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BELL GRAPHS ARE DETERMINED BY THEIR LAPLACIAN SPECTRA

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Abstract. A graph G is said to be determined by the spectrum of its Laplacian spectrum (DLS, for short) if every graph with the same spectrum is isomorphic to G. An ∞-graph is a graph consisting of two cycles with just a vertex in common. Consider the coalescence of an ∞-graph and the star graph $K_{1,s}$, with respect to their unique maximum degree. We call this a bell graph. In this paper, we aim to prove that all bell graphs are DLS.

1. Introduction

As usual $G = (V(G), E(G))$ is a simple graph having n vertices and m edges, with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The complement of G is denoted by $\overline{G}$.

The degree sequence of G, denoted by $\deg(G)$, is the sequence of vertex degrees; in fact $\deg(G) = (d_1, d_2, \ldots, d_n)$ in which $d_i = d_i(G) = d_G(v_i)$ for $i = 1, \ldots, n$, is the degree of the vertex v_i so that $d_1 \geq d_2 \geq \cdots \geq d_n$.

Let $A(G)$ and $D(G) = \mathrm{Diag}(d_1, d_2, \ldots, d_n)$ denote the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian matrix* of G is defined as $L(G) = A(G) - D(G)$. The polynomial $\varphi_{L(G)}(x) = \det(x\mathbb{I}_n - L(G))$, where $\mathbb{I}_n$ is the identity matrix of order n, is called the Laplacian characteristic polynomial of G. Any root of $\varphi_{L(G)}(x)$ is called a *Laplacian eigenvalue* of G. The multi-set of Laplacian eigenvalue of G is called the *Laplacian spectrum* or *L-spectrum* of G. Note that $L(G)$ is a symmetric, positive semidefinite matrix, and thus its eigenvalues are all real non-negative numbers. We denote its eigenvalues in the non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

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Although, the spectral graph theory originated with the eigenvalues of the adjacency matrices, but Laplacian matrices have come to have comparable importance.

The coalescence of two graphs G_1 and G_2, with respect to $u_1 \in V(G_1)$ and $u_2 \in V(G_1)$, is the graph obtained by identifying u_1 and u_2 in the disjoint union of G_1 and G_2. We denote it by $(G_1 \circ G_2)(u_1, u_2)$. In the case when it does not make deference which vertex in G_1 and G_2 is identified to obtain a coalescence, we denote this graph by $G_1 \circ G_2$. This operation is extended, inductively, to any arbitrary number of graphs. For example, the coalescence of k arbitrary cycles is called a k-rose graph; in fact, this is a graph with $k \geq 1$ cycles meeting in one vertex. For $i, j \geq 3$, $C_i \circ C_j$ is a 2-rose graph called an ∞-graph.

Van Dam and Haemers [12] conjectured that almost all graphs are determined by their Laplacian spectrum, that is, they are the only graph (up to isomorphism) with that spectrum. However, very few graphs are known to have that property, and so discovering new classes of such graphs is an interesting problem. Formally, we define two graphs G and H to be *L-cospectral* if they have the same L-spectrum, and a graph G is *determined by its Laplacian spectrum*, abbreviated by DLS, if no other graphs are L-cospectral with G. Let us mention some known DLS graphs obtained by coalescence of other DLS graphs:

- Liu et al. [10] proved that any rose graph, each cycle of which is a triangle, is DLS;
- Wang et al. [14] showed that triangle-free 2-rose graphs are almost DLS (notice that not all 2-rose graphs are DLS (see [9]);
- Wang et al. [15] proved that all 3-rose graphs, having at least one triangle, are DLS.

It is known that the Laplacian eigenvalues of a graph give the Laplacian eigenvalues of its complement. Therefore, complement of a DLS graph, is also DLS. Hence, all the complements of the above graphs are DLS.

In the current article, we consider a new graph being coalescence of a 2-rose graph and a star graph with respect to their vertices of maximum degree. In fact, this graph is the coalescence of $C_i \circ C_j$, with the vertex v_1 of maximum degree 4 and the star graph $K_{1,s}$ with the vertex v_2 of maximum degree s. Let us call this graph a *bell graph* and denote it by $BG(C_i, C_j, s)$, $i \leq j$, see Figure 1.

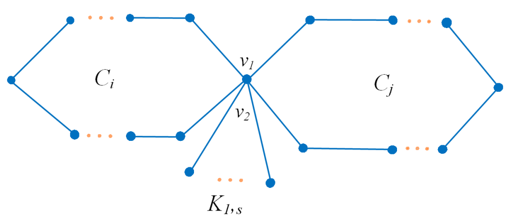

FIGURE 1. The bell graph $BG(C_i, C_j, s)$

In this paper, it is proved that bell graphs and their complements are DLS. The rest of this article is organized as follows: In Section 2, we recall some previously established results playing a crucial role throughout this paper. In Section 3, we fisrt prove that no two non-isomorphic bell graphs are L-cospectral, and then we determine the degree sequence of graphs L-cospectral with the bell graphs. Finally, we obtain all bell graphs are DLS.

2. Preliminaries

In this section, we recall some previously established results playing a crucial role throughout this paper.

Theorem 2.1 ([12,13]). *The following can be obtained from the Laplacian spectrum of a graph:*

(i) *the number of vertices;*
(ii) *the number of edges;*
(iii) *the number of spanning trees;*
(iv) *the number of components;*
(v) *the sum of the squares of the degrees of the vertices.*

Lemma 2.1 ([3]). *For a graph G, we have $\mu_{n-1} \geq 0$ with equality if and only if G is connected.*

Theorem 2.2 ([7]). *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of G and $\overline{G}$, respectively. Then $\overline{\mu}_i = n - \mu_{n-i}$ for $i = 1, 2, \ldots, n-1$.*

For any two graphs G and H, we denote by $\mathcal{N}_G(H)$ and $\mathcal{W}_G(i)$, the number of subgraphs of G being isomorphic to H, and the number of closed walks of length i in G, respectively. Note that the trace of a matrix M is denoted by $\operatorname{tr}(M)$.

Theorem 2.3 ([1,13]). *Suppose G is a graph with m edges. The number of closed walks of lengths 2, 3, and 4 in G can be computed by the following formulas:*

(a) $\mathcal{W}_G(2) = 2m$;
(b) $\mathcal{W}_G(3) = \operatorname{tr}(A^3(G)) = 6\mathcal{N}_G(C_3)$;
(c) $\mathcal{W}_G(4) = 2m + 4\mathcal{N}_G(P_3) + 8\mathcal{N}_G(C_4)$.

Theorem 2.4 ([6]). *If G is a non-empty graph with n vertices, then*

$$\mu_1(G) \geq d_1(G) + 1. \tag{2.1}$$

Furthermore, if G is connected, then the equality in (2.1) holds if and only if $d_1(G) = n - 1$.

A graph G is called *regular* if $d_1(G) = \cdots = d_n(G)$. A bipartite graph is called *semi-regular* if the degrees of vertices in each part, are constant.

The next result uses the quantity $\theta_G(u) = \sum_{v \in N_G(u)} \frac{d_G(v)}{d_G(u)}$, where $N_G(u)$ denotes the set of neighbors of the vertex u in G.

Theorem 2.5 ([13]). *For a connected graph G, we have*

$$\mu_1(G) \leq \max\{d_G(u) + \theta_G(u) \mid u \in V(G)\}. \tag{2.2}$$

Besides, the equality in (2.2) *holds if and only if G is either regular or semi-regular, bipartite graph.*

Theorem 2.6 ([1,8]). *Let G be a non-empty graph. Then $\mu_1(G) \leq d_1(G) + d_2(G)$. Moreover, G is connected only if $\mu_2(G) \geq d_2(G)$.*

Cevetcovic et al. in [2] obtained the first three coefficients of the Laplacian characteristic polynomials, while the forth one, was obtained by Oliveira et al. in [11].

Theorem 2.7 ([2,11]). *Let G be a graph with n vertices and m edges with the degree set* $\deg(G) = (d_1, d_2, \ldots, d_n)$. *Then we have the following: $\varphi_{L(G)}(x) = \sum_{i=0}^{n} l_i(G)x^i$, are obtained as follows:*

$$\begin{aligned} l_0(G) =&1, \quad l_1(G) = -2m, \quad l_2(G) = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2, \\ l_3(G) =&\frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6\mathcal{N}_G(C_3)\right). \end{aligned}$$

As an immediate consequence of Theorem 2.7, we have following result.

Corollary 2.1. *If G and H are L-cospectral graphs such that* $\deg(H) = \deg(G)$, *then they have the same number of triangles, i.e., $\mathcal{N}_G(C_3) = \mathcal{N}_H(C_3)$.*

Let G and H be two L-cospectral graphs. It follows from Theorem 2.1 (i), (ii), (iv), (v) and Theorem 2.7 and Corollary 2.1 that

$$\operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} d_i^3(G) = \operatorname{tr}(A^3(H)) - \sum_{i=1}^{n} d_i^3(H).$$

Based on this, Liu and Huang [9] defined the following invariant for a graph G:

$$\varepsilon(G) = \operatorname{tr}(A^3(G)) - \sum_{i=1}^{n} (d_i(G) - 2)^3.$$

Theorem 2.8 ([13]). *If G and H are L-cospectral, then $\varepsilon(G) = \varepsilon(H)$.*

Theorem 2.9 ([3]). *If u is a vertex of G and $G - u$ is the subgraph obtained from G by deleting u, then $\mu_i(G) \geq \mu_i(G - u) \geq \mu_{i+1}(G) - 1$, $i = 1, 2, \ldots, n - 1$.*

3. Main Results

In this section, we establish bound on the first and the second largest Laplacian eigenvalues of bell graphs.

Lemma 3.1. *For a bell graph G with s pendent vertices, we have*

(i) $5 + s \leq \mu_1(G) < 6 + s$;

(ii) $\mu_2(G) < 5$.

Proof. (i) It follows from Theorems 2.4 and 2.5 that

$$5+s \leq \mu_1(G) \leq 4+s+\frac{4+s+4}{4+s} = 5+s+\frac{4}{4+s} < 6+s.$$

(ii) This is a direct consequence of Theorem 2.9 and this fact that the greatest eigenvalue of a path is less than 4. □

Let G be a connected graph with n vertices and m edges. Then G is called *k-cyclic* if $m = n+k-1$. For a bell graph $G = BG(C_i, C_j, s)$, we have $n = n(G) = (i+j)-1+s$ and $m = m(G) = (i+j)+s$ and so $m = m(G) = n+1 = n+2-1$, implying that G is a 2-cyclic graph.

Lemma 3.2. *If H is L-cospectral with $G = BG(C_i, C_j, s)$, then H is connected, and*

$$\deg(H) = \deg(G) = (s+4, \underbrace{2, \ldots, 2}_{i+j-2 \ times}, \underbrace{1, \ldots, 1}_{s \ times}).$$

Proof. Connectedness of H is clear by Theorem 2.1 (iv) and Lemma 3.1 (iii). Let us determine its degree sequence. By Lemma 3.1, $\mu_2(H) < 5$, and thus, it follows from Theorem 2.6 that $d_2(H) \leq 4$. Since H and G are L-cospectral, by Theorem 2.1, H is also connected, and has the same order, size, and sum of the squares of its degrees as G. Let n_i denote the number of vertices of degree i in H for $i = 1, 2, \ldots, d_1(H)$. Then

$$\sum_{i=1}^{d_1(H)} n_i = n(G) = (i+j)-1+s, \tag{3.1}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(G) = 2((i+j)+s), \tag{3.2}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + d_1^2(G), \tag{3.3}$$

where n_i' is the number of vertices of G of degree i for $i = 1, 2$. By adding up (3.1), (3.2) and (3.3) with coefficients 2, -3, 1, respectively, we get:

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2) n_i = (s+2)(s+3). \tag{3.4}$$

By Lemma 3.1, $5+s \leq \mu_1(G) < 6+s$. From Theorem 2.4 it follows that

$$d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) < 6+s,$$

which leads to $d_1(H) \leq 4+s$. On the other hand, by Lemma 3.1 and Theorem 2.6, one can conclude that

$$5+s \leq \mu_1(G) = \mu_1(H) \leq d_1(H) + d_2(H) \leq d_1(H) + 4,$$

from which we have $d_1(H) \geq s+1$. Therefore, we have $s+1 \leq d_1(H) \leq s+4$. From Theorem 2.8 it follows that

$$6\mathcal{N}_H(C_3) - \sum_{i=1}^{n} (d_i(H)-2)^3 = 6\mathcal{N}_G(C_3) - ((s+2)^3 - s). \tag{3.5}$$

Therefore,

$$\mathcal{N}_H(C_3) = \frac{1}{6}\left(\sum_{i=1}^{n} (d_i(H)-2)^3 + 6\mathcal{N}_G(C_3) - ((s+2)^3 - s)\right). \tag{3.6}$$

We consider the following three main cases.

Case A. $d_1(H) = s+4$. By (3.4) one can deduce that

$$((s+4)^2 - 3(s+4) + 2) + 2n_3 + 6n_4 = (s+2)(s+3), \tag{3.7}$$

from which it follows that $n_3 = 0$. Combining (3.2) and (3.3), we find that $n_1 = s$ and $n_2 = n-(s+1)$. Therefore, $\deg(H) = \deg(G)$. In this case, it follows from (3.6) that $\mathcal{N}_H(C_3) = \mathcal{N}_G(C_3)$. Obviously, $d_1(H) \geq 5 > 4 \geq d_2(H)$ or $n_{s+4} = 1$.

Case B. $s+3 = d_1(H)$. Then $n_{s+3} = 1$. By an argument similar to that of (3.6), we have the following:

$$((s+3)^2 - 3(s+3) + 2) + 2n_3 = (s+2)(s+3), \tag{3.8}$$

By (3.1), (3.2) and (3.4) we get

$$\begin{cases} n_1 = 2s+11, \\ n_2 = -3s+n-20, \\ n_3 = s+8. \end{cases}$$

It follows from (3.6) that $\mathcal{N}_H(C_3) = \frac{6\mathcal{N}_G(C_3)-s^3-6s^2-12s-11}{6}$. Therefore, $\mathcal{N}_H(C_3) < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$. We assume that $n_{s+3} \geq 2$. Then

$$s+3 = d_1(H) = d_2(H) \leq 3,$$

which is a contradiction, since $s \geq 1$.

Case C. $d_1(H) = s+2$. We first assume that $n_{s+2} = 1$. In this case, $s+2 = d_1(H) > 3 \geq d_2(H)$ and as a result $s \geq 2$. From (3.4) and by a straightforward calculation, we get:

$$((s+2)^2 - 3(s+2) + 2) + 2n_3 = (s+2)(s+3). \tag{3.9}$$

By (3.1), (3.2) and (3.4) we get:

$$\begin{cases} n_1 = 3s+1, \\ n_2 = -5s+n-5, \\ n_3 = 2s+3. \end{cases}$$

It follows from (3.6) that

$$\mathcal{N}_H(C_3) = \frac{6\mathcal{N}_G(C_3) - s^3 - 6s^2 - 12s - 10}{6}.$$

Therefore, $\mathcal{N}_H(C_3) < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$. Next we assume that $n_{s+2} \geq 2$. Then $s+2 = d_1(H) = d_2(H) \leq 3$ implying that $s = 1$. By (3.1), (3.2) and (3.4) we get

$$\begin{cases} n_1 = 4, \\ n_2 = n - 10, \\ n_3 = 6. \end{cases}$$

It follows from (3.6) that $\mathcal{N}_H(C_3) = \mathcal{N}_G(C_3) - \frac{14}{3} < 0$, since $0 \leq \mathcal{N}_G(C_3) \leq 2$, which is a contradiction.

Case D. $d_1(H) = s + 1$. By a similar argument, we will have a contradiction. □

In the following, we show that any graph L-cospectral with a bell graph G, is also a bell graph with the same degree sequence as G.

Corollary 3.1. *Let H be a graph L-cospectral with a bell graph $G = BG(C_i, C_j, s)$. Then H is a bell graph with the same degree sequence as G.*

Proof. By Lemma 3.2, $\deg(H) = \deg(G) = (s+4, \underbrace{2, \ldots, 2}_{i+j-2\,\text{times}}, \underbrace{1, \ldots, 1}_{s\,\text{times}})$. So, H has a unique vertex of degree greater than 2, say $d_H(v) = s + 4 > 2$. It is clear that the maximum degree of $H - v$ is most 2, i.e., $d_1(H - v) \leq 2$. Moreover, $H - v$ contains no cycles, otherwise, since it is connected, there would be another vertex of degree greater than 2. Consequently, $H - v$ must be a forest each component of which is a path. Therefore, H consists of exactly 2 cycles intersecting in a single vertex. Hence, H must be a bell graph. □

Before proving our main result, we state some essential lemmas and notations.

Lemma 3.3 ([4])**.** *Let G be a graph with a set of vertices $X = \{u_1, u_2, \ldots, u_k\}$ such that*

$$N_G(u_1) = N_G(u_2) = \cdots = N_G(u_k) = \{w_1, w_2, \ldots, w_p\}.$$

If G^* is the graph obtained from G by adding any q, $1 \leq q \leq \frac{k(k-1)}{2}$, edges among $\{u_1, u_2, \ldots, u_k\}$, then the eigenvalues of $L(G^*)$ are as follows: those eigenvalues of $L(G)$ which are equal to p are incremented by $\lambda_i(G^*[X])$, $i = 1, 2, \ldots, k-1$, and the remaining eigenvalues are the same.

Lemma 3.4 ([5])**.** *No two non-isomorphic starlike trees are L-cospectral.*

Suppose that $H = BG(C_i, C_j, s)$ is a bell graph, and let v be the vertex of H such that $d_H(v) = s + 4$. Now, we remove an arbitrary edge not being adjacent to v, from cycle C_i and C_j. Then we obtain a starlike tree, say, $S(H)$. Hereafter, $S(H) = (s, l_1, l_2)$ means $S(H) - v = P_{l_1} \cup P_{l_2} \cup \overline{K_s}$ such that $l_1 + l_2 = (i+j) - 1$.

Note that in the proof of Lemma 3.4, it was shown that if $S_1 = S(l_1, \ldots, l_t)$ and $S_2 = S(j_1, \ldots, j_t)$ are two non-isomorphic starlike trees, then $\mu_1(S_1) \neq \mu_1(S_2)$, where $l_1 \geq l_2 \geq \cdots \geq l_t \geq 1$ and $j_1 \geq j_2 \geq \cdots \geq j_t \geq 1$.

Corollary 3.2. *If $S(G) = (s, l_1, l_2)$ and $S(H) = (s, j_1, j_2)$ are two non-isomorphic starlike trees, then $\mu_1(S(G)) \neq \mu_1(S(H))$.*

Now we express our main result.

Theorem 3.1. *Bell graphs are determined by their Laplacian spectrum.*

Proof. Let H be a graph L-cospectral with a bell graph $G = BG(C_{t_1}, C_{t_2}, s)$. It follows from Corollary 3.1 that H is also a bell graph with the same degree sequence as G. Assuming that $H = BG(C_{k_1}, C_{k_2}, s)$ we need to prove that $\{t_1, t_2\} = \{k_1, k_2\}$. To do so, consider the corresponding starlike trees $S(G) = (s, l_1, l_2)$ and $S(H) = (s, j_1, j_2)$. We claim that H and G are isomorphic, otherwise, $\mu_1(S(G)) \neq \mu_1(S(H))$ and so $\mu_1(G) \neq \mu_1(H)$, contradicting Lemma 3.2. $\square$

From Theorem 2.2, it follows that the Laplacian eigenvalues of a graph give the Laplacian eigenvalues of its complement. Therefore, the complement of a DLS graph, is also DLS. Hence, the following fact is immediately follows from Theorem 3.1.

Corollary 3.3. *The complements of bell graphs are also DLS.*

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