Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 245–257.

GENERALIZED MIXED TYPE BERNOULLI-GEGENBAUER POLYNOMIALS

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Abstract. The generalized mixed type Bernoulli-Gegenbauer polynomials of order $\alpha > -\frac{1}{2}$ are special polynomials obtained by use of the generating function method. These polynomials represent an interesting mixture between two classes of special functions, namely generalized Bernoulli polynomials and Gegenbauer polynomials. The main purpose of this paper is to discuss some of their algebraic and analytic properties.

1. Introduction

Bernoulli and Gegenbauer polynomials are among classical families of algebraic polynomials whose history goes back centuries. Each one of these polynomials, as well as their natural generalizations, have showed their useful in several disciplines [1–3, 6–9, 16, 17, 19–21, 23–25, 27–29]. In this paper we shall be concerned with the some of the main properties of the generalized mixed type Bernoulli-Gegenbauer polynomials $\mathscr{V}_n^{(\alpha)}(x)$ of order $\alpha \in (-1/2, \infty)$, $n \geq 0$ (GBG polynomials, in short). This is a special family of polynomials defined through the generating functions and series expansions as follows:

$$\left[\frac{z}{(e^z - 1)\left(1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}\right)}\right]^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathscr{V}_n^{(\alpha)}(x) \frac{z^n}{n!}, \tag{1.1}$$

Key words and phrases. Generalized Bernoulli polynomials, Gegenbauer polynomials, GBG polynomials, inversion formula, matrix representations, matrix-inversion formula.

2010 *Mathematics Subject Classification.* Primary: 11B83. Secondary: 11C08, 33B99.

DOI 10.46793/KgJMat2302.245Q

Received: April 22, 2020.

Accepted: August 13, 2020.

where $|z| < 2\pi$, $|x| \leq 1$ and $\alpha \in (-1/2, \infty) \setminus \{0\}$,

$$\left[\frac{2\pi - xz}{1 - \frac{xz}{\pi} + \frac{z^2}{4\pi^2}}\right] e^{xz} = \sum_{n=0}^{\infty} \mathscr{V}_n^{(0)}(x)\frac{z^n}{n!}, \quad |z| < 2\pi,\ |x| \leq 1. \tag{1.2}$$

The polynomials $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ represent an interesting mixture between two classes of special functions, namely generalized Bernoulli polynomials and Gegenbauer polynomials. The separate emergence of these families of polynomials in different fields as such as physical mathematics, information theory, combinatorics, approximation theory, number theory, numerical analysis and partial differential equations and so on, has been a well-known fact and documented [1, 3, 4, 6, 7, 12, 14, 18–20, 27, 28]. However, in recent years new connections between these families of polynomials have been given (see, for instance [2, 9, 29]). The aim of this note is to investigate some properties of the GBG polynomials, focusing our attention on their explicit expressions, derivatives formulas, matrix representations, matrix-inversion formulas, and other relations connecting them with Gegenbauer polynomials.

The paper is organized as follows. In Section 2 some relevant properties of the generalized Bernoulli polynomials and the Gegenbauer polynomials are given. Section 3 contains the main algebraic and analytic properties of the GBG polynomials (see e.g., Proposition 3.1, Lemmas 3.1 and 3.2, and Theorem 3.1), as well as, some illustrative examples.

2. Basic Facts: Generalized Bernoulli Polynomials and Gegenbauer Polynomials

This section is devoted to present some structural properties of the generalized Bernoulli polynomials and Gegenbauer polynomials which will be useful in the sequel. We will begin with the generalized Bernoulli polynomials. As is well known, these polynomials play an important role in the calculus of finite differences since the coefficients in all the usual central-difference formulas for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of them (see e.g., [10] and the references therein).

Recent and interesting works dealing with generalized Bernoulli and Euler polynomials, Appell and Apostol type polynomials, their properties and applications in several areas can be found by reviewing the current literature on this subject. For a broad information on old literature and new research trends about these classes of polynomials we strongly recommend to the interested reader see [8, 10, 13, 14, 16, 17, 20, 21, 24, 25].

From now on, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n.

2.1. Generalized Bernoulli Polynomials.

Generalized Bernoulli Polynomials. The classical Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined as follows (see, for details, [3, 14, 20, 23]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z|<2\pi,\ 1^{\alpha}:=1, \tag{2.1}$$

and

$$B_n(x) := B_n^{(1)}(x), \quad n\in\mathbb{N}_0, \tag{2.2}$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

The numbers $B_n^{(\alpha)} := B_n^{(\alpha)}(0)$ are called generalized Bernoulli numbers of order α, $n\in\mathbb{N}_0$. Clearly, we have

$$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(x-\alpha),$$

so that

$$B_n^{(\alpha)}(\alpha) = (-1)^n B_n^{(\alpha)}. \tag{2.3}$$

From the generating relation (2.1), it is fairly straightforward to deduce the addition formula:

$$B_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{(\alpha)}(x) B_{n-k}^{(\beta)}(y). \tag{2.4}$$

Making the substitution $\beta=0$ into (2.4) and interchanging x and y, we obtain the well known representation:

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k} B_k^{(\alpha)} x^{n-k}. \tag{2.5}$$

The following theorem summarizes some properties of the generalized Bernoulli polynomials.

Theorem 2.1. (a) ([26, (3)]) *Explicit formula for the generalized Bernoulli polynomials in terms of the Gaussian hypergeometric function:*

$$\begin{aligned} B_n^{(\alpha)}(x) \ &= \ \sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} j^{2k}(x+j)^{n-k} \\ &\quad \times {}_2F_1(k-n, k-\alpha; 2k+1; j/(x+j)), \end{aligned} \tag{2.6}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function given by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}, \quad c\notin\{0,-1,-2,\dots\},$$

with $(a)_0=1$, $(a)_n = a(a+1)\cdots(a+n-1)$, $n\in\mathbb{N}$, being the Pochhammer's symbol.

(b) ([26, (13)]) *The substitution $x=0$ into* (2.6) *yields the following representation for the generalized Bernoulli numbers:*

$$B_n^{(\alpha)} = \sum_{k=0}^{n}\binom{\alpha+n}{n-k}\binom{\alpha+k-1}{k}\frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} j^{n+k}. \tag{2.7}$$

The interested reader also may consult [20, 22, 26] for detailed proofs of the above assertions.

In addition to (2.2) classical Bernoulli polynomials $B_n(x)$ admit a variety of different representations. For instance, we recall that the classical Bernoulli polynomials $B_n(x)$ may be inverted in order to give a representation of the monomial basis (cf., [17, Eq. (4)] and the references therein). This resulting representation is commonly called inversion formula:

$$\begin{aligned} x^n &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x) \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B_{n-k}(x), \quad n \geq 0. \end{aligned} \tag{2.8}$$

Consequently, the set $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ is a basis for $\mathbb{P}_n$.

In the next lemma we show an inversion formula for a subfamily of generalized Bernoulli polynomials.

Lemma 2.1. *For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{(m)}(x)\right\}_{n\geq 0}$ be the sequence of generalized Bernoulli polynomials of order m. Then we have*

$$x^n = \frac{1}{(n+1)_m}\sum_{r=0}^{n}\binom{n+m}{r+m}a_r(m)B_{n-r}^{(m)}(x), \quad n \geq 0, \tag{2.9}$$

where the coefficients $a_r(m)$ are given by

$$a_r(m) = \sum_{k_1=0}^{r}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}}\binom{r+m}{k_1+m-1}\binom{k_1+m-1}{k_2+m-2}\cdots\binom{k_{m-2}+2}{k_{m-1}+1}, \quad r = 0, \ldots, n.$$

Proof. From (2.1) it follows that

$$z^m e^{xz} = (e^z-1)^m\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!}. \tag{2.10}$$

It is not difficult to show by repeated application of the Cauchy product of series that

$$(e^z-1)^m = \sum_{n=0}^{\infty}a_n(m)\frac{z^{n+m}}{(n+m)!},$$

where

$$a_n(m) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}}\binom{n+m}{k_1+m-1}\binom{k_1+m-1}{k_2+m-2}\cdots\binom{k_{m-2}+2}{k_{m-1}+1}.$$

Thus, the right-hand side of (2.10) becomes

$$\begin{aligned} (e^z-1)^m\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!} &= \left[\sum_{n=0}^{\infty}a_n(m)\frac{z^{n+m}}{(n+m)!}\right]\left[\sum_{n=0}^{\infty}B_n^{(m)}(x)\frac{z^n}{n!}\right] \\ &= \sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n+m}{r+m}a_r(m)B_{n-r}^{(m)}(x)\right]\frac{z^{n+m}}{(n+m)!}. \end{aligned} \tag{2.11}$$

Likewise, the left-hand side of (2.10) can be expressed by use of the Cauchy product of series as follows

$$z^m e^{xz} = \sum_{n=0}^{\infty} x^n \frac{z^{n+m}}{n!} = \sum_{n=0}^{\infty} (n+1)_m \, x^n \frac{z^{n+m}}{(n+m)!}. \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\sum_{n=0}^{\infty} (n+1)_m \, x^n \frac{z^{n+m}}{(n+m)!} = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n+m}{r+m} a_r(m) B_{n-r}^{(m)}(x) \right] \frac{z^{n+m}}{(n+m)!}, \tag{2.13}$$

and comparing the coefficients on both sides of (2.13), we get the desired inversion formula (2.9). □

As a straightforward consequence of the inversion formula (2.9) we obtain an expected algebraic property.

Corollary 2.1. *For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\left\{ B_0^{(m)}(x), \ldots, B_n^{(m)}(x) \right\}$ is a basis for $\mathbb{P}_n$, i.e.,*

$$\mathbb{P}_n = \operatorname{span} \left\{ B_0^{(m)}(x), B_1^{(m)}(x), \ldots, B_n^{(m)}(x) \right\}.$$

2.2. **Gegenbauer polynomials.** For $\alpha > -\frac{1}{2}$ we denote by $\{\hat{C}_n^{(\alpha)}\}_{n \geq 0}$ the sequence of Gegenbauer polynomials, orthogonal on $[-1, 1]$ with respect to the measure $d\mu(x) = (1 - x^2)^{\alpha - \frac{1}{2}} \, dx$ (cf., [27, Chapter IV]), normalized by

$$\hat{C}_n^{(\alpha)}(1) = \frac{\Gamma(n + 2\alpha)}{n! \Gamma(2\alpha)}.$$

More precisely,

$$\int_{-1}^{1} \hat{C}_n^{(\alpha)}(x) \hat{C}_m^{(\alpha)}(x) \, d\mu(x) = \int_{-1}^{1} \hat{C}_n^{(\alpha)}(x) \hat{C}_m^{(\alpha)}(x) (1 - x^2)^{\alpha - \frac{1}{2}} \, dx = M_n^{\alpha} \, \delta_{n,m}, \quad n, m \geq 0,$$

where the constant M_n^{α} is positive. It is clear that the normalization above does not allow α to be zero or a negative integer. Nevertheless, the following limits exist for every $x \in [-1, 1]$ (see [27, (4.7.8)])

$$\lim_{\alpha \to 0} \hat{C}_0^{(\alpha)}(x) = T_0(x), \quad \lim_{\alpha \to 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n} T_n(x),$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence $\{\hat{C}_n^{(0)}(x)\}_{n \geq 0}$ as follows

$$\hat{C}_0^{(0)}(1) = 1, \quad \hat{C}_n^{(0)}(1) = \frac{2}{n}, \quad \hat{C}_n^{(0)}(x) = \frac{2}{n} T_n(x), \quad n \geq 1.$$

We denote the nth monic Gegenbauer orthogonal polynomial by

$$C_n^{(\alpha)}(x) = (k_n^{\alpha})^{-1} \hat{C}_n^{(\alpha)}(x),$$

where the constant k_n^α (cf., [27, formula (4.7.31)]) is given by

$$\begin{aligned} k_n^\alpha &= \frac{2^n\Gamma(n+\alpha)}{n!\Gamma(\alpha)}, \quad \alpha \neq 0, \\ k_n^0 &= \lim_{\alpha\to 0}\frac{k_n^\alpha}{\alpha} = \frac{2^n}{n}, \quad n \geq 1. \end{aligned}$$

Then for $n \geq 1$, we have $C_n^{(0)}(x) = \lim_{\alpha\to 0}(k_n^\alpha)^{-1}\hat{C}_n^{(\alpha)}(x) = \frac{1}{2^{n-1}}T_n(x)$.

It is well known that the Gegenbauer polynomials are closely connected with axially symmetric potentials in n dimensions and contain the Legendre and Chebyshev polynomials as special cases [6, 7]. Furthermore, they inherit practically all the formulae known in the classical theory of Legendre polynomials.

Proposition 2.1. ([15, cf., Proposition 2.1]) *Let $\{C_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.*

(*a*) *Three-term recurrence relation.*

$$xC_n^{(\alpha)}(x) = C_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)}C_{n-1}^{(\alpha)}(x), \quad \alpha > -\frac{1}{2}, \alpha \neq 0, \tag{2.14}$$

with initial conditions $C_0^{(\alpha)}(x) = 1$, $C_1^{(\alpha)}(x) = x$ and recurrence coefficient $\gamma_n^{(\alpha)} = \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}$.

(*b*) *For every $n \in \mathbb{N}$ (see [27, (4.7.15)])*

$$h_n^\alpha := \|C_n^{(\alpha)}\|_\mu^2 = \int_{-1}^1 [C_n^{(\alpha)}(x)]^2 d\mu(x) = \pi 2^{1-2\alpha-2n}\frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}. \tag{2.15}$$

(*c*) *Rodrigues formula.*

$$(1-x^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(x) = \frac{(-1)^n\Gamma(n+2\alpha)}{\Gamma(2n+2\alpha)}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-\frac{1}{2}}\right], \quad x \in (-1,1).$$

(*d*) *Structure relation (see [27, (4.7.29)]). For every $n \geq 2$*

$$C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) + \xi_{n-2}^{(\alpha)}C_{n-2}^{(\alpha)}(x),$$

where

$$\xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, \quad n \geq 0.$$

(*e*) *For every $n \in \mathbb{N}$ (see [27, formula (4.7.14)])*

$$\frac{d}{dx}C_n^{(\alpha)}(x) = nC_{n-1}^{(\alpha+1)}(x).$$

As is well known the monic Gegenbauer orthogonal polynomials admit other different definitions [1, 4, 27, 28]. In order to deal with the definitions (1.1) and (1.2) of the

GBG polynomials, we also are interested in the definition of the monic Gegenbauer orthogonal polynomials by means of the following generating functions:

(2.16)
$$\left(1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}\right)^{-\alpha}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x)\frac{z^n}{n!},\quad |z|<2\pi, |x|\le 1, \alpha\in(-1/2,\infty)\setminus\{0\},$$

and

(2.17)
$$\frac{2\pi-xz}{1-\frac{xz}{\pi}+\frac{z^2}{4\pi^2}}=\sum_{n=0}^{\infty}\frac{1}{\pi^{n-1}}C_n^{(0)}(x)z^n=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\pi^{n-1}}C_n^{(0)}(x)\frac{z^n}{n!},\quad |z|<2\pi,\ |x|\le 1.$$

Remark 2.1. Note that (2.16) and (2.17) are suitable modifications of the generating functions for the Gegenbauer polynomials $\hat{C}_n^{(\alpha)}(x)$:

$$\begin{aligned}\left(1-2xz+z^2\right)^{-\alpha} &= \sum_{n=0}^{\infty}\hat{C}_n^{(\alpha)}(x)z^n,\quad |z|<1,\ |x|\le 1,\ \alpha\in(-1/2,\infty)\setminus\{0\},\\ \frac{1-xz}{1-xz+z^2} &= 1+\sum_{n=1}^{\infty}\frac{n}{2}\hat{C}_n^{(0)}(x)z^n,\quad |z|<1,\ |x|\le 1.\end{aligned}$$

3. Some Algebraic and Analytic Properties of the GBG Polynomials

Now we are in a position to investigate some properties of the GBG polynomials as follows.

Proposition 3.1. *For $\alpha\in(-1/2,\infty)$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\ge 0}$ be the sequence of GBG polynomials of order α. Then the following explicit formulas hold.*

$$\mathscr{V}_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(k+\alpha)}{\pi^k\Gamma(\alpha)}C_k^{(\alpha)}(x)B_{n-k}^{(\alpha)}(x),\quad n\ge 0,\ \alpha\ne 0, \tag{3.1}$$

$$\begin{aligned}\mathscr{V}_n^{(0)}(x) &= \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}C_k^{(0)}(x)B_{n-k}^{(0)}(x)\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{\pi^{k-1}}C_k^{(0)}(x)x^{n-k},\quad n\ge 0.\end{aligned} \tag{3.2}$$

Proof. On account of the generating functions (1.1) and (2.16), it suffices the appropriate use of Cauchy product of series in order to deduce the expression (3.1).

Similarly, taking into account the generating functions (1.2) and (2.17), we can use an analogous reasoning to the previous one for getting the expression (3.2). □

Thus, the suitable use of (2.3), (2.5), (2.7), (2.14) and (3.1) allow us to check that for $\alpha \in (-1/2, \infty) \setminus \{0\}$ the first four GBG polynomials are:

$$\begin{aligned}
\mathscr{V}_0^{(\alpha)}(x) =& 1,\\
\mathscr{V}_1^{(\alpha)}(x) =& \left(1+\frac{\alpha}{\pi}\right)x-\frac{\alpha}{2},\\
\mathscr{V}_2^{(\alpha)}(x) =& \left(1+\frac{2\alpha}{\pi}+\frac{(\alpha+1)\alpha}{\pi^2}\right)x^2-\left(\alpha+\frac{\alpha^2}{\pi}\right)x+\frac{\alpha(3\alpha-1)}{12}-\frac{\alpha}{2\pi^2},\\
\mathscr{V}_3^{(\alpha)}(x) =& \left(\sum_{k=0}^{3}\binom{3}{k}\frac{(\alpha)_k}{\pi^k}\right)x^3-\frac{3\alpha}{2}\left(\sum_{k=0}^{2}\binom{2}{k}\frac{(\alpha)_k}{\pi^k}\right)x^2\\
&+\left(\frac{(3\alpha-1)\alpha}{4}+\frac{(3\alpha-1)\alpha^2}{4\pi}-\frac{3\alpha}{2\pi^2}-\frac{3(\alpha+1)\alpha}{2\pi^3}\right)x+\frac{\alpha^2(1-\alpha)}{8}+\frac{3\alpha^2}{4\pi^2}.
\end{aligned}$$

It is worth pointing out that the left hand side of (1.1) can be expressed as $G^{(\alpha)}(z)(1-xg(z))^{-\alpha}e^{xz}$, where

$$G^{(\alpha)}(z)=\left[\frac{4\pi^2 z}{(e^z-1)(z^2+4\pi^2)}\right]^{\alpha} \quad \text{and} \quad g(z)=\frac{2\pi z}{z^2+4\pi^2},$$

hence the polynomials $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 1}$ are not generalized Appell polynomials (cf., [5, Chapters I, III]). Also, in contrast to the generalized Bernoulli polynomials and Gegenbauer polynomials, the GBG polynomials neither satisfy a Hanh condition nor an Appell condition. More precisely, we have the following result.

Lemma 3.1. *For $\alpha \in (-1/2, \infty) \setminus \{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then we have*

$$\frac{d}{dx}\mathscr{V}_{n+1}^{(\alpha)}(x)=(n+1)!\sum_{k=0}^{n}\frac{\mathscr{V}_k^{(\alpha)}(x)}{k!}A_{n-k}^{(\alpha)}(x), \quad n\geq 0, \tag{3.3}$$

where

$$A_n^{(\alpha)}(x)=\begin{cases}1+\dfrac{\alpha}{\pi}, & n=0,\\[2mm] \dfrac{\alpha}{\pi^{n+1}}C_n^{(1)}(x), & n\geq 1.\end{cases}$$

Proof. The identity (3.3) it is a straightforward consequence of (1.1) and (2.16). □

Also, it is possible to obtain some integral relations between the GBG polynomials and monic Gegenbauer polynomials.

Lemma 3.2. *For $\alpha \in (-1/2, \infty) \setminus \{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then the following formula holds.*

$$\int_{-1}^{1}\mathscr{V}_n^{(\alpha)}(x)C_n^{(\alpha)}(x)d\mu(x)=\frac{n!\Gamma(n+2\alpha)}{\pi^{2\alpha+2n}\Gamma(n+\alpha+1)\Gamma(n+\alpha)}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k}{\pi^{k-1}}, \tag{3.4}$$

whenever $n\geq 0$.

Proof. In order to obtain (3.4) it suffices to use the orthogonality of the monic Gegenbauer polynomials, (2.5), (2.15) and (3.1). $\square$

Finally, from a matrix framework we can use the expression (3.1) in order to obtain a matrix form of $\mathscr{V}_r^{(\alpha)}(x)$, $r = 0, 1, \ldots, n$, as follows.

The expression (3.1) yields

$$\mathscr{V}_r^{(\alpha)}(x) = \mathbf{C}_r^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x), \tag{3.5}$$

where

$$\mathbf{C}_r^{(\alpha)}(x) = \left[\binom{r}{r}\frac{\Gamma(r+\alpha)}{\pi^r\Gamma(\alpha)}C_r^{(\alpha)}(x) \quad \binom{r}{r-1}\frac{\Gamma(r-1+\alpha)}{\pi^{r-1}\Gamma(\alpha)}C_{r-1}^{(\alpha)}(x) \quad \cdots \quad C_0^{(\alpha)}(x) \quad 0 \quad \cdots \quad 0\right],$$

the null entries of the matrix $\mathbf{C}_r^{(\alpha)}(x)$ appear $(n-r)$-times and the matrix $\mathbf{B}^{(\alpha)}(x)$ is given by $\mathbf{B}^{(\alpha)}(x) = \left(B_0^{(\alpha)}(x) \quad B_1^{(\alpha)}(x) \quad \cdots \quad B_r^{(\alpha)}(x) \quad \cdots \quad B_n^{(\alpha)}(x)\right)^T$.

Then, by (3.5) the matrix $\mathbf{V}^{(\alpha)}(x) = \left(\mathscr{V}_0^{(\alpha)}(x) \quad \mathscr{V}_1^{(\alpha)}(x) \quad \cdots \quad \mathscr{V}_n^{(\alpha)}(x)\right)^T$, can be expressed as follows:

$$\mathbf{V}^{(\alpha)}(x) \quad = \quad \mathbf{C}^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x), \tag{3.6}$$

where $\mathbf{C}^{(\alpha)}(x)$ is the following $(n+1)\times(n+1)$ matrix

$$\mathbf{C}^{(\alpha)}(x) = \begin{bmatrix} C_0^{(\alpha)}(x) & 0 & 0 & \cdots & 0 \\ \binom{1}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & C_0^{(\alpha)}(x) & 0 & \cdots & 0 \\ \binom{2}{2}\frac{\Gamma(2+\alpha)}{\pi^2\Gamma(\alpha)}C_2^{(\alpha)}(x) & \binom{2}{1}\frac{\Gamma(1+\alpha)}{\pi\Gamma(\alpha)}C_1^{(\alpha)}(x) & C_0^{(\alpha)}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}C_n^{(\alpha)}(x) & \binom{n}{n-1}\frac{\Gamma(n-1+\alpha)}{\pi^{n-1}\Gamma(\alpha)}C_{n-1}^{(\alpha)}(x) & \binom{n}{n-2}\frac{\Gamma(n-2+\alpha)}{\pi^{n-2}\Gamma(\alpha)}C_{n-2}^{(\alpha)}(x) & \cdots & C_0^{(\alpha)}(x) \end{bmatrix}.$$

The following theorem summarizes the ideas described above.

Theorem 3.1. *For $\alpha \in (-1/2, \infty)\setminus\{0\}$, let $\left\{\mathscr{V}_n^{(\alpha)}(x)\right\}_{n\geq 0}$ be the sequence of GBG polynomials of order α. Then, the matrix $\mathbf{V}^{(\alpha)}(x) = \left(\mathscr{V}_0^{(\alpha)}(x) \quad \cdots \quad \mathscr{V}_n^{(\alpha)}(x)\right)^T$ has the following matrix form:*

$$\mathbf{V}^{(\alpha)}(x) \quad = \quad \mathbf{C}^{(\alpha)}(x)\mathbf{B}^{(\alpha)}(x).$$

Remark 3.1. Note that according to (3.5) the rows of the matrix $\mathbf{C}^{(\alpha)}(x)$ are precisely the matrices $\mathbf{C}_r^{(\alpha)}(x)$ for $r = 0, \ldots, n$. Furthermore, the matrix $\mathbf{C}^{(\alpha)}(x)$ is an $(n+1)\times(n+1)$ lower triangular matrix for each $x \in \mathbb{R}$, so that

$$\det\left(\mathbf{C}^{(\alpha)}(x)\right) = \left(C_0^{(\alpha)}(x)\right)^{n+1} = (1)^{n+1} = 1.$$

Therefore, $\mathbf{C}^{(\alpha)}(x)$ is an invertible matrix for each $x \in \mathbb{R}$.

The following example shows how Theorem 3.1 can be used.

Example 3.1. Let us consider $n = 3$ and $\alpha = 1$. From (2.14), (3.1), (3.6) and a standard computation we obtain

$$\mathbf{B}(x) := \mathbf{B}^{(1)}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{\pi} & 1 & 0 & 0 \\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0 \\ \frac{6x^3-3x}{\pi^3} & \frac{3(4x^2-1)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1} \mathbf{V}^{(1)}(x), \tag{3.7}$$

where

$$\mathbf{V}^{(1)}(x) = \begin{bmatrix} 1 \\ \left(1+\frac{1}{\pi}\right)x - \frac{1}{2} \\ \left(1+\frac{2}{\pi}+\frac{2}{\pi^2}\right)x^2 - \left(1+\frac{1}{\pi}\right)x + \frac{1}{6} - \frac{1}{2\pi^2} \\ \left(1+\frac{3}{\pi}+\frac{6}{\pi^2}+\frac{6}{\pi^3}\right)x^3 - \frac{3}{2}\left(1+\frac{2}{\pi}+\frac{2}{\pi^2}\right)x^2 + \frac{1}{2}\left(1+\frac{1}{\pi}-\frac{3}{\pi^2}-\frac{6}{\pi^3}\right)x + \frac{3}{4\pi^2} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{\pi} & 1 & 0 & 0 \\ \frac{4x^2-1}{2\pi^2} & \frac{2x}{\pi} & 1 & 0 \\ \frac{6x^3-3x}{\pi^3} & \frac{3(4x^2-1)}{2\pi^2} & \frac{3x}{\pi} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{x}{\pi} & 1 & 0 & 0 \\ \frac{1}{2\pi^2} & -\frac{2x}{\pi} & 1 & 0 \\ 0 & \frac{3}{2\pi^2} & -\frac{3x}{\pi} & 1 \end{bmatrix},$$

then (3.7) becomes

$$\mathbf{B}(x) = \begin{bmatrix} 1 \\ x - \frac{1}{2} \\ x^2 - x + \frac{1}{6} \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{bmatrix}.$$

That is the entries of the matrix $\mathbf{B}(x)$ are the first four classical Bernoulli polynomials (2.2).

Another interesting algebraic property of the GBG polynomials is related to the inversion formula satisfied by the classical Bernoulli polynomials (2.8). The following example shows the inversion formula for the GBG polynomials $\mathscr{V}_n(x) := \mathscr{V}_n^{(1)}(x)$, $n \geq 0$.

Example 3.2. Making the substitution $\alpha = 1$ into (2.5), we obtain the well known representation:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Then the matrix $\mathbf{B}(x)$ can be expressed as follows (cf., [17, (8)]):

$$\mathbf{B}(x) \;=\; \mathbf{M}\mathbf{T}(x),$$

where

$$\mathbf{M} = \begin{bmatrix} B_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{1}B_1 & \binom{1}{0}B_0 & 0 & 0 & \cdots & 0 \\ \binom{2}{2}B_2 & \binom{2}{1}B_1 & \binom{2}{0}B_0 & 0 & \cdots & 0 \\ \binom{3}{3}B_3 & \binom{3}{2}B_2 & \binom{3}{1}B_1 & \binom{3}{0}B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}B_n & \binom{n}{n-1}B_{n-1} & \binom{n}{n-2}B_{n-2} & \binom{n}{n-3}B_{n-3} & \cdots & \binom{n}{0}B_0 \end{bmatrix},$$

and $\mathbf{T}(x) = \begin{pmatrix} 1 & x & \cdots & x^n \end{pmatrix}^T$. It is clear that $\det(\mathbf{M}) = (B_0)^{n+1} = (1)^{n+1} = 1$. So, $\mathbf{M}$ is an invertible matrix.

Making the substitution $\alpha = 1$ into (3.6), we get the matrix representation:

$$\mathbf{V}(x) := \mathbf{V}^{(1)}(x) = \mathbf{C}^{(1)}(x)\mathbf{B}(x) = \mathbf{C}^{(1)}(x)\mathbf{M}\mathbf{T}(x).$$

It follows that

$$\mathbf{T}(x) = \left[\mathbf{C}^{(1)}(x)\mathbf{M}\right]^{-1}\mathbf{V}(x) = \mathbf{M}^{-1}\left(\mathbf{C}^{(1)}(x)\right)^{-1}\mathbf{V}(x).$$

On the account of (2.8), we can deduce the following matrix equation

$$\mathbf{T}(x) = \mathbf{Q}\mathbf{B}(x), \tag{3.8}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{2!}{3!} & \frac{2!}{2!} & 1 & 0 & \cdots & 0 \\ \frac{3!}{4!} & \frac{3!}{3!} & \frac{3}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n!}{(n+1)!} & \frac{n!}{n!} & \frac{n!}{2!(n-1)!} & \frac{n!}{3!(n-2)!} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & 1 & 1 & 0 & \cdots & 0 \\ \frac{1}{4} & 1 & \frac{3}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & 1 & \frac{n}{2} & \frac{n(n-1)}{6} & \cdots & 1 \end{bmatrix}.$$

Notice that $\mathbf{M}^{-1} = \mathbf{Q}$. Consequently, from (3.8) we deduce a matrix-inversion formula for $\mathbf{V}(\mathbf{x})$ as follows

$$\mathbf{T}(x) \;=\; \mathbf{Q}\mathbf{B}(x) = \mathbf{Q}\left(\mathbf{C}^{(1)}(x)\right)^{-1}\mathbf{V}(x). \tag{3.9}$$

Also, the matrix identity (3.9) allows us to conclude that the set $\{\mathscr{V}_0(x), \ldots, \mathscr{V}_n(x)\}$ is a basis for $\mathbb{P}_n$, i.e.,

$$\mathbb{P}_n = \operatorname{span}\left\{\mathscr{V}_0(x), \mathscr{V}_1(x), \ldots, \mathscr{V}_n(x)\right\}.$$

Remark 3.2. In view of (2.9) it is possible to deduce a matrix-inversion formula for $\mathbf{B}^{(m)}(x)$ as follows

$$\mathbf{T}(x) \quad = \quad \mathbf{Q}^{(m)}\mathbf{B}^{(m)}(x),$$

where $\mathbf{Q}^{(m)}$ is an $(n+1)\times(n+1)$ lower triangular and invertible matrix, for $m\in\mathbb{N}$ fixed.

Applying Theorem 3.1 (or equivalently, making the substitution $\alpha = m$ into (3.6)) we obtain the following matrix-inversion formula for $\mathbf{V}^{(m)}(x)$

$$\mathbf{T}(x) \quad = \quad \mathbf{Q}^{(m)}\left[\mathbf{C}^{(m)}(x)\right]^{-1}\mathbf{V}^{(m)}(x).$$

Finally, we leave to the reader the formulation of the analogous identities for the GBG polynomials $\mathscr{V}_n^{(0)}(x)$, $n\geq 0$.

Acknowledgements. The author would like to thank the reviewers for their valuable comments and advice. This research was partially supported by Decanato de Investigación y Desarrollo, Universidad Simón Bolívar, Venezuela, grant DID-USB (S1-IC-CB-004-17).

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