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COMPOSITIONS OF COSPECTRALITY GRAPHS OF SMITH GRAPHS

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Abstract. Graphs whose spectrum belongs to the interval $[-2, 2]$ are called Smith graphs. Vertices of the cospectrality graph $C(H)$ of a Smith graph H are all graphs cospectral with H with two vertices adjacent if there exists a certain transformation transforming one to another. We study how the cospectrality graph of the union of two Smith graphs can be composed starting from cospectrality graphs of starting graphs.

1. Introduction

In this section we present standard basic facts on graph spectra and on Smith graphs.

Let G be a graph with n vertices and adjacency matrix A. The characteristic polynomial $\det(xI - A)$ of A is also called the *characteristic polynomial* of G. The eigenvalues and the spectrum of A (which consists of n eigenvalues) are called the *eigenvalues* and the *spectrum* of G, respectively. Since A is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1, \ldots, \lambda_n$. In particular, λ_1, as the largest eigenvalue of G, will be called the *spectral radius* (or *index*) of G. For general information on spectra of graphs see, for example, [2].

The spectrum of G (as a family of reals) will be denoted by $\widehat{G}$. The disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while the union of their spectra (i.e., the spectrum of $G_1 + G_2$) will be denoted by $\widehat{G}_1 + \widehat{G}_2$. In addition, kG ($k\widehat{G}$) stands for the union of k copies of G (resp. $\widehat{G}$).

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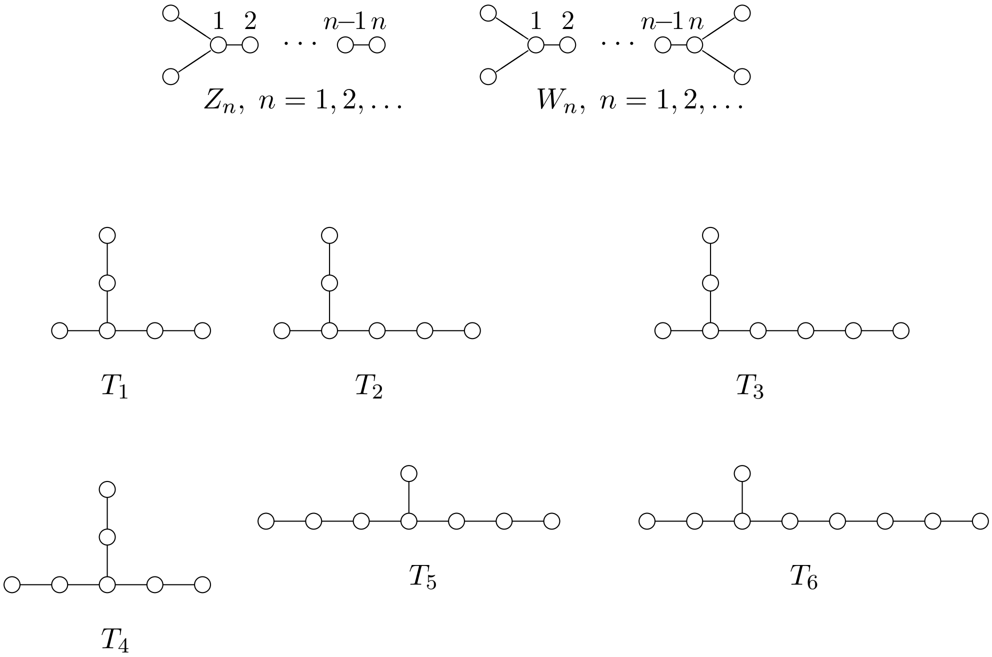

Figure 1. Some of the Smith graphs

We say that two (non-isomorphic) graphs are *cospectral* if their spectra coincide. They are also called *cospectral mates*. On the other hand, we say that a graph is determined by its spectrum if it is a unique graph having this spectrum.

The *cospectral equivalence class* of a graph G is the set of all graphs cospectral to G (including G itself).

We consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle.

All graphs with the spectral radius at most 2 have been constructed by J. H. Smith [5].

A path (cycle) on n vertices will be denoted by P_n (resp. C_n).

A connected graph with index ≤ 2 is either a cycle C_n ($n = 3, 4, \dots$), or a path P_n ($n = 1, 2, \dots$), or one of the graphs depicted in Fig. 1 (see [5]). Note that W_1 coincide with the star $K_{1,4}$, while Z_1 with P_3. In addition, the graphs C_n, W_n, T_4, T_5, and T_6 are connected graphs with index equal to 2. All other graphs, namely, P_n, Z_n, T_1, T_2 and T_3 are the induced subgraphs of these graphs (so the index of each of them is less than 2). The graph Z_n is called a *snake* while W_n is a *double snake*. The trees T_1, T_2, T_3, T_4, T_5, and T_6 will be called *exceptional Smith graphs*.

The spectrum of each of these graphs can be found (in an explicit form) in [3].

A Smith graph has connected Smith graphs as components.

We denote the set of all Smith graphs by $\mathcal{S}^*$. The set of those which are bipartite, so odd cycles are excluded, will be denoted by $\mathcal{S}$.

Let G be any graph each component of which belongs to $\mathcal{S}^*$, we can write

$$G = \sum_{H \in \mathcal{S}^*} r(H)H, \tag{1.1}$$

where $r(H) \geq 0$ is a repetition factor (tells how many times H is appearing as a component in G).

The repetition factor $r(S_i)$ of some of the graph $S_i \in \mathcal{S}^*$ for any relevant index i will be denoted by s_i. So we have non-negative integers

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since $Z_1 = P_3$ and the variable p_3 is relevant. We shall use $c_2, c_3, \ldots,$ for repetition factors of the even cycles $C_4, C_6, \ldots$

For non-bipartite graphs from $\mathcal{S}^*$ we have to introduce variables $o_3, o_5, o_7, \ldots$ counting the numbers of odd cycles $C_3, C_5, C_7, \ldots$

For a given graph $G \in \mathcal{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G. Parameters of a graph indicate the actual number of components of particular types present in G.

The rest of the paper is organized as follows.

Section 2 contains some earlier results on Smith graphs necessary for handling the phenomenon of cospectrality of Smith graphs by means of the so called cospectrality graphs. In Section 3 we present some properties of cospectrality graphs. Section 4 contains description of some compositions of cospectrality graphs. At the end, in Section 5, we describe a computer program for generating cospectral Smith graphs and include some examples of the work of the program.

2. Preliminary Results

Let $H \in \mathcal{S}$. Let

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

be the canonical representation (as defined in [1]) of the spectrum $\widehat{H}$ of a bipartite Smith graph H. Here $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_m$ are integers with $\sigma_0 \geq 0$. This representation always exists and is unique. The expression

$$\sigma_0 C_4 + \sum_{i=1}^{m} \sigma_i P_i,$$

is called *canonical representation* of H. It defines a graph if $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_m$ are non-negative, otherwise it is just a formal expression. In the first case H is cospectral to its canonical representation but not necessarily isomorphic.

If all quantities σ_i are non-negative, the graph H is called a *Smith graph of type* A, otherwise it is *of type* B. Let I (resp. J) be the set of indices i for which σ_i in a graph of type B is negative (resp. positive).

Obviously, cospectral Smith graphs are of the same type.

Let $P_H = \sum_{i \in I} |\sigma_i| P_i$. Components of the graph P_H are paths whose spectra appear with a negative sign in the canonical representation of the spectrum of H. The graph P_H is called the *basis* of H. The basis of a graph of type A is empty. If we add components from its basis to a graph of type B, it becomes a graph of type A.

The graph $K_H = \sigma_0 C_4 + \sum_{i \in J} \sigma_i P_i$ is called the *kernel* of H.

Following [1] we shall consider the corresponding component transformations:

$$
\begin{array}{lccc}
 & (\gamma_1) & W_n \rightleftarrows C_4 + P_n, & (\delta_1) \\
 & (\gamma_2) & Z_n + P_n \rightleftarrows P_{2n+1} + P_1, & (\delta_2) \\
 & (\gamma_3) & C_{2n} + 2P_1 \rightleftarrows C_4 + 2P_{n-1}, n \geq 3 & (\delta_3) \\
 & (\gamma_4) & T_1 + P_5 + P_3 \rightleftarrows P_{11} + P_2 + P_1, & (\delta_4) \\
(2.1) & (\gamma_5) & T_2 + P_8 + P_5 \rightleftarrows P_{17} + P_2 + P_1, & (\delta_5) \\
 & (\gamma_6) & T_3 + P_{14} + P_9 + P_5 \rightleftarrows P_{29} + P_4 + P_2 + P_1, & (\delta_6) \\
 & (\gamma_7) & T_4 + P_1 \rightleftarrows C_4 + 2P_2, & (\delta_7) \\
 & (\gamma_8) & T_5 + P_1 \rightleftarrows C_4 + P_3 + P_2, & (\delta_8) \\
 & (\gamma_9) & T_6 + P_1 \rightleftarrows C_4 + P_4 + P_2. & (\delta_9)
\end{array}
$$

They are of the form $A \to B$ or $B \to A$ meaning that in a graph the group of components A is replaced with the group of components B or vice versa. These transformations are called *G-transformations*. Those of the form $A \to B$ are denoted by $\gamma_1, \gamma_2, \ldots, \gamma_9$ and are called *C-transformations*. For each C-transformation $A \to B$ we define the corresponding opposite transformation $B \to A$, also denoted by $A \leftarrow B$. Transformations $A \leftarrow B$ are called *D-transformations* and are denoted by $\delta_1, \delta_2, \ldots, \delta_9$.

Graphs $C_4, P_1, P_2, \ldots,$ appearing in canonical representations of bipartite Smith graphs, are called *basic* graphs. All other connected bipartite Smith graphs are called *non-basic* graphs. Non-basic graphs are of two types. Graphs W_n $(n = 1, 2, \ldots)$, C_{2k} $(k = 3, 4, \ldots)$ and T_4, T_5, T_6 are non-basic graphs of type I while graphs Z_n $(n = 2, 3, \ldots)$, T_1, T_2, T_3 are non-basic graphs of type II. Note that non-basic graphs of type I have spectral radius equal to 2 while for those of type II spectral radius is less than 2.

G-transformations $\gamma_1, \gamma_2, \gamma_3$ and their opposite transformations $\delta_1, \delta_2, \delta_3$ are not unique since they depend on the index n of the involved non-basic graphs W_n, Z_n, C_{2n}. If we want to specify this index in the name of the G-transformation, we shall use superscripts (for example, γ_1^n or δ_2^n).

Application of any G-transformation does not change the spectrum of the corresponding graph. Moreover, we have the following theorem from [1].

Theorem 2.1. *Let H_1 and H_2 be bipartite Smith graphs with corresponding bases P_{H_1} and P_{H_2}. If graphs H_1 and H_2 are cospectral, then the graph $H_1 + P_{H_1}$ can be transformed into $H_2 + P_{H_2}$ by a finite number of G-transformations.*

Cospectrality graphs have been introduced in [4] as follows.

For any A-type graph G we define its *cospectrality graph* $C(G)$ in the following way. Vertices of $C(G)$ are all graphs cospectral with G, i.e. the set of vertices of $C(G)$ is the cospectral equivalence class of G. Two vertices x and y are adjacent if there exists a G-transformation transforming one to another. Of course, if x can be transformed into y by a G-transformation, then y can be transformed into x by the opposite transformation. Hence, $C(G)$ is an undirected graph without multiple edges or loops.

By Theorem 2.1 the cospectrality graph is connected.

We shall also consider *general cospectrality graphs*. Such graphs have mutually cospectral vertex weights, the adjacency relation being defined as above.

It can be easily seen that *identifying* two vertices with same weights in a general cospectrality graph leads again to a regular general cospectrality graph. When identifying such vertices, all edges which were going to particular vertices, go now to the new single vertex.

3. Some Properties of Cospectrality Graphs

Let G be an A-type graph and let G^* be its canonical representation. We have $C(G) = C(G^*)$ and the later will be considered as a standard denotation for a cospectrality graph. Let $C(G^*) = C$.

Cospectrality graph C is a double weighted graph. Both vertices and edges carry some weights. Weights of vertices are some Smith graphs while weights of edges are pairs of mutually opposite G-transformations. Vertex weights determine edge weights since weights of adjacent vertices determine the pair of mutually opposite G-transformations transforming one vertex to another.

A cospectrality graph C, which is considered as an undirected graph, defines the following two directed weighted graphs: C_γ obtained from C by replacing edges with arcs with corresponding γ-transformations as weights and corresponding orientations, and C_δ, defined analogously.

Note that C_γ and C_δ, as digraphs, are mutually converse.

In considering cospectrality problems for Smith graphs we can treat together C, C_γ and C_δ and pass from one to another as appropriate.

Also we can treat incomplete cospectrality graphs, i.e., double weighted graphs in which the vertex set does not contain all mutually cospectral graphs. Sometimes we allow in such graphs vertices with the same weights.

Next theorem characterizes Smith graphs whose cospectrality graphs have just one vertex.

Theorem 3.1. *If the cospectrality graph of a Smith graph G of type* A *consists just of one vertex, then G is one of the following graphs:*

- *multiple cycles kC_4, $(k \in \mathbb{N})$;*
- *kP_1 $(k \in \mathbb{N})$ in the union with any collection of paths $P_2, P_3, P_4, P_6, P_8, \ldots$;*
- *any collection of paths without P_1.*

Proof. Clearly, graph G is characterized by the spectrum. Since any Smith graph is cospectral to its canonical representation, graph G must be itself in the form of canonical representation. If G is one of the graphs kC_4 ($k \in \mathbb{N}$), then there is no G-transformation producing a cospectral mate. G cannot contain C_4 and a path because of the transformation δ_1. In the remaining cases G is just a collection of paths. If P_1 is present, transformation δ_2 prevents the presence of any path P_{2k+1} for $k \geq 2$. If P_1 is excluded, any collection of other paths is feasible. □

One can also classify graphs whose cospectrality graphs consist of two vertices. In fact, for each of nine types of D-transformations one can consider cospectrality graphs in which exactly this transformation appears.

4. Building Cospectrality Graphs

We present several ways in which new cospectrality graphs can be obtained from starting ones.

Let G_1 and G_2 be two Smith graphs and let C and D be cospectrality graphs such that $C = C(G_1^*)$ and $D = C(G_2^*)$. Corresponding directed graphs with arcs whose weights are $\delta-$transformations will be denoted C_δ and D_δ, respectively. Let $V(C_\delta) = \{1, 2, \dots, m\}$ with weights $\{c_1, c_2, ..., c_m\}$ and $V(D_\delta) = \{1, 2, \dots, n\}$ with weights $\{d_1, d_2, ..., d_n\}$ be the corresponding vertex sets. Note that c_i are graphs cospectral with G_1 and d_j are graphs cospectral with G_2.

Given cospectrality graphs $C(G_1)$ and $C(G_2)$ of graphs G_1 and G_2 we want to construct the cospectrality graph $C(G_1 + G_2)$ of the graph $G_1 + G_2$. The construction is not straightforward and we need several definitions. In particular, we shall define the sum of cospectrality graphs, merging vertices in a cospectrality graph and extending cospectrality graphs. All these operations can occur when constructing $C(G_1 + G_2)$.

First, we use Cartesian product $\times$ of sets to define, similarly as in the sum of graphs (see, for example, [2], page 65), the *sum* $C_\delta \oplus D_\delta$ of cospectrality graphs C_δ and D_δ. The operation $\oplus$ is called the *cospectrality sum*.

The vertex set $V(C_\delta \oplus D_\delta)$ of $C_\delta \oplus D_\delta$ is $V(C_\delta) \times V(D_\delta)$ and vertices (i, j) and (k, l) are adjacent if $i = k$ and j and l are adjacent in D_δ or $j = l$ and i and k are adjacent in C_δ. Weights $w(a)$ of arcs (or vertices) a are defined as follows (with subscript indicating the actual graph):

$$w((i,j),(i,l)) = w_D(j,l),$$

$$w((i,j),(k,j)) = w_C(i,k),$$

for $i, k \in \{1, \dots, m\}$ and $j, l \in \{1, \dots, n\}$ and

$$w(i,j) = c_i + d_j, \quad i \in \{1, \dots, m\}, j \in \{1, \dots, n\}.$$

The definition of $C_\gamma \oplus D_\gamma$ is analogous and leads to a directed graph converse to $C_\delta \oplus D_\delta$ with weights being the corresponding γ-transformations.

We might also consider the corresponding undirected graph $C \oplus D$ obtained from considered digraphs by replacing arcs with edges with corresponding pairs of opposite G-transformations as weights.

All three objects $C \oplus D$, $C_\gamma \oplus D_\gamma$ and $C_\delta \oplus D_\delta$ will be considered as the sum of cospectrality graphs C and D.

A subgraph of a cospectrality graph is called a *partial cospectrality graph.*

Theorem 4.1. *Let $C = C(G_1^*)$ and $D = C(G_2^*)$. The sum $C_\delta \oplus D_\delta$ of cospectrality graphs C_δ and D_δ, after merging vertices wth the same weights, is a partial cospectrality graph of the graph $C(G_1 + G_2)$.*

Proof. By definition of the sum, the weight $c_i + d_j$ of a vertex (i, j) is transformed either in the part c_i or in the part d_j giving in both cases the weight of a vertex cospectral to $c_i + d_j$. □

Let us introduce the notion of an empty graph G_ϕ. It is a graph without vertices or edges and represents a neutral element for the operation of union of graphs. For any (non-weighted) graph G let also $Q(G)$ be a weighted graph consisting of a single vertex with vertex weight G.

It can easily be verified that $Q(G_\phi)$ behaves as a neutral element for the cospectrality sum $\oplus$, i.e, for any (partial) cospectrality graph C we have $Q(G_\phi) \oplus C = C \oplus Q(G_\phi) = C$.

Let S be any bipartite Smith graph and consider the cospectrality sum $Q(S) \oplus C$. The resulting cospectrality graph is isomorphic to C with each vertex weight being the union of the weight of the corresponding vertex in C and S.

We shall also consider *extending* - finding new vertices and arcs in a general cospectrality graph.

It happens sometimes that the weight $c_i + d_j$ of a vertex (i, j) of a sum $C_\delta \oplus D_\delta$ contains a Smith graph S which is contained neither in c_i nor in d_j and such that a D-transformation can be applied to it. This means that $c_i + d_j$ can be transformed in some additional ways. Let $c_i + d_j = S + S'$ for some Smith graph S' In fact, if $C(S)$ is a (partial) cospectrality graph for S, then the graph $Q(S') \oplus C(S)$ has a vertex with the weight $c_i + d_j$. The vertex of $Q(S') \oplus C(S)$ and the vertex in $C_\delta \oplus D_\delta$ with the same weight $c_i + d_j$ could be identified. In this way, $C_\delta \oplus D_\delta$ is *extended* by $Q(S') \oplus C(S)$ at vertex (i, j).

5. A Computer Program

We have implemented a computer program generating all graphs cospectral to a given bipartite Smith graph G of type A and the corresponding cospectrality graph $C(G)$.

The input contains a bipartite Smith graph of type A in its canonical form.

The vertex v_0 representing the canonical representation of G is called the *c-center* of $C(G)$ [4].

For any vertex v of $C(G)$ we define $H(v)$ to be the graph which is represented by v, i.e., the weight of v. The rank *rank* H of a Smith graph H is the number of non-basic components of H.

Vertices of $C(G)$ are partitioned into *layers* according to ranks of corresponding graphs. Layer k contains vertices v such that *rank* $H(v) = k$. The largest rank of a vertex in $C(G)$ is called the *c-radius* of $C(G)$. The vertices with largest rank are called *peripheral* vertices. Their rank is equal to the c-radius.

Applying a D-transformation on a vertex enhances its rank while C-transformations diminish the rank. Using C-transformations we are approaching the c-center while by D-transformations we go from c-center to peripheral vertices.

When considering a current graph the program tries to apply a D-transformation and if this is done the program forms a new vertex of the search tree. The depth first search is applied. Repeated graphs are not considered again.

The program is realized as a console application. The following tools are used: .NET Framework v4.7.2, $C\#$, XML, LinQ and Visual Studio 2019.

Example 5.1. Our program has been applied to the graph $T_5 + T_6 + 2P_1$. The program produced 25 graphs in the corresponding cospectrality graph. This shows that the cospectrality graph of $T_5 + T_6 + 2P_1$, given in [4], Figure 3, is not complete.

The program output is presented in Table 1.

TABLE 1.

Layer 0	2C4 2P2 P3 P4
Layer 1	1: $C4\ P2\ P3\ P4\ W2$, 1: $C4\ 2P2\ P4\ W3$, 1: $C4\ 2P2\ P3\ W4$, 3: $C6\ C4\ 2P1\ P3\ P4$, 7: $C4\ P3\ P4\ T4\ P1$, 8: $P2\ C4\ P4\ T5\ P1$, 9: $P2\ C4\ P3\ T6\ P1$
Layer 2	1: $P3\ P4\ 2W2$, 1: $P2\ P4\ W2\ W3$, 1: $P2\ P3\ W2\ W4$, 8: $P4\ W2\ T5\ P1$, 9: $P3\ W2\ T6\ P1$ | 1: $P2\ P4\ W3\ W2$, 1: $2P2\ W3\ W4$, 3: $P4\ W3\ C6\ 2P1$, 7: $P4\ W3\ T4\ P1$, 9: $P2\ W3\ T6\ P1$ | 1: $P2\ P3\ W4\ W2$, 1: $2P2\ W4\ W3$, 3: $P3\ W4\ C6\ 2P1$, 7: $P3\ W4\ T4\ P1$, 8: $P2\ W4\ T5\ P1$ | 1: $P3\ P4\ C6\ P1\ W1$, 1: $P4\ C6\ 2P1\ W3$, 1: $P3\ C6\ 2P1\ W4$ | 1: $P3\ P4\ T4\ W1$, 1: $P4\ T4\ P1\ W3$, 1: $P3\ T4\ P1\ W4$ | 1: $P2\ P4\ T5\ W1$, 1: $P4\ T5\ P1\ W2$, 1: $P2\ T5\ P1\ W4$, 9: $T5\ 2P1\ T6$ | 1: $P2\ P3\ T6\ W1$, 1: $P3\ T6\ P1\ W2$, 1: $P2\ T6\ P1\ W3$, 8: $T6\ 2P1\ T5$

Generated graphs are classified within layers. Starting from layer 1, graphs in a layer are listed in order as they are generated from the previous layer and the index i of the used D-transformation δ_i is indicated. The symbol + of the union of graphs is omitted. Graphs generated by different (and neighboring) graphs from the previous layer are separated by a vertical line |. Repeated graphs are underlined. There are exactly 25 graphs in the table which are not underlined.

Example 5.2. When applied to $T_4 + T_5 + T_6 + 3P_1$, the program produced 86 mutually cospectral graphs.

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