKANT DEO[1]

This paper is dedicated to Prof. Dr. Gradimir V. Milovanović

Abstract. Gupta operators are a modified form of Srivastava-Gupta operators and we are concerned about investigating the difference of operators and we estimate the difference of Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. We also study the weighted approximation of functions and obtain the rate of convergence with the help of the moduli of continuity as well as Peetre's K-functional of Gupta operators.

1. Introduction and preliminaries

Acu-Rasa [3], Aral et al. [4] and Gupta [17] studied some fascinating results for the difference of operators in general sense. Several results on this topic are compiled in the recent book of Gupta et al. [19]. We extend here the study for some important operators. The Mastroianni operators [23] are mentioned below:

$$\mathcal{M}_{n,c}(f;x) = \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{F}_{n,i}(f), \tag{1.1}$$

where

$$v_{n,i}(x,c) = \frac{(-x)^i}{i!}\tau_{n,c}^{(i)}(x), \quad \mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right),$$

with individual cases, which are mentioned below.

(i) If $\tau_{n,0}(x) = exp(-nx)$, then $v_{n,i}(x,0) = exp(-nx)\frac{(nx)^i}{i!}$ and the operators $\mathcal{M}_{n,0}$ becomes Szász operators.

Key words and phrases. Mastroianni operator, modulus of continuity, Gupta operator.
2010 *Mathematics Subject Classification.* Primary: 41A25. Secondary: 41A35.
DOI 10.46793/KgJMat2302.259D
Received: March 12, 2020.
Accepted: August 13, 2020.

(*ii*) If $c \in \mathbb{N}$ and $\tau_{n,c}(x) = \frac{1}{(1+cx)^{n/c}}$, then we have $v_{n,i}(x,c) = \frac{(n/c)_i}{i!} \cdot \frac{(cx)^i}{(1+cx)^{\frac{n}{c}+i}}$ and we obtain classical Baskakov operators.

(*iii*) If $\tau_{n,-1}(x) = (1-x)^n$, then $v_{n,i}(x,-1) = \binom{n}{i} x^i (1-x)^{n-i}$ and the operators (1.1) reduce to Bernstein polynomials,

where $\mathcal{F}_{n,i} : \mathcal{S} \to \mathbb{R}$ is a functional (linear and positive) defined on $\mathcal{S}$ and $\mathcal{S} \subset C[0,\infty)$. Case (*iii*) has not been considered here, we will continue with this case in our next upcoming paper.

Srivastava-Gupta operator (see [10, 29]) reproduce only constant functions, recently Gupta in [16] studied few examples of the genuine operators (operators preserving linear functions), we consider here following operators

$$\mathcal{G}_{n;c}(f;x) = \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{H}_{n,i}(f), \tag{1.2}$$

where $v_{n,i}(x,c)$ is defined in (1.1) and

$$\mathcal{H}_{n,i}(f) = (n+c)\int_0^\infty v_{n+2c,i-1}(t,c) f(t)dt, \quad 1 \le i < \infty, \quad \mathcal{H}_{n,0}(f) = f(0).$$

Remark 1.1. For operators (1.1), we have $\mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right)$ such that

$$\mathcal{F}_{n,i}(e_0) = 1 \quad \text{and} \quad b^{\mathcal{F}_{n,i}} := \mathcal{F}_{n,i}(e_1).$$

If we denote $T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}} e_0)^r,\ r \in \mathbb{N}$, then by simple computation, we have

$$T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}} e_0)^r = 0, \quad r = 2, 4.$$

2. Preliminaries

Remark 2.1. For the Gupta type operators (1.2), by simple computation, we have

$$\mathcal{H}_{n,i}(e_r) = \frac{(i+r-1)!}{(i-1)!} \cdot \frac{\Gamma\left(\frac{n}{c} - r + 1\right)}{c^r \Gamma\left(\frac{n}{c}+1\right)},$$

where $\mathcal{H}_{n,i}(e_0) = 1,\ b^{\mathcal{H}_{n,i}} := \mathcal{H}_{n,i}(e_1) = \frac{i}{n}$. If we denote $T_r^{\mathcal{H}_{n,i}} = \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^r,\ r \in \mathbb{N}$, then after simple computation, we have

$$T_2^{\mathcal{H}_{n,i}} := \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^2 = \frac{ci^2 + ni}{n^2(n-c)}$$

and

$$\begin{aligned} T_4^{\mathcal{H}_{n,i}} :=& \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^4 \\ =& \mathcal{H}_{n,i}(e_4, x) - 4\mathcal{H}_{n,i}(e_3, x)\left(\frac{i}{n}\right) + 6\mathcal{H}_{n,i}(e_2, x)\left(\frac{i}{n}\right)^2 \\ & - 4\mathcal{H}_{n,i}(e_1, x)\left(\frac{i}{n}\right)^3 + \mathcal{H}_{n,i}(e_0, x)\left(\frac{i}{n}\right)^4 \end{aligned}$$

$$=\frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)}-4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)}+6\frac{(i+1)i^3}{n^3(n-c)}-\frac{3i^4}{n^4}.$$

Lemma 2.1. *Few moments of Mastroianni operators are given by*

$$\begin{aligned}
\mathcal{M}_n(e_0;x)=&1,\\
\mathcal{M}_n(e_1;x)=&x,\\
\mathcal{M}_n(e_2;x)=&\frac{x}{n}[x(n+c)+1],\\
\mathcal{M}_n(e_3;x)=&\frac{x}{n^2}[x^2(n+c)(n+2c)+3x(n+c)+1],\\
\mathcal{M}_n(e_4;x)=&\frac{x}{n^3}[x^3(n+c)(n+2c)(n+3c)+6x^2(n+c)(n+2c)+7x(n+c)+1],\\
\mathcal{M}_n(e_5;x)=&\frac{x}{n^4}[x^4(n+c)(n+2c)(n+3c)(n+4c)+10x^3(n+c)(n+2c)(n+3c)\\
&+25x^2(n+c)(n+2c)+15x(n+c)+1],\\
\mathcal{M}_n(e_6;x)=&\frac{x}{n^5}[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)+15x^4(n+c)(n+2c)\\
&\times(n+3c)(n+4c)+65x^3(n+c)(n+2c)(n+3c)+90x^2(n+c)(n+2c)\\
&+31x(n+c)+1].
\end{aligned}$$

Lemma 2.2. *Let $f(t)=e_i,\ i=0,1,2,3,4,$ and c is the element of the set $\{0,1,2\}$, then we have*

$$\begin{aligned}
\mathcal{G}_{n,c}(e_0;x)=&1,\\
\mathcal{G}_{n,c}(e_1;x)=&x,\\
\mathcal{G}_{n,c}(e_2;x)=&\frac{(n+c)}{(n-c)}x^2+\frac{2}{(n-c)}x,\quad n>c,\\
\mathcal{G}_{n,c}(e_3;x)=&\frac{(n+c)\,(n+2c)}{(n-c)\,(n-2c)}x^3+\frac{6\,(n+c)}{(n-c)\,(n-2c)}x^2+\frac{6}{(n-c)\,(n-2c)}x,\quad n>2c,\\
\mathcal{G}_{n,c}(e_4;x)=&\frac{(n+c)\,(n+2c)\,(n+3c)}{(n-c)\,(n-2c)\,(n-3c)}x^4+\frac{12\,(n+c)\,(n+2c)}{(n-c)\,(n-2c)\,(n-3c)}x^3\\
&+\frac{36\,(n+c)}{(n-c)\,(n-2c)\,(n-3c)}x^2+\frac{24}{(n-c)\,(n-2c)\,(n-3c)}x,\quad n>3c.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{G}_{n,c}\left((e_1-x);x\right)=&0,\\
\mathcal{G}_{n,c}\left((e_1-x)^2;x\right)=&\frac{2x\,(1+cx)}{n-c},\quad n>c,\\
\mathcal{G}_{n,c}\left((e_1-x)^4;x\right)=&\frac{12c^2\,(n+7c)}{(n-c)\,(n-2c)\,(n-3c)}x^4+\frac{24c^2\,(13n+c)}{(n-c)\,(n-2c)\,(n-3c)}x^3\\
&+\frac{12c^2\,(n+9c)}{(n-c)\,(n-2c)\,(n-3c)}x^2
\end{aligned}$$

$$+\frac{24}{(n-c)\,(n-2c)\,(n-3c)}x, \quad n>3c.$$

Very recently, Pratap and Deo [28] considered genuine Gupta-Srivastava operators and studied fundamental properties, the rate of convergence, Voronovskaya type estimates, convergence estimates and weighted approximation. In the year 2018, Garg et al. [13] studied the weighted approximation properties for Stancu generalized Baskakov operators. In the same year, Acu et al. [2] also studied the order of approximation for Srivastava-Gupta operators via Peetre's K-functional and weighted approximation properties and some numerical considerations regarding the approximation properties, were considered. Several researchers studied approximation operators and its variants, and they were given some impressive results like asymptotic formula, Voronovskaya-type formula, rate of convergence and bounded variation (see [1, 2, 4–9, 11, 12, 14, 18, 24–27]).

The purpose of this paper to study the approximation properties of Gupta operators and the approximation of difference of operators and find an estimate for the difference of Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. In the third section, we give the rate of convergence with the help of the moduli of continuity and the Peetre's K-functional and the last section of this paper the weighted approximation of functions are studied.

3. Difference of Operators

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on the interval $[0,\infty)$ equipped with the norm $||\cdot|| = \sup_{x\in[0,\infty)} |f(x)| < \infty$.

Theorem 3.1 (Theorem A)**.** ([15, 17]). *Let $f^{(s)} \in C_B[0,\infty)$, s is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get*

$$|(\mathcal{G}_{n,c}-\mathcal{M}_{n,c})(f,x)| \le ||f''||\alpha(x) + \omega(f'',\delta_1)(1+\alpha(x)) + 2\omega(f,\delta_2(x)),$$

where

$$\alpha(x) = \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}} + T_2^{\mathcal{H}_{n,i}}),$$

and

$$\delta_1^2 = \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}} + T_4^{\mathcal{H}_{n,i}}), \quad \delta_2^2 = \sum_{i=0}^{\infty} v_{n,i}(x,c)(b^{\mathcal{F}_{n,i}} - b^{\mathcal{H}_{n,i}})^2.$$

We give the quantitative estimate for difference of Mastroianni and Gupta type operators as an application of Theorem A.

Theorem 3.2. *Let $f^{(j)} \in C_B[0,\infty)$, j is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get*

$$|(\mathcal{G}_{n,c}-\mathcal{M}_{n,c})(f;x)| \le ||f''||\beta(x) + \omega(f'',\delta_1)(1+\beta(x)),$$

where

$$\beta(x) = \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)}$$

and

$$\begin{aligned}\delta_1^2 = & \frac{1}{2n^4(n-c)(n-2c)(n-3c)}\left[\left\{3c^2\left(n+c\right)\left(n+2c\right)\left(n+3c\right)\left(n+6c\right)\right\}x^4\right.\\ & + 6c\left(n+c\right)\left(n+2c\right)\left\{3c\left(n+6c\right)+2n\left(n+2c\right)\right\}x^3\\ & + \left(n+c\right)\left\{21c^2\left(n+6c\right)+36nc\left(n+2c\right)+n^2\left(3n+c\right)\right\}x^2\\ & \left.+\left\{3c^2\left(n+6c\right)+12nc\left(n+2c\right)+n^2\left(3n+c\right)+6n^3\right\}x\right].\end{aligned}$$

Proof. First using Remark 1.1, Remark 2.1 and applying Lemma 2.1, we get

$$\begin{aligned}\beta(x) = & \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}} + T_2^{\mathcal{H}_{n,i}})\\ = & \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\frac{ci^2+ni}{n^2(n-c)}\\ = & \frac{c}{2(n-c)}\mathcal{M}_n(e_2,x) + \frac{n}{2n(n-c)}\mathcal{M}_n(e_1,x)\\ = & \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)}.\end{aligned}$$

Next, by Remark 1.1 and Remark 2.1, we get

$$\begin{aligned}\delta_1^2 = & \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}} + T_4^{\mathcal{H}_{n,i}})\\ = & \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)T_4^{\mathcal{H}_{n,i}}\\ = & \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\left[\frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)} - 4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)}\right.\\ & \left. + 6\frac{(i+1)i^3}{n^3(n-c)} - \frac{3i^4}{n^4}\right]\\ = & \frac{1}{2}\sum_{i=0}^{\infty}\frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\left[\left(i^4+6i^3+11i^2+6i\right)n^3\right.\\ & - 4\left(i^4+3i^3+2i^2\right)n^2(n-3c) + 6(i^4+i^3)n(n-2c)(n-3c)\\ & \left.-3i^4(n-c)(n-2c)(n-3c)\right]\\ = & \frac{1}{2}\sum_{i=0}^{\infty}\frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\left[i^4\left\{n^3-4n^2(n-3c)+6n(n-2c)(n-3c)\right.\right.\\ & \left.-3(n-c)(n-2c)(n-3c)\right\} + i^3\left\{6n^3-12n^2\left(n-3c\right)+6n\left(n-2c\right)\left(n-3c\right)\right\}\end{aligned}$$

$$+ i^2 \left\{11n^3 - 8n^2 (n-3c)\right\} + 6in^3\Big]$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} \frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}$$

$$\times \left[3i^4c^2 (n+6c) + 12i^3nc(n+2c) + i^2n^2(3n+c) + 6in^3\right]$$

$$= \frac{1}{2n^4(n-c)(n-2c)(n-3c)} \Big[3n^4c^2 (n+6c) \mathcal{M}_n(e_4,x) + 12n^4c(n+2c) \mathcal{M}_n(e_3,x)$$

$$+ n^4 (3n+c) \mathcal{M}_n(e_2,x) + 6n^4 \mathcal{M}_n(e_1,x)\Big]$$

$$= \frac{3xc^2 (n+6c) \{x^3(n+c)(n+2c)(n+3c) + 6x^2(n+c)(n+2c) + 7x(n+c) + 1\}}{2n^4(n-c)(n-2c)(n-3c)}$$

$$+ \frac{6nc(n+2c)x \{x^2(n+c)(n+2c) + 3x(n+c) + 1\}}{n^4(n-c)(n-2c)(n-3c)}$$

$$+ \frac{n^2 (3n+c) x \{x(n+c)+1\}}{2n^4(n-c)(n-2c)(n-3c)} + \frac{3n^3x}{n^4(n-c)(n-2c)(n-3c)}$$

$$= \frac{1}{2n^4(n-c)(n-2c)(n-3c)} \Big[\left\{3c^2 (n+c)(n+2c)(n+3c)(n+6c)\right\} x^4$$

$$+ 6c(n+c)(n+2c)\{3c(n+6c) + 2n(n+2c)\} x^3$$

$$+ (n+c)\left\{21c^2 (n+6c) + 36nc(n+2c) + n^2(3n+c)\right\} x^2$$

$$+ \left\{3c^2 (n+6c) + 12nc(n+2c) + n^2(3n+c) + 6n^3\right\} x\Big]$$

and

$$\delta_2^2 = \sum_{i=0}^{\infty} v_{n,i}(x,c)(b^{\mathcal{F}_{n,i}} - b^{\mathcal{H}_{n,i}})^2 = 0. \qquad \square$$

4. Weighted Approximation

The usual first order of modulus of continuity of f on bounded interval $[0,b]$ is defined as:

$$\omega_b(f;\delta) = \sup_{0<|t-x|\le\delta} \sup_{t,x\in[0,b]} |f(t) - f(x)|.$$

Let

$$B_2[0,\infty) := \left\{f : [0,\infty) \to \mathbb{R} : |f(x)| \le M_f \left(1+x^2\right)\right\},$$

where M_f is a constant dependant on f, with the norm

$$\|f\|_2 = \sup_{x\ge0} \frac{|f(x)|}{1+x^2}.$$

Let

$$C_2[0,\infty) = C[0,\infty) \cap B_2[0,\infty).$$

In [20], Ispir acquainted the weighted modulus of continuity $\Omega(f;\delta)$ as:

$$\Omega(f;\delta)=\sup_{0\le|k|<\delta,x\ge0}\frac{|f(x+k)-f(x)|}{(1+k^2)(1+x^2)},\quad f\in C_2[0,\infty).\tag{4.1}$$

Let

$$C_2'[0,\infty)=\left\{f\in C_2[0,\infty):\lim_{t\to\infty}\frac{|f(x)|}{1+t^2}<\infty\right\}.$$

From [20, 21], if $f\in C_2'[0,\infty)$, then $\lim\limits_{\delta\to0}\Omega(f,\delta)=0$ and

$$\Omega(f;p\delta)\le2(1+p)\left(1+\delta^2\right)\Omega(f;\delta),\quad p>0.\tag{4.2}$$

From (4.1) and (4.2) and for $f\in C_2'[0,\infty)$, we have

$$\begin{aligned}|f(t)-f(x)|\le&\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|)\\ \le&2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(t-x)^2\right)\left(1+x^2\right).\end{aligned}$$

Now we give rate of approximation of unbounded functions in theorem of first order of modulus of continuity.

Theorem 4.1. *Let $f\in C_2[0,\infty)$, then we get*

$$|\mathcal{G}_{n,c}(f,x)-f(x)|\le4M_f\left(1+b^2\right)\delta_n^2(x)+2\omega_{b+1}(f,\delta),$$

where $\delta=\delta_n(x)=\sqrt{\mathcal{G}_{n,c}\left((t-x)^2,x\right)}$.

Proof. For $x\in[0,b]$ and $t\ge0$, we have

$$|f(t)-f(x)|\le4M_f\left(1+b^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),\quad\delta>0.$$

Applying operator $\mathcal{G}_{n,c}$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{G}_{n,c}(f;x)-f(x)|\le&4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2,x\right)\\&+\left(1+\frac{\mathcal{G}_{n,c}(|t-x|,x)}{\delta}\right)\omega_{b+1}(f,\delta)\\ \le&4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2,x\right)\\&+\left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,c}\left((t-x)^2,x\right)}\right)\omega_{b+1}(f,\delta).\end{aligned}$$

After choosing $\delta=\sqrt{\mathcal{G}_{n,c}\left((t-x)^2,x\right)}$, we obtain the required result. □

Theorem 4.2. *Let $f\in C_2'[0,\infty)$, then we have*

$$\lim_{n\to\infty}\|\mathcal{G}_{n,c}(f)-f\|_2=0.$$

Proof. From [22], it is sufficient to verify the following by well-known Bohman-Korovkin theorem as:

$$\lim_{n\to\infty}\left\|\mathcal{G}_{n,c}\left(t^i;x\right)-x^i\right\|_2=0,\quad i=0,1,2.$$

From Lemma 2, the result is true for $i=0,1$. Again using Lemma 2, we get

$$\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=\sup_{x\geq 0}\left|\frac{(n+c)}{(n-c)}x^2+\frac{2}{(n-c)}x-x^2\right|.$$

Finally, we have

$$\lim_{n\to\infty}\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=0.$$

Thus, we get the desired result. ☐

Theorem 4.3. *Let $g\in C_2'[0,\infty)$ and $\eta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\mathcal{G}_{n,c}(g;x)-g(x)|}{(1+x^2)^{1+\eta}}=0,\quad x_0\in(0,\infty].$$

Proof. Let $x_0>0$ be any arbitrary fixed value and $x_0\in(0,\infty]$ then, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|\mathcal{G}_{n,c}(g;x)-g(x)|}{(1+x^2)^{1+\eta}}&\leq\sup_{x\leq x_0}\frac{|\mathcal{G}_{n,c}(g;x)-g(x)|}{(1+x^2)^{1+\eta}}+\sup_{x>x_0}\frac{|\mathcal{G}_{n,c}(g;x)-g(x)|}{(1+x^2)^{1+\eta}}\\&\leq|\mathcal{G}_{n,c}(g)-g|_{C[0,x_0]}+\|g\|_2\sup_{x>x_0}\frac{|\mathcal{G}_{n,c}(1+t^2;x)|}{(1+x^2)^{1+\eta}}\\&\quad+\sup_{x>x_0}\frac{|g(x)|}{(1+x_0^2)^{1+\eta}}.\end{aligned}$$

From Theorem 4.2, the first term of the above inequality tends to zero.

Since $|g(x)|\leq\|g\|_2(1+x^2)$, we have

$$\sup_{x>x_0}\frac{|g(x)|}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{(1+x_0^2)^{\eta}}.$$

Let $\varepsilon>0$ be arbitrary and if we choose x_0 very big then

$$\frac{\|g\|_2}{(1+x_0^2)^{\eta}}<\frac{\varepsilon}{2}.\tag{4.3}$$

Since $\lim_{n\to\infty}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{1+x^2}=1$, we have

$$\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{1+x^2}\leq\frac{(1+x_0^2)^{\eta}}{\|g\|_2}\cdot\frac{\varepsilon}{2}+1\quad\text{as}\quad n\to\infty.$$

Therefore,

$$\|g\|_2\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{(1+{x_0}^2)^{\eta}}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{(1+x^2)}\leq\frac{\varepsilon}{2}+\frac{\|g\|_2}{(1+x^2)^{\eta}}.$$

From Theorem 4.1, and for sufficient large n, we have

$$\|\mathcal{G}_{n,c}(g)-g\|_{C[0,x_0]}<\varepsilon.\tag{4.4}$$

Estimates from (4.3) to (4.4), the theorem is proved. □

Theorem 4.4. *Let* $f \in C_2'[0,\infty)$. *For sufficient large* n, *we have*

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{G}_{n,c}(f;x) - f(x)\right|}{(1+x^2)^{5/2}} \leq \hat{C}\Omega\left(f;n^{-1/2}\right),$$

where $\hat{C} > 0$ *is constant.*

Proof. For x is a point of interval $\in [0,\infty)$ and δ is a positive number and by using definition of the weighted modulus of continuity and Lemma 2.2, we obtain

$$\begin{aligned} |f(t) - f(x)| \leq & \left(1 + (x + |t-x|)^2\right)\Omega(f;|t-x|) \\ \leq & 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Applying operator $\mathcal{G}_{n,c}$ both sides, we get

$$\begin{aligned} \left|\mathcal{G}_{n,c}(f;x) - f(x)\right| \leq & 2\left(1+x^2\right)\Omega(f;\delta)\left\{1 + \mathcal{G}_{n,c}\left((t-x)^2;x\right)\right. \\ & \left. + \mathcal{G}_{n,c}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right\}. \end{aligned}$$

Applying Cauchy-Schwarz inequality, Lemma 2.2 and choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain the required result. □

Acknowledgements. The authors are grateful to the referee for several helpful suggestions and comments. The present work is supported by the Council of Scientific and Industrial Research (CSIR) with reference no. 08/133(0021)/2018-EMR-I for the first author.

References

[1] U. Abel, V. Gupta and M. Ivan, *Asymptotic approximation of functions and their derivatives by generalized Baskakov-Szász-Durrmeyer operators*, Anal. Theory Appl. **21**(1) (2005), 15–26. https://doi.org/10.1007/BF02835246

[2] A. M. Acu and C. V. Muraru, *Certain approximation properties of Srivastava-Gupta operators*, J. Math. Inequal. **12**(2) (2018), 583–595. https://doi.org/10.7153/jmi-2018-12-44

[3] A. M. Acu and I. Raşa, *New estimates for the differences of positive linear operators*, Numer. Algorithms **73** (2016), 775–789. https://doi.org/10.1007/s11075-016-0117-8

[4] A. Aral, D. Inoan and I. Raşa, *On differences of linear positive operators*, Anal. Math. Phys. **9** (2019), 1227–1239. https://doi.org/10.1007/s13324-018-0227-7

[5] D. Barbosu and N. Deo, *Some Bernstein-Kantorovich operators*, Automat. Comput. Appl. Math. **22**(1) (2013), 15–21.

[6] D. Barbosu and D. Miclăuş, *On the Voronovskaja-type formula for the Bleimann, Butzer and Hahn bivariate operators*, Carpathian J. Math. **33**(1) (2017), 35–42.

[7] N. Deo, *Voronovskaya type asymptotic formula for Lupaş-Durrmeyer operators*, Rev. Un. Mat. Argentina **48**(1) (2007), 47–54.

[8] N. Deo, *A note on equivalent theorem for Beta operators*, Mediterr. J. Math. **4**(2) (2007), 245–250. https://doi.org/10.1007/s00009-007-0115-0

[9] N. Deo and N. Bhardwaj, *Some approximation results for Durrmeyer operators*, Appl. Math. Comput. **217**(12) (2011), 5531–5536. https://doi.org/10.1016/j.amc.2010.12.026

[10] N. Deo, *Faster rate of convergence on Srivastava-Gupta operators*, Appl. Math. Comput. **218**(21) (2012), 10486–10491. https://doi.org/10.1016/j.amc.2012.04.012

[11] N. Deo and M. Dhamija, *Generalized positive linear operators based on PED and IPED*, Iran. J. Sci. Technol. Trans. A Sci. **43** (2019), 507–513. https://doi.org/10.1007/s40995-017-0477-5

[12] N. Deo and R. Pratap, *The family of Szász-Durrmeyer type operators involving Charlier polynomials*, Kragujevac J. Math. (2023) (to appear).

[13] T. Garg, P. N. Agrawal and A. Kajla, *Jain-Durrmeyer operators involving inverse Polya-Eggenberger distribution*, Proc. Nat. Acad. Sci. India Sect. A **89** (2019), 547–557. https://doi.org/10.1007/s40010-018-0492-8

[14] I. Gavrea and M. Ivan, *Asymptotic behaviour of the iterates of positive linear operators*, Abstr. Appl. Anal. (2011). https://doi.org/10.1155/2011/670509

[15] V. Gupta, *On difference of operators with applications to Szász type operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(3) (2019), 2059–2071. https://doi.org/10.1007/s13398-018-0605-x

[16] V. Gupta, *Some examples of Genuine approximation operators*, General Math. **26**(1-2) (2018), 3–9.

[17] V. Gupta, *Difference of operators of Lupaş type*, Constructive Mathematical Analysis **1**(1) (2018), 9–14. https://doi.org/10.33205/cma.452962

[18] V. Gupta and R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, Switzerland AG, 2014. https://doi.org/10.1007/978-3-319-02765-4

[19] V. Gupta, T. M. Rassias, P. N. Agrawal and A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Springer Optimization and Its Applications **138**, Springer, Cham, 2018. https://doi.org10.1007/978-3-319-92165-5

[20] N. Ispir, *On modified Baskakov operators on weighted spaces*, Turkish J. Math. **25**(3) (2001), 355–365.

[21] N. Ispir and C. Atakut, *Approximation by modified Szász-Mirakjan operators on weighted spaces*, Proc. Indian Acad. Sci. Math. Sci. **112**(4) (2002), 571–578.

[22] E. Ibikli and E. A. Gadjieva, *The order of approximation of some unbounded function by the sequences of positive linear operators*, Turkish J. Math. **19**(3) (1995), 331–337.

[23] A. J. López-Moreno and J. M. Latorre-Palacios, *Localization results for generalized Baskakov/Mastroianni and composite operators*, J. Math. Anal. appl. **380**(2) (2011), 425–439. https://doi.org/10.1016/j.jmaa.2011.03.060

[24] N. Malik, *Some approximation properties for generalized Srivastava-Gupta operators*, Appl. Math. Comput. **269** (2015), 747–758. https://doi.org/10.1016/j.amc.2015.07.113

[25] N. S. Mishra and N. Deo, *Kantorovich Variant of Ismail-May Operators*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), 739–748. https://doi.org/10.1007/s40995-020-00863-x

[26] T. Neer, N. Ispir and P. N. Agrawal, *Bezier variant of modified Srivastava-Gupta operators*, Rev. Un. Mat. Argentina **58**(2) (2017), 199–214.

[27] M. A. Özarslan and H. Aktŭglu, *Local approximation for certain King type operators*, Filomat **27** (2013), 173–181. https://doi.org/0.2298/FIL1301173O

[28] R. Pratap and N. Deo, *Approximation by genuine Gupta-Srivastava operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **113**(3) (2019), 2495–2505. https://doi.org/10.1007/s13398-019-00633-4

[29] H. M. Srivastava and V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **37** (2003), 1307–1315. https://doi.org/10.1016/S0895-7177(03)90042-2

[1]Department of Applied Mathematics,
Delhi Technological University,
Bawana Road, Delhi-110042, India.
Email address: nehadtu03@gmail.com
Email address: naokantdeo@dce.ac.in

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 271–279.

COMPOSITIONS OF COSPECTRALITY GRAPHS OF SMITH GRAPHS

DRAGOŠ M. CVETKOVIĆ[1] AND MARIJA JEROTIJEVIĆ[2]

Abstract. Graphs whose spectrum belongs to the interval $[-2,2]$ are called Smith graphs. Vertices of the cospectrality graph $C(H)$ of a Smith graph H are all graphs cospectral with H with two vertices adjacent if there exists a certain transformation transforming one to another. We study how the cospectrality graph of the union of two Smith graphs can be composed starting from cospectrality graphs of starting graphs.

1. Introduction

In this section we present standard basic facts on graph spectra and on Smith graphs.

Let G be a graph with n vertices and adjacency matrix A. The characteristic polynomial $\det(xI - A)$ of A is also called the *characteristic polynomial* of G. The eigenvalues and the spectrum of A (which consists of n eigenvalues) are called the *eigenvalues* and the *spectrum* of G, respectively. Since A is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1, \dots, \lambda_n$. In particular, λ_1, as the largest eigenvalue of G, will be called the *spectral radius* (or *index*) of G. For general information on spectra of graphs see, for example, [2].

The spectrum of G (as a family of reals) will be denoted by $\widehat{G}$. The disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while the union of their spectra (i.e., the spectrum of $G_1 + G_2$) will be denoted by $\widehat{G}_1 + \widehat{G}_2$. In addition, kG ($k\widehat{G}$) stands for the union of k copies of G (resp. $\widehat{G}$).

Key words and phrases. Spectral graph theory, Smith graphs, cospectrality graphs.
2010 *Mathematics Subject Classification.* Primary: 05C50.
DOI 10.46793/KgJMat2302.271C
Received: December 27, 2019.
Accepted: August 15, 2020.

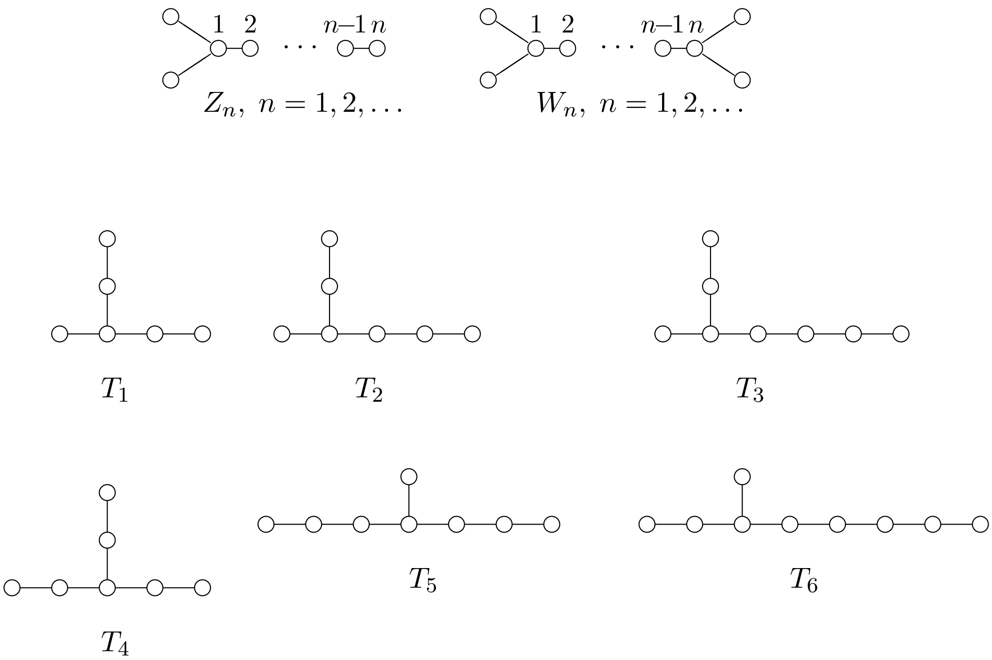

FIGURE 1. Some of the Smith graphs

We say that two (non-isomorphic) graphs are *cospectral* if their spectra coincide. They are also called *cospectral mates*. On the other hand, we say that a graph is determined by its spectrum if it is a unique graph having this spectrum.

The *cospectral equivalence class* of a graph G is the set of all graphs cospectral to G (including G itself).

We consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle.

All graphs with the spectral radius at most 2 have been constructed by J. H. Smith [5].

A path (cycle) on n vertices will be denoted by P_n (resp. C_n).

A connected graph with index ≤ 2 is either a cycle C_n ($n = 3, 4, \ldots$), or a path P_n ($n = 1, 2, \ldots$), or one of the graphs depicted in Fig. 1 (see [5]). Note that W_1 coincide with the star $K_{1,4}$, while Z_1 with P_3. In addition, the graphs C_n, W_n, T_4, T_5, and T_6 are connected graphs with index equal to 2. All other graphs, namely, P_n, Z_n, T_1, T_2 and T_3 are the induced subgraphs of these graphs (so the index of each of them is less than 2). The graph Z_n is called a *snake* while W_n is a *double snake*. The trees T_1, T_2, T_3, T_4, T_5, and T_6 will be called *exceptional Smith graphs*.

The spectrum of each of these graphs can be found (in an explicit form) in [3].

A Smith graph has connected Smith graphs as components.

We denote the set of all Smith graphs by $\mathcal{S}^*$. The set of those which are bipartite, so odd cycles are excluded, will be denoted by $\mathcal{S}$.

Let G be any graph each component of which belongs to $\mathcal{S}^*$, we can write

$$G = \sum_{H \in \mathcal{S}^*} r(H)H, \tag{1.1}$$

where $r(H) \geq 0$ is a repetition factor (tells how many times H is appearing as a component in G).

The repetition factor $r(S_i)$ of some of the graph $S_i \in \mathcal{S}^*$ for any relevant index i will be denoted by s_i. So we have non-negative integers

$$p_1, p_2, p_3, \dots, z_2, z_3, \dots, w_1, w_2, w_3, \dots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since $Z_1 = P_3$ and the variable p_3 is relevant. We shall use $c_2, c_3, \dots,$ for repetition factors of the even cycles $C_4, C_6, \dots$

For non-bipartite graphs from $\mathcal{S}^*$ we have to introduce variables $o_3, o_5, o_7, \dots$ counting the numbers of odd cycles $C_3, C_5, C_7, \dots$

For a given graph $G \in \mathcal{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G. Parameters of a graph indicate the actual number of components of particular types present in G.

The rest of the paper is organized as follows.

Section 2 contains some earlier results on Smith graphs necessary for handling the phenomenon of cospectrality of Smith graphs by means of the so called cospectrality graphs. In Section 3 we present some properties of cospectrality graphs. Section 4 contains description of some compositions of cospectrality graphs. At the end, in Section 5, we describe a computer program for generating cospectral Smith graphs and include some examples of the work of the program.

2. Preliminary Results

Let $H \in \mathcal{S}$. Let

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

be the canonical representation (as defined in [1]) of the spectrum $\widehat{H}$ of a bipartite Smith graph H. Here $\sigma_0, \sigma_1, \sigma_2, \dots \sigma_m$ are integers with $\sigma_0 \geq 0$. This representation always exists and is unique. The expression

$$\sigma_0 C_4 + \sum_{i=1}^{m} \sigma_i P_i,$$

is called *canonical representation* of H. It defines a graph if $\sigma_0, \sigma_1, \sigma_2, \dots \sigma_m$ are non-negative, otherwise it is just a formal expression. In the first case H is cospectral to its canonical representation but not necessarily isomorphic.

If all quantities σ_i are non-negative, the graph H is called a *Smith graph of type* A, otherwise it is *of type* B. Let I (resp. J) be the set of indices i for which σ_i in a graph of type B is negative (resp. positive).

Obviously, cospectral Smith graphs are of the same type.

Let $P_H = \sum_{i\in I} |\sigma_i| P_i$. Components of the graph P_H are paths whose spectra appear with a negative sign in the canonical representation of the spectrum of H. The graph P_H is called the *basis* of H. The basis of a graph of type A is empty. If we add components from its basis to a graph of type B, it becomes a graph of type A.

The graph $K_H = \sigma_0 C_4 + \sum_{i\in J} \sigma_i P_i$ is called the *kernel* of H.

Following [1] we shall consider the corresponding component transformations:

$$\begin{array}{ccc}
(\gamma_1) & W_n \rightleftarrows C_4 + P_n, & (\delta_1)\\
(\gamma_2) & Z_n + P_n \rightleftarrows P_{2n+1} + P_1, & (\delta_2)\\
(\gamma_3) & C_{2n} + 2P_1 \rightleftarrows C_4 + 2P_{n-1}, n \geq 3 & (\delta_3)\\
(\gamma_4) & T_1 + P_5 + P_3 \rightleftarrows P_{11} + P_2 + P_1, & (\delta_4)\\
(\gamma_5) & T_2 + P_8 + P_5 \rightleftarrows P_{17} + P_2 + P_1, & (\delta_5)\\
(\gamma_6) & T_3 + P_{14} + P_9 + P_5 \rightleftarrows P_{29} + P_4 + P_2 + P_1, & (\delta_6)\\
(\gamma_7) & T_4 + P_1 \rightleftarrows C_4 + 2P_2, & (\delta_7)\\
(\gamma_8) & T_5 + P_1 \rightleftarrows C_4 + P_3 + P_2, & (\delta_8)\\
(\gamma_9) & T_6 + P_1 \rightleftarrows C_4 + P_4 + P_2. & (\delta_9)
\end{array} \tag{2.1}$$

They are of the form $A \to B$ or $B \to A$ meaning that in a graph the group of components A is replaced with the group of components B or vice versa. These transformations are called *G-transformations*. Those of the form $A \to B$ are denoted by $\gamma_1, \gamma_2, \dots, \gamma_9$ and are called *C-transformations*. For each C-transformation $A \to B$ we define the corresponding opposite transformation $B \to A$, also denoted by $A \leftarrow B$. Transformations $A \leftarrow B$ are called *D-transformations* and are denoted by $\delta_1, \delta_2, \dots, \delta_9$.

Graphs $C_4, P_1, P_2, \dots,$ appearing in canonical representations of bipartite Smith graphs, are called *basic* graphs. All other connected bipartite Smith graphs are called *non-basic* graphs. Non-basic graphs are of two types. Graphs W_n $(n = 1, 2, \dots)$, C_{2k} $(k = 3, 4, \dots)$ and T_4, T_5, T_6 are non-basic graphs of type I while graphs Z_n $(n = 2, 3, \dots)$, T_1, T_2, T_3 are non-basic graphs of type II. Note that non-basic graphs of type I have spectral radius equal to 2 while for those of type II spectral radius is less than 2.

G-transformations $\gamma_1, \gamma_2, \gamma_3$ and their opposite transformations $\delta_1, \delta_2, \delta_3$ are not unique since they depend on the index n of the involved non-basic graphs W_n, Z_n, C_{2n}. If we want to specify this index in the name of the G-transformation, we shall use superscripts (for example, γ_1^n or δ_2^n).

Application of any G-transformation does not change the spectrum of the corresponding graph. Moreover, we have the following theorem from [1].

Theorem 2.1. *Let H_1 and H_2 be bipartite Smith graphs with corresponding bases P_{H_1} and P_{H_2}. If graphs H_1 and H_2 are cospectral, then the graph $H_1 + P_{H_1}$ can be transformed into $H_2 + P_{H_2}$ by a finite number of G-transformations.*

Cospectrality graphs have been introduced in [4] as follows.

For any A-type graph G we define its *cospectrality graph* $C(G)$ in the following way. Vertices of $C(G)$ are all graphs cospectral with G, i.e. the set of vertices of $C(G)$ is the cospectral equivalence class of G. Two vertices x and y are adjacent if there exists a G-transformation transforming one to another. Of course, if x can be transformed into y by a G-transformation, then y can be transformed into x by the opposite transformation. Hence, $C(G)$ is an undirected graph without multiple edges or loops.

By Theorem 2.1 the cospectrality graph is connected.

We shall also consider *general cospectrality graphs*. Such graphs have mutually cospectral vertex weights, the adjacency relation being defined as above.

It can be easily seen that *identifying* two vertices with same weights in a general cospectrality graph leads again to a regular general cospectrality graph. When identifying such vertices, all edges which were going to particular vertices, go now to the new single vertex.

3. Some Properties of Cospectrality Graphs

Let G be an A-type graph and let G^* be its canonical representation. We have $C(G) = C(G^*)$ and the later will be considered as a standard denotation for a cospectrality graph. Let $C(G^*) = C$.

Cospectrality graph C is a double weighted graph. Both vertices and edges carry some weights. Weights of vertices are some Smith graphs while weights of edges are pairs of mutually opposite G-transformations. Vertex weights determine edge weights since weights of adjacent vertices determine the pair of mutually opposite G-transformations transforming one vertex to another.

A cospectrality graph C, which is considered as an undirected graph, defines the following two directed weighted graphs: C_γ obtained from C by replacing edges with arcs with corresponding γ-transformations as weights and corresponding orientations, and C_δ, defined analogously.

Note that C_γ and C_δ, as digraphs, are mutually converse.

In considering cospectrality problems for Smith graphs we can treat together C, C_γ and C_δ and pass from one to another as appropriate.

Also we can treat incomplete cospectrality graphs, i.e., double weighted graphs in which the vertex set does not contain all mutually cospectral graphs. Sometimes we allow in such graphs vertices with the same weights.

Next theorem characterizes Smith graphs whose cospectrality graphs have just one vertex.

Theorem 3.1. *If the cospectrality graph of a Smith graph G of type* A *consists just of one vertex, then G is one of the following graphs:*

- *multiple cycles kC_4, $(k \in \mathbb{N})$;*
- *kP_1 $(k \in \mathbb{N})$ in the union with any collection of paths $P_2, P_3, P_4, P_6, P_8, \ldots$;*
- *any collection of paths without P_1.*

Proof. Clearly, graph G is characterized by the spectrum. Since any Smith graph is cospectral to its canonical representation, graph G must be itself in the form of canonical representation. If G is one of the graphs kC_4 ($k \in \mathbb{N}$), then there is no G-transformation producing a cospectral mate. G cannot contain C_4 and a path because of the transformation δ_1. In the remaining cases G is just a collection of paths. If P_1 is present, transformation δ_2 prevents the presence of any path P_{2k+1} for $k \geq 2$. If P_1 is excluded, any collection of other paths is feasible. $\square$

One can also classify graphs whose cospectrality graphs consist of two vertices. In fact, for each of nine types of D-transformations one can consider cospectrality graphs in which exactly this transformation appears.

4. Building Cospectrality Graphs

We present several ways in which new cospectrality graphs can be obtained from starting ones.

Let G_1 and G_2 be two Smith graphs and let C and D be cospectrality graphs such that $C = C(G_1^*)$ and $D = C(G_2^*)$. Corresponding directed graphs with arcs whose weights are $\delta-$transformations will be denoted C_δ and D_δ, respectively. Let $V(C_\delta) = \{1, 2, \ldots, m\}$ with weights $\{c_1, c_2, ..., c_m\}$ and $V(D_\delta) = \{1, 2, \ldots, n\}$ with weights $\{d_1, d_2, ..., d_n\}$ be the corresponding vertex sets. Note that c_i are graphs cospectral with G_1 and d_j are graphs cospectral with G_2.

Given cospectrality graphs $C(G_1)$ and $C(G_2)$ of graphs G_1 and G_2 we want to construct the cospectrality graph $C(G_1 + G_2)$ of the graph $G_1 + G_2$. The construction is not straightforward and we need several definitions. In particular, we shall define the sum of cospectrality graphs, merging vertices in a cospectrality graph and extending cospectrality graphs. All these operations can occur when constructing $C(G_1 + G_2)$.

First, we use Cartesian product $\times$ of sets to define, similarly as in the sum of graphs (see, for example, [2], page 65), the *sum* $C_\delta \oplus D_\delta$ of cospectrality graphs C_δ and D_δ. The operation $\oplus$ is called the *cospectrality sum*.

The vertex set $V(C_\delta \oplus D_\delta)$ of $C_\delta \oplus D_\delta$ is $V(C_\delta) \times V(D_\delta)$ and vertices (i, j) and (k, l) are adjacent if $i = k$ and j and l are adjacent in D_δ or $j = l$ and i and k are adjacent in C_δ. Weights $w(a)$ of arcs (or vertices) a are defined as follows (with subscript indicating the actual graph):

$$w((i,j),(i,l)) = w_D(j,l),$$

$$w((i,j),(k,j)) = w_C(i,k),$$

for $i, k \in \{1, \ldots, m\}$ and $j, l \in \{1, \ldots, n\}$ and

$$w(i,j) = c_i + d_j, \quad i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}.$$

The definition of $C_\gamma \oplus D_\gamma$ is analogous and leads to a directed graph converse to $C_\delta \oplus D_\delta$ with weights being the corresponding γ-transformations.

We might also consider the corresponding undirected graph $C \oplus D$ obtained from considered digraphs by replacing arcs with edges with corresponding pairs of opposite G-transformations as weights.

All three objects $C \oplus D$, $C_\gamma \oplus D_\gamma$ and $C_\delta \oplus D_\delta$ will be considered as the sum of cospectrality graphs C and D.

A subgraph of a cospectrality graph is called a *partial cospectrality graph.*

Theorem 4.1. *Let $C = C(G_1^*)$ and $D = C(G_2^*)$. The sum $C_\delta \oplus D_\delta$ of cospectrality graphs C_δ and D_δ, after merging vertices wth the same weights, is a partial cospectrality graph of the graph $C(G_1 + G_2)$.*

Proof. By definition of the sum, the weight $c_i + d_j$ of a vertex (i, j) is transformed either in the part c_i or in the part d_j giving in both cases the weight of a vertex cospectral to $c_i + d_j$. $\square$

Let us introduce the notion of an empty graph G_ϕ. It is a graph without vertices or edges and represents a neutral element for the operation of union of graphs. For any (non-weighted) graph G let also $Q(G)$ be a weighted graph consisting of a single vertex with vertex weight G.

It can easily be verified that $Q(G_\phi)$ behaves as a neutral element for the cospectrality sum $\oplus$, i.e, for any (partial) cospectrality graph C we have $Q(G_\phi) \oplus C = C \oplus Q(G_\phi) = C$.

Let S be any bipartite Smith graph and consider the cospectrality sum $Q(S) \oplus C$. The resulting cospectrality graph is isomorphic to C with each vertex weight being the union of the weight of the corresponding vertex in C and S.

We shall also consider *extending* - finding new vertices and arcs in a general cospectrality graph.

It happens sometimes that the weight $c_i + d_j$ of a vertex (i, j) of a sum $C_\delta \oplus D_\delta$ contains a Smith graph S which is contained neither in c_i nor in d_j and such that a D-transformation can be applied to it. This means that $c_i + d_j$ can be transformed in some additional ways. Let $c_i + d_j = S + S'$ for some Smith graph S' In fact, if $C(S)$ is a (partial) cospectrality graph for S, then the graph $Q(S') \oplus C(S)$ has a vertex with the weight $c_i + d_j$. The vertex of $Q(S') \oplus C(S)$ and the vertex in $C_\delta \oplus D_\delta$ with the same weight $c_i + d_j$ could be identified. In this way, $C_\delta \oplus D_\delta$ is *extended* by $Q(S') \oplus C(S)$ at vertex (i, j).

5. A Computer Program

We have implemented a computer program generating all graphs cospectral to a given bipartite Smith graph G of type A and the corresponding cospectrality graph $C(G)$.

The input contains a bipartite Smith graph of type A in its canonical form.

The vertex v_0 representing the canonical representation of G is called the *c-center* of $C(G)$ [4].

For any vertex v of $C(G)$ we define $H(v)$ to be the graph which is represented by v, i.e., the weight of v. The rank *rank* H of a Smith graph H is the number of non-basic components of H.

Vertices of $C(G)$ are partitioned into *layers* according to ranks of corresponding graphs. Layer k contains vertices v such that *rank* $H(v) = k$. The largest rank of a vertex in $C(G)$ is called the *c-radius* of $C(G)$. The vertices with largest rank are called *peripheral* vertices. Their rank is equal to the c-radius.

Applying a D-transformation on a vertex enhances its rank while C-transformations diminish the rank. Using C-transformations we are approaching the c-center while by D-transformations we go from c-center to peripheral vertices.

When considering a current graph the program tries to apply a D-transformation and if this is done the program forms a new vertex of the search tree. The depth first search is applied. Repeated graphs are not considered again.

The program is realized as a console application. The following tools are used: .NET Framework v4.7.2, $C\#$, XML, LinQ and Visual Studio 2019.

Example 5.1. Our program has been applied to the graph $T_5 + T_6 + 2P_1$. The program produced 25 graphs in the corresponding cospectrality graph. This shows that the cospectrality graph of $T_5 + T_6 + 2P_1$, given in [4], Figure 3, is not complete.

The program output is presented in Table 1.

TABLE 1.

Layer 0	2C4 2P2 P3 P4
Layer 1	1: $C4\ P2\ P3\ P4\ W2$, 1: $C4\ 2P2\ P4\ W3$, 1: $C4\ 2P2\ P3\ W4$, 3: $C6\ C4\ 2P1\ P3\ P4$, 7: $C4\ P3\ P4\ T4\ P1$, 8: $P2\ C4\ P4\ T5\ P1$, 9: $P2\ C4\ P3\ T6\ P1$
Layer 2	1: $P3\ P4\ 2W2$, 1: $P2\ P4\ W2\ W3$, 1: $P2\ P3\ W2\ W4$, 8: $P4\ W2\ T5\ P1$, 9: $P3\ W2\ T6\ P1$ | 1: $P2\ P4\ W3\ W2$, 1: $2P2\ W3\ W4$, 3: $P4\ W3\ C6\ 2P1$, 7: $P4\ W3\ T4\ P1$, 9: $P2\ W3\ T6\ P1$ | 1: $P2\ P3\ W4\ W2$, 1: $2P2\ W4\ W3$, 3: $P3\ W4\ C6\ 2P1$, 7: $P3\ W4\ T4\ P1$, 8: $P2\ W4\ T5\ P1$ | 1: $P3\ P4\ C6\ P1\ W1$, 1: $P4\ C6\ 2P1\ W3$, 1: $P3\ C6\ 2P1\ W4$ | 1: $P3\ P4\ T4\ W1$, 1: $P4\ T4\ P1\ W3$, 1: $P3\ T4\ P1\ W4$ | 1: $P2\ P4\ T5\ W1$, 1: $P4\ T5\ P1\ W2$, 1: $P2\ T5\ P1\ W4$, 9: $T5\ 2P1\ T6$ | 1: $P2\ P3\ T6\ W1$, 1: $P3\ T6\ P1\ W2$, 1: $P2\ T6\ P1\ W3$, 8: $T6\ 2P1\ T5$

Generated graphs are classified within layers. Starting from layer 1, graphs in a layer are listed in order as they are generated from the previous layer and the index i of the used D-transformation δ_i is indicated. The symbol $+$ of the union of graphs is omitted. Graphs generated by different (and neighboring) graphs from the previous layer are separated by a vertical line |. Repeated graphs are underlined. There are exactly 25 graphs in the table which are not underlined.

Example 5.2. When applied to $T_4 + T_5 + T_6 + 3P_1$, the program produced 86 mutually cospectral graphs.

Acknowledgements. This work is supported by the Serbian Ministry for Education, Science and Technological Development, Grants ON174033 and F-159. We are thankful to Miljan Jerotijević for his help in implementing the computer program.

References

[1] D. M. Cvetković, *Spectral theory of Smith graphs*, Bull. Cl. Sci. Math. Nat. Sci. Math. **150** (2017), 19–40.

[2] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[3] D. M. Cvetković and I. Gutman, *On the spectral structure of graphs having the maximal eigenvalue not greater than two*, Publ. Inst. Math. (Beograd) **18** (1975), 39–45.

[4] D. M. Cvetković and V. Todorčević, *Cospectrality graphs of Smith graphs*, Filomat **33** (2019), 3269–3276.

[5] J. H. Smith, *Some properties of the spectrum of a graph*, Combinatorial Structures and Their Applications (1970), 403–406.

[1]Mathematical Institute SANU,
Kneza Mihaila 36, Belgrade, Serbia
Email address: ecvetkod@etf.rs

[2]School of computing,
Union University,
Kneza Mihaila 6/VI, Belgrade, Serbia
Email address: marijajerotijevic90@gmail.com

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 281–295.

A STABILITY RESULT FOR A TIMOSHENKO SYSTEM WITH INFINITE HISTORY AND DISTRIBUTED DELAY TERM

ZINEB KHALILI[1] AND DJAMEL OUCHENANE[1]

Abstract. This manuscript is mainly focusing on a general stability of solution for one-dimensional Timoshenko system with infinite history and distributed delay term regardless also of the speeds of wave propagation. We prove our result by using the energy method combined with some properties of convex functions.

1. Introduction

In this paper, we consider the following Timoshenko system with infinite history and distributed delay term

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\,\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds = 0, \end{cases} \tag{1.1}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, and ρ_1, ρ_2, μ_1, K are positive constants, $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds < \mu_1, \tag{1.2}$$

where τ_1 and τ_2 two real numbers satisfying $0 \le \tau_1 \le \tau_2$ and the relaxation function g satisfies the folowing assumptions.

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad b - \int_0^\infty g(s)\,ds = b - g_0 = L > 0.$$

Key words and phrases. Timoshenko system, distributed delay, energy methode, infinite history.
2010 *Mathematics Subject Classification.* Primary: 35B40. Secondary: 74H40, 74H55, 93D20.
DOI 10.46793/KgJMat2302.281K
Received: April 07, 2020.
Accepted: August 24, 2020.

(G2) There exists a positive constant ζ such that

$$g'(t) \leq -\zeta g(t), \quad \text{for all } t \geq 0. \tag{1.3}$$

System (1.1) is provided with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \psi_t(x,-t) = f_0(x,t) \text{ in } (0,1) \times (0,\tau_2), \end{cases}$$

and

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \text{for all } t \geq 0,$$

where $x \in (0,1)$ and f_0 is the history function.

Let us first recall some result related to the problem we address. Said-Houari and Rahali [12] considered the following Timoshenko system with infinite history and a delay term in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{1.4}$$

They established the well-posedness of problem (1.4) and the exponential stability of solution. In the absence of the viscoelastic damping ($g \equiv 0$), problem (1.4) has been studied recently by Said-Houari and Laskri [11]. Under some assumption, they proved the well-posedness and established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.,

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}.$$

Subsequently, the work in [11] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari and Anwar [6]. First, by using the variable norm technique of Kato and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$, the system has been shown to be well-posed. Second, under relationship between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

In [6, 11], the authors have extended some works on the wave equation with delay to the Timoshenko system with delay. The stability of the wave equation with delay has become recently an active area of research and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [2] for more details).

Kafini et al. [5] considered the following Timoshenko system of thermoelasticity of type III with delay

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-s) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\ \rho_3\theta_{tt} - k\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0. \end{cases}$$

The authors established well-posedness and stability of the system for the cases of equal and nonequal speeds of wave propagation, they showed that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay and in the opposite case it decays polynomially. Also, Kafini et al. [4] concerned with the following Timoshenko system of thermoelasticity of type III with distributive delay

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - \kappa\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\,\theta_{txx}(x,t-s)\,ds + \gamma\psi_{tx} = 0, \end{cases}$$

where $\tau_1 < \tau_2$ are non-negative constants. They proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data. Very recently, Hao and Wang [3] considered the following Timoshenko-type system with distributed delay and past history

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x,\psi)_x + \beta\theta_{tx} = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) - \beta\theta_x + \int_0^{\infty} g(s)\,\psi_{xx}(x,t-s)\,ds + f(\psi) = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{tx} - l\theta_{txx} + \gamma\psi_t + \int_{\tau_1}^{\tau_2} \mu(\zeta)\,\psi_t(x,t-\zeta)\,d\zeta = 0. \end{cases} \tag{1.5}$$

The authors proved well-posedness and stability of the system (1.5) for the cases of equal and nonequal speeds of wave propagation. Their results show that the damping effect is strong enough to uniformly stabilize the system even in the existence of time delay under suitable conditions.

Motivated by the works mentioned above, we investigate system (1.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), we can establish a general energy decay regardless also of the speeds of wave propagation. To achieve our goals we make use the energy method combined with some properties of convex functions. The arguments of convexity were introduced by Lasiecka and Tataru [7] and used by Liu and Zuazua [8] and others.

2. Preliminaries

The main aim in this section is to present some materials needed in the proof of our result. We also state, without proof, a local existence result for problem (1.1). The proof can be established by using Faedo-Galerkin method as in [9]. Let us introduce the following new dependent variable

$$z(x,\rho,s,t) = \psi_t(x,t-s\rho), \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Then, we get the following system

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,\tau,t) = \psi_t(x,t). \end{cases}$$

We then set an auxiliary variable as in [1]

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s), \quad s \geq 0.$$

Then

$$\eta^t_t(x,s) + \eta^t_s(x,s) = \psi_t(x,t).$$

Hence, we can rewrite the problem (1.1) as

$$\left\{\begin{array}{l}\rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0,\\ \rho_2\psi_{tt}-b\psi_{xx}+K(\varphi_x+\psi)+\int_0^\infty g(s)\,\eta^t_{xx}(x,s)\,ds\\ +\mu_1\psi_t(x,t)+\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds=0,\\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0,\\ \eta^t_t(x,s)+\eta^t_s(x,s)=\psi_t(x,t),\end{array}\right. \tag{2.1}$$

where $x\in(0,1)$, $\rho\in(0,1)$ and $t>0$. System (2.1) subjected to the following initial conditions

$$\left\{\begin{array}{l}\left.\begin{array}{l}\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x),\end{array}\right. \quad x\in(0,1),\\ z(x,\rho,s,0)=f_0(x,\rho s),\quad \text{in } (0,1)\times(0,1)\times(0,\tau_2),\\ \eta^t(x,0)=0,\quad \text{for all } t\geq 0,\\ \eta^0(x,s)=\eta_0(s)=0,\quad \text{for all } s\geq 0.\end{array}\right. \tag{2.2}$$

In addition, we consider the following boundary conditions

$$\begin{aligned}\varphi(0,t)=&\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\quad \text{for all } t\geq 0,\\ \eta^t(0,s)=&\eta^t(1,s)=0,\quad \text{for all } s\geq 0.\end{aligned} \tag{2.3}$$

We now define the energy space

$$\boldsymbol{H}:=\left[H^1_0(0,1)\times L^2(0,1)\right]^2\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\times L^2_g\left(\mathbb{R}^+,H^1_0(0,1)\right),$$

where $L^2_g(\mathbb{R}^+,H^1_0(0,1))$ denotes the Hilbert space of H^1_0-valued functions on $\mathbb{R}^+$.

3. Exponential Stability

The functional energy of the solution of problem (2.1)–(2.3) is given by

$$\begin{aligned}E(t)=&E\left(t,\varphi,\psi,z,\eta^t\right)\\ =&\frac{1}{2}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx+\frac{1}{2}\int_0^1\left\{K(\varphi_x+\psi)^2+b\psi_x^2\right\}dx\\ &+\frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2 dsdx\\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2(s)\right|z^2(x,\rho,s,t)\,dsd\rho dx.\end{aligned} \tag{3.1}$$

We multiply $(2.1)_1$ by φ_t, $(2.1)_2$ by ψ_t and $(2.1)_3$ by $|\mu_2(s)|\,z$, integrating by parts over $(0,1)$, using Young and Cauchy-Schwarz's inequality we get

$$\begin{aligned}\frac{dE(t)}{dt}\leq&\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta^t_x(x,s)\right|^2 dsdx\\ &-C\left\{\int_0^1\psi_t\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,s,t)+\int_0^1\psi_t^2(x,t)\,dx\right\},\end{aligned} \tag{3.2}$$

where $C>0$, which implies that the energy E is a non-increasing function with respect to t.

Our main stability result reads as follows.

Theorem 3.1. *Let $U_0 \in D(A)$. Assume that $\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu_1$ and*

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}.$$

Then there exist two positive constants C and γ independent of t such that

$$E(t) \le Ce^{-\gamma t}, \quad \text{for all } t > 0. \tag{3.3}$$

Remark 3.1. To derive the exponential decay of the solution, it is enough to construct a functional $L(t)$, equivalent to the energy $E(t)$, satisfying

$$\frac{dL(t)}{dt} \le -\Lambda L(t), \quad \text{for all } t > 0,$$

where Λ is a positive constant. In order to obtain such a functional L, we need several lemmas.

Let us first define the following functional

$$I_1(t) := -\int_0^1 (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi)\, dx - \frac{\mu_1}{2}\int_0^1 \psi^2 dx. \tag{3.4}$$

Then we have the following estimate.

Lemma 3.1. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.1)–(2.3), then for any $\varepsilon, \delta_1 > 0$, we have*

$$\begin{aligned} \frac{dI_1(t)}{dt} \le & -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx \\ & + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \frac{c\varepsilon_2}{2}\int_0^1 \psi^2 dx \\ & + (b+\delta_1)\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\, \psi_t^2(x, t-s)\, dsdx \\ & + K\int_0^1 (\varphi_x + \psi)^2 dx, \end{aligned} \tag{3.5}$$

where $c = 1/\pi^2$ is the Poincaré's constant.

Proof. Taking the derivative of (3.4), integrating by parts, we obtain

$$\frac{dI_1(t)}{dt} = -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx - \int_0^1 (\rho_1 \varphi_{tt}\varphi_t + \rho_2 \psi_{tt}\psi_t)\, dx - \mu_1 \int_0^1 \psi_t \psi dx. \tag{3.6}$$

Therefore, by using $(2.1)_1$, $(2.1)_2$, integration by parts, we obtain from (3.6)

$$\begin{aligned} \frac{dI_1(t)}{dt} = & -\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + K\int_0^1 (\varphi_x + \psi)^2 dx + b\int_0^1 \psi_x^2 dx \\ & + \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s)\, \psi_t(x, t-s)\, dsdx \\ & + \int_0^1 \psi_x(x,t) \int_0^\infty g(s)\left|\eta_x^t(x,s)\right| dsdx. \end{aligned} \tag{3.7}$$

By exploiting Young and Poincaré's inequalities, we get for any $\varepsilon > 0$

$$\begin{aligned}
&\int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s)\, \psi_t(x, t-s)\, ds dx \\
\leq & \frac{c\varepsilon_2}{2} \int_0^1 \psi^2 dx + \frac{1}{2\varepsilon_2} \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s)\, \psi_t^2(x, t-s)\, ds dx.
\end{aligned} \tag{3.8}$$

Moreover, Young, Hölder's inequalities and (1.3) imply that for any $\delta_1 > 0$

$$\begin{aligned}
&\int_0^1 \psi_x(x,t) \int_0^\infty g(s) \left|\eta_x^t(x,s)\right| ds dx \\
\leq & \delta_1 \int_0^1 \psi_x^2(x,t)\, dx + \frac{g_0}{4\delta_1} \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 ds dx.
\end{aligned} \tag{3.9}$$

Inserting the estimates (3.8) and (3.9) into (3.7), then (3.5) is fulfilled. $\square$

Now, let w be the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0, \tag{3.10}$$

then

$$w(x,t) = -\int_0^x \psi(y,t)\, dy + x \left(\int_0^1 \psi(y,t)\, dy \right).$$

We have the following inequalities.

Lemma 3.2. *The solution of* (3.10) *satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply (3.10) by w, integrate by parts and use the Cauchy-Schwarz's inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (3.10) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx. \qquad \square$$

Let w be the solution of (3.10). We introduce the following functional

$$I_2(t) := \int_0^1 (\rho_2 \psi_t \psi + \rho_1 \varphi_t w)\, dx + \frac{\mu_1}{2} \int_0^1 \psi^2 dx. \tag{3.11}$$

Then, we have the following estimate.

Lemma 3.3. *Let* $(\varphi, \psi, z, \eta^t)$ *be the solution of* (2.1)–(2.3). *Then we have for any* $\varepsilon_3 > 0$,

$$\begin{aligned}\frac{dI_2(t)}{dt} \leq& (\delta_1 - b)\int_0^1 \psi_x^2 dx + \rho_1\lambda_2 \int_0^1 \varphi_t^2 dx + \frac{c\varepsilon_3}{2}\int_0^1 \psi^2 dx \\ &+ \left(\rho_2 + \frac{\rho_1}{4\lambda_2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_3} + \frac{\delta\gamma}{2\kappa\varepsilon_3}\right)\int_0^1 q^2 dx \\ &+ \frac{1}{2\varepsilon_3}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.\end{aligned} \tag{3.12}$$

Proof. By taking the derivative of (3.11), we conclude

$$\begin{aligned}\frac{dI_2(t)}{dt} =& -b\int_0^1 \psi_x^2 dx - K\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx + K\int_0^1 w_x^2 dx \\ &+ \rho_1\int_0^1 \varphi_t w_t dx + \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx \\ &- \int_0^1 \psi\int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds dx.\end{aligned}$$

We apply Young and Poincaré's inequalities, we find

$$\int_0^1 \psi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx \leq \delta_1\int_0^1 \psi_x^2(x,t) + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx,$$

and for any $\lambda_2 > 0$ we have

$$\rho_1\int_0^1 \varphi_t\psi_t dx \leq \rho_1\lambda_2\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\lambda_2}\int_0^1 \psi_t^2 dx. \qquad \square$$

Now, we define the functional I_3

$$I_3(t) := \rho_2\int_0^1 \psi_t(\varphi_x + \psi)\, dx + \frac{\rho_1 b}{K}\int_0^1 \psi_x\varphi_t dx + \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx. \tag{3.13}$$

Lemma 3.4. *Let* $(\varphi, \psi, z, \eta^t)$ *be the solution of* (2.1)–(2.3). *Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b + g_0} = \frac{\rho_2}{b}. \tag{3.14}$$

Then, for any $\varepsilon_4 > 0$, *we have*

$$\begin{aligned}\frac{dI_3(t)}{dt} \leq& \left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\right)\right]_{x=0}^{x=1} - (K - 2\varepsilon_4)\int_0^1 (\varphi_x + \psi)^2 dx \\ &+ \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_4}\right)\int_0^1 \psi_t^2 dx + \varepsilon_4\int_0^1 \varphi_t^2 dx + \frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\, z^2(x,1,s,t)\, ds dx \\ &- g_0 C(\varepsilon_4)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 ds dx.\end{aligned} \tag{3.15}$$

Proof. Differentiating $I_3(t)$, we obtain

$$\frac{dI_3(t)}{dt} = \rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi)\, dx + \rho_2\int_0^1 \psi_t(\varphi_x + \psi)_t\, dx$$

$$
\begin{aligned}
&+\frac{\rho_1 b}{K}\int_0^1 \psi_x\varphi_{tt}dx+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx\\
&+\frac{\rho_1 b}{K}\int_0^1 \psi_{xt}\varphi_{t}dx+\frac{\rho_1}{K}\int_0^1\varphi_{tt}\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx.
\end{aligned}
$$

Then, by using (2.1), we find

$$
\begin{aligned}
\frac{dI_3(t)}{dt}=&\rho_2\int_0^1(\varphi_x+\psi)\left(b\psi_{xx}(x,t)-K(\varphi_x+\psi)(x,t)\right.\\
&\left.-\mu_1\psi_t(x,t)-\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\right)dx\\
&+\int_0^1(\varphi_x+\psi)\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,dsdx+\rho_2\int_0^1\psi_t^2dx\\
&+b\int_0^1(\varphi_x+\psi)_x\,\psi_xdx+\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\psi_{tx}\varphi_tdx\\
&+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\left(\psi_{tx}(t,x)-\eta_{tx}^t(x,s)\right)dsdx\\
&+\frac{\rho_1}{K}\int_0^1(\varphi_x+\psi)_x\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx.
\end{aligned}
$$

By (3.14), we obtain

$$
\begin{aligned}
\frac{dI_3(t)}{dt}=&-K\int_0^1(\varphi_x+\psi)^2dx-\mu_1\int_0^1(\varphi_x+\psi)\,\psi_tdx+\rho_2\int_0^1\psi_t^2dx\\
&-\int_0^1(\varphi_x+\psi)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,dsdx\\
&+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx\\
&+\left[b\psi_x\varphi_xdx\right]_{x=0}^{x=1}+\left[\varphi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right].
\end{aligned}
\tag{3.16}
$$

For any $\varepsilon_4>0$, Young's inequality leads to

$$
\left|\mu_1\int_0^1(\varphi_x+\psi)\,\psi_t(x,t)\right|\le\varepsilon_4\int_0^1(\varphi_x+\psi)^2dx+\frac{\mu_1^2}{4\varepsilon_4}\int_0^1\psi_t^2dx
\tag{3.17}
$$

and

$$
\begin{aligned}
&\left|\int_0^1(\varphi_x+\psi)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,dsdx\right|\\
\le&\frac{c\varepsilon_4}{2}\int_0^1(\varphi_x+\psi)^2dx+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
&\left|\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx\right|\\
\le&\frac{\rho_1^2}{4K\varepsilon_4}\int_0^1\left(\int_0^\infty g'(s)\,\eta_x^t(x,s)\,ds\right)^2dx+\varepsilon_4\int_0^1\varphi_t^2dx
\end{aligned}
\tag{3.19}
$$

$$\leq -g(0)\,C(\varepsilon_4)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx+\varepsilon_4\int_0^1\varphi_t^2dx.$$

Plugging (3.17), (3.18) and (3.19) into (3.16), then inequality (3.15) holds. $\square$

Next, in order to handle the boundary terms appearing in (3.15) we use, as in [10], the function

$$q(x)=2-4x,\quad x\in(0,1).$$

So, we have the following result.

Lemma 3.5. *Let* (φ,ψ,z,η^t) *be the solution of* (2.1). *Then we have that for a positive constant* ε_6

(3.20)

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x-\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1}\\
\leq&-\frac{\varepsilon_6}{K}\frac{d}{dt}\int_0^1\rho_1 q(x)\,\varphi_t\varphi_x dx+K^2\varepsilon_6\int_0^1(\varphi_x+\psi)^2dx\\
&-\frac{\rho_2}{4\varepsilon_6}\frac{d}{dt}\int_0^1 q(x)\,\psi_t\left(b\psi_x-\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)dx+3\varepsilon_6\int_0^1\varphi_x^2dx\\
&+\left(\varepsilon_6+\frac{b}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\right)\int_0^1\psi_x^2dx+\frac{1}{4\varepsilon_6}\left(2\rho_2(b+g_0)+4\mu_1^2\varepsilon_6^2+\rho_2\varepsilon_6\right)\int_0^1\psi_t^2dx\\
&-\frac{\rho_2 g(0)\,C(\varepsilon_6)}{4\varepsilon_6}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx+\frac{2\rho_1\varepsilon_6}{K}\int_0^1\varphi_t^2dx\\
&+\frac{g_0}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx\\
&+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx.
\end{aligned}$$

Proof. By using Young and Poincaré inequalities, we obtain for any $\varepsilon_6>0$

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x+\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1} \qquad (3.21)\\
=&\varphi_x(1)\left(b\psi_x(1)+\int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right)\\
&-\varphi_x(0)\left(b\psi_x(0)+\int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)\\
\leq&\frac{1}{4\varepsilon_6}\left[\left(b\psi_x(1)+\int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right)^2\right.\\
&\left.+\left(b\psi_x(0)+\int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)^2\right]+\varepsilon_6\left[\varphi_x(1)^2+\varphi_x(0)^2\right].
\end{aligned}$$

On the other hand, it is clear that

$$\frac{d}{dt}\int_0^1\rho_2 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx$$

$$
\begin{aligned}
&= \int_0^1 \rho_2 q(x)\,\psi_{tt}\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&\quad + \int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx} + \int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx.
\end{aligned}
$$

Now, using $(2.1)_2$, we find

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&= \int_0^1 q(x)\Big(b\psi_{xx} - k(\varphi_x + \psi) - \mu_1\psi_t \\
&\quad - \int_{\tau_1}^{\tau_2} \mu_2(s)\,z(x,1,t)\,ds + \int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\Big) \\
&\quad \times \left(b\psi_x - \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&\quad + \int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx} + \int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx.
\end{aligned}
\tag{3.22}
$$

By the fact that

$$
\begin{aligned}
&\int_0^1 q(x)\left(b\psi_{xx} + \int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\right)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&= -\frac{1}{2}\int_0^1 q'(x)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2 dx \\
&\quad + \left[\frac{q(x)}{2}\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2\right]_{x=0}^{x=1}.
\end{aligned}
\tag{3.23}
$$

The last term in (3.22) can be treated as follows

$$
\begin{aligned}
&\int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx} + \int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx \\
=&\rho_2 b\int_0^1 q(x)\,\psi_t\psi_{tx}dx + \rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx \\
=&-\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2dx + \rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx \\
=&-\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2dx + \rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\left(\psi_t - \eta_s^t\right)_x dsdx \\
=&-\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2dx + \rho_2 g_0\int_0^1 q(x)\,\psi_t\psi_{tx}dx - \rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\,\eta_{sx}^t dsdx \\
=&-\frac{\rho_2(b+g_0)}{2}\int_0^1 q'(x)\,\psi_t^2dx + \rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g'(s)\,\eta_x^t dsdx.
\end{aligned}
\tag{3.24}
$$

Inserting (3.23) and (3.24) in (3.22), we arrive at

$$
\left(b\psi_x(0,t) + \int_0^\infty g(s)\,\eta_x^t(0,s)\,ds\right)^2 + \left(b\psi_x(1,t) + \int_0^\infty g(s)\,\eta_x^t(1,s)\,ds\right)^2
\tag{3.25}
$$

$$
\begin{aligned}
=&-\frac{d}{dt}\int_0^1 \rho_2 q\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx+2\rho_2(b+g_0)\int_0^1\psi_t^2dx\\
&-K\int_0^1 q(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\
&+\rho_2\int_0^1 q\psi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx\\
&-\mu_1\int_0^1 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\
&+2\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx\\
&-\int_0^1 q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx.
\end{aligned}
$$

Now, we estimate terms in the RHS of (3.25) as follows.

First, using Minkowski and Young's inequalities, we have

$$
\begin{aligned}
&2\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx\\
\le&4b^2\int_0^1\psi_x^2dx+4g_0\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned}
\tag{3.26}
$$

Second, by Young's inequality and (3.26), we have for any $\lambda>0$

$$
\begin{aligned}
&\left|K\int_0^1 q(x)(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right|\\
\le&2K\left|\int_0^1(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right|\\
\le&4K^2\lambda\int_0^1(\varphi_x+\psi)^2dx+\frac{1}{4\lambda}\int_0^1\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx\\
\le&4K^2\lambda\int_0^1(\varphi_x+\psi)^2dx+\frac{b^2}{2\lambda}\int_0^1\psi_x^2dx+\frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\left|\mu_1\int_0^1 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right|\\
\le&4\mu_1\lambda\int_0^1\psi_t^2dx+\frac{b^2}{2\lambda}\int_0^1\psi_x^2dx+\frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|-\int_0^1 q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right|\\
\le&b\int_0^1 q(x)\,\psi_x\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,dsdx\\
&+\int_0^1\left(q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx
\end{aligned}
$$

$$\leq 4\delta_0\lambda \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, ds + \frac{b^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx.$$

For any $\varepsilon_2 > 0$, we have

$$\begin{aligned}&\left|\rho_2 \int_0^1 q\psi_t \int_0^\infty g'(s)\,\eta_x^t(x,s)\, dsdx\right| \\ \leq &\rho_2\varepsilon_2 \int_0^1 \psi_t^2 dx - \rho_2 g(0)\, C(\varepsilon_2)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx.\end{aligned}$$

Inserting all the above estimates into (3.25), we obtain

$$\begin{aligned}(3.27)\quad &\left(b\psi_x(0,t) + \int_0^\infty g(s)\,\eta_x^t(0,s)\, ds\right)^2 + \left(b\psi_x(1,t) + \int_0^\infty g(s)\,\eta_x^t(1,s)\, ds\right)^2 \\ &\leq -\frac{d}{dt}\int_0^1 \rho_2 q\psi_t\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\, ds\right) dx \\ &+ \left(2\rho_2(b+g_0) + 4\mu_1^2\lambda + \rho_2\varepsilon_2\right)\int_0^1 \psi_t^2 dx \\ &+ b^2\left(4 + \frac{3}{2\lambda}\right)\int_0^1 \psi_x^2 dx + 4K^2\lambda\int_0^1 (\varphi_x+\psi)^2 dx \\ &- \rho_2 g(0)\, C(\varepsilon_2)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ &+ g_0\left(4+\frac{3}{2\lambda}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + 4\delta_0\lambda\int_{\tau_1}^{\tau_2}\mu_2(s) z^2(x,1,s,t)\, ds.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(3.28)\qquad \left[\varphi_x^2(1) - \varphi_x^2(0)\right] \leq &-\frac{d}{dt}\frac{1}{k}\int_0^1 \rho_1 q(x)\,\varphi_t\varphi_x dx \\ &+ 3\int_0^1 \varphi_x^2 dx + \int_0^1 \psi_x^2 dx + \frac{2\rho_1}{k}\int_0^1 \varphi_t^2 dx.\end{aligned}$$

Consequently, substituting (3.27) and (3.28) into (3.21), our desired estimate (3.20) holds. □

Now, we define the functional

$$(3.29)\qquad I_4(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsd\rho dx.$$

Then the following result holds.

Lemma 3.6. *Let (φ,ψ,z,η^t) be the solution of (2.1)–(2.3). Then for $C_1 > 0$ we have*

$$\begin{aligned}(3.30)\qquad \frac{dI_4(t)}{dt} \leq &-C_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsd\rho dx \\ &- C_1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsdx + \mu_1\int_0^1 \psi_t^2 dx,\end{aligned}$$

where C_1 is a positive constant.

Proof. Differentiating (3.29) and using $z\,(x,0,s,t) = \psi_t$, $e^{-s} \leq e^{-s\rho}$, we get for all $\rho \in [0,1]$

$$\begin{aligned}\frac{dI_4\,(t)}{dt} \leq & \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s}\,|\mu_2\,(s)|\,z^2\,(x,1,s,t)\,dsdx + \int_{\tau_1}^{\tau_2} |\mu_2\,(s)|\,ds \int_0^1 \psi_t^2 dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s}\,|\mu_2\,(s)|\,z^2\,(x,1,s,t)\,dsd\rho dx.\end{aligned}$$

Since $s \to -e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$. Finally, setting, $C_1 = -e^{-\tau_2}$ and recalling (1.2), we obtain (3.30). $\square$

Proof of Theorem 3.1. We are now ready to define the Lyapunov functional $L(t)$ as follows

$$\begin{aligned}L\,(t) :=& NE\,(t) + \frac{1}{4}I_1\,(t) + N_2 I_2\,(t) + I_3\,(t) + \frac{\varepsilon_2}{K}\int_0^1 \rho_1 q\varphi_t\varphi_x dx \\ & + \frac{1}{4\varepsilon_2}\int_0^1 \rho_2 q\,(x)\,\psi_t\left(b\psi_x + \int_0^\infty g\,(s)\,\eta_x^t\,(x,s)\,ds\right)dx + N_4 I_4\,(t)\,,\end{aligned}$$

where N, N_2, N_4 are positive real numbers which will be chosen later.

Consequently, the estimates (3.2), (3.5), (3.12), (3.15), (3.20) and (3.30) together with (1.3) and the following inequality

$$\int_0^1 \varphi_x^2 dx \leq 2\int_0^1 (\varphi_x + \psi)^2\,dx + 2\int_0^1 \psi_x^2 dx,$$

lead to

(3.31)

$$\begin{aligned}\frac{d}{dt}L\,(t) \leq & \Bigg\{-MC - \frac{\rho_1}{4} + N_2\left(\rho_2 + \frac{\rho_1}{4\lambda_2}\right) + \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_1}\right) \\ & + \frac{1}{4\varepsilon_2}\left(2\rho_2\,(b+g_0) + 4\mu_1^2\varepsilon_2^2 + \rho_2\varepsilon_2\right) + N_4\mu_1 + \frac{1}{2\tau}\Bigg\}\int_0^1 \psi_t^2 dx \\ & + \left\{\frac{1}{8\varepsilon_2} + \frac{N_2}{2\varepsilon_4} + \frac{1}{2\varepsilon_4} - C_1 N_4\right\}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2\,(s)|\,z^2\,(x,1,s,t)\,dsdx \\ & + \left\{-\frac{\rho_1}{4} + N_2\rho_1\lambda_2 + \frac{2\rho_1\varepsilon_2}{K} + \varepsilon_1\right\}\int_0^1 \varphi_t^2 dx \\ & + \left\{-\left(\frac{3K}{4} - 2\varepsilon\right) + K^2\varepsilon_2 + 6\varepsilon_2 + \frac{\varepsilon_4 c}{2}\right\}\int_0^1 (\varphi_x + \psi)^2\,dx - I_3\,(t) \\ & + \Bigg\{\frac{1}{4}\,(b+\delta_1) + N_2\,(\delta_1 + \mu_2 C^*\lambda_2 - b) + 7\varepsilon_2 \\ & + \frac{b^2}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right)\Bigg\}\int_0^1 \psi_x^2 dx + \left\{\left(\frac{c\varepsilon_2}{8} - \frac{cN_2\varepsilon_3}{2}\right)\right\}\int_0^1 \psi^2 dx \\ & + \Bigg\{\frac{g_0}{4\delta_1}\left(\frac{1}{4} + N_2\right) + \frac{g_0}{4\varepsilon_2}\left(4 + \frac{2}{2\varepsilon_2^2}\right)\end{aligned}$$

$$-\zeta\left(\frac{M}{2}-g_0C\left(\varepsilon_1\right)-\frac{\rho_2 g\left(0\right)C\left(\varepsilon_2\right)}{4\varepsilon_2}\right)\Bigg\}\int_0^1\int_0^\infty g\left(s\right)\left|\eta_x^t\left(x,s\right)\right|^2 dsdx$$
$$-C_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_2\left(s\right)\right|z^2\left(x,1,s,t\right)dsd\rho dx.$$

At this point, we have to choose our constants very carefully.

First, let us choose ε small enough such that

$$\varepsilon\leq\frac{3K}{8}.$$

Then, we take $\varepsilon_2=\varepsilon_1$ and choose ε_2 small enough such that

$$\varepsilon_2\leq\min\left\{\frac{K/8}{\left(K^2+6\right)},\frac{\rho_1/8}{\left(2\rho_1/K\right)+1}\right\}.$$

Then, we choose $\lambda_2=\delta_1$ and choose ε_2 small enough such that

$$\lambda_2\leq\frac{b/2}{1+\mu_2 C^*}.$$

Once all the above constants are fixed, we fix N_2 large enough such that

$$N_2\frac{b}{4}\geq\frac{1}{4}\left(b+\delta_1\right)+7\varepsilon_2+\frac{b}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right).$$

After that, we pick λ_2 so small that

$$\lambda_2\leq\frac{1}{32N_2}.$$

Finally, we choose M large enough so that, there exists a positive constant η_1, such that (3.31) becomes

$$\begin{aligned}\frac{d}{dt}L\left(t\right)\leq&-\eta_1\int_0^1\left(\psi_t^2+\psi_x^2+\varphi_t^2+\left(\varphi_x+\psi\right)^2+\psi^2\right)dx\\&-\eta_1\int_0^1\int_0^\infty g\left(s\right)\left|\eta_x^t\left(x,s\right)\right|^2 dsdx\\&+\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|z^2\left(x,1,s,t\right)dsdx\\&-\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu_2\left(s\right)\right|z^2\left(x,1,s,t\right)dsd\rho dx,\end{aligned}$$

which implies by (3.1), that there exists also $\eta_2>0$, such that

$$\frac{d}{dt}L\left(t\right)\leq-\eta_2E\left(t\right),\quad\text{for all } t\geq0. \tag{3.32}$$

In addition, we can choose M large enough so that

$$\beta_1E\left(t\right)\leq L\left(t\right)\leq\beta_2E\left(t\right),\quad\text{for all } t\geq0. \tag{3.33}$$

Combining (3.32) and (3.33), we conclude that there exists $\Lambda > 0$ such that

$$\frac{d}{dt}L(t) \leq -\Lambda L(t), \quad \text{for all } t \geq 0. \tag{3.34}$$

A simple integration of (3.34) leads to

$$L(t) \leq L(0)\, e^{-\Lambda t}, \quad \text{for all } t \geq 0. \tag{3.35}$$

Again, (3.33) and (3.35) yeilds the desired result (3.3). This completes the proof of Theorem 3.1. □

References

[1] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal **37** (1970), 297–308. http://dx.doi.org/10.1007/BF00251609

[2] R. Datko, J. Lagnese and M. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), 152–156. http://dx.doi.org/10.1109/CDC.1985.268529

[3] J. Hao and F. Wang, *Energy decay in a Timoshenko-type system for thermoelasticity of type III with distributed delay and past history*, Electron. J. Differential Equations **2018**(75) (2018), 1–27.

[4] M. Kafini, S. A. Messaoudi and M. I. Mustafa, *Energy decay result in a Timoshenko-type system of thermoelasticity of type III with distributive delay*, J. Math. Phys **54** (2013), Paper ID 101503. http://dx.doi.org/10.1063/1.4826102

[5] M. Kafini, S. A. Messaoudi, M. I. Mustafa and T. Apalara, *Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay*, Z. Angew. Math. Phys **66** (2015), 1499–1517. http://dx.doi.org/10.1007/s00033-014-0475-9

[6] M. Kirane, B. Said-Houari and M. N. Anwar, *Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks*, Comm. Pure Appl. Anal **10** (2011), 667–686. http://dx.doi.org/10.3934/cpaa.2011.10.667

[7] I. Lasiecka, D. Tataru et al., *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equations **6** (1993), 507–533.

[8] W.-J. Liu and E. Zuazua, *Decay rates for dissipative wave equations*, Ric. Mat. **48** (1999), 61–75.

[9] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), 1561–1585, http://dx.doi.org/10.1137/060648891

[10] J. E. M. Rivera and R. Racke, *Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability*, J. Math. Anal. Appl. **276** (2002), 248–278. http://dx.doi.org/10.1016/j.amc.2010.08.021

[11] B. Said-Houari and Y. Laskri, *A stability result of a Timoshenko system with a delay term in the internal feedback*, Appl. Math. Comput. **217** (2010), 2857–2869. http://dx.doi.org/10.1016/S0022-247X(02)00436-5

[12] B. Said-Houari and R. Rahali, *A stability result for a Timoshenko system with past history and a delay term in the internal feedback*, Dynam. Systems Appl. **20** (2011), 327.

[1]Laboratory of Pure and Applied Mathematics,
University of Amar TELIDJI,
Laghouat, Algeria
Email address: z.Khalili@lagh-univ.dz
Email address: d.ouchenane@lagh-univ.dz

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(2) (2023), PAGES 297–308.

LIGHTLIKE HYPERSURFACES IN SEMI-RIEMMANIAN MANIFOLDS ADMITTING AFFINE CONFORMAL VECTOR FIELDS

SAMUEL SSEKAJJA[1]

ABSTRACT. Lightlike hypersurfaces with integrable screen distributions are very important as far as lightlike geometry is concerned. They include, among others, screen conformal and screen totally umbilic ones. In this paper, we show that any lightlike hypersurface of a semi-Riemannian manifold admitting a certain closed affine conformal vector field has an integrable screen distribution. Several examples are furnished in support of the main results.

1. INTRODUCTION

Lightlike submanifolds are very important and their numerous applications, particularly to mathematical physics-like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [3] and [4] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in $(4+m)$-dimensional spacetime manifold, where m is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [3] and [4], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [3], Duggal-Sahin [4] and Kupeli [15]. It is upon those books that many other researchers, including but not limited to $[1, 5, 7–10, 12, 13]$ have extended their theories.

Key words and phrases. Lightlike hypersurfaces, affine conformal vector fields, Lorentzian manifolds.
2010 *Mathematics Subject Classification.* Primary: 53C25. Secondary: 53C40, 53C50.
DOI 10.46793/KgJMat2302.297S
Received: April 30, 2020.
Accepted: August 25, 2020.

Although a lot has been done on the geometry of lightlike submanifolds of semi-Riemannian manifolds, we remark that very little, see [3, page 259], efforts has been dedicated towards understanding what affine conformal vector fields, on semi-Riemmanian manifolds, can offer as far as characterising lightlike hypersurfaces. The present paper is directed towards achieving a characterisation of lightlike hypersurfaces in such spaces. The paper is arranged as follows. In Section 2, we quote some basic notions required in the rest of the paper. In Section 3, we prove some preliminary results on affine conformal vector fields, and Section 4 is dedicated to the main results of the study.

2. Preliminaries

An $(n+2)$-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\overline{g}$, that is, $\overline{M}$ admits a smooth tensor field $\overline{g}$ of type $(0,2)$ such that, for each point $p \in \overline{M}$, the tensor $\overline{g}_p : T_p\overline{M} \times T_p\overline{M} \longrightarrow \mathbb{R}$ is a *non-degenerate* inner product of signature $(-,+,\dots,+)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector field $v \in T_p\overline{M}$ is said to be timelike (resp., non-spacelike, null and spacelike) if it satisfies $\overline{g}_p(v,v) < 0$ (resp., ≤ 0, $= 0$ and > 0) [11]. Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional semi-Riemannian manifold and let M be a hypersurface of $\overline{M}$. Let g be the induced tensor field by $\overline{g}$ on M. Then, M is called a *lightlike hypersurface* of $\overline{M}$ if g is of constant rank n [3]. Consider the vector bundle $TM^{\perp}$ whose fibers are defined by $T_xM^{\perp} = \{Y_x \in T_x\overline{M} : \overline{g}_x(X_x, Y_x) = 0 \text{ for all } X_x \in T_xM\}$, for any $x \in M$. Hence, a hypersurface M of $\overline{M}$ is lightlike if and only if $TM^{\perp}$ is a distribution of rank 1 on M. Let M be a lightlike hypersurface. We consider the complementary distribution $S(TM)$ to $TM^{\perp}$ in TM, which is called a *screen distribution*. It is well-known that $S(TM)$ is non-degenerate (see [3]). Thus, we have the decomposition $TM = S(TM) \perp TM^{\perp}$.

As $S(TM)$ is non-degenerate with respect to $\overline{g}$, we have $T\overline{M} = S(TM) \perp S(TM)^{\perp}$, where $S(TM)^{\perp}$ is the complementary vector bundle to $S(TM)$ in $T\overline{M}|_M$. Let (M, g) be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$, called the *lightlike transversal bundle* [3] of M with respect to $S(TM)$, of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, Z) = 0, \tag{2.1}$$

for any section Z of $S(TM)$. Consequently, we have the following decomposition of $T\overline{M}$

$$T\overline{M}|_M = S(TM) \perp \{TM^{\perp} \oplus tr(TM)\} = TM \oplus tr(TM).$$

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [3]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \quad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$B(\xi, X) = 0, \quad g(A^*_\xi X, Y) = B(X,Y), \quad g(A_N X, PY) = C(X, PY),$$

for any $X, Y \in \Gamma(TM)$. Moreover, $\overline{g}(A^*_\xi X, N) = 0$ and $\overline{g}(A_N X, N) = 0$ for all $X \in \Gamma(TM)$. From these relations, we notice that A^*_ξ and A_N are both screen-valued operators. Moreover, it is easy to show that

$$(\nabla_X g)(Y, Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \tag{2.4}$$

for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection.

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is *screen conformal* [4, Definition 2.2.1, p. 51] if the shape operators A_N and A^*_ξ of M and $S(TM)$, respectively, are related by $A_N = \psi A^*_\xi$, where ψ is a non-vanishing smooth function on a neighbourhood $\mathcal{U}$ in M. In particular, if ψ is a non-zero constant, M is called *screen homothetic*. When A_N and A^*_ξ are instead linked by $A_N = \psi_1 A^*_\xi + \psi_2 P$, for some smooth functions ψ_1 and ψ_2, then M is called *quasi screen conformal* [12]. It is easy to see that a quasi screen conformal lightlike hypersurface is screen conformal when $\psi_2 \equiv 0$. A semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature c is called a semi-Riemannian space form (see [11, p. 80]) and denoted by $\overline{M}(c)$. The curvature tensor field $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \quad \text{for all } X, Y, Z \in \Gamma(T\overline{M}). \tag{2.5}$$

3. Some Basic Results

A smooth vector field V on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *affine conformal* vector (ACV) field if there exists a smooth function $\rho : \overline{M} \to \mathbb{R}$, called the *potential*, on $\overline{M}$ that satisfies

$$(\pounds_V \overline{\nabla})(X,Y) = (X\rho)Y + (Y\rho)X - \overline{g}(X,Y)\,\mathrm{grad}\,\rho, \tag{3.1}$$

where $\pounds_V$ is the Lie derivative with respect V and the affinity tensor $(\pounds_V \overline{\nabla})$ of V defined by

$$(\pounds_V \overline{\nabla})(X,Y) = \pounds_V \overline{\nabla}_X Y - \overline{\nabla}_{\pounds_L X} Y - \overline{\nabla}_X \pounds_Z Y,$$

for all $X, Y \in \Gamma(T\overline{M})$. In particular, V is an *affine* vector field if ρ is constant, that is if $\pounds_V \overline{\nabla} = 0$. The following result is well-known for an ACV field V.

Theorem 3.1 ([2,3])**.** *A vector field V on $\overline{M}$ is an ACV if and only if*

$$\pounds_V \overline{g} = 2\rho\overline{g}(X,Y) + K, \quad \overline{\nabla} K = 0, \tag{3.2}$$

where K is a covariant constant ($\overline{\nabla}K = 0$) symmetric and therefore, Killing tensor (abbreviated K-tensor) of second order.

A sub case is the conformal killing vector (CKV) when $K = 0$ and $\rho_{;a} = X_a\rho \neq 0$, $a = 0, 1, 2, \dots, n + 1$. This also includes *homothetic* vector fields (HV) and *killing* vector fields (KV) when $\rho_{;a} = 0$ and $\rho = 0$, respectively. See [2, p. 276] or [3, p. 264], and many more references cited therein, for more details.

Example 3.1 (K. L. Duggal [2]). Let $\overline{M}$ be a four-dimensional Einstein static fluid spacetime with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - r^2)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}, \phi^2)$$

and the fluid 4-velocity vector $u^a = \delta^a_a$, $a = 0, 1, 2, 3$. This spacetime admits a CKV

$$V_1^a = (1 - r^2)^{1/2}\cos t\delta^1_0 - r(1 - r^2)^{1/2}\sin t\delta^a_0$$

and a proper affine vector $V_2^a = t\delta^a_0$. As the spacetime metric is reducible, the combination $V^a = V_1^a + V_2^a$ is a proper ACV [2, p. 279] such that

$$\begin{aligned} V =&(t + (1 - r^2)^{1/2}\cos t)\delta^a_0 - r(1 - r^2)^{1/2}\sin t\delta^a_1, \\ \rho =& -(1 - r^2)^{1/2}\sin t, \quad K_{ab} = -2t_{;a}t_{;b}. \end{aligned}$$

Utilising Koszul's formula [11, Theorem 11, p. 61], we have

$$2\overline{g}(\overline{\nabla}_X V, Y) = (\pounds_V g)(X,Y) + d\eta(X,Y), \tag{3.3}$$

for all $X, Y \in \Gamma(T\overline{M})$, where η is the 1-form dual to V, that is, $\eta(X) = \overline{g}(V,X)$, $X \in \Gamma(T\overline{M})$. Define a skew symmetric tensor field φ of type $(1,1)$ on $\overline{M}$ by

$$d\eta(X,Y) = 2\overline{g}(\varphi X, Y), \tag{3.4}$$

for all $X, Y \in \Gamma(T\overline{M})$. The skew symmetric tensor field φ in the above equation is called the *associate tensor* field [6] of the affine conformal vector field V. We say that V is a *closed affine conformal vector* field if η is closed, that is $d\eta = 0$. Also, define a symmetric tensor field A_K of type $(1,1)$ on $\overline{M}$ by

$$K(X,Y) = \overline{g}(A_K X, Y), \tag{3.5}$$

for all $X, Y \in \Gamma(T\overline{M})$, where K is the symmetric $(0,2)$ tensor of Theorem 3.1. Then, using (3.2)–(3.5), and the fact that $\overline{g}$ is nondegenerate, we get the following result.

Lemma 3.1. *A vector field V on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is an ACV field if and only if*

$$\overline{\nabla}_X V = \rho X + \frac{1}{2}A_K X + \varphi X \quad \text{and} \quad (\overline{\nabla}_X A_K) = 0, \tag{3.6}$$

for all $X \in \Gamma(T\overline{M})$, where A_K and φ are tensor fields of type $(1,1)$ on $\overline{M}$, in which A_K is symmetric and φ is skew-symmetric.

Proof. From (3.2)–(3.5), we have

$$\overline{g}(\overline{\nabla}_X V, Y) = \rho\overline{g}(X,Y) + \frac{1}{2}\overline{g}(A_K X, Y) + \overline{g}(\varphi X, Y),$$

from which the first relation of (3.6) follows by utilising the fact that $\overline{g}$ is non-degenerate. On the other hand, using the second condition of (3.2), that is $\overline{\nabla}K = 0$, together with (3.5), we get

$$X\overline{g}(A_K Y, Z) = \overline{g}(A_K\overline{\nabla}_X Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z), \tag{3.7}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. As $\overline{\nabla}$ is Levi-Civita, it then follows from (3.7) that

$$\overline{g}(\overline{\nabla}_X A_K Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z) = \overline{g}(A_K\overline{\nabla}_X Y, Z) + \overline{g}(A_K Y, \overline{\nabla}_X Z),$$

from which one gets

$$\overline{g}((\overline{\nabla}_X A_K)Y, Z) = 0, \quad \text{for all } X, Y, Z \in \Gamma(T\overline{M}). \tag{3.8}$$

Then (3.8) shows that $(\overline{\nabla}A_K) = 0$, as $\overline{g}$ is non-degenerate, which proves the second relation in (3.6), and completing the proof. □

Lemma 3.2. *Let V be an ACV field on a semi-Riemannian manifold $(\overline{M}, \overline{g})$, then the covariant derivative of φ satisfies*

$$(\overline{\nabla}_X\varphi)(Y) = \overline{R}(X,V)Y + (Y\rho)X - \overline{g}(X,Y)\,\mathrm{grad}\,\rho, \tag{3.9}$$

where $(\overline{\nabla}_X\varphi)(Y) = \overline{\nabla}_X\varphi Y - \varphi\overline{\nabla}_X Y$ for any $X, Y \in \Gamma(T\overline{M})$.

Proof. Note, from (3.4), that the smooth 2-form $\overline{g}(\varphi X, Y)$ is closed. Thus, a direct calculation gives

$$\overline{g}((\overline{\nabla}_X\varphi)(Y), Z) + \overline{g}((\overline{\nabla}_Y\varphi)(Z), X) + \overline{g}((\overline{\nabla}_Z\varphi)(X), Y) = 0, \tag{3.10}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. Then, using Lemma 3.1, we derive

$$\begin{aligned}\overline{R}(X,Y)V =&(X\rho)Y - (Y\rho)X + \frac{1}{2}(\overline{\nabla}_X A_K)Y - \frac{1}{2}(\overline{\nabla}_Y A_K)X + (\overline{\nabla}_X\varphi)Y - (\overline{\nabla}_Y\varphi)X\\ =&(X\rho)Y - (Y\rho)X + (\overline{\nabla}_X\varphi)Y - (\overline{\nabla}_Y\varphi)X,\end{aligned} \tag{3.11}$$

in which we have used the fact that $\overline{\nabla}A_K = 0$ (see second relation of (3.6)). Substituting (3.11) in (3.10) and noting that $\overline{\nabla}\varphi$ is skew-symmetric, we get

$$\overline{g}(\overline{R}(X,Y)V - (X\rho)Y + (Y\rho)X, Z) + \overline{g}((\overline{\nabla}_Z\varphi)X, Y) = 0,$$

which reduces to

$$\overline{g}(\overline{R}(Z,V)X + (X\rho)Z - \overline{g}(X,Z)\mathrm{grad}\rho - (\overline{\nabla}_Z\varphi)X, Y) = 0, \tag{3.12}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. Finally, our result follows from (3.12) using the non-degeneracy of $\overline{g}$, which completes the proof. □

Lemma 3.3. *Let V be an ACV on a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then Hessian of the function $\alpha := \overline{g}(V, V)$ is given by*

$$\begin{aligned} \mathrm{Hess}_\alpha(X, Y) = &- 2\overline{g}(\overline{R}(X, V)V, Y) - 2(V\rho)\overline{g}(X, Y) \\ &+ 2\overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V) + 2(X\rho)\eta(Y) + 2(Y\rho)\eta(X), \end{aligned} \tag{3.13}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Proof. By virtue of (3.6), we derive

$$\begin{aligned} \overline{g}(\overline{\nabla}_X\overline{\nabla}_Y V - \overline{\nabla}_{\overline{\nabla}_X Y} V, V) &= \overline{g}((X\rho)Y + \frac{1}{2}(\overline{\nabla}_X A_K)Y + (\overline{\nabla}_X \varphi)Y, V) \\ &= \overline{g}((X\rho)Y + (\overline{\nabla}_X \varphi)Y, V), \end{aligned} \tag{3.14}$$

for any $X, Y \in \Gamma(T\overline{M})$, in which we have used the fact $\overline{\nabla} A_K = 0$. On the other hand,

$$\overline{g}(\overline{\nabla}_X\overline{\nabla}_Y V, V) = \frac{1}{2}X(Y\alpha) - \overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V), \tag{3.15}$$

$$\overline{g}(\overline{\nabla}_{\overline{\nabla}_X Y} V, V) = \frac{1}{2}(\overline{\nabla}_X Y)\alpha. \tag{3.16}$$

Replacing (3.15) and (3.16) in (3.14), leads to

$$\mathrm{Hess}_\alpha(X, Y) = 2\overline{g}(\overline{\nabla}_X V, \overline{\nabla}_Y V) + 2(X\rho)\eta(Y) + 2\overline{g}(\overline{\nabla}_X \varphi)Y, V). \tag{3.17}$$

Hence, the result follows from (3.17) and Lemma 3.2. □

4. Main Results

Consider a complementary vector bundle E of $TM^\perp$ in $S(TM)^\perp$ and take $V \in \Gamma(E_{|\mathcal{U}})$. Then $\overline{g}(V, \xi) \neq 0$ on $\mathcal{U}$ otherwise $S(TM)^\perp$ will be degenerate at a point of $\mathcal{U}$. Define on $\mathcal{U}$, a vector field

$$N = \frac{1}{\overline{g}(V, \xi)}\left\{V - \frac{\overline{g}(V, V)}{2\overline{g}(V, \xi)}\xi\right\}, \tag{4.1}$$

where $V \in \Gamma(E_{|\mathcal{U}})$, such that $\overline{g}(V, \xi) \neq 0$. It is easy to see that N, given by (4.1), satisfies (2.1). See more details in [4, p. 45] on the construction of N.

The vector field V, appearing in (4.1), is fundamental to the study of lightlike hypersyrfaces, and submanifolds in general. Its choice on $\overline{M}$ determines, to some extent, the geometry of the underlying lightlike hypersurface. For example, it has been proved in [4, Theorem 2.3.5, p. 63] that if E admits a *covariant constant timelike* vector field V, then with respect to a section $\xi \in TM^\perp$, $(M, g, S(TM))$ is screen conformal. Thus, M can admit an integrable unique screen distribution. A concrete example in this category include the lightlike Monge hypersurface (see Example 6 in [4, p. 62]). Thus, we ask the following general question.

Problem 1. Classify lightlike hypersurfaces $(M, g, S(TM))$ of a semi-riemannian manifold $(\overline{M}, \overline{g})$ relative to the geometry of the vector field $V \in \Gamma(E_{|\mathcal{U}})$.

We partially respond to the above problem by considering a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, admitting an *affine conformal vector* field (ACV), $V \in \Gamma(E_{|\mathcal{U}})$. To that end, let us set

$$\alpha := \overline{g}(V, V) \quad \text{and} \quad \beta := \overline{g}(V, \xi). \tag{4.2}$$

Then, using (4.2), we see that (4.1) give V as

$$V = \beta N + \frac{\alpha}{2\beta}\xi. \tag{4.3}$$

Then, we have the following result.

Theorem 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, admitting a closed ACV field, $V \in \Gamma(E_{|\mathcal{U}})$ given by (4.3). Then, M admits an integrable screen distribution and, therefore, locally isometric to product manifold $\xi_c \times M^*$, where ξ_c is a lightlike curve tangent to $TM^\perp$ and M^* a leaf of its screen distribution. Moreover, M is quasi screen conformal lightlike hypersurface if $A_K \circ P = 0$ or $A_K = 0$.*

Proof. First note that when V is a closed ACV field, then $\varphi = 0$ which follows from (3.4). It then follows from Lemma 3.1 that

$$\overline{\nabla}_X V = \rho X + \frac{1}{2}A_K X \quad \text{and} \quad (\overline{\nabla}_X A_K) = 0. \tag{4.4}$$

Using (4.3) and (4.4), together with the Weingarten formulae (4.16) and (2.3), we get

$$\begin{aligned} &-\beta A_N X - \frac{\alpha}{2\beta}A^*_\xi X + \left\{X\left(\frac{\alpha}{2\beta}\right) - \frac{\alpha}{2\beta}\tau(X)\right\}\xi + \{X(\beta) + \beta\tau(X)\}N \\ &= \rho X + \frac{1}{2}A_K X, \end{aligned} \tag{4.5}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (4.5) with $Y \in \Gamma(S(TM))$, one gets

$$\beta C(X, Y) + \frac{\alpha}{2\beta}B(X, Y) = -\rho g(X, Y) - \frac{1}{2}K(X, Y), \tag{4.6}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$. As B and K are symmetric, it follows from (4.6) that C is symmetric on $S(TM)$ too. Hence, by a direct calculation, using (2.3), we get

$$\theta([X, Y]) = C(X, Y) - C(Y, X) = 0,$$

for all $X, Y \in \Gamma(S(TM))$, from which we conclude that $S(TM)$ is an integrable distribution over M. Then, the product assertions follows from [5, Remark 5, p. 215]. Finally, when $A_K Y = 0$ for any $Y \in \Gamma(S(TM))$, then $K(X, Y) = 0$ for any $X \in \Gamma(TM)$. This shows that $\beta C(X, Y) + \frac{\alpha}{2\beta}B(X, Y) = -\rho g(X, Y)$. The case $A_K = 0$ follows in the similar manner. Hence, M is locally quasi screen conformal, which completes the proof. □

Example 4.1 (Lightlike hypersurface of generalised Robertson-Walker space). Consider (F, g_F) to be an $(n+1)$-dimensional, connected, Riemmanian manifold, $(I, -dt^2)$ an open interval of $\mathbb{R}$ with its usual metric reversed, and $f = e^{\lambda}$ (> 0) a smooth function on I. A Generalized Robertson-Walker (GRW) spacetime with base $(I, -dt^2)$, and fibre (F, g_F) and warping function f is the product manifold $\overline{M}(k, f) = I \times_f F$ endowed with the Lorentz metric

$$\overline{g} = -\pi_I^* dt^2 + (f \circ \pi_I)^2 \pi_F^* g_F \equiv -dt^2 + f^2(t) g_F, \tag{4.7}$$

where π_I and π_F are the natural projections of $I \times F$ onto I and F, respectively, and k the constant sectional curvature of F. The the GRW metric (4.7) can be rewritten as

$$\overline{g} = f^2(t)\{-f^{-2}(t)dt^2 + g_F\} = f^2(s)\{-ds^2 + g_F\}, \tag{4.8}$$

where the variable t is changed by s, define by $ds = dt/f(t)$. Thus, the warped metric $\overline{g}$ is conformal to the product metric $\tilde{g} = -ds^2 + F_F$. One of the consequences of this simple fact is: the vector field $V = f\partial t$ is parallel for $\tilde{g}$. That is $\tilde{\nabla} V = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection for g. So, this vector filed is conformal for any metric conformal to $\tilde{g}$. Thus, for $\overline{g}$, we have

$$\pounds_V \overline{g} = 2\rho \overline{g}, \tag{4.9}$$

where $\rho = f' \circ \pi_I \equiv f'$. From [11, Corollary 8, p. 344], we get

$$\overline{\nabla}_X V = f' X, \quad \text{for all } X \in \Gamma(T\overline{M}). \tag{4.10}$$

It then follows from (4.9), (4.10), (3.4) and Lemma 3.1 that $V = f\partial t$ is CKV and the 1-form η dual to V is closed, that is V is a closed CKV vector field. Next, consider a lightlike hypersurface (M, g) of $(\overline{M}, \overline{g})$. Along M, consider the timelike section $V = f\partial t \in \Gamma(T\overline{M})$ such that $\overline{g}(V, \xi) = 1$, where $\xi \in \Gamma(TM^{\perp})$. This means that V is not tangent to M. Therefore, the vector bundle H spanned by V and ξ is non-degenerate on M. The complementary orthogonal vector bundle $S(TM)$ to H in $T\overline{M}$ is a non-degenerate distribution on M and is complementary to $TM^{\perp}$. Thus, $S(TM)$ is a screen distribution on M. The unique lightlike transversal vector bundle $tr(TM)$ is spanned by $N = V + \frac{1}{2} f^2 \xi$. By direct calculation, using (4.10), we have

$$A_N X - \frac{1}{2} f^2 A_{\xi}^* X = -f' P X, \quad \tau(X) = 0, \quad Xf = -(\ln f)' \theta(X), \tag{4.11}$$

for all $X \in \Gamma(TM)$. Then from the first relation in (4.11) we see that M is a quasi screen conformal lightlike hypersurface.

When $\overline{M}$ has constant curvature c, we have the following.

Theorem 4.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}(c), \overline{g})$ of constant curvature c, admitting a closed ACV* *field* $V \in \Gamma(E_{|\mathfrak{U}})$*, given by* (4.3)*. Then,* $\overline{M}$ *is flat if and only if* V *is an affine vector field. Moreover, the function* $\alpha := \overline{g}(V,V)$ *satisfies the differential equations*

$$\alpha V(V\alpha) - (V\alpha)^2 + 2c\alpha^3 - \alpha V K(V,V) + (V\alpha)K(V,V) = 0. \tag{4.12}$$

Proof. From (3.4), the closure of η implies that $\varphi = 0$. Therefore, Lemma 3.2 leads to

$$\overline{R}(X,V)Y + (Y\rho)X - \overline{g}(X,Y)\mathrm{grad}\rho = 0, \tag{4.13}$$

for all $X, Y \in \Gamma(T\overline{M})$. As $\overline{M}$ has constant curvature c, (4.13) and (2.5) leads to

$$c\{\overline{g}(V,Y)\overline{g}(X,Z) - \overline{g}(X,Y)\overline{g}(V,Z)\} + (Y\rho)\overline{g}(X,Z) - (Z\rho)\overline{g}(X,Y) = 0,$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. From the above relation, one gets

$$\{Y\rho + c\overline{g}(V,Y)\}X = \{X\rho + c\overline{g}(V,X)\}Y. \tag{4.14}$$

Then it follows from (4.14) that

$$X\rho + c\overline{g}(V,X) = 0, \quad \text{for all } X \in \Gamma(T\overline{M}), \tag{4.15}$$

which proves the first assertion in the theorem. Letting $X = V$ in (4.15) and using the obvious fact that $\rho = \frac{V\alpha - K(V,V)}{2\alpha}$ (comes from the first relation in (3.6) of Lemma 3.1), we get

$$V\left(\frac{V\alpha - K(V,V)}{\alpha}\right) + 2c\alpha = 0,$$

from which (4.12) follows by differentiation, which end the proof. □

Example 4.2. For $\overline{M}(c) = \overline{M}(k,f)$, the GRW of Example 4.1, we have $\rho = f'$, $V = f\partial t$, $\alpha = \overline{g}(V,V) = -f^2$ and $A_K = 0$. Then, from these quantities, we have

$$V\alpha = -2f^2 f' \quad \text{and} \quad V(V\alpha) = -2f^2\{2(f')^2 + ff''\}. \tag{4.16}$$

Replacing (4.16) in (4.12), we get

$$2f'' - 2cf = 0. \tag{4.17}$$

Multiplying (4.17) by f' leads to

$$\frac{d}{dt}((f')^2 - cf^2) = 0. \tag{4.18}$$

Integrating (4.2) gives

$$(f')^2 + k = cf^2, \tag{4.19}$$

where k is a some constant. It then follows from (4.19) that

$$c = \frac{(f')^2 + k}{f^2}. \tag{4.20}$$

Indeed, relation (4.20) gives the constant sectional curvature of a GRW manifold as seen in [11, Corollary 9, p. 345]. The parameter k represents the constant sectional curvature of F.

Corollary 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$ dimensional semi-Riemannian manifold $(\overline{M}(c), \overline{g})$ of constant curvature c, admitting a closed affine vector field $V \in \Gamma(E_{|\mathcal{U}})$, given by (4.3). Then $\overline{M}$ is a flat.*

When the smooth function $\alpha = \overline{g}(V, V)$ has a critical point on $\overline{M}$, we prove the following result, analogous to [14, Theorem 3.1, p. 98] for projective vector fields.

Theorem 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional Lorentzian manifold, $(\overline{M}, \overline{g})$, admitting a timelike ACV field $V_p \in \Gamma(E_{|\mathcal{U}})$, given by (4.3). Assume that $\alpha := \overline{g}(V, V)$ attains a local maximum at $p \in \overline{M}$. Then*

$$\overline{g}(\overline{R}(X, V_p)V_p, X) + (V_p\rho)\overline{g}(X, X) \geq 0, \tag{4.21}$$

for all $X \in T_p\overline{M}$ orthogonal to V_p. Hence,

$$\overline{\mathrm{R}}\mathrm{ic}(V_p, V_p) + (n + 1)V_p\rho \geq 0, \tag{4.22}$$

where $\overline{\mathrm{R}}\mathrm{ic}$ is the Ricci tensor of $\overline{M}$. Furthermore, the sectional curvature, $\kappa(\pi)$, of any non-degenerate plane π containing V_p satisfies

$$\kappa(\pi) + \frac{V_p\rho}{\alpha} \leq 0. \tag{4.23}$$

Moreover, if the equality holds for all such planes, then V is an affine vector field, that is ρ is constant. The underlying lightlike hypersurface M has an integrable screen distribution $S(TM)$, and therefore locally isometric to product manifold $\xi_c \times M^*$, where ξ_c is a lightlike curve tangent to $TM^{\perp}$ and M^* a leaf of its screen distribution. In case $A_N\xi = 0$, then M is locally screen conformal.

Proof. For the function $\alpha = \overline{g}(V, V)$ having a critical point means that $Y\alpha = 0$, for any $Y \in T_p\overline{M}$. This means that $\overline{g}(\overline{\nabla}_Y V, V_p) = 0$. Since V_p is timelike, it then follows that

$$\overline{g}(\overline{\nabla}_Y V, \overline{\nabla}_Y V) \geq 0, \quad \text{for all } Y \in T_p\overline{M}. \tag{4.24}$$

On the other hand, $(\mathrm{Hess}_\alpha)_p$ must be negative semi-definite if p is assumed to be a local maximum. Therefore, from (3.3) and (4.24), we get

$$\overline{g}(\overline{R}(X, V_p)V_p, X) + (V_p\rho)\overline{g}(X, X) \geq \overline{g}(\overline{\nabla}_X V, \overline{\nabla}_X V) \geq 0, \tag{4.25}$$

for all $X \in T_p\overline{M}$, orthogonal to V_p at $p \in \overline{M}$. Then (4.21) and (4.22) follows directly from (4.25). Furthermore, as V is timelike, we divide (4.25) by $\alpha\overline{g}(X, X)$ to get (4.23). If equality holds for all such planes, it easy to see, from (4.24), that

$$\overline{\nabla}_X V = 0, \tag{4.26}$$

for all X orthogonal to V_p. Thus, from (3.6) of Lemma 3.1 and (4.26), we get

$$2\rho X + A_K X + 2\varphi X = 0. \tag{4.27}$$

Applying the second condition of Lemma 3.1 to (4.27), we get

$$(\overline{\nabla}_Y\varphi)(X) + (Y\rho)X = 0, \tag{4.28}$$

for all $Y \in \Gamma(T\overline{M})$. The inner product of (4.28) with X and noting that $(\overline{\nabla}_Y \varphi)$ is skew-symmetric, leads to $Y\rho = 0$. Hence, V is an affine vector field. On the other hand, we know that V is orthorgonal to any $X \in \Gamma(S(TM))$. Hence, using (4.26), (4.3), (4.16) and (2.3), we derive

$$\beta A_N X + \frac{\alpha}{2\beta} A^*_\xi X - \left\{ X\left(\frac{\alpha}{2\beta}\right) - \frac{\alpha}{2\beta}\tau(X) \right\} \xi - \{X(\beta) + \beta\tau(X)\} N = 0,$$

for all $X \in \Gamma(S(TM))$. It then follows that

$$\beta A_N X + \frac{\alpha}{2\beta} A^*_\xi X = 0, \tag{4.29}$$

and $X(\beta) + \beta\tau(X) = 0$, and thus A_N is symmetric on $S(TM)$. Thus, $S(TM)$ is integrable and therefore a product manifold by Remark 5 of [5, p. 215]. Finally if $A_N\xi = 0$, we see, from (4.29) that M is locally screen conformal, which completes the proof. □

The following is a direct consequence of Theorem 4.3.

Corollary 4.2. *Under the assumptions of Theorem* 4.3*, there exist no any Einstein manifold* $\overline{M}^{n+2}$*,* $n \geq 1$*, that is* $\overline{\mathrm{Ric}} = \gamma\overline{g}$*, such that* $\alpha := \overline{g}(V, V)$ *attains a maximum,* $\gamma > 0$ *and* $V\rho \leq 0$.

Acknowledgements. At this point, we would like to thank the referees for their valuable comments and suggestions which greatly improved the quality of this paper.

References

[1] C. Atindogbé, *Scalar curvature on lightlike hypersurfaces*, Appl. Sci. **11** (2009), 9–18.

[2] K. L. Duggal, *Affine conformal vector fields in semi-Riemannian manifolds*, Acta Appl. Math. **23**(3) (1991), 275–294. https://doi.org/10.1007/BF00047139

[3] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and Its Applications **364**, Kluwer Academic Publishers, Dordrecht, 1996.

[4] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Frontiers in Mathematics, Birkhauser, Basel, 2010.

[5] K. L. Duggal and D. H. Jin, *Lightlike Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific Publishing, Hackensack, NJ, 2007.

[6] S. Deshmukh, *Geometry of conformal vector fields*, Arab. J. Math. Sci. **23** (2017), 44–73. https://doi.org/10.1016/j.ajmsc.2016.09.003

[7] D. H. Jin, *Ascreen lightlike hypersurfaces of an indefinite Sasakian manifold*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **20**(1) (2013), 25–35. https://doi.org/10.7468/jksmeb.2013.20.1.25

[8] D. H. Jin, *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. Pure Appl. Math. **41**(4) (2010), 569–581. https://doi.org/10.1007/s13226-010-0032-y

[9] S. Ssekajja, *Some results on lightlike hyperurfaces in* (LCS)*-manifolds*, Kyungpook Math. J. **59**(4) (2019), 783–795. https://doi.org/10.5666/KMJ.2019.59.4.783

[10] S. Ssekajja, *Geometry of isoperametric lightlike hypersurfaces of Lorentzian manifolds*, Korean Math. Soc. **57**(1) (2020), 195–213. https://doi.org/10.4134/JKMS.j190001

[11] B. O'Neill, *Semi-Riemannian Geometry, with Applications to Relativity*, Academic Press, New York, NY, 1983.

[12] M. Navarro, O. Palmas and D. A. Solis, *Null screen isoparametric hypersurfaces in Lorentzian space forms*, Mediterr. J. Math. **15** (2018), Article ID 215. https://doi.org/10.1007/s00009-018-1262-1

[13] T. H. Kang, *On lightlike hypersurfaces of a GRW space-time*, Bull. Korean Math. Soc. **49**(4) (2012), 863–874. https://doi.org/10.4134/BKMS.2012.49.4.863

[14] A. Romero and M. Sanchez, *Projective vector fields on Lorentzian manifolds*, Geom. Dedicata **93** (2002), 95–105. https://doi.org/10.1023/A:1020308012870

[15] D. N. Kupeli, *Singular Semi-Riemannian Geometry,* Mathematics and Its Applications **366**, Kluwer Academic Publishers, Kluwer, 1996.

[1]School of Mathematics,
University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg
Email address: samuel.ssekajja@wits.ac.za
Email address: ssekajja.samuel.buwaga@aims-senegal.org

Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 309–329.

ON THE SIMPLICIAL COMPLEXES ASSOCIATED TO THE CYCLOTOMIC POLYNOMIAL

ALEKSANDRA KOSTIĆ[1]

Abstract. Musiker and Reiner in [9] studied coefficients of cyclotomic polynomial in terms of topology of associated simplicial complexes. They determined homotopy type of associated complexes for all cyclotomic polynomials, except for cyclotomic polynomials whose degree is a product of three prime numbers. Using discrete Morse theory for simplicial complexes we partially answer a question posed by the two authors regarding homotopy type of the associated complexes when degree of the cyclotomic polynomial is a product of three prime numbers.

1. Introduction

Cyclotomic polynomials are an important type of polynomials in algebraic number theory, Galois theory and geometry. If n is a positive integer, then the n^{th} cyclotomic polynomial is defined as the unique monic, irreducible polynomial having all n^{th} primitive roots of unity as its zeros. It has degree given by Euler phi function $\phi(n)$, with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \xi^j),$$

where ξ is the n^{th} root of unity in $\mathbb{C}$. Additionally, cyclotomic polynomial $\Phi_n(x)$ has integer coefficients which are well-studied. Musiker and Reiner in [9] interpreted these coefficients topologically, as the torsion in the homology of a certain simplicial complex associated with the degree of the cyclotomic polynomial. The idea for these simplicial complexes originally appeared in [3] and reappeared in [1, 7]. In what follows, we give a review of associated simplicial complexes. It is sufficient to interpret the coefficients

Key words and phrases. Cyclotomic polynomial, simplicial complexes, discrete Morse theory, homotopy type.

2010 *Mathematics Subject Classification.* Primary: 55U10. Secondary: 55P10, 55P15.

DOI 10.46793/KgJMat2302.309K

Received: July 16, 2020.

Accepted: August 25, 2020.

of the cyclotomic polynomial for squarefree n. Therefore, we fix such a squarefree $n = p_1 \cdots p_d$. Let

$$K_{p_1,\ldots,p_d} := K_{p_1} * \cdots * K_{p_d}$$

be the simplicial join of $K_{p_1}, \ldots, K_{p_d}$, where K_{p_i} is a 0-dimensional abstract simplicial complex with p_i vertices which are labeled by residues $\{0 \pmod{p_i}, 1 \pmod{p_i}, \ldots, (p_i - 1) \pmod{p_i}\}$. The facets of $K_{p_1,\ldots,p_d}$ are labeled by a sequence of residues $(j_1 \pmod{p_1}, \ldots, j_d \pmod{p_d})$ and by the Chinese Reminder Theorem, they can be denoted by residue $j \pmod{n}$ (denote this facet by $F_{j \pmod{n}}$). Let $A \subseteq \{0, 1, \ldots, \phi(n)\}$. We denote by K_A the subcomplex of $K_{p_1,\ldots,p_d}$ which is generated by the facets $\{F_{j \pmod{n}}\}$, where

$$j \in A \cup \{\phi(n) + 1, \phi(n) + 2, \ldots, n - 2, n - 1\}.$$

It turns out that subcomplexes $K_\emptyset$ and $K_{\{j\}}$, where $j \in \{0, \ldots, \phi(n)\}$ have a very nice feature, which Musiker and Reiner proved in the next two theorems. Let $[z_{j \pmod{n}}] := \partial[F_{j \pmod{n}}]$ denote the $(d-2)$-cycle which is its image under the simplical boundary map ∂.

Theorem 1.1. ([9, Theorem 7.1.]). *Let $n = p_1 \cdots p_d$ be squarefree.*

(i) *One has a homology isomorphism*

$$\tilde{H}_*(K_\emptyset) \cong \tilde{H}_*(\mathbb{S}^{d-2}),$$

with $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$ generated by the cycle $[z_{\phi(n) \pmod{n}}]$.

(ii) *If $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, then for $j = 0, 1, \ldots, \phi(n)$, one has*

$$[z_{j \pmod{n}}] = c_j [z_{\phi(n) \pmod{n}}] \text{ in } \tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$$

and a homology isomorphism

$$\tilde{H}_*(K_{\{j\}}) \cong \tilde{H}_*(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2}),$$

where $\deg(f_j) = c_j$.

Theorem 1.2. ([9, Theorem 7.5.]). *For $d \geq 4$ and every $A \subseteq \{0, 1, \ldots, \phi(n)\}$, the complex K_A is simply-connected. Consequently, for $d \neq 3$, one has the following.*

(i) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^{d-2}$ and contains $[z_{\phi(n) \pmod{n}}]$ as a fundamental $(d-2)$-cycle.*

(ii) *For $j = 0, 1, \ldots, \phi(n)$, the cyclotomic polynomial coefficient c_j gives the degree of the attaching map from the oriented boundary $[z_{j \pmod{n}}]$ of the facet $F_{j \pmod{n}}$ into the homotopy $(d-2)$-sphere $K_\emptyset$, with respect to the choice of $[z_{\phi(n) \pmod{n}}]$ as the fundamental cycle.*

(iii) *In particular, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S}^{d-2} \cup_{f_j} \mathbb{B}^{d-1}$, where* $\deg(f_j) = c_j$.

For $d \geq 4$ the fundamental group of K_A is determined by its 2-skeleton, which is the same as 2-skeleton of $K_{p_1,\ldots,p_d}$ since the subcomplex $K_\emptyset$, and consequently every subcomplex K_A, contains the full $(d-2)$-skeleton of $K_{p_1,\ldots,p_d}$ [9, Proposition 5.5.].

This skeleton is shellable [9, Proposition 5.1.], hence homotopy equivalent to a wedge of $(d-2)$-spheres.

The homotopy types of $K_\emptyset$ and $K_{\{j\}}$ remain as opened question when $d = 3$. Namely, in Question 7.6, Musiker and Rainer ask the following.

1) Is $K_\emptyset$ homotopy equivalent to the circle $\mathbb{S}^1$?
2) Is $K_{\{j\}}$ homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$, where $\deg(f_j) = c_j$, $j = 0, 1, \ldots, \phi(n)$?

In [8], authors show, giving a counter-example, that Theorem 1.2 does not follow generally when $d = 3$. For $n = 3 \cdot 5 \cdot 7$, $K_\emptyset$ is not homotopy equivalent to the circle $\mathbb{S}^1$ and $K_{\{j\}}$ is not homotopy equivalent to $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$ for $j = 7$.

In this paper, by using discrete Morse theory, we prove that for $n = 3 \cdot 5 \cdot p$, where $p \geq 7$ is an arbitrary prime number, Theorem 1.2 holds for certain classes of prime p modulo 15, while for the others we show it does not hold. This result is given in the following two theorems.

Theorem 1.3. *Let $p \equiv k \pmod{15}$, where $k \in \{1, 2, 13, 14\}$, and $n = 3 \cdot 5 \cdot p$.*

(1) *The complex $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.*
(2) *If $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, then for $j \in \{0, 1, \ldots, \phi(n)\}$, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S} \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.*

Theorem 1.4. *Let $p \equiv k \pmod{15}$, where $k \in \{4, 7, 8, 11\}$, and $n = 3 \cdot 5 \cdot p$. The complex $K_\emptyset$ is not homotopy equivalent to $\mathbb{S}^1$.*

The paper is organized as follows. In Section 2, we briefly introduce notation of simplicial complexes, define an acyclic discrete vector field and its critical elements. In Section 3 we study the structure of the subcomplex $K_\emptyset$. Additionally, we construct an appropriate acyclic discrete vector field on $K_\emptyset$. In Section 4, we prove Theorem 1.3 by using results from Section 3. Finally, in Section 5 we prove Theorem 1.4.

2. Basic Concepts

2.1. **Simplicial complex.** Here, we present the basic notation and terminology concerning simplicial complexes which we will use intensively in this paper. For more details see [10].

An *abstract simplicial complex* K is a collection of finite non-empty sets such that, if $\sigma \in K$ and $\emptyset \neq \tau \subseteq \sigma$, then $\tau \in K$. If $\sigma \in K$, and σ has $n+1$ elements, we refer to σ as an n-simplex. If we want to emphasize that σ is n-dimensional simplex, i.e., n-simplex, we use notation $\sigma^{(n)}$.

A non-empty subset τ of σ is called a *face* of σ. Those simplices that are not faces of any other simplex in K are called *facets*.

Definition 2.1. Let K be any simplicial complex and let σ be any face of K. The *star* $\mathrm{St}(\sigma)$ of σ is the subcomplex of K consisting of all faces τ containing σ and of all faces of τ, i.e.,

$$\mathrm{St}(\sigma) = \{s \in K \mid (\exists \tau \in K)\ (\sigma \subseteq \tau \text{ and } s \subseteq \tau)\}.$$

Definition 2.2. The *link* of σ, denoted by $\mathrm{Lk}(\sigma)$, is the subcomplex of K consisting of all faces in $\mathrm{St}(\sigma)$ that do not intersect σ, i.e.,

$$\mathrm{Lk}(\sigma) = \{\tau \in K \mid \tau \in \mathrm{St}(\sigma) \text{ and } \sigma \cap \tau = \emptyset\}.$$

To simplify notation, if $\sigma = \{v\}$, where v is a vertex, we write $\mathrm{St}(v)$ for $\mathrm{St}(\{v\})$ and $\mathrm{Lk}(v)$ for $\mathrm{Lk}(\{v\})$. From the previous, it is clear that $\mathrm{St}(v) = v * \mathrm{Lk}(v)$. In order to simplify notation we denote the union $\bigcup_{i=1}^{k} \mathrm{St}(v_i)$ by $\mathrm{St}(v_1, \dots, v_k)$.

2.2. **Discrete Morse theory.** This subsection aims to give a brief introduction and some of the main results from Forman's discrete Morse theory. Discrete Morse theory (shorter DMT) is based on pairing faces of the complex, which actually represent forming sequences of collapses on the complex. We will use this theory in order to prove homotopical equivalence between certain simplicial complexes. For a more thorough background concerning DMT, we refer the reader to [4–6].

Definition 2.3. A function $F : K \to \mathbb{R}$ is discrete Morse function if, for every $\alpha^{(p)} \in K$,

(1) $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$ for at most one $\beta^{(p+1)} \supset \alpha^{(p)}$, and
(2) $f(\gamma^{(p-1)}) \geq f(\alpha^{(p)})$ for at most one $\gamma^{(p-1)} \subset \alpha^{(p)}$.

Definition 2.4. Simplex $\alpha^{(p)}$ is critical simplex if $f(\beta^{(p+1)}) > f(\alpha^{(p)})$ for all $\beta^{(p+1)} \supset \alpha^{(p)}$ and $f(\gamma^{(p-1)}) < f(\alpha^{(p)})$ for all $\gamma^{(p-1)} \subset \alpha^{(p)}$.

Forman proved that the topology of a simplicial complex is related to its critical simplex in a very strong way. This connection is given in the next theorem.

Theorem 2.1 ([5])**.** *Suppose K is a simplicial complex with a discrete Morse function. Then, K is homotopy equivalent to a* CW *complex with exactly one cell of dimension p for each critical simplex of dimension p.*

The number of critical simplices is not a topological invariant as it depends on the discrete Morse function. According to the previous theorem, the goal is to find Morse function with as small critical simplicies as possible. For this purpose, we introduce discrete vector field, which is (under some conditions) an equivalent concept.

Definition 2.5. Discrete vector field V on a finite simplicial complex K is the set of pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$, where $\alpha^{(p)} \subset \beta^{(p+1)}$, and each simplex is in at most one pair. We say that $\{\alpha^{(p)}, \beta^{(p+1)}\}$ is a matching in V. Simplex γ in K is critical or unmatched with respect to V if γ is not contained in any pair in V.

For a simplicial complex K and a discrete vector field V on K, let $\mathcal{C}_k(K, V)$ denote the set of all critical k-simplices in the simplicial complex K with respect to V and let

$$\mathcal{C}(K, V) = \bigcup_{k=0}^{\dim K} \mathcal{C}_k(K, V).$$

Definition 2.6. Given a discrete vector field V on a finite simplicial complex K, a V-path is a sequence of simplicies

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \ldots, \alpha_{r+1}^{(p)}, \beta_{r+1}^{(p+1)},$$

such that, for each $i \in \{0, \ldots, r\}$, pair $\{\alpha_i, \beta_i\} \in V$ and $\beta_i \supset \alpha_{i+1} \neq \alpha_i$. This path is non-trivially closed if $r > 0$ and $\alpha_0 = \alpha_{r+1}$.

If a discrete vector field V does not contain a non-trivial closed V-path we say that V is acyclic.

Theorem 2.2 ([5]). *A discrete vector field V on a finite simplicial complex K is a discrete vector field of some Morse function if and only if V is acyclic.*

Namely, for a discrete Morse function f, we can easily define a discrete vector field in the following way: $\{\alpha^{(p)}, \beta^{(p+1)}\} \in V$ whenever $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$. Previous theorem give a condition when we can do the converse process.

On the other hand, matching $\{\alpha^{(p)}, \beta^{(p+1)}\}$ in a discrete vector field V on a finite simplicial complex K can be represent by an arrow from a simplex $\alpha^{(p)}$ to a simplex $\beta^{(p+1)}$ of K. According to this, a modified Hasse (directed) diagram of the complex K corresponds to V. Hasse diagram is modified in the following way: arrows are reversed each time when for $\beta^{(p+1)}$ and its face $\alpha^{(p)}$ one has $\{\alpha^{(p)}, \beta^{(p+1)}\} \in V$. We denote this diagram by $D(K, V)$. Directed path from α to β in $D(K, V)$ we denote by $\alpha \to \beta$. It turns out that if V is an acyclic discrete vector field then $D(K, V)$ is acyclic directed graph, that is, $\alpha \to \beta$ and $\beta \to \alpha$ implies $\alpha = \beta$. The symbol $\alpha \not\to \beta$ we use to denote that a directed path from α to β does not exist in $D(K, V)$. Generally, for families K_1 and K_2, we write $K_1 \to K_2$ if there are $\alpha \in K_1$ and $\beta \in K_2$ such that $\alpha \to \beta$. The symbol $K_1 \not\to K_2$ is used to denote the non-existence of such a directed path.

We will use the next theorem in further work in order to prove the existence of a homotopical equivalence between a certain simplicial complexes.

Theorem 2.3. ([6, Theorem 4.4]). *Suppose that K_0 is a subcomplex of K such that $K_0 \not\to K \setminus K_0$ and such that all critical faces belong to K_0. Then it is possible to collapse K to K_0. In particular, K and K_0 are homotopy equivalent. Hence, K has no homology in dimensions strictly greater than* $\dim K_0$.

3. Complex $K_\emptyset$

Let $n = 3 \cdot 5 \cdot p$, where p is a prime. As the case when $p = 7$ was investigated in [8], we can assume that $p > 7$. For $n = 3 \cdot 5 \cdot p$ simplicial complex $K_{3,5,p}$ has $p + 8$ vertices:

$$0 \pmod 3,\ 1 \pmod 3,\ 2 \pmod 3,\ 0 \pmod 5,\ 1 \pmod 5,\ 2 \pmod 5,$$
$$3 \pmod 5,\ 4 \pmod 5,\ 0 \pmod p,\ 1 \pmod p,\ \ldots,\ p-1 \pmod p.$$

In order to simplify the notation we label these vertices by numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, p + 7$, respectively.

Subcomplex $K_\emptyset$ is a two-dimensional complex built of facets:

$$\begin{array}{lllllll}
F_{(8p-7) \pmod n}, & \dots, & F_{(8p-1) \pmod n}, & F_{8p \pmod n}, & \dots, & F_{(9p-8) \pmod n}, \\
F_{(9p-7) \pmod n}, & \dots, & F_{(9p-1) \pmod n}, & F_{9p \pmod n}, & \dots, & F_{(10p-8) \pmod n}, \\
\vdots & & \vdots & \vdots & & \vdots \\
F_{(14p-7) \pmod n}, & \dots, & F_{(14p-1) \pmod n}, & F_{14p \pmod n}, & \dots, & F_{(15p-8) \pmod n}, \\
F_{(15p-7) \pmod n}, & \dots, & F_{(15p-1) \pmod n}. & & &
\end{array}$$

Note that facet $F_{(dp-i) \pmod n}$ contains vertex $(p-i) \pmod p$ which is labeled by number $p-i+8$ for all $d \in \{8,\dots,15\}$, $i \in \{1,\dots,7\}$. Similarly, facet $F_{(dp+i) \pmod n}$ contains vertex $i \pmod p$ which is labeled by number $i+8$ for all $d \in \{8,\dots,14\}$, $i \in \{0,\dots,p-8\}$. Let

$$\left[a_j^i, b_j^i, i\right] = \begin{cases} F_{((8+j-1)p+i-8) \pmod n}, & \text{for } i \in \{8,\dots,p\}, \\ \\ F_{((8+j-1)p+i-8-p) \pmod n}, & \text{for } i \in \{p+1,\dots,p+7\}. \end{cases}.$$

Then, the above set of facets are:

$$\begin{array}{lllllll}
\left[a_1^{p+1}, b_1^{p+1}, p+1\right], & \dots, & \left[a_1^{p+7}, b_1^{p+7}, p+7\right], & \left[a_1^8, b_1^8, 8\right], & \dots, & \left[a_1^p, b_1^p, p\right], \\
\left[a_2^{p+1}, b_2^{p+1}, p+1\right], & \dots, & \left[a_2^{p+7}, b_2^{p+7}, p+7\right], & \left[a_2^8, b_2^8, 8\right], & \dots, & \left[a_2^p, b_2^p, p\right], \\
\vdots & & \vdots & \vdots & & \vdots \\
\left[a_7^{p+1}, b_7^{p+1}, p+1\right], & \dots, & \left[a_7^{p+7}, b_7^{p+7}, p+7\right], & \left[a_7^8, b_7^8, 8\right], & \dots, & \left[a_7^p, b_7^p, p\right], \\
\left[a_8^{p+1}, b_8^{p+1}, p+1\right], & \dots, & \left[a_8^{p+7}, b_8^{p+7}, p+7\right], & & &
\end{array}$$

respectively.

As every facet of $K_\emptyset$ contains exactly one vertex from the set of vertices $\{8, 9, \dots, p+7\}$ it is clear that

$$K_\emptyset = \bigcup_{i=8}^{p+7} \mathrm{St}(i).$$

Therefore, we begin our analysis of the complex $K_\emptyset$ with analysis of its subcomplexes $\mathrm{St}(8), \dots, \mathrm{St}(p+7)$.

As the number of 2-simplicies of $K_\emptyset$ is $7p+7$, we can notice that the subcomplex $\mathrm{St}(i)$ is built of facets $\left\{[a_j^i, b_j^i, i]\right\}_{j=1}^{7}$ when $i \in \{8,\dots,p\}$ and facets $\left\{[a_j^i, b_j^i, i]\right\}_{j=1}^{8}$ when $i \in \{p+1,\dots,p+7\}$. Furthermore, it follows that

$$a_1^i = a_4^i = a_7^i, \quad a_2^i = a_5^i, \quad a_3^i = a_6^i, \quad \text{for } i \in \{8,\dots,p\},$$

and

$$a_1^i = a_4^i = a_7^i, \quad a_2^i = a_5^i = a_8^i, \quad a_3^i = a_6^i, \quad \text{for } i \in \{p+1,\dots,p+7\},$$

because $8p \equiv 11p \equiv 14p$, $9p \equiv 12p \equiv 15p$ and $10p \equiv 13p$ modulo 3. Similarly, as $8p \equiv 13p$, $9p \equiv 14p$ and $10p \equiv 15p$ modulo 5, we can conclude that

$$b_1^i = b_6^i, \quad b_2^i = b_7^i, \quad \text{for } i \in \{8, \dots, p\},$$

and

$$b_1^i = b_6^i, \quad b_2^i = b_7^i, \quad b_3^i = b_8^i, \quad \text{for } i \in \{p+1, \dots, p+7\}.$$

Figure 1 shows subcomplex $\mathrm{St}(i)$ depending on the index $i \in \{8, \dots, p+7\}$.

(A) $i \in \{8, \dots, p\}$ (B) $i \in \{p+1, \dots, p+7\}$

FIGURE 1. Simplicial complex $\mathrm{St}(i)$

3.1. **Discrete vector field on $K_\emptyset$.** In order to examine the topology of the complex $K_\emptyset$, we will look for a discrete vector field such that the number of critical 2-simplices are as small as possible. We will see below that finding an appropriate discrete vector field on $K_\emptyset$ can be reduced to finding an appropriate discrete vector field on its subcomplex $\mathrm{St}(p+1, \dots, p+7)$.

The simplicial subcomplex $\mathrm{St}(i)$, $i \in \{8, \dots, p\}$, is built of facets:

$$\begin{aligned}
&[a_1^i, b_1^i, i],\ [a_2^i, b_2^i, i],\ [a_3^i, b_3^i, i],\\
&[a_1^i, b_4^i, i],\ [a_2^i, b_5^i, i],\ [a_3^i, b_1^i, i],\\
&[a_1^i, b_2^i, i].
\end{aligned}$$

As a_1^i, a_2^i, a_3^i and $b_1^i, b_2^i, b_3^i, b_4^i, b_5^i$ are different vertices, we can define acyclic discrete vector field on $\mathrm{St}(i)$ as follows:

$$\begin{aligned}
S_i = \{&\{[b_1^i, i], [a_1^i, b_1^i, i]\},\ \{[b_2^i, i], [a_2^i, b_2^i, i]\},\ \{[b_3^i, i], [a_3^i, b_3^i, i]\},\\
&\{[b_4^i, i], [a_1^i, b_4^i, i]\},\ \{[b_5^i, i], [a_2^i, b_5^i, i]\},\ \{[a_3^i, i], [a_3^i, b_1^i, i]\},\\
&\{[a_1^i, i], [a_1^i, b_2^i, i]\},\ \{[i], [a_2^i, i]\}\}.
\end{aligned}$$

Discrete vector field S_i is an acyclic discrete vector field on $\mathrm{St}(i)$, as we can see on Figure 2 (A). Note that $\mathcal{C}(\mathrm{St}(i), S_i) = \mathrm{Lk}(i) \subset K_{3,5}$ and $\mathcal{C}_2(\mathrm{St}(i), S_i) = \emptyset$ (see Figure 2 (B)).

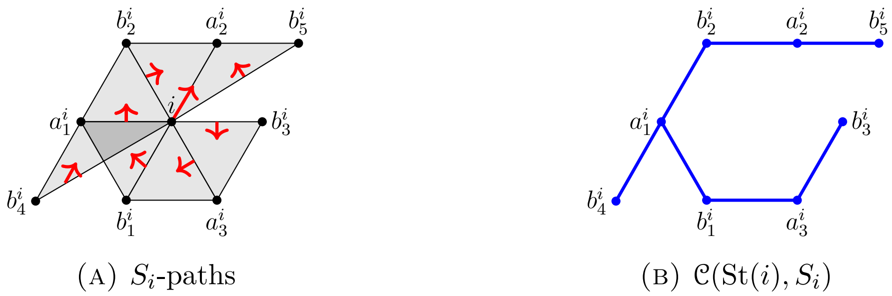

FIGURE 2. Discrete vector field S_i on complex $\mathrm{St}(i)$, $i \in \{8, \ldots, p\}$

Lemma 3.1. *Let $n = 3 \cdot 5 \cdot p$, where $p > 7$ is a prime. If C is an arbitrary acyclic discrete vector field on* $\mathrm{St}(p+1, \ldots, p+7)$*, then*

$$V = \left(\bigcup_{i=8}^{p} S_i \right) \cup C$$

is an acyclic discrete vector field on $K_\emptyset$.

Proof. It follows that

$$\left(\bigcup_{k \in \{8, \ldots, \hat{i}, \ldots, p+7\}} \mathrm{St}(k) \right) \cap \mathrm{St}(i) \subseteq \mathrm{Lk}(i).$$

As $\mathcal{C}(\mathrm{St}(i), S_i) = \mathrm{Lk}(i)$ for all $i \in \{8, \ldots, p\}$, V is a well-defined discrete vector field on $K_\emptyset$ as each simplex is in at most one pair. Additionally, for all $i \in \{8, \ldots, p\}$,

$$\bigcup_{k \in \{8, \ldots, \hat{i}, \ldots, p+7\}} \mathrm{St}(k) \not\longrightarrow \mathrm{St}(i) \setminus \mathrm{Lk}(i).$$

Hence, there are no non-trivial closed V-paths which contain simplices from the set $\bigcup_{i=8}^{p} (\mathrm{St}(i) \setminus \mathrm{Lk}(i))$. Note that

$$\bigcup_{i=8}^{p} \mathrm{Lk}(i) \setminus \mathrm{St}(p+1, \ldots, p+7) \subseteq \mathcal{C}(K_\emptyset, V).$$

As

$$K_\emptyset = \left(\bigcup_{i=8}^{p} (\mathrm{St}(i) \setminus \mathrm{Lk}(i)) \right) \cup \left(\bigcup_{i=8}^{p} \mathrm{Lk}(i) \setminus \mathrm{St}(p+1, \ldots, p+7) \right) \cup \mathrm{St}(p+1, \ldots, p+7),$$

the discrete vector field V is acyclic on $K_\emptyset$. $\square$

According to the previous lemma, in what follows, we will focus on finding an appropriate discrete vector field on the subcomplex $\mathrm{St}(p+1, \ldots, p+7)$.

3.2. **Discrete vector field on** $\mathrm{St}(p+1,\dots,p+7)$**.** As $\mathrm{St}(p+1,\dots,p+7) = \bigcup_{i=p+1}^{p+7}\mathrm{St}(i)$, we will find acyclic discrete vector fields without unpaired 2-simplices for the subcomplexes $\mathrm{St}(i)$, $i \in \{p+1,\dots,p+7\}$. We know that 2-simplices in $\mathrm{St}(i)$ are

$$[a_1^i, b_1^i, i],\ [a_2^i, b_2^i, i],\ [a_3^i, b_3^i, i],$$
$$[a_1^i, b_4^i, i],\ [a_2^i, b_5^i, i],\ [a_3^i, b_1^i, i],$$
$$[a_1^i, b_2^i, i],\ [a_2^i, b_3^i, i].$$

First, we consider the following discrete vector field on $\mathrm{St}(i)$:

$$\begin{aligned} V_i = \{&\{[b_1^i, i],\ [a_1^i, b_1^i, i]\},\ \{[b_2^i, i],\ [a_2^i, b_2^i, i]\},\ \{[b_3^i, i], [a_3^i, b_3^i, i]\},\\ &\{[b_4^i, i],\ [a_1^i, b_4^i, i]\},\ \{[b_5^i, i],\ [a_2^i, b_5^i, i]\},\ \{[a_3^i, i],\ [a_3^i, b_1^i, i]\},\\ &\{[a_1^i, i],\ [a_1^i, b_2^i, i]\},\ \{[a_2^i, i],\ [a_2^i, b_3^i, i]\}\}. \end{aligned}$$

Discrete vector field V_i is well-defined and pairs all facets of $\mathrm{St}(i)$, but it is not acyclic (see Figure 3).

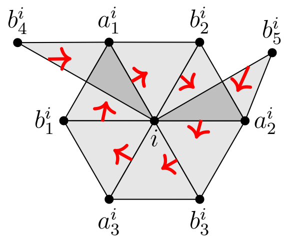

FIGURE 3. V_i-paths on complex $\mathrm{St}(i)$, $i \in \{p+1,\dots,p+7\}$

Namely, there is exactly one non-trivial closed V_i-path. This path contains facets $[a_1^i, b_1^i, i]$, $[a_2^i, b_2^i, i]$, $[a_3^i, b_3^i, i]$, $[a_3^i, b_1^i, i]$, $[a_1^i, b_2^i, i]$ and $[a_2^i, b_3^i, i]$ (see Figure 4).

Note that if $\alpha \supset [b_4^i, i]$, then $\alpha = [a_1^i, b_4^i, i]$. Similarly, if $\beta \supset [b_5^i, i]$, then $\beta = [a_2^i, b_5^i, i]$. Hence, there are no facets $\alpha, \beta \in \mathrm{St}(i)$ such that $\alpha \to [b_4^i, i]$ and $\beta \to [b_5^i, i]$ in V_i. Consequently, simplices $[b_4^i, i]$, $[b_5^i, i]$, $[a_1^i, b_4^i, i]$ and $[a_2^i, b_5^i, i]$ cannot be a part of a non-trivial closed V_i-path.

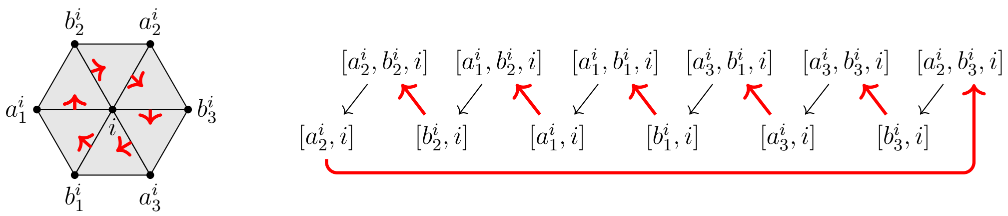

FIGURE 4. Non-trivial closed V_i-path

However, if we perform certain changes, we can make V_i acyclic. Namely, for some $(l,k) \in \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,3)\}$, if we modify V_i in a way that we pair $[a_l^i, b_k^i]$ with $[a_l^i, b_k^i, i]$, instead paring $[b_k^i, i]$ or $[a_l^i, i]$, it becomes acyclic (see Figure 5).

If $\{[b^i_k,\, i],\, [a^i_l,\, b^i_k, i]\} \in V_i$, we replace the matching $\{[b^i_k,\, i],\, [a^i_l,\, b^i_k,\, i]\}$ by the matching $\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\}$. Thus, 1-simplex $[b^i_k,\, i]$ becomes unmatched, so we can add matching $\{[i],\, [b^i_k,\, i]\}$. Analogously, if $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\} \in V_i$ we replace the matching $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\}$ by the matching $\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\}$ and 1-simplex $[a^i_l,\, i]$ becomes unmatched. Then, we can add matching $\{[i],\, [a^i_l, i]\}$.

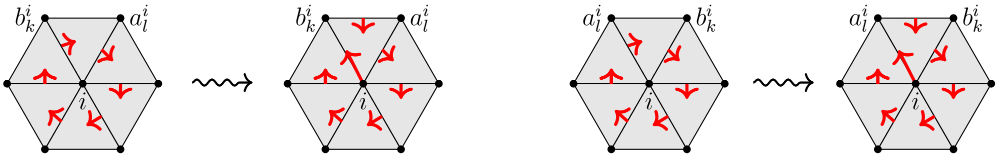

FIGURE 5. The two type of modification in discrete vector field V_i

Note that 0-simplex $[i]$ is not part of any non-trivial closed path in the mentioned modification of V_i. Namely, if $\alpha \to [i]$ for some 1-simplex $\alpha \supset [i]$, then α is not paired with any 0-simplex. Actually, α is pared with a 2-simplex.

Let

$$V_i([a^i_l, b^i_k]) := \Big(V_i \setminus \Big\{\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\}\Big\}\Big) \cup \Big\{\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\},\ \{[i],\, [a^i_l,\, i]\}\Big\}$$

if $\{[a^i_l,\, i],\, [a^i_l,\, b^i_k,\, i]\} \in V_i$ and

$$V_i([a^i_l, b^i_k]) := \Big(V_i \setminus \Big\{\{[b^i_k,\, i],\, [a^i_l,\, b^i_k,\, i]\}\Big\}\Big) \cup \Big\{\{[a^i_l,\, b^i_k],\, [a^i_l,\, b^i_k,\, i]\},\ \{[i],\, [b^i_k,\, i]\}\Big\}$$

if $\{[b^i_k, i],\, [a^i_l, b^i_k, i]\} \in V_i$.

According to the previous considerations, $V_i([a^i_l, b^i_k])$ is an acyclic discrete vector field on $\mathrm{St}(i)$, without critical 2-simplices, for all $i \in \{p+1, \dots, p+7\}$. Actually, $\mathcal{C}(\mathrm{St}(i), V_i([a^i_l, b^i_k]) = \mathrm{Lk}(i) \setminus [a^i_l, b^i_k] \subset K_{3,5}$.

The choice of (l, k) will depend on the rest of the complex $\mathrm{St}(p+1, \dots, p+7)$. Let $V_i([a^i_{l_i}, b^i_{k_i}])$ be the corresponding acyclic discrete vector field on $\mathrm{St}(i)$ for $i \in \{p+1, \dots, p+7\}$. If $[a^{p+1}_{l_{p+1}}, b^{p+1}_{k_{p+1}}]$, $[a^{p+2}_{l_{p+2}}, b^{p+2}_{k_{p+2}}], \dots, [a^{p+7}_{l_{p+7}}, b^{p+7}_{k_{p+7}}]$ are distinct 1-simplices then

$$C := \bigcup_{i=p+1}^{p+7} V_i([a^i_{l_i}, b^i_{k_i}])$$

is a well-defined discrete vector field on $\mathrm{St}(p+1, \dots, p+7)$. Generally, C does not have to be acyclic. Namely, there are no non-trivial closed C-paths which contain simplices from the only one subcomplex $\mathrm{St}(i)$, but there may be non-trivial closed paths containing simplices from the various subcomplexes $\mathrm{St}(i)$, $i \in \{p+1, \dots, p+7\}$.

For $i \in \{p+1, \dots, p+7\}$, note that the only "entrance" in $\mathrm{St}(i)$ with respect to C from $\mathrm{St}(p+1, \dots, \widehat{i}, \dots p+7)$ is through the 1-simplex $[a^i_{l_i}, b^i_{k_i}]$, whereas the set of "exits" are

$$\Big(\mathrm{Lk}(i) \setminus \{[a^i_{l_i}, b^i_{k_i}]\}\Big) \cap \{[a^j_{l_j}, b^j_{k_j}]\}_{j \in \{p+1, \dots, \widehat{i}, \dots, p+7\}}.$$

The only way to reach 1-simplex $[a_1^i, b_4^i]$ from the rest of the complex $\mathrm{St}(i)$ is through the 1-simplex $[b_4^i, i]$, i.e., $[b_4^i, i] \longrightarrow [a_1^i, b_4^i, i] \rightarrow [a_1^i, b_4^i]$. Similarly, 1-simplex $[a_2^i, b_5^i]$ can be reached from the rest of the complex $\mathrm{St}(i)$ through the 1-simplex $[b_5^i, i]$ only, i.e., $[b_5^i, i] \longrightarrow [a_2^i, b_5^i, i] \rightarrow [a_2^i, b_5^i]$. As $[b_4^i, i]$ and $[b_5^i, i]$ do not have entrance arrows in C, we can ignore 1-simplicies $[a_1^i, b_4^i]$ and $[a_2^i, b_5^i]$ as exits from $\mathrm{St}(i)$.

We form a directed graph $\mathrm{Flow}(C)$ of the "entrances/exits" through the subcomplexes $\mathrm{St}(i)$, $i \in \{p+1, \dots, p+7\}$, with respect to C. The graph $\mathrm{Flow}(C) = (A \sqcup B, E)$ is bipartite, where:

$$\begin{aligned} A =& \{\mathrm{St}(i) \mid i \in \{p+1, \dots, p+7\}\}, \\ B =& \{[a_{l_i}^i,\, b_{k_i}^i] \mid i \in \{p+1, \dots, p+7\}, \end{aligned}$$

and

$$\begin{aligned} E = \bigcup_{i=p+1}^{p+7} & \Big\{(\mathrm{St}(i), \alpha) \mid \alpha \in B \cap \mathrm{Lk}(i) \setminus \{[a_{l_i}^i, b_{k_i}^i], [a_1^i, b_4^i], [a_2^i, b_5^i]\}\Big\} \\ & \cup \Big\{([a_{l_i}^i,\, b_{k_i}^i], \mathrm{St}(i)) \mid i \in \{p+1, \dots, p+7\}\Big\}. \end{aligned}$$

Therefore, if $[a_{l_j}, b_{k_j}] \in \mathrm{St}(i) \setminus \{[a_1^i, b_4^i], [a_2^i, b_5^i]\}$ it follows that

$$[a_{l_i}, b_{k_i}] \rightarrow \mathrm{St}(i) \rightarrow [a_{l_j}, b_{k_j}] \rightarrow \mathrm{St}(j)$$

is path in $\mathrm{Flow}(C)$, for all distinct $i, j \in \{p+1, \dots, p+7\}$ (see Figure 6).

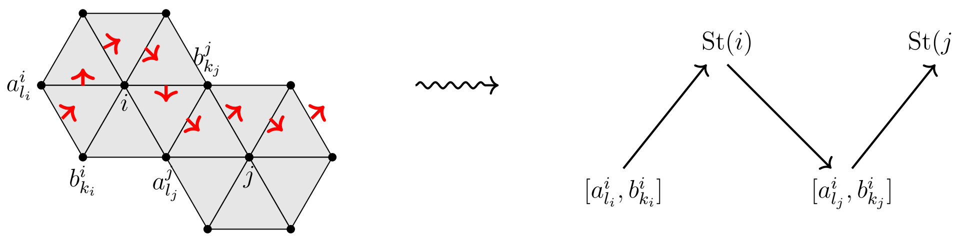

FIGURE 6. Forming a directed graph $\mathrm{Flow}(C)$

It is clear that if digraph $\mathrm{Flow}(C)$ is acyclic then C is an acyclic discrete vector field. In order to make C acyclic, we will choose appropriate 1-simplices $\{[a_{l_i}^i, b_{k_i}^i]\}_{i=p+1}^{p+7}$.

4. Proof of Theorem 1.3

Let $p_1, p_2 \geq 7$ be two distinct primes and $n_1 = 3 \cdot 5 \cdot p_1$ and $n_2 = 3 \cdot 5 \cdot p_2$. We consider subcomplexes $\mathrm{St}(p_1+1, \dots, p_1+7)$ and $\mathrm{St}(p_2+1, \dots, p_2+7)$ of $K_\emptyset$ for n_1 and n_2. The subcomplex $\mathrm{St}(p_1+1, \dots, p_1+7)$ consists of facets

$$\{F_{(dp_1 - i) \pmod{n_1}}\}_{d=\overline{8,15},\, i=\overline{1,7}},$$

while $\mathrm{St}(p_2+1, \dots, p_2+7)$ consists of facets

$$\{F_{(dp_2 - i) \pmod{n_2}}\}_{d=\overline{8,15},\, i=\overline{1,7}}.$$

It turns out that for certain primes p_1 and p_2, complexes $\mathrm{St}(p_1+1,\dots,p_1+7)$ and $\mathrm{St}(p_2+1,\dots,p_2+7)$ are isomorphic. Namely, when $p_1 \equiv p_2 \pmod{15}$, we can define the map $\pi : \mathrm{St}(p_1+1,\dots,p_1+7) \to \mathrm{St}(p_2+1,\dots,p_2+7)$ such that

$$p_1 + k \overset{\pi}{\mapsto} p_2 + k, \quad \text{for } k \in \{1,\dots,7\},$$

and π fixes every other vertex. Note that $dp_1 - i \equiv_{15} dp_2 - i$ when $p_1 \equiv_{15} p_2$ for all $i \in \{1,\dots,7\}$, $d \in \{8,\dots,15\}$. Additionally, $dp_1 - i \equiv_{p_1} p_1 - i$ and $dp_2 - i \equiv_{p_2} p_2 - i$. As $\pi(p_1 - i + 8) = p_2 - i + 8$, we can conclude that

$$\pi(F_{(dp_1-i) \pmod{n_1}}) = F_{(dp_2-i) \pmod{n_2}}.$$

Therefore, π is an isomorphism of the complexes.

If p is a prime number, then potential reminders modulo 15 are 1, 2, 4, 7, 8, 11, 13 and 14. According to the above, for a fixed reminder r modulo 15, we do not have to examine complexes $\mathrm{St}(p+1,\dots,p+7)$ for all primes $p \equiv r \pmod{15}$, it is enough to examine just for one of them.

Proof of Theorem 1.3. In order to show that $K_\emptyset \simeq \mathbb{S}^1$, we will construct an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, with one critical 1-simplex and one critical 0-simplex. If such discrete vector field exists, by Theorem 2.1, $K_\emptyset$ is homotopy equivalent to a CW complex with exactly one 1-cell and one 0-cell. According to Theorem 1.1, $H_1(K_\emptyset) = \mathbb{Z}$. Therefore, $K_\emptyset$ is homotopy equivalent to $\mathbb{S}^1$.

Similarly, to show that $K_{\{j\}} \simeq \mathbb{S} \cup_{f_j} \mathbb{B}^2$ we will construct an acyclic discrete vector field on $K_{\{j\}}$ with one critical 2-simplex, one critical 1-simplex and one critical 0-simplex. Then, by Theorem 2.1, $K_{\{j\}}$ is homotopy equivalent to a CW complex with exactly one 2-cell, one 1-cell and one 0-cell. Consequently, as $\pi_1(K_{\{j\}})$ has a presentation where the generators are the 1-cells and the relations come from the 2-cells,

$$\pi_1(K_{\{j\}}) = \langle g \mid g^d = 1 \rangle,$$

where d is the degree of the attaching map from the boundary of 2-cell into the 1-cell. By Theorem 1.1, $H_1(K_{\{j\}}) = \mathbb{Z}/c_j\mathbb{Z}$. As $H_1(K_{\{j\}})$ is the abelianization of $\pi_1(K_{\{j\}})$, it follows that $d = c_j$. Finally, the complex $K_{\{j\}}$ is homotopy equivalent to $\mathbb{S} \cup_{f_j} \mathbb{B}^2$, where $\deg(f_j) = c_j$.

Note that if V is an acyclic discrete vector field on $K_\emptyset$ with one critical 1-simplex, one critical 0-simplex and without critical 2-simplices, then V is an acyclic discrete vector field on $K_{\{j\}}$ with one critical 2-simplex, one critical 1-simplex and one critical 0-simplex. The complex $K_{\{j\}}$ is obtained by adding the facets $F_{j \pmod{n}}$ to the complex $K_\emptyset$, so the critical 2-simplex with respect to V is facet $F_{j \pmod{n}}$.

Now, we divide analysis in several cases, depending on the remainder of the prime p modulo 15. We will focus on finding an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, one critical 1-simplex and one critical 0-simplex for each case. For each case we will find an acyclic vector field C on $\mathrm{St}(p+1,\dots,p+7)$, hence, by

Lemma 3.1,

$$V = \left(\bigcup_{i=8}^{p} S_i \right) \cup C,$$

is an acyclic discrete vector field on $K_\emptyset$. For such defined discrete vector field V on $K_\emptyset$ it follows that

$$\mathcal{C}(K_\emptyset, V) = (\mathrm{Lk}(8, \dots, p) \setminus \mathrm{St}(p+1, \dots, p+7)) \ \cup \ \mathcal{C}(\mathrm{St}(p+1, \dots, p+7), C).$$

Case 1: $p \equiv 1 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1, \dots, p+7)$ consists of the following 2-simplices:

$$\begin{array}{lllllll}
[1,4,p+1], & [2,5,p+2], & [0,6,p+3], & [1,7,p+4], & [2,3,p+5], & [0,4,p+6], & [1,5,p+7], \\
[2,5,p+1], & [0,6,p+2], & [1,7,p+3], & [2,3,p+4], & [0,4,p+5], & [1,5,p+6], & [2,6,p+7], \\
[0,6,p+1], & [1,7,p+2], & [2,3,p+3], & [0,4,p+4], & [1,5,p+5], & [2,6,p+6], & [0,7,p+7], \\
[1,7,p+1], & [2,3,p+2], & [0,4,p+3], & [1,5,p+4], & [2,6,p+5], & [0,7,p+6], & [1,3,p+7], \\
[2,3,p+1], & [0,4,p+2], & [1,5,p+3], & [2,6,p+4], & [0,7,p+5], & [1,3,p+6], & [2,4,p+7], \\
[0,4,p+1], & [1,5,p+2], & [2,6,p+3], & [0,7,p+4], & [1,3,p+5], & [2,4,p+6], & [0,5,p+7], \\
[1,5,p+1], & [2,6,p+2], & [0,7,p+3], & [1,3,p+4], & [2,4,p+5], & [0,5,p+6], & [1,6,p+7], \\
[2,6,p+1], & [0,7,p+2], & [1,3,p+3], & [2,4,p+4], & [0,5,p+5], & [1,6,p+6], & [2,7,p+7].
\end{array}$$

We define discrete vector field C on $\mathrm{St}(p+1, \dots, p+7)$ in the following way:

$$\begin{aligned}
C = & V_{p+1}([1,4]) \cup V_{p+2}([2,5]) \cup V_{p+3}([0,6]) \cup V_{p+4}([1,7]) \\
& \cup V_{p+5}([0,5]) \cup V_{p+6}([1,6]) \cup V_{p+7}([2,7]) \\
& \cup \{\{[1],[1,3]\}, \{[2],[2,4]\}, \{[3],[2,3]\}, \{[4],[0,4]\}, \{[5],[1,5]\}, \{[6],[2,6]\}, \\
& \{[7],[0,7]\}\}.
\end{aligned}$$

Discrete vector field C is well-defined (see Figure 7). Additionally, Figure 7 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices. Graph Flow(C) is acyclic (see Figure 8), thus, there are no non-trivial closed C-paths which consist of 2-simplices and 1-simplices as well. Consequently, C is an acyclic discrete vector field on $\mathrm{St}(p+1, \dots, p+7)$.

Note that the only critical simplex in $\mathrm{St}(p+1, \dots, p+7)$ with respect to C is $[0]$, i.e., $\mathcal{C}(\mathrm{St}(p+1, \dots, p+7), C) = \{[0]\}$. Additionally, it follows that

$$K_{3,5} \setminus \mathrm{St}(p+1, \dots, p+7) = \{[0,3]\}.$$

On the other hand, $[0,3] \in K_{3,5}$ and

$$[0,3] \in F_{15p \pmod{n}} \subset \mathrm{St}(8).$$

Consequently,

$$[0,3] \in \mathrm{Lk}(8) \subset \mathrm{Lk}(8, \dots, p).$$

Since $\mathrm{Lk}(8, \dots, p) \subset K_{3,5}$, we conclude that

$$\mathrm{Lk}(8, \dots, p) \setminus \mathrm{St}(p+1, \dots, p+7) = \{[0,3]\}.$$

Finally,

$$\mathcal{C}(K_\emptyset, V) = \mathcal{C}(\mathrm{St}(p+1,\dots,p+7), C) \cup \{[0,3]\} = \{[0],[0,3]\},$$

so V is an acyclic discrete vector field on $K_\emptyset$ without critical 2-simplices, with one critical 1-simplex and one critical 0-simplex.

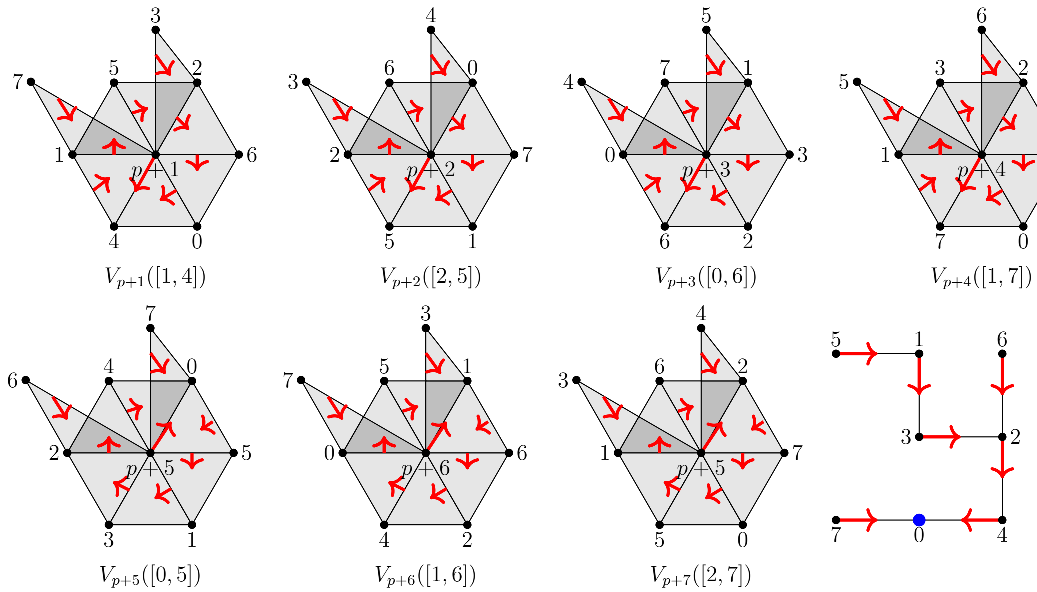

FIGURE 7. Gradient vector field C

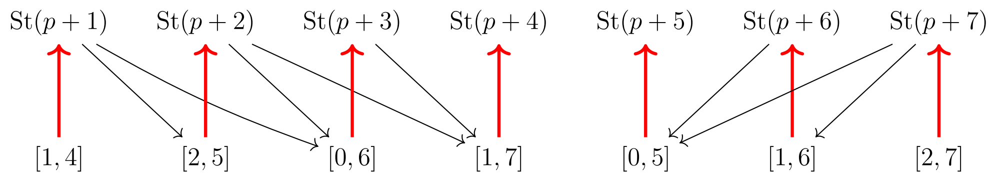

FIGURE 8. Digraph Flow(C)

Case 2: $p \equiv 2 \pmod{15}$.

Again, we look for an acyclic discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$ such that the number of critical simplices are as small as possible. The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ consists of facets:

$[0,7,p+1]$, $[1,3,p+2]$, $[2,4,p+3]$, $[0,5,p+4]$, $[1,6,p+5]$, $[2,7,p+6]$, $[0,3,p+7]$,
$[2,4,p+1]$, $[0,5,p+2]$, $[1,6,p+3]$, $[2,7,p+4]$, $[0,3,p+5]$, $[1,4,p+6]$, $[2,5,p+7]$,
$[1,6,p+1]$, $[2,7,p+2]$, $[0,3,p+3]$, $[1,4,p+4]$, $[2,5,p+5]$, $[0,6,p+6]$, $[1,7,p+7]$,
$[0,3,p+1]$, $[1,4,p+2]$, $[2,5,p+3]$, $[0,6,p+4]$, $[1,7,p+5]$, $[2,3,p+6]$, $[0,4,p+7]$,
$[2,5,p+1]$, $[0,6,p+2]$, $[1,7,p+3]$, $[2,3,p+4]$, $[0,4,p+5]$, $[1,5,p+6]$, $[2,6,p+7]$,
$[1,7,p+1]$, $[2,3,p+2]$, $[0,4,p+3]$, $[1,5,p+4]$, $[2,6,p+5]$, $[0,7,p+6]$, $[1,3,p+7]$,
$[0,4,p+1]$, $[1,5,p+2]$, $[2,6,p+3]$, $[0,7,p+4]$, $[1,3,p+5]$, $[2,4,p+6]$, $[0,5,p+7]$,
$[2,6,p+1]$, $[0,7,p+2]$, $[1,3,p+3]$, $[2,4,p+4]$, $[0,5,p+5]$, $[1,6,p+6]$, $[2,7,p+7]$.

Consider a discrete vector field

$$\begin{aligned} C = &V_{p+1}([0,7]) \cup V_{p+2}([2,3]) \cup V_{p+3}([0,4]) \cup V_{p+4}([1,5]) \\ &\cup V_{p+5}([2,5]) \cup V_{p+6}([0,6]) \cup V_{p+7}([1,7]) \\ &\cup \{\{[0],[0,3]\},\{[5],[0,5]\},\{[3],[1,3]\},\{[4],[1,4]\},\{[1],[1,6]\},\{[6],[2,6]\}, \\ &\{[7],[2,7]\}\} \end{aligned}$$

on $\mathrm{St}(p+1,\ldots,p+7)$. This discrete vector field is well-defined and acyclic. Namely, there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices (see Figure 9) and graph $\mathrm{Flow}(C)$ is acyclic (see Figure 10).

It follows that $\mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),C) = \{[2],[2,4]\}$. Additionally,

$$K_{3,5} \subset \mathrm{St}(p+1,\ldots,p+7).$$

As $\mathrm{Lk}(8,\ldots,p) \subset K_{3,5}$, we can finally conclude that

$$\mathcal{C}(K_\emptyset,V) = \mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),C) = \{[2],[2,4]\}.$$

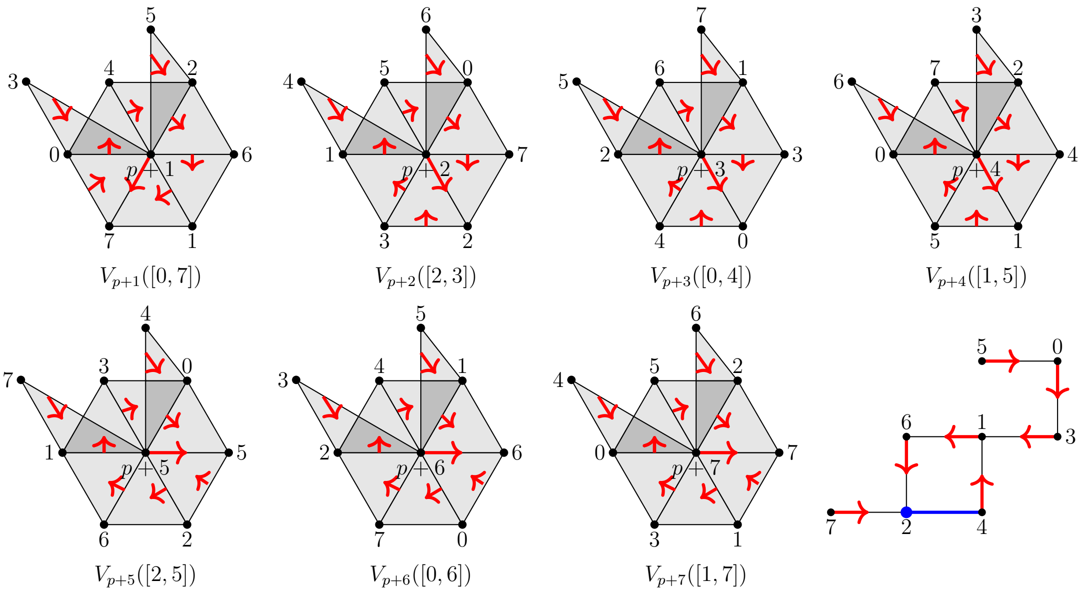

FIGURE 9. Gradient vector field C

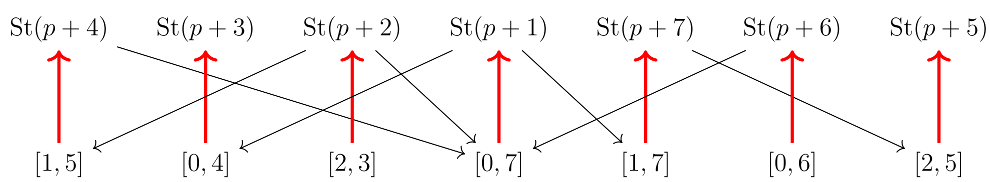

FIGURE 10. Digraph $\mathrm{Flow}(C)$

Case 3: $p \equiv 13 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ consists of the following 2-simplices:

$$\begin{array}{lllllll}
[1,5,p+1]\,, & [2,6,p+2]\,, & [0,7,p+3]\,, & [1,3,p+4]\,, & [2,4,p+5]\,, & [0,5,p+6]\,, & [1,6,p+7]\,,\\
[2,3,p+1]\,, & [0,4,p+2]\,, & [1,5,p+3]\,, & [2,6,p+4]\,, & [0,7,p+5]\,, & [1,3,p+6]\,, & [2,4,p+7]\,,\\
[0,6,p+1]\,, & [1,7,p+2]\,, & [2,3,p+3]\,, & [0,4,p+4]\,, & [1,5,p+5]\,, & [2,6,p+6]\,, & [0,7,p+7]\,,\\
[1,4,p+1]\,, & [2,5,p+2]\,, & [0,6,p+3]\,, & [1,7,p+4]\,, & [2,3,p+5]\,, & [0,4,p+6]\,, & [1,5,p+7]\,,\\
[2,7,p+1]\,, & [0,3,p+2]\,, & [1,4,p+3]\,, & [2,5,p+4]\,, & [0,6,p+5]\,, & [1,7,p+6]\,, & [2,3,p+7]\,,\\
[0,5,p+1]\,, & [1,6,p+2]\,, & [2,7,p+3]\,, & [0,3,p+4]\,, & [1,4,p+5]\,, & [2,5,p+6]\,, & [0,6,p+7]\,,\\
[1,3,p+1]\,, & [2,4,p+2]\,, & [0,5,p+3]\,, & [1,6,p+4]\,, & [2,7,p+5]\,, & [0,3,p+6]\,, & [1,4,p+7]\,,\\
[2,6,p+1]\,, & [0,7,p+2]\,, & [1,3,p+3]\,, & [2,4,p+4]\,, & [0,5,p+5]\,, & [1,6,p+6]\,, & [2,7,p+7]\,.
\end{array}$$

Let

$$\begin{aligned}
C =&V_{p+1}([0,6]) \cup V_{p+2}([1,7]) \cup V_{p+3}([2,3]) \cup V_{p+4}([0,4])\\
&\cup V_{p+5}([1,4]) \cup V_{p+6}([2,5]) \cup V_{p+7}([1,6])\\
&\cup \{\{[5],[0,5]\},\{[7],[0,7]\},\{[3],[1,3]\},\{[1],[1,5]\},\{[4],[2,4]\},\{[6],[2,6]\},\\
&\{[2],[2,7]\}\}
\end{aligned}$$

be a discrete vector field on $\mathrm{St}(p+1,\dots,p+7)$. As each simplex is in at most one pair, this discrete vector field is well-defined. Additionally, C is acyclic on $\mathrm{St}(p+1,\dots,p+7)$ and such that

$$\mathcal{C}(\mathrm{St}(p+1,\dots,p+7),C) = \{[0],[0,3]\}.$$

Namely, corresponding digraph $\mathrm{Flow}(C)$ is acyclic (see Figure 12), so there are no non-trivial closed C-paths which consist of 1-simplices and 2-simplices. Figure 11 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices as well.

Like in the previous case, $K_{3,5} \subset \mathrm{St}(p+1,\dots,p+7)$, and consequently,

$$\mathrm{Lk}(8,\dots,p) \subset \mathrm{St}(p+1,\dots,p+7),$$

because $\mathrm{Lk}(8,\dots,p) \subset K_{3,5}$. According to this, all critical simplices in $K_\emptyset$ with recpect to V are in $\mathrm{St}(p+1,\dots,p+7)$. Hence, we conclude

$$\mathcal{C}(K_\emptyset,V) = \mathcal{C}(\mathrm{St}(p+1,\dots,p+7),C) = \{[0],[0,3]\}.$$

Case 4: $p \equiv 14 \pmod{15}$.

The subcomplex $\mathrm{St}(p+1,\dots,p+7)$ is generated by facets:

$$\begin{array}{lllllll}
[0,3,p+1]\,, & [1,4,p+2]\,, & [2,5,p+3]\,, & [0,6,p+4]\,, & [1,7,p+5]\,, & [2,3,p+6]\,, & [0,4,p+7]\,,\\
[2,7,p+1]\,, & [0,3,p+2]\,, & [1,4,p+3]\,, & [2,5,p+4]\,, & [0,6,p+5]\,, & [1,7,p+6]\,, & [2,3,p+7]\,,\\
[1,6,p+1]\,, & [2,7,p+2]\,, & [0,3,p+3]\,, & [1,4,p+4]\,, & [2,5,p+5]\,, & [0,6,p+6]\,, & [1,7,p+7]\,,\\
[0,5,p+1]\,, & [1,6,p+2]\,, & [2,7,p+3]\,, & [0,3,p+4]\,, & [1,4,p+5]\,, & [2,5,p+6]\,, & [0,6,p+7]\,,\\
[2,4,p+1]\,, & [0,5,p+2]\,, & [1,6,p+3]\,, & [2,7,p+4]\,, & [0,3,p+5]\,, & [1,4,p+6]\,, & [2,5,p+7]\,,\\
[1,3,p+1]\,, & [2,4,p+2]\,, & [0,5,p+3]\,, & [1,6,p+4]\,, & [2,7,p+5]\,, & [0,3,p+6]\,, & [1,4,p+7]\,,\\
[0,7,p+1]\,, & [1,3,p+2]\,, & [2,4,p+3]\,, & [0,5,p+4]\,, & [1,6,p+5]\,, & [2,7,p+6]\,, & [0,3,p+7]\,,\\
[2,6,p+1]\,, & [0,7,p+2]\,, & [1,3,p+3]\,, & [2,4,p+4]\,, & [0,5,p+5]\,, & [1,6,p+6]\,, & [2,7,p+7]\,.
\end{array}$$

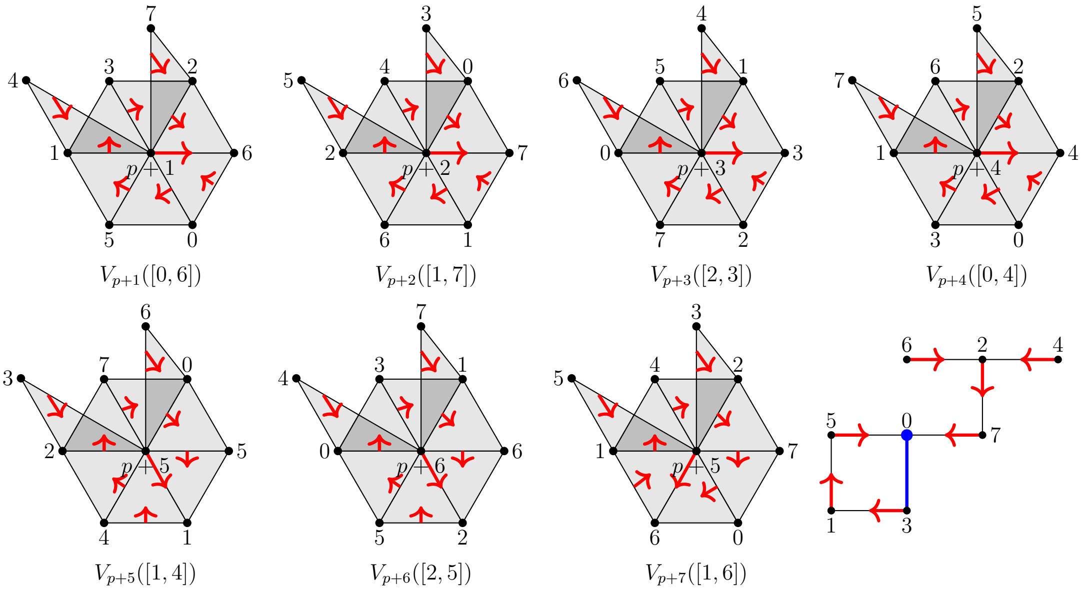

FIGURE 11. Gradient vector field C

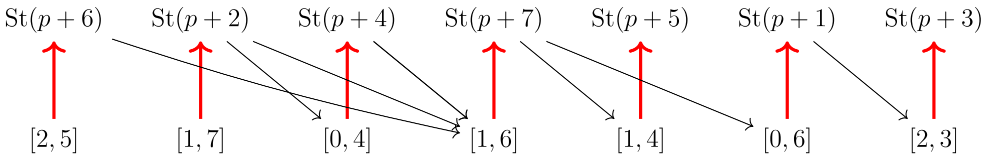

FIGURE 12. Digraph Flow(C)

Let us define discrete vector field on $\mathrm{St}(p+1,\ldots,p+7)$ as it follows

$$\begin{aligned} C =&V_{p+1}([2,6] \cup V_{p+2}([0,7]) \cup V_{p+3}([1,3]) \cup V_{p+4}([2,4]) \\ &\cup V_{p+5}([1,7]) \cup V_{p+6}([2,3]) \cup V_{p+7}([0,4]) \\ &\cup \{\{[3],[0,3]\},\{[0],[0,5]\},\{[6],[0,6]\},\{[4],[1,4]\},\{[1],[1,6]\}, \\ &\{[2],[2,5]\},\{[7],[2,7]\}\}. \end{aligned}$$

Discrete vector field C is well-defined (see Figure 13). Figure 14 shows that digraph Flow(C) is acyclic. In addition, Figure 13 shows that there are no non-trivial closed C-paths consisting of 0-simplices and 1-simplices. Therefore, C is acyclic on $\mathrm{St}(p+1,\ldots,p+7)$ and $\mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),V)=\{[5]\}$.

It follows that

$$K_{3,5} \setminus \mathrm{St}(p+1,\ldots,p+7)=\{[1,5]\}.$$

As $[1,5] \in F_{8p \pmod n} \subset \mathrm{St}(8)$ we conclude that $[1,5] \in \mathrm{St}(8) \cap K_{3,5} = \mathrm{Lk}(8)$. Hence,

$$\mathrm{Lk}(8,\ldots,p) \setminus \mathrm{St}(p+1,\ldots,p+7) = [1,5].$$

From the previous considerations, we conclude

$$\mathcal{C}(K_\emptyset,V)=\mathcal{C}(\mathrm{St}(p+1,\ldots,p+7),C) \cup \{[1,5]\} = \{[5],[1,5]]\}.$$

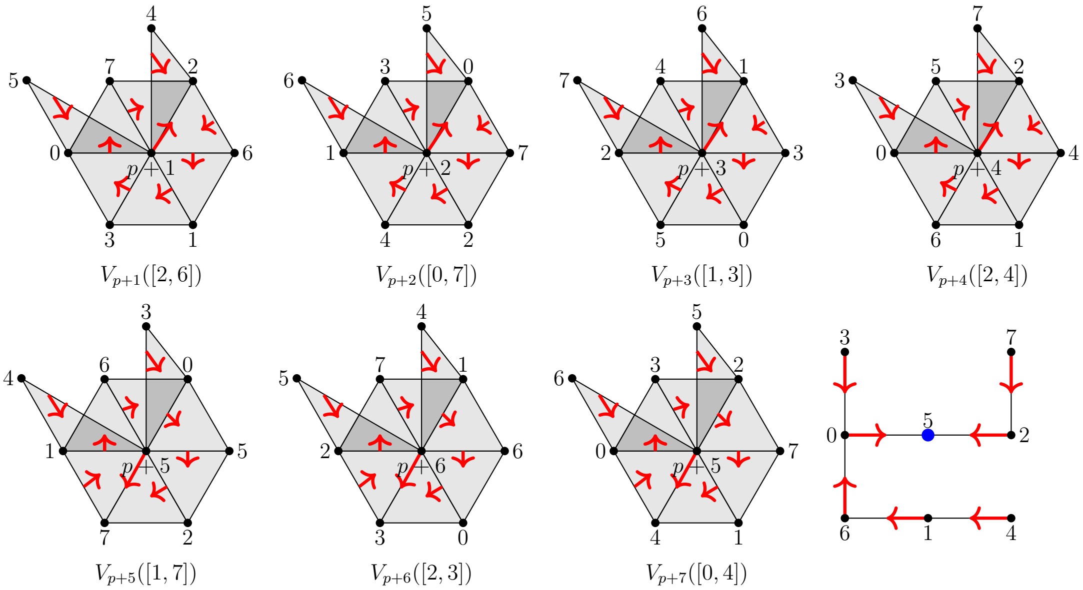

FIGURE 13. Gradient vector field C

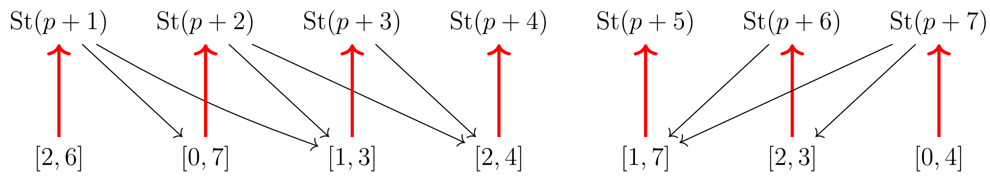

FIGURE 14. Digraph Flow(C)

$\square$

5. Proof of Theorem 1.4

In order to prove Theorem 1.4, we will need next theorem which points out an interesting feature of the complex $K_\emptyset$. Namely, under some conditions, complex $K_\emptyset$ is completely determined by its subcomplex $\mathrm{St}(p+1,\ldots,p+7)$.

Theorem 5.1. *Let $n = 3\cdot 5\cdot p$, where $p \geq 7$ is a prime and $p \equiv k \pmod{15}$. If $k \in \{2,4,7,8,11,13\}$, then complex $K_\emptyset$ is homotopy equivalent to its subcomplex* $\mathrm{St}(p+1,\ldots,p+7)$.

Proof. Obviously, $K_\emptyset = \mathrm{St}(p+1,\ldots,p+7)$ when $p=7$, therefore we consider $p>7$. Let C be an acyclic discrete vector field on $\mathrm{St}(p+1,\ldots,p+7)$. Then, by Lemma 3.1, $V = (\bigcup_{i=8}^{p} S_i) \cup C$ is an acyclic vector field on $K_\emptyset$. In order to prove this theorem, we show that there are no critical simplices in $K_\emptyset \setminus \mathrm{St}(p+1,\ldots,p+7)$ with respect to V and $\mathrm{St}(p+1,\ldots,p+7) \not\to K_\emptyset \setminus \mathrm{St}(p+1,\ldots,p+7)$. Then the theorem follows by Theorem 2.3.

Recall that S_i is an acyclic vector field on $\mathrm{St}(i)$ for $i \in \{8, \dots, p\}$ (see Figure 2). Additionally, all 1-simplices from the set $\mathrm{Lk}(i)$, $i \in \{8, \dots, p\}$, are unmatched with respect to S_i. Consequently,

$$\mathrm{St}(p+1, \dots, p+7) \not\to K_\emptyset \setminus \mathrm{St}(p+1, \dots, p+7).$$

It follows that $\alpha \in K_\emptyset \setminus \mathrm{St}(p+1, \dots, p+7)$ is critical with respect to V, when $\alpha \in \mathrm{Lk}(i) \setminus \mathrm{St}(p+1, \dots, p+7)$ for some $i \in \{8, \dots, p\}$. However, $\mathrm{St}(p+1, \dots, p+7)$ is built of facets $F_{j \pmod{15p}}$, where $j \in \{dp - i\}_{d=\overline{8,15},\, i=\overline{1,7}}$. There are 15 numbers which are distinct modulo 15 between numbers $\{dp - i\}_{j=\overline{8,15},\, i=\overline{1,7}}$ when $k \in \{2, 4, 7, 8, 11, 13\}$ (see Table 1). Thus, it follows that

$$K_{3,5} \subset \mathrm{St}(p+1, \dots, p+7).$$

Table 1

k	15 numbers among $\{dp - i\}_{j=\overline{8,15},\, i=\overline{1,7}}$ which are distinct modulo 15
2	$8p-1$, $9p-2$, $9p-1$, $10p-2$, $10p-1$, $11p-2$, $11p-1$, $12p-2$,$12p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$
4	$8p-2$, $8p-1$, $9p-4$, $9p-3$, $9p-2$, $9p-1$, $10p-4$, $10p-3$, $10p-2$, $10p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$
7	$9p-3$, $9p-2$, $9p-1$, $10p-7$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $9p-7$, $9p-6$, $9p-5$, $9p-4$
8	$8p-3$, $8p-2$, $8p-1$, $10p-1$, $9p-7$, $9p-6$, $9p-5$, $9p-4$, $9p-3$, $9p-2$, $9p-1$, $8p-7$, $8p-6$, $8p-5$, $8p-4$
11	$10p-5$, $10p-4$, $9p-7$, $9p-6$, $9p-5$, $9p-4$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $10p-7$, $10p-6$
13	$12p-6$, $11p-7$, $11p-6$, $10p-7$, $10p-6$, $9p-7$, $9p-6$, $8p-7$, $8p-6$, $8p-5$, $8p-4$, $8p-3$, $8p-2$, $8p-1$, $12p-7$

As $\mathrm{Lk}(i) \subset K_{3,5}$, we can conclude that

$$\mathrm{Lk}(i) \subseteq \mathrm{St}(p+1, \dots, p+7),$$

for all $i \in \{8, \dots, p\}$. Therefore, there are no simplices in $K_\emptyset \setminus \mathrm{St}(p+1, \dots, p+7)$ which are critical with respect to V. $\square$

Remark 5.1. It is not always true that $K_{3,5} \subset \mathrm{St}(p+1, \dots, p+7)$. Namely, if $p \equiv 1 \pmod{15}$ then $[0,3] \in F_{15p \pmod n} \subset \mathrm{St}(8)$ and $[0,3] \notin \mathrm{St}(p+1, \dots, p+7)$ (see Case 1 in the proof of Theorem 1.3). Therefore,

$$[0,3] \in \mathrm{Lk}(8) \setminus \mathrm{St}(p+1, \dots, p+7) \subseteq K_{3,5} \setminus \mathrm{St}(p+1, \dots, p+7).$$

Similarly, when $p \equiv 14 \pmod{15}$, it follows that $[1,5] \in F_{8p \pmod n} \subset \mathrm{St}(8)$. On the other hand, $[1,5] \notin \mathrm{St}(p+1, \dots, p+7)$ (see Case 4 in the proof of Theorem 1.3), so it follows that

$$[1,5] \in \mathrm{Lk}(8) \setminus \mathrm{St}(p+1, \dots, p+7) \subseteq K_{3,5} \setminus \mathrm{St}(p+1, \dots, p+7).$$

We are now prove Theorem 1.4.

Proof of Theorem 1.4. According to the above theorem and consideration from the beginning of the previous section, it is enough to examine the smallest possible cases: $p = 19$, $p = 7$, $p = 23$ and $p = 11$.

In [8], it was calculated that $\pi_1(K_\emptyset) = \langle a, b \mid ab^2a^{-1}b^{-2}a^{-1}bab^{-1}a^{-1}b^{-1}\rangle$ when $p = 7$. Further, it was proved that $\pi_1(K_\emptyset)$ is not commutative, and consequently, $K_\emptyset \not\simeq \mathbb{S}^1$. Here, we will use a similar idea for the remaining three cases.

Using Algorithm 1 from [8] for computing the fundamental group, for the listed maximal spanning trees, we obtain the following results.

p	maximal tree	$\pi_1(K_\emptyset)$
19	{[0,3], [0,4], [0,5], [0,6], [0,7], [0,8], [0,9], [0,10], [0,11], [0,12], [0,13], [0,14], [0,15], [0,16], [0,17], [0,18], [0,19], [0,20], [0,21], [0,22], [0,23], [0,24], [0,25], [0,26], [1,3], [2,3]}	$\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$
23	{[2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [2,10], [2,11], [2,12], [2,13], [2,14], [2,15], [2,16], [2,17], [2,18], [2,19], [2,20], [2,21], [2,22], [2,23], [2,24], [2,25], [2,26], [2,27], [2,28], [2,29], [2,30], [0,3], [1,3]}	$\langle a, b \mid b^{-1}a^{-1}b^2ab^{-1}aba^{-1}b^{-1}a = 1\rangle$
11	{[0,3], [0,4], [0,5], [0,6], [0,7], [0,8], [0,9], [0,10], [0,11], [0,12], [0,13], [0,14], [0,15], [0,16], [0,17], [0,18], [1,3], [2,3]}	$\langle a, b \mid a^{-1}bab^{-2} = 1\rangle$

Note that $\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$ and $\langle a, b \mid a^{-1}bab^{-2} = 1\rangle$ are distinct presentations of the same group. Namely, starting with $\langle a, b \mid a^{-1}b^{-1}a^{-2}b^{-1}ab = 1\rangle$ and letting $x = a^{-1}b^{-1}a^{-1}, y = a^{-1}$ we obtain new presentation $\langle x, y \mid x^2y^{-1}x^{-1}y = 1\rangle$ for the same group. This group is Baumslag-Solitar group $\mathrm{BS}(1, 2)$ (for more details see [2]). The group $\mathrm{BS}(1, 2)$ is not commutative. Namely, we can define an epimorphism $f : \mathrm{BS}(1, 2) \to \mathrm{S}_3$, such that $f(x) = (123)$, $f(y) = (23)$. Since permutation group S_3 is not commutative, $\mathrm{BS}(1, 2)$ cannot be commutative. Consequently, $\pi_1(K_\emptyset) \not\simeq \mathrm{S}^1$ when $p = 11$ and $p = 19$.

Now, we consider $\pi_1(K_\emptyset)$ when $p = 23$. Letting $x = b^{-1}a^{-1}$, relation

$$b^{-1}a^{-1}b^2ab^{-1}aba^{-1}b^{-1}a = 1$$

transforms into relation $xb^2x^{-1}b^{-2}x^{-1}bxb^{-1}x^{-1}b^{-1} = 1$. Thus, $\pi_1(K_\emptyset)$ is the same group for $p = 7$ and $p = 23$. Therefore, $K_\emptyset \not\simeq \mathbb{S}^1$ when $p = 23$. □

References

[1] R. M. Adin, *Counting colorful multi-dimensional trees*, Combinatorica **12** (1992), 247–260. https://doi.org/10.1007/BF01285814

[2] G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68**(3) (1962), 199–201. https://doi.org/10.1090/S0002-9904-1962-10745-9

[3] E. D. Bolker, *Simplicial geometry and transportation polytopes*, Trans. Amer. Math. Coc. **217** (1976), 121–142. `https://doi.org/10.2307/1997562`

[4] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134**(1) (1998), 90–145. `https://doi.org/10.1006/aima.1997.1650`

[5] R. Forman, *A user's guide to discrete Morse theory*, Sém. Lothar. Combin. **48** (2002), Article ID B48c, 35 pages.

[6] J. Jonsson, *Simplicial Complexes of Graphs*, Lecture Notes in Mathematics **1928**, Springer-Verlag, Berlin, 2008.

[7] G. Kalai, *Enumeration of Q-acyclic simplicial complexes*, Israel J. Math. **45** (1983), 337–351. `https://doi.org/10.1007/BF02804017`

[8] A. Kostić, N. Milošević and Z. Z. Petrović, *Note on the Cyclotomic Polynomial Topologically*, Exp. Math. (to appear). `https://doi.org/10.1080/10586458.2019.1680464`

[9] G. Musiker and V. Reiner, *The cyclotomic polynomial topologically*, J. Reine Angew. Math. **687** (2014), 113–132. `https://doi.org/10.1515/crelle-2012-0051`

[10] J. R. Munkres, *Elements of Algebraic Topology*, The Benjamin/Cummings Publishing Company, Menlo Park, California 1984.

[1]Department of Algebra and Mathematical Logic,
Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, Belgrade
Email address: `alex@matf.bg.ac.rs`

KRAGUJEVAC JOURNAL OF MATHEMATICS

About this Journal

The *Kragujevac Journal of Mathematics* (KJM) is an international journal devoted to research concerning all aspects of mathematics. The journal's policy is to motivate authors to publish original research that represents a significant contribution and is of broad interest to the fields of pure and applied mathematics. All published papers are reviewed and final versions are freely available online upon receipt. Volumes are compiled and published and hard copies are available for purchase. From 2018 the journal appears in one volume and four issues per annum: in March, June, September and December. From 2021 the journal appears in one volume and six issues per annum: in February, April, June, August, October and December.

During the period 1980–1999 (volumes 1–21) the journal appeared under the name *Zbornik radova Prirodno–matematičkog fakulteta Kragujevac* (Collection of Scientific Papers from the Faculty of Science, Kragujevac), after which two separate journals—the *Kragujevac Journal of Mathematics* and the *Kragujevac Journal of Science*—were formed.

Instructions for Authors

The journal's acceptance criteria are originality, significance, and clarity of presentation. The submitted contributions must be written in English and be typeset in TEX or LATEX using the journal's defined style (please refer to the Information for Authors section of the journal's website `http://kjm.pmf.kg.ac.rs`). Papers should be submitted using the online system located on the journal's website by creating an account and following the submission instructions (the same account allows the paper's progress to be monitored). For additional information please contact the Editorial Board via e-mail (`krag_j_math@kg.ac.rs`).