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GRAPHS WITH AT MOST FOUR SEIDEL EIGENVALUES

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Abstract. Let G be a graph of order n with adjacency matrix $A(G)$. The eigenvalues of matrix $S(G) = J_n - I_n - 2A(G)$, where J_n is the n by n matrix with all entries 1, are called the Seidel eigenvalues of G. Let $\mathcal{G}(n, r)$ be the set of all graphs of order n with a single Seidel eigenvalue with multiplicity r. In the present work, we will characterize all graphs in the class $\mathcal{G}(n, n-i)$ for $i = 1, 2$ and for the case $i = 3$ our characterization is done by this condition that the nullity of $S(G)$ is zero. If the nullity of $S(G)$ is not zero the problem is solved in special cases.

1. Introduction

Let G be a simple graph on n vertices with adjacency matrix $A(G)$. The roots of the characteristic polynomial $P_G(\lambda) = \det(\lambda I_n - A(G))$ of G, where I_n is the identity matrix of order n, are called the eigenvalues of G. The spectrum of an adjacency matrix $A(G)$ of G is the multiset of its eigenvalues and forms the spectrum of G denoted by $\mathrm{Spec}(G)$.

Lint and Seidel in [13] introduced a symmetric $(0, -1, 1)$-adjacency matrix for a graph G called the Seidel matrix of G as $S(G) = J_n - I_n - 2A(G)$, where J_n is the n by n matrix with entries 1 in every position.

The rank of the matrix $S(G)$ denoted by $\mathrm{rank}(S(G))$ is equal to the maximum number of linearly independent columns of $S(G)$. The multiplicity of the eigenvalue zero of $A(G)$ is called the nullity of G denoted by $\eta(G)$.

Let $\mu_1(G), \dots, \mu_n(G)$ be the Seidel eigenvalues of G, namely the roots of $\det(\mu I - S(G))$, arranged in non-increasing order. The multiset of distinct Seidel eigenvalues of G composes the Seidel spectrum of G and we denote it by $\mathrm{Spec}_S(G)$. If G has exactly s distinct Seidel eigenvalues $\mu_1(G), \dots, \mu_s(G)$ with multiplicities $t_1, \dots, t_s$,

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respectively, then we write $\mathrm{Spec}_S(G) = \{[\mu_1(G)]^{t_1}, \dots, [\mu_s(G)]^{t_s}\}$. We encourage the interested readers to consult papers [7,9] for more information about the mathematical properties of this matrix.

A Seidel switching of graph G can be constructed as follows. Let $V(G) = U_1 \cup U_2$ be a partition of vertices of G and G' be a graph obtained from G by removing all edges between U_1 and U_2 and adding all edges between them not presented in G. We say that G' is a Seidel switching of G with respect to U_1 and in this case G' and G are Seidel co-spectral, see [8]. Two graphs G and G' are called switching equivalent, if G' is constructed by a sequence of Seidel switching from G.

The Figure 1 contains the class of graphs of order n, $2 \leq n \leq 6$, and their Seidel switching together with their Seidel spectra, see [13]. For example, in Figure 1 three switching equivalent classes of all graphs of order 4 are presented.

We proceed as follows. In the rest of this section, further definition are given and known results needed are stated. In Section 2, we provide some preparatory results. Section 3 contains the main results of this paper. In other words, in this section, we give the characterization of some graphs in $\mathcal{G}(n, n-i)$ for $i = 1, 2, 3$ in terms of their Seidel eigenvalues.

The complement of graph G is denoted by $\overline{G}$. Also, the complete graph, cycle graph and path graph on n vertices are denoted by K_n, C_n and P_n, respectively. A complete bipartite graph with a bipartition of sizes a and b is denoted by $K_{a,b}$, where $a + b = n$.

A graph obtained by removing a perfect matching from $K_{a,b}$ is denoted by $K_{a,b}^-$.

The union of two disjoint graphs G and H is denoted by $G \cup H$. The join $G + H$ is the graph obtained from $G \cup H$ by connecting all vertices from $V(G)$ with all vertices from $V(H)$.

The graph $G + e$ is a new graph obtained from G by adding an edge e.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ be two arbitrary matrices. A new $mp \times nq$ product matrix constructed from A by replacing each element a_{ij} with the block $a_{ij}B$ is called as Kronecker product or Tensor product of them and we denote it by $A \otimes B$.

2. Auxiliary Results

Lemma 2.1 ([3]). *For any graph G with n vertices, where $n \geq 2$, we have*

- **i)** $\sum_{i=1}^{n} \mu_i(G) = 0$;
- **ii)** $\sum_{i=1}^{n} \mu_i^2(G) = n(n-1)$.

Lemma 2.2 ([3]). *If G is a graph on n vertices, then* $\mathrm{rank}(S(G)) = n - 1$ *or n.*

Theorem 2.1 (Interlacing Theorem, [3]). *Let G be a graph of order n with induced subgraph H of order m. Let $\mu_1(G) \geq \cdots \geq \mu_n(G)$ and $\mu_1(H) \geq \cdots \geq \mu_m(H)$ be eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, we have $\mu_i(G) \geq \mu_i(H) \geq \mu_{n-m+i}(G)$.*

Let $\mathcal{G}(n, r)$ be the set of a graphs on n vertices which has a single Seidel eigenvalue with multiplicity r. Here, we give the characterization of some graphs in $\mathcal{G}(n, n-i)$ for $i = 1, 2, 3$ in terms of their Seidel eigenvalues.

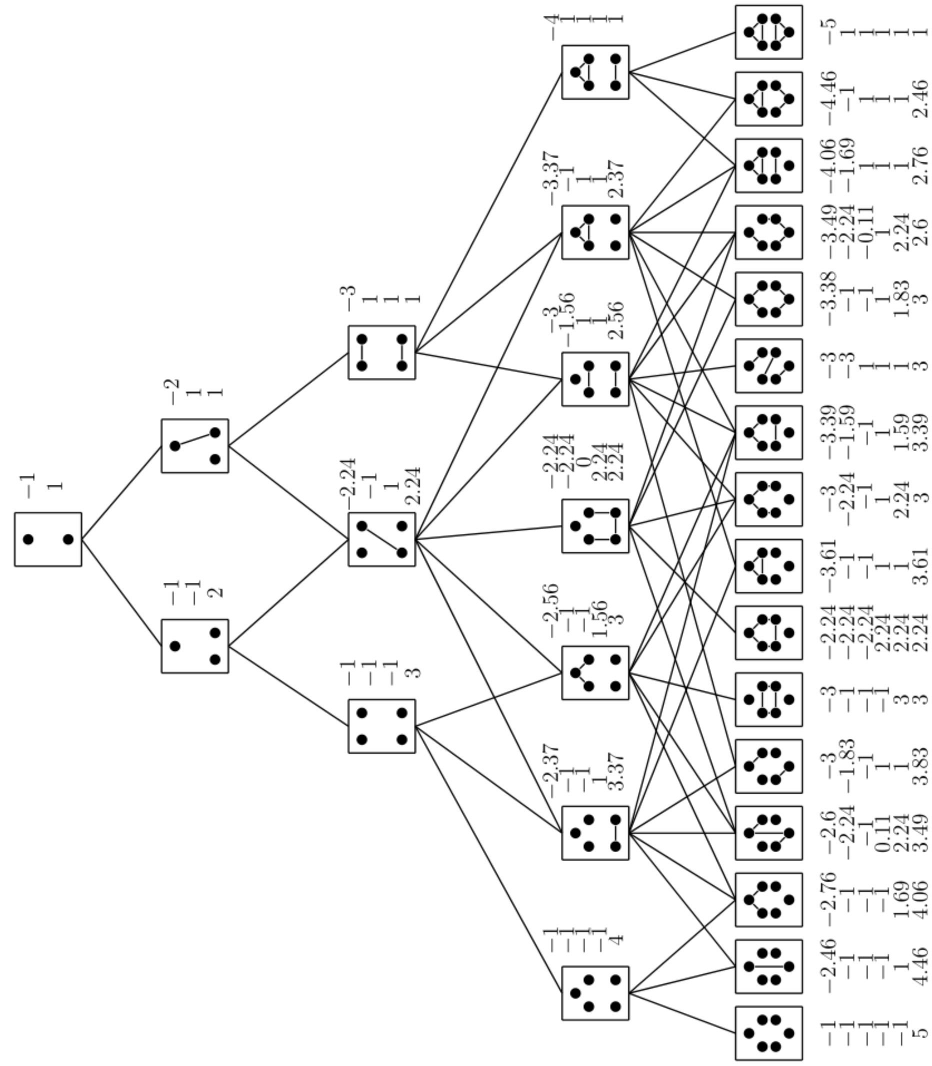

FIGURE 1. Each graph in above diagram is a representative of a class of graphs of order n $(2 \leq n \leq 6)$ together with their switching equivalent graphs which have the same Seidel spectra. Also, all Seidel eigenvalues are written in the right hand side of each graph.

Theorem 2.2. *A graph of order $n \geq 2$ has exactly one positive Seidel eigenvalue if and only if it is a complete bipartite graph or an empty graph.*

Proof. Let G be a graph of order n. If $G \neq K_{n_1,n_2}$, where $n_1 + n_2 = n$ and $n \geq 2$, then we have

$$\mathrm{Spec}_S(K_{n_1,n_2}) = \{[-1]^{n-1}, [n-1]^1\},$$

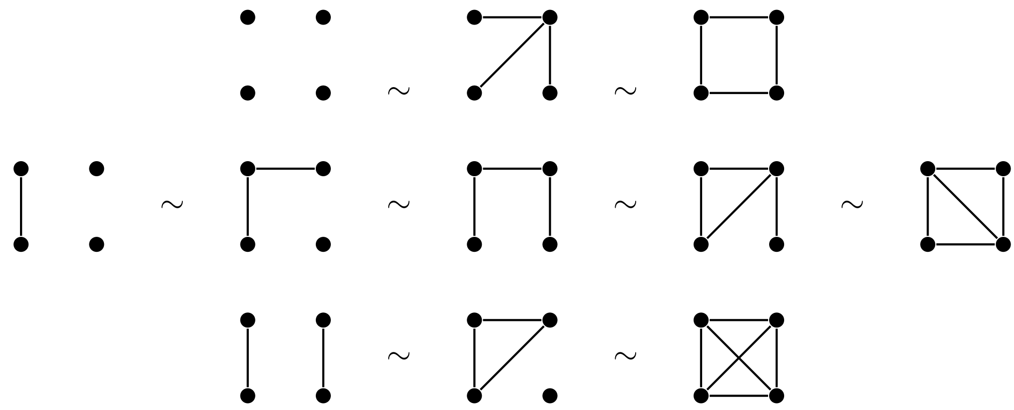

FIGURE 2. Three switching equivalent classes of graphs of order 4

thus $\mu_2(G) = -1$. Now suppose that G is connected graph, where $\mu_2(G) < 0$ and $G \neq K_{n_1,n_2}$. Hence, either K_3 or P_4 are as an induced subgraph of G. Since

$$\mathrm{Spec}_S(K_3) = \{[-2]^1, [1]^2\}$$

and

$$\mathrm{Spec}_S(P_4) = \{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\},$$

the interlacing theorem yields that $\mu_2(G) \geq \mu_2(K_3) = 1$ or $\mu_2(G) \geq \mu_2(P_4) = 1$, a contradiction. If G is a disconnected graph with exactly one positive Seidel eigenvalue, then G is Seidel equivalent to a connected graph (e.g., by letting U_1 be the vertex set of a component), thus G is Seidel equivalent to a complete bipartite graph (by the first part of the proof) and consequently to an empty graph. □

Corollary 2.1. *If $G \neq K_{n_1,n_2}$, $n_1 + n_2 = n$, is a connected graph with at least two vertices, then $\mu_2 \geq 1$.*

Corollary 2.2. *A connected graph G has exactly two positive Seidel eigenvalues if and only if it has K_3 or P_4 as an induced subgraph.*

Theorem 2.3. *A graph of order $n \geq 3$ has exactly one negative Seidel eigenvalue if and only if it is a complete graph or it is isomorphic with $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$.*

Proof. By regarding $S(\overline{G}) = -S(G)$, one can see that if G has exactly one negative Seidel eigenvalue then $\overline{G}$ has exactly one positive Seidel eigenvalue. By Theorem 2.2 the proof is complete. □

Corollary 2.3. *If $G \neq K_n$ is a connected graph with at least three vertices, then $\mu_{n-1}(G) \leq -1$.*

Corollary 2.4. *The connected graph G has exactly two negative Seidel eigenvalues if and only if it has graph P_3 as an induced subgraph.*

3. Main Results

The main goal of this paper is to classify some classes of graphs $G \in \mathcal{G}(n, n-i)$ for $i = 1, 2, 3$. For $i = 1, 2$ the problem is completely solved but for $i = 3$, in the case that $\eta(G) = 0$, we are done. But if $\eta > 0$, we characterized the graphs in special cases. At first, suppose G is a graph with a single eigenvalue with multiplicity $n-1$ or $n-2$. The following result can be obtained.

Theorem 3.1.

(i) *For $n \geq 2$, $\mathcal{G}(n, n-1) = \{K_n, \overline{K}_n, K_{n_1,n_2}, K_{n_1} \cup K_{n_2}\}$, where $n_1 + n_2 = n$.*

(ii) *For $n \geq 3$, $\mathcal{G}(n, n-2) = \{K_3, \overline{K}_3, P_3, K_2 \cup K_1\}$.*

Proof. **(i)** If $G \in \mathcal{G}(n, n-1)$, then G has exactly two distinct Seidel eigenvalues and so G has one single positive or one single negative Seidel eigenvalue. By Theorems 2.2 and 2.3, G is Seidel equivalent to one of graphs $K_n, \overline{K}_n, K_{n_1,n_2}$ or $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$. This completes the proof of the first claim.

(ii) If $G \in \mathcal{G}(n, n-2)$, then G has at most three distinct Seidel eigenvalues and thus we can consider the following cases.

Case 1. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-2}, [\beta]^2\}$, where $\alpha \neq \beta$ are two real numbers.

Subcase 1. If $\beta < 0 < \alpha$, then by Lemma 2.1, we obtain

$$\alpha = \frac{1}{n-2}\sqrt{2(n-1)(n-2)} \quad \text{and} \quad \beta = -\frac{1}{2}\sqrt{2(n-1)(n-2)}. \tag{3.1}$$

Suppose G is a graph of order greater than 2. If K_3 or $K_2 \cup K_1$ is an induced subgraph of G, then by interlacing theorem we have $\alpha = 1$ and $\beta \leq -2$. Hence (3.1) implies that $n = 0$, a contradiction. If $\overline{K}_3$ or P_3 is an induced subgraph of G, then interlacing theorem yields that $\alpha = 2$ and $\beta \leq -1$. Thus, (3.1) implies that $n = 3$ and so G is Seidel equivalent to one of graphs $\overline{K}_3$ or P_3.

Subcase 2. If $\alpha < 0 < \beta$, then a similar argument shows that G is isomorphic to one of graphs K_3 or $K_2 \cup K_1$.

Case 2. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-2}, [\beta]^1, [\gamma]^1\}$ and α, β, γ are distinct Seidel eigenvalues. Lemma 2.2 implies that the multiplicity of the Seidel eigenvalue zero is at most 1. If $[0]^1 \in \mathrm{Spec}_S(G)$, then G has a single positive or a single negative Seidel eigenvalue and by Theorem 2.2 and 2.3 we conclude that G is Seidel equivalent to one of graphs K_n, $\overline{K}_n$, K_{n_1,n_2} or $K_{n_1} \cup K_{n_2}$, where $n_1 + n_2 = n$, both of them are contradictions. By a similar argument, the cases $\beta < 0 < \alpha < \gamma$ and $\gamma < \alpha < 0 < \beta$ and $\beta < 0 < \gamma < \alpha$ and $\alpha < \gamma < 0 < \beta$ are impossible. Also, if either $\alpha < 0 < \beta < \gamma$ or $\gamma < \beta < 0 < \alpha$, then G is Seidel equivalent to one of graphs K_3, $K_2 \cup K_1$, $\overline{K}_3$ or P_3, all of which are impossible, and we are done. $\square$

For the graph G in $\mathcal{G}(n, n-3)$, we know that G has at most four distinct Seidel eigenvalues. In terms of the number and multiplicity of Seidel eigenvalues, we can divide all graphs in $\mathcal{G}(n, n-3)$ into three classes:

$$\mathcal{G}_1(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^3\}\Big\},$$

$$\mathcal{G}_2(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^2, [\gamma]^1\}\Big\},$$
$$\mathcal{G}_3(n, n-3) = \Big\{G \in \mathcal{G}(n, n-3) \;\Big|\; \mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^1, [\gamma]^1, [\rho]^1\}\Big\}.$$

Theorem 3.2 ([6]). *Let G be a graph of order n. Let $d \geq 1$ and $S(G)$ be a Seidel matrix of order $n \geq 2$ with smallest eigenvalue $\mu_n(G)$ of multiplicity $n-d \geq 1$ and suppose $\mu_n^2(G) \geq d+2$. Then*

$$n \leq \frac{d(\mu_n^2(G) - 1)}{\mu_n^2(G) - d},$$

with equality holds if and only if the spectrum of $S(G)$ is $\Big\{[\mu_n(G)]^{n-d}, [\frac{\mu_n(G)}{d}(n-d)]^d\Big\}$.

Example 3.1. Suppose $G \in \mathcal{G}(5,2)$ and $\mathrm{rank}(S(G)) = 4$. Then by Figure 1, $\mathcal{G}(5,2) = \{G_1, G_2, C_5, P_4 \cup K_1\}$, where G_1 and G_2 are as depicted in Figure 3. Furthermore, their Seidel spectra are $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^2, [0]^1, [\sqrt{5}]^2\}$.

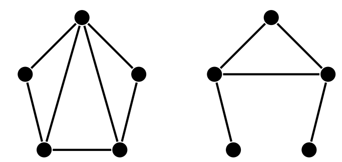

FIGURE 3. Two graphs G_1 and G_2 in Theorem 3.3

Theorem 3.3. *Let $G \in \mathcal{G}(n, n-3)$ be a graph of order $n \geq 6$ and* $\mathrm{rank}(S(G)) = n-1$. *Then $\mathcal{G}(n, n-3)$ is empty.*

Proof. If $\mathrm{rank}(S(G)) = n-1$, then $[0]^1 \in \mathrm{Spec}_S(G)$. Hence, we have the following cases.

Case 1. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [0]^1, [\beta]^2\}$, $\alpha \neq \beta \neq 0$. If $\alpha < \beta$, then by Lemma 2.1, obtain

$$\alpha = -\frac{1}{n-3}\sqrt{2n(n-3)} \quad \text{and} \quad \beta = \frac{1}{2}\sqrt{2n(n-3)}. \tag{3.2}$$

Suppose G is a graph of order at least 6 and contains one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ as an induced subgraph. First, notice that

$$\mathrm{Spec}_S(K_4) = \mathrm{Spec}_S(K_2 \cup K_2) = \mathrm{Spec}_S(K_1 \cup K_3) = \{[-3]^1, [1]^3\}.$$

Hence, interlacing theorem, yields that $\alpha = -3$, $\beta \geq 1$ and $1 \leq 0$, a contradiction. Suppose $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Since

$$\mathrm{Spec}_S(\overline{K}_4) = \mathrm{Spec}_S(C_4) = \mathrm{Spec}_S(K_{1,3}) = \{[-1]^3, [3]^1\},$$

the interlacing theorem implies that $\alpha = -1$ and $\beta \geq 3$. Hence, by (3.2), we find $n = -3$ which contradicts this fact that $n \geq 6$. If there is no graph with either Seidel eigenvalues $\{[-1]^3, [3]^1\}$ or $\{[-3]^1, [1]^3\}$ as an induced subgraph of G, then, by Figure 1, every induced subgraph on 4 vertices has the Seidel spectrum $\{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\}$. Hence, the interlacing theorem implies that $\alpha = -\sqrt{5}$, $\beta \geq \sqrt{5}$ and so by (3.2) we obtain $n = 5$ and $\beta = \sqrt{5}$, a contradiction. Now, suppose $\alpha > \beta$. By a similar argument, we can show that this case is also impossible.

Case 2. $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [0]^1, [\beta]^1, [\gamma]^1\}$, where α, β, γ are three distinct non-zero real numbers. Suppose that G is not a graph with a single negative (positive) Seidel eigenvalue. Then we yield $n \geq 6$ and the following cases hold.

Subcase 2.1. If $\alpha < 0 < \beta < \gamma$, then we can suppose either K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G. The interlacing theorem yields that $\alpha = -3$ and $\gamma > \beta \geq 1$, a contradiction. Also, we may assume one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Again, interlacing theorem implies that $\alpha = -1$, $\beta > 0$ and $\gamma \geq 3$. Thus by Lemma 2.1 we find $\beta = \gamma = \frac{1}{2}(n - 3 + \sqrt{n^2 + 2n - 3})$, a contradiction. This means that P_4, $C_4 + e$, $K_3 + e$, $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G. Then interlacing theorem implies that $\alpha = -\sqrt{5}$, $\beta \geq 1$ and $\gamma \geq \sqrt{5}$. Hence, by Lemma 2.1, we obtain $\beta = \gamma = \frac{1}{2}(\sqrt{5}(n-3) + \sqrt{-3n^2 + 18n - 15})$ which contradicts this fact that $\beta < \gamma$.

Subcase 2.2. Let $\beta < \gamma < 0 < \alpha$. Since $S(\overline{G}) = -S(G)$ a similar argument with Subcase 2.1 shows that this case is also impossible. This completes the proof. $\square$

Theorem 3.4. *Let $G \in \mathcal{G}_1(n, n-3)$ be a graph of order $n \geq 4$. Then*

$$\mathcal{G}_1(n, n-3) = \{K_4, \overline{K}_4, C_4, K_{1,3}, K_2 \cup K_2, K_3 \cup K_1, C_5 \cup K_1, H_1, H_2, H_3\},$$

where H_i, $1 \leq i \leq 3$, are as depicted in Figure 4.

Proof. Let $\mathrm{Spec}_S(G) = \{[\alpha]^{n-3}, [\beta]^3\}$. If $\alpha < \beta$, then by Lemma 2.1, we get

$$\alpha = \frac{-1}{n-3}\sqrt{3(n-1)(n-3)} \quad \text{and} \quad \beta = \frac{1}{3}\sqrt{3(n-1)(n-3)}. \tag{3.3}$$

Similar to the Theorem 3.3, we can show that one of graphs K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G and thus $\alpha = -3$ and $\beta \geq 1$. Hence, (3.3) implies that $n = 4$, $\beta = 1$ and G has either K_4, $K_2 \cup K_2$ or $K_1 \cup K_3$ as an induced subgraph of G. If $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then we have $\alpha = -1$, $\beta \geq 3$ and so by (3.3), we find $n = 0$ or $n = 3$, a contradiction with $n \geq 4$. If G has one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ as an induced subgraph, by interlacing theorem, we conclude that $\alpha = -\sqrt{5}$, $\beta \geq \sqrt{5}$ and (3.3) yields $n = 6$. Hence, $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$. By Figure 1, G is Seidel equivalent to one of graphs $C_5 \cup K_1$, H_1, H_2 or H_3. Next suppose that $\beta < \alpha$. It is not difficult to see that G is Seidel equivalent to one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ or $C_5 \cup K_1$ or H_1 or H_2 or H_3. This completes the proof. $\square$

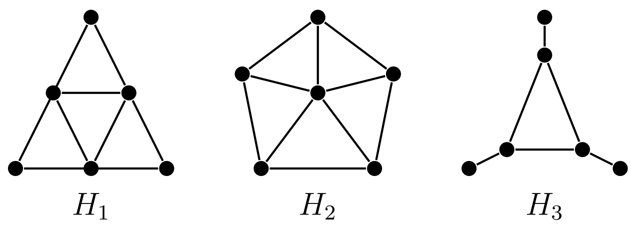

FIGURE 4. Three graphs H_1, H_2 and H_3 in Theorem 3.4

Theorem 3.5. *There is no graph in $\mathcal{G}_2(n, n-3)$ of order $n \geq 4$ with Seidel spectrum $\{[\alpha]^{n-3}, [\beta]^2, [\gamma]^1\}$, where α, β and γ satisfy in the following conditions:*

(i) $\gamma < 0 < \alpha < \beta$ *or* $\gamma < 0 < \beta < \alpha$ *or* $\beta < \alpha < 0 < \gamma$ *or* $\alpha < \beta < 0 < \gamma$;
(ii) $\alpha < 0 < \beta < \gamma$ *or* $\gamma < \beta < 0 < \alpha$;
(iii) $\beta < 0 < \gamma < \alpha$ *or* $\alpha < \gamma < 0 < \beta$;
(iv) $\beta < \gamma < 0 < \alpha$ *or* $\alpha < 0 < \gamma < \beta$.

Proof. **(i)** If G has a single positive or a single negative Seidel eigenvalue with multiplicity 1, then Theorems 2.2 and 2.3 yield that $\mathcal{G}_2(n, n-3)$ is empty.

(ii) Suppose that $\alpha < 0 < \beta < \gamma$ and $n \geq 5$ (if $n = 4$ then G has only one negative Seidel eigenvalue and $\mathcal{G}_2(n, n-3)$ is empty). If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then by interlacing theorem, we get $\alpha = -3$ and $\gamma > \beta \geq 1$. Thus Lemma 2.1 implies that

$$\begin{cases} -3(n-3) + 2\beta + \gamma = 0, \\ 9(n-3) + 2\beta^2 + \gamma^2 = n(n-1). \end{cases}$$

Consequently, $\gamma = 3(n-3) - 2\beta$ and so $\beta = \frac{1}{3}(3n - 3 \pm \sqrt{-3n(n-4)})$. Thus, $-3n(n-4) \geq 0$ if and only if $n = 4$. This means that $\beta = 1$ and $\gamma = 1$, a contradiction. Now, suppose one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G. Thus, $\alpha = -1$, $\beta > 0$ and $\gamma \geq 3$. Thus, by Lemma 2.1, we find $\gamma = n - 1$ and $\beta = -1$, a contradiction. If one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G, again one can prove that $\alpha = -\sqrt{5}$, $\beta \geq 1$ and $\gamma \geq \sqrt{5}$. Hence, Theorem 3.2 implies that $n \leq 6$. There is no graph with these conditions and thus in this case $\mathcal{G}_2(n, n-3)$ is empty.

By a similar argument, we can show that in all cases **(ii)**-**(iv)**, $\mathcal{G}_2(n, n-3)$ is empty and the proof is complete. $\square$

Theorem 3.6 ([3]). *Let G be a k-regular graph of order n. Then the Seidel spectrum of G is $\{[n-1-2k]^1, [-1-2\lambda_{n-1}]^1, \ldots, [-1-2\lambda_1]^1\}$, where λ_i $(1 \leq i \leq n)$ are eigenvalues of adjacency matrix $A(G)$.*

Theorem 3.7 ([4]). *Suppose that G is a graph of order n without isolated vertices. Then $\eta(G) = n - 3$ if and only if G is isomorphic to the complete tripartite graph K_{n_1,n_2,n_3}, where $n_1 + n_2 + n_3 = n$, $n_1, n_2, n_3 > 0$.*

In continuing by $\mathcal{Q}_n(4,2)$ we mean the collection of all connected regular graphs of order n with spectrum $\{[\lambda_1]^1, [\lambda_2]^1, [\lambda_3]^{t_1}, [\lambda_4]^{t_2}\}$, $t_1 + t_2 = n - 2$. Also, $\mathcal{Q}_n(4,2,-1)$ (resp. $\mathcal{Q}_n(4,2,0)$) denotes the set of all graphs in $\mathcal{Q}_n(4,2)$, in which -1 (resp. 0) is an eigenvalue.

Let G be a graph of order n and adjacency matrix A. By $G \circledast J_m$ we mean a new graph obtained from G by replacing every vertex of G with a clique K_m and two such cliques are adjacent (namely for two cliques Q_1 and Q_2 all vertices of Q_1 are adjacent with all vertices of Q_2) if and only if their corresponding vertices are joined in G, see [11]. One can see that the adjacency matrix of $G \circledast J_m$ is $A \circledast J_m = (A + I_n) \otimes J_m - I_{nm}$.

Theorem 3.8 ([11])**.** *The connected regular graph G is in $\mathcal{Q}_n(4,2,0)$ if and only if $G = \overline{K^-_{s,s} \circledast J_t}$, where $n = 2st$, $s \geq 3$ and $t \geq 1$.*

Theorem 3.9 ([11])**.** *The connected regular graph G is in $\mathcal{Q}_n(4,2,-1)$ if and only if $G = K_{s,s} \circledast J_t$, where $s, t \geq 2$, or $G = K^-_{s,s} \circledast J_t$, where $n = 2st$, $s \geq 3$ and $t \geq 1$.*

Theorem 3.10 ([11])**.** *There is no connected k-regular graph of order $n \geq 4$ with adjacency spectrum $\{[k]^1, [\lambda_2]^1, [\lambda_3]^1, [\lambda_4]^{n-3}\}$.*

Theorem 3.11. *Let $G \in \mathcal{G}_2(n, n-3)$ be a connected regular graph of order $n \geq 4$. Then the following cases hold.*

- **(i)** *If $\gamma < \alpha < \beta$, then G is isomorphic to the one of graphs $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$, $n \equiv 0 \pmod 3$ or $\overline{K^-_{3,3} \circledast J_{\frac{n}{6}}}$, $n \equiv 0 \pmod 6$.*
- **(ii)** *If $\beta < \alpha < \gamma$, then G is isomorphic to $K^-_{3,3} \circledast J_{\frac{n}{6}}$, $n \equiv 0 \pmod 6$.*

Proof. **(i)** Let G be a graph of order n. By Theorem 3.5, we can assume that $\gamma < \alpha < 0 < \beta$. If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then by interlacing theorem we get $\alpha = 1$, a contradiction. If G has one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ as an induced subgraph of G, then we obtain $\gamma \leq -\sqrt{5}$, $\alpha = -1$, $\beta \geq \sqrt{5}$ and so by Lemma 2.1, we get $\beta = \frac{2n}{3} - 1$ and $\gamma = -\frac{n}{3} - 1$. As well as, if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, Lemma 2.1 implies that $\beta = \frac{2n}{3} - 1$ and $\gamma = -\frac{n}{3} - 1$. Therefore,

$$\mathrm{Spec}_S(G) = \left\{ \left[\frac{-n}{3} - 1 \right]^1, [-1]^{n-3}, \left[\frac{2n}{3} - 1 \right]^2 \right\},$$

where $n \equiv 0 \pmod 3$. By Theorem 3.6, the adjacency spectrum of G is

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{3} \right]^2, [0]^{n-3}, \left[\frac{2n}{3} \right]^1 \right\}$$

or

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{3} \right]^2, [0]^{n-4}, \left[\frac{n}{6} \right]^1, \left[\frac{n}{2} \right]^1 \right\}.$$

Suppose $\mathrm{Spec}(G) = \{[\frac{-n}{3}]^2, [0]^{n-3}, [\frac{2n}{3}]^1\}$, since $\eta(G) = n - 3$, by Theorem 3.7, G is isomorphic to $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$. If $\mathrm{Spec}(G) = \{[\frac{-n}{3}]^2, [0]^{n-4}, [\frac{n}{6}]^1, [\frac{n}{2}]^1\}$, then Theorem 3.8 implies that G is isomorphic to $\overline{K^-_{3,3} \circledast J_{\frac{n}{6}}}$, where $n \equiv 0 \pmod 6$.

(ii) Assume that $\beta < 0 < \alpha < \gamma$. It is not difficult to see that $\alpha = 1$ and Lemma 2.1 yields that $\gamma = \frac{n}{3} + 1$ and $\beta = 1 - \frac{2n}{3}$. By Theorem 3.6, we obtain

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{6} - 1\right]^1, [-1]^{n-3}, \left[\frac{n}{3} - 1\right]^1, \left[\frac{5n}{6} - 1\right]^1 \right\} \tag{3.4}$$

or

$$\mathrm{Spec}(G) = \left\{ \left[\frac{-n}{6} - 1\right]^1, [-1]^{n-4}, \left[\frac{n}{3} - 1\right]^2, \left[\frac{n}{2} - 1\right]^1 \right\}. \tag{3.5}$$

Theorem 3.10 implies that (3.4) is impossible. If (3.5) holds, then Theorem 3.9 yields that G is isomorphic to the graph $K_{3,3}^{-} \circledast J_{\frac{n}{6}}$, $n \equiv 0 \pmod 6$ and this completes the proof. $\square$

Example 3.2. Suppose $n \equiv 0 \pmod 3$. For two graphs $G_1 = K_{\frac{n}{3},\frac{n}{3}} \cup \overline{K}_{\frac{n}{3}}$ and $G_2 = \overline{K_{\frac{n}{3},\frac{n}{3}} \cup \overline{K}_{\frac{n}{3}}}$, we obtain

$$\mathrm{Spec}_S(G_1) = \left\{ \left[\frac{-n}{3} - 1\right]^1, [-1]^{n-3}, \left[\frac{2n}{3} - 1\right]^2 \right\},$$
$$\mathrm{Spec}_S(G_2) = \left\{ \left[\frac{-2n}{3} + 1\right]^2, [1]^{n-3}, \left[\frac{n}{3} + 1\right]^1 \right\}.$$

This implies that both graphs G_1 and G_2 are in $\mathcal{G}_2(n, n-3)$.

Example 3.3. Suppose $n = 6$. By using a program in SageMath software [12], we conclude that all graphs in $\mathcal{G}_2(6,3)$ are as depicted in Figures 5 and 6.

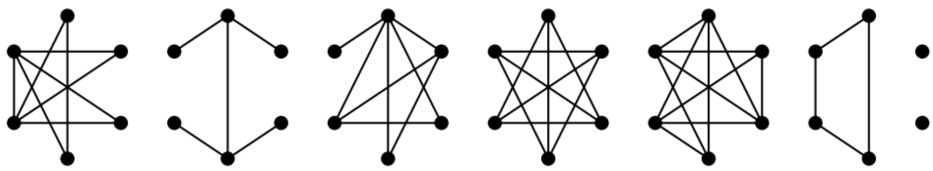

FIGURE 5. All graphs in $\mathcal{G}_2(6,3)$ with Seidel spectrum $\{[3]^2, [-1]^3, [-3]^1\}$

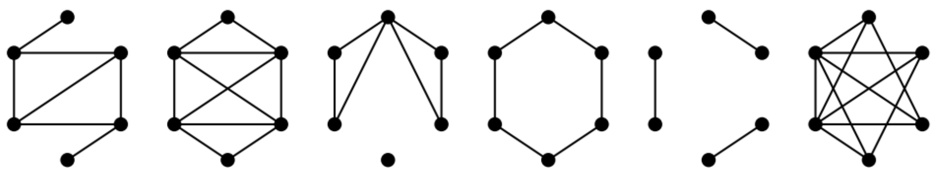

FIGURE 6. All graphs in $\mathcal{G}_2(6,3)$ with Seidel spectrum $\{[3]^1, [1]^3, [-3]^2\}$

In what follows, by mG we mean the disjoint union of m copies of G, namely $\underbrace{G \cup \cdots \cup G}_{m \text{ times}}$.

Theorem 3.12 ([3]). **(i)** *A graph G with the smallest Seidel eigenvalue larger than -3 is switching equivalent to graphs $\overline{K}_n$ or $K_2 \cup \overline{K}_{n-2}$ or one of graphs depicted in Figure 7.*

(ii) *A graph G with smallest Seidel eigenvalue greater than or equal with -3 is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$ (namely the complement of the line graph of K_8).*

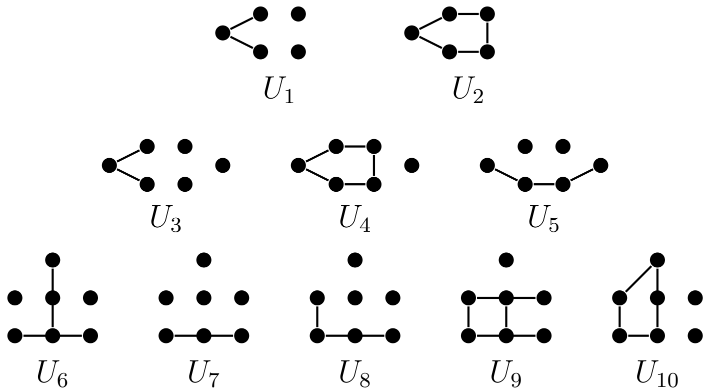

FIGURE 7. Ten graphs with the smallest Seidel eigenvalue larger than -3.

TABLE 1. Graphs together with the Seidel spectra in Theorem 3.12.

Graphs	Seidel spectrum
$\overline{K}_n$	$\{[-1]^{n-1}, [n-1]^1\}$
$K_2 \cup \overline{K}_{n-2}$	$\{[\frac{n}{2} - 2 - \frac{1}{2}\sqrt{(n+6)(n-2)}]^1, [-1]^{n-3}, [1]^1, [\frac{n}{2} - 2 + \frac{1}{2}\sqrt{(n+6)(n-2)}]^1\}$
U_1	$\{[-2.56]^1, [-1]^2, [1.56]^1, [3]^1\}$
U_2	$\{[-\sqrt{5}]^2, [0]^1, [\sqrt{5}]^2\}$
U_3	$\{[-2.75]^1, [-1]^3, [1.69]^1, [4.06]^1\}$
U_3	$\{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$
U_5	$\{[-2.6]^1, [-2.24]^1, [-1], [0.11]^1, [2.24]^1, [3.49]^1\}$
U_6	$\{[-2.78]^1, [-2.46]^1, [-1]^2, [0.29]^1, [2.49]^1, [4.46]^1\}$
U_7	$\{[-2.9]^1, [-1]^4, [1.74]^1, [5.15]^1\}$
U_8	$\{[-2.83]^1, [-2.24]^1, [-1]^2, [0.15]^1, [2.24]^1, [4.68]^1\}$
U_9	$\{[-2.6]^2, [-2]^1, [0.11]^2, [3.49]^2\}$
U_{10}	$\{[-2.7]^1, [-2.24]^1, [-1]^1, [2.24]^2, [3.7]^1\}$
mK_2 $(m \geq 2)$	$\{[-3]^{m-1}, [1]^m, [n-3]^1\}$
$\overline{T(8)}$	$\{[-3]^{21}, [9]^7\}$

Example 3.4. Suppose $G \in \mathcal{G}_3(4,1)$, then we have $\mathcal{G}_3(4,1) = \{K_2 \cup \overline{K}_2, P_3 \cup K_1, K_3 + e, P_4, C_4 + e\}$ and $\mathrm{Spec}_S(G) = \{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\}$.

Theorem 3.13. *Let $G \in \mathcal{G}_3(n, n-3)$ be a graph of order $n \geq 5$. Then the following cases hold.*

(i) *There is no graph in $\mathcal{G}_3(n, n-3)$ which satisfies in the following conditions:*

$\alpha < \beta < \gamma < 0 < \rho$ *or* $\beta < \alpha < \gamma < 0 < \rho$ *or* $\beta < \gamma < \alpha < 0 < \rho$ *or*
$\rho < 0 < \gamma < \beta < \alpha$ *or* $\rho < 0 < \gamma < \alpha < \beta$ *or* $\rho < 0 < \alpha < \gamma < \beta$ *or*
$\alpha < \beta < 0 < \gamma < \rho$ *or* $\rho < \gamma < 0 < \beta < \alpha$.

(ii) *If $\alpha < 0 < \beta < \gamma < \rho$ or $\rho < \gamma < \beta < 0 < \alpha$, then G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$.*

Proof. **(i)** If G has a single positive or a single negative Seidel eigenvalue with multiplicity 1, then by Theorems 2.2 and 2.3, $\mathcal{G}_3(n, n-3)$ is empty. Now, suppose $\alpha < \beta < 0 < \gamma < \rho$ and $n \geq 5$. If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then by interlacing theorem, we get $\alpha = -3$ and $\beta \geq 1$, a contradiction. If G has one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ as an induced subgraph, then we yield $\alpha = -\sqrt{5}$, $-1 \leq \beta < 0$, $\gamma \geq 1$ and $\rho \geq \sqrt{5}$. As well as, if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then we obtain $\alpha = -1$, $-1 \leq \beta < 0$, $\gamma > 0$ and $\rho \geq 3$. Since, $\alpha > -3$, applying Theorem 3.12 (i) and Table 1, we achieve a contradiction. By a similar argument the case $\rho < \gamma < 0 < \beta < \alpha$ is impossible.

(ii) Suppose $\alpha < 0 < \beta < \gamma < \rho$ and $n \geq 5$. If one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ is an induced subgraph of G, then $\alpha = -\sqrt{5}$, $\beta > 0$, $\gamma \geq 1$ and $\rho \geq \sqrt{5}$ and if one of graphs $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G, then $\alpha = -1$, $\beta, \gamma > 0$ and $\rho \geq 3$, a contradiction with $\alpha > -3$. If one of graphs K_4 or $K_2 \cup K_2$ or $K_1 \cup K_3$ is an induced subgraph of G, then interlacing theorem, yields $\alpha = -3$ and $\beta, \gamma, \rho \geq 1$. Theorem 3.12 (ii) implies that G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$. Let $\rho < \gamma < \beta < 0 < \alpha$. Since $S(G) = -S(\overline{G})$, a similar argument shows that in this case G is Seidel equivalent to a subgraph of mK_2, $m \geq 2$, or of $\overline{T(8)}$. □

Example 3.5. Suppose $n = 5$. By Figure using a method described in 1, we conclude that all graphs in $\mathcal{G}_3(5, 2)$, where $\beta < \alpha < 0 < \gamma < \rho$ and $\rho < \gamma < 0 < \alpha < \beta$, are as depicted in Figures 8 and 9, respectively.

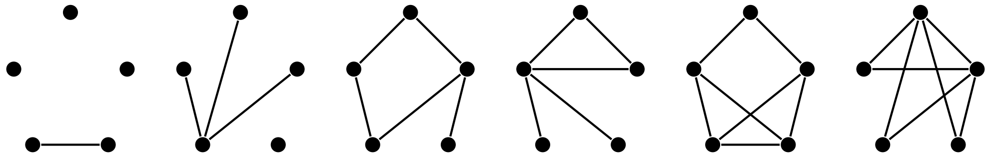

FIGURE 8. All graphs in $\mathcal{G}_3(5, 2)$ with Seidel spectrum $\{[-2.37]^1, [-1]^2, [1]^1, .37]^1\}$

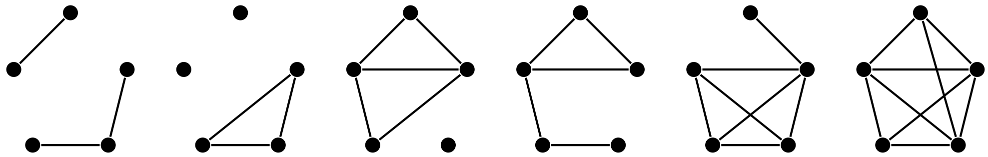

FIGURE 9. All graphs in $\mathcal{G}_3(5,2)$ with Seidel spectrum $\{[-3.37]^1, [-1]^1, [1]^2, [2.37]^1\}$

Conjecture 3.1. Let $G \in \mathcal{G}_3(n, n-3)$ be a graph of order $n \geq 6$. Then the following cases hold:

i) if $\beta < \alpha < 0 < \gamma < \rho$, then G is Seidel equivalent to $K_{i,j} \cup \overline{K}_p$;
ii) if $\rho < \gamma < 0 < \alpha < \beta$, then G is Seidel equivalent to $\overline{K_{i,j} \cup \overline{K}_p}$,

where $1 \leq i \leq [\frac{n}{3}]$, $i \leq j \leq n-3$ and $3 \leq p \leq n-(i+j)$ unless $n \equiv 0 \pmod 3$ and $i = j = p = \frac{n}{3}$.

Remark 3.1. Suppose $G \in \mathcal{G}_3(n, n-3)$ is a graph of order $n \geq 6$. If the Seidel eigenvalues of G are ordered as $\beta < \alpha < 0 < \gamma < \rho$, then it is not difficult to see that one of graphs P_4 or $C_4 + e$ or $K_3 + e$ or $K_2 \cup \overline{K}_2$ or $P_3 \cup K_1$ or $\overline{K}_4$ or C_4 or $K_{1,3}$ is an induced subgraph of G and by interlacing theorem, we have $\alpha = -1$. Also, if the Seidel eigenvalues of G satisfy in $\rho < \gamma < 0 < \alpha < \beta$, by a similar argument we can show that $\alpha = 1$.

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