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# ON THE SEMIGROUP OF BI-IDEALS OF AN ORDERED SEMIGROUP

SUSMITA MALLICK[1] AND KALYAN HANSDA[2]

ABSTRACT. The purpose of this paper is to characterize an ordered semigroup $S$ in terms of the properties of the associated semigroup $\mathcal{B}(S)$ of all bi-ideals of $S$. We show that an ordered semigroup $S$ is a Clifford ordered semigroup if and only if $\mathcal{B}(S)$ is a semilattice. The semigroup $\mathcal{B}(S)$ is a normal band if and only if the ordered semigroup $S$ is both regular and intra regular. For each subvariety $\mathcal{V}$ of bands, we characterize the ordered semigroup $S$ such that $\mathcal{B}(S) \in \mathcal{V}$.

## 1. INTRODUCTION AND PRELIMINARIES

The passage from semigroup without order to ordered semigroup is not straightforward. Regular rings and semigroups have been influenced many authors to study the order structure on regular semigroups as well as to introduce a natural notion of regularity which arises out of a combination of the partial order and binary operation on an ordered semigroup. Bhuniya and Hansda [1] presented a natural analogy between these two regularities. Thus it is quite obvious to explore a natural analogy between the subclasses of these two regularities.

An ordered semigroup $(S, \cdot, \leq)$ is a partially ordered set $(S, \leq)$ and at the same time a semigroup $(S, \cdot)$ such that for all $a, b$ and $x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. Let $(S, \cdot, \leq)$ be an ordered semigroup, $(\emptyset \neq)A \subseteq S$ is called a subsemigroup of $S$ if for every $a, b \in A$, $ab \in A$. Every subsemigroup $A$ of $S$ with the relation $\leq_A$ on $A$ defined by $\leq_A = \leq \cap \{(a, b) \in A \times A\}$ is an ordered semigroup (called an ordered

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subsemigroup of $S$). Clearly, $\leq_A = \leq \cap A \times A$. For an ordered semigroup $S$ and $H \subseteq S$, denote $(H] := \{t \in H : t \leq h \text{ for some } h \in H\}$.

Let $I$ be a non-empty subset of an ordered semigroup $S$. $I$ is a left(right) ideal of $S$, if $SI \subseteq I(IS \subseteq I)$ and $(I] = I$. We call $I$ is an ideal of $S$ if it is both a left and a right ideal of $S$. We denote the set of all left and right ideals of $S$ by $\mathcal{L}(S)$ and $\mathcal{R}(S)$ respectively. Following Kehayopulu and Tsingelis [9], a subsemigroup $B$ of $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$ and $(B] = B$. We denote the set of all bi-ideals of $S$ by $\mathcal{B}(S)$. The principal left ideal, right ideal, ideal and bi-ideal generated by $a \in S$ are denoted by $L(a), R(a), I(a)$ and $B(a)$ respectively and defined by $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$, $B(a) = (a \cup a^2 \cup aSa]$.

Characterizations of a semigroup (without order) $S$ by the set of all bi-ideals of $S$, were beautifully presented by S. Lajos [11]. Here our approach allows one to characterize an ordered semigroup $S$ by the set $\mathcal{B}(S)$ of all bi-ideals of $S$ as a semigroup without order. We show that product of two bi-ideals in an ordered semigroup $S$ is again a bi-ideal of $S$. Thus, $\mathcal{B}(S)$ is closed under this product. The main object of this paper is to study the semigroup $\mathcal{B}(S)$ of all bi-ideals of $S$ whenever $S$ is in different important subclasses of the regular ordered semigroups.

Kehayopulu [6] defined Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup $S$ in the following way: for $a, b \in S$ $a\mathcal{L}b$ if $L(a) = L(b)$; $a\mathcal{R}b$ if $R(a) = R(b)$; $a\mathcal{J}b$ if $I(a) = I(b)$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These four are equivalence relations on $S$. An ordered semigroup $S$ is said to be *regular* if for every $a \in S$, $a \in (aSa]$ and is intra-regular if for every $a \in S$, $a \in (Sa^2S]$. An ordered semigroup $S$ is group like ordered semigroup [1] if for all $a, b \in S$ there are $x, y \in S$ such that $a \leq xb$ and $a \leq by$. A regular ordered semigroup $S$ is called a left group like ordered semigroup [1] if for all $a, b \in S$ there is $x \in S$ such that $a \leq xb$. Right group like ordered semigroup defined dually. Class of *Clifford* [4] as well as *left Clifford* [4] ordered semigroups are subclasses of class of regular ordered semigroups. A regular ordered semigroup $S$ is called a *Clifford* (*left Clifford*) [4] ordered semigroup if for all $a, b \in S$ there is $x \in S$ such that $ab \leq bxa$ ($ab \leq xa$). Following results have been given for the sake of convenience of general readers.

**Theorem 1.1.** *Let $S$ be an ordered semigroup. Then following conditions hold in $S$.*

(1) *If $S$ is regular, then $B = (BSB]$ for every bi-ideal $B$ of $S$ (see [8]).*
(2) *If $S$ is regular, then a nonempty subset $B$ of $S$ is a bi-ideal of $S$ if and only if $B = (RL]$ for some right ideal $R$ and left ideal $L$ of $S$ (see [5]).*

**Theorem 1.2** ([1])**.** *An ordered semigroup $S$ is a group like ordered semigroup if and only if it is both left group like and right group like ordered semigroup.*

For the sake of convenience of general readers we give some definitions and results from semigroup theory. By a band $F$ we mean a semigroup $(F, \cdot)$ with the property $a^2 = a$ for every $a \in F$. A band $(F, \cdot)$ is called rectangular if for every $a, b \in F$ $aba = a$. A left(right) zero band is a band $(F, \cdot)$ with the property $ab = a$ $(ba = a)$ for every $a, b \in F$. A band $(F, \cdot)$ is said to be left (right) normal band if for every $a, b, c \in F$,

$abc = acb$ ($abc = bac$) and $F$ is said to be normal if $abca = acba$. A commutative band is called a semilattice. A semigroup in which every finitely generated subsemigroup is finite called locally finite. A locally finite semigroup $S$ is called locally testable [3] if for every idempotent $f$ of $S$, $fSf$ is a semilattice.

## 2. Semigroup of Bi-Ideals in Regular Ordered Semigroups

First we define a product of two bi-ideals of an ordered semigroup $S$. Let $(S, \cdot, \leq)$ be an ordered semigroup and $P(S)$ be the set of all subsets of $S$. We define a binary operation $*$ on $S$ as follows: For $A, B \in P(S)$, $A * B = (AB]$, where $AB = \{ab : a \in A, b \in B\}$. It is easy to check that $(P(S), *)$ forms semigroup. Throughout the paper $A * A$ will be denoted by $A^2$, for every bi-ideal $A$ of $S$. It is also noted that $A^2$ is not $AA$ rather $A^2 = (AA]$. Followed by above, it is a routine task to verify that $\mathcal{L}(S)$, $\mathcal{R}(S)$ and $\mathcal{B}(S)$ are semigroups with respect to $*$.

In the following proposition we show that regularity of an ordered semigroup is equivalent to the regularity of the semigroup $B(S)$.

**Proposition 2.1.** *Let $S$ be an ordered semigroup. Then $S$ is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideals is regular.*

*Proof.* First assume that $\mathcal{B}(S)$ is a regular semigroup. Let $a \in S$. Then $B(a) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is regular, there is $C \in \mathcal{B}(S)$ such that $B(a) = B(a) * C * B(a) = (B(a)CB(a)]$. Since $a \in B(a)$, there are $b \in B(a)$, $x \in C$ and $c \in B(a)$ such that $a \leq bxc$. Also, for $b, c \in B(a)$ there are $s_1, s_2 \in S$ such that $b \leq a$ or $b \leq as_1a$ and $c \leq a$ or $c \leq as_2a$. Thus, in either case $a \leq bxc$ gives that $a \in (aSa]$ and therefore $S$ is a regular ordered semigroup.

The converse follows directly from Theorem 1.1. □

**Theorem 2.1.** *Let $S$ be a regular ordered semigroup. Then $\mathcal{R}(S)(\mathcal{L}(S))$ is a band and $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

*Proof.* Let $R \in \mathcal{R}(S)$ and $a \in R$. Since $S$ is regular there exist $x \in S$ such that $a \leq axa$. Also $ax \in R$ which gives that $a \in (RR] = R * R = R^2$ and so $R \subseteq R^2$. Thus, $R^2 = R$. Hence, $\mathcal{R}(S)$ is a band. Similarly, $\mathcal{L}(S)$ is a band.

Choose $R \in \mathcal{R}(S)$ and $L \in \mathcal{L}(S)$. Let $B = R * L$. Then $B = (RL]$ and $B$ is a subsemigroup of $S$. Now $BSB = (RL]S(RL] \subseteq (RLSRL] \subseteq (RL] = B$, by Theorem 1.1. This shows that $B \in \mathcal{B}(S)$ and so $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$. Next choose $D \in \mathcal{B}(S)$. Now $D \in \mathcal{B}(S) \subseteq \mathcal{R}(S)\mathcal{L}(S)$. Thus, $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$. Hence, the theorem is proved. □

**Theorem 2.2.** *An ordered semigroup $S$ is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

*Proof.* Suppose $S$ is both regular and intra-regular ordered semigroup. Let $B \in \mathcal{B}(S)$ and $a \in B$. Then $a \leq axa \leq axaxa$ for some $x \in S$. Since $S$ is intra-regular there are $s_1, s_2 \in S$ such that $a \leq s_1a^2s_2$ which implies that $a \leq axs_1a^2s_2xa \leq (axs_1a)(as_2xa)$.

Since $axs_1a \in BSB \subseteq B$, $axs_1a^2s_2xa \in B^2$ so that $a \in (BB] = B * B = B^2$. Also, $B^2 \subseteq B$ and thus $B^2 = B$.

Conversely, assume that $\mathcal{B}(S)$ is a band. Let $a \in S$. Then $B(a) \in B(S)$ and so $a \in B(a) = B(a)^2 = B(a) * B(a) = (B(a)B(a)]$. Thus, $a \leq bc$ for some $b, c \in B(a)$. This gives that $b \leq a$ or $b \leq asa$ for some $s \in S^1$ and $c \leq a$ or $c \leq ata$ for some $t \in S^1$. Then $a \leq bc$ implies that either $a \leq a^2$ or $a \in (aSa^2Sa]$ which gives that $a$ is both regular and intra-regular. Thus, $S$ is both regular and intra-regular. □

**Lemma 2.1.** *Let $S$ is a both regular and intra-regular ordered semigroup. Then*

(1) *for every $B, C, D \in \mathcal{B}(S)$, $((BCB](BDB]] = (BCB] \cap (BDB]$;*

(2) *$\mathcal{B}(S)$ is locally testable semigroup.*

*Proof.* (1) We have, $((BCB](BDB]] \subseteq ((BCB](B]] \subseteq ((BCB]] \subseteq (BCB]$. Similarly, $((BCB](BDB]] \subseteq (BDB]$. Thus, $((BCB](BDB]] \subseteq (BCB] \cap (BDB]$. Now let $u \in (BCB] \cap (BDB]$. Then there are $b \in B$, $c \in C$, $d \in D$ such that $u \leq bcb$ and $u \leq bdb$. Since $S$ is both regular and intra-regular, then there are $x, t, s \in S$ such that $u \leq uxu, b \leq btb$ and $b \leq s_1b^2s_2$ this implies $u \leq bcbxbdb \leq bcbtbxbdb \leq bcbts_1b^2s_2xbdb \leq (bcbts_1b)(bs_2xbdb)$. So, $u \in ((BCB](BDB]]$. Hence, $(BCB] \cap (BDB] \subseteq ((BCB](BDB]]$. Thus, $((BCB](BDB]] = (BCB] \cap (BDB]$.

(2) Consider $B \in \mathcal{B}(S)$. Then $B\mathcal{B}(S)B$ is a subsemigroup of $\mathcal{B}(S)$ and so a band. Now for every $C, D \in \mathcal{B}(S)$, $(BCB] * (BDB] = ((BCB](BDB]] = (BCB] \cap (BDB] = (BDB] \cap (BCB] = ((BDB](BCB]] = (BDB] * (BCB]$ shows that $B\mathcal{B}(S)B$ is a semilattice. Thus, $\mathcal{B}(S)$ is locally testable. □

Nambooripad [3] proved that a regular semigroup $S$ is locally testable if and only if for every $f \in E(S)$, $fSf$ is a semilattice. Also, following Zalcstein [12] a locally testable semigroup is a band if and only if it is a normal band.

**Corollary 2.1.** *Let $S$ be an ordered semigroup. If $S$ is both regular and intra-regular then $\mathcal{B}(S)$ is a band if and only if $\mathcal{B}(S)$ is a normal band.*

This follows from Theorem 2.2, Lemma 2.1 and Theorem 5 of [12].

**Theorem 2.3.** *Let $S$ be an ordered semigroup. Then $\mathcal{B}(S)$ is a rectangular band if and only if $S$ is regular and simple.*

*Proof.* First suppose that $\mathcal{B}(S)$ is a rectangular band. Let $a, b \in S$. Then $B(a), B(b) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is rectangular band, we have $B(a) = B(a) * B(b) * B(a)$ and $B(b) = B(b) * B(a) * B(b)$. Also, by Theorem 2.2, $S$ is regular. Since $a \in B(a) = B(a) * B(b) * B(a) = (B(a)B(b)B(a)]$, there are $w, z \in B(a)$, $u \in B(b)$ such that $a \leq zuw$. Since $w, z \in B(a)$, $z \leq as_1a$ and $w \leq as_2a$ for some $s_1, s_2 \in S$. Also, for $u \in B(b)$ there is $s_3 \in S$ such that $u \leq bs_3b$. Thus, $a \leq (as_1abs_3)b(as_2a)$, i.e., $a \leq xby$ for some $x, y \in S$. Hence, $S$ is simple.

Conversely, let $S$ is a regular and simple ordered semigroup. Consider, $a \in S$. Now by given condition we have $a \in (Sa^2S]$ so that $S$ is intra-regular. So by Theorem 2.2, $\mathcal{B}(S)$ is a band. Next let $A, B \in \mathcal{B}(S)$. We show that $A = A * B * A$. For this let $a \in A$ and $b \in B$. Since $a, aba \in S$ and $a\mathcal{J}b$ so $a \leq y_1 aba y_2$ for some $y_1, y_2 \in S$. The regularity of $S$ yields that $a \leq axa \leq axaxa$ for some $x \in S$. Then $a \leq (axy_1a)b(ay_2xa)$ so that $a \in ((ASA)B(ASA)] \subseteq (ABA] = A * B * A$ that is, $A \subseteq A * B * A$. Again $A * B * A \subseteq (ASA] = A$. Thus, $A = A * B * A$ hence $\mathcal{B}(S)$ is a rectangular band. □

**Theorem 2.4.** *Let $S$ be an ordered semigroup. Then $\mathcal{B}(S)$ is a left (right) zero band if and only if $S$ is a left (right) group like ordered semigroup.*

*Proof.* Let $\mathcal{B}(S)$ is a left zero band. Then by Proposition 2.2, $S$ is regular. Let $a, b \in S$. Then $B(a), B(b) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is a left zero band, $B(a) = B(a) * B(b)$, so $a \in (B(a)B(b)]$. Then there are $z \in B(a)$ and $w \in B(b)$ such that $a \leq zw$. Also, $w \leq bsb$ for some $s \in S$. Therefore, $a \leq (zbs)b$ and hence $S$ is a left group like ordered semigroup.

Conversely, let $S$ be a left group like ordered semigroup. Let $B, C \in \mathcal{B}(S)$. Let $u \in B * C$, then there are $b \in B$ and $c \in C$ such that $u \leq bc$. Since $S$ is a left group like ordered semigroup we have $c \leq tb$ for some $t \in S$. Then for $c \leq tb$ together with $u \leq bc \leq btb$ gives $u \in B$. Thus, $B * C \subseteq B$. Now for any $d \in B$, $d \leq dtd$ for some $t \in S$. Since $d$, $dc \in S$, $d \leq t_1dc$ for some $t_1 \in S$. So, $d \leq dtt_1dc$. Clearly $d \in BSB \subseteq B$ so that $d \in (BC] = B * C$. Hence, $B = B * C$ and so $B$ is a left zero band. □

Thus, it is very logical step to study the set of all bi-ideals $\mathcal{B}(S)$ for a group like ordered semigroup $S$.

**Theorem 2.5.** *Let $S$ be an ordered semigroup. Then $\mathcal{B}(S)$ is both left zero and right zero band if and only if $S$ is a group like ordered semigroup.*

*Proof.* This is similar to the proof of the Theorem 2.4. □

We now focus on the characterization of *Clifford* and *left Clifford* ordered semigroup $S$ by the semigroup $\mathcal{B}(S)$.

**Theorem 2.6.** *Let $S$ be an ordered semigroup. Then the following statements are equivalent:*

(1) *$S$ is a Clifford ordered semigroup;*
(2) *$B_1 * B_2 = B_1 \cap B_2$ for all $B_1, B_2 \in \mathcal{B}(S)$;*
(3) *$(\mathcal{B}(S), *)$ is a semilattice.*

*Proof.* (1) ⇒ (2) First suppose that $S$ is a Clifford ordered semigroup. Let $B_1, B_2 \in \mathcal{B}(S)$ and $u \in B_1 * B_2$. Then $u \leq b_1b_2$ for $b_1 \in B_1$ and $b_2 \in B_2$. Since $S$ is regular there is $x \in S$ such that $u \leq uxu \leq b_1b_2xb_1b_2$. Since $S$ is Clifford, there is $x_1 \in S$ such that $b_1b_2 \leq b_2x_1b_1$, so that $u \leq b_1b_2xb_2x_1b_1$. This implies $u \in B_1$. Similarly $u \in B_2$. Hence, $B_1 * B_2 \subseteq B_1 \cap B_2$. Next let $b \in B_1 \cap B_2$. Since $S$ is regular, there is $y \in S$ such that $b \leq byb \leq bybyb$. Since $S$ is Clifford, $yb \leq bzy$ for some $z \in S$. Thus,

$b \leq bbzy^2b$. Since $b \in B_2$ and $B_2$ is a bi-ideal of $S$ it yields that $bzy^2b \in B_2SB_2 \subseteq B_2$. Also, $b \in B_1$ so that $b \in (B_1B_2] = B_1 * B_2$. Hence, $B_1 * B_2 = B_1 \cap B_2$.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Assume that $(B(S), *)$ is a semilattice. Then $S$ is a regular ordered semigroup (by Theorem 2.2). Consider $a, b \in S$. Then $ab \in B(a) * B(b) = B(b) * B(a)$ implies that $ab \leq vu$ for some $u \in B(a)$ and $v \in B(b)$. Since $S$ is regular, there are $s, t \in S$ such that $u \leq asa$ and $v \leq btb$. Thus, $ab \leq btbasa = bza$ where $z = tbas \in S$. Hence, $S$ is a Clifford ordered semigroup. $\square$

**Theorem 2.7.** *Let $S$ be an ordered semigroup. Then $\mathcal{B}(S)$ is a left normal band if and only if $S$ is a left Clifford ordered semigroup.*

*Proof.* First suppose that $S$ is a left Clifford ordered semigroup. Let $A, B$ and $C \in \mathcal{B}(S)$ and $x \in A * B * C$. Then $x \in (ABC]$ so $x \leq abc$ for some $a \in A$, $b \in B$ and $c \in C$. Since $S$ is regular, there is $s \in S$ such that $x \leq xsx$ so that $x \leq abcsabc$. Since $S$ is a left Clifford ordered semigroup, it follows $bc \leq s_1b$ for some $s_1 \in S$, so $x \leq abc(sas_1)b \leq abs_2cb$ for $s_2 \in S$. Since $S$ is regular there is $t \in S$ such that $a \leq ata$ implies $x \leq atabs_2cb$. Also there are $s_3, s_4 \in S$, $x \leq ats_3as_2cb \leq ats_3s_4acb$ implies $x \in A * C * B$. Therefore, $A * B * C \subseteq A * C * B$. Similarly it can be shown that $A * C * B \subseteq A * B * C$. Hence, $A * B * C = A * C * B$ and so $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then $S$ is regular, by Theorem 2.2. Let $a, b \in S$. Then there is $x \in S$ such that $ab \leq abxab$ which implies $ab \in (B(abx)B(a)B(b)] = (B(abx)B(b)B(a)]$, since $\mathcal{B}(S)$ is a left normal band. Then $ab \leq uvw$, where $u \in B(abx), v \in B(b), w \in B(a)$. Again, $w \leq asa$ for some $s \in S$. Now $ab \leq uvw \leq (uvas)a \leq s_1a$, where $s_1 = uvas \in S$. Thus, $S$ is left Clifford ordered semigroup. $\square$

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## References

[1] A. K. Bhuniya and K. Hansda, *On completely regular and Clifford ordered semigroups*, Afrika Matematika (2020), `https://doi.org/10.1007/s13370-020-00778-1`

[2] J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford, 1995.

[3] K. S. S. Nambooripad, *Pseudo-semilattices and biordered sets I*, Simon Stevin **55** (1981), 103–110.

[4] K. Hansda, *Bi-ideals in Clifford ordered semigroup*, Discuss. Math. Gen. Algebra Appl. **33** (2013), 73–84. `https://doi.org/10.7151/dmgaa.1195`

[5] K. Hansda, *Minimal bi-ideals in regular and completely regular ordered semigroups*, Quasigroup and Related Systems **27**(1), 63–72.

[6] N. Kehayopulu, *Note on Green's relation in ordered semigroup*, Math. Japonica **36** (1991), 211–214.
[7] N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japonica **37** (1992), 123–130.
[8] N. Kehayopulu, J. S. Ponizovskii and M. Tsingelis, *Bi-ideals in ordered semigroups and ordered groups*, J. Math. Sci. **112** (4) (2002), 4353–4354. https://doi.org/10.1023/A:1020347003781
[9] N. Kehayopulu and M. Tsingelis, *Fuzzy bi-ideals in ordered semigroups*, Inform. Sci. **171** (2005), 13–28. https://doi.org/10.1016/j.ins.2004.03.015
[10] R. A. Good and D. R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc. **58** (1952), 624–625.
[11] S. Lajos, *On the bi-ideals in semigroups*, Proc. Japan Acad. **45**(8) (1969), 710–712. https://doi.org/10.3792/pja/1195520625
[12] Y. Zalcstein, *Locally testable semigroups*, Semigroup Forum **5**(1973), 216–227. https://doi.org/10.1007/BF02572893

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# $L^\infty-$ASYMPTOTIC BEHAVIOR OF A FINITE ELEMENT METHOD FOR A SYSTEM OF PARABOLIC QUASI-VARIATIONAL INEQUALITIES WITH NONLINEAR SOURCE TERMS

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Abstract. This paper is an extension and a generalization of the previous results, cf. [3,6,8,11]. It is devoted to studying the finite element approximation of the non coercive system of parabolic quasi-variational inequalities related to the management of energy production problem. Specifically, we prove optimal $L^\infty$-asymptotic behavior of the system of evolutionary quasi-variational inequalities with nonlinear source terms using the finite element spatial approximation and the subsolutions method.

## 1. Introduction

This paper is concerned with the semi-implicit time scheme combined with a finite element spatial approximation for a system of parabolic quasi-variational inequalities with nonlinear source terms: Find $(u^1,\ldots,u^J)\in\left(L^2\left((0,T),H_0^1(\Omega)\right)\right)^J$ satisfying

$$\begin{cases}\dfrac{\partial u^i}{\partial t}+A^iu^i\le f^i\left(u^i\right)\ \text{ in }\Phi,\\ u^i\le Mu^i,\quad i=1,\ldots,J,\\ \left(\dfrac{\partial u^i}{\partial t}+A^iu^i-f^i\left(u^i\right)\right)\left(u^i-Mu^i\right)=0\text{ in }\Phi,\\ u^i(x,0)=u_0^i\text{ in }\Omega,\quad u^i=0\text{ on }\Sigma.\end{cases}\tag{1.1}$$

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Here $A^i$ denotes uniformly second order elliptic operators on a bounded convex domain $\Omega$ in $\mathbb{R}^J$, $J \geq 1$ with smooth boundary $\partial\Omega$ and $\Phi$ set in $\mathbb{R}^J \times \mathbb{R}$ defined as $\Phi = \Omega \times [0, T]$, with $T < +\infty$, $\Sigma = \partial\Omega \times [0, T]$.

$f^i(u^i)$ are $J$ nonlinear and Lipschitz functions with Lipschitz constant $\alpha < \beta$ and satisfying the following condition

(1.2)
$$f^i \in \left(L^2\left((0,T), L^\infty(\Omega)\right) \cap C^1\left((0,T), H^{-1}(\Omega)\right)\right)^J, \quad f^i > 0, \text{ also is increasing.}$$

This system arises from the management of energy production problems (see [4] and the references therein). In the case studied here, $Mu^i$ represents a "cost function" and the prototype encountered is

$$Mu^i(x) = \mathbf{k} + \inf_{\mu \neq i} u^\mu, \quad \text{where } \mathbf{k} > 0 \text{ and } \mu > 0, \tag{1.3}$$

and we know by [25] on page 243 that $M$ satisfies some proprieties as $M$ is a concave operator, i.e.,

$$M(\delta u + (1-\delta) v) \geq \delta M(u) + (1-\delta) M(v), \quad \text{for all } u, v \in C(\Omega),$$

and it also satisfies

$$M(u + \eta) = M(u) + \eta, \quad \text{for all } \eta \in \mathbb{R},$$

where $\mathbf{k}$ represents the switching cost. It is positive when the unit is "turned on" and equal to zero when the unit is "turned off".

Many results on error estimates for the classical obstacle problems, system of stationary and evolutionary quasi-variational and variational inequalities have been achieved in this norm, (cf., e.g., [1–3, 5, 9, 18, 20, 22]).

Moreover, in [11] Boulaaras, Bencheikh and Haiour established quasi-optimal $L^\infty$-asymptotic behavior of the system of parabolic quasi-variational inequality related the management of energy production problems with mixed boundary condition using a discrete algorithm based on a $\theta$-scheme combined with a finite element spatial approximation, that is, for $\theta \geq \frac{1}{2}$

$$\| U_h^n - U^\infty \|_\infty \leq C\left[h^2 \mid \log h \mid^3 + \left(\frac{1}{1+\theta\Delta t}\right)^n\right],$$

and for $0 \leq \theta < \frac{1}{2}$

$$\|U_h^n - U^\infty\|_\infty \leq C\left[h^2 \mid \log h \mid^3 + \left(\frac{2}{2+\beta\theta(1-2\theta)\rho(A^i)}\right)^n\right],$$

where $\rho(A^i)$ is the spectral radios of the elliptic operator $A^i$ and $U_h^n$, the discrete solution of the system of QVIs calculated at the moment-end $T = n\Delta t$ for an index of the time discretization $k = 1, \ldots, n$, and $U^\infty$, the asymptotic continuous solution of the system of QVIs.

Also, in [8] Boulaaras, Haiour proved quasi-optimal $L^\infty$-asymptotic behavior of the evolutionary Hamilton-Jacobi-Bellman equations using the semi-implicit scheme with respect to the $t$-variable combined with a finite element spatial approximation where $k = \Delta t$, that is

$$\|U_h^n - U^\infty\|_\infty \le C^* \left[ h^2 \left|\log h\right|^3 + \left(\frac{1+kc}{1+k\beta}\right)^n \right],$$

where $U_h^n$, the discrete solution of the evolutionary Hamilton-Jacobi-Bellman equations calculated at the moment-end $T = n\Delta t$ for an index of the time discretization $k = 1, \dots, n$ and $U^\infty$, the asymptotic continuous solution of the evolutionary Hamilton-Jacobi-Bellman equations.

In [14] Boulbrachene, Cortey Dumont established optimal $L^\infty$-error estimate of a finite element approximation of the Hamilton-Jacobi-Bellman (HJB) equations using the discrete regularity introduced by Cortey Dumont in [20], that is

$$\|u - u_h\|_\infty \le Ch^2 \left|\log h\right|^2,$$

where $u$, the continuous solution of the Hamilton-Jacobi-Bellman (HJB) equations, and $u_h$, the discrete solution of the Hamilton-Jacobi-Bellman (HJB) equations.

In a recent work in [7] Bencheikh, Boulaaras and Haiour also established optimal $L^\infty$-asymptotic behavior for a system of parabolic quasi-variational inequalities related to stochastic control problems using the regularization of the obstacles appearing in the discrete system of QVIs “the discrete regularity”, they have the following estimation

$$\|U_h(T,\cdot) - U^\infty(\cdot)\|_\infty \le C \left[ h^2 \left|\log h\right|^2 + \left(\frac{1}{1+\theta\Delta t}\right)^N \right],$$

where $U_h(T,\cdot)$, the discrete solution of the system of parabolic quasi-variational inequalities related to stochastic control problems calculated at the moment-end $T = N\Delta t$ for an index of the time discretization $k = 1, \dots, N$, and $U^\infty(\cdot)$, the asymptotic continuous solution of the system of parabolic quasi-variational inequalities related to stochastic control problems.

In this paper we propose a new proof to get the optimal $L^\infty$-asymptotic behavior of the system of parabolic QVIs with nonlinear source terms without going through the discrete regularity of the obstacles appearing in the discrete system of QVIs and we improve the convergence order in works of Boulaaras, Haiour [8,9] and Boulaaras, Bencheikh and Haiour [11] for the system of parabolic quasi-variational inequalities.

The subsolutions method (see [14,17,21]) characterizes the continuous solution (resp. the discrete solution) as the least upper bound of the set of continuous subsolutions (resp. the discrete subsolution) will also be crucial to determine the convergence order.

The approximation method developed in this paper stands on the construction a sequence of continuous subsolution denoted $\beta^k = (\beta^{1,k}, \dots, \beta^{J,k})$ such that

$$\beta^{i,k} \le u^{i,k} \quad \text{and} \quad \left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \le Ch^2 |\ln h|^2, \quad \text{for all } k \ge 1,\ i = 1, 2, \dots, J,$$

and the construction of a sequence of discrete subsolution $\alpha_h^k = (\alpha_h^{1,k}, \ldots, \alpha_h^{J,k})$ such that

$$\alpha_h^{i,k} \leq u_h^{i,k} \quad \text{and} \quad \left\| \alpha_h^{i,k} - u^{i,k} \right\|_\infty \leq Ch^2 |\ln h|^2, \quad \text{for all } k \geq 1,\ i = 1, 2, \ldots, J,$$

to obtain

$$\max_{1 \leq i \leq J} \left\| u^{i,k} - u_h^{i,k} \right\|_\infty \leq Ch^2 |\ln h|^2, \quad \text{for all } k \geq 1.$$

In this situation, we establish the optimal $L^\infty$-asymptotic behavior of the system of parabolic QVIs, that is

$$\| U_h^N - U^\infty \|_\infty = \max_{1 \leq i \leq J} \| u_h^{i,N} - u^{i,\infty} \|_\infty \leq C \left[ h^2 \ \ln h|^2 + \left( \frac{1 + \alpha \Delta t}{1 + \beta \Delta t} \right)^N \right].$$

The paper is organized as follows. In Section 2, we consider system of continuous quasi-variational inequalities and we give some related qualitative properties. In Section 3, we characterize the discrete solution as a fixed point of a contraction. In Section 4, we introduce two auxiliary problems which allow us to define sequences of continuous and discrete subsolutions. In Section 5, we present the main result of the paper.

## 2. The Continuous Problem

2.1. **Notations, Assumptions.** Let $a_{jp}^i(x)$, $a_p^i(x)$, $a_0^i(x)$ in $L^\infty(\Omega) \cap C^2(\overline{\Omega})$, $x \in \bar{\Omega}$, $j, p = 1, \ldots, S$, are sufficiently smooth coefficients and satisfying the following conditions:

$$\sum_{j,p=1}^{S} a_{jp}^i(x)\, \zeta_j \zeta_p \geqq \gamma\, |\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^S,\ \gamma > 0,\ x \in \overline{\Omega},$$

and

$$a_{jp}^i = a_{pj}^i, \quad a_0^i(x) \geqslant \beta > 0, \quad \beta \text{ is a constant.} \tag{2.1}$$

We define the second order differential operators $A^i$:

$$A^i = -\sum_{j,p=1}^{S} a_{jp}^i(x) \frac{\partial^2}{\partial x_j \partial x_p} + \sum_{p=1}^{S} a_p^i(x) \frac{\partial}{\partial x_p} + a_0^i(x),$$

and the associated variational forms for any $u, v \in H_0^1(\Omega)$

$$a^i(u,v) = \int_\Omega \left( \sum_{j,p=1}^{S} a_{jp}^i(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_p} + \sum_{p=1}^{S} a_p^i(x) \frac{\partial u}{\partial x_p} v + a_0^i(x) uv \right) dx.$$

We shall also need the following notations

$$\|W\|_\infty = \max_{1 \leq i \leq J} \left\| w^i \right\|_\infty, \quad \text{for all } W = \left( w^1, w^2, \ldots, w^J \right) \in \prod_{i=1}^{J} L^\infty(\Omega),$$

where $\|\cdot\|_\infty$ denotes the well-known $L^\infty$-norm, $(\cdot, \cdot)$ be the inner product in $L^2(\Omega)$.

2.2. **The system of continuous parabolic quasi-variational inequalities.** The problem (1.1) can be approximated by the following system of continuous parabolic quasi-variational inequalities: Find $U = (u^1, u^2, \dots, u^J) \in (L^2((0,T), H_0^1(\Omega)))^J$ solution for:

$$(2.2)\qquad \begin{cases} \left(\dfrac{\partial u^i}{\partial t}, v^i - u^i\right) + a^i\left(u^i, v^i - u^i\right) \geqq \left(f^i\left(u^i\right), v^i - u^i\right), \\ u^i \le Mu^i, \quad v^i \le Mu^i, \quad 1 \le i \le J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases}$$

Now, we apply the semi-implicit scheme of the system to the continuous parabolic quasi-variational inequalities (2.2). Therefore, we seek a sequence of elements $u^{i,k} \in (H_0^1(\Omega))^J$, $1 \le i \le J$, which approaches $u^i(t_k)$, $t_k = k\Delta t$, with initial data $u^{i,0}$. Thus, we have $k = 1, \dots, N$,

$$(2.3)\qquad \begin{cases} \left(\dfrac{u^{i,k} - u^{i,k-1}}{\Delta t}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k}\right), v^i - u^{i,k}\right), \\ u^{i,k} \le Mu^{i,k}, \quad v^i \le Mu^{i,k}, \quad 1 \le i \le J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases}$$

2.2.1. *Existence and uniqueness of continuous solution of the system of parabolic QVIs.* Let us recall just the main steps leading to the existence of a unique solution to system (2.3). For more details, we refer the reader to [4].

**A fixed point mapping associated with the continuous problem.**

Let $\mathbb{H}^+ = (L_+^\infty(\Omega))^J = \{\, V = (v^1, \dots, v^J) \text{ such that } v^i \in L_+^\infty(\Omega)\}$, where $L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$.

We introduce the following mapping:

$$(2.4)\qquad \begin{aligned} &T : \mathbb{H}^+ \to (L^\infty(\Omega))^J, \\ &W \to TW = \zeta^k = (\zeta^{1,k}, \dots, \zeta^{J,k}), \end{aligned}$$

we note $\zeta^{i,k} = \partial(F^{i,k}(w^i), Mw^i) \in (H_0^1(\Omega))^J$ for all $i = 1, \dots, J$, the solution of the following problem:

$$\begin{cases} b^i\left(\zeta^{i,k}, v^i - \zeta^{i,k}\right) \geqq \left(f^{i,k}\left(w^i\right) + \lambda w^i, v^i - \zeta^{i,k}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \zeta^{i,k} \le Mw^i, \quad v^i \le Mw^i, \end{cases}$$

where $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$.

**An iterative continuous algorithm.**

Let us also define the vector $U^0 = (u^{1,0}, \dots, u^{J,0})$, where $u^{i,0}$ is the solution to the continuous equation:

$$b^i\left(u^{i,0}, v^i\right) = \left(g^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J,$$

where $g^{i,0}$ is a linear and a regular function.

Now we give the following continuous algorithm

$$u^{i,k} = Tu^{i,k-1}, \quad k = 1, \ldots, N,\ i = 1, \ldots, J, \tag{2.5}$$

or

$$U^k = TU^{k-1},$$

where $U^k = \left(u^{1,k}, \ldots, u^{J,k}\right)$ is the solution of the problem (2.3).

*Remark* 2.1. We denote

$$\mathbb{C} = \left\{W \in \mathbb{H}^+ \mid 0 \leq W \leq U^0\right\},$$

where $U^0 = U_0 = \left(u_0^1, \ldots, u_0^J\right)$, $\mathbb{H}^+ = (L_+^\infty(\Omega))^J$. Since $f^{i,k}(\cdot) \geq 0$, combining comparison results in variational inequalities with simple induction, we obtain $U^k = \left(u^{1,k}, \ldots, u^{J,k}\right) \geq 0$ for all $k = 1, \ldots, N$ and $TW \geq 0$.

Similarly as in [12], the mapping $T$ is monotone increasing for the stationary free boundary problem with nonlinear source term. Then it can be easily verified that

$$U^2 = TU^1 \leq TU^0 = U^1 \leq U^0,$$

thus, inductively,

$$U^{k+1} = TU^k \leq U^k \leq \cdots \leq U^0, \quad \text{for all } k = 1, \ldots, N,$$

and also it can be seen that the sequence $U^k$ stays in $\mathbb{C}$.

According to assumption (1.2), $f$ is increasing, for $k = 1, \ldots, N$, $i = 1, \ldots, J$, and using the Remark 2.1, we have

$$f\left(U^k\right) \leq f\left(U^{k-1}\right)$$

or

$$f\left(u^{i,k}\right) \leq f\left(u^{i,k-1}\right),$$

which implies

$$\begin{cases} \left(\dfrac{u^{i,k}}{\Delta t}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k-1}\right) + \dfrac{u^{i,k-1}}{\Delta t}, v^i - u^{i,k}\right), \\ u^{i,k} \leq Mu^{i,k}, \quad v^i \leq Mu^{i,k}, \quad 1 \leq i \leq J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.6}$$

Then, the problem (2.6) can be reformulated into the following coercive continuous system of elliptic quasi-variational inequalities (EQVIs)

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v^i - u^{i,k}\right), \quad u^{i,k} \in \left(H_0^1(\Omega)\right)^J, \\ u^{i,k} \leq Mu^{i,k}, \quad v^i \leq Mu^{i,k}, \quad 1 \leq i \leq J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.7}$$

where

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) = \lambda\left(u^{i,k}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right), \quad u^{i,k} \in \left(H_0^1(\Omega)\right)^J, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{N}{T}, \quad k = 1, \ldots, N. \end{cases}$$

Then the bilinear form $b(\cdot,\cdot)$ is strongly coercive see [26]. There exist two constants $\lambda > 0$ and $\gamma > 0$ such that:

$$b^i(v,v) = a^i(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geqq \gamma \|v\|^2_{H_0^1(\Omega)}, \quad \text{for all } v \in H_0^1(\Omega).$$

Let $\mathbb{C} = \{W \in \mathbb{H}^+ \mid 0 \le W \le U^0\}$, where $U^0 = U_0 = \left(u_0^1, \ldots, u_0^J\right)$ and $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$, $\tilde{F}^{i,k}(\tilde{w}^i) = f^{i,k}(\tilde{w}^i) + \lambda \tilde{w}^i \in (L^\infty(\Omega))^J$ be the corresponding right-hand sides to the continuous PQVIs and $\mathbf{k}$ and $\tilde{\mathbf{k}}$ be two parameters that are defined in (1.2) and (1.3).

**A monotonicity property**

**Proposition 2.1** ([16,20])**.** *If* $F^{i,k}(w^i) \le F^{i,k}(\tilde{w}^i)$ *and* $\mathbf{k} \le \tilde{\mathbf{k}}$*, then*

$$u^{i,k} = \partial\left(F^{i,k}\left(w^i\right), \mathbf{k}\right) \le \tilde{u}^{i,k} = \partial\left(F^{i,k}\left(\tilde{w}^i\right), \tilde{\mathbf{k}}\right).$$

**Proposition 2.2** ([8,12])**.** *Under the previous assumption and notations* (1.2), (2.1), (2.4)*, the mapping* $T$ *is a contraction in* $\mathbb{H}^+$ *with contraction constant* $\frac{\alpha+\lambda}{\beta+\lambda}$*. Therefore,* $T$ *admits a unique fixed point which coincides with the continuous solution of the system of parabolic QVIs* (2.7)*.*

**Proposition 2.3** ([8])**.** *Under the conditions of Proposition* 2.2 *and notations* (1.2), (2.1), (2.4)*, we have the following estimate of geometric convergence*

$$\| U^k - U^\infty \|_\infty = \max_{1\le i\le J} \| u^{i,k} - u^{i,\infty} \|_\infty \le \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^k \| U^\infty - U^0 \|_\infty,$$

*where* $U^\infty$ *is an asymptotic continuous solution of the following system of QVIs*

$$\begin{cases} b^i\left(u^{i,\infty}, v^i - u^{i,\infty}\right) \ge \left(f^i\left(u^{i,\infty}\right) + \lambda u^{i,\infty}, v^i - u^{i,\infty}\right), \quad \textit{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ u^{i,\infty} \le M u^{i,\infty}, \quad i = 1, \ldots, J. \end{cases}$$

**Lipschitz dependence with respect to the right-hand sides and the parameter k**

**Proposition 2.4** ([14,21])**.** *Under the conditions of Proposition* 2.1*. Then we have:*

$$\max_{1\le i\le J} \| u^{i,k} - \tilde{u}^{i,k} \|_\infty \le C \max_{1\le i\le J} \left(\left|\mathbf{k} - \tilde{\mathbf{k}}\right| + \| F^{i,k} - \tilde{F}^{i,k} \|_\infty\right).$$

**Characterization of the solution of the system (2.7) as the envelope of continuous subsolutions**

**Definition 2.1** ([4])**.** $Z = (z^1, \dots, z^J) \in (H_0^1(\Omega))^J$ is said to be a continuous subsolution for the system of quasi-variational inequalities (2.7) if

$$\begin{cases} b^i\left(z^{i,k}, v^i\right) \leq \left(f^{i,k}\left(z^{i,k-1}\right) + \lambda z^{i,k-1}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \ v^i \geqq 0, \\ z^{i,k} \leq M z^{i,k}, \quad i = 1, \dots, J, \ k = 1, \dots, N. \end{cases}$$

Let $\mathbb{Y}$ denote the set of such continuous subsolutions.

**Theorem 2.1** ([4,21])**.** *The solution of the system* (2.7) *is the maximum element of the set* $\mathbb{Y}$.

## 3. The Discrete Problem

Let $\Omega$ be decomposed into triangles and let $\tau_h$ denote the set of all those elements, $h > 0$ is the mesh size. We assume the family $\tau_h$ is regular and quasi-uniform. We consider $\varphi_l$, $l = 1, 2, \dots, m(h)$, are the nodal basis functions defined by $\varphi_l(M_s) = \delta_{ls}$ where $M_s$, $s = 1, \dots, m(h)$, is a vertex of the considered triangulation and $r_h$ is the usual interpolation operator.

Let $\mathbb{V}_h$ denote the standard piecewise linear finite element space

$$\begin{aligned} \mathbb{V}_h = &\{u^i \in (L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\bar{\Omega})\right))^J \mid u^i \mid_{k_i} \in P_1, \\ &k_i \in \tau_h^i \text{ and } u^i(\cdot, 0) = u_0^i \text{ in } \Omega, u^i = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

$P_1$ denotes the space of polynomials with degree no more than 1 and $\mathbb{B}^i$, $1 \leq i \leq J$, denote the finite element matrices defined by

$$(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s), \quad 1 \leq l, s \leq m(h).$$

**The Discrete Maximum Principle Assumption (dmp)** (cf. [19]). We assume that the matrices $(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s) = a^i(\varphi_l, \varphi_s) + \lambda(\varphi_l, \varphi_s)$ are M-matrices.

Under the **dmp**, we shall achieve a similar study to that devoted to the continuous problem.

We discretize in space the problem (2.2), i.e., that we approach the space $H_0^1$ by a space discretization of finite dimensional $\mathbb{V}_h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we seek a sequence of elements $u_h^{i,k} \in (\mathbb{V}_h)^J$, $1 \leq i \leq J$, which approaches $u_h^i(t_k)$, $t_k = k\Delta t$ with initial data $u^{i,0}$. Thus, we have $k = 1, \dots, N$,

$$\begin{cases} \left(\dfrac{u_h^{i,k} - u_h^{i,k-1}}{\Delta t}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right), v_h^i - u_h^{i,k}\right), \\ u_h^{i,k} \leq r_h M u_h^{i,k}, \quad v_h^i \leq r_h M u_h^{i,k}, \quad 1 \leq i \leq J. \end{cases} \tag{3.1}$$

Then we can write (3.1) as follows:

$$\begin{cases} \left(\dfrac{u_h^{i,k}}{\Delta t}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right) + \dfrac{u_h^{i,k-1}}{\Delta t}, v_h^i - u_h^{i,k}\right). \\ u_h^{i,k} \leq r_h M u_h^{i,k}, \quad v_h^i \leq r_h M u_h^{i,k}, \quad 1 \leq i \leq J. \end{cases} \tag{3.2}$$

The problem (3.2) can be reformulated into the following coercive system of discrete elliptic quasi-variational inequalities:

(3.3)

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right) + \lambda u_h^{i,k-1}, v_h^i - u_h^{i,k}\right), \quad \text{for all } v_h^i \in (\mathbb{V}_h)^J, \\ u_h^{i,k} \leq r_h M u_h^{i,k}, \quad v_h^i \leq r_h M u_h^{i,k}, \end{cases}$$

such that

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) = \lambda\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right), \quad u_h^{i,k} \in (\mathbb{V}_h)^J, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{N}{T}, \quad k = 1, \ldots, N. \end{cases}$$

3.0.1. *Existence and uniqueness for discrete solution of the system of PQVI.* As in the continuous problem, we shall characterize the discrete solution of system of PQVI as the unique fixed point of a contraction.

**A fixed point mapping associated with discrete problem**

We introduce the following mapping:

$$T_h : \mathbb{H}^+ \to (\mathbb{V}_h)^J, \tag{3.4}$$

$$W \to T_h W = \zeta_h^k = (\zeta_h^{1,k}, \ldots, \zeta_h^{J,k}),$$

we keep the precedent notation, i.e., $\zeta_h^{i,k} = \partial_h(F^{i,k}(w^i), r_h M w^i) \in (\mathbb{V}_h)^J$ for all $i = 1, \ldots, J$, the solution to the following problem:

$$\begin{cases} b^i\left(\zeta_h^{i,k}, v_h^i - \zeta_h^{i,k}\right) \geqq \left(f^{i,k}\left(w^i\right) + \lambda w^i, v_h^i - \zeta_h^{i,k}\right), \quad \text{for all } v_h^i \in (\mathbb{V}_h)^J, \\ \zeta_h^{i,k} \leq r_h M w^i, \quad v_h^i \leq r_h M w^i, \end{cases}$$

where $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$.

**An iterative discrete algorithm**

Let us also define the vector $U_h^0 = (u_h^{1,0}, \ldots, u_h^{J,0})$, where $u_h^{i,0}$ is the solution of the continuous equation:

$$b^i\left(u_h^{i,0}, v^i\right) = \left(g^{i,0}, v^i\right), \quad \text{for all } v^i \in (\mathbb{V}_h)^J,$$

where $g^{i,0}$ is a linear and a regular function.

Now we give the following discrete algorithm

$$u_h^{i,k} = T_h u_h^{i,k-1}, \quad k = 1, \ldots, N,\ i = 1, \ldots, J, \tag{3.5}$$

or

$$U_h^k = T_h U_h^{k-1},$$

where $U_h^k = \left(u_h^{1,k}, \dots, u_h^{J,k}\right)$ is the solution of the problem (3.3).

We denote $\mathbb{C}_h = \{W \in \mathbb{H}^+ \mid 0 \le W \le U_h^0\}$, where $U_h^0 = \left(u_{h0}^1, \dots, u_{h0}^J\right)$ and $F^{i,k}\left(w^i\right) = f^{i,k}\left(w^i\right) + \lambda w^i$ $\tilde{F}^{i,k}\left(\tilde{w}^i\right) = f^{i,k}\left(\tilde{w}^i\right) + \lambda \tilde{w}^i \in \left(L^\infty\left(\Omega\right)\right)^J$ are the corresponding right-hand sides to the discrete PQVIs and $\mathbf{k}$ and $\tilde{\mathbf{k}}$ be two parameters.

As in the continuous case, we give some related qualitative properties of the discrete solution of the system of parabolic QVIs (3.3).

**A monotonicity property**

**Proposition 3.1** ([16, 20])**.** *If $F^{i,k}\left(w^i\right) \le F^{i,k}\left(\tilde{w}^i\right)$ and $\mathbf{k} \le \widetilde{\mathbf{k}}$, then*

$$u_h^{i,k} = \partial_h\left(F^{i,k}\left(w^i\right), \mathbf{k}\right) \le \tilde{u}_h^{i,k} = \partial_h\left(F^{i,k}\left(\tilde{w}^i\right), \tilde{\mathbf{k}}\right).$$

**Proposition 3.2** ([8, 12])**.** *Under the previous assumption, notations* (1.2), (2.1), (3.4) *and the* ***dmp****, the mapping $T_h$ is a contraction in $\mathbb{H}^+$ with contraction constant $\frac{\alpha+\lambda}{\beta+\lambda}$. Therefore, $T_h$ admits a unique fixed point which coincides with the discrete solution of the system of parabolic QVIs* (3.3)*.*

**Proposition 3.3** ([8])**.** *Under the conditions of Proposition* 3.2 *and notations* (1.2), (2.1), (3.4)*, we have the following estimate of geometric convergence*

$$\| U_h^k - U_h^\infty \|_\infty = \max_{1\le i\le J} \| u_h^{i,k} - u_h^{i,\infty} \|_\infty \le \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^k \| U_h^\infty - U_h^0 \|_\infty,$$

*where $U_h^\infty$ is an asymptotic discrete solution of the following system of QVIs*

$$\begin{cases} b^i(u_h^{i,\infty}, v_h^i - u_h^{i,\infty}) \ge \left(f^i\left(u_h^{i,\infty}\right) + \lambda u_h^{i,\infty}, v_h^i - u_h^{i,\infty}\right), \quad \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J, \\ u_h^{i,\infty} \le r_h M u_h^{i,\infty}, \quad i = 1, \dots, J. \end{cases}$$

**Lipschitz dependence with respect to the right-hand sides and the parameter k**

**Proposition 3.4** ([14, 21])**.** *Under the* ***dmp*** *and the Proposition* 3.1*, we have:*

$$\max_{1\le i\le J} \| u_h^{i,k} - \tilde{u}_h^{i,k} \|_\infty \le C \max_{1\le i\le J}\left(\left|\mathbf{k} - \tilde{\mathbf{k}}\right| + \| F^{i,k} - \tilde{F}^{i,k} \|_\infty\right).$$

**Characterization of the solution of system** (3.3) **as the envelope of discrete subsolutions**

**Definition 3.1** ([4])**.** $Z_h = (z_h^1, \dots, z_h^J) \in (\mathbb{V}_h)^J$ is said to be a discrete subsolution for the system of quasi-variational inequalities (3.3) if

$$\begin{cases} b^i\left(z_h^{i,k}, \varphi_l\right) \le \left(f^{i,k}\left(z_h^{i,k-1}\right) + \lambda z_h^{i,k-1}, \varphi_l\right), \quad \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J, \varphi_l \ge 0, \\ l = 1, \dots, m\left(h\right), \\ z_h^{i,k} \le r_h M z_h^{i,k}, \quad i = 1, \dots, J, \, k = 1, \dots, N. \end{cases}$$

Let $\mathbb{Y}_h$ denote the set of such discrete subsolutions.

**Theorem 3.1** ([4, 21])**.** *The discrete solution of the system* (3.3) *is the maximum element of the set $\mathbb{Y}_h$.*

## 4. $L^\infty$-Error Estimates

In this section, we first introduce the following two auxiliary systems of variational inequalities and next we prove a fundamental lemma of the subsolutions method.

4.1. **Two auxiliary sequences of system of variational inequalities.** We define the sequence $\left\{\bar{U}^k\right\}_{k\geq 1} = \left(\bar{u}^{1,k}, \ldots, \bar{u}^{J,k}\right)$ such that $\bar{U}^k$ solves the continuous system of V.I.

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i - \bar{u}^{i,k}\right) \geqq \left(f^i\left(u_h^{i,k-1}\right) + \lambda u_h^{i,k-1}, v^i - \bar{u}^{i,k}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k} \leq M u_h^{i,k-1}, \quad v^i \leq M u_h^{i,k-1}, \end{cases}$$

where $U_h^{k-1} = \left(u_h^{1,k-1}, \ldots, u_h^{J,k-1}\right)$ is defined in (3.5), and the sequence $\left\{\bar{U}_h^k\right\}_{k\geq 1} = \left(\bar{u}_h^{1,k}, \ldots, \bar{u}_h^{J,k}\right)$ is such that $\bar{U}_h^k$ solves the discrete system of V.I.

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, v_h^i - \bar{u}_h^{i,k}\right) \geqq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v_h^i - \bar{u}_h^{i,k}\right), \quad \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J, \\ \bar{u}_h^{i,k} \leq r_h M u^{i,k-1}, \quad v_h^i \leq r_h M u^{i,k-1}, \end{cases}$$

where $U^{k-1} = \left(u^{1,k-1}, \ldots, u^{J,k-1}\right)$ is defined in (2.5).

**Lemma 4.1** ([20, 21]). *There exists a constant $C$ independent of $h$ and $k$ such that*

$$\max_{1\leq i\leq J} \| \bar{u}^{i,k} - u_h^{i,k} \|_\infty \leq Ch^2 |\ln h|^2$$

*and*

$$\max_{1\leq i\leq J} \| \bar{u}_h^{i,k} - u^{i,k} \|_\infty \leq Ch^2 |\ln h|^2.$$

4.2. **Optimal $L^\infty$-error estimates.** Now, we obtain the optimal $L^\infty$-error estimate between the $k$-th continuous iterates $u^{i,k}$ and $k$-th discrete iterates $u_h^{i,k}$ defined in (2.7) and (3.3), respectively.

In this theorem, we exploit the idea of Boulbrachene in [13] given for variational inequalities with noncoercive operators, where we have adapted it to a system of QVIs related to the management of energy production problem.

**Theorem 4.1.**

$$\left\|U^k - U_h^k\right\|_\infty = \max_{1\leq i\leq J} \left\|u^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2.$$

The following lemma plays crucial role in proving the Theorem 4.1.

**Lemma 4.2.** *There exists a sequence of continuous subsolutions $\left(\beta^k\right)_{k\geq 1} = \left(\beta^{1,k}, \ldots, \beta^{J,k}\right)$, such that*

$$\beta^{i,k} \leq u^{i,k}, \quad 1 \leq k \leq N, 1 \leq i \leq J,$$

*and*

$$\left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2,$$

*and a sequence of discrete subsolutions* $\left(\alpha_h^k\right)_{k\geq 1} = \left(\alpha_h^{1,k}, \ldots, \alpha_h^{J,k}\right)$*, such that*

$$\alpha_h^{i,k} \leq u_h^{i,k}, \quad 1 \leq k \leq N,\, 1 \leq i \leq J,$$

*and*

$$\left\|\alpha_h^{i,k} - u^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2.$$

*Proof.* Let $\bar{U}^1$ be continuous solution of the system of V.I.

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i - \bar{u}^{i,1}\right) \geqq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0}, v^i - \bar{u}^{i,1}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf\limits_{\mu\neq i} u_h^{\mu,0}, \quad v^i \leq \mathbf{k} + \inf\limits_{\mu\neq i} u_h^{\mu,0}. \end{cases}$$

Then, as $\bar{U}^1 = (\bar{u}^{i,1})_{1\leq i\leq J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i\right) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf\limits_{\mu\neq i} u_h^{\mu,0}, \quad v^i \leq \mathbf{k} + \inf\limits_{\mu\neq i} u_h^{\mu,0}, \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i\right) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0} - \lambda u^{i,0} + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf\limits_{\mu\neq i} u_h^{\mu,0} - \inf\limits_{\mu\neq i} u^{\mu,0} + \inf\limits_{\mu\neq i} u^{\mu,0}. \end{cases}$$

We have

$$\left\|u_h^{i,0} - u^{i,0}\right\|_\infty \leq Ch^2 \left|\ln h\right|^{\frac{3}{2}} \quad \text{(see [23])}, \tag{4.1}$$

then

$$\begin{cases} b^i(\bar{u}^{i,1}, v^i) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda\left\|u_h^{i,0} - u^{i,0}\right\|_\infty + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \left\|\inf\limits_{\mu\neq i} u_h^{\mu,0} - \inf\limits_{\mu\neq i} u^{\mu,0}\right\|_\infty + \inf\limits_{\mu\neq i} u^{\mu,0}, \end{cases}$$

and using (4.1), we get

$$\begin{cases} b^i(\bar{u}^{i,1}, v^i) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \inf\limits_{\mu\neq i} u^{\mu,0}. \end{cases}$$

As $\bar{U}^1 = (\bar{u}^{i,1})_{1\leq i\leq J}$ is a subsolution for the system of V.I., where the solution is $\tilde{U}^1 = (\tilde{u}^{i,1})_{1\leq i\leq J} = \partial\left(f^i\left(u_h^{i,0}\right) + \lambda Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0}, \mathbf{k} + Ch^2 \left|\ln h\right|^{\frac{3}{2}}\right)$.

Let $U^1 = \left(u^{i,1}\right)_{1\leq i\leq J} = \partial\left(f^i\left(u_h^{i,0}\right) + \lambda u^{i,0}, \mathbf{k}\right)$ using the Proposition 2.4, we get

$$\begin{aligned}\left\|\tilde{u}^{i,1} - u^{i,1}\right\|_\infty \leq & C\left(\left\|f^i\left(u_h^{i,0}\right) + \lambda Ch^2\left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0} - f^i\left(u_h^{i,0}\right) - \lambda u^{i,0}\right\|_\infty\right.\\ & \left. + \left\|\mathbf{k} + Ch^2\left|\ln h\right|^{\frac{3}{2}} - \mathbf{k}\right\|_\infty\right)\\ \leq & C\left(\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}} + Ch^2\left|\ln h\right|^{\frac{3}{2}}\right)\\ \leq & Ch^2\left|\ln h\right|^{\frac{3}{2}},\end{aligned}$$

and using the Theorem 2.1, we have

$$\bar{u}^{i,1} \leq \tilde{u}^{i,1} \leq u^{i,1} + Ch^2\left|\ln h\right|^{\frac{3}{2}}.$$

Now taking $\beta^{i,1} = \bar{u}^{i,1} - Ch^2\left|\ln h\right|^{\frac{3}{2}}$, we have

$$\beta^{i,1} \leq u^{i,1}, \tag{4.2}$$

and

$$\begin{aligned}\left\|\beta^{i,1} - u_h^{i,1}\right\|_\infty \leq & \left\|\bar{u}^{i,1} - Ch^2\left|\ln h\right|^{\frac{3}{2}} - u_h^{i,1}\right\|_\infty\\ \leq & \left\|\bar{u}^{i,1} - u_h^{i,1}\right\|_\infty + Ch^2\left|\ln h\right|^{\frac{3}{2}}\\ \leq & Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^{\frac{3}{2}}\\ \leq & Ch^2\left|\ln h\right|^2.\end{aligned} \tag{4.3}$$

Let $\bar{U}_h^1$ be the discrete solution of the system of V.I.

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, v_h^i - \bar{u}_h^{i,1}\right) \geqq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0}, v_h^i - \bar{u}_h^{i,1}\right), & \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J,\\ \bar{u}_h^{i,1} \leq r_h\left(k + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \quad v_h^i \leq r_h\left(k + \inf\limits_{\mu\neq i} u^{\mu,0}\right). \end{cases}$$

Then, as $\bar{U}_h^1 = \left(\bar{u}_h^{i,1}\right)_{1\leq i\leq J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \quad v_h^i \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0} - \lambda u_h^{i,0} + \lambda u_h^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right). \end{cases}$$

Then

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0} - \lambda u_h^{i,0} + \lambda u_h^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq \mathbf{k} + r_h\left(\inf\limits_{\mu\neq i} u^{\mu,0}\right) - r_h\left(\inf\limits_{\mu\neq i} u_h^{\mu,0}\right) + r_h\left(\inf\limits_{\mu\neq i} u_h^{\mu,0}\right), \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}_h^{i,1},\varphi_s\right)\leq\left(f^i\left(u^{i,0}\right)+\lambda\left\|u^{i,0}-u_h^{i,0}\right\|_\infty+\lambda u_h^{i,0},\varphi_s\right), & \text{for all } \varphi_s,\ s=1,\ldots,m(h),\\ \bar{u}_h^{i,1}\leq\mathbf{k}+\left\|r_h\left(\inf\limits_{\mu\neq i}u^{\mu,0}\right)-r_h\left(\inf\limits_{\mu\neq i}u_h^{\mu,0}\right)\right\|_\infty+r_h\left(\inf\limits_{\mu\neq i}u_h^{\mu,0}\right), \end{cases}$$

using (4.1) , we get

$$\begin{cases} b^i\left(\bar{u}_h^{i,1},\varphi_s\right)\leq\left(f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0},\varphi_s\right), & \text{for all } \varphi_s,\ s=1,\ldots,m(h),\\ \bar{u}_h^{i,1}\leq\mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}+r_h\left(\inf\limits_{\mu\neq i}u_h^{\mu,0}\right). \end{cases}$$

As $\bar{U}_h^1=\left(\bar{u}_h^{i,1}\right)_{1\leq i\leq J}$ is a subsolution for the system of V.I., where the solution is $\tilde{U}_h^1=\left(\tilde{u}_h^{i,1}\right)_{1\leq i\leq J}=\partial_h\left(f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0},\mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}\right)$.

Let $U_h^1=\left(u_h^{i,1}\right)_{1\leq i\leq J}=\partial_h\left(f^i\left(u^{i,0}\right)+\lambda u_h^{i,0},\mathbf{k}\right)$. Using Proposition 3.4, we have

$$\begin{aligned}\left\|\tilde{u}_h^{i,1}-u_h^{i,1}\right\|_\infty\leq&C\left(\left\|f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0}-f^i\left(u^{i,0}\right)-\lambda u_h^{i,0}\right\|_\infty\right.\\&\left.+\left\|\mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}-\mathbf{k}\right\|_\infty\right)\\\leq&C\left(\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+Ch^2\left|\ln h\right|^{\frac{3}{2}}\right)\\\leq&Ch^2\left|\ln h\right|^{\frac{3}{2}},\end{aligned}$$

and using Theorem 3.1, we get

$$\bar{u}_h^{i,1}\leq\tilde{u}_h^{i,1}\leq u_h^{i,1}+Ch^2\left|\ln h\right|^{\frac{3}{2}}.$$

Now taking $\alpha_h^{i,1}=\bar{u}_h^{i,1}-Ch^2\left|\ln h\right|^{\frac{3}{2}}$, we get

$$\alpha_h^{i,1}\leq u_h^{i,1}\tag{4.4}$$

and

$$\begin{aligned}\left\|\alpha_h^{i,1}-u^{i,1}\right\|_\infty\leq&\left\|\bar{u}_h^{i,1}-Ch^2\left|\ln h\right|^{\frac{3}{2}}-u^{i,1}\right\|_\infty\\\leq&\left\|\bar{u}_h^{i,1}-u^{i,1}\right\|_\infty+Ch^2\left|\ln h\right|^{\frac{3}{2}}\\\leq&Ch^2\left|\ln h\right|^2+Ch^2\left|\ln h\right|^{\frac{3}{2}}\\\leq&Ch^2\left|\ln h\right|^2.\end{aligned}\tag{4.5}$$

Then, according to (4.2), (4.3) and (4.4), (4.5), we get

$$\begin{aligned}u^{i,1}\leq&\alpha_h^{i,1}+Ch^2\left|\ln h\right|^2\leq u_h^{i,1}+Ch^2\left|\ln h\right|^2,\\u_h^{i,1}\leq&\beta^{i,1}+Ch^2\left|\ln h\right|^2\leq u^{i,1}+Ch^2\left|\ln h\right|^2.\end{aligned}$$

Thus,

$$\left\|u^{i,1}-u_h^{i,1}\right\|_\infty\leq Ch^2\left|\ln h\right|^2.$$

Therefore,

$$\max_{1\le i\le J}\left\|u^{i,1}-u_h^{i,1}\right\|_\infty \le Ch^2\left|\ln h\right|^2.$$

For $k$ we assume that

$$\left\|u^{i,k-1}-u_h^{i,k-1}\right\|_\infty \le Ch^2\left|\ln h\right|^2 \tag{4.6}$$

and we prove that

$$\left\|u^{i,k}-u_h^{i,k}\right\|_\infty \le Ch^2\left|\ln h\right|^2.$$

For that, consider the following system of continuous V.I.

$$\begin{cases} b^i\left(\bar u^{i,k}, v^i-\bar u^{i,k}\right) \geqq \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}, v^i-u^{i,k}\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar u^{i,k}\le \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}, \quad v^i\le \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

Then, as $\bar U^k=\left(\bar u^{i,k}\right)_{1\le i\le J}$ be a solution to a system of V.I. it is also a subsolution i.e.,

$$\begin{cases} b^i\left(\bar u^{i,k}, v^i\right) \le \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}, v^i\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar u^{i,k}\le \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}, \quad v^i\le \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

Then

$$\begin{cases} b^i\left(\bar u^{i,k}, v^i\right) \le \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}-\lambda u^{i,k-1}+\lambda u^{i,k-1}, v^i\right), \\ \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar u^{i,k}\le \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}-\inf\limits_{\mu\neq i} u^{\mu,k-1}+\inf\limits_{\mu\neq i} u^{\mu,k-1}, \end{cases}$$

and

$$\begin{cases} b^i\left(\bar u^{i,k}, v^i\right) \le \left(f^i\left(u_h^{i,k-1}\right)+\lambda\left\|u_h^{i,k-1}-u^{i,k-1}\right\|_\infty+\lambda u^{i,k-1}, v^i\right), \\ \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar u^{i,k}\le \mathbf{k}+\left\|\inf\limits_{\mu\neq i} u_h^{\mu,k-1}-\inf\limits_{\mu\neq i} u^{\mu,k-1}\right\|_\infty+\inf\limits_{\mu\neq i} u^{\mu,k-1}. \end{cases}$$

Using (4.6), we get

$$\begin{cases} b^i\left(\bar u^{i,k}, v^i\right) \le \left(f^i\left(u_h^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u^{i,k-1}, v^i\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar u^{i,k}\le \mathbf{k}+Ch^2\left|\ln h\right|^2+\inf\limits_{\mu\neq i} u^{\mu,k-1}. \end{cases}$$

Let $\bar U^k=\left(\bar u^{i,k}\right)_{1\le i\le J}$ be a subsolution for the system of V.I. whose solution is $\tilde U^k=\left(\tilde u^{i,k}\right)_{1\le i\le J}=\partial\left(f^i\left(u_h^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u^{i,k-1}, \mathbf{k}+Ch^2\left|\ln h\right|^2\right)$.

Then, as $U^k = \left(u^{i,k}\right)_{1\leq i\leq J} = \partial\left(f^i\left(u_h^{i,k-1}\right) + \lambda u^{i,k-1}, \mathbf{k}\right)$ making use of Proposition 2.4, we have

$$\begin{aligned}\left\|\tilde{u}^{i,k} - u^{i,k}\right\|_\infty \leq & C\left(\left\|f^i\left(u_h^{i,k-1}\right) + \lambda Ch^2\left|\ln h\right|^2 - f^i\left(u_h^{i,k-1}\right)\right\|_\infty\right.\\ & \left. + \left\|\mathbf{k} + Ch^2\left|\ln h\right|^2 - \mathbf{k}\right\|_\infty\right)\\ \leq & C(\lambda Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2)\\ \leq & Ch^2\left|\ln h\right|^2,\end{aligned}$$

and, using Theorem 2.1, we have

$$\bar{u}^{i,k} \leq \tilde{u}^{i,k} \leq u^{i,k} + Ch^2\left|\ln h\right|^2.$$

Now putting $\beta^{i,k} = \bar{u}^{i,k} - Ch^2\left|\ln h\right|^2$, we get

$$\beta^{i,k} \leq u^{i,k} \tag{4.7}$$

and

$$\begin{aligned}\left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \leq & \left\|\bar{u}^{i,k} - Ch^2\left|\ln h\right|^2 - u_h^{i,k}\right\|_\infty\\ \leq & \left\|\bar{u}^{i,k} - u_h^{i,k}\right\|_\infty + Ch^2\left|\ln h\right|^2\\ \leq & Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2\\ \leq & Ch^2\left|\ln h\right|^2.\end{aligned}\tag{4.8}$$

Let $\bar{U}_h^k$ be the discrete solution of the following system of V.I.

$$\begin{cases}b^i\left(\bar{u}_h^{i,k}, v_h^i - \bar{u}_h^{i,k}\right) \geqq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v_h^i - \bar{u}_h^{i,k}\right), & \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J,\\ \bar{u}_h^{i,k} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,k-1}\right), \quad v \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,k-1}\right).\end{cases}$$

Then, as $\bar{U}_h^k = \left(\bar{u}_h^{i,k}\right)_{1\leq i\leq J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases}b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \leq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,k} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,k-1}\right), \quad v \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,k-1}\right).\end{cases}$$

Then we have

$$\begin{cases}b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \leq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1} - \lambda u_h^{i,k-1} + \lambda u_h^{i,k-1}, \varphi_s\right), & \text{for all } \varphi_s,\\ \bar{u}_h^{i,k} \leq \mathbf{k} + r_h \inf\limits_{\mu\neq i} u^{\mu,k-1} - r_h \inf\limits_{\mu\neq i} u_h^{\mu,k-1} + r_h \inf\limits_{\mu\neq i} u_h^{\mu,k-1},\end{cases}$$

and

$$\begin{cases}b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \leq \left(f^i\left(u^{i,k-1}\right) + \lambda\left\|u^{i,k-1} - u_h^{i,k-1}\right\|_\infty + \lambda u_h^{i,k-1}, \varphi_s\right), & \text{for all } \varphi_s,\\ \bar{u}_h^{i,k} \leq \mathbf{k} + \left\|r_h \inf\limits_{\mu\neq i} u^{\mu,k-1} - r_h \inf\limits_{\mu\neq i} u_h^{\mu,k-1}\right\|_\infty + r_h \inf\limits_{\mu\neq i} u_h^{\mu,k-1}.\end{cases}$$

Using (4.6), we obtain

$$\begin{cases} b^i\left(\bar{u}_h^{i,k},\varphi_s\right) \le \left(f^i\left(u^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u_h^{i,k-1},\varphi_s\right), \quad \text{for all } \varphi_s, \\ \bar{u}_h^{i,k} \le \mathbf{k}+Ch^2\left|\ln h\right|^2+r_h\inf_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

So, $\bar{U}_h^k = \left(\bar{u}_h^{i,k}\right)_{1\le i\le J}$ is a subsolution for the system of V.I. whose solution is $\tilde{U}_h^k = \left(\tilde{u}_h^{i,k}\right)_{1\le i\le J} = \partial_h\left(f^i\left(u^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u_h^{i,k-1},\mathbf{k}+Ch^2\left|\ln h\right|^2\right)$. Then, as $U_h^k = \left(u_h^{i,k}\right)_{1\le i\le J} = \partial_h\left(f^i\left(u^{i,k-1}\right)+\lambda u_h^{i,k-1},\mathbf{k}\right)$ making use of Proposition 3.4, we have

$$\begin{aligned}\left\|\tilde{u}_h^{i,k}-u_h^{i,k}\right\|_\infty \le & C\left(\left\|f^i\left(u^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2-f^i\left(u^{i,k-1}\right)\right\|_\infty\right. \\ & \left.+\left\|\mathbf{k}+Ch^2\left|\ln h\right|^2-\mathbf{k}\right\|_\infty\right) \\ \le & C(\lambda Ch^2\left|\ln h\right|^2+Ch^2\left|\ln h\right|^2) \\ \le & Ch^2\left|\ln h\right|^2,\end{aligned}$$

and, using Theorem 3.1, we have

$$\bar{u}_h^{i,k} \le \tilde{u}_h^{i,k} \le u_h^{i,k}+Ch^2\left|\ln h\right|^2.$$

Now, putting $\alpha_h^{i,k} = \bar{u}_h^{i,k}-Ch^2\left|\ln h\right|^2$, we have

$$\alpha_h^{i,k} \le u_h^{i,k} \tag{4.9}$$

and

$$\begin{aligned}\left\|\alpha_h^{i,k}-u^{i,k}\right\|_\infty \le & \left\|\bar{u}_h^{i,k}-Ch^2\left|\ln h\right|^2-u^{i,k}\right\|_\infty \\ \le & \left\|\bar{u}_h^{i,k}-u^{i,k}\right\|_\infty+Ch^2\left|\ln h\right|^2 \\ \le & Ch^2\left|\ln h\right|^2+Ch^2\left|\ln h\right|^2 \\ \le & Ch^2\left|\ln h\right|^2.\end{aligned} \tag{4.10}$$

Then, combining (4.7), (4.8) and (4.9), (4.10), we get

$$\begin{aligned} u^{i,k} \le & \alpha_h^{i,k}+Ch^2\left|\ln h\right|^2 \le u_h^{i,k}+Ch^2\left|\ln h\right|^2, \\ u_h^{i,k} \le & \beta^{i,k}+Ch^2\left|\ln h\right|^2 \le u^{i,k}+Ch^2\left|\ln h\right|^2.\end{aligned}$$

Thus,

$$\left\|u^{i,k}-u_h^{i,k}\right\|_\infty \le Ch^2\left|\ln h\right|^2.$$

Therefore,

$$\left\|U^k-U_h^k\right\|_\infty = \max_{1\le i\le J}\left\|u^{i,k}-u_h^{i,k}\right\|_\infty \le Ch^2\left|\ln h\right|^2. \qquad \square$$

## 5. Asymptotic Behavior in $L^\infty$-Norm

This section is devoted to the proof of the main result of the present paper, where we prove the optimal $L^\infty$-asymptotic behavior for the system of parabolic quasi-variational inequalities with nonlinear source terms. More precisely, we evaluate the variation in $L^\infty$ between $U_h^N$, the discrete solution calculated at the moment $T = N\Delta t$ and $U^\infty$, the stationary continuous solution of the system of QVIs.

**Theorem 5.1.** *Under the results of the Proposition* 2.3 *and Theorem* 4.1, *we have*

$$\left\|U_h^N - U^\infty\right\|_\infty \leq C\left[h^2\left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N\right], \tag{5.1}$$

*where $C$ is a constant independent of $h$ and $N$, $\beta > 0$ is constant and $\alpha < \beta$ Lipschitz constant.*

*Proof.* We have

$$u_h^{i,k} = u_h^i\left(t, x\right), \quad \text{for } t \in \left](k-1)\,t, kt\right[.$$

Thus,

$$u_h^{i,N} = u_h^i\left(T, x\right),$$

then

$$\begin{aligned}\left\|u_h^{i,N} - u^{i,\infty}\right\|_\infty &= \left\|u_h^{i,N} - u^{i,N} + u^{i,N} - u^{i,\infty}\right\|_\infty \\ &\leq \left\|u_h^{i,N} - u^{i,N}\right\|_\infty + \left\|u^{i,N} - u^{i,\infty}\right\|_\infty.\end{aligned}$$

Using Theorem 4.1 and Proposition 2.3, we get,

$$\left\|u_h^{i,N} - u^{i,\infty}\right\|_\infty \leq C\left[h^2\left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N\right],$$

which yields the following estimate:

$$\left\|U_h^N - U^\infty\right\|_\infty = \max_{1\leq i\leq J}\left\|u_h^{i,N} - u^{i,\infty}\right\|_\infty \leq C\left[h^2\left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N\right]. \qquad \square$$

*Remark* 5.1. In the previous estimate (5.1), $\left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N$ tends to 0 when $N \to +\infty$. Then, we obtain the optimal $L^\infty$-error estimate for the system of elliptic quasi-variational inequalities related to management of energy production problems (cf. [16]):

$$\left\|U_h^\infty - U^\infty\right\|_\infty \leq Ch^2\left|\ln h\right|^2.$$

If we replace $Mu^i$ in (1.3) by $Mu = \mathbf{k} + \inf\limits_{\xi\geq 0, x+\xi\in\bar{\Omega}}\left(u+\xi\right)$ and $f\left(u\right)$ by $f$, the problem (2.2) reduces to the parabolic quasi-variational inequalities related to impulse control problem with linear source term (cf. [10]). Find $u \in K(u)$

$$\left(\frac{\partial u}{\partial t}, v - u\right) + a\left(u, v-u\right) \geq \left(f, v-u\right), \quad \text{for all } v \in K(u),$$

with
$$K(u) = \left\{ u \in L^2\left(0, T; H_0^1(\Omega)\right) \mid u \leq Mu,\ u(0,x) = u_0 \text{ in } \Omega \right\}.$$
In this case, the error estimate given in (5.1) becomes
$$\left\| u_h^N - u^\infty \right\|_\infty \leq C \left[ h^2 \left| \ln h \right|^2 + \left( \frac{1}{1 + \beta \Delta t} \right)^N \right].$$

If we replace $Mu^i$ in (1.3) by $Mu^i = l + u^{i+1}$, where $Mu^i = l + u^{i+1}$ represents the obstacle of Hamilton Jacobi Bellman equation, the problem (2.2) reduces to the system of evolutionary Hamilton Jacobi Bellman (HJB) equation with nonlinear source terms (cf [8]): Find a victor $U = \left(u^1, \ldots, u^J\right) \in \left(L^2\left(0, T; H_0^1(\Omega)\right)\right)^J$ such that
$$\begin{cases} \left( \dfrac{\partial u^i}{\partial t}, v^i - u^i \right) + a^i\left(u^i, v^i - u^i\right) \geqq \left(f^i\left(u^i\right), v^i - u^i\right), \\ u^i \leq l + u^{i+1}, \quad v^i \leq l + u^{i+1}, \quad u^{J+1} = u^1, \quad 1 \leq i \leq J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases}$$

In this case, we get the following error estimate:
$$\max_{1 \leq i \leq J} \left\| u_h^{i,N} - u^{i,\infty} \right\|_\infty \leq C \left[ h^2 \left| \ln h \right|^2 + \left( \frac{1 + \alpha \Delta t}{1 + \beta \Delta t} \right)^N \right].$$

*Conclusion* 1. We have introduced a new approach and we have obtained the optimal $L^\infty$-asymptotic behavior for the finite element approximation of the system of parabolic quasi-variational inequalities with nonlinear source terms. This method stands on the Bensoussan-Lions algorithm and the concept of subsolutions. A future work will consolidate our theoretical results by numerical simulation, where efficient numerical monotone algorithms will be treated.

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## References

[1] C. Baiocchi, *Estimation d'erreur dans $L^\infty$ pour les inequations a obstacle*, Mathematical Aspects of Finite Methods **606** (1977), 27–34. https://doi.org/10.1007/BFb0064453

[2] D. C. Benchettah and M. Haiour, *$L^\infty$-asymptotic behavior of the variational inequality related to American options problem*, Applied Mathematics **5**(8) (2014), 1299–1309. https://doi.org/10.4236/am.2014.58122

[3] D. C. Benchettah and M. Haiour, *Sub-solution approach for the asymptotic behavior of a parabolic variational inequality related to American options problem*, Global Journal of Pure and Applied Mathematics, **11**(4) (2015), 1727–1745.

[4] A. Bensoussan and J. L. Lions, *Impulse Control and Quasi-Variational Inequalities*, Gauthier Villars, Paris, 1984.

[5] A. Bensoussan and J. L. Lions, *Applications des inèquations variationnelles en contrôle stochastique*, Dunod, Paris, 1978.

[6] M. A. Bencheikh le Hocine, S. Boulaaras and M. Haiour, *An optimal $L^\infty$-error estimate for an approximation of a parabolic variational inequality*, Numer. Funct. Anal. Optim. **37**(1) (2015), 1–18. https://doi.org/10.1080/01630563.2015.1109520

[7] M. A. Bencheikh le Hocine, S. Boulaaras and M. Haiour, *On finite element approximation of system of parabolic quasi-variationnal inequalities related to stochastic control problems*, Cogent Math. **3**(1) (2016), Paper ID 1251386. https://doi.org/10.1080/23311835.2016.1251386

[8] S. Boulaaras and M. Haiour, *The finite element approximation of evolutionary Hamilton-Jacobi-Bellman equations with nonlinear source terms*, Indag. Math. **24**(1) (2013), 161–173. https://doi.org/10.1016/j.indag.2012.07.005

[9] S. Boulaaras and M. Haiour, *The theta time scheme combined with a finite-element spatial approximation in the evolutionary Hamilton-Jacobi-Bellman equation with linear source terms*, Comput. Math. Model. **25**(3) (2014), 423–438.

[10] S. Boulaaras and M. Haiour, *$L^\infty$-asymptotic behavior for a finite element approximation in parabolic quasi-variational inequalities related to impulse control problem*, Appl. Math. Comput. **217**(14) (2011), 6443–6450. https://doi.org/10.1016/j.amc.2011.01.025

[11] S. Boulaaras, M. A. Bencheikh le Hocine and M. Haiour, *The finite element approximation in a system of parabolic quasi-variationnal inequalities related to management of energy production with mixed boundary condition*, Comput. Math. Model. **25**(4) (2014), 530–543.

[12] M. Boulbrachene, *Pointwise error estimates for a class of elliptic quasi-variational inequalities with nonlinear source terms*, Appl. Math. Comput. **161**(1) (2005), 129–138.

[13] M. Boulbrachene, *On the finite element approximation of variational inequalities with noncoercive operators*, Numer. Funct. Anal. Optim. **36**(9) (2015), 1107–1121. https://doi.org/10.1080/01630563.2015.1056913

[14] M. Boulbrachene and P. Cortey Dumont, *Optimal $L^\infty$-error estimate of a finite element method for Hamilton Jacobi Bellman equations*, Numer. Funct. Anal. Optim. **30**(5–6) (2009), 421–435. https://doi.org/10.1080/01630560902987683

[15] M. Boulbrachene, *On the finite element approximation of variational inequalities with noncoercive operators*, Numer. Funct. Anal. Optim. **36**(9) (2015), 1107–1121.

[16] M. Boulbrachene, *Pointwise error estimate for a noncoercive system of quasi-variational inequalities related to the management of energy production*, Journal of Inequalities in Pure and Applied Mathematics **3**(5) (2002), Article ID 79.

[17] M. Boulbrachene, *On variational inequalities with vanishing zero term*, J. Inequal. Appl. (2013), Article ID 438. https://doi.org/10.1186/1029-242X-2013-438

[18] F. Brezzi and L. A. Caffarelli, *Convergence of the discrete free boundaries for finite element approximations*, RAIRO Analyse Numérique **17**(4) (1983), 385–395.

[19] P. G. Ciarlet and P. A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Comput. Methods Appl. Mech. Engrg. **2**(1) (1973), 17–31.

[20] P. Cortey-Dumont, *Sur l'analyse numérique des équations de Hamilton-Jacobi-Bellman*, Math. Methods Appl. Sci. **9**(1) (1987), 198–209. https://doi.org/10.1002/mma.1670090115

[21] P. Cortey-Dumont, *Sur les inéquations variationnelles à opérateurs non coercif*, Modélisation mathématique et analyse numérique **19**(2) (1985), 195–212.

[22] J. Hannouzet and P. Joly, *Convergence uniforme des iteres definissant la solution d'une inéquation quasi-variationnelle*, C. R. Math. Acad. Sci. Paris, Serie *A* **286** (1978), 1 page. https://10.5802/jedp.172

[23] J. Nitsche, *$L^\infty$-convergence of finite element approximations, mathematical aspects of finite element methods*, Lecture Notes in Math. **606** (1977), 261–274.

[24] R. H. Nochetto, *A note on the approximation of free boundaries by finite element methods*, Modelisation Mathématique et Analyse Numérique **20**(2) (1986), 355–368.

[25] B. Perthame, *Some remarks on quasi-variational inequalities and the associated impulsive control problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2**(3) (1985), 237–260.

[26] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, Heidelberg, 1994.

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# DENUMERABLY MANY POSITIVE SOLUTIONS FOR ITERATIVE SYSTEM OF BOUNDARY VALUE PROBLEMS WITH N-SINGULARITIES ON TIME SCALES

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Abstract. In this paper we consider a iterative system of two-point boundary value problems with integral boundary conditions having $n$ singularities and involve an increasing homeomorphism, positive homomorphism operator. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally we provide an example to check validity of our obtained results.

## 1. Introduction

Theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [7, 8].

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact

*Key words and phrases.* Iterative system, time scale, singularity, homeomorphism, homomorphism, cone, Krasnoselskii's fixed point theorem, positive solutions.

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heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [9] and drying of different products such as iron ore [15]. To study such type of problems, Leibenson [13] introduced the following $p$-Laplacian equation

$$\Big(\varphi_p(\varpi'(t))\Big)' = f\Big(t, \varpi(t), \varpi'(t)\Big),$$

where $\varphi_p(\varpi) = |\varpi|^{p-2}\varpi$, $p > 1$, is the $p$-Laplacian operator its inverse function is denoted by $\varphi_q(\tau)$, with $\varphi_q(\tau) = |\tau|^{q-2}\tau$ and $p, q$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. It is well known fact that the $p$-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with $p$-Laplacian operator.

In this paper, we consider an operator $\varphi$ called increasing homeomorphism and positive homomorphism operator (IHPHO), which generalizes and improves the $p$-Laplacian operator for some $p > 1$ and $\varphi$ is not necessarily odd. Liang and Zhang [14] studied countably many positive solutions for nonlinear singular $m$–point boundary value problems on time scales with IHPHO,

$$\begin{aligned}
&\Big(\varphi(\varpi^{\Delta}(t))\Big)^{\nabla} + a(t) f\Big(\varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\
&\varpi(0) = \sum_{i=1}^{m-2} a_i \varpi(\xi_i), \quad \varpi^{\Delta}(T) = 0,
\end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [10], Dogan considered second order $p$-boundary value problem on time scales,

$$\begin{aligned}
&\Big(\varphi_p(\varpi^{\Delta}(t))\Big)^{\nabla} + \omega(t) f\Big(t, \varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\
&\varpi(0) = \sum_{i=1}^{m-2} a_i \varpi(\xi_i), \quad \varphi_p(\varpi^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \varphi_p(\varpi^{\Delta}(\xi_i)),
\end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Inspired by aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of denumerably many positive solutions for dynamical iterative system of two-point boundary value problem with $n$ singularities and involving IHPHO on time scales,

$$\left.\begin{aligned}
&\varphi\Big(\varpi_j^{\Delta\nabla}(t)\Big) + \chi(t) f_j\Big(\varpi_{j+1}(t)\Big) = 0, \quad 1 \le j \le \ell, \quad t \in [0,1]_{\mathbb{T}},\\
&\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0,1]_{\mathbb{T}},
\end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned}
\alpha\varpi_j(0) - \beta\varpi_j^{\Delta}(0) &= \int_0^1 \kappa_1(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell,\\
\gamma\varpi_j(1) + \delta\varpi_j^{\Delta}(1) &= \int_0^1 \kappa_2(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell,
\end{aligned}\right\} \tag{1.2}$$

where $\ell \in \mathbb{N}$, $\chi(t) = \prod_{i=1}^{n} \chi_i(t)$ and each $\chi_i(t) \in L^{p_i}_{\nabla}([0,1]_{\mathbb{T}})$, $p_i \geq 1$, has a singularity in the interval $\left(0, \frac{1}{2}\right)_{\mathbb{T}}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an IHPHO with $\varphi(0) = 0$.

A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called a IHPHO, if the following three conditions are fulfilled:

(*a*) $\varphi(\tau_1) \leq \varphi(\tau_2)$ whenever $\tau_1 \leq \tau_2$, for any real numbers $\tau_1, \tau_2$;
(*b*) $\varphi$ is a continuous bijection and its inverse $\varphi^{-1}$ is continuous;
(*c*) $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$ for any real numbers $\tau_1, \tau_2$.

We use following notations in the entire paper: $i = 1, 2,\ \mathfrak{z} \in (0, 1/2)_{\mathbb{T}}$,

$$a(t) = \gamma + \delta - \gamma t, \quad b(t) = \beta + \alpha t, d = \alpha\gamma + \alpha\delta + \beta\gamma,$$

$$\aleph_0(t,\tau) = \frac{1}{d}\begin{cases} a(\tau)b(t), & t \leq \tau, \\ a(t)b(\tau), & \tau \leq t, \end{cases} \quad c_i = \int_0^1 \left[\int_0^1 \aleph_0(\tau_1,\tau_2)\kappa_i(\tau_1)\nabla\tau_1\right]\chi(\tau_2)\nabla\tau_2,$$

$$u_a = \frac{1}{d}\int_0^1 \kappa_1(\tau)a(\tau)\nabla\tau, \quad u_b = \frac{1}{d}\int_0^1 \kappa_1(\tau)b(\tau)\nabla\tau, \quad \kappa_i^* = \int_0^1 \kappa_i(\tau)\nabla\tau,$$

$$v_a = \frac{1}{d}\int_0^1 \kappa_2(\tau)a(\tau)\nabla\tau, \quad v_b = \frac{1}{d}\int_0^1 \kappa_2(\tau)b(\tau)\nabla\tau, \quad \kappa_i(\mathfrak{z}) = \int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_i(\tau)\nabla\tau,$$

$$\eta(t) = \frac{(1-v_b)a(t) + v_a b(t)}{d[(1-u_a)(1-v_b) - u_b v_a]}, \quad \lambda(t) = \frac{(1-u_a)b(t) + u_b a(t)}{d[(1-u_a)(1-v_b) - u_b v_a]},$$

$$\eta^* = \max_{t\in[0,1]_{\mathbb{T}}} \eta(t), \quad \eta(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \eta(t), \quad \lambda^* = \max_{t\in[0,1]_{\mathbb{T}}} \lambda(t), \quad \lambda(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \lambda(t).$$

We assume the following conditions are true in the entire paper:

($H_1$) $f_j : [0,+\infty) \to [0,+\infty)$ and $\kappa_1, \kappa_2 : [0,1]_{\mathbb{T}} \to [0,+\infty)$ are continuous;
($H_2$) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \frac{1}{2}$,

$$\lim_{r\to\infty} t_r = t^* < \frac{1}{2}, \quad \lim_{t\to t_r} \chi_i(t) = +\infty, \quad i = 1, 2, \ldots, n,\ r \in \mathbb{N},$$

and each $\chi_i(t)$ does not vanish identically on any subinterval of $[0,1]_{\mathbb{T}}$. Moreover, there exists $\delta_i > 0$ such that

$$\delta_i < \varphi^{-1}\left(\chi_i(t)\right) < \infty \quad \text{a.e. on } [0,1]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.$$

## 2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions; for details, see [3–5, 7, 12, 17, 18].

**Definition 2.1.** A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$ and $\mu(t) = \rho(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}_k = \mathbb{T}$.

- If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.

**Definition 2.2.** Let $\mu_\Delta$ and $\mu_\nabla$ be the Lebesgue $\Delta$-measure and the Lebesgue $\nabla$-measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call $A$ is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of $A$. Let $P$ denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that $P$ holds on $A\backslash\Gamma_1$, then $P$ is said to hold $\Delta$-a.e. on $A$.
(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that $P$ holds on $A\backslash\Gamma_2$, then $P$ is said to hold $\nabla$-a.e. on $A$.

**Definition 2.3.** Let $E \subset \mathbb{T}$ be a $\nabla$-measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be $\nabla$-measurable function. We say that $f$ belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if } p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M \quad \nabla\text{-a.e. on } E, \text{ if } p = +\infty.$$

**Lemma 2.1.** *Let $E \subset \mathbb{T}$ be a $\nabla$-measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a $\nabla$-integrable on $E$, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i\in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

*where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i\in I}$, $I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.*

**Lemma 2.2.** *For any $\varrho(t) \in C([0, 1]_{\mathbb{T}})$, the boundary value problem,*

$$-\varphi(\varpi_1^{\Delta\nabla}(t)) = \varrho(t), \quad t \in [0, 1]_{\mathbb{T}}, \tag{2.1}$$

$$\left.\begin{aligned} \alpha\varpi_1(0) - \beta\varpi_1^\Delta(0) &= \int_0^1 \kappa_1(\tau)\varpi_1(\tau)\nabla, \\ \gamma\varpi_1(1) + \delta\varpi_1^\Delta(1) &= \int_0^1 \kappa_2(\tau)\varpi_1(\tau)\nabla, \end{aligned}\right\} \tag{2.2}$$

*has a unique solution*

$$\varpi_1(t) = \int_0^1 \aleph(t, \tau)\varphi^{-1}(\varrho(\tau))\nabla\tau,$$

*where*

$$\aleph(t,\tau) = \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1.$$

*Proof.* Suppose $\varpi_1$ is a solution of (2.1), then

$$\begin{aligned}\varpi_1(t) &= -\int_0^t\int_0^\tau \varphi^{-1}(\varrho(\tau_1))\nabla\tau_1\Delta\tau + At + B\\ &= -\int_0^t (t-\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + A_1t + A_2,\end{aligned}$$

where $A_1 = \varpi_1^{\Delta}(0)$ and $A_2 = \varpi_1(0)$. By the conditions (2.2), we get

$$A_1 = \frac{1}{d}\int_0^1 [\alpha\kappa_2(\tau) - \gamma\kappa_1(\tau)]\vartheta_1(\tau)\nabla\tau + \frac{1}{d}\int_0^1 \alpha[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau$$

and

$$A_2 = \frac{1}{d}\int_0^1 [(\gamma+\delta)\kappa_1(\tau) + \beta\kappa_2(\tau)]\vartheta_1(\tau)\nabla\tau + \frac{1}{d}\int_0^1 \beta[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau.$$

So, we have

(2.3)
$$\varpi_1(t) = \int_0^1 \aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + \frac{a(t)}{d}\int_0^1 \kappa_1(\tau)\vartheta_1(\tau)\nabla\tau + \frac{b(t)}{d}\int_0^1 \kappa_2(\tau)\vartheta_1(\tau)\nabla\tau.$$

By simple computations, we find that

$$\int_0^1 \kappa_1(\tau)\vartheta_1(\tau)\nabla\tau = \frac{c_1(1-v_b) + c_2u_b}{(1-u_a)(1-v_b) - u_bv_a}, \tag{2.4}$$

$$\int_0^1 \kappa_2(\tau)\vartheta_1(\tau)\nabla\tau = \frac{c_2(1-u_a) + c_1v_a}{(1-u_a)(1-v_b) - u_bv_a}. \tag{2.5}$$

Plugging (2.4) and (2.5) into (2.3), we received

$$\begin{aligned}\varpi_1(t) &= \int_0^1 \aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau + c_1\eta(t) + c_2\lambda(t)\\ &= \int_0^1 \left[\aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\right]\\ &\quad \times \varphi^{-1}(\varrho(\tau))\nabla\tau\\ &= \int_0^1 \aleph(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau.\end{aligned}$$

This completes the proof. □

**Lemma 2.3.** *Suppose* $(H_1)$*-*$(H_2)$ *hold. For* $\mathfrak{z} \in (0,\frac{1}{2})_{\mathbb{T}}$, *let*

$$\mathcal{L}(\mathfrak{z}) = \min\left\{\frac{\alpha\mathfrak{z}+\beta}{\alpha+\beta}, \frac{\gamma\mathfrak{z}+\delta}{\gamma+\delta}\right\} < 1.$$

*Then* $\aleph_0(t,\tau)$ *have the following properties:*

(i) $0 \le \aleph_0(t,\tau) \le \aleph_0(\tau,\tau)$ *for all* $t,\tau \in [0,1]_{\mathbb{T}}$*;*

(ii) $\mathcal{L}(\mathfrak{z})\aleph_0(\tau,\tau) \le \aleph_0(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

*Proof.* (i) is evident. We establish (ii), for this, let $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{b(t)}{b(\tau)} = \frac{\alpha t + \beta}{\alpha\tau + \beta} \ge \frac{\alpha\mathfrak{z} + \beta}{\alpha + \beta} \ge \mathcal{L}(\mathfrak{z}).$$

For $\tau \le t$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{a(t)}{a(\tau)} = \frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma\mathfrak{z}} \ge \frac{\gamma\mathfrak{z} + \delta}{\gamma + \delta} \ge \mathcal{L}(\mathfrak{z}).$$

This completes the proof. □

**Lemma 2.4.** *Suppose* $(H_1)$-$(H_2)$ *hold. Then* $\aleph(t,\tau)$ *satisfies properties:*

(i) $0 \le \aleph(t,\tau) \le \Xi\,\aleph_0(\tau,\tau)$ *for all* $t,\tau \in\in [0,1]_{\mathbb{T}}$*;*

(ii) $0 \le \Xi_{\mathfrak{z}}\aleph_0(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$, *where*

$$\Xi = 1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*$$

*and*

$$\Xi_{\mathfrak{z}} = \mathcal{L}(\mathfrak{z})\Big[1 + \eta(\mathfrak{z})\kappa_1(\mathfrak{z}) + \lambda(\mathfrak{z})\kappa_2(\mathfrak{z})\Big].$$

*Proof.* From Lemma 2.3, we get

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1 \\
&\le \aleph_0(\tau,\tau) + \eta(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_2(\tau_1)\nabla\tau_1 \\
&\le \left[1 + \eta(t)\int_0^1 \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \kappa_2(\tau_1)\nabla\tau_1\right]\aleph_0(\tau,\tau) \\
&\le \Big[1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

On the other hand, for $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $\tau \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1 \\
&\ge \aleph_0(t,\tau) + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1 \\
&\ge \mathcal{L}(\mathfrak{z})\left[1 + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_2(\tau_1)\nabla\tau_1\right]\aleph_0(\tau,\tau) \\
&\ge \mathcal{L}(\mathfrak{z})\Big[1 + \eta^{**}\kappa_1^{**} + \lambda^{**}\kappa_2^{**}\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

This completes the proof. □

Notice that an $\ell$-tuple $(\varpi_1(t), \varpi_2(t), \varpi_3(t), \ldots, \varpi_\ell(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}
&\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau)f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ 1 \le j \le \ell, \\
&\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0,1]_{\mathbb{T}},
\end{aligned}$$

i.e.,

$$
\begin{aligned}
\varpi_1(t) = & \int_0^1 \aleph(t, \tau_1)\varphi^{-1}\Bigg[\chi(\tau_1) f_1\Bigg(\int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\Bigg[\chi(\tau_2) f_2\Bigg(\int_0^1 \aleph(\tau_2, \tau_3) \\
& \times \varphi^{-1}\Bigg[\chi(\tau_3) f_3\Bigg(\int_0^1 \aleph(\tau_3, \tau_4) \cdots \\
& \times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg) \cdots \nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.
\end{aligned}
$$

Let $\mathtt{B}$ be the Banach space $C_{ld}([0,1]_{\mathbb{T}}, \mathbb{R})$ with the norm $\|\varpi\| = \max_{t\in[0,1]_{\mathbb{T}}} |\varpi(t)|$. For $\mathfrak{z} \in \left(0, \frac{1}{2}\right)$, we define the cone $\mathtt{K}_{\mathfrak{z}} \subset \mathtt{B}$ as

$$
\mathtt{K}_{\mathfrak{z}} = \left\{\varpi \in \mathtt{B} : \varpi(t) \text{ is nonnegative and } \min_{t\in[\mathfrak{z},\, 1-\mathfrak{z}]_{\mathbb{T}}} \varpi(t) \geq \frac{\Xi_{\mathfrak{z}}}{\Xi}\|\varpi(t)\|\right\}.
$$

For any $\varpi_1 \in \mathtt{K}_{\mathfrak{z}}$, define an operator $\Omega : \mathtt{K}_{\mathfrak{z}} \to \mathtt{B}$ by

$$
\begin{aligned}
(\Omega\varpi_1)(t) = & \int_0^1 \aleph(t, \tau_1)\varphi^{-1}\Bigg[\chi(\tau_1) f_1\Bigg(\int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\Bigg[\chi(\tau_2) f_2\Bigg(\int_0^1 \aleph(\tau_2, \tau_3) \\
& \times \varphi^{-1}\Bigg[\chi(\tau_3) f_3\Bigg(\int_0^1 \aleph(\tau_3, \tau_4) \cdots \\
& \times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg) \cdots \nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.
\end{aligned}
$$

**Lemma 2.5.** *Assume that* $(H_1)$-$(H_2)$ *hold. Then for each* $\mathfrak{z} \in \left(0, \frac{1}{2}\right)$, $\Omega(\mathtt{K}_{\mathfrak{z}}) \subset \mathtt{K}_{\mathfrak{z}}$ *and* $\Omega : \mathtt{K}_{\mathfrak{z}} \to \mathtt{K}_{\mathfrak{z}}$ *is completely continuous.*

*Proof.* From Lemma 2.3, $\aleph(t, \tau) \geq 0$ for all $t, \tau \in [0,1]_{\mathbb{T}}$. So, $(\Omega\varpi_1)(t) \geq 0$. Also, for $\varpi_1 \in \mathtt{K}$, we have

$$
\begin{aligned}
(\Omega\varpi_1)(t) \leq & \,\Xi \int_0^1 \aleph_0(\tau_1, \tau_1)\varphi^{-1}\Bigg[\chi(\tau_1) f_1\Bigg(\int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\Bigg[\chi(\tau_2) \\
& \times f_2\Bigg(\int_0^1 \aleph(\tau_2, \tau_3)\varphi^{-1}\Bigg[\chi(\tau_3) f_3\Bigg(\int_0^1 \aleph(\tau_3, \tau_4) \cdots \\
& \times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg) \cdots \nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.
\end{aligned}
$$

So,

$$
\begin{aligned}
\|\Omega\varpi_1\| \leq & \,\Xi \int_0^1 \aleph_0(\tau_1, \tau_1)\varphi^{-1}\Bigg[\chi(\tau_1) f_1\Bigg(\int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\Bigg[\chi(\tau_2) \\
& \times f_2\Bigg(\int_0^1 \aleph(\tau_2, \tau_3)\varphi^{-1}\Bigg[\chi(\tau_3) f_3\Bigg(\int_0^1 \aleph(\tau_3, \tau_4) \cdots
\end{aligned}
$$

$$\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.$$

Again from Lemma 2.3, we get

$$\begin{aligned}&\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\\ \geq&\Xi_{\mathfrak{z}}\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\geq\frac{\Xi_{\mathfrak{z}}}{\Xi}\|\Omega\varpi_1\|.$$

So, $\Omega\varpi_1\in\mathtt{K}_{\mathfrak{z}}$ and thus $\Omega(\mathtt{K}_{\mathfrak{z}})\subset\mathtt{K}_{\mathfrak{z}}$. Next, by standard methods and Arzela-Ascoli theorem, it can be proved easily that the operator $\Omega$ is completely continuous. The proof is complete. □

## 3. Denumerably Many Positive Solutions

For the existence of denumerably many positive solutions for iterative system of boundary value problem (1.1), we apply following theorems.

**Theorem 3.1** ([11]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and $\mathtt{M}_1$, $\mathtt{M}_2$ are open sets with $0\in\mathtt{M}_1,\overline{\mathtt{M}}_1\subset\mathtt{M}_2$. Let $\mathcal{A}:\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)\to\mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$, *or*
(b) $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$.

*Then $\mathcal{A}$ has a fixed point in $\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)$.*

**Theorem 3.2** ([8, 16]). *Let $f\in L^p_\nabla(J)$, with $p>1$, $g\in L^q_\nabla(J)$, with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $fg\in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla}\leq\|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$, where*

$$\|f\|_{L^p_\nabla}:=\begin{cases}\left[\displaystyle\int_J|f|^p(s)\nabla s\right]^{\frac{1}{p}}, & p\in\mathbb{R},\\ \inf\Big\{M\in\mathbb{R}\,/\,|f|\leq M\ \nabla\text{-}a.e.\ on\ J\Big\}, & p=\infty,\end{cases}$$

*and $J=(a,b]_{\mathbb{T}}$.*

**Theorem 3.3** (Hölder's). *Let $f \in L^{p_i}_{\nabla}(J)$, with $p_i > 1$, for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then $\prod_{i=1}^{n} f_i \in L^1_{\nabla}(J)$ and*

$$\left\| \prod_{i=1}^{n} f_i \right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}.$$

*Further, if $f \in L^1_{\nabla}(J)$ and $g \in L^{\infty}_{\nabla}(J)$, then $fg \in L^1_{\nabla}(J)$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_{\infty}$.*

Consider the following three possible cases for $\chi_i \in L^{p_i}_{\nabla}([0,1]_{\mathbb{T}})$ :

(i) $\sum_{i=1}^{n} \frac{1}{p_i} < 1$;
(ii) $\sum_{i=1}^{n} \frac{1}{p_i} = 1$;
(iii) $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Firstly, we seek denumerably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

**Theorem 3.4.** *Suppose $(H_1)$-$(H_3)$ hold, let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{\Gamma_r\}_{r=1}^{\infty}$ and $\{\Theta_r\}_{r=1}^{\infty}$ be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

*where*

$$\mathfrak{Z} = \max\left\{ \left[ \Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau \right]^{-1},\ 1 \right\}.$$

*Assume that $f$ satisfies*

$(C_1)$ *$f_j(\varpi) \leq \varphi(\mathfrak{N}_1\Gamma_r)$ for all $t \in [0,1]_{\mathbb{T}}$, $0 \leq \varpi \leq \Gamma_r$, where*

$$\mathfrak{N}_1 < \left[ \Xi \, \|\aleph_0\|_{L^q_{\nabla}} \prod_{i=1}^{n} \left\| \varphi^{-1}(\chi_i) \right\|_{L^{p_i}_{\nabla}} \right]^{-1};$$

$(C_2)$ *$f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ for all $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$.*

*Then the iterative boundary value problem* (1.1)–(1.2) *has denumerably many solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \ldots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \geq 0$ on $[0,1]_{\mathbb{T}}$, $j = 1, 2, \ldots, \ell$ and $r \in \mathbb{N}$.*

*Proof.* Let

$$\mathtt{M}_{1,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Gamma_r\},$$

$$\mathtt{M}_{2,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Theta_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \mathfrak{z}_r < t_r < \frac{1}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\mathfrak{z}_r}$ by

$$\mathtt{K}_{\mathfrak{z}_r} = \left\{ \varpi \in \mathtt{B} : \varpi(t) \geq 0,\ \min_{t \in [\mathfrak{z}_r,\, 1-\mathfrak{z}_r]_{\mathbb{T}}} \varpi(t) \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi} \|\varpi(t)\| \right\}.$$

Let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{1,r}$. Then $\varpi_1(\tau) \leq \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By $(C_1)$ and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\left[\prod_{i=1}^n \chi_i(\tau_\ell)\right]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n \varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell.\end{aligned}$$

There exists a $q > 1$ such that $\frac{1}{q} + \sum_{i=1}^n \frac{1}{p_i} = 1$. So,

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\,\mathfrak{N}_1\Gamma_r\Big\|\aleph_0\Big\|_{L^q_\nabla}\left\|\prod_{i=1}^n \varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n \Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ &\leq \Gamma_r.\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\begin{aligned}&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\left[\chi(\tau_{\ell-1})f_{\ell-1}\left(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\right)\right]\nabla\tau_{\ell-1}\\ \leq&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\left[\prod_{i=1}^n \chi_i(\tau_{\ell-1})\right]\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n \varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1}\\ \leq&\Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n \Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ \leq&\Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$(\Omega\varpi_1)(t) = \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\left[\chi(\tau_1)f_1\left(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\left[\chi(\tau_2)f_2\left(\int_0^1 \aleph(\tau_2,\tau_3)\right.\right.\right.\right.$$

$$\times \varphi^{-1}\Big[\chi(\tau_3) f_3\Big(\int_0^1 \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\leq \Gamma_r.$$

Since $\Gamma_r = \|\varpi_1\|$ for $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{1,r}$ we get

$$\|\Omega\varpi_1\| \leq \|\varpi_1\|. \tag{3.1}$$

Let $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$. Then

$$\Theta_r = \|\varpi_1\| \geq \varpi_1(t) \geq \min_{t\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}} \varpi_1(t) \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\|\varpi_1\| \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r.$$

By $(C_2)$ and for $\tau_{\ell-1} \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell &\geq \Xi_{\mathfrak{z}_r}\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}(\chi(\tau_\ell))\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r} \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n \varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell\\
&\geq \Xi_{\mathfrak{z}_1}\mathfrak{Z}\Theta_r\prod_{i=1}^n \delta_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau_\ell,\tau_\ell)\nabla\tau_\ell\\
&\geq \Theta_r.
\end{aligned}$$

Continuing with bootstrapping argument we get

$$(\Omega\varpi_1)(t) = \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Big[\chi(\tau_1) f_1\Big(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Big[\chi(\tau_2) f_2\Big(\int_0^1 \aleph(\tau_2,\tau_3)$$
$$\times \varphi^{-1}\Big[\chi(\tau_3) f_3\Big(\int_0^1 \aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\geq \Theta_r.$$

Thus, if $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{K}_{2,r}$, then

$$\|\Omega\varpi_1\| \geq \|\varpi_1\|. \tag{3.2}$$

It is evident that $0 \in \mathtt{M}_{2,k} \subset \overline{\mathtt{M}}_{2,k} \subset \mathtt{M}_{1,k}$. From (3.1) and (3.2), it follows from Theorem 3.1 that the operator $\Omega$ has a fixed point $\varpi_1^{[r]} \in \mathtt{K}_{\mathfrak{z}_r} \cap \left(\overline{\mathtt{M}}_{1,r}\backslash\mathtt{M}_{2,r}\right)$ such that $\varpi_1^{[r]}(t) \geq 0$

on $[0,1]_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain denumerably many positive solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau) f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ j = \ell, \ell-1, \dots, 1.$$

The proof is completed. □

For $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

**Theorem 3.5.** *Suppose* $(H_1)$-$(H_3)$ *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Theta_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

*where*

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

*Assume that* $f$ *satisfies*

$(C_3)$ $f_j(\varpi) \le \varphi(\mathfrak{N}_2\Gamma_r)$ *for all* $t \in [0,1]_{\mathbb{T}}$, $0 \le \varpi \le \Gamma_r$, *where*

$$\mathfrak{N}_2 < \min\left\{ \left[\Xi\, \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1}, \mathfrak{Z}\right\};$$

$(C_4)$ $f_j(\varpi) \ge \varphi(\mathfrak{Z}\Theta_r)$ *for all* $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \le \varpi \le \Theta_r$.

*Then the iterative boundary value problem* (1.1)–(1.2) *has denumerably many solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) \ge 0$ *on* $[0,1]_{\mathbb{T}}$, $j = 1, 2, \dots, \ell$, *and* $r \in \mathbb{N}$.

*Proof.* For a fixed $r$, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{2,r}$. Again $\varpi_1(\tau) \le \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By $(C_3)$ and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\le \Xi \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\left[\prod_{i=1}^{n}\chi_i(\tau_\ell)\right]\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^{n}\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell \\
&\le \Xi\, \mathfrak{N}_2\Gamma_r \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla} \\
&\le \Gamma_r.
\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\left[\chi(\tau_{\ell-1}) f_{\ell-1}\left(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\right)\right]\nabla\tau_{\ell-1}$$

$$
\begin{aligned}
&\leq \int_0^1 \aleph(\tau_{\ell-2}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1} \\
&\leq \Xi \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1} \\
&\leq \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1} \\
&\leq \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\left[\prod_{i=1}^n \chi_i(\tau_{\ell-1})\right]\nabla\tau_{\ell-1} \\
&\leq \Xi\, \mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\prod_{i=1}^n \varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1} \\
&\leq \Xi\, \mathfrak{N}_2\Gamma_r \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^n \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla} \\
&\leq \Gamma_r.
\end{aligned}
$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3) \\
&\times \varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots \\
&\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1 \\
\leq &\,\Gamma_r.
\end{aligned}
$$

Since $\Gamma_r = \|\varpi_1\|$ for $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{1,r}$, we get $\|\Omega\varpi_1\| \leq \|\varpi_1\|$. Now define $\mathtt{M}_{2,r} = \{\varpi_1 \in \mathtt{B} : \|\varpi_1\| < \Theta_r\}$. Let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{2,r}$ and let $\tau \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$. Then the argument leading to (3.2) can be done to the present case. Hence, the theorem is proved. $\square$

Lastly, the case $\sum_{i=1}^n \frac{1}{p_i} > 1$.

**Theorem 3.6.** *Suppose $(H_1)$-$(H_2)$ hold, let $\{\mathfrak{z}_r\}_{r=1}^\infty$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{\Gamma_r\}_{r=1}^\infty$ and $\{\Theta_r\}_{r=1}^\infty$ be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

*where*

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1}\prod_{i=1}^n \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

*Assume that $f$ satisfies*

$(C_5)$ *$f_j(\varpi) \leq \varphi(\mathfrak{N}_3\Gamma_r)$ for all $t \in [0,1]_{\mathbb{T}}$, $0 \leq \varpi \leq \Gamma_r$, where*

$$\mathfrak{N}_3 < \min\left\{\left[\Xi\,\|\aleph_0\|_{L^\infty_\nabla}\prod_{i=1}^n \left\|\varphi^{-1}(\chi_i)\right\|_{L^1_\nabla}\right]^{-1}, \mathfrak{Z}\right\};$$

$(C_6)$ *$f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ for all $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$.*

*Then the iterative boundary value problem* (1.1)–(1.2) *has denumerably many solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \geq 0$ on $[0,1]_{\mathbb{T}}$, $j = 1, 2, \dots, \ell$, and $r \in \mathbb{N}$.*

*Proof.* The proof is similar to the proof of Theorem 3.1. So, we omit the details here. □

## 4. Examples

In this section, we present an example to check validity of our main results.

**Example** 4.1. Consider the following boundary value problem on $\mathbb{T} = [0,1]$

$$\left.\begin{aligned} &\varphi(\varpi_j''(t)) + \chi(t) f_j(\varpi_{j+1}(t)) = 0, \quad j = 1,2,\ t \in [0,1],\\ &\varpi_3(t) = \varpi_1(t), \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \varpi_j(0) - \varpi_j'(0) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau,\\ \varpi_j(1) + \varpi_j'(1) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau, \end{aligned}\right\} \tag{4.2}$$

where

$$\varphi(\varpi) = \begin{cases} \dfrac{\varpi^3}{1+\varpi^2}, & \varpi \leq 0,\\ \varpi^2, & \varpi > 0, \end{cases}$$

$$\chi(t) = \chi_1(t) \cdot \chi_2(t),$$

in which

$$\chi_1(t) = \frac{1}{|t-\frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \chi_2(t) = \frac{1}{|t-\frac{1}{3}|^{\frac{1}{2}}},$$

$$f_1(\varpi) = f_2(\varpi)$$
$$= \begin{cases} 0.05\times 10^{-8}, & \varpi \in (10^{-4}, +\infty),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{10^{-(4r+3)} - 10^{-4r}}(\varpi - 10^{-4r}) & \\ +0.05\times 10^{-8r}, & \varpi \in \left[10^{-(4r+3)}, 10^{-4r}\right],\\ 5604\times 10^{-(8r+6)}, & \varpi \in \left(0.98\times 10^{-(4r+3)}, 10^{-(4r+3)}\right),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{0.98\times 10^{-(4r+3)} - 10^{-(4r+4)}}(\varpi - 10^{-(4r+4)}) & \\ +0.05\times 10^{-8r}, & \varpi \in \left(10^{-(4r+4)}, 0.98\times 10^{-(4r+3)}\right]. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \mathfrak{z}_r = \frac{1}{2}(t_r + t_{r+1}), \quad \text{for } r = 1,2,3,\dots,$$

then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32} \quad \text{and} \quad t_{r+1} < \mathfrak{z}_r < t_r, \quad \mathfrak{z}_r > \frac{1}{5}, \quad \text{for } r = 1,2,3,\dots$$

Therefore, $\frac{\mathfrak{z}_r}{1} = \mathfrak{z}_r > \frac{1}{5}$, $j = 1, 2, 3, \ldots$ It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \ldots$$

Since $\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$\chi_1, \chi_2 \in L^p[0,1]$ for $0 < p < 2$, so $\delta_1 = \delta_2 = \frac{1}{\sqrt{3}}$,

$$a(t) = 2 - t, \quad b(t) = 1 + t, \quad d = 3, \quad \aleph_0(t,\tau) = \frac{1}{3}\begin{cases} (2-\tau)(1+t), & t \le \tau, \\ (2-t)(1+\tau), & \tau \le t, \end{cases}$$

$$c_i = \int_0^1 \Big[\int_0^1 \aleph_0(\tau_1,\tau_2)\kappa_i(\tau_1)\nabla\tau_1\Big]\chi(\tau_2)\nabla\tau_2 = 2.774076198,$$

$$u_a = u_b = v_a = v_b = \frac{1}{4}, \quad \kappa_1^* = \kappa_2^* = \frac{1}{2}, \quad \kappa_1(\mathfrak{z}_1) = \kappa_2(\mathfrak{z}_1) = 0.06558641976,$$

$$\mathcal{L}(\mathfrak{z}_1) = \min\left\{\frac{\alpha\mathfrak{z}_1 + \beta}{\alpha+\beta}, \frac{\gamma\mathfrak{z}_1 + \delta}{\gamma+\delta}\right\} = \frac{1+\mathfrak{z}_1}{2} = 0.7336033950,$$

$$\eta(t) = \frac{(1-v_b)a(t) + v_a b(t)}{d[(1-u_a)(1-v_b) - u_b v_a]} = \frac{7-2t}{6}, \quad \eta^* = \frac{7}{6}, \quad \eta(\mathfrak{z}_1) = 1.010931070,$$

$$\lambda(t) = \frac{(1-u_a)b(t) + u_b a(t)}{d[(1-u_a)(1-v_b) - u_b v_a]} = \frac{5-2t}{6}, \quad \lambda^* = \frac{5}{6}, \quad \lambda(\mathfrak{z}_1) = 0.6775977366,$$

$$\Xi = 1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^* = 2,$$

$$\Xi_{\mathfrak{z}_1} = \mathcal{L}(\mathfrak{z}_1)\big[1 + \eta(\mathfrak{z}_1)\kappa_1(\mathfrak{z}_1) + \lambda(\mathfrak{z}_1)\kappa_2(\mathfrak{z}_1)\big] = 0.8148459802.$$

Note that $\Xi_{\mathfrak{z}}$ is increasing, it follows that $1.969391539 = \Xi_{\mathfrak{z}\infty} < \Xi_{\mathfrak{z}_r} < \Xi_{\mathfrak{z}_1} = 2$, $0.9846957695 \le \frac{\Xi_{\mathfrak{z}_r}}{\Xi} \le 2$ and

$$\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197800.$$

Thus, we get

$$\mathfrak{Z} = \max\left\{\left[\Xi_{\mathfrak{z}_1}\prod_{i=1}^{n}\delta_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1}\aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\} = \max\Big\{74.85826138, 1\Big\}$$
$$= 74.85826138$$

and

$$\|\aleph_0\|_{L^q_\nabla} = \left[\int_0^1 |\aleph_0(\tau,\tau)|^q d\tau\right]^{\frac{1}{q}} < 1, \quad \text{for } 0 < q < 2.$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\chi_1, \chi_2 \in L^{1+\mathfrak{a}}[0,1]$. It follows that

$$\|\varphi^{-1}(\chi_1)\|_{1+\mathfrak{a}} = \left[\frac{1}{3-\mathfrak{a}}\left(3^{\frac{3-\mathfrak{a}}{4}} + 1\right)2^{\frac{1+\mathfrak{a}}{2}}\right]^{\frac{1}{1+\mathfrak{a}}}$$

and

$$\|\varphi^{-1}(\chi_2)\|_{1+\mathfrak{a}} = \left[\frac{4}{3-\mathfrak{a}}\left(2^{\frac{3-\mathfrak{a}}{4}}+1\right)(1/3)^{\frac{3-\mathfrak{a}}{4}}\right]^{\frac{1}{1+\mathfrak{a}}}.$$

So, for $0<\mathfrak{a}<1$, we have

$$0.2509961333 \leq \left[\Xi \|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^{n}\left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1} \leq 0.2856331500.$$

Taking $\mathfrak{N}_1 = 0.2$. In addition, if we take

$$\Gamma_r = 10^{-4r}, \quad \Theta_r = 10^{-(4r+3)},$$

then

$$\begin{aligned}\Gamma_{r+1} &= 10^{-(4r+4)} < 0.9846957695\times 10^{-(4r+3)} < \frac{\Xi_{\mathfrak{z}r}}{\Xi}\Theta_r < \Theta_r = 10^{-(4r+3)}\\ &\quad< \Gamma_r = 10^{-4r},\\ \mathfrak{Z}\Theta_r &= 74.85826138\times 10^{-(4r+3)} < 0.2\times 10^{-4r} = \mathfrak{N}_1\Gamma_r, \quad r=1,2,3,\dots,\end{aligned}$$

and $f_1, f_2$ satisfies the following growth conditions:

$$\begin{aligned}f_1(\varpi) = f_2(\varpi) &\leq \varphi(\mathfrak{N}_1\Gamma_r) = \mathfrak{N}_1^2\Gamma_r^2 = 0.04\times 10^{-8r}, \quad \varpi\in\left[0, 10^{-4r}\right]\\ f_1(\varpi) = f_2(\varpi) &\geq \varphi(\mathfrak{Z}\Theta_r) = \mathfrak{Z}^2\Theta_r^2\\ &= 5603.759297\times 10^{-(8r+6)}, \quad \varpi\in\left[0.98\times 10^{-(4r+3)}, 10^{-(4r+3)}\right].\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (4.1)–(4.2) has denumerably many solutions $\left\{\left(\varpi_1^{[r]}, \varpi_2^{[r]}\right)\right\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t)\geq 0$ on $[0,1]$, $j=1,2$ and $r\in\mathbb{N}$.

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## References


[1] R. P. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3–22. https://doi.org/10.1007/BF03322019

[2] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), 1–26. https://doi.org/10.1016/S0377-0427(01)00432-0

[3] R. P. Agarwal, V. Otero-Espinar, K. Perera and D. R. Vivero, *Basic properties of Sobolev's spaces on time scales*, Adv. Differ. Equ. **2006**(1) (2006), 1–14. https://doi.org/10.1155/ADE/2006/38121

[4] G. A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-17098-0

[5] M. Bohner and H. Luo, *Singular second-order multi-point dynamic boundary value problems with mixed derivatives*, Adv. Differ. Equ. **2006**(1) (2006), 1–15. https://doi.org/10.1155/ADE/2006/54989

[6] Z. He, *Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales*, J. Comput. Appl. Math. **182** (2005), 304–315. https://doi.org/10.1016/j.cam.2004.12.012
[7] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001. https://doi.org/10.1007/978-1-4612-0201-1
[8] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003. https://doi.org/10.1007/978-0-8176-8230-9
[9] M. J. De Lemos, *Turbulence in Porous Media: Modeling and Applications*, Elsevier, London, 2012. https://doi.org/10.1016/C2011-0-06981-8
[10] A. Dogan, *Positive solutions of the p-Laplacian dynamic equations on time scales with sign changing nonlinearity*, Electron. J. Differential Equations **39** (2018), 1–17.
[11] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988. https://doi.org/10.1016/C2013-0-10750-7
[12] G. S. Guseinov, *Integration on time scales*, J. Math. Anal. App. **285**(1) (2003), 107–127. https://doi.org/10.1016/S0022-247X(03)00361-5
[13] L. S. Leibenson, *General problem of the movement of a compressible fluid in a porous medium*, Izvestiia Akademii Nauk Kirgizskoi SSR **9** (1983), 7–10.
[14] S. Liang and J. Zhang, *The existence of countably many positive solutions for nonlinear singular $m$-point boundary value problems on time scales*, J. Comput. Appl. Math. **223**(1) (2009), 291–303. https://doi.org/10.1016/j.cam.2008.01.021
[15] A. L. Ljung, V. Frishfelds, T. S. Lundstrom and B. D. Marjavaara, *Discrete and continuous modeling of heat and mass transport in drying of a bed of iron ore pellets*, Dry. Technol. **30**(7) (2012), 760–773. https://doi.org/10.1080/07373937.2012.662567
[16] U. M. Ozkan, M. Z. Sarikaya and H. Yildirim, *Extensions of certain integral inequalities on time scales*, Appl. Math. Lett. **21**(10) (2008), 993–1000. https://doi.org/10.1016/j.aml.2007.06.008
[17] B. P. Rynne, *$L^2$ spaces and boundary value problems on time scales*, J. Math. Anal. App. **328**(2) (2007), 1217–1236. https://doi.org/10.1016/j.jmaa.2006.06.008
[18] P. A. Williams, *Unifying fractional calculus with time scales*, PhD thesis, University of Melbourne, 2012.

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# ESTIMATES FOR INITIAL COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-CLOSE-TO-CONVEX ANALYTIC FUNCTIONS

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Abstract. In this paper we find bounds on the modulii of the second, third and fourth Taylor-Maclaurin's coefficients for functions in a subclass of *bi-close-to-convex* analytic functions, which includes the class studied by Srivastava et al. as particular case. Our estimates on the second and third coefficients improve upon earlier bounds. The result on the fourth coefficient is new. Our bounds are obtained by refining well known estimates for the initial coefficients of the Carthéodory functions.

## 1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ represented by the following *normalized* Taylor-Maclaurin's series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. The function $f \in \mathcal{S}$ has a *compositional inverse* $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (w \in \text{range of } f).$$

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It is well known that for every function $f \in \mathcal{S}$ the compositional inverse function $f^{-1}(w)$ is analytic in some disc $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. Moreover, $f^{-1}(w)$ has the Taylor-Maclaurin series expansion of the form:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad (|w| < r_0(f)),$$

where

$$b_n = \frac{(-1)^{n+1}}{n!}|A_{ij}|$$

and $|A_{ij}|$ is the $(n-1)^{\text{th}}$ order determinant whose entries are defined, in terms of the coefficients of $f(z)$, by the following:

$$|A_{ij}| = \begin{cases} [(i-j+1)n + j - 1]a_{i-j+2}, & \text{if } i+1 \geq j, \\ 0, & \text{if } i+1 < j. \end{cases}$$

For initial values of $n$ we, therefore, have:

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3, \quad b_4 = 5a_2a_3 - 5a_2^3 - a_4, \tag{1.2}$$

and so on.

The function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f \in \mathcal{S}$ and $f^{-1}(w)$ has univalent *analytic continuation* to the unit disk $\mathbb{U}$. For example, the function

$$f(z) = ze^{-Az}$$

is bi-univalent in $\mathbb{U}$ if $|A| \leq \frac{1}{e}$ [10]. For some more examples see [12,15,19]. We denote by $\sigma$, the class of analytic bi-univalent functions in $\mathbb{U}$ given by (1.1). Investigation on the class $\sigma$ was initiated by Lewin [14]. He showed that $|a_2| \leq 1.51$ for every $f \in \sigma$. Subsequently, Brannan and Clunie [3] surmised that $|a_2| \leq \sqrt{2}$. Netanyahu [16] finally proved that $|a_2| \leq \frac{4}{3}$ $(f \in \sigma)$. Later Brannan and Taha [4] introduced and studied new sub-classes of bi-univalent functions (also see Taha [20]). For a detailed history of the developments on the class of functions $\sigma$ see [2,13].

In this paper we shall also investigate bi-univalent functions defined on

$$\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}.$$

Let $\Sigma$ denote the class of analytic functions of the form:

$$h(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \quad (z \in \Delta), \tag{1.3}$$

which are univalent in $\Delta$. The inverse of a function in $\Sigma$ is represented by

$$h^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \quad (M < |w| < \infty, M > 1). \tag{1.4}$$

We say that the function $h \in \Sigma$ is bi-univalent in $\Delta$ if $h^{-1}(w)$ has analytic continuation to $\Delta$.

In order to describe certain sub classes of $\mathcal{S}$ and $\Sigma$ we shall also need the class $\mathcal{P}$ consisting of functions $P(z)$ which are analytic in $\mathbb{U}$, satisfy $|\arg(P(z))| \leq \frac{\pi}{2}$ $(z \in \mathbb{U})$ and $P(0) = 1$. The functions $P(z) \in \mathcal{P}$ are named after Carthéodory.

It is well known that if $f$ in $\mathcal{A}$ is such that $f' \in \mathcal{P}$, then $f \in \mathcal{S}$. In fact, $f$ is *close-to-convex* [7, 10]. We denote the class of these functions by $Q$. Chichra [6] studied the class of functions $Q_\lambda$ $(\lambda \geq 0)$ consisting of functions $f \in \mathcal{A}$ and satisfying $(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \in \mathcal{P}$. We observe that in the particular case $\lambda = 1$, we have $Q_1 := Q$. Chichra [6] further more, proved that

$$Q_{\lambda_1} \subseteq Q_{\lambda_2} \quad (0 \leq \lambda_2 \leq \lambda_1) \quad \text{(also see [8])}.$$

Therefore,

$$Q_\lambda \subseteq Q_1 := Q \subset \mathcal{S} \quad (\lambda \geq 1).$$

Frasin and Aouf [9] introduced the following subclass of *bi-close-to-convex* analytic functions analogous to the subclass $Q_\lambda$ studied by Chichra [6].

**Definition 1.1** (See [9]). The function $f(z)$ given by (1.1) is said to be in the class $\sigma Q_\lambda^\alpha$ $(0 < \alpha \leq 1, \lambda \geq 1)$ if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \tag{1.5}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}), \tag{1.6}$$

where $g$ is the analytic continuation of $f^{-1}$ to $\mathbb{U}$.

We observe that in the particular case $\lambda = 1$, the class $\sigma Q_1^\alpha := \sigma Q^\alpha$ was earlier studied by Srivastava et al. [19]. More recently, Çağlar et al. [5] introduced a more general class of bi-univalent analytic functions than the class $\sigma Q_\lambda^\alpha$ (also see Srivastava et al. [1, 18, 21]). However, in this paper we shall restrict our attention to the class $\sigma Q_\lambda^\alpha$.

In addition to the class $\sigma Q_\lambda^\alpha$, in this paper we shall also study the following subclass of $\Sigma$.

**Definition 1.2.** The function $h(z)$ given by (1.3) is said to be in the class $\Sigma\Theta_\lambda^\alpha$ $(0 < \alpha \leq 1, \lambda \geq 1)$ if $h \in \Sigma$ and the following conditions are satisfied:

$$\left|\arg\left((1-\lambda)\frac{h(z)}{z} + \lambda h'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta) \tag{1.7}$$

and

$$\left|\arg\left((1-\lambda)\frac{H(w)}{w} + \lambda H'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta), \tag{1.8}$$

where $H$ is the analytic continuation of $h^{-1}$ to $\Delta$.

In the present paper we develop an elementary method to find new estimates for $|a_2|$ and $|a_3|$ for $f \in \sigma Q_\lambda^\alpha$, which improve upon bounds of Frasin and Aouf [9] and afortiori, the bounds obtained earlier by Srivastava et al. [19]. We also extend a result of Hayami and Owa [12] and find estimate on $|a_4|$ for $f \in \sigma Q_\lambda^\alpha$. Further more, we find estimates on the initial coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for functions in the class $\Sigma\Theta_\lambda^\alpha$. We note that very recently Hamidi et al. [11] studied coefficient estimate problem for a class of functions similar to our class $\Sigma\Theta_\lambda^\alpha$ under the additional restriction that the initial coefficients of the functions are missing. Thus our Theorem 2.2, proved below, on bounds of initial coefficients attempts to bridge this gap and supplements the work in [11]. The methods adopted and developed in this paper are applicable for finding improved coefficient estimates for the several sub-classes of bi-univalent functions studied in [5, 17] and [18].

## 2. Coefficient Bounds for the Function Classes $\sigma Q_\lambda^\alpha$ and $\Sigma\Theta_\lambda^\alpha$

In this section we denote by $g(w)$ the analytic continuation of the function $f^{-1}(w)$ to the unit disc $\mathbb{U}$. We state and prove the following.

**Theorem 2.1.** *Let the function $f(z)$ given by* (1.1)*, be in the class $\sigma Q_\lambda^\alpha$ $(0 < \alpha \leq 1$ and $\lambda \geq 1)$. Then*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}}, & 1 \leq \lambda \leq 1+\sqrt{2}, \\ \frac{2\alpha}{(1+\lambda)}, & \lambda > 1+\sqrt{2}, \end{cases} \tag{2.1}$$

$$|a_3| \leq \frac{2\alpha}{1+2\lambda} \tag{2.2}$$

*and*

$$|a_4| \leq \frac{2\alpha}{1+3\lambda} \begin{cases} 1 + \frac{2(1-\alpha)(1+\lambda)\left\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\right\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]^{\frac{3}{2}}}, & 1 \leq \lambda \leq 1+\sqrt{2},\ 0<\alpha\leq 1, \\ 1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+(5-4\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}, & 1+\sqrt{2} < \lambda \leq \lambda_0, 0<\alpha\leq 1 \\ & \text{or } \lambda > \lambda_0,\ 0 < \alpha \leq \frac{1}{2}, \\ 1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+4(2-\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}, & \lambda > \lambda_0,\ \frac{1}{2} < \alpha \leq 1, \end{cases} \tag{2.3}$$

*where $\lambda_0$ is the positive root of the quadratic equation*

$$2(1-2\alpha)\lambda^2 + 3(1+3\alpha)\lambda + (1+3\alpha) = 0.$$

*Proof.* Let the function $f(z)$ be a member of the class $\sigma Q_\lambda^\alpha$ $(\lambda \geq 1,\ 0 < \alpha \leq 1)$. Then by Definition 1.1, we have the following:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = [P(z)]^\alpha \tag{2.4}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda f'(w)=[Q(w)]^{\alpha}, \tag{2.5}$$

respectively, where $P(z)$ and $Q(w)$ are members of the Carthéodory class $\mathcal{P}$ and have the forms:

$$P(z)=1+c_1z+c_2z^2+c_3z^3+\cdots \quad (z\in\mathbb{U}) \tag{2.6}$$

and

$$Q(w)=1+l_1w+l_2w^2+l_3w^3+\cdots \quad (w\in\mathbb{U}), \tag{2.7}$$

respectively. Now, equating the coefficients of $(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)$ with the coefficients of $[P(z)]^{\alpha}$, we get

$$(1+\lambda)a_2=\alpha c_1 \quad \text{or} \quad a_2=\frac{\alpha}{1+\lambda}c_1, \tag{2.8}$$

$$(1+2\lambda)a_3=\alpha c_2+\frac{\alpha(\alpha-1)}{2}c_1^2, \tag{2.9}$$

$$(1+3\lambda)a_4=\alpha c_3+\alpha(\alpha-1)c_1c_2+\frac{\alpha(\alpha-1)(\alpha-2)}{6}c_1^3. \tag{2.10}$$

Similarly, a comparison of coefficients of both sides of (2.5) yields:

$$(1+\lambda)a_2=-\alpha l_1, \tag{2.11}$$

$$(1+2\lambda)(2a_2^2-a_3)=\alpha l_2+\frac{\alpha(\alpha-1)}{2}l_1^2 \tag{2.12}$$

and

$$-(1+3\lambda)(5a_2^3-5a_2a_3+a_4)=\alpha l_3+\alpha(\alpha-1)l_1l_2+\frac{\alpha(\alpha-1)(\alpha-2)}{6}l_1^3. \tag{2.13}$$

In order to find improved estimates on $|a_2|$ and $|a_3|$, we first establish certain relations involving $c_1, l_1, c_2$ and $l_2$. To this end we observe that (2.8) and (2.11), together give

$$c_1=-l_1. \tag{2.14}$$

We add (2.9) with (2.12), then use the relation $c_1=-l_1$ and get the following:

$$2(1+2\lambda)a_2^2=\alpha\,(c_2+l_2)+\alpha(\alpha-1)c_1^2.$$

Putting $a_2=\frac{\alpha}{(1+\lambda)}c_1$ from (2.8) we have after simplification:

$$c_1^2=\frac{(1+\lambda)^2}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}(c_2+l_2). \tag{2.15}$$

The relation (2.15) also gives the following refined estimates:

$$|c_1|\le\frac{2\,(1+\lambda)}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}} \quad (1\le\lambda\le1+\sqrt{2}) \tag{2.16}$$

and

$$|c_2 + l_2| \leq \frac{4\ [2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2]}{(1+\lambda)^2} \quad (\lambda \geq 1 + \sqrt{2}). \tag{2.17}$$

Now using the estimates (2.16) for the range $1 \leq \lambda \leq 1+\sqrt{2}$ and $|c_1| \leq 2$ for the range $\lambda > 1+\sqrt{2}$ in the expression (2.8) we get:

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}}, & 1 \leq \lambda \leq 1+\sqrt{2}, \\ \frac{2\alpha}{(1+\lambda)}, & \lambda > 1+\sqrt{2}. \end{cases}$$

We thus get the claimed bound of (2.1).

We now express $a_3$ in terms of the coefficients of the functions $P(z)$ and $Q(w)$. For this, we subtract (2.12) from (2.9) and get

$$2(1+2\lambda)(a_3 - a_2^2) = \alpha(c_2 - l_2) + \frac{\alpha(\alpha-1)}{2}(c_1^2 - l_1^2).$$

The relation $c_1^2 = l_1^2$ from (2.14), reduces the above expression to

$$a_3 = a_2^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2). \tag{2.18}$$

Next putting $a_2 = \frac{\alpha}{1+\lambda}c_1$ and then using (2.15) for $c_1^2$, we obtain

$$\begin{aligned} a_3 =& \frac{\alpha^2}{(1+\lambda)^2}c_1^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2), \\ =& \frac{\alpha^2}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}(c_2 + l_2) + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2), \\ =& \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2} + \frac{1}{2(1+2\lambda)}\right)c_2 \\ &+ \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2} - \frac{1}{2(1+2\lambda)}\right)l_2. \end{aligned}$$

Since

$$\begin{aligned} &\frac{\alpha}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2} - \frac{1}{2(1+2\lambda)} \\ =& \frac{-(1-\alpha)(1+\lambda)^2}{2(1+2\lambda)(2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2)} < 0, \end{aligned}$$

an application of triangle inequality gives the following

$$\begin{aligned} |a_3| \leq& \alpha\left(\frac{\alpha}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2} + \frac{1}{2(1+2\lambda)}\right)|c_2| \\ &+ \alpha\left(\frac{1}{2(1+2\lambda)} - \frac{\alpha}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}\right)|l_2|. \end{aligned}$$

Therefore, the well known estimates $|c_2| \leq 2$ and $|l_2| \leq 2$ (cf. [4]), give the following:

$$|a_3| \leq \frac{2\alpha}{(1+2\lambda)}. \tag{2.19}$$

This is precisely our assertion at (2.2).

We next derive a relation between $c_1(c_2 - l_2)$ and $c_3 + l_3$ for our future use. For this purpose we add (2.13) and (2.10). After simplification we get the following:

$$-(1+3\lambda)(5a_2^3 - 5a_2a_3) = \alpha(c_3 + l_3) + \alpha(\alpha - 1)c_1(c_2 - l_2). \tag{2.20}$$

By substituting $a_3 = a_2^2 + \frac{\alpha}{2(1+2\lambda)}(c_2 - l_2)$ from (2.18) and $a_2 = \frac{\alpha}{1+\lambda}c_1$ in the above equation (2.20) we have

$$c_1(c_2 - l_2) = \mu_0(c_3 + l_3), \tag{2.21}$$

where

$$\mu_0 = \frac{2(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda) + 2(1-\alpha)(1+\lambda)(1+2\lambda)}.$$

We observe that $0 < \mu_0 \leq 2$ for every $\lambda \geq 1$ if $0 \leq \alpha \leq \frac{1}{2}$. However, if $\frac{1}{2} \leq \alpha < 1$, then $0 < \mu_0 \leq 2$ for $1 < \lambda \leq \lambda_0$, where $\lambda_0$ is the positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(1+3\alpha)\lambda + (1+3\alpha) = 0.$$

Moreover, $\lambda_0 > \frac{97}{16}$.

We are now ready to find a bound for $|a_4|$. As in our estimate for $|a_3|$ in this case also we shall express $a_4$ in terms of the first three coefficients of $P(z)$ and $Q(w)$. For this purpose we subtract (2.13) from (2.10) and get

$$\begin{aligned} 2(1+3\lambda)a_4 = &-(1+3\lambda)(5a_2^3 - 5a_2a_3) + \alpha(c_3 - l_3) + \alpha(\alpha-1)(c_1c_2 - l_1l_2) \\ &+ \frac{\alpha(\alpha-1)(\alpha-2)}{6}(c_1^3 - l_1^3). \end{aligned}$$

The relation $c_1 = -l_1$ reduces the above expression to

$$\begin{aligned} 2(1+3\lambda)a_4 = &-(1+3\lambda)(5a_2^3 - 5a_2a_3) \\ &+ \alpha(c_3 - l_3) + \alpha(\alpha-1)c_1(c_2 + l_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3. \end{aligned} \tag{2.22}$$

In (2.22) we replace $-(1+3\lambda)(5a_2^3 - 5a_2a_3)$ by the expression on the right hand side of the equality of (2.20) and use the relation $c_1 = -l_1$. This gives on simplification the following:

$$2(1+3\lambda)a_4 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2-l_2) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3. \tag{2.23}$$

First suppose that $\lambda$ and $\alpha$ are constrained by the requirement $1 \leq \lambda \leq \lambda_0$ and $0 < \alpha \leq 1$ or $\lambda > \lambda_0$ and $0 < \alpha \leq \frac{1}{2}$. Then in the equation (2.23) replacing $c_1(c_2 - l_2)$ by

$\mu_0(c_3+l_3)$ from (2.21) we get:

$$\begin{aligned}
2(1+3\lambda)a_4 =& 2\alpha c_3+\alpha(\alpha-1)\frac{2(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}(c_3+l_3)\\
&+\alpha(\alpha-1)c_1\ (c_2+l_2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1^3\\
=&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\
&-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3\\
&-\alpha(1-\alpha)c_1(c_2+l_2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3}c_1c_1^2.
\end{aligned}\tag{2.24}$$

Suppose that we furthermore restrict $\lambda$ in the range $1\leq\lambda\leq 1+\sqrt{2}<\lambda_0,\ 0<\alpha\leq 1$. Then in (2.24) we substitute the expression in the right hand side of the equality of (2.15) in place of $c_1^2$ and get

$$\begin{aligned}
2(1+3\lambda)a_4 =& \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\
&-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3-\alpha(1-\alpha)c_1(c_2+l_2)\\
&+\frac{\alpha(1-\alpha)(2-\alpha)}{3}\frac{(1+\lambda)^2}{2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2}c_1(c_2+l_2)\\
=&\frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}c_3\\
&-\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}l_3\\
&-\alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right]c_1(c_2+l_2).
\end{aligned}\tag{2.25}$$

Now, we apply the triangle inequality in (2.25) and get the following:

$$\begin{aligned}
2(1+3\lambda)|a_4| \leq& \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|c_3|\\
&+\frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|l_3|\\
&+\alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right]|c_1(c_2+l_2)|.
\end{aligned}$$

Note that we made use of the fact that if $1\leq\lambda\leq 1+\sqrt{2}$ and $0<\alpha\leq 1$ then

$$6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2>0.$$

The well known estimates $|c_n| \le 2$, $|l_n| \le 2\,(n = 2,3)$, and the refined bound (2.16) for $|c_1|$ yields the following:

$$\begin{aligned}
2(1+3\lambda)|a_4| \le & \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}2 \\
& + \frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}2 \\
& + \alpha(1-\alpha)\left[\frac{\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}\right] \\
& \times \frac{2(1+\lambda)}{\sqrt{[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}}4
\end{aligned}$$

or

$$\begin{aligned}
|a_4| \le & \frac{2\alpha}{1+3\lambda}\left(1+\frac{2(1-\alpha)(1+\lambda)\{6\alpha(1+2\lambda)+(1-2\alpha)(1+\lambda)^2\}}{3[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]^{\frac{3}{2}}}\right) \\
& (1 \le \lambda \le 1+\sqrt{2}, 0 < \alpha \le 1).
\end{aligned}$$

We get the first bound of (2.3).

Next, suppose that $1+\sqrt{2} < \lambda \le \lambda_0$ and $0 < \alpha \le 1$ or $\lambda > \lambda_0$ and $0 < \alpha \le \frac{1}{2}$. We apply the triangle inequality in (2.24) and get

$$\begin{aligned}
2(1+3\lambda)|a_4| \le & \frac{10\alpha^2(1+3\lambda)+2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|c_3| \\
& + \frac{2\alpha(1-\alpha)(1+\lambda)(1+2\lambda)}{5\alpha(1+3\lambda)+2(1-\alpha)(1+\lambda)(1+2\lambda)}|l_3| \\
& + \alpha(1-\alpha)|c_1(c_2+l_2)| + \frac{\alpha(\alpha-1)(\alpha-2)}{3}|c_1^3|.
\end{aligned}$$

The estimates $|c_n| \le 2\,(n = 1,3)$, $|l_3| \le 2$ together with the estimate (2.17) for $|c_2+l_2|$ yields the following:

$$\begin{aligned}
|a_4| \le & \frac{2\alpha}{(1+3\lambda)}\left(1+\frac{2(1-\alpha)\{6\alpha(1+2\lambda)+(5-4\alpha)(1+\lambda)^2\}}{3(1+\lambda)^2}\right) \\
& \left(1+\sqrt{2} \le \lambda \le \lambda_0,\ 0 \le \alpha < 1 \,\text{or}\, \lambda > \lambda_0,\ 0 \le \alpha \le \frac{1}{2}\right).
\end{aligned}$$

We get the second estimate in (2.3).

Lastly, if $\lambda > \lambda_0$ and $\frac{1}{2} < \alpha < 1$, then we apply the triangle inequality in (2.23) and get

$$2(1+3\lambda)|a_4| \le 2\alpha|c_3| + \alpha(1-\alpha)|c_1||(c_2-l_2)| + \alpha(1-\alpha)|c_1||(c_2+l_2)| + \frac{\alpha(1-\alpha)(2-\alpha)}{3}|c_1^3|.$$

By using the well bounds $|c_n| \leq 2\,(n = 1, 2, 3)$, $|l_2| \leq 2$ and the refined estimate (2.17) for $|c_2 + l_2|$ we have

$$2(1+3\lambda)|a_4| \leq 4\alpha + 8\alpha(1-\alpha) + \frac{8\alpha(1-\alpha)[2\alpha(1+2\lambda)+(1-\alpha)(1+\lambda)^2]}{(1+\lambda^2}$$
$$+ \frac{8\alpha(1-\alpha)(2-\alpha)}{3}$$

or

$$|a_4| \leq \frac{2\alpha}{(1+3\lambda)}\left(1 + \frac{2(1-\alpha)\left\{6\alpha(1+2\lambda)+4(2-\alpha)(1+\lambda)^2\right\}}{3(1+\lambda)^2}\right)$$
$$\left(\lambda > \lambda_0,\, \frac{1}{2} < \alpha < 1\right).$$

This is precisely the third estimate in (2.3). Thus, the proof of Theorem 2.1 is completed. □

**Theorem 2.2.** *Let the function* $h(z)$*, given by* (1.3)*, be in the class* $\Sigma\Theta_\lambda^\alpha\,(\lambda \geq 1,\ 0 < \alpha \leq 1)$*. Then*

$$|b_1| \leq \frac{2\alpha}{2\lambda - 1} \tag{2.26}$$

*and*

(2.27)
$$|b_2| \leq \begin{cases} \frac{2\alpha}{3\lambda-1}\left(1 + \frac{2(1-\alpha)|1-2\alpha|}{3}\right) & 1 \leq \lambda \leq \lambda_1,\ 0 < \alpha \leq 1 \; or \; \lambda > \lambda_1,\ 0 < \alpha \leq \frac{1}{2}, \\ \frac{2\alpha}{3\lambda-1}\left(1 + \frac{4(1-\alpha)(1+\alpha)}{3}\right) & \lambda > \lambda_1,\ \frac{1}{2} < \alpha \leq 1, \end{cases}$$

*where* $\lambda_1$ *is the largest positive root of the quadratic equation*

$$2(1-2\alpha)\lambda^2 + 3(3\alpha-1)\lambda + 1 - 3\alpha = 0.$$

*Proof.* We adopt and suitably modify the lines of proof of Theorem 2.1 here. Therefore, we choose to omit the details and sketch only the main steps. In this case we have the following:

$$(1-\lambda)\frac{h(z)}{z} + \lambda h'(z) = [P(z)]^\alpha \tag{2.28}$$

and

$$(1-\lambda)\frac{H(w)}{w} + \lambda H'(w) = [Q(w)]^\alpha, \tag{2.29}$$

respectively, where

$$P(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \tag{2.30}$$

and

$$Q(w) = 1 + \frac{l_1}{w} + \frac{l_2}{w^2} + \frac{l_3}{w^3} + \cdots \quad (w \in \Delta) \tag{2.31}$$

are functions with positive real part in $\Delta$. By comparing coefficients we get:

$$(1-\lambda)b_0 = \alpha c_1, \tag{2.32}$$

$$(1-2\lambda)b_1 = \alpha c_2 + \frac{1}{2}\alpha(\alpha-1)c_1^2, \tag{2.33}$$

$$(1-3\lambda)b_2 = \alpha c_3 + \alpha(\alpha-1)c_1c_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3, \tag{2.34}$$

$$(1-\lambda)b_0 = -\alpha l_1, \tag{2.35}$$

$$(1-2\lambda)b_1 = -\alpha l_2 - \frac{1}{2}\alpha(\alpha-1)l_1^2 \tag{2.36}$$

and

$$-(1-3\lambda)(b_2+b_0b_1) = \alpha l_3 + \alpha(\alpha-1)l_1l_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)l_1^3. \tag{2.37}$$

The equations (2.32) and (2.35) give the following relation between $c_1$ and $l_1$:

$$c_1 = -l_1. \tag{2.38}$$

Similarly, the equations (2.33) and (2.36) provide the following relation among $c_1, c_2$ and $l_2$

$$c_2 + l_2 = (1-\alpha)c_1^2. \tag{2.39}$$

We add (2.36) and (2.33) which yields, after simplification, the following:

$$2(1-2\lambda)b_1 = \alpha(c_2 - l_2). \tag{2.40}$$

By applying the triangle inequality and using the well known estimates $|c_2| \leq 2$ and $|l_2| \leq 2$ we obtain

$$|b_1| \leq \frac{2\alpha}{2\lambda-1}. \tag{2.41}$$

This is precisely our assertion at (2.26).

In order to find a bound for $|b_2|$ we subtract (2.37) from (2.34) and after simplification get

$$2(1-3\lambda)b_2 = -(1-3\lambda)b_0b_1 + \alpha(c_3-l_3) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3. \tag{2.42}$$

Similarly addition of (2.37) and (2.34) yields:

$$-(1-3\lambda)b_0b_1 = \alpha(c_3+l_3) + \alpha(\alpha-1)c_1(c_2-l_2). \tag{2.43}$$

By substituting $b_1 = \frac{\alpha(c_2-l_2)}{2(1-2\lambda)}$ from (2.40) and $b_0 = \frac{\alpha c_1}{1-\lambda}$ in the above equation (2.43) we have

$$c_1(c_2-l_2) = \mu_1(c_3+l_3), \tag{2.44}$$

where

$$\mu_1 = \frac{2(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}.$$

We notice that $0 < \mu_1 \leq 2$ for every $\lambda \geq 1$ if $0 < \alpha \leq \frac{1}{2}$. However, if $\frac{1}{2} < \alpha \leq 1$, then $0 < \mu_1 \leq 2$ for $1 \leq \lambda \leq \lambda_1$, where $\lambda_1$ is the largest positive root of the quadratic equation

$$2(1-2\alpha)\lambda^2 + 3(3\alpha-1)\lambda + 1 - 3\alpha = 0.$$

In (2.42) we replace $-(1-3\lambda)b_0b_1$ by the expression on the right hand side of the equality (2.43) and use the relation $c_1 = -l_1$. This gives on simplification the following:

$$2(1-3\lambda)b_2 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2-l_2) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3.$$

By replacing $c_2 + l_2$ by $(1-\alpha)c_1^2$ from the relation (2.39) we obtain

$$2(1-3\lambda)b_2 = 2\alpha c_3 + \alpha(\alpha-1)c_1(c_2-l_2) + \frac{1}{3}\alpha(\alpha-1)(1-2\alpha)c_1^3. \tag{2.45}$$

We first suppose that $\lambda$ and $\alpha$ are constrained by the requirement that $1 \leq \lambda \leq \lambda_1$ and $0 < \alpha \leq 1$ or $\lambda > \lambda_1$ and $0 < \alpha \leq \frac{1}{2}$. Now, we replace $c_1(c_2-l_2)$ by $\mu_1(c_3+l_3)$ from (2.44) and get:

$$\begin{aligned}2(1-3\lambda)b_2 =& 2\alpha c_3 - \frac{2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}(c_3+l_3)\\ &+ \frac{1}{3}\alpha(\alpha-1)(1-2\alpha)c_1^3\\ =& \frac{2\alpha^2(3\lambda-1) + 2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}c_3\\ &- \frac{2\alpha(1-\alpha)(\lambda-1)(2\lambda-1)}{(3\lambda-1)\alpha + 2(1-\alpha)(\lambda-1)(2\lambda-1)}l_3 + \frac{\alpha(1-\alpha)(2\alpha-1)}{3}c_1^3.\end{aligned}$$

By applying the triangle inequality together with the estimates $|c_n| \leq 1\,(n = 1, 3)$, $|l_3| \leq 2$ we have after simplification the following:

$$\begin{aligned}|b_2| \leq& \frac{2\alpha}{3\lambda-1}\left(1 + \frac{2(1-\alpha)|1-2\alpha|}{3}\right)\\ &\left(1 \leq \lambda \leq \lambda_1 \text{ and } 0 < \alpha \leq 1 \text{ or } \lambda > \lambda_1 \text{ and } 0 < \alpha \leq \frac{1}{2}\right).\end{aligned}$$

We get the first estimate in (2.27). Lastly, suppose that $\lambda > \lambda_1$ and $\frac{1}{2} < \alpha \leq 1$. We apply the triangle inequality and the familiar estimates $|c_n| \leq 2\,(n = 1, 2, 3)$ in (2.45) and get

$$|b_2| \leq \frac{2\alpha}{3\lambda-1}\left(1 + \frac{4(1-\alpha)(1+\alpha)}{3}\right) \quad \left(\lambda > \lambda_1,\, \frac{1}{2} < \alpha \leq 1\right).$$

This is precisely the second estimate in (2.27). The proof Theorem 2.2 is thus completed. □

## 3. Concluding Remarks

By Definition 1.1, to each function $f \in \sigma Q_\lambda^\alpha$ we associate a function of the Carthéodory class which is of the the form:

$$P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

Similarly, its compositional inverse function $g$ is also associated with a function $Q(z)$ of the Carthéodory class which is given by:

$$Q(z) = 1 + l_1 z + l_2 z^2 + l_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

The correspondence in both cases is one-to-one. In the present paper we derived suitable relationships between $c_1$ and $l_1$, and also among $c_1$, $c_2$, $c_3$ and $l_3$. These relations yielded refined bounds on $|c_1|$, $|c_1(c_2 - l_2)|$, $|c_2 + l_2|$ and $|c_3 + l_3|$, in suitable ranges of $\alpha$ and $\lambda$. Using the refined bounds we found estimates on $|a_3|$ and $|a_4|$ for functions in the class $\sigma Q_\lambda^\alpha$. We suitably adopted and amended the lines of proof of our Theorem 2.1 and found estimates on $|b_1|$ and $|b_2|$ for functions in the class $\Sigma\Theta_\lambda^\alpha$.

Recently Hayami and Owa [12] found bounds on $|a_4|$ and improved upon the bounds of Srivastava et al. [19] for $|a_3|$ for the class $\sigma Q^\alpha$. Thus, we have

(3.1)
$$|a_3| \le \frac{2\alpha}{3}, \quad |a_4| \le \frac{\alpha}{2}\left(1 + \frac{2(1-\alpha)(2+5\alpha)}{3(2+\alpha)}\sqrt{\frac{2}{2+\alpha}}\right) \quad (f \in \sigma Q^\alpha,\ 0 < \alpha \le 1).$$

Also Frasin and Aouf [9] extended the work of Srivastava et al. [19] as follows:
(3.2)
$$|a_2| \le \frac{2\alpha}{\sqrt{2\alpha(1+2\lambda) + (1-\alpha)(1+\lambda)^2}}, \quad |a_3| \le \frac{2\alpha}{1+2\lambda} + \frac{4\alpha^2}{(1+\lambda)^2} \quad (f \in \sigma Q_\lambda^\alpha).$$

A comparison of (2.1) and (2.2) with (3.2) shows that our estimates on $|a_2|$ and $|a_3|$, for the class $\sigma Q_\lambda^\alpha$ found in Theorem 2.1, improve upon the earlier bound obtained by Frasin and Aouf [9]. Also taking $\lambda = 1$ in Theorem 2.1 we get the estimates of Hayami and Owa [12] mentioned at (3.1).

In a recent paper Hamidi et al. [11] found bounds for functions in a class closely related to the function class $\Sigma\Theta_\lambda^\alpha$ studied in this paper, but under the restriction that initial coefficients are missing. Our work in Theorem 2.2 on coefficient bounds for initial coefficients supplements the results in [11].

## References


[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl. Math. Lett. **25** (2012), 344–351. `https://doi.org/10.1016/j.aml.2011.09.012`

[2] S. Barik, *Estimates for initial coefficients in subclasses of bi-univalent analytic functions*, Ph. D. Thesis, Berhampur University, India, 2015.

[3] D. A. Brannan, and J. G. Cluine (Eds.), *Aspects of Contemporary Complex Analysis*, in: *Proceedings of the NATO Advanced Study Institute*, University of Durham, Durham, July 1979, 1–20, Academic Press, New York, London, 1980.

[4] D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, in: S. M. Mazhar, A. Hamoui, N. S. Faour (Eds.), *Mathematical Analysis and its Applications*, Kuwait, February 1985, 18–21, KFAS Proceedings Series **3**, Pergamon Press, Elsevier Science Limited, Oxford, 1998, 53–60. See also Studia Univ. Babes-Bolyai Math. **31**(2) (1986), 70–77.

[5] M. Çağlar, H. Orhan and N. Yağmur, *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat **27**(7) (2013), 1165–1171. `https://doi.org/10.101610.2298/FIL1307165C`

[6] P. N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc. **62** (1997), 37–43.

[7] P. L. Duren, *Univalent Functions*, Grundlheren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[8] S. S. Ding, Y. Ling and G. J. Bao, *Some properties of a class of analytic functions*, J. Math. Anal. Appl. **195**(1) (1995), 71–81.

[9] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011), 1569–1573. `https://doi.org/10.1016/j.aml.2011.03.048`

[10] A. W. Goodman, *Univalent Functions*, Vol. I, Mariner Publishing Co. Inc, Tampa, FL, 1983.

[11] S. G. Hamidi, T. Janani and G. Murugasundaramoorthy, *Coefficient estimates for certain classes of meromorphic bi-univalent functions*, C. R. Math. Acad. Sci. Paris **352**(4) (2014), 287–282. `https://doi.org/10.1016/j.crma.2014.01.010`

[12] T. Hayami, and S. Owa, *Coefficient bounds for bi-univalent functions*, PanAmer. Math. J. **22**(4) (2012), 15–26.

[13] S. S. Kumar, V. Kumar and V. Ravichandran, *Estimates for the initial coefficients of bi-univalent functions*, Tamsui Oxf. J. Inf. Math. Sci. **29**(4) (2013), 487–504.

[14] M. Lewin, *On the coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68. `https://doi.org/10.1090/S0002-9939-1967-0206255-1`

[15] A. K. Mishra and M. M. Soren, *Coefficient bounds for bi-starlike analytic functions*, Bull. Belg. Math. Soc. Simon Stevin **21**(2) (2014), 157–161.

[16] E. Natanyahu, *The minimal distance of image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Ration. Mech. Anal. **32** (1969), 100–112. `https://doi.org/10.1007/BF00247676`

[17] Z. Peng and Q. Han, *On the coefficient of several classes of bi-univalent functions*, Acta Math. Sci. Ser. B **34**(1) (2014), 228–240. `https://doi.org/10.1016/S0252-9602(13)60140-X`

[18] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat **27**(5) (2013), 831–842. `https://doi.org/DOI10.2298/FIL1305831S`

[19] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192. `https://doi.org/10.1016/j.aml.2010.05.009`

[20] T. S. Taha, *Topics in univalent function theory*, Ph. D. Thesis, University of London, 1981.

[21] Q. H. Xu, Y. C. Gui, and H. M. Srivastava, *Coefficients for certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **25** (2012), 990–994. `https://doi.org/10.1016/j.aml.2011.11.013`

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# ON ZERO FREE REGIONS FOR DERIVATIVES OF A POLYNOMIAL

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Abstract. Let $P_n$ denote the set of polynomials of the form

$$p(z) = (z-a)^m \prod_{k=1}^{n-m} (z - z_k),$$

with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$. For the polynomials of the form $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$, Brown [2] stated the problem "Find the best constant $C_n$ such that $p'(z)$ does not vanish in $|z| < C_n$". He also conjectured in the same paper that $C_n = \frac{1}{n}$. This problem was solved by Aziz and Zarger [1]. In this paper, we obtain the results which generalizes the results of Aziz and Zarger.

## 1. Introduction and statement of results

Let $p(z) = \prod_{k=1}^{n}(z - z_k)$ be a complex polynomial of degree $n$. The classical Gauss-Lucas theorem states that every critical point of a complex polynomial $p$ of degree at least 2 lies in the convex hull of its zeros. This theorem has been further investigated and developed. About the location of critical point relative to each individual zero, a possible answer is given by the famous conjecture known in literature as Sendov's conjecture.

*Conjecture* 1 (Sendov's Conjecture). If all the zeros of a polynomial $p(z)$ lie in $|z| \leq 1$, then for any zero $z_0$ of $p$, the disc $|z - z_0| \leq 1$ contains at least one critical point of $p$.

This conjecture has attracted much attention. About 100 papers have been published related to this conjecture. This conjecture has so far been verified for general

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polynomials of degree less than or equal to 8. However the problem is still unproved in general.

In connection with this conjecture, Brown [2] observed that, if $p(z) = z(z-1)^{n-1}$, then $p'(\frac{1}{n}) = 0$ and posed the following problem.

"Let $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$. Find the best constant $C_n$ such that $p'(z)$ does not vanish in $|z| < C_n$".

However, Brown himself conjectured that $C_n = \frac{1}{n}$. This problem has been settled by Aziz and Zarger [1], in fact they proved the following.

**Theorem 1.1.** *If $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$ is a polynomial of degree $n$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in $|z| < \frac{1}{n}$.*

As a generalization of Theorem 1.1, N. A. Rather and F. Ahmad [3] have proved the following result.

**Theorem 1.2.** *Let $p(z) = (z-a)\prod_{k=1}^{n-1}(z - z_k)$ with $|a| \leq 1$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in the region*

$$\left|z - \left(\frac{n-1}{n}\right)a\right| < \frac{1}{n}.$$

*The result is best possible as is shown by the polynomial*

$$p(z) = (z-a)(z - e^{i\alpha})^{n-1}, \quad 0 \leq \alpha < 2\pi.$$

N. A. Rather and F. Ahmad also proved the following result in the same paper.

**Theorem 1.3.** *Let $p(z) = (z-a)^m\prod_{k=1}^{n-m}(z - z_k)$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then $p'(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros of $p'(z)$ lie in the region*

$$\left|z - \left(\frac{n-m}{n}\right)a\right| \geq \frac{m}{n}.$$

*The result is best possible as is shown by the polynomial*

$$p(z) = (z-a)^m(z - e^{i\alpha})^{n-m}, \quad 0 \leq \alpha < 2\pi.$$

Zarger and Manzoor [4] have extended Theorem 1.1 to the second derivative $p''(z)$ of a polynomial of the form $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$, with $|z_k| \geq 1$ for $1 \leq k \leq n-m$. In fact they proved the following.

**Theorem 1.4.** *If $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p''(z)$ does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$.*

Zarger and Manzoor [4] also obtained the following result for the polynomial $p^{(m)}(z)$, $m \geq 1$.

**Theorem 1.5.** *If $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$ is a polynomial of degree $n$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, does not vanish in $|z| < \frac{m!}{n(n-1)\cdots(n-m+1)}$.*

In this paper, we first prove the following theorem which generalize the result of Theorem 1.4.

**Theorem 1.6.** *Let $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree $n$ with $|a| \le 1$, and $|z_k| \ge 1$ for $1 \le k \le n-m$, then $p''(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region*

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

*Proof.* We can write

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, then by Theorem 1.3, the polynomial

$$p'(z) = (z-a)^{m-1} R(z),$$

where $R(z) = (z-a)Q'(z) + mQ(z)$ has $(m-1)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{n-m}{n}\right) a \right| \ge \frac{m}{n}.$$

Now, consider the polynomial

$$S(z) = p'\left(\frac{m}{n}z + \frac{n-m}{n}a\right) \tag{1.1}$$

or

$$S(z) = \left(\frac{m}{n}\right)^{m-1} (z-a)^{m-1} R\left(\frac{m}{n}z + \frac{n-m}{n}a\right),$$

then $S(z)$ is a polynomial of degree $n-1$ with $(m-1)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in $|z| \ge 1$.

Now, applying Theorem 1.3 to the polynomial $S(z)$, the derivative $S'(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{(n-1)-(m-1)}{n-1}\right) a \right| \ge \frac{m-1}{n-1},$$

which is equivalent to

$$\left| z - \left(\frac{n-m}{n-1}\right) a \right| \ge \frac{m-1}{n-1}.$$

Replacing $z$ by $\frac{n}{m}z + \left(\frac{m-n}{m}\right)a$, in equation (1.1) and differentiating, we obtain

$$p''(z) = (z-a)^{m-2} T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$.

Applying above, we see $p''(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

This completes the proof. □

*Remark* 1.1. For $a = 0$, it reduces to Theorem 1.4.

Our next result generalizes Theorem 1.5 to the polynomial of the form $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$.

**Theorem 1.7.** *If $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, has all its zeros in the region*

$$\left|z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)}\right)a\right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

*Proof.* We can write

$$p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$$

or

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, $|z_k| \geq 1$, $1 \leq k \leq n-m$.

From the proof of Theorem 1.6, we can write

$$p''(z) = (z-a)^{m-2}T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$. Also, $p''(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \frac{n(n-1) - m(m-1)}{n(n-1)}a\right| \geq \frac{m(m-1)}{n(n-1)}.$$

Now, consider the polynomial

$$U(z) = p''\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1) - m(m-1)}{n(n-1)}a\right) \tag{1.2}$$

or

$$U(z) = \left(\frac{m(m-1)}{n(n-1)}\right)^{m-2}(z-a)^{m-2}T\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1) - m(m-1)}{n(n-1)}a\right).$$

Then $U(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in $|z| \geq 1$.

Again, applying Theorem 1.3 to $U(z)$, which is a polynomial of degree $n-2$, the derivative $U'(z)$ has $(m-3)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(\frac{n-2-(m-2)}{n-2}\right)a\right| \geq \frac{m-2}{n-2},$$

which is equivalent to

$$\left|z - \left(\frac{n-m}{n-2}\right)a\right| \geq \frac{m-2}{n-2}.$$

Replacing $z$ by $\frac{n(n-1)}{m(m-1)}z + \frac{m(m-1)-n(n-1)}{m(m-1)}a$, in (1.2) and differentiating, we obtain

$$p'''(z) = (z-a)^{m-3}V(z),$$

where $V(z) = (z-a)T'(z) + (m-2)T(z)$ has $(m-3)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie

$$\left|z - \left(1 - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}\right)a\right| \geq \frac{m(m-1)(m-2)}{n(n-1)(n-2)}.$$

Proceeding similarly, for any positive integer $m = 1, 2, \ldots, n-1$, we see that the polynomial $p^{(m)}(z)$ has all its zeros in the region

$$\left|z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)}\right)a\right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

This completes the proof. □

*Remark* 1.2. For $a = 0$, it reduces to Theorem 1.5.

*Remark* 1.3. For $m = 1$, it reduces to Theorem 1.2.

*Remark* 1.4. For $a = 0$ and $m = 1$, it reduces to the result of Aziz and Zarger.

## References

[1] A. Aziz and B. A. Zarger, *On the critical points of a polynomial,* Bull. Aust. Math. Soc. **57** (1998), 173–174. https://doi.org/10.1017/S000497270003152X
[2] J. E. Brown, *On the Ilief-Sendov conjecture,* Pacific J. Math. **135** (1988), 223–232.
[3] N. A. Rather and F. Ahmad, *On the critical points of a polynomial*, BIBECHANA **9** (2013), 28–32. https://doi.org/10.1017/S000497270003152X
[4] B. A. Zarger and A. W. Manzoor, *On zero free regions for derivative of a polynomial*, BIBECHANA **14** (2017), 48–53. https://doi.org/10.3126/bibechana.v14i0.15521

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# A NEW EXTENSION OF BANACH-CARISTI THEOREM AND ITS APPLICATION TO NONLINEAR FUNCTIONAL EQUATIONS

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Abstract. In this paper, we present a new extension of Banach-Caristi type theorem for multivalued mappings. We show that our result is not a consequence of multivalued version of Banach contraction principle due to Nadler. We provide an application of our result to the solution of functional equations.

## 1. Preliminaries

Caristi [4] introduced an important generalization of the Banach contraction principle as follows.

**Theorem 1.1** ([4]). *Let* $(\Lambda, \eta)$ *be a complete metric space* (*MS, in short*) *and* $\Im : \Lambda \to \Lambda$ *be a self-map satisfying*

$$\eta(\varsigma, \Im(\varsigma)) \leq \phi(\varsigma) - \phi(\Im(\varsigma)),$$

*for all* $\varsigma \in \Lambda$, *where* $\phi : \Lambda \to [0, \infty)$ *is a lower semicontinous mapping. Then* $\Im$ *admits a fixed point.*

Caristi's theorem has a close connection with Ekeland's variational principle [7, 8]. Weston [20] established a characterization for the metric completeness in terms of Caristi's theorem. Agarwal and Khamsi [1] extended Caristi's result to vector valued metric spaces.

In 1969, Nadler [17] established a number of very significant fixed point results for multivalued maps using the Hausdorff concept, i.e., by considering the distance

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between two arbitrary sets. Khan [12] studied some interesting common fixed points for multivalued maps.

Let $(\Lambda, \eta)$ be a complete MS and let $CB(\Lambda)$ denote the class of all nonempty closed and bounded subsets of $\Lambda$. Then for $\mathcal{A}, \mathcal{B} \in CB(\Lambda)$, define the map $\mathcal{H} : CB(\Lambda) \times CB(\Lambda) \to [0, \infty)$ by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{\xi \in \mathcal{B}} \Delta(\xi, \mathcal{A}), \sup_{\delta \in \mathcal{A}} \Delta(\delta, \mathcal{B}) \right\},$$

where $\Delta(\delta, \mathcal{B}) = \inf_{\xi \in \mathcal{B}} \eta(\delta, \xi)$. $\mathcal{H}$ is called the Pompeiu-Hausdorff metric induced by $\eta$.

**Definition 1.1** ([17]). $\varsigma \in \Lambda$ is said to be a fixed point of the multivalued map $\Im : \Lambda \to CB(\Lambda)$ if $\varsigma \in \Im(\varsigma)$. The set of all fixed points of $\Im$ is denoted by $Fix(\Im)$.

*Remark* 1.1. In the MS $(CB(\Lambda), \mathcal{H})$, $\varsigma \in \Lambda$, is a fixed point of $\Im$ if and only if $\Delta(\varsigma, \Im(\varsigma)) = 0$.

The following results are important in the present context.

**Lemma 1.1** ([3, 5]). *Let $(\Lambda, \eta)$ be a MS and $U, V, W \in CB(\Lambda)$. Then*

(a) $\Delta(\mu, V) \leq \eta(\mu, \gamma)$ *for any* $\gamma \in V$ *and* $\mu \in \Lambda$;
(b) $\Delta(\mu, V) \leq \mathcal{H}(U, V)$ *for any* $\mu \in U$;
(c) $\Delta(\mu, U) \leq [\eta(\mu, \nu) + \Delta(\nu, U)]$ *for all* $\mu, \nu \in \Lambda$.

**Lemma 1.2** ([17]). *Let $U, V \in CB(\Lambda)$ and let $\varsigma \in U$. Then for any $p > 0$ there exists $\xi \in V$ such that*

$$\eta(\varsigma, \xi) \leq \mathcal{H}(U, V) + p.$$

*However, there may not be a point $\xi \in V$ such that*

$$\eta(\varsigma, \xi) \leq \mathcal{H}(U, V).$$

*If $V$ is compact, then such a point $\xi$ exists, i.e., $\eta(\varsigma, \xi) \leq \mathcal{H}(U, V)$.*

**Lemma 1.3** ([17]). *Let $\{U_n\}$ be a sequence in $CB(\Lambda)$ and $\lim_{n\to\infty} \mathcal{H}(U_n, U) = 0$ for some $U \in CB(\Lambda)$. If $\upsilon_n \in U_n$ and $\lim_{n\to\infty} \eta(\upsilon_n, \upsilon) = 0$ for some $\upsilon \in \Lambda$, then $\upsilon \in U$.*

**Lemma 1.4** ([16]). *If $\{\varsigma_n\}$ is a sequence in a MS $(\Lambda, \eta)$ such that there exists a constant $\lambda \in [0, 1)$ satisfying*

$$\eta(\varsigma_{n+1}, \varsigma_n) \leq \lambda \eta(\varsigma_n, \varsigma_{n-1}), \quad \text{for all } n \geq 1,$$

*then the sequence $\{\varsigma_n\}$ is Cauchy.*

Caristi type conditions have been applied to multivalued mappings by Jachymski [10], Feng and Liu [9], Latif and Kutbi [15] and many more. Also, generalized Caristi's fixed point theorems have been studied by Latif [14], Suzuki [19], and several others.

Recently, Khojateh et al. [13] gave some applications of Caristi's theorem in MS, whereas Karapinar et al. [11] extended the Banach and Caristi type theorems to $b$-metric spaces. In the present paper, we introduce a new extension of Banach and

Caristi type theorem to a complete MS for multivalued mappings. In Section 2, we present the main result and in Section 3 we provide an application of our result to the solution of a particular type of nonlinear functional equations. For some recent work on the application of multivalued fixed point results to the solution of functional/integral equations, we refer to [2, 6, 18].

## 2. Main Results

In this section, we present our main result which is a new extension of Banach-Caristi theorem.

**Theorem 2.1.** *Let* $(\Lambda, \eta)$ *be a complete MS and* $\Im : \Lambda \to CB(\Lambda)$ *be a multivalued map such that* $\Im(\varsigma)$ *is compact for each* $\varsigma \in \Lambda$. *Suppose that the function* $\phi : \Lambda \to \mathbb{R}$ *satisfies the following conditions:*

(a) $\phi$ *is bounded below* (*i.e.,* $\inf \phi(\varsigma) > -\infty$)*;*

(b) $\Delta(\varsigma, \Im(\varsigma)) > 0$ *implies* $\mathcal{H}(\Im(\varsigma), \Im(\xi)) \leq (\phi(\varsigma) - \phi(\xi))\eta(\varsigma, \xi)$ *for all* $\xi \in \Lambda$.

*Then* $\Im$ *has a fixed point.*

*Proof.* Consider $\varsigma_0 \in \Lambda$ and choose $\varsigma_1 \in \Im(\varsigma_0)$. Since $\Im(\varsigma_1)$ is compact, by Lemma 1.2, we can select $\varsigma_2 \in \Im(\varsigma_1)$ satisfying $\eta(\varsigma_1, \varsigma_2) \leq \mathcal{H}(\Im(\varsigma_0), \Im(\varsigma_1))$. Similarly, we can choose $\varsigma_3 \in \Im(\varsigma_2)$ satisfying $\eta(\varsigma_2, \varsigma_3) \leq \mathcal{H}(\Im(\varsigma_1), \Im(\varsigma_2))$ and so on.

Continuing in this way, we construct a sequence $\{\varsigma_n\}_{n=0}^{\infty}$ satisfying

$$\eta(\varsigma_n, \varsigma_{n+1}) \leq \mathcal{H}(\Im(\varsigma_{n-1}), \Im(\varsigma_n)).$$

We assume that $\varsigma_n \notin \Im(\varsigma_n)$ (i.e., $\Delta(\varsigma_n, \Im(\varsigma_n)) > 0$) for all $n \geq 0$, since otherwise we obtain a fixed point and the proof is completed.

Let $\alpha_n = \eta(\varsigma_{n-1}, \varsigma_n)$. Using condition (b), we have

$$\begin{aligned}\alpha_{n+1} = \eta(\varsigma_n, \varsigma_{n+1}) &\leq \mathcal{H}(\Im(\varsigma_{n-1}), \Im(\varsigma_n)) \\ &\leq (\phi(\varsigma_{n-1}) - \phi(\varsigma_n))\eta(\varsigma_{n-1}, \varsigma_n) \\ &= (\phi(\varsigma_{n-1}) - \phi(\varsigma_n))\alpha_n. \end{aligned} \tag{2.1}$$

So, $0 < \frac{\alpha_{n+1}}{\alpha_n} \leq \phi(\varsigma_{n-1}) - \phi(\varsigma_n)$ for each $n \in \mathbb{N}$.

Thus, the sequence $\{\phi(\varsigma_n)\}$ is positive and non-increasing (i.e., bounded and monotone). Hence, it converges to some $r \geq 0$.

Further, for each $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=1}^{n} \frac{\alpha_{k+1}}{\alpha_k} &\leq \sum_{k=1}^{n} (\phi(\varsigma_{k-1}) - \phi(\varsigma_k)) \\ &= (\phi(\varsigma_0) - \phi(\varsigma_1)) + (\phi(\varsigma_1) - \phi(\varsigma_2)) + \cdots + (\phi(\varsigma_{n-1}) - \phi(\varsigma_n)) \\ &= (\phi(\varsigma_0) - \phi(\varsigma_n)) \\ &\to \phi(\varsigma_0) - r \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{\alpha_n} < \infty$, which implies that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 0$. Thus, for $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\alpha_{n+1}}{\alpha_n} \leq \lambda$ for all $n \geq n_0$. This implies that $\eta(\varsigma_{n+1}, \varsigma_n) \leq$

$\lambda\eta(\varsigma_n, \varsigma_{n-1})$ for all $n \geq n_0$. Using Lemma 1.4, we see that the sequence $\{\varsigma_n\}$ is Cauchy and since $(\Lambda, \eta)$ is complete, $\varsigma_n \to \varsigma$ as $n \to \infty$ for some $\varsigma \in \Lambda$.

We claim that $\varsigma$ is a fixed point of $\Im$. We have

$$\begin{aligned}\Delta(\varsigma, \Im(\varsigma)) &\leq [\eta(\varsigma, \varsigma_{n+1}) + \Delta(\varsigma_{n+1}, \Im(\varsigma))] \quad \text{(using (c) of Lemma 1.1)}\\ &\leq [\eta(\varsigma, \varsigma_{n+1}) + \mathcal{H}(\Im(\varsigma_n), \Im(\varsigma))] \quad \text{(using (b) of Lemma 1.1)}\\ &\leq [\eta(\varsigma, \varsigma_{n+1}) + (\phi(\varsigma_n) - \phi(\varsigma))\eta(\varsigma_n, \varsigma)] \quad \text{(using (b) of the hypothesis)}\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, $\Delta(\varsigma, \Im(\varsigma)) = 0$, i.e., $\varsigma \in \Im(\varsigma)$. □

*Remark* 2.1. It should be noted that the right hand side of Caristi's condition from Theorem 1.1 does not depend on the distance function, whereas in our condition from Theorem 2.1, the right hand side depends on the distance function. As such, Theorem 2.1 should better be treated as a variant of a Caristi type result instead of a generalization of the same. This is more so, because one can observe that when $\Im$ is a single-valued mapping, our result does not reduce to original Caristi's result. For these reasons our result does not bear a direct connection with the multivalued results studied in [9, 10, 15].

Next, we provide an example to validate Theorem 2.1.

*Example* 2.1. Consider $\Lambda = \{0, 1, 2\}$ and $\eta : \Lambda \times \Lambda \to [0, \infty)$ be defined as $\eta(0, 1) = 1$, $\eta(0, 2) = 2$, $\eta(1, 2) = 1$, $\eta(\varsigma, \varsigma) = 0$ and $\eta(\varsigma, \xi) = \eta(\xi, \varsigma)$ for all $\varsigma, \xi \in \Lambda$. Then $(\Lambda, \eta)$ is a complete MS. Define the multivalued map $\Im : \Lambda \to CB(\Lambda)$ by

$$\Im(\varsigma) = \begin{cases} \{0\}, & \text{if } \varsigma \neq 2,\\ \{0, 2\}, & \text{if } \varsigma = 2.\end{cases}$$

Also, define $\phi : \Lambda \to \mathbb{R}$ by $\phi(0) = 0$, $\phi(1) = 5$ and $\phi(2) = 3$. Clearly, $(\Lambda, \eta)$ is a complete MS and $\Im(\varsigma)$ is compact for each $\varsigma \in \Lambda$. Further, we observe that $\Delta(\varsigma, \Im(\varsigma)) > 0$ for $\varsigma = 1$. Indeed, we have

$$\begin{aligned}\Delta(0, \Im 0) =& \Delta(0, \{0\}) = 0,\\ \Delta(1, \Im 1) =& \Delta(1, \{0\}) = 1,\\ \Delta(2, \Im 2) =& \Delta(2, \{0, 2\}) = 0.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{H}(\Im 1, \Im 0) =& \mathcal{H}(\{0\}, \{0\}) = 0,\\ \mathcal{H}(\Im 1, \Im 1) =& 0,\\ \mathcal{H}(\Im 1, \Im 2) =& \mathcal{H}(\{0\}, \{0, 2\}) = 2.\end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned}\mathcal{H}(\Im 1, \Im 0) &\leq (\phi(1) - \phi(0))\eta(1, 0),\\ \mathcal{H}(\Im 1, \Im 1) &\leq (\phi(1) - \phi(0))\eta(1, 1),\\ \mathcal{H}(\Im 1, \Im 2) &\leq (\phi(1) - \phi(0))\eta(1, 2).\end{aligned}$$

Therefore, all conditions of Theorem 2.1 are satisfied and we see that $\Im$ has a fixed point. Here, $Fix(\Im) = \{0, 2\}$.

*Remark* 2.2. Note that $\mathcal{H}(\Im 1, \Im 2) = 2 > \eta(1, 2) = 1$. Hence, Theorem 2.1 is not a consequence of the multivalued version of the Banach contraction principle due to Nadler [17].

## 3. Application to Functional Equations

Mathematical optimization problems often make use of dynamic programming to obtain the optimal solution. In many such optimal problems, the corresponding dynamical program gets boiled down to solve a functional equation of the form:

$$u(t) = \sup_{s\in V}\{g(t,s) + G(t,s,u(h(t,s)))\}, \quad t \in W, \tag{3.1}$$

where $h : W \times V \to W$, $g : W \times V \to \mathbb{R}$ and $G : W \times V \times \mathbb{R} \to \mathbb{R}$. Let $M$ and $N$ be Banach spaces. $W \subset M$ is called a state space and $V \subset N$ is called a decision space. The process under study is a multistage process. We define the following:

•$B(W) :=$ the collection of all bounded and closed real functions on $W$;

•$\|f\| := \sup_{t\in V} |f(t)|$, $f \in B(W)$.

The metric induced by $\|\cdot\|$ is given by

$$\eta(f_1, f_2) = \sup_{t\in W} |f_1(t) - f_2(t)|, \quad f_1, f_2 \in B(W). \tag{3.2}$$

Then $(B(W), \|\cdot\|)$ is a Banach space. Further, define the operator $\Im : B(W) \to B(W)$ by

$$\Im(f)(t) = \sup_{s\in V} g(t,s) + G(t,s,f(h(t,s))), \tag{3.3}$$

for all $f \in B(W)$ and $t \in W$. To prove an existence result, we need the following theorem.

**Theorem 3.1.** *Let $\Im : B(W) \to B(W)$ be defined by* (3.3). *Let $\Im$ be upper semi-continuous satisfying:*

(a) *$g$ and $G$ are bounded and continuous;*

(b) *for all $f_1, f_2 \in B(W)$ we have*

$$0 < \eta(f_1, f_2) < 1 \Rightarrow |G(t,s,f_1(t)) - G(t,s,f_2(t))| \le \frac{1}{2}\eta^2(f_1, f_2),$$

$$\eta(f_1, f_2) \ge 1 \Rightarrow |G(t,s,f_1(t)) - G(t,s,f_2(t))| \le \frac{2}{3}\eta^2(f_1, f_2), \tag{3.4}$$

*where $t \in W$ and $s \in V$.*

*Then* (3.1) *has a bounded solution.*

*Proof.* Let $\epsilon > 0$ and $t \in W$. Since $(B(W), \eta)$ is complete for $f_1, f_2 \in B(W)$ and $\epsilon > 0$ there exist $s_1, s_2 \in V$ such that

$$\Im(f_1)(t) \ge g(t,s_1) + G(t,s_1,f_1(h(t,s_1))), \tag{3.5}$$

but

$$\Im(f_1)(t) < g(t,s_1) + G(t,s_1,f_1(h(t,s_1))) + \epsilon \tag{3.6}$$

and

$$\Im(f_2)(t) \geq g(t,s_2) + G(t,s_2,f_2(h(t,s_2))). \tag{3.7}$$

But,

$$\Im(f_2)(t) < g(t,s_2) + G(t,s_2,f_2(h(t,s_2))) + \epsilon. \tag{3.8}$$

Consider the function $\delta : (0,\infty) \to (0,\infty)$ defined by

$$\delta(\varsigma) = \begin{cases} \dfrac{\varsigma^2}{2}, & \text{if } 0 < \varsigma < 1, \\ \dfrac{2}{3}\varsigma, & \text{if } \varsigma \geq 1. \end{cases}$$

Then (3.4) reduces to

$$|G(t,s,f_1(t)) - G(t,s,f_2(t))| \leq \delta(\eta(f_1,f_2)). \tag{3.9}$$

We observe that $\delta(\varsigma) < \varsigma$ for all $\varsigma \in (0,\infty)$. From (3.6), (3.7) and (3.8), we have that

$$\begin{aligned} \Im(f_1)(t) - \Im(f_2)(t) &< |G(t,s_1,f_1(h(t,s_1))) - G(t,s_2,f_2(h(t,s_2)))| + \epsilon \\ &\leq \delta(\eta(f_1,f_2)) + \epsilon. \end{aligned} \tag{3.10}$$

Similarly, we have

$$\Im(f_2)(t) - \Im(f_1)(t) < \delta(\eta(f_1,f_2)) + \epsilon. \tag{3.11}$$

From (3.10) and (3.11)

$$|\Im(f_2)(t) - \Im(f_1)(t)| < \delta(\eta(f_1,f_2)) + \epsilon. \tag{3.12}$$

Thus,

$$\eta(\Im(f_1),\Im(f_2)) < \delta(\eta(f_1,f_2)) + \epsilon. \tag{3.13}$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\eta(\Im(f_1),\Im(f_2)) \leq \delta(\eta(f_1,f_2)) < \eta(f_1,f_2), \tag{3.14}$$

(since $\delta(\varsigma) < \varsigma$, for each $\varsigma \in (0,\infty)$). Now, define $\phi : B(W) \to \mathbb{R}$ such that $\phi(f) = [\|f\|]^2$, where $f \in B(W)$ and $[\cdot]$ denotes the greatest integer function. Now, all such functions $f_i \in B(W)$ which satisfy $\eta(f_i,(f_j)) > 0$, $i \neq j$, we observe that

$$\eta(\Im(f_1),\Im(f_2)) \leq \delta(\eta(f_1,f_2)) < \eta(f_1,f_2) \leq |\phi(f_1) - \phi(f_2)|\eta(f_1,f_2),$$

(since in this case $|\phi(f_1)-\phi(f_2)| \geq 1$). Thus, we observe that Theorem 2.1 is applicable to the operator $\Im$, so $\Im$ has a fixed point $f^* \in B(W)$, which in turn is a bounded solution of the functional equation (3.1). □

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## References


[1] R. P. Agarwal and M. A. Khamsi, *Extension of Caristi's fixed point theorem to vector valued metric space*, Nonlinear Anal. **74** (2011), 141–145. https://doi.org/10.1016/j.na.2010.08.025

[2] M. U. Ali and T. Kamran, *Multi-valued $F$-contractions and related fixed point theorems with an application*, Filomat **30** (2010), 3779–3793. https://doi.org/10.2298/FIL1614779A

[3] M. Boriceanu, A. Petrusel and I. Rus, *Fixed point theorems for some multivalued generalized contraction in $b$-metric spaces*, Int. J. Math. Stat. **6** (2010), 65–76.

[4] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251. https://doi.org/10.1090/S0002-9947-1976-0394329-4

[5] S. Czerwik, *Nonlinear set-valued contraction mappings in $b$-metric spaces*, Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia **46** (1998), 263–276.

[6] P. Debnath and H. M. Srivastava, *New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations*, Symmetry **12**(7) (2020), 9 pages. https://doi.org/10.3390/sym12071090

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0

[8] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. **1** (1979), 443–474. https://doi.org/10.1090/S0273-0979-1979-14595-6

[9] Y. Feng and S. Liu, *On multivalued Caristi's type fixed point theorems*, J. Math. Anal. Appl. **317** (2006), 103–112. https://doi.org/10.1016/j.jmaa.2005.12.004

[10] J. Jachymski, *Caristi's fixed point theorem and selection of set-valued contractions*, J. Math. Anal. Appl. **227** (1998), 55–67. https://doi.org/10.1006/jmaa.1998.6074

[11] E. Karapinar, F. Khojateh and Z. Mitrović, *A proposal for revisiting Banach and Caristi type theorems in $b$-metric spaces*, Mathematics **7**(308) (2019), 4 pages. https://doi.org/10.3390/math7040308

[12] M. S. Khan, *Common fixed point theorems for multivalued mappings*, Pac. J. Math **95** (1981), 337–347. https://doi.org/10.2140/pjm.1981.95.337

[13] F. Khojateh, E. Karapinar and H. Khandani, *Some applications of Caristis fixed point theorem in metric spaces*, Fixed Point Theory Appl. (2016). https://doi.org/10.1186/s13663-016-0501-z

[14] A. Latif, *Generalized Caristi's fixed point theorems*, Fixed Point Theory Appl. **2009**, Article ID 170140, 7 pages. https://doi.org/10.1155/2009/170140

[15] A. Latif and M. A. Kutbi, *On multivalued Caristi's type fixed point theorems*, Acta Univ. Apulensis Math. Inform. **28** (2011), 179–188.

[16] Z. Mitrović, *A note on the result of Suzuki, Miculescu and Mihail*, J. Fixed Point Theory Appl.**21** (2019), Article ID 24. https://doi.org/10.1007/s11784-019-0663-5

[17] S. B. Nadler, *Multi-valued contraction mappings*, Pac. J. Math. **30**(2) (1969), 475–488. https://doi.org/10.2140/pjm.1969.30.475

[18] M. Sgroi and C. Vetro, *Multi-valued $F$-contractions and the solution of certain functional and integral equations*, Filomat **27** (2013), 1259–1268. https://doi.org/10.2298/FIL1307259S

[19] T. Suzuki, *Generalized Caristi's fixed point theorem by Bae and others*, J. Math. Anal. Appl. **302** (2005), 502–508. https://doi.org/10.1016/j.jmaa.2004.08.019

[20] J. D. Weston, *A characterization of metric completeness*, Proc. Amer. Math. Soc. **64** (1977), 186–188. https://doi.org/10.1090/S0002-9939-1977-0458359-2

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# CHARACTERIZATION OF ORDERED SEMIHYPERGROUPS BY COVERED HYPERIDEALS

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Abstract. After introducing the notions of the Green's relation $\mathcal{J}$, hyper $\mathcal{J}$-class and covered hyperideal in an ordered semihypergroup, some important properties of the hyper $\mathcal{J}$-class and covered hyperideals are studied. Then maximal and minimal hyperideals of an ordered semihypergroup are defined and some vital results have been proved. We also define a hyperbase of an ordered semihypergroup and prove the existence of a hyperbase under certain conditions in an ordered semihypergroup. In an ordered semihypergroup, after defining the greatest covered hyperideal and the greatest hyperideal, some results about these hyperideals are proved. Finally, in a regular ordered semihypergroup, we show that, under some conditions, each hyperideal is also a covered hyperideal.

## 1. Introduction and Prelinimaries

In 1934, Marty [16] introduced the concept of a hyperstructure, in particular, the hypergroup theory in the 8th Congress of Scandinavian Mathematicians. The beauty of hyperstructure is that in hyperstructures, composition of two elements is a set. Thus the notion of algebraic hyperstructures is a generalization of classical notion of algebraic structures. The concept of ordered semihypergroup is a generalization of the concept of ordered semigroup and was introduced by Heidari and Davvaz in [11]. Thereafter it was studied by several authors. Davvaz et al. [1, 2, 11, 17] studied some properties of hyperideals, bi-hyperideals and quasi-hyperideals in ordered semihypergroups. In [7, 9], Fabrici introduced the notion of a covered ideal and, in

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[20], Xie generalized the notion of a covered ideal for ordered semigroups. Thereafter, Thawhat Changphas and Pisan Summaprab [4] discussed the structure of an ordered semigroup containing covered ideals. Later on Saber Omidi and Bijan Davvaz [18] discussed the notion of a covered $\gamma$- hyperideal in an ordered $\gamma$-semihypergroup.

A hyperoperation on a set $S(\neq \emptyset)$ is a map $\circ : S \times S \to \mathcal{P}^{\star}(S)$, where $\mathcal{P}^{\star}(S)$ denotes the power set of $S$ except $\{\emptyset\}$. Then $(S, \circ)$ is a hypergroupoid. The image of the pair $(a, b)$ in $S \times S$ is denoted by $a \circ b$.

A hypergroupoid $(S, \circ)$ is called a semihypergroup if for all $x_1, x_2, x_3 \in S$

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3).$$

It means that $\bigcup\limits_{t \in x_1 \circ x_2} t \circ x_3 = \bigcup\limits_{r \in x_2 \circ x_3} x_1 \circ r$.

For any $T_1, T_2 \in \mathcal{P}^{\star}(S)$, we denote

$$T_1 \circ T_2 = \bigcup_{t \in T_1, t' \in T_2} t \circ t'.$$

Instead of $\{x_1\} \circ T_1$ and $T_2 \circ \{x_1\}$ we shall write, in whatever follows, $x_1 \circ T_1$ and $T_2 \circ x_1$, respectively. We shall write $A^n$ for $A \circ A \circ A \circ \cdots \circ A$ ($n$-copies of $A$) in the sequel without further mention.

**Definition 1.1.** Let $\leq$ be an ordered relation on a set $S(\neq \emptyset)$. The triplet $(S, \circ, \leq)$ is called an ordered semihypergroup if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for all $t_1, t_2, t \in S$, $t_1 \leq t_2$ implies $t_1 \circ t \leq t_2 \circ t$ and $t \circ t_1 \leq t \circ t_2$. Here $t_1 \circ t \leq t_2 \circ t$ means that for any $w \in t_1 \circ t$ there exists $w' \in t_2 \circ t$ such that $w \leq w'$.

A subset $H(\neq \emptyset)$ of an ordered semihypergroup $S$ is called a subsemihypergroup of $S$ if $H \circ H \subseteq H$. We note that for every $x, y, z, u, v, w \in S$ such that $x \circ y \leq z \circ w$ and $u \leq v$, we obtain $x \circ y \circ u \leq z \circ w \circ v$.

For $L \subseteq S$, let $(L] = \{t \in S \mid t \leq h \text{ for some } h \in L\}$. Throughout this paper $S$ denotes an ordered semihypergroup until or unless it is mentioned.

**Definition 1.2.** A subset $W(\neq \emptyset)$ of $S$ is called a right (resp. left) hyperideal of $S$ if

(a) $W \circ S \subseteq W$ (resp. $S \circ W \subseteq W$);
(b) $(W] \subseteq W$.

$W$ becomes a hyperideal if it is both a right hyperideal and a left hyperideal of $S$. The set of all hyperideals of $S$ shall be denoted, in whatever follows, by $I^{\star}$.

**Definition 1.3.** A proper hyperideal $W$ of $S$ is called minimal if $W$ does not contain any hyperideal of $S$. Equivalently, if for any $U \in I^{\star}$ such that $U \subseteq W$, we have $U = W$. The proper hyperideal $W$ of $S$ is called maximal if for any $V \in I^{\star}$ such that $W \subset V$, we have $V = S$. Equivalently, if for any $V \in I^{\star}$ such that $W \subseteq V$, we have $V = W$. Finally, $S$ is called simple if $S$ has no proper hyperideals. The ordered semihypergroup $S$ is called regular if for any $a_1 \in S$ there exists $t \in S$ such that $a_1 \in (a_1 \circ t \circ a_1]$. Equivalently, $W \subseteq (W \circ S \circ W]$ for every $W \subseteq S$.

For an ordered semihypergroup $S$, the hyperideal $J(a)$ generated by the element $a$ of $S$ is equal to $(a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]$.

In Section 2 of the paper, after defining the notions of Green's relation $\mathcal{J}$ and covered hyperideal, some important properties of Green's relation $\mathcal{J}$ and covered hyperideals of an ordered semihypergroup are obtained as the main results while Section 3 deals with the structural properties of ordered semihypergroups containing covered hyperideals.

## 2. Basic Properties of Covered Hyperideals

The Green's relation $\mathcal{J}$ on $S$, an ordered semihypergroup, is defined by, for $t, t' \in S$,

$$t \,\mathcal{J}\, t' \text{ if and only if } J(t) = J(t').$$

For any $t \in S$, $T_t$ is $\mathcal{J}$-hyperclass of $t$. Let $D$ be the collection of all $\mathcal{J}$-hyperclasses of $S$. Define an order '$\preccurlyeq$' on $D$ by: for any $t, t' \in S$,

$$T_t \preccurlyeq T_t' \text{ if and only if } J(t) \subseteq J(t').$$

Then it is easy to verify that $(D, \preccurlyeq)$ is a quasi-ordered set.

The following result easily follows.

**Lemma 2.1.** *Let $t$ be any element of $S$ such that $T(t) \nsubseteq K$ for any principal hyperideal $K$ of $S$. Then the $\mathcal{J}$-hyperclass $T_t$ is maximal.*

**Lemma 2.2.** *Let $K$ be any subset of $S$. Then $K$ is a maximal $\mathcal{J}$-hyperclass of $S$ if and only if $S \setminus K$ is a maximal hyperideal of $S$.*

*Proof.* First we consider that $K$ is a maximal $\mathcal{J}$-hyperclass of $S$. Then $K = T_t$ for some $t \in S$. We now show that $S \setminus T_t$ is a hyperideal of $S$. For this let $h \in S$ and $t' \in S \setminus T_t$, then $t' \notin T_t \;\Rightarrow\; J(t) \neq J(t')$. Let $y \in h \circ t'$. Then either $J(y) = J(t)$ or $J(y) \neq J(t)$. If $J(y) \neq J(t)$ then the proof is obvious. If $J(y) = J(t)$, then we have $y \in h \circ t' \subseteq S \circ J(t') \subseteq J(t')$ and $J(y) = J(t) \neq J(t')$. Since $T_{t'}$ and $T_t$ are disjoint $\mathcal{J}$-hyperclasses of $S$, we have $y \notin T_t \;\Rightarrow\; y \in S \setminus T_t$. Thus $S \circ S \setminus T_t \subseteq S \setminus T_t$. Similarly, we may show that $(S \setminus T_t) \circ S \subseteq S \setminus T_t$. Let $u \in S \setminus T_t$ and $v \in S$ be such that $v \leq u$. So we have $v \in (v] \subseteq (u] \subseteq J(u)$ and thus, $J(v) \subseteq J(u) \;\Rightarrow\; T_v \preccurlyeq T_u$. If $v \in T_t$, since $T_t$ is maximal, so $T_v$ is also maximal $\mathcal{J}$-hyperclass of $S$. Thus we have $T_t = T_u$. So $u \in T_t$, a contradiction. Hence, $v \in S \setminus T_t$ and $S \setminus T_t$ is a hyperideal of $S$. Now it remains to show the maximality of $S \setminus T_t$. For this take any hyperideal $L$ of $S$ such that $S \setminus T_x \subset L$. Then there exists $w \in L \setminus (S \setminus T_t)$. Thus $w \in T_t$. Now, for any $y \in T_t$, we have

$$J(y) = J(x) = J(w) \subseteq L,$$

and, so, $T_t \subseteq L$. Hence, $S = L$. This shows that $S \setminus T_t$ is a maximal hyperideal of $S$.

Conversely suppose that $S \setminus K$ is a maximal hyperideal of $S$. Take $z \in S \setminus (S \setminus K)$. So $z \in K$. If $t \in T_z$, then $J(t) = J(z) \subseteq K$. Thus $t \in K$. Hence, $T_z \subseteq K$. Since $S \setminus K \subset (S \setminus K) \cup J(z)$ and $S \setminus K$ is a maximal hyperideal of $S$, we have $(S \setminus K) \cup J(z) = S$. It now follows that for any $t' \in K$, $J(t') = J(z)$. Thus, for $t' \in K$, $t' \in T_z \;\Rightarrow\; K \subseteq T_z$. Hence, $K = T_z$. If $T_z$ is not maximal $\mathcal{J}$-hyperclass of $S$, then there exists $e \in S$ such that $T_z \precneqq T_e$. This implies that $J(z) \subset J(e)$ and, so, by hypothesis, $J(e) \subseteq S \setminus K$. As $e \notin T_z = K \Rightarrow e \in S \setminus K$. Thus, $z \in S \setminus K$. This is a contradiction as $z \in T_z$. Hence, $T_z$ is a maximal $\mathcal{J}$-hyperclass of $S$. □

**Definition 2.1.** Any proper hyperideal $K$ of an ordered semihypergroup $S$ is called a covered hyperideal of $S$ if $K \subseteq (S \circ (S \setminus K) \circ S]$. The set of all covered hyperideals of $S$ shall be denoted, in whatever follows, by $\mathcal{C}_{\mathcal{H}}$.

*Example* 2.1. Let $S = \{u, v, w, x\}$. Define the hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $u$ | $v$ | $w$ | $x$ |
|---|---|---|---|---|
| $u$ | $\{u\}$ | $\{u,v\}$ | $\{u,w\}$ | $\{u\}$ |
| $v$ | $\{u\}$ | $\{u,v\}$ | $\{u,w\}$ | $\{u\}$ |
| $w$ | $\{u\}$ | $\{u,v\}$ | $\{u,w\}$ | $\{u\}$ |
| $x$ | $\{u\}$ | $\{u,v\}$ | $\{u,w\}$ | $\{u\}$ |

Define order on $S$ as $\leq = \{(u,u),(v,v),(w,w),(x,x),(v,u),(w,u)\}$. Then $(S, \circ, \leq)$ is an ordered semihypergroup. Now, it may easily be verified that $B = \{u, v, w\}$ is a covered hyperideal of $S$.

*Example* 2.2. Let $S = \{u, v, w, x, y\}$. Define the hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $u$ | $v$ | $w$ | $x$ | $y$ |
|---|---|---|---|---|---|
| $u$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ |
| $v$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ | $\{u,v\}$ |
| $w$ | $\{u,v\}$ | $\{u,v\}$ | $\{w\}$ | $\{w\}$ | $\{y\}$ |
| $x$ | $\{u,v\}$ | $\{u,v\}$ | $\{w\}$ | $\{x\}$ | $\{y\}$ |
| $y$ | $\{u,v\}$ | $\{u,v\}$ | $\{w\}$ | $\{w\}$ | $\{y\}$ |

Define order on $S$ as $\leq = \{(u,u),(v,v),(w,w),(x,x),(y,y),(u,w),(u,x),(u,y),(v,w),$ $(v,x),(v,y),(w,x),(w,y)\}$. Then $(S, \circ, \leq)$ becomes an ordered semihypergroup. One may easily verify that the sets $A_1 = \{u, v\}$ and $A_2 = \{u, v, w, y\}$ are covered hyperideals of $S$.

**Proposition 2.1.** *Let $A_1, A_2$ be different proper hyperideals of $S$ such that $A_1 \cup A_2 = S$. Then none of them is covered hyperideal of $S$.*

*Proof.* On contrary, assume that $A_1$ is a covered hyperideal of $S$. Since $A_1 \cup A_2 = S$, we have $S \setminus A_1 \subseteq A_2$ and $S \setminus A_2 \subseteq A_1$. Thus we have

$$A_1 \subseteq (S \circ (S \setminus A_1) \circ S] \subseteq (S \circ A_2 \circ S] \subseteq A_2.$$

Therefore, $S = A_2$. This is a contradiction. By the similar argument, we may show that if $A_2$ is a covered hyperideal of $S$, then $S = A_1$. This is again a contradiction. Hence, the result hold. □

The following corollary follows easily from Proposition 2.1.

**Corollary 2.1.** *If an ordered semihypergroup $S$ contains two or more maximal hyperideal, then none of them is a covered hyperideal of $S$.*

**Proposition 2.2.** *Let $A_1, A_2$ be covered hyperideals of $S$. Then $A_1 \cup A_2 \in \mathcal{C}_{\mathcal{H}}$.*

*Proof.* Let $A_1$ and $A_2 \in \mathcal{C}_{\mathcal{H}}$. Then $A_1 \subseteq (S \circ (S \setminus A_1) \circ S]$ and $A_2 \subseteq (S \circ (S \setminus A_2) \circ S]$. Clearly $A_1 \cup A_2$ is a hyperideal of $S$. To show that $A_1 \cup A_2 \in \mathcal{C}_{\mathcal{H}}$, take any $z \in (A_1 \cup A_2)$. If $z \in A_1$, then $z \in (s_1 \circ t \circ s_2]$ for some $s_1, s_2 \in S$ and $t \in S \setminus A_1$. In case $t \in S \setminus (A_1 \cup A_2)$, then $z \in (S \circ (S \setminus (A_1 \cup A_2)) \circ S]$. Again, if $t \in (A_1 \cup A_2)$, then $t \in A_2 \subseteq (s_1' \circ t' \circ s_2']$ for $s_1', s_2' \in S$ and $t' \in S \setminus A_2$. Now $z \in (s_1 \circ t \circ s_2] \subseteq (s_1 \circ (s_1' \circ t' \circ s_2'] \circ s_2] \subseteq (S \circ S \circ t' \circ S \circ S] \subseteq (S \circ t' \circ S]$. If $t' \in A_1$, then $t \in (s_1' \circ t' \circ s_2'] \subseteq (S \circ A_1 \circ S] \subseteq A_1$. This is a contradiction. Hence, $t' \in S \setminus (A_1 \cup A_2)$ and, so, $z \in (S \circ (S \setminus (A_1 \cup A_2)) \circ S]$. In a similar way we may show that if $z \in A_2$, then $z \in (S \circ (S \setminus (A_1 \cup A_2)) \circ S]$. Hence, the result follows. □

**Proposition 2.3.** *Let $A_1$ be any hyperideal of $S$ and $A_2 \in \mathcal{C}_{\mathcal{H}}$. Then $A_1 \cap A_2 \in \mathcal{C}_{\mathcal{H}}$.*

*Proof.* First we prove that $A_1 \cap A_2$ is a non-empty hyperideal of $S$. For this, let $t \in A_1$ and $t' \in A_2$, then we have $t \circ t' \subseteq A_1 \circ A_2 \subseteq A_1 \circ S \subseteq A_1$. Also, $t \circ t' \subseteq A_1 \circ A_2 \subseteq S \circ A_2 \subseteq A_2$. Thus, $t \circ t' \subseteq A_1 \cap A_2 \subseteq S$. Clearly, $(A_1 \cap A_2) \circ S \subseteq A_1 \circ S \subseteq A_1$ and $(A_1 \cap A_2) \circ S \subseteq A_2 \circ S \subseteq A_2$. Thus $(A_1 \cap A_2) \circ S \subseteq A_1 \cap A_2$. In a similar way we may show that $S \circ (A_1 \cap A_2) \subseteq A_1 \cap A_2$. Also, as $(A_1 \cap A_2] \subseteq (A_1] = A_1$ and $(A_1 \cap A_2] \subseteq (A_2] = A_2$, we have $(A_1 \cap A_2] \subseteq A_1 \cap A_2$. Now, as $A_1 \cap A_2 \subseteq A_2 \subseteq (S \circ (S \setminus A_2) \circ S] \subseteq (S \circ (S \setminus (A_1 \cap A_2)) \circ S]$, $A_1 \cap A_2$ is a covered hyperideal of $S$. □

**Corollary 2.2.** *If $A_1$ and $A_2 \in \mathcal{C}_{\mathcal{H}}$, then $A_1 \cap A_2 \in \mathcal{C}_{\mathcal{H}}$.*

Combining Proposition 2.2 and Corollary 2.2, we have the following.

**Theorem 2.1.** *For an ordered semihypergroup $S$, $\mathcal{C}_{\mathcal{H}}$ is a sublattice of the lattice of all hyperideals of $S$.*

## 3. Covered Hyperideals in Ordered Semihypergroups

**Theorem 3.1.** *An ordered semihypergroup $S$ contains a covered hyperideal if it is not simple.*

*Proof.* Proof of this theorem is similar to the proof of Theorem 3.10 of [18]. □

**Theorem 3.2.** *If an ordered semihypergroup $S$ contains covered hyperideals, then every covered hyperideal of $S$ is minimal if and only if any two distinct covered hyperideals of $S$ are disjoint.*

*Proof.* Proof of this theorem is similar to the proof of Theorem 3.9 of [18]. □

**Corollary 3.1.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If $S$ is not simple, then each covered hyperideal of $S$ is minimal if and only if any two distinct covered hyperideals of $S$ are disjoint.*

**Theorem 3.3.** *Let $K$ be any proper hyperideal of a regular ordered semihypergroup $S$. If for every $J(t) \subseteq K$, there exists $t' \in S \setminus K$ such that $J(t) \subseteq J(t')$, then every proper hyperideal of $S$ is a covered hyperideal of $S$.*

*Proof.* Clearly $(S \circ S] \subseteq S$. As $S$ is regular, $S \subseteq (S \circ S \circ S] \subseteq (S \circ S] \subseteq S \Rightarrow S = (S \circ S]$. Now suppose that for any hyperideal $K$ of $S$ and $t \in K$ such that $J(t) \subseteq K$, there exists $t' \in S \setminus K$ such that $J(t) \subseteq J(t')$. As $S = (S \circ S]$, we get $S = (S \circ S \circ S]$. Then $t' \leq s_1 \circ s_2 \circ s_3$ for some $s_1, s_2, s_3 \in S$. If $s_2 \in K$, then $t' \in (S \circ K \circ S] \subseteq (K] = K$. This is a contradiction. Therefore, $t' \in S \setminus K$. Also, $t' \in (S \circ (S \setminus K) \circ S] \Rightarrow J(t') \subseteq (S \circ (S \setminus K) \circ S]$. Now $t \in J(t) \subseteq J(t') \subseteq (S \circ (S \setminus K) \circ S]$. Hence, $K \in \mathcal{C}_{\mathcal{H}}$. □

The following example illustrates Theorem 3.3.

*Example* 3.1. In Example 2.2, one may easily check that $(S, \circ, \leq)$ is a regular ordered semihypergroup. Consider the subset $K = \{u, v, w, y\}$ of $S$. Then $K$ is a hyperideal of $S$ such that $J(u) \subseteq K$. For $x \in S \setminus K$, $J(t) \subseteq J(x)$ for all $t \in S$. Then, by the hypothesis of the Theorem 3.3, $K$ becomes covered hyperideal of $S$.

**Proposition 3.1.** *Let $K$ be any hyperideal of a regular order semihypergroup $S$. Then any covered hyperideal $L$ of $K$ is also a covered hyperideal of $S$.*

*Proof.* As being a hyperideal of $S$, $K$ is also a subsemihypergroup of $S$. Let $h \in K \subseteq S$. Since $S$ is regular, there exists $t' \in S$ such that $h \leq h \circ t' \circ h \leq h \circ t' \circ (h \circ t' \circ h) = h \circ (t' \circ h \circ t') \circ h$. As $K$ is a hyperideal of $S$, we have $t' \circ h \circ t' \subseteq S \circ K \circ S \subseteq K$. Therefore, $h \in (h \circ K \circ h]$. Hence, $K$ is a regular subsemihypergroup of $S$.

Now we show that $L$ is a hyperideal of $S$. For this, take any $u \in L \subseteq K$ and $s \in S$. Then $u \circ s \subseteq K$. For any $v \in u \circ s \subseteq K$, there exists $h \in K$ such that

$$\begin{aligned} v \leq v \circ h \circ v &\subseteq (u \circ s) \circ h \circ (u \circ s) \\ &\subseteq L \circ (S \circ K \circ S) \circ S \\ &\subseteq L \circ K \circ S \\ &\subseteq L \circ K \subseteq L \quad (\text{ as L is a hyperideal of K}). \end{aligned}$$

Therefore, $u \circ s \subseteq L$. By the similar argument we may show that $s \circ u \subseteq L$. Also, if $l \in L \subseteq K$ and $t \in S$ such that $t \leq l \Rightarrow t \in K$. As $L$ is a hyperideal of $K$, it follows that $t \in L$. Hence $L$ is a hyperideal of $S$. Again, by hypothesis, we have $L \subseteq (K \circ (K \setminus L) \circ K] \subseteq (S \circ (K \setminus L) \circ S] \subseteq (S \circ (S \setminus L) \circ S]$ (since $\phi \neq K \setminus L \subseteq S \setminus L$). Hence, $K \in \mathcal{C}_{\mathcal{H}}$. □

The following example shows that the condition of the Proposition 3.1 on $S$ to be regular ordered semihypergroup is a sufficient condition.

*Example* 3.2. Let $S = \{v, w, x, t\}$. Define a hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $v$ | $w$ | $x$ | $t$ |
|---|---|---|---|---|
| $v$ | $\{v\}$ | $\{v,w\}$ | $\{v,x\}$ | $\{v\}$ |
| $w$ | $\{v\}$ | $\{v,w\}$ | $\{v,x\}$ | $\{v\}$ |
| $x$ | $\{v\}$ | $\{v,w\}$ | $\{v,x\}$ | $\{v\}$ |
| $t$ | $\{v\}$ | $\{v,w\}$ | $\{v,x\}$ | $\{v\}$ |

Define an order on $S$ as $\leq = \{(v,v),(w,w),(x,x),(t,t),(w,v),(x,v)\}$. Then $(S,\circ,\leq)$ is an ordered semihypergroup but not a regular ordered semihypergroup. For the subsets $K = \{v,w,x\}$, $L_1 = \{w\}$ and $L_2 = \{x\}$ of $S$, one may easily verify that $K$ is a hyperideal of $S$ and each $L_i$ $(i = 1,2)$ is a covered hyperideal of both $K$ and $S$.

**Definition 3.1.** A non-empty subset $H_B$ of $S$ is called a two-sided hyperbase of $S$ if
(a) $S = (H_B \cup H_B \circ S \cup S \circ H_B \cup S \circ H_B \circ S]$;
(b) If $D \subseteq H_B$ such that $S = (D \cup D \circ S \cup S \circ D \cup S \circ D \circ S]$, then $D = H_B$.

Maximal $\mathcal{J}$-hyperclasses of $S$ may be realized as the compliments of maximal hyperideals of $S$. The complement of a maximal hyperideal $H_t$ of $S$, in the sequel, will be denoted by $H^t$.

In the followings, to provide examles of hyperbases of ordered semihypergroups, examples of ordered semihypergroups are taken from [19] and [2], respectively.

*Example* 3.3. Let $S = \{u,v,w,x,y,z\}$. Define a hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|
| $u$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ |
| $v$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ |
| $w$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ |
| $x$ | $\{x,y\}$ | $\{y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ |
| $y$ | $\{x,y\}$ | $\{y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ | $\{x,y\}$ |
| $z$ | $\{u\}$ | $\{v\}$ | $\{w\}$ | $\{x\}$ | $\{y\}$ | $\{z\}$ |

Define an order on $S$ as $\leq = \{(u,u),(v,v),(w,w),(x,x),(x,u),(x,w),(x,z),(y,y),$ $(y,u),(y,v),(y,w),(y,x),(y,z),(z,z)\}$. Then $(S,\circ,\leq)$ is an ordered semihypergroup. Consider the subset $H_B = \{z\}$ of $S$. Then, clearly, $S \circ H_B = S$ and, hence, $S = (H_B \cup H_B \circ S \cup S \circ H_B \cup S \circ H_B \circ S]$. So $H_B$ is a hyperbase of $S$.

*Example* 3.4. Let $S = \{u,v,w,x,t\}$. Define a hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $u$ | $v$ | $w$ | $x$ | $t$ |
|---|---|---|---|---|---|
| $u$ | $\{u\}$ | $\{u\}$ | $\{u\}$ | $\{u\}$ | $\{u\}$ |
| $v$ | $\{u\}$ | $\{u,v\}$ | $\{u\}$ | $\{u,x\}$ | $\{u\}$ |
| $w$ | $\{u\}$ | $\{u,t\}$ | $\{u,w\}$ | $\{u,w\}$ | $\{u,t\}$ |
| $x$ | $\{u\}$ | $\{u,v\}$ | $\{u,x\}$ | $\{u,x\}$ | $\{u,v\}$ |
| $t$ | $\{u\}$ | $\{u,t\}$ | $\{u\}$ | $\{u,w\}$ | $\{u\}$ |

Define an order on $S$ as $\leq = \{(u,u),(v,v),(w,w),(x,x),(t,t),(u,v),(u,w),(u,x),$ $(u,t)\}$. $(S,\circ,\leq)$ is an ordered semihypergroup may easily be checked. Consider the subsets $H_B = \{v\}$ and $H'_B = \{x\}$ of $S$. It is easy to verify that both $H_B$ and $H'_B$ are hyperbases of $S$.

A covered hyperideal $A$ of an ordered semihypergroup $S$ is called the greatest covered hyperideal of $S$ if it contains every covered hyperideal of $S$. The greatest covered hyperideal $A$ of $S$ will be denoted by $A^g$ in the sequel.

**Theorem 3.4.** *If $S$ is not hypersimple and contains a two-sided hyperbase $H_B$ of $S$, then $S$ has the greatest covered hyperideal $A^g$. Moreover, $A^g = (S^3] \cap \hat{H}$, where $\hat{H} = \bigcap_{t\in\alpha} H_t$, where $\{H_t\}_{t\in\alpha}$ is the family of all maximal hyperideals of $S$.*

*Proof.* Containment of hyperbase $H_B$ implies the existence of maximal hyperideals in $S$ and $H_t = S \setminus H^t$, where $H^t$ is a maximal $\mathcal{J}$-hyperclass. Since $\phi \neq \hat{H} = \bigcap_{t\in\alpha} H_t = \bigcap_{t\in\alpha} S \setminus H^t = S \setminus \bigcup_{t\in\alpha} H^t$. It is easy to verify that $\hat{H}$ and $(S^3]$ are hyperideals of $S$. Let $K = (S^3] \cap \hat{H}$. We show that $K \in \mathcal{C}_{\mathcal{H}}$. For this, let $h \in K$ be any element. Then $h \in (S^3] \Rightarrow h \in (S \circ t' \circ S]$ for some $t' \in S$. If $t' \in H_B$, then $\exists\ c \in H_B$ such that $t' \in J(c)$ and, hence, $t' \in (S \circ c \cup c \circ S \cup S \circ c \circ S]$ i.e. $t'$ is at least in one of the subsets: $(S \circ c]$, $(c \circ S]$, $(S \circ c \circ S]$. Then, for all these subsets, we have $(S \circ t' \circ S] \subseteq (S \circ c \circ S]$ and, hence, $h \in (S \circ c \circ S]$ for $c \in H_B$. Thus, for any $h \in K$, $\exists\ c \in H_B$ such that $h \in (S \circ c \circ S] \subseteq (S \circ H_B \circ S] \subseteq (S \circ (S \setminus \hat{H}) \circ S] \subseteq (S \circ (S \setminus K) \circ S]$. Therefore $K \subseteq (S \circ (S \setminus K) \circ S]$. It now remains to show that $K$ is the greatest covered hyperideal of $S$. To show this, let $L$ be any covered hyperideal of $S$. Then $L \subseteq (S \circ (S \setminus L) \circ S] \subseteq (S^3]$. Since $L \in \mathcal{C}_{\mathcal{H}}$, $L$ can not contain any maximal $\mathcal{J}$-hyperclass. So $L \subseteq S \setminus H^t$ for every $t \in \alpha$. Therefore, $L \subseteq \bigcap_{t\in\alpha} S \setminus H^t = \bigcap_{t\in\alpha} H_t = \hat{H}$. Hence, $L \subseteq (S^3] \cap \hat{H} = K$. Therefore, any covered hyperideal is contained in $K$, i.e., $K = A^g$. ☐

**Lemma 3.1.** *Let $S$ be any ordered semihypergroup having the greatest covered hyperideal $A^g$. If $A^g \subseteq (S \circ S \circ S]$, then*

(a) *every $\mathcal{J}$-hyperclass in $(S^3] \setminus A^g$ is maximal;*
(b) *$J(t) = (S \circ t \circ S]$ for all $t \in (S^3] \setminus A^g$.*

*Proof.* First we assume that $A^g \subset (S^3]$. Then, we have $(S^3] \setminus A^g \neq \phi$. To show the second part let $t \in (S^3] \setminus A^g$. Since $A^g$ is a hyperideal of $S$, the $\mathcal{J}$-hyperclass $T_t \subseteq (S^3] \setminus A^g$. Thus $t \in (S \circ t' \circ S]$ for some $t' \in S$ and $(S \circ t \circ S] \subseteq (S \circ t' \circ S]$. Since $(S \circ t' \circ S] \subseteq J(t')$, we have $J(t) \subseteq J(t')$. Now suppose to the contrary that $t' \notin T_t$. So $T_t \neq T'_t$. We claim that $t' \in S \setminus J(t)$. For this, if $t' \in J(t)$, then $J(t) = J(t') \Rightarrow T_t = T'_t$, which is impossible. Thus we have $J(t) \subseteq (S \circ (S \setminus J(t)) \circ S]$ and, so, $J(t) \in \mathcal{C}_{\mathcal{H}}$. By Proposition 2.2, $A^g \cup J(t) \in \mathcal{C}_{\mathcal{H}}$. As $t \notin A^g$, we, thus, have $A^g \subset A^g \cup J(t)$. This is a contradiction. Hence, $t' \in T_t$ and $J(t) \subseteq (S \circ t' \circ S] \subseteq J(t') = J(t)$.

Thus, $J(t) = (S \circ t' \circ S] = J(t')$. So, obviously $(S \circ t \circ S] \subseteq J(t)$. Now there are two possibilities: if $t' \leq t$, then $J(t) = (S \circ t' \circ S] \subseteq (S \circ t \circ S] \Rightarrow J(t) \subseteq (S \circ t \circ S]$. If

$t' \leq t$ is not true, then $t' \in (S \circ t \cup t \circ S \cup S \circ t \circ S]$. Now, if $t' \in (S \circ t]$, then we have

$$S \circ t' \circ S \subseteq S \circ (S \circ t] \circ S \subseteq (S \circ (S \circ t] \circ S] \subseteq (S \circ S \circ t \circ S] \subseteq (S \circ t \circ S].$$

Similarly, for $t' \in (t \circ S] \cup (S \circ t \circ S]$, we may show that $(S \circ t' \circ S] \subseteq (S \circ t \circ S]$. Therefore, $J(t) = J(t') = (S \circ t' \circ S] \subseteq (S \circ t \circ S]$.

To prove the reverse part, let $T_t$ be a $\mathcal{J}$-hyperclass in $(S^3] \setminus A^g$. On contrary assume that $T_t$ is not maximal. Then, by Lemma 2.1, $J(t) \subset J(t')$ for some $t' \in S$. So $t \in J(t')$. This implies that $t \in (t'] \cup (S \circ t'] \cup (t' \circ S] \cup (S \circ t' \circ S]$. For such $t$, we may easily prove that $(S \circ t \circ S] \subseteq (S \circ t' \circ S] \Rightarrow J(t) \subseteq (S \circ t' \circ S]$. Now, as $t' \in S \setminus J(t)$, $J(t)$ is a covered hyperideal of $S$. Hence $A^g \subset A^g \cup J(t)$, a contradiction. Therefore, every $\mathcal{J}$-hyperclass in $(S^3] \setminus A^g$ is maximal. $\square$

**Theorem 3.5.** *Let $S$ be any ordered semihypergroup having the greatest covered hyperideal $A^g$. If*

(a) $A^g \subset (S \circ S \circ S]$;

(b) *neither $T_t \preccurlyeq T_{t'}$ nor $T_{t'} \preccurlyeq T_t$ for any $t, t' \in S \setminus (S^2]$,*

*then $S$ contains a hyperbase.*

*Proof.* Suppose that $A^g \subset (S^3]$ and $t, t' \in S \setminus (S^2]$ such that they are incomparable. Since $A^g$ is a covered hyperideal of $S$, we have

$$A^g \subseteq (S \circ (S \setminus A^g) \circ S] \subseteq (S^3] \subseteq (S^2] \subseteq S.$$

Let $C_1 = \{T_t \mid t \in S \setminus (S^2]\}$, $C_2 = \{T_t \mid t \in (S^2] \setminus (S^3]\}$and $C_3 = \{T_t \mid t \in (S^3] \setminus A^g\}$. Let $K$ be the set containing all the elements from the members of $C_1$ and $C_3$. Then, it is easy to verify that $K$ is a hyperbase of $S$. To show that $S = J(K) = (K \cup S \circ K \cup K \circ S \cup S \circ K \circ S]$, we only need to show that $A^g, (S^3] \setminus A^g, (S^2] \setminus (S^3],$ and $S \setminus (S^2]$ are subsets of $J(K)$.

(i) Let $z \in A^g$. Then $z \in (S \circ (S \setminus A^g) \circ S] \Rightarrow z \in (S \circ y \circ S]$ for some $y \in S \setminus A^g$. Clearly $y \in T_t$ for some $t \in (S \setminus (S^2]) \cup ((S^2] \setminus (S^3]) \cup ((S^3] \setminus A^g)$. Now, by the construction of $K$, if $t \in (S \setminus (S^2]) \cup ((S^3] \setminus A^g)$, then we have $y \in J(K)$. Hence $z \in J(K)$. If $t \in (S^2] \setminus (S^3]$, then $t \leq u_1 \circ u_2$ for some $u_1, u_2 \in S$ Since $t \notin (S^3]$, we have $u_1, u_2 \in S \setminus (S^2]$. It implies that $t \in J(K)$ and, so, $y \in J(K)$. Thus, we have $z \in J(K)$.

(ii) If $z \in (S^3] \setminus A^g$. Then there exists $x_1 \in K$ such that $z \in J(x_1)$. Therefore $z \in J(K)$.

(iii) If $z \in (S^2] \setminus (S^3]$, then one may prove in a similar way as in the Case (i).

(iv) If $z \in S \setminus (S^2]$, then there exists $x_2 \in K$ such that $z \in J(x_2) \subseteq J(K)$.

Now, we show the minimality of $K$ satisfying $S = J(K)$. By Lemma 3.1, every $T_t \in C_3$ is maximal. Also every $T_t \in C_1$ is maximal since for any elements $t, t' \in S \setminus (S^2]$, neither $T_t \preccurlyeq T_{t'}$ nor $T_{t'} \preccurlyeq T_t$. Let $L \subset K$ such that $S = (L \cup S \circ L \cup L \circ S \cup S \circ L \circ S]$ and let $z \in K \setminus L$. Then $z \leq z'$ for some $z' \in (L \cup S \circ L \cup L \circ S \cup S \circ L \circ S] \Rightarrow z' \in J(l)$ for some $l \in L$. Thus, $J(z) \subset J(l)$, a contradiction to the construction of $K$. Hence, the proof is completed. $\square$

A hyperideal $A$ of $S$ is called the greatest hyperideal of $S$ if every proper hyperideal of $S$ is contained in $A$. The greatest hyperideal $A$, if exists, will be denoted by $A^\star$ in the sequel.

**Theorem 3.6.** *The greatest hyperideal $A^\star$ of $S$ is a covered hyperideal of $S$ if and only if $(S^2] = (S^3]$.*

*Proof.* First we assume that $A^\star \in \mathcal{C}_{\mathcal{H}}$. So $A^\star \subseteq (S\circ(S\setminus A^\star)\circ S]$. Since $A^\star$ is a maximal hyperideal of $S$, it follows that $S \setminus A^\star = T_a$ is the unique maximal $\mathcal{I}$-hyperclass of $S$. Then either $(S^2] \subset S$ or $(S^2] = S$. If $(S^2] = S$, then the proof is obvious. If $(S^2] \subset S$, then either $(S^3] = (S^2]$ or $(S^3] \subset (S^2]$.

If $(S^3] \subset (S^2]$, then $A^\star \subseteq (S \circ (S \setminus A^\star) \circ S] \subseteq (S^3] \subset (S^2]$. Hence $S \setminus A^\star$ would contain at least two different $\mathcal{I}$-hyperclasses, each from $(S^2] \setminus (S^3]$ and $S \setminus (S^2]$. This is a contradiction to the fact that $S \setminus A^\star$ contains only one maximal $\mathcal{J}$-class. Thus $(S^2] = (S^3]$.

Conversely, suppose that $S$ contains $A^\star$ and $(S^2] = (S^3]$. Then show that $A^\star$ is a covered hyperideal of $S$. For this, take any $z \in A^\star$. Then, for any element $c \in T_a = S \setminus A^\star$, we have $J(c) = S$. Thus $z \in J(c)$. However, $z \in A^\star$ and $c \in T_a = S \setminus A^\star$, hence $z \neq c$. Therefore, $z \in (c \circ S \cup S \circ c \cup S \circ c \circ S]$. If $z \in (c \circ S]$ or $z \in (S \circ c]$, then, clearly $z \in (S^2]$. If $z \in (S \circ c \circ S]$, then $z \in (S^3]$. But, by hypothesis, $(S^2] = (S^3]$. Therefore, $z \in (S^3]$, i.e., $z \in (S \circ d \circ S]$ for some $d \in S = J(c)$. If $d = c$, then, clearly $d \in (S \circ c \circ S]$. If $d \neq c$, then $d \in (c \circ S \cup S \circ c \cup S \circ c \circ S]$. Again, if $d \in (c \circ S]$, then, clearly $(S \circ d \circ S] \subseteq (S \circ c \circ S \circ S] \subseteq (S \circ c \circ S]$. The same relation may be shown if $d \in ((S \circ c]) \cup ((S \circ c \circ S])$. Thus, $z \in (S \circ d \circ S] \subseteq (S \circ c \circ S]$ and $c \in T_a = S \setminus A^\star$. This shows that, for any $z \in A^\star$, we have $z \in (S \circ c \circ S]$ and $c \in T_a = S \setminus A^\star$. Hence, $A^\star \subseteq (S \circ (S \setminus A^\star) \circ S]$ i.e. $A^\star \in \mathcal{C}_{\mathcal{H}}$. ☐

**Theorem 3.7.** *Suppose $S$ has only one maximal hyperideal $K$. If $K \in \mathcal{C}_{\mathcal{H}}$, then $K = A^\star$.*

*Proof.* Let $L$ be any proper hyperideal of $S$. Then it is easy to verify that $L \subseteq K$, otherwise we shall get a contradiction to the Proposition 2.1. Hence, $K = A^\star$. ☐

The following example illustrates that the converse of the Theorem 3.7 is not be true in general.

*Example* 3.5. Let $S = \{u, v, w, x\}$. Define a hyper operation $(\circ)$ on $S$ by the following table:

| $\circ$ | $u$ | $v$ | $w$ | $x$ |
|---|---|---|---|---|
| $u$ | $\{u\}$ | $\{v\}$ | $\{u\}$ | $\{v\}$ |
| $v$ | $\{v\}$ | $\{u\}$ | $\{v\}$ | $\{u\}$ |
| $w$ | $\{u\}$ | $\{v\}$ | $\{u\}$ | $\{v\}$ |
| $x$ | $\{v\}$ | $\{u\}$ | $\{v\}$ | $\{u\}$ |

Define an order on $S$ as $\leq = \{(u,u), (v,v), (w,w), (x,x), (u,w)\}$. The proof that $(S, \circ, \leq)$ is an ordered semihypergroup is an easy exercise. Consider the subset $K =$

$\{u, v, w\}$ of $S$. Then it is easy to verify that $K$ is the only maximal hyperideal of $S$. As any proper hyperideal of $S$ is contained in $K$, Thus, $K = A^\star$. Now $S \setminus A^\star = \{x\}$, $S \circ x \circ S = \{u, v\}$. So, $A^\star \nsubseteq (S \circ (S \setminus A^\star) \circ S]$. Hence, $A^\star$ is not a covered hyperideal of $S$.

**Theorem 3.8.** *If every proper hyperideal of $S$ is a covered hyperideal of $S$, then either of the followings is true:*

(1) *$S$ contains $A^\star$;*
(2) *$S = (S \circ S]$ and for any proper hyperideal $K$ and for every hyperideal $J(t)$ of $S$ such that $J(t) \subseteq K$, there exists $y \in S \setminus K$ such that $J(t) \subset J(y) \subset S$.*

*Proof.* Take any $a, b \in S$. If $T_a$ and $T_b$ are maximal $\mathcal{J}$-hyperclasses of $S$ such that $T_a \neq T_b$, then, by Lemma 2.2, $A_a = S \setminus T_a$ and $A_b = S \setminus T_b$ are maximal proper hyperideals of $S$. So, by Corollary 2.1, none of them is a covered hyperideal of $S$. This is a contradiction. Thus $S$ has no different maximal $\mathcal{J}$-hyperclasses. Hence either $S$ contains one maximal $\mathcal{J}$-hyperclass or $S$ does not contain any maximal $\mathcal{J}$-hyperclass. Let the only maximal $\mathcal{J}$-hyperclass $T_a$ be contained in $S$. Then $A_a = S \setminus T_a$ is a maximal hyperideal of $S$. By hypothesis, $A_a \in \mathcal{C}_{\mathcal{H}}$. Thus, by Theorem 3.7, $A_a = A^\star$.

For the second possibility, suppose that $S$ does not contain any maximal $\mathcal{J}$-hyperclass. We need to show that $S = (S \circ S]$. For this, suppose that $(S \circ S] \subset S$. Then $\exists$ $c \in S \setminus (S \circ S]$. We claim that the principal hyperideal $J(c) \subsetneq S$. If $J(c) = S$, then $S$ has a maximal $\mathcal{J}$-hyperclass which is impossible. Hence $J(c) \subset S$. By hypothesis, $J(c) \in \mathcal{C}_{\mathcal{H}}$, i.e., $J(c) \subseteq (S \circ (S \setminus J(c)) \circ S]$. Then $c \in (S \circ S \circ S] \subseteq (S \circ S]$. This is a contradiction.

Now let $K$ be any proper hyperideal of $S$ and let the principal hyperideal $J(t) \subseteq K$. By hypothesis, $K \subseteq (S \circ (S \setminus K) \circ S]$. So $\exists$ $y \in S \setminus K$ such that $t \in (S \circ y \circ S] \Rightarrow J(t) \subseteq J(y) \subseteq S$. As $y \in S \setminus K$, $J(t) \subset J(y)$. Since $S$ contains no maximal $\mathcal{J}$-hyperclass, we have $J(y) \subset S$, as required. ☐

**Theorem 3.9.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If*

(1) *$S$ contains the greatest hyperideal $A^\star$ such that $A^\star \in \mathcal{C}_{\mathcal{H}}$ or*
(2) *$S = (S^2]$ and for any proper hyperideal $K$ and for every hyperideal $J(t)$ of $S$ such that $J(t) \subseteq K$, there exists $y \in S \setminus K$ such that $J(t) \subseteq J(y)$,*

*then every proper hyperideal of $S$ is a covered hyperideal of $S$.*

*Proof.* Let $K$ be any proper hyperideal of $S$. First, suppose that the condition (1) holds. Then $K \subseteq A^\star$ and $S \setminus A^\star \subseteq S \setminus K$. Since $A^\star \in \mathcal{C}_{\mathcal{H}}$, we have

$$K \subseteq A^\star \subseteq (S \circ (S \setminus A^\star) \circ S] \subseteq (S \circ (S \setminus K) \circ S].$$

Therefore, $K \in \mathcal{C}_{\mathcal{H}}$.

Secondly we assume that $S$ satisfies the condition (2). Let $h \in K \Rightarrow J(h) \subseteq K$. By the condition (2), we have $J(h) \subset J(y)$ for some $y \in S \setminus K$. Since $S = (S^2] \Rightarrow S = (S^3]$. Thus, $y \in (S \circ b \circ S]$ for some $b \in S$. As $y \in S \setminus K$, we thus have $b \in S \setminus K$.

Therefore, $h \in (S \circ b \circ S] \subseteq (S \circ (S \setminus K) \circ S]$ and, so, $K \subseteq (S \circ (S \setminus K) \circ S]$. Hence, $K \in \mathcal{C}_{\mathcal{H}}$. $\square$

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## 4. Open Problems

(1) Is it true that the greatest hyperideal $A^*$ of an ordered semihypergroup $S$ is a covered hyperideal of $S$ if and only if $S = (S \circ S]$?

(2) Suppose an ordered semihypergroup $(S, \circ, \leq)$ contains only one maximal hyperideal $K$. Does $K \in \mathcal{C}_{\mathcal{H}}$ if $K = A^*$, the greatest hyperideal of $S$?

## 5. Conclusion

In ordered semigroups and ordered semihypergroups, ideals and hyperideals, play an important role to discuss the nature of the structure of ordered semigroups and ordered semihypergroups. Nowadays the hyperideal theory has been extensively studied by several authors. In ordered semihypergroups different types of hyperideals such as bi-hyperideals, quasi-hyperideals have been studied. These notions had been widely studied by several authors in different algebraic structures (see [1, 2, 12, 19]). In this paper, we have enhanced the understanding of ordered semihypergroups by introducing the concept of a covered hyperideal in an ordered semihypergroup. We have also introduced the notion of a hyperbase in an ordered semihypergroup and proved some vital results.

## References

[1] T. Changphas and B. Davvaz, *Properties of hyperideals in ordered semihypergroups*, Italian Journal of Pure and Applied Mathematics **33** (2014), 425–432.

[2] T. Changphas and B. Davvaz, *Bi-hyperdeals and quasi-hyperideals in ordered semihypergroups*, Italian Journal of Pure and Applied Mathematics **35** (2015), 493–508.

[3] T. Changphas and P Summaprab, *On two-sided bases of an ordered semigroup*, Quasigroups Related Systems **22** (2014), 59–66.

[4] T. Changphas and P. Summaprab, *Ordered semigroups containing covered ideals*, Comm. Algebra **44** (2016), 4104–4113. `https://doi.org/10.1080/00927872.2015.1087015`

[5] J. Chvalina, *Commutative hypergroups in the sense of Marty and ordered sets*, in: *Proceedings of the Summer School on General Algebra and Ordered Sets*, Olomouc, 1994, 19–30.

[6] B. Davvaz, P. Corsini and T. Changphas, *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder*, European J. Combin. **44**(PB) (2015), 208–217. `https://doi.org/10.1016/j.ejc.2014.08.006`

[7] I. Fabrici, *Semigroups containing covered one-sided ideals*, Math. Slovaca **31**(3) (1981), 225–231.

[8] I. Fabrici and T. Macko, *On bases and maximal ideals in semigroups*, Math. Slovaca **31**(2) (1981), 115–120.

[9] I. Fabrici, *Semigroups containing covered two-sided ideals*, Math. Slovaca **34**(4) (1984), 355–363.

[10] Z. Gu and X. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra **450** (2016), 384–397. https://doi.org/10.1016/j.jalgebra.2015.11.026

[11] D. Heidari and B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **73**(2) (2011), 85–96.

[12] K. Hila, B. Davvaz and K. Naka, *On quasi-hyperideals in semihypergroups*, Comm. Algebra **39** (2011), 4183–4194. https://doi.org/10.1080/00927872.2010.521932

[13] N. Kehayopulu, *Note on Green's relations in ordered semigroups*, Mathematica Japonica **36** (1991), 211–214.

[14] N. Kehayopulu, *On regular, intra-regular ordered semigroups*, Pure Math. Appl. (PU.M.A.) **4** (1993), 447–461.

[15] N. Kehayopulu, *Ideals and Green's relations in ordered semigroups*, Int. J. Math. Math. Sci. (2006), 1–8. https://doi.org/10.1155/IJMMS/2006/61286

[16] F. Marty, *Sur une generalization de la notion de group*, 8th Congres Math. Scandinaves, Stockholm, 1934, 45–49.

[17] S. Omidi and B. Davvaz, *A short note on the relation* $\mathcal{N}$ *in ordered semihypergroups*, Gazi University Journal of Science **29**(3) (2016), 659–662.

[18] S. Omidi and B. Davvaz, *$C$-$\gamma$-hyperideal theory in ordered $\gamma$-semihypergroups*, J. Math. Fundam. Sci. **49**(2) (2017), 181–192.

[19] J. Tang and X. Xie, *An investigation on left hyperideals of ordered semihypergroups*, J. Math. Res. App. **37**(4) (2017), 419–434.

[20] X. Xie, *On po-semigroups containing no maximal ideals*, Southeast Asian Bull. Math. **20**(4) (1996), 31–36.

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# THE FAMILY OF SZÁSZ-DURRMEYER TYPE OPERATORS INVOLVING CHARLIER POLYNOMIALS

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Abstract. In this paper, we consider Szász-Durrmeyer type operators based on Charlier polynomials associated with Srivastava-Gupta operators [17]. For the considered operators, we discuss error of estimation by using first and second order modulus of continuity, Lipchtiz-type space, Ditzian-Totik modulus of smoothness, Voronovskaya type asymptotic formula and weighted modulus of continuity.

## 1. Introduction

For the Charlier polynomials [8], the generating functions are as follows:

$$e^u\left(1-\frac{u}{a}\right)^t=\sum_{j=0}^{\infty}C_j^{[a]}(t)\frac{u^j}{j!}, \tag{1.1}$$

where $C_j^{[a]}(t)=\sum\limits_{r=0}^{j}\binom{j}{r}(-t)_r\frac{1}{a^r}$ and $(j)_0=1$, $(j)_i=j(j+1)(j+2)\cdots(j+i-1)$ for $i\geq 1$.

Suppose $\gamma>0$, the space $C_\gamma[0,\infty):=\left\{g\in C[0,\infty):|g(t)|\leq Me^{\gamma t}\right\}$ for some $M>0$.

In view of Charlier polynomials, Varma and Tasdelen [19] proposed a sequence of linear positive operators for $g\in C_\gamma[0,\infty)$ as follows:

$$L_n(g;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}g\left(\frac{j}{n}\right), \tag{1.2}$$

*Key words and phrases.* Charlier polynomials, Srivastava-Gupta operators, modulus of continuity, Ditzian-Totik modulus of smoothness, weighted modulus of continuity.

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where $a>1$ and $x\in[0,\infty)$. For sufficiently large $a$, if we replace $x$ by $x-\frac{1}{n}$ the above operators reduce to well-known Szász-Mirakyan operators [18].

In [17], Srivastava and Gupta introduced a new family of linear positive operators as follows:

$$G_n^c(g;x)=(n-c)\sum_{j=0}^{\infty}p_{n,j}(x;c)\int_0^{\infty}p_{n+c,j-1}(u;c)du+p_{n,0}(x;c)g(0), \tag{1.3}$$

where $p_{n,j}(x;c)=\frac{(-x)^j}{j!}\phi_{n,c}^{(j)}(x)$ and

$$\phi_{n,c}(x)=\begin{cases} e^{-nx}, & c=0,\\ (1+cx)^{\frac{-n}{c}}, & c=1,2,3,\dots\end{cases}$$

For the operators (1.3), they also studied the rate of convergence for the functions of bounded variation. Ispir and Yüksel [10] defined the Bézier varient of the Srivastava-Gupta operators and discussed rate of convergence for the functions of bounded variation. Srivastava-Gupta [17] contains several well-known operators for different values of $c$. Many authors have proposed various forms and modifications of the above operators and studied several local and global approximation results. For more (see [1,3,7,12,14,16,20,21]).

Motivated from the above stated work, we define a linear positive operators for $g\in C_B[0,\infty)$ as follows:

$$\begin{aligned}G_{n,c}^{[a]}(g;x)=&(n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx}\Bigg[\sum_{j=1}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\int_0^{\infty}p_{n+c,j-1}(u;c)g(u)du\\ &+C_0^{[a]}g(0)\Bigg].\end{aligned}\tag{1.4}$$

In above operators, it can easily be seen that if we take $c=1$, we obtain Szász-Durrmeyer type operators involving Charlier Polynomials which were proposed by Kajla and Agrawal [11] and studied several approximation results like Vorovskaya type asymptotic theorem, local approximation, statistical rate of convergence and functions of bounded variation. For more articles based on Charlier polynomials (see [2,4]).

The main purpose of this article is to define the operators (1.4) and discuss the approximation results using the first and second order modulus of continuity, Lipschitz-type space, Ditzian-Totik modulus of smoothness, Voronovskaya-type formula and weighted approximation.

## 2. Auxiliary Results

**Lemma 2.1** ([19]). *For the operators $L_n(\cdot;x,a)$, we have*

(i) $L_n(1;x,a)=1$;
(ii) $L_n(u;x,a)=x+\frac{1}{n}$;

(iii) $L_n(u^2;x,a) = x^2 + \frac{x}{n}\left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2}$;

(iv) $L_n(u^3;x,a) = x^3 + \frac{x^2}{n}\left(6 + \frac{3}{a-1}\right) + \frac{x}{n^2}\left(5 + \frac{3}{a-1} + \frac{1}{(a-1)^2}\right) + \frac{5}{n^3}$;

(v)

$$\begin{aligned}L_n(u^4;x,a) =& x^4 + \frac{x^3}{n}\left(10 + \frac{6}{a-1}\right) + \frac{x^2}{n^2}\left(31 + \frac{30}{a-1} + \frac{11}{(a-1)^2}\right)\\ &+ \frac{x}{n^3}\left(37 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}\right) + \frac{15}{n^4}.\end{aligned}$$

**Lemma 2.2.** *The moments of the operators $G_{n,c}^{[a]}(u^i;x)$, $i = 0,1,2,3,4$, are as follows:*

(i) $G_{n,c}^{[a]}(1;x) = 1$;

(ii) $G_{n,c}^{[a]}(u;x) = \frac{1}{(n-2c)}(nx+2)$;

(iii) $G_{n,c}^{[a]}(u^2;x) = \frac{1}{(n-2c)(n-3c)}\left(n^2x^2 + n\left(6 + \frac{1}{a-1}\right)x + 7\right)$;

(iv)

$$\begin{aligned}G_{n,c}^{[a]}(u^3;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)}\left(n^3\, x^3 + 3n^2\left(4 + \frac{1}{a-1}\right)x^2\right.\\ &\left.+n\left(28 + \frac{12}{a-1} + \frac{2}{(a-1)^2}\right)x + 34\right);\end{aligned}$$

(v)

$$\begin{aligned}G_{n,c}^{[a]}(u^4;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)(n-5c)}\left(n^4x^4 + 2n^3\left(10 + \frac{3}{a-1}\right)x^3\right.\\ &+ n^2\left(126 + \frac{60}{a-1} + \frac{11}{(a-1)^2}\right)x^2\\ &\left.+ n\left(292 + \frac{126}{a-1} + \frac{40}{(a-1)^2} + \frac{6}{(a-1)^3}\right)x + 209\right).\end{aligned}$$

**Lemma 2.3.** *The central moments for the defined operators:*

(i) $G_{n,c}^{[a]}(u-x;x) = \frac{2}{(n-2c)}(1+cx)$;

(ii) $G_{n,c}^{[a]}((u-x)^2;x) = \frac{1}{(n-2c)(n-3c)}\left(c(n+6c)x^2 + \left(n\left(2 + \frac{1}{a-1}\right) + 12c\right)x + 7\right)$;

(iii)

$$\begin{aligned}G_{n,c}^{[a]}((u-x)^4;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)(n-5c)}\Bigg((3n^2 + 86cn + 126c^2)c^2x^4\\ &+ \frac{2c(3(2a-1)n^2 + 4c(43a-28)n + 240(a-1)c^2)}{(a-1)}x^3\\ &+ \frac{(56a^2 - 100a + 47)n^2 + 2c(91a^2 - 62a - 9)n + 840(a-1)^2c^2}{(a-1)^2}x^2\\ &+\frac{2(78a^3 - 171a^2 + 128a - 32)n + 680c(a-1)^3}{(a-1)^3}x + 209\Bigg).\end{aligned}$$

**Lemma 2.4.** *For sufficiently large $n$, we have*

(i) $\lim_{n\to\infty} nG^{[a]}_{n,c}((u-x);x) = 2(1+cx)$;

(ii) $\lim_{n\to\infty} nG^{[a]}_{n,c}((u-x)^2;x) = x\left(cx+\frac{1}{a-1}+2\right)$;

(iii) $\lim_{n\to\infty} n^2G^{[a]}_{n,c}((u-x)^4;x) = x^2\left(3c^2x^2+\frac{3(2a-1)}{(a-1)}x+\frac{56a^2-100a+47}{(a-1)^2}\right)$.

## 3. Main Result

**Theorem 3.1.** *Let $g \in C_\gamma[0,\infty)$ and for sufficiently large $n$ the operators $G^{[a]}_{n,c}(g(u);x)$ converges to $g(x)$ uniformly in each compact subset of $[0,\infty)$.*

*Proof.* From Lemma 2.2, $\lim_{n\to\infty} G^{[a]}_{n,c}(1;x) = 1$, $\lim_{n\to\infty} G^{[a]}_{n,c}(u;x) = x$ and $\lim_{n\to\infty} G^{[a]}_{n,c}(u^2;x) = x^2$. Then by Bohman-Korovokin theorem, $G^{[a]}_{n,c}(g(u);x)$ converges to $g(x)$ uniformly in each compact subset of $[0,\infty)$. □

**Theorem 3.2.** *For $g \in C_\gamma[0,\infty)$ and $g'(x)$, $g''(x)$ exist in $[0,\infty)$, we have*

$$\left[G^{[a]}_{n,c}(g(u);x)-g(x)\right] = 2(1+cx)g'(x)+\frac{x}{2!}\left(cx+\frac{1}{a-1}+2\right)g''(x).$$

*Proof.* From Taylor's expansion, we have

$$g(u) = g(x)+(u-x)g'(x)+\frac{(u-x)^2g''(x)}{2!}+r(u,x)(u-x)^2,$$

where $r(u,x)$ converges to 0 when $u\to x$.

Applying $G^{[a]}_{n,c}(\cdot;x)$ in above expression, we have

$$\begin{aligned} n\left[G^{[a]}_{n,c}(g(u);x)-g(x)\right] =& nG^{[a]}_{n,c}((u-x);x)g'(x)+\frac{nG^{[a]}_{n,c}((u-x)^2;x)g''(x)}{2!}\\ &+nG^{[a]}_{n,c}(r(u,x)(u-x)^2;x). \end{aligned} \tag{3.1}$$

Using Cauchy-Schwarz inequality and Lemma 2.4 in last term of above equality, we obtain

$$\lim_{n\to\infty} nG^{[a]}_{n,c}(r(u,x)(u-x)^2;x) = 0. \tag{3.2}$$

From (3.1), using (3.2) and Lemma 2.4, we get the required result. □

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions $g$ on $[0,\infty)$, provided with norm

$$\|g\| = \sup_{x\in[0,\infty)} |g(x)|,$$

and Peetre's K-functional for $g\in C_B[0,\infty)$ is given as:

$$K_2(g;\delta) = \inf_{x\in W^2_\infty}\{\|g-h\|+\delta\|h''\|\}, \quad \delta>0,$$

where $W_\infty^2[0,\infty)=\{h\in C_B[0,\infty): h',h''\in C_B[0,\infty)\}$. Devore and Lorentz [5, Theorem 2.4, page 177], provided relation between Peetre's $K$ functional and second order modulus of continuity as follows:

$$K_2(g;\delta)\leq C\omega_2(g;\sqrt{\delta}), \tag{3.3}$$

and the second order modulus of continuity $\omega_2(g;\sqrt{\delta})$ is given as

$$\omega_2(g;\sqrt{\delta})=\sup_{0<i\leq\delta}\sup_{x\in[0,\infty)}\left|g(x+2i)-2g(x+i)+g(x)\right|.$$

The usual modulus of continuity $\omega(g;\delta)$ for $g\in C_B[0,\infty)$

$$\omega(g;\delta)=\sup_{0<i\leq\delta}\sup_{x\in[0,\infty)}\left|g(x+i)-g(x)\right|.$$

**Theorem 3.3.** *For $g\in C_B[0,\infty)$ and $a>1$, we have*

$$\left|G_{n,c}^{[a]}(g(u);x)-g(x)\right|\leq C\omega_2\left(g\sqrt{\delta_{n,c}^a(x)}\right)+\omega\left(g;\left|\frac{2(1+cx)}{(n-2c)}\right|\right),$$

*where $C$ is positive constant and* $\delta_{n,c}^a(x)=\left[G_{n,c}^{[a]}((u-x)^2;x)+\frac{2(1+cx)^2}{(n-2c)^2}\right]$.

*Proof.* We consider an auxiliary operators:

$$\tilde{G}_{n,c}^{[a]}(g(u);x)=G_{n,c}^{[a]}(g(u);x)-g\left(x+\frac{2(1+cx)}{n-2c}\right)+g(x).$$

The Taylor's expansion for the function $h\in W_\infty^2[0,\infty)$ is given as

$$h(u)=h(x)+(u-x)h'(x)+\int_x^u(u-x)h''(u)du.$$

Applying $\tilde{G}_{n,c}^{[a]}(\cdot;x)$ in above expression

$$\tilde{G}_{n,c}^{[a]}(h(u);x)-h(x)=\tilde{G}_{n,c}^{[a]}((u-x);x)h'(x)+\tilde{G}_{n,c}^{[a]}\left(\int_x^u(u-x)h''(u)du;x\right).$$

Since $\tilde{G}_{n,c}^{[a]}(1;x)=1$, $\tilde{G}_{n,c}^{[a]}(u;x)=x$ and $\tilde{G}_{n,c}^{[a]}(u-x;x)=0$, we get

$$\begin{aligned}\left|\tilde{G}_{n,c}^{[a]}(h(u);x)-h(x)\right|&=\left|\tilde{G}_{n,c}^{[a]}\left(\int_x^u(u-x)h''(u)du;x\right)\right|\\&\leq\left|G_{n,c}^{[a]}\left(\int_x^u(u-x)h''(u)du;x\right)\right|\\&\quad+\left|\int_x^{x+\frac{2(1+cx)}{n-2c}}\left(x+\frac{2(1+cx)}{n-2c}-u\right)h''du\right|\end{aligned}$$

$$\leq \left[G_{n,c}^{[a]}((u-x)^2;x) + \frac{2(1+cx)^2}{(n-2c)^2}\right] \|h''\|$$

$$\leq \delta_{n,c}^{a}(x) \|h''\| . \tag{3.4}$$

Using auxiliary operators we can write

$$\left|G_{n,c}^{[a]}(g(u);x) - g(x)\right| \leq \left|\tilde{G}_{n,c}^{[a]}(g-h;x) - (g-h)(x)\right| + \left|\tilde{G}_{n,c}^{[a]}(h(u);x) - h(x)\right| + \left|g\left(x + \frac{2(1+cx)}{n-2c}\right) - g(x)\right|$$
$$\leq 2\left|g-h\right| + \delta_{n,c}^{a}(x) \|h''\| + \omega\left(g; \frac{2(1+cx)}{n-2c}\right).$$

Taking infimum on the right hand side of the above inequality for $g \in W_\infty^2[0,\infty)$, we have

$$\left|G_{n,c}^{[a]}(g(u);x) - g(x)\right| = 2K_2\left(g; \delta_{n,c}^{a}(x)\right) + \omega\left(g, \left|\frac{2(1+cx)}{n-2c}\right|\right).$$

From (3.3), we obtain

$$\left|G_{n,c}^{[a]}(g(u);x) - g(x)\right| = C\omega_2\left(g; \sqrt{\delta_{n,c}^{a}(x)}\right) + \omega\left(g, \left|\frac{2(1+cx)}{n-2c}\right|\right).$$

Hence, the proof. □

In the next theorem, we estimate global rate of convergence by using Ditzian-Totik modulus of smoothness $\omega_{\phi^\alpha}(g;\delta)$ for $g \in C_B[0,\infty)$, $0 < \alpha \leq 1$ and $\phi(x) = \sqrt{x(1+cx)}$ which is defined as:

$$\omega_{\phi^\alpha}(g;\delta) = \sup_{0\leq s\leq\delta} \sup_{x\pm\frac{s\phi^\alpha(x)}{2}\in[0,\infty)} \left|g\left(x + \frac{s\phi^\alpha(x)}{2}\right) - g\left(x - \frac{s\phi^\alpha(x)}{2}\right)\right|,$$

and the Peetre $K$-functional is defined as:

$$K_{\phi^\alpha}(g;\delta) = \inf_{g\in W_\alpha} \left\{\|g-h\| - \delta\left\|\phi^\alpha g'\right\|\right\},$$

where $W_\alpha$ is subspaces of those functions which are locally absolutely continuous on $g \in [0,\infty)$ with the normed $\|\phi^\alpha g'\| \leq \infty$. In [6], there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\alpha}(g;\delta) \leq K_{\phi^\alpha}(g;\delta) \leq C\omega_{\phi^\alpha}(g;\delta).$$

**Theorem 3.4.** *Suppose $g \in C_B[0,\infty)$ and for sufficiently large $n$, we have*

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq C\omega_{\phi^\alpha}\left(g; \frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right).$$

*Proof.* For $h \in W_\alpha$, we have

$$h(u) = h(x) + \int_x^u h'(t)dt.$$

Applying $G_{n,c}^{[a]}(\cdot;x)$ in the above equality and using Hölder's inequality, we obtain

$$\begin{aligned}\left|G_{n,c}^{[a]}(h;x)-h(x)\right| \leq & G_{n,c}^{[a]}\left(\int_x^u h'(t)dt;x\right)\\ \leq & \|\phi^\alpha h'\|\, G_{n,c}^{[a]}\left(\int_x^u \frac{dt}{\phi^\alpha(t)};x\right)\\ \leq & \|\phi^\alpha h'\|\, G_{n,c}^{[a]}\left(|u-x|^{1-\alpha}\left|\int_x^u \frac{dt}{\phi(t)}\right|^\alpha;x\right). \end{aligned} \tag{3.5}$$

Let $p(u,x)=\left|\int_x^u \frac{dt}{\phi(t)}\right|$, we have

$$\begin{aligned} p(u,x) \leq & \left|\int_x^u \frac{dt}{\phi(t)}\right|\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right)\\ \leq & \frac{2\,|u-x|}{\sqrt{x}+\sqrt{u}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right)\\ \leq & \frac{2\,|u-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right). \end{aligned}$$

Since $|a+b|^\alpha \leq |a|^\alpha+|b|^\alpha$, $0\leq\alpha\leq 1$, and from the above inequality, we obtain

$$\left|\int_x^u \frac{dt}{\phi(t)}\right|^\alpha \leq \frac{2^\alpha|u-x|^\alpha}{x^{\frac{\alpha}{2}}}\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}+\frac{1}{(1+cu)^{\frac{\alpha}{2}}}\right). \tag{3.6}$$

From (3.5), (3.6) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(h;x)-h(x)\right| \leq & \frac{2^\alpha\,\|\phi^\alpha h'\|}{x^{\frac{\alpha}{2}}} G_{n,c}^{[a]}\left(|u-x|\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}+\frac{1}{(1+cu)^{\frac{\alpha}{2}}}\right);x\right)\\ \leq & \frac{2^\alpha\,\|\phi^\alpha h'\|}{x^{\frac{\alpha}{2}}}\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}\left(G_{n,c}^{[a]}((u-x)^2;x)\right)^{\frac{1}{2}}\right.\\ & \left. +\left(G_{n,c}^{[a]}((u-x)^2;x)\right)^{\frac{1}{2}}\times\left(G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3.7}$$

From Theorem 3.1, $G_{n,c}^{[a]}((1+cu)^{-\alpha}$ converges to $(1+cx)^{-\alpha}$ for sufficiently large $n$. Thus, for $\epsilon>0$, there exist $n_0\in N$ such that $G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\leq(1+cx)^{-\alpha}+\epsilon$ for all $n\geq n_0$.

Choosing $\varepsilon=(1+cx)^{-\alpha}$, we get

$$G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\leq 2(1+cx)^{-\alpha},\quad \text{for all } n\geq n_0. \tag{3.8}$$

For sufficiently large $n$ there exists a constant $C>0$, such that

$$G_{n,c}^{[a]}((u-x)^2;x)\leq C\frac{\phi^2(x)}{n}. \tag{3.9}$$

From (3.7) to (3.9), we obtain

$$\left|G_{n,c}^{[a]}(h;x)-h(x)\right|\leq 2^{\alpha+1}C\left\|\phi^{\alpha}h'\right\|\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}. \tag{3.10}$$

We can write

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| &\leq \left|G_{n,c}^{[a]}(g-h;x)\right|+\left|G_{n,c}^{[a]}(h;x)-h(x)\right|+|h(x)-g(x)|\\ &\leq 2\left\|g-h\right\|+\left|G_{n,c}^{[a]}(h;x)-h(x)\right|.\end{aligned}$$

From (3.10), we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| &\leq 2\left\|g-h\right\|+2^{\alpha+1}C\left\|\phi^{\alpha}h'\right\|\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\\ &\leq C\left\{\left\|g-h\right\|+\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\left\|\phi^{\alpha}h'\right\|\right\}\\ &\leq CK_{\phi}^{\alpha}\left(g;\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right)\\ &\leq C{\omega_{\phi}}^{\alpha}\left(g;\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right).\end{aligned}$$

Hence, the proof. □

In [15], the Lipschitz-type space for positive real numbers $\beta_1$, $\beta_2$ is defined as:

$$Lip_M^{\beta_1,\beta_2}(\lambda)=\left\{g\in C_{\mathbf{B}}[0,\infty):|g(u)-g(x)|\leq M_g\frac{|u-x|^{\lambda}}{(u+\beta_1x^2+\beta_2x)^{\frac{\lambda}{2}}};x,u\in[0,\infty)\right\},$$

where $M_g>0$ and $0<\lambda\leq 1$.

**Theorem 3.5.** *Let $g\in Lip_M^{\beta_1,\beta_2}(\lambda)$ and $0<\lambda\leq 1$, then for $x\geq 0$ we have*

$$\left|G_{n,c}^{[a]}(g;x)-g(x)\right|\leq M_g\left(\frac{\mu_{n,2}^{a,c}(x)}{(\beta_1x^2+\beta_2x)}\right)^{\frac{\lambda}{2}},$$

*where $\mu_{n,2}^{a,c}(x)=G_{n,c}^{[a]}((u-x)^2;x)$.*

*Proof.* First, we discuss the result for $\lambda=1$. For $g\in Lip_M^{\beta_1,\beta_2}(\lambda)$, we have

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| &\leq (n-c)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\\ &\quad\times\int_0^{\infty}p_{n,j}(u;c)\left|g(u)-g(x)\right|du\\ &\leq (n-c)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\end{aligned}$$

$$\times \int_0^\infty p_{n,j}(u;c) M_g \frac{|u-x|}{(u+\beta_1 x^2+\beta_2 x)^{\frac{1}{2}}} du.$$

Since $\frac{1}{\sqrt{u+\beta_1 x^2+\beta_2 x}} < \frac{1}{\sqrt{\beta_1 x^2+\beta_2 x}}$, applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|G_{n,c}^{[a]}(g;x)-g(x)\right| \leq & \frac{M_g}{\sqrt{\beta_1 x^2+\beta_2 x}}(n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx} \sum_{j=0}^{\infty} \frac{C_j^{[a]}(-(a-1)nx)}{j!} \\
& \times \int_0^\infty p_{n,j}(x;c)\,|u-x|\,du \\
\leq & \frac{M_g}{\sqrt{\beta_1 x^2+\beta_2 x}} \sqrt{G_{n,c}^{[a]}((u-x)^2;x)} \\
\leq & M_g \sqrt{\frac{\mu_{n,2}^{a,c}(x)}{\beta_1 x^2+\beta_2 x}}.
\end{aligned}$$

The result is true for $\lambda = 1$. Now we prove for $0 < \lambda < 1$. Using Hölder's inequality with $p = \frac{2}{\lambda}$ and $q = \frac{2}{2-\lambda}$, we have

$$\begin{aligned}
& \left|G_{n,c}^{[a]}(g;x)-g(x)\right| \\
\leq & \left\{ n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx} \sum_{j=0}^{\infty} \frac{C_j^{[a]}(-(a-1)nx)}{j!} \right. \\
\leq & M_g \left\{ (n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx} \sum_{j=0}^{\infty} \frac{C_j^{[a]}(-(a-1)nx)}{j!} \right. \\
& \left. \times \int_0^\infty p_{n,j}(x;c) \frac{(u-x)^2}{(u+\beta_1 x^2+\beta_2 x)} du \right\}^{\frac{\lambda}{2}} \\
\leq & \frac{M_g}{(\beta_1 x^2+\beta_2 x)^{\frac{\lambda}{2}}} \left\{ (n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx} \sum_{j=0}^{\infty} \frac{C_j^{[a]}(-(a-1)nx)}{j!} \right. \\
& \left. \times \int_0^\infty p_{n,j}(x;c)(u-x)^2 du \right\}^{\frac{\lambda}{2}} \\
\leq & M_g \left( \frac{\mu_{n,2}^{a,c}(x)}{(\beta_1 x^2+\beta_2 x)} \right)^{\frac{\lambda}{2}}.
\end{aligned}$$

Hence, the proof. □

**Theorem 3.6.** *If $g(x)$ is continuously differentiable function on $[0,\infty)$ and $|g'(x)| \le D$ for some $D > 0$, then we have*

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le D\left|\frac{2(1+cx)}{n-2c}\right| + 2\sqrt{\mu_{n,2}^{a,c}(x)}\,\omega_b\left(g'; \sqrt{\mu_{n,2}^{a,c}(x)}\right),$$

*where $\omega_b(g;\delta)$, $\delta > 0$, is usual modulus of continuity on $[0,b]$ and*

$$\mu_{n,2}^{a,c}(x) = G_{n,c}^{[a]}((u-x)^2;x).$$

*Proof.* From Lagrange's mean value theorem, we get

$$g(u) - g(x) = (u-x)g'(\eta) = (u-x)g'(x) + (u-x)(g'(\eta) - g'(x)),$$

where $\eta$ lies between $x$ and $u$.

now, we apply $G_{n,c}^{[a]}(\cdot;x)$ on both side of the above equation. Since $x < \eta < u$ we have $|\eta - x| < |u-x|$ and

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le |g'(x)|\left|G_{n,c}^{[a]}((u-x);x)\right| + \omega_b(g';\delta)\left(|u-x| + \frac{(u-x)^2}{\delta}\right).$$

Applying Cauchy-Schwarz inequality, we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le D\left|\frac{2(1+cx)}{n-2c}\right| + \sqrt{\mu_{n,2}^{a,c}(x)}\,\omega_b(g';\delta)\left(1 + \frac{\sqrt{\mu_{n,2}^{a,c}(x)}}{\delta}\right).$$

Taking $\delta = \sqrt{\mu_{n,2}^{a,c}(x)}$, we get required result. □

In our next theorem, we study the rate of convergence for the operators (1.4) based on Lipscitz maximal function of order $r$ given by Lenze [13] as

$$\varpi_r(g;x) = \sup_{u\neq x,\, x,u\in[0,\infty)} \frac{|g(u)-g(x)|}{|u-x|^r}, \tag{3.11}$$

where $0 < r \le 1$.

**Theorem 3.7.** *For $g \in C_B[0,\infty)$, we have*

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le \varpi_r(g;x)\left(\mu_{n,2}^{a,c}(x)\right)^r.$$

*Proof.* From (3.11), we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le \varpi_r(g;x) G_{n,c}^{[a]}\left(|u-x|^r;x\right).$$

Using Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \le \varpi_r(g;x)\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^r \le \varpi_r(g;x)\left(\mu_{n,2}^{a,c}(x)\right)^r.$$

Hence, the proof. □

Let $C_2[0,\infty)$ be the space of all continuous functions on $[0,\infty)$ and defined as:

$$C_2[0,\infty) := \left\{g : |g| \leq M_g(1+x^2)\right\},$$

where $M_g$ is positive constant which may depends on $g$ with the norm

$$\|g\|_2 = \sup_{x>0} \frac{|g(x)|}{1+x^2}.$$

Let $C_2^*[0,\infty) := \left\{g \in C_2[0,\infty) : \lim_{x\to\infty} \frac{g(x)}{1+x^2} \text{ exists and finite}\right\}$. The weighted modulus of continuity [9] $\Omega(g;\delta)$ is given as

$$\Omega(g;\delta) = \sup_{0\leq\beta<\delta} \frac{|g(x+\beta)-g(x)|}{(1+\beta^2)(1+x^2)}.$$

**Lemma 3.1.** *For every $g \in C_2^*[0,\infty)$, $\Omega(g;\delta)$ has the properties:*

(i) *$\Omega(g;\delta)$ is a monotonically increasing function of $\delta$;*
(ii) *$\lim_{\delta\to 0^+} \Omega(g;\delta) = 0$;*
(iii) *$\Omega(g;k\delta) \leq 2(1+k)(1+\delta^2)\Omega(g;\delta)$, $k>0$ and $\delta>0$.*

**Theorem 3.8.** *For $g \in C_2^*[0,\infty)$, we have*

$$\sup_{x\in[0,\infty)} \frac{\left|G_{n,c}^{[a]}(g;x)-g(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq C\Omega\left(g;\frac{1}{\sqrt{n}}\right),$$

*where $C$ is positive constant depends on $a$ and $c$.*

*Proof.* For $x,u \in [0,\infty)$ and from (3.11), we can write

$$\begin{aligned}|g(u)-g(x)| \leq&(1+(u-x)^2)(1+x^2)\Omega\left(g;\frac{|u-x|\,\delta}{\delta}\right)\\ \leq&2(1+\delta^2)(1+x^2)\left(1+\frac{|u-x|}{\delta}\right)(1+(u-x)^2)\Omega(g;\delta).\end{aligned}$$

Applying $G_{n,c}^{[a]}(\cdot;x)$ in the above inequality, we have

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| \leq&2(1+\delta^2)(1+x^2)\Omega(g;\delta)G_{n,c}^{[a]}\left(\left(1+\frac{|u-x|}{\delta}\right)(1+(u-x)^2);x\right)\\ \leq&2(1+\delta^2)(1+x^2)\Omega(g;\delta)\Big\{G_{n,c}^{[a]}(1;x)+G_{n,c}^{[a]}((u-x)^2;x)\\ &+\frac{1}{\delta}G_{n,c}^{[a]}\left(|u-x|\,;x\right)+\frac{1}{\delta}G_{n,c}^{[a]}\left(|u-x|\,(u-x)^2;x\right)\Big\}\\ \leq&2(1+\delta^2)(1+x^2)\Omega(g;\delta)\Big\{1+G_{n,c}^{[a]}((u-x)^2;x)\\ &+\frac{1}{\delta}\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^{\frac{1}{2}}\\ &+\frac{1}{\delta}\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^{\frac{1}{2}}\Big(G_{n,c}^{[a]}\left((u-x)^4;x\right)\Big)^{\frac{1}{2}}\Big\}.\end{aligned} \tag{3.12}$$

From Lemma 2.3, we have

$$G_{n,c}^{[a]}((u-x)^2;x) \leq \frac{C_1(1+x^2)}{n}$$

and

$$G_{n,c}^{[a]}((u-x)^4;x) \leq \frac{C_2(1+x^2)^2}{n^2},$$

where $C_1$ and $C_2$ are positive constants depend on $a$ and $c$.

Using the above inequality in (3.12) and taking $\delta = \frac{1}{\sqrt{n}}$, we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq & 2\left(1+\frac{1}{n}\right)\Omega\left(g;\frac{1}{\sqrt{n}}\right)(1+x^2)\left\{1 + C_1\,(1+x^2)\right.\\ & \left. + \sqrt{C_1(1+x^2)} + \sqrt{C_1C_2}(1+x^2)^{\frac{3}{2}}\right\}.\end{aligned}$$

Taking $C = 4\left(1 + C_1 + \sqrt{C_1} + \sqrt{C_1C_2}\right)$, we obtain the result. □

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## References

[1] A. M. Acu and C. Muraru, *Certain approximation properties of Srivastava-Gupta operators*, J. Math. Inequal. **12**(2) (2018), 583–595. https://dx.doi.org/10.7153/jmi-2018-12-44

[2] R. Chauhan, B. Baxhaku and P. N. Agrawal, *Szász type operators involving Charlier polynomials of blending type*, Complex Anal. Oper. Theory **13** (2019), 1197–1226. https://doi.org/10.1007/s11785-018-0854-x

[3] N. Deo, *Faster rate of convergence on Srivastava-Gupta operators*, Appl. Math. Comput. **218**(21) (2012), 10486–10491. https://doi.org/10.1016/j.amc.2012.04.012

[4] N. Deo and M. Dhamija, *Charlier-Szász-Durrmeyer type linear positive operators*, Afr. Mat. **19**(1) (2018), 223–232, https://doi.org/10.1007/s13370-017-0537-1

[5] R. A. Devore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, Heidelberg, 1993.

[6] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer series in Computational Mathematics **9**, Springer, New York, 1987.

[7] V. Gupta and H. M. Srivastava, *A General family of the Srivastava-Gupta operators preserving linear functions*, Eur. J. Pure Appl. Math. **11**(3) (2018), 575–579, https://doi.org/10.29020/nybg.ejpam.v11i3.3314

[8] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge, New York, 2005.

[9] N. Ispir, *On modified Baskakov operators on weighted spaces*, Turkish J. Math. **26**(3) (2001), 355–365.

[10] N. Ispir and I. Yüksel, *On the Bézier variant of Srivastava-Gupta operators*, Appl. Math. E-Notes **5** (2005), 129–137.

[11] A. Kajla and P. N. Agrawal, *Szász-Durrmeyer type operators based on Charlier polynomials*, Appl. Math. Comput. **268** (2015), 1001–1014, https://doi.org/10.1016/j.amc.2015.06.126

[12] A. Kumar, V. N. Mishra and D. Tapiawala, *Stancu type generalization of modified Srivastava-Gupta operators*, Eur. J. Pure Appl. Math. **10**(3) (2017), 890–897.

[13] B. Lenze, *On Lipschitz maximal functions and their smoothness spaces*, Indag. Math. **50**(1) (1988), 53–63. https://doi.org/10.1016/1385-7258(88)90007-8

[14] P. Maheswari (Sharma), *On modified Srivastava-Gupta operators*, Filomat **29**(6) (2015), 1173–1177. https://doi.org/10.2307/24898197

[15] M. A. Özarslan and O. Duman, *Local approximation properties for certain king type operators*, Filomat **27**(1) (2013), 173–181. https://doi.org/10.2307/24896345

[16] R. Pratap and N. Deo, *Approximation by Genuine Gupta-Srivastava Operators* Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **113** (2019), 2495–2505. https://doi.org/10.1007/s13398-019-00633-4

[17] H. M. Srivastava and V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **37** (2003), 1307–1315. https://doi.org/10.1016/S0895-7177(03)90042-2

[18] O. Szász, *Generalisation of S. Bernstein polynomials to the infinite interval*, Journal of Research of the National Bureau of Standards **56** (1950), 239–245.

[19] S. Varma and F. Tasdelen, *Szász type operators involving Charlier polynomials*, Math. Comput. Modelling **56** (2012), 118–122.

[20] D. K. Verma and P. N. Agrawal, *Convergence in simultaneous approximation for Srivastava-Gupta operators*, Math. Sci. (Springer) **6** (2012), Article ID 22, 8 pages. https://doi.org/10.1186/2251-7456-6-22

[21] R. Yadav, *Approximation by modified Srivastava-Gupta operators*, Appl. Math. Comput. **226** (2014), 61–66. https://doi.org/10.1016/j.amc.2013.10.039

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# A PARAMETER-BASED OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS WHOSE DERIVATIVES BELONGS TO $L_p([a,b])$ INVOLVING MULTIPLE POINTS

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Abstract. A new generalization of Ostrowski's inequality for functions whose derivatives belong to $L_p([a,b])$ $(1 \leq p < \infty)$ for $k$ points via a parameter is provided. Some particular integral inequalities are derived as by products. Our results generalize some results in the literature.

## 1. Introduction

In 1938, Ostrowski [17] obtained the following inequality which is known in the literature as Ostrowski's inequality.

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in $(a,b)$ and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in $(a,b)$. If $M := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left( \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)M,$$

*for all $x \in [a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.*

Due to the numerous applications of the Ostrowski's inequality, many authors have studied and generalized the inequality in several different ways. For more information

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about the Ostrowski's inequality and its associates, we refer the interested reader to the papers [1–16, 18].

In [5], Dragomir and Wang provided the following extension of Theorem 1.1 for functions whose derivatives belong to $L_1$ as follows.

**Theorem 1.2.** *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[ \frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \|f'\|_1$$

*holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.*

The same authors in [7], obtained an Ostrowski type inequality for differentiable mappings whose derivatives belong to $L_p$-spaces as follows.

**Theorem 1.3.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $a, b \in I^\circ$ with $a < b$. If $f' \in L_p(a,b)$ $(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{b-a} \left[ \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right] \|f'\|_p,$$

*for all $x \in [a,b]$, where $\|\cdot\|_p$ is the $L_p([a,b])$-norm.*

In [2], Dragomir obtained the following generalization of Theorem 1.2 as follows.

**Theorem 1.4.** *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and $\alpha_i$ $(i = 0, 1, \ldots, k+1)$ be $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the inequality*

$$\begin{aligned}
&\left| \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)dt \right| \\
\leq &\left[ \frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| : i = 0, \ldots, k-1 \right\} \right] \|f'\|_1 \\
\leq &\nu(h)\|f'\|_1,
\end{aligned}$$

*where $\nu(h) := \max\{h_i | i = 0, 1, \ldots, k-1\}$, $h_i = x_{i+1} - x_i$ $(i = 0, \ldots, k-1)$.*

In [3], Dragomir obtained the following generalization of Theorem 1.3 as follows.

**Theorem 1.5.** *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and $\alpha_i$ $(i = 0, 1, \ldots, k+1)$ be $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the inequality*

$$\left| \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)dt \right|$$

$$
\begin{aligned}
&\leq \frac{1}{(q+1)^{1/q}} \|f'\|_p \left\{ \sum_{i=0}^{k-1} \left[ (\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right\}^{1/q} \\
&\leq \frac{1}{(q+1)^{1/q}} \|f'\|_p \left\{ \sum_{i=0}^{k-1} h_i^{q+1} \right\}^{1/q} \\
&\leq \frac{\nu(h)(b-a)^{1/q}}{(q+1)^{1/q}} \|f'\|_p,
\end{aligned}
$$

*where* $\nu(h) := \max\{h_i | i = 0, 1, \ldots, k-1\}$, $h_i = x_{i+1} - x_i$ $(i = 0, \ldots, k-1)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\| \cdot \|_p$ *is the ususal* $L_p([a,b])$*-norm.*

Motivated by the numerous research on the Ostrowski's inequality in the past years, our main goal in this paper is to provide a generalization of Theorem 1.1 involving multiple points by introducing a parameter $\lambda \in [0,1]$ for functions whose derivative belongs to $L_p$ for $1 \leq p < \infty$ such that when $\lambda = 0$, we recapture Theorem 1.4 and Theorem 1.5.

## 2. Main Results

To prove our main results, we need the following lemma which is the case when the time scale $\mathbb{T} = \mathbb{R}$ in [18, Lemma 1] but the proof is provided here for completion.

**Lemma 2.1** (Montgomery Identity)**.** *Let*

(a) $a, b \in \mathbb{R}$, $\lambda \in [0,1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a,b]$*;*

(b) $\alpha_i \in \mathbb{R}$ $(i = 0, 1, \ldots, k+1)$ *is* $k+2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = b$*;*

(c) $f : [a,b] \to \mathbb{R}$ *is a differentiable function;*

(d) *define the kernel function* $K(\cdot, I_k) : [a,b] \to \mathbb{R}$ *as follows*

$$
K(t, I_k) = \begin{cases}
t - \left(\alpha_1 - \lambda \frac{\alpha_1 - a}{2}\right), & t \in [a, \alpha_1), \\
t - \left(\alpha_1 + \lambda \frac{\alpha_2 - \alpha_1}{2}\right), & t \in [\alpha_1, x_1), \\
t - \left(\alpha_2 - \lambda \frac{\alpha_2 - \alpha_1}{2}\right), & t \in [x_1, \alpha_2), \\
\vdots \\
t - \left(\alpha_{k-1} + \lambda \frac{\alpha_k - \alpha_{k-1}}{2}\right), & t \in [\alpha_{k-1}, x_{k-1}), \\
t - \left(\alpha_k - \lambda \frac{\alpha_k - \alpha_{k-1}}{2}\right), & t \in [x_{k-1}, \alpha_k), \\
t - \left(\alpha_k + \lambda \frac{\alpha_{k+1} - \alpha_k}{2}\right), & t \in [\alpha_k, b],
\end{cases}
$$

*for all* $t \in [a,b]$.

*Then we have the identity*

$$
\begin{aligned}
\int_a^b K(t,I_k)f'(t)dt =&(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\\
&+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt.
\end{aligned}
\tag{2.1}
$$

*Proof.* We observe that

$$
\begin{aligned}
\int_a^b K(t,I_k)f'(t)dt =&\sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}\left[t-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f'(t)dt\right.\\
&\left.+\int_{\alpha_{i+1}}^{x_{i+1}}\left[t-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f'(t)dt\right].
\end{aligned}
$$

By integrating by parts, we have

$$
\begin{aligned}
\int_a^b K(t,I_k)f'(t)dt =&\sum_{i=0}^{k-1}\left[\left[\alpha_{i+1}-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f(\alpha_{i+1})\right.\\
&-\left[x_i-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right]f(x_i)-\int_{x_i}^{\alpha_{i+1}}f(t)dt\\
&+\left[x_{i+1}-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f(x_{i+1})\\
&\left.-\left[\alpha_{i+1}-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right]f(\alpha_{i+1})-\int_{\alpha_{i+1}}^{x_{i+1}}f(t)dt\right]\\
=&\sum_{i=0}^{k-1}\left[\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(\alpha_{i+1})-(x_i-\alpha_{i+1})f(x_i)-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i)\right.\\
&+(x_{i+1}-\alpha_{i+1})f(x_{i+1})-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})\\
&\left.+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(\alpha_{i+1})-\int_{x_i}^{x_{i+1}}f(t)dt\right].
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
\int_a^b K(t,I_k)f'(t)dt =&\sum_{i=0}^{k-1}\left[\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_{x_i}^{x_{i+1}}f(t)dt\right]\\
&+\sum_{i=0}^{k-1}\left[-(x_i-\alpha_{i+1})f(x_i)+(x_{i+1}-\alpha_{i+1})f(x_{i+1})\right]\\
&+\sum_{i=0}^{k-1}\left[-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i)-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})\right]\\
=&\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_a^b f(t)dt
\end{aligned}
$$

$$- x_0 f(x_0) + x_k f(x_k) + \sum_{i=0}^{k-1} \alpha_{i+1}(f(x_i) - f(x_{i+1}))$$
$$+ \sum_{i=0}^{k-1} -\frac{\lambda}{2}\left[(\alpha_{i+1} - \alpha_i) f(x_i) + (\alpha_{i+2} - \alpha_{i+1}) f(x_{i+1})\right].$$

That is,

$$\begin{aligned}\int_a^b K(t, I_k) f'(t)dt =& \sum_{i=0}^{k-1} \lambda \frac{\alpha_{i+2} - \alpha_i}{2} f(\alpha_{i+1}) - \int_a^b f(t)dt \\ &+ (\alpha_1 - a)f(a) + (b - \alpha_k)f(b) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \\ &- \frac{\lambda}{2}\left[(\alpha_1 - a)f(a) + (b - \alpha_k)f(b) + 2\sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i) f(x_i)\right] \\ =& \sum_{i=0}^{k-1} \lambda \frac{\alpha_{i+2} - \alpha_i}{2} f(\alpha_{i+1}) - \int_a^b f(t)dt + (1 - \lambda)\sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i) f(x_i) \\ &+ \left(1 - \frac{\lambda}{2}\right)\left[(\alpha_1 - a)f(a) + (b - \alpha_k)f(b)\right]. \end{aligned} \tag{2.2}$$

Now, consider the following

$$\begin{aligned}&\sum_{i=0}^{k-1}(\alpha_{i+2} - \alpha_i) f(\alpha_{i+1}) \\ &= \sum_{i=0}^{k-1}(\alpha_{i+2} - \alpha_{i+1}) f(\alpha_{i+1}) + \sum_{i=0}^{k-1}(\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) \\ &= \sum_{i=1}^{k}(\alpha_{i+1} - \alpha_i) f(\alpha_i) + \sum_{i=0}^{k-1}(\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) \\ &= \sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i) f(\alpha_i) - (\alpha_1 - \alpha_0) f(\alpha_0) \\ &\quad + \sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i) f(\alpha_{i+1}) - (\alpha_{k+1} - \alpha_k) f(\alpha_{k+1}) \\ &= \sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)(f(\alpha_i) + f(\alpha_{i+1})) - \left[(\alpha_1 - \alpha_0) f(\alpha_0) + (\alpha_{k+1} - \alpha_k) f(\alpha_{k+1})\right]. \end{aligned} \tag{2.3}$$

So,

$$\begin{aligned}\sum_{i=0}^{k-1} \frac{\lambda}{2}(\alpha_{i+2} - \alpha_i) f(\alpha_{i+1}) =& \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)\left(f(\alpha_i) + f(\alpha_{i+1})\right) \\ &- \frac{\lambda}{2}\left[(\alpha_1 - a)f(a) + (b - \alpha_k)f(b)\right]. \end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.2) gives the identity

$$\begin{aligned}\int_a^b K(t,I_k)f'(t)dt =&(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\\ &+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt. \qquad \square\end{aligned}$$

**Lemma 2.2.** *Under the conditions of Lemma* 2.1*, we have the following inequality:*

$$\begin{aligned}\left|\int_a^b K(t,I_k)f'(t)dt\right| \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}|\,|f'(t)|\,dt\right]\\ &+\frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\dots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\int_a^b|f'(t)|dt.\end{aligned}\tag{2.5}$$

*Proof.* First, we observe that for any real numbers $\delta$ and $\gamma$, the following holds:

$$\max\{\gamma,\delta\}=\frac{\gamma+\delta}{2}+\frac{|\gamma-\delta|}{2}.\tag{2.6}$$

Now, by using the property of the absolute value and the definition of $K(\cdot,I_k)$, we have that

$$\begin{aligned}\left|\int_a^b K(t,I_k)f'(t)dt\right| \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}|K(t,I_k)|\,|f'(t)|\,dt+\int_{\alpha_{i+1}}^{x_{i+1}}|K(t,I_k)|\,|f'(t)|\,dt\right]\\ \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}\left|t-\left(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)\right|\,|f'(t)|\,dt\right.\\ &\left.+\int_{\alpha_{i+1}}^{x_{i+1}}\left|t-\left(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)\right|\,|f'(t)|\,dt\right]\\ \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}|t-\alpha_{i+1}|\,|f'(t)|\,dt+\frac{\lambda}{2}(\alpha_{i+1}-\alpha_i)\int_{x_i}^{\alpha_{i+1}}|f'(t)|\,dt\right.\\ &\left.+\int_{\alpha_{i+1}}^{x_{i+1}}|t-\alpha_{i+1}|\,|f'(t)|\,dt+\frac{\lambda}{2}(\alpha_{i+2}-\alpha_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}}|f'(t)|\,dt\right]\\ \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt\right.\\ &\left.+\frac{\lambda}{2}\max\left\{\alpha_{i+1}-\alpha_i,\alpha_{i+2}-\alpha_{i+1}\right\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\right].\end{aligned}$$

Thus,

$$\begin{aligned}\left|\int_a^b K(t,I_k)f'(t)dt\right| \le &\sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt\right]\\ &+\frac{\lambda}{2}\sum_{i=0}^{k-1}\left[\max\left\{\alpha_{i+1}-\alpha_i,\alpha_{i+2}-\alpha_{i+1}\right\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\right].\end{aligned}\tag{2.7}$$

By using (2.10), we deduce that

$$\begin{aligned}
&\sum_{i=0}^{k-1}\max\left\{\alpha_{i+1}-\alpha_i,\alpha_{i+2}-\alpha_{i+1}\right\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&=\sum_{i=0}^{k-1}\left(\frac{1}{2}(\alpha_{i+2}-\alpha_i)+\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right)\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&\le\max_{i=0,1,\ldots,k-1}\left\{\frac{1}{2}\tau_i+\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|f'(t)|dt\\
&\le\left(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\int_a^b|f'(t)|dt. \qquad (2.8)
\end{aligned}$$

Using (2.7) and (2.8) yields the desired result. Hence, the proof is complete. □

We now state and prove our first theorem which is for the case $p=1$.

**Theorem 2.1.** *Under the conditions of Lemma* 2.1*, suppose that* $f'\in L_1[a,b]$*, then the following inequalities hold:*

$$\begin{aligned}
&\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt\right|\\
&\le\left[\frac{1}{2}\nu(h)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right\}\right]\|f'\|_1\\
&\quad+\left[\frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\right]\|f'\|_1\\
&\le\left(\nu(h)+\frac{\lambda}{2}\nu(\tau)\right)\|f'\|_1, \qquad (2.9)
\end{aligned}$$

*where* $h_i=x_{i+1}-x_i$, $\tau_i=\alpha_{i+2}-\alpha_i$ $(i=0,1,\ldots,k-1)$, $\nu(h)=\max\{h_i:i=0,1,\ldots,k-1\}$ *and* $\nu(\tau)=\max\left\{\tau_i:i=0,1,\ldots,k-1\right\}$ *.*

*Proof.* First, we observe that for any real numbers $\beta,\delta$ and $\gamma$, the following holds:

$$\sup_{t\in[\gamma,\delta]}|t-\beta|=\max\{|\gamma-\beta|,|\delta-\beta|\}. \qquad (2.10)$$

By using (2.10), we have

$$\begin{aligned}
&\sum_{i=0}^{k-1}\left[\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt\right]\\
&\le\sum_{i=0}^{k-1}\left[\sup_{t\in[x_i,x_{i+1}]}|t-\alpha_{i+1}|\int_{x_i}^{x_{i+1}}|f'(t)|dt\right]\\
&=\sum_{i=0}^{k-1}\left[\max\left\{|x_i-\alpha_{i+1}|,|x_{i+1}-\alpha_{i+1}|\right\}\int_{x_i}^{x_{i+1}}|f'(t)|dt\right]
\end{aligned}$$

$$= \sum_{i=0}^{k-1} \left[ \max \{\alpha_{i+1} - x_i, x_{i+1} - \alpha_{i+1}\} \int_{x_i}^{x_{i+1}} |f'(t)| dt \right].$$

That is,

$$\sum_{i=0}^{k-1} \left[ \int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}||f'(t)| dt \right]$$
$$\leq \sum_{i=0}^{k-1} \left[ \max\{\alpha_{i+1} - x_i, x_{i+1} - \alpha_{i+1}\} \int_{x_i}^{x_{i+1}} |f'(t)| dt \right]. \tag{2.11}$$

Now, by using (2.6) in (2.11), we have that

$$\sum_{i=0}^{k-1} \left[ \int_{x_i}^{x_{i+1}} |t - \alpha_{i+1}||f'(t)| dt \right]$$
$$\leq \sum_{i=0}^{k-1} \left[ \left( \frac{1}{2}(x_{i+1} - x_i) + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right) \int_{x_i}^{x_{i+1}} |f'(t)| dt \right]$$
$$= \sum_{i=0}^{k-1} \left[ \left( \frac{1}{2} h_i + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right) \int_{x_i}^{x_{i+1}} |f'(t)| dt \right]$$
$$\leq \max_{i=0,1,\ldots,k-1} \left\{ \frac{1}{2} h_i + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |f'(t)| dt$$
$$\leq \left[ \frac{1}{2}\nu(h) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \right] \|f'\|_1. \tag{2.12}$$

Using (2.5) and (2.12), we have

$$\left| \int_a^b K(t, I_k) f'(t) dt \right| \leq \left[ \frac{1}{2}\nu(h) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \right.$$
$$\left. + \frac{\lambda}{2} \left( \frac{1}{2}\nu(\tau) + \max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2} \right| \right\} \right) \right] \|f'\|_1. \tag{2.13}$$

By using (2.1) and (2.13), we obtained the first inequality in (2.9). To obtain the second inequality, we observe that

$$\left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2}(x_{i+1} - x_i) = \frac{1}{2} h_i$$

and

$$\left| \alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2} \right| \leq \frac{1}{2}(\alpha_{i+2} - \alpha_i) = \frac{1}{2}\tau_i.$$

So, it follows that

$$\max_{i=0,1,\ldots,k-1} \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right\} \leq \frac{1}{2}\nu(h)$$

and

$$\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\leq\frac{1}{2}\nu(\tau).$$

This completes the proof of the theorem. □

*Remark* 2.1. If we take $\lambda = 0$ in Theorem 2.1, then we recover Theorem 1.4.

**Lemma 2.3.** *Under the conditions of Lemma* 2.1 *with $f' \in L_p([a,b])$, we have the following inequalities*

$$\begin{aligned}\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}|t-\alpha_{i+1}||f'(t)|dt &\leq \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}}\\ &\leq \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{\frac{1}{q}}\\ &\leq \frac{\nu(h)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p,\end{aligned}\tag{2.14}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Remark* 2.2. The inequalities in Lemma 2.3 were established in the proof of [3, Theorem 3], and hence the proof is omited.

**Theorem 2.2.** *Under the conditions of Lemma* 2.1*, suppose that $f' \in L_p([a,b])$, for $1 < p < \infty$, then the following inequalities hold:*

$$\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\left(f(\alpha_i)+f(\alpha_{i+1})\right)-\int_a^b f(t)dt\right|$$

$$\begin{aligned}&\leq\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{\frac{1}{q}}+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p\\ &\leq\frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{\frac{1}{q}}+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p\\ &\leq\frac{\nu(h)(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_p+\frac{\lambda}{2}\nu(\tau)(b-a)^{\frac{1}{q}}\|f'\|_p,\end{aligned}$$

*where* $h_i = x_{i+1} - x_i$, $\tau_i = \alpha_{i+2} - \alpha_i$ $(i = 0, 1, \ldots, k-1)$, $\nu(h) = \max_{i=0,1,\ldots,k-1} h_i$, $\nu(\tau) = \max_{i=0,1,\ldots,k-1}\tau_i$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* By using Lemma 2.2 and the fact that

$$\max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\leq\frac{1}{2}\nu(\tau),$$

we deduce that

$$\left|\int_a^b K(t,I_k)f'(t)dt\right| \leq \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\,|f'(t)|\,dt + \frac{\lambda}{2}\nu(\tau)\int_a^b |f'(t)|dt. \tag{2.15}$$

Using the Hölder's inequality, we obtain

$$\int_a^b |f'(t)|dt \leq (b-a)^{\frac{1}{q}}\|f'\|_p. \tag{2.16}$$

We obtain the desired results by using (2.1), (2.14), (2.15) and (2.16). □

*Remark* 2.3. If we choose $\lambda = 0$ in Theorem 2.2, then we recover Theorem 1.5.

## 3. Some Particular Cases

In this section, we consider some particular cases of our main results.

**Corollary 3.1.** *Under the conditions of Theorem* 2.1*, if we choose* $\alpha_0 = a$*,* $\alpha_{i+1} = \frac{x_i+x_{i+1}}{2}$ $(i=0,\ldots,k-1)$ *and* $\alpha_{k+1} = b$*, then we have the inequalities*

$$\begin{aligned}
&\left|\frac{1-\lambda}{2}\left[(x_1-a)f(a) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i) + (b-x_{k-1})f(b)\right]\right.\\
&\quad + \frac{\lambda}{4}\Bigg[(x_1-a)\left(f(a)+f\left(\frac{x_1+a}{2}\right)\right) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})\Bigg(f\left(\frac{x_i+x_{i-1}}{2}\right)\\
&\quad \left.+ f\left(\frac{x_{i+1}+x_i}{2}\right)\Bigg) + (b-x_{k-1})\left(f(b)+f\left(\frac{b+x_{k-1}}{2}\right)\right)\Bigg] - \int_a^b f(t)dt\right|\\
&\leq \left[\frac{1}{2}\nu(h) + \frac{\lambda}{2}\left(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\ldots,k-1}\left\{\left|\alpha_{i+1} - \frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\right)\right]\|f'\|_1\\
&\leq \frac{\nu(h)+\lambda\nu(\tau)}{2}\|f'\|_1,
\end{aligned}$$

*where* $h_i = x_{i+1}-x_i$*,* $\tau_i = \alpha_{i+2}-\alpha_i$ $(i=0,1,\ldots,k-1)$*,* $\nu(h) = \max\{h_i : i=0,1,\ldots,k-1\}$ *and* $\nu(\tau) = \max\{\tau_i : i=0,1,\ldots,k-1\}$ *.*

*Proof.* In this case, we have $\alpha_1-\alpha_0 = \frac{x_1-a}{2}$, $\alpha_{i+1}-\alpha_i = \frac{x_{i+1}-x_{i-1}}{2}$ $(i=1,\ldots,k-1)$, $\alpha_{k+1}-\alpha_k = \frac{b-x_{k-1}}{2}$ and $\alpha_{i+1} - \frac{x_i+x_{i+1}}{2} = 0$ $(i=0,\ldots,k-1)$. □

Now, if we choose $I_k$ to be the equidistant partition of $[a,b]$, then we have the following corollary.

**Corollary 3.2.** *Let* $I_k : x_i = a+(b-a)\frac{i}{k}$ $(i=0,1,\ldots,k)$ *be the equidistant partitioning of* $[a,b]$ *and the* $\alpha_i'$*s be as in Corollary* 3.1*. Then the following inequality holds*

$$\begin{aligned}
&\left|\frac{1-\lambda}{2}\left[\frac{b-a}{k}\left(f(a)+f(b)\right) + \frac{2(b-a)}{k}\sum_{i=1}^{k-1} f\left(\frac{(k-i)a+bi}{k}\right)\right]\right.\\
&\quad + \frac{\lambda}{4}\left[\frac{b-a}{k}\left(f(a)+f(b)+f\left(\frac{(2k-1)a+b}{2k}\right)+f\left(\frac{a+(2k-1)b}{2k}\right)\right)\right.
\end{aligned}$$

$$+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}\left(f\left(\frac{(2k-2i+1)a+(2i-1)b}{2k}\right)\right.$$
$$\left.\left.+f\left(\frac{(2k-2i-1)a+(2i+1)b}{2k}\right)\right)\right]-\int_a^b f(t)dt\Bigg|$$
$$\leq\frac{b-a}{2k}(1+2\lambda)\|f'\|_1.$$

*Proof.* This follows from the second inequality in Corollary 3.1 and the following computations:

$$x_1-a=\frac{b-a}{k},\quad x_1+a=\frac{(2k-1)a+b}{k},$$
$$x_{i+1}-x_{i-1}=\frac{2(b-a)}{k}\quad(i=1,\dots,k-1),\quad \frac{x_i+x_{i-1}}{2}=\frac{(2k-2i+1)a+(2i-1)b}{k},$$
$$\frac{x_i+x_{i+1}}{2}=\frac{(2k-2i-1)a+(2i+1)b}{k}\quad(i=1,\dots,k-1),$$
$$b-x_{k-1}=\frac{b-a}{k},\quad b+x_{k-1}=\frac{a+(2k-1)b}{k},\quad h_i=\frac{b-a}{k},$$
$$(i=0,\cdots,k-1),\quad \tau_0=\frac{3(b-a)}{2k},\quad \tau_i=\frac{2(b-a)}{k}\quad(i=1,\dots,k-2)$$
$$\text{and}\quad \tau_{k-1}=\frac{3(b-a)}{2k}.$$

Thus, we deduce that $\nu(h)=\frac{b-a}{k}$ and $\nu(\tau)=\dfrac{2(b-a)}{k}$. ☐

**Corollary 3.3.** *Under the conditions of Theorem 2.2, if we choose $I_k$ to be the equidistant partitioning of $[a,b]$ and the $\alpha_i's$ be as in Corollary 3.1, then the following inequality holds;*

$$\left|\frac{1-\lambda}{2}\left[\frac{b-a}{k}(f(a)+f(b))+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}f\left(\frac{(k-i)a+bi}{k}\right)\right]\right.$$
$$+\frac{\lambda}{4}\left[\frac{b-a}{k}\left(f(a)+f(b)+f\left(\frac{(2k-1)a+b}{2k}\right)+f\left(\frac{a+(2k-1)b}{2k}\right)\right)\right.$$
$$+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}\left(f\left(\frac{(2k-2i+1)a+(2i-1)b}{2k}\right)\right.$$
$$\left.\left.+f\left(\frac{(2k-2i-1)a+(2i+1)b}{2k}\right)\right)\right]-\int_a^b f(t)dt\Bigg|$$
$$\leq\left[\frac{1}{(q+1)^{\frac{1}{q}}}+\lambda\right]\frac{(b-a)^{\frac{1}{q}+1}}{k}\|f'\|_p.$$

*Proof.* The result follows from Theorem 2.2 and using the computations in the proves of Corollaries 3.1 and 3.2. ☐

In what follows, we consider some special cases of Theorem 2.1. Similar results could also be derived from Theorem 2.2 as well.

**Corollary 3.4.** *Let $a, b \in \mathbb{R}$, $a < b$, $\lambda \in [0,1]$, $a \leq \alpha_1 \leq x \leq \alpha_2 \leq b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$ such that $f' \in L_1([a,b])$. Then the following inequalities hold*

$$\begin{aligned}
&\Big|(1-\lambda)\left[(\alpha_1 - a)f(a) + (\alpha_2 - \alpha_1)f(x) + (b-\alpha_2)f(b)\right] \\
&\quad + \frac{\lambda}{2}[(\alpha_1 - a)(f(a) + f(\alpha_1)) + (\alpha_2 - \alpha_1)(f(\alpha_1) + f(\alpha_2)) \\
&\quad + (b-\alpha_2)(f(\alpha_2) + f(b))] - \int_a^b f(t)dt\Big| \\
&\leq \Bigg[\frac{1}{2}\max\{x-a, b-x\} + \max\left\{\left|\alpha_1 - \frac{a+x}{2}\right|, \left|\alpha_2 - \frac{x+b}{2}\right|\right\} \\
&\quad + \frac{\lambda}{2}\left(\frac{1}{2}\max\{\alpha_2 - a, b-\alpha_1\} + \max\left\{\left|\alpha_1 - \frac{a+\alpha_2}{2}\right|, \left|\alpha_2 - \frac{\alpha_1+b}{2}\right|\right\}\right)\Bigg]\|f'\|_1 \\
&\leq \left(\max\{x-a, b-x\} + \frac{\lambda}{2}\max\{\alpha_2 - a, b-\alpha_1\}\right)\|f'\|_1.
\end{aligned}$$

*Proof.* The proof follows directly from Theorem 2.1 by choosing $k = 2$. □

**Corollary 3.5.** (a) *If we choose $\alpha_1 = a$ and $\alpha_2 = b$ in Corollary* 3.4*, then we have the inequality*

$$\begin{aligned}
&\left|(b-a)\left[(1-\lambda)f(x) + \frac{\lambda}{2}(f(a)+f(b))\right] - \int_a^b f(t)dt\right| \\
&\leq \left(\max\{x-a, b-x\} + \frac{\lambda}{2}(b-a)\right)\|f'\|_1 \\
&= \left(\frac{(1+\lambda)(b-a)}{2} + \left|x - \frac{a+b}{2}\right|\right)\|f'\|_1
\end{aligned}$$

*for all $x \in [a,b]$.*

(b) *If we choose $x = \frac{a+b}{2}$ in part* (a)*, then we have the following perturbed "midpoint inequality":*

$$\begin{aligned}
&\left|(b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \frac{\lambda}{2}(f(a)+f(b))\right] - \int_a^b f(t)dt\right| \\
&\leq \frac{(1+\lambda)(b-a)}{2}\|f'\|_1.
\end{aligned}$$

(c) *If we choose $\alpha_1 = \frac{5a+b}{6}$, $\alpha_2 = \frac{a+5b}{6}$ and $x_1 = x$ in Corollary* 3.4*, then we have*

$$\Bigg|(1-\lambda)\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x)\right]$$

$$
\begin{aligned}
&+\frac{\lambda(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+\frac{5}{2}f\left(\frac{5a+b}{6}\right)+\frac{5}{2}f\left(\frac{a+5b}{6}\right)\right]-\int_a^b f(t)dt\Bigg| \\
\leq&\left[\frac{1}{2}\max\{x-a,b-x\}+\frac{1}{2}\max\left\{\left|x-\frac{2a+b}{3}\right|,\left|x-\frac{a+2b}{3}\right|\right\}+\frac{\lambda}{3}(b-a)\right]\|f'\|_1 \\
\leq&\left(\max\{x-a,b-x\}+\frac{5\lambda(b-a)}{12}\right)\|f'\|_1 \\
=&\left[\frac{(b-a)(6+5\lambda)}{12}+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_1.
\end{aligned}
$$

(d) *In particular, if we choose* $x=\frac{a+b}{2}$ *in the first inequality in part* (c), *then we have the following perturbed "Simpson's inequality":*

$$
\begin{aligned}
&\Bigg|(1-\lambda)\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]+\frac{\lambda(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+\frac{5}{2}f\left(\frac{5a+b}{6}\right)\right.\\
&\left.+\frac{5}{2}f\left(\frac{a+5b}{6}\right)\right]-\int_a^b f(t)dt\Bigg|\leq\frac{(b-a)(1+\lambda)}{3}\|f'\|_1.
\end{aligned}
$$

## 4. Conclusion

Some new integral inequalities of Ostrowski type involving a parameter $\lambda\in[0,1]$ for functions whose derivatives belong to $L_p$ involving multiple points have been established. Some particular cases have be considered as examples. By considering different partitions, different points and/or different values of the parameter we will obtain several interesting inequalities. For $\lambda=0$, our results reduces to some results in the literature and for $\lambda\in(0,1]$, we obtain new results. It is worth noting that the Ostrowski inequality plays a very important role in numerical integration such as applications to the numerical quadrature rule. So, we believe that the inequalities obtained in this paper could be applied in numerical integration and other areas of Mathematical Analysis.

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## References

[1] G. A. Anastassiou, *Ostrowski type inequalities*, Proc. Amer. Math. Soc. **123**(12) (1995), 3775–3781. https://doi.org/10.2307/2161906

[2] S. S. Dragomir, *A generalization of Ostrowski integral inequality for mappings whose derivatives belong to* $L_1[a,b]$ *and applications in numerical integration*, J. Comput. Anal. Appl. **3** (2001), 343–360. https://doi.org/10.1023/A:1012050307412

[3] S. S. Dragomir, *A generalization of the Ostrowski integral inequality for mappings whose derivatives belong to* $L_p[a,b]$ *and applications in numerical integration*, J. Math. Anal. Appl. **255**(2) (2001), 605–626. https://doi.org/10.1006/jmaa.2000.7300

[4] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. **91** (2008), 450–460. https://doi.org/10.1007/s00013-008-2879-2

[5] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in $L_1$-norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. Math. **28** (1997), 239–244.

[6] S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett. **11** (1998), 105–109. https://doi.org/10.1016/S0893-9659(97)00142-0

[7] S. S. Dragomir and S. Wang, *A new inequality of Ostrowski's type in $L_p$-norm*, Indian J. Math. **40**(3) (1998), 299–304.

[8] M. A. Khan, S. Begum, Y. Khurshid and Y.-M. Chu, *Ostrowski type inequalities involving conformable fractional integrals*, J. Inequal. Appl. **2018**(70) (2018). https://doi.org/10.1186/s13660-018-1664-4

[9] S. Kermausuor, *A generalization of Ostrowski's inequality for functions of bounded variation via a parameter*, Australian J. Math. Anal. Appl. **16**(1) (2019), Article ID 16, 1–12.

[10] S. Kermausuor and E. R. Nwaeze, *A parameter-based Ostrowski type inequality on time scales for k points for functions with bounded second derivatives*, J. Math. Inequal. **12**(4) (2018), 1159–1172. https://dx.doi.org/10.7153/jmi-2018-12-88

[11] S. Kermausuor, E. R. Nwaeze and D. F. M. Torres, *Generalized weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales via a parameter function*, J. Math. Inequal. **11**(4) (2017), 1185–1199. https://dx.doi.org/10.7153/jmi-2017-11-88

[12] Y. Khurshid, M. A. Khan and Y.-M. Chu, *Ostrowski type inequalities involving conformable integrals via preinvex functions*, AIP Advances **10** (2020), Article ID 055204.

[13] W. J. Liu, Q. A. Ngô and W. Chen, *A new generalization of Ostrowski type inequality on time scales*, An. St. Univ. Ovidius Constanta **17**(2) (2009), 101–114.

[14] W. J. Liu and Q. A. Ngô, *A generalization of Ostrowski inequality on time scales for k points*, Appl. Math. Comput. **203**(2) (2008), 754–760. https://doi.org/10.1016/j.amc.2008.05.124

[15] E. R. Nwaeze and S. Kermausuor, *New Bounds of Ostrowski–Grüss type inequality for $(k+1)$ points on time scales*, Int. J. Anal. Appl. **15**(2) (2017), 211–221. https://doi.org/10.28924/2291-8639-15-2017-211

[16] E. R. Nwaeze, S. Kermausuor and A. M. Tameru, *New time scale generalizations of the Ostrowski–Grüss type inequality for k points*, J. Inequal. Appl. **2017** (2017), Article ID 245. https://doi.org/10.1186/s13660-017-1525-6

[17] A. M. Ostrowski, *Über die Absolutabweichung einer differentiebaren funktion von ihrem Integralmitelwert,* Comment. Math. Helv. **10** (1938), 226–227.

[18] G. Xu and Z. B. Fang, *A generalization of Ostrowski type inequality on time scales with k points*, J. Math. Inequal. **11**(1) (2017), 41–48. https://dx.doi.org/10.7153/jmi-11-04

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# ON MINIMAXITY AND LIMIT OF RISKS RATIO OF JAMES-STEIN ESTIMATOR UNDER THE BALANCED LOSS FUNCTION

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Abstract. The problem of estimating the mean of a multivariate normal distribution by different types of shrinkage estimators is investigated. Under the balanced loss function, we establish the minimaxity of the James-Stein estimator. When the dimension of the parameters space and the sample size tend to infinity, we study the asymptotic behavior of risks ratio of James-Stein estimator to the maximum likelihood estimator. The positive-part of James-Stein estimator is also treated.

## 1. Introduction

Stein [22] showed that the maximum likelihood estimator (MLE) of the mean $\theta = (\theta_1, \ldots, \theta_p)^\top$ of a multivariate Gaussian distribution $N_p(\theta, \sigma^2 I_p)$ is inadmissible in mean squared sense when the dimension of the parameters space $p \geq 3$. In particular, he proved the existence of a class of estimators which achieve the smaller total mean squared error regardless of the true $\theta$. Perhaps the best known estimator of such kind is James-Stein's estimator introduced by James and Stein [16]. This one is a special case of a larger class of estimators known as shrinkage estimators which is a combination of a model with low bias and high variance, and a model with high bias but low variance. Interestingly, the James-Stein estimator is itself inadmissible, and there exists a wide class of estimators that outperform the MLE, see for example, Lindley [18], Baranchik [1], Bhattacharya [7], Bock [8], Berger [5] and Berger and Wolpert [6]. Some of them, found some particular minimax estimators. Selahattin et al. [19] provided

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several alternative methods for derivation of the restricted ridge regression estimator (RRRE). Hansen [15] compared the mean-squared error of ordinary least squares (OLS), James-Stein, and least absolute shrinkage and selection operator (LASSO) shrinkage estimators and showed that neither James-Stein nor LASSO dominates uniformly the other. Xie et al. [24] introduced a class of semi-parametric/parametric shrinkage estimators and established their asymptotic optimality properties.

Casella and Hwang [9] have studied the estimation of the mean $\theta$ of the random variable $X \sim N_p(\theta, I_p)$ when the dimension of parameters space $p$ tends to infinity. They showed that if the limit of the ratio $\|\theta\|^2/p$ is a constant $c > 0$, then the risks ratio of the James-Stein estimator $\delta^{JS}$ and the positive-part of the James-Stein estimator $\delta^{JS+}$, to the MLE, tends to a constant value $c/(1+c)$. Benmansour and Hamdaoui [3] have taken the model $X \sim N_p(\theta, \sigma^2 I_p)$ where the parameter $\sigma^2$ is unknown and estimated by $S^2$ ($S^2 \sim \sigma^2\chi^2_n$). They established the same results given by Casella and Hwang [9]. Hamdaoui and Benmansour [12] considered the same model given by Benmansour and Hamdaoui [3], but this time, they studied the following class of shrinkage estimators $\delta_\phi = \delta^{JS} + l(S^2\phi(S^2, \|X\|^2)/\|X\|^2)X$ with $l$ is a real parameter. The authors showed that, when the sample size $n$ and the dimension of parameters space $p$ tend to infinity, the estimators $\delta_\phi$ have a lower bound $B_m = c/(1+c)$ and if the shrinkage function $\phi$ satisfies some conditions, the risks ratio $R(\delta_\phi, \theta)/R(X, \theta)$ attains this lower bound $B_m$, in particular the risks ratios $R(\delta^{JS}, \theta)/R(X, \theta)$ and $R(\delta^{JS+}, \theta)/R(X, \theta)$. Hamdaoui et al. [14] studied the limit of risks ratio of two forms of shrinkage estimators. The first one has been introduced by Benmansour and Mourid [4], $\delta_\psi = \delta^{JS} + l\left(S^2\psi\left(S^2, \|X\|^2\right)/\|X\|^2\right)X$, where $\psi(\cdot, u)$ is a function with support $[0, b]$, $b \in \mathbb{R}_+$ and satisfies some different conditions from the one given by Hamdaoui and Benmansour [12]. The second is the polynomial form of shrinkage estimator introduced by Li and Kio [17]. Hamdaoui and Mezouar [13] have treated the general class of shrinkage estimators $\delta_\phi = \left(1 - S^2\phi\left(S^2, \|X\|^2\right)/\|X\|^2\right)X$. They showed the same results given by Hamdaoui and Benmansour [12], with different conditions on the shrinkage function $\phi$. Benkhaled and Hamdaoui [2] have considered the model $X \sim N_p(\theta, \sigma^2 I_p)$ where $\sigma^2$ is unknown. They studied two different forms of shrinkage estimators of $\theta$: estimators of the form $\delta^\psi = (1 - \psi(S^2, \|X\|^2)S^2/\|X\|^2)X$, and estimators of Lindley-Type given by $\delta^\varphi = (1 - \varphi(S^2, T^2)S^2/T^2)(X - \overline{X}) + \overline{X}$ with $\overline{X} = (1/p)\sum_{i=1}^p X_i$ and $T^2 = \sum_{i=1}^p \left(X_i - \overline{X}\right)^2$, that shrink the components of the MLE $X$ to the random variable $\overline{X}$. The authors showed that if the shrinkage function $\psi$ (respectively $\varphi$) satisfies the new conditions different from the known results in the literature, then the estimator $\delta^\psi$ (respectively $\delta^\varphi$) is minimax. When the sample size and the dimension of parameters space tend to infinity, they studied the behavior of risks ratio of these estimators to the MLE. Hamdaoui et al. [11] have studied the minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the Bayesian case. The authors have considered the model $X \sim N_p(\theta, \sigma^2 I_p)$ where $\sigma^2$ is unknown and have taken the prior law $\theta \sim N_p(\upsilon, \tau^2 I_p)$. They constructed a modified Bayes estimator $\delta_B^*$ and an empirical modified Bayes estimator $\delta_{EB}^*$. When $n$ and $p$ are finite, they showed that the estimators $\delta_B^*$ and $\delta_{EB}^*$ are minimax. The authors have also interested in studying the limits of risks ratios of these estimators, to the MLE $X$, when $n$ and $p$ tend to infinity. The majority of these works has been considered under the quadratic loss function.

In the field of the estimation of a multivariate normal mean under the balanced loss function we cite for example, Farsipour and Asgharzadeh [10] have considered the model: $X_1, \ldots, X_n$ to be a random sample from a $N_p(\theta, \sigma^2)$ with $\sigma^2$ known and the aim is to estimate the parameter $\theta$. They studied the admissibility of the estimator of the form $a\overline{X} + b$ under the balanced loss function. Selahattin and Issam [20] introduced and derived the optimal extended balanced loss function (EBLF) estimators and predictors and discussed their performances.

In this work, we deal with the model $X \sim N_p(\theta, \sigma^2 I_p)$, where the parameter $\sigma^2$ is unknown and estimated by $S^2$ ($S^2 \sim \sigma^2 \chi_n^2$). Our aim is to estimate the unknown parameter $\theta$ by shrinkage estimators deduced by the MLE. The criterion adopted for comparing two estimators is the risk associated to the balanced loss function. The paper is organized as follows. In Section 2, we recall some preliminaries that are useful for our main results. In the first part of the Section 3, we study the minimaxity of the James-Stein estimator and the positive-part of James-Stein estimator. In the second part of this Section, we show that the positive-part of James-Stein estimator is not only minimax but also dominates the James-Stein estimator. In Section 4, we treat the asymptotic behavior of risks ratios of James-Stein estimator and the positive-part of the James-Stein estimator to the MLE, when the dimension $p$ tends to infinity and the sample size $n$ is fixed on one hand, and on the other hand when $p$ and $n$ tend simultaneously to infinity. We compute lower and upper bounds of each risks ratio, that allow us to calculate the limit of risks ratio. In Section 5, we graphically illustrate some obtained results. We end the manuscript by giving an Appendix which contains technical lemmas that are used in the proofs of our main results.

## 2. Preliminaries

We recall that if $X$ is a multivariate Gaussian random $N_p(\theta, \sigma^2 I_p)$ in $\mathbb{R}^p$, then $\frac{\|X\|^2}{\sigma^2} \sim \chi_p^2(\lambda)$ where $\chi_p^2(\lambda)$ denotes the non-central chi-square distribution with $p$ degrees of freedom and non-centrality parameter $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$.

In the next we also recall the following results that are useful in our proofs.

**Definition 2.1.** Let $U \sim \chi_p^2(\lambda)$. For any measurable function $f : \mathbb{R}_+ \to \mathbb{R}$, $\chi_p^2(\lambda)$ integrable, we have

$$\begin{aligned} E[f(U)] &= E_{\chi_p^2(\lambda)}[f(U)] \\ &= \int_{\mathbb{R}_+} f(u)\chi_p^2(\lambda)\, du \end{aligned}$$

$$= \sum_{k=0}^{+\infty} \left[ \int_{\mathbb{R}_+} f(u) \chi^2_{p+2k}(0)\, du \right] e^{-\dfrac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^k}{k!}$$
$$= \sum_{k=0}^{+\infty} \left[ \int_{\mathbb{R}_+} f(u) \chi^2_{p+2k} du \right] P\left(\frac{\lambda}{2}; dk\right),$$

where $P\left(\frac{\lambda}{2}\right)$ is a Poisson random variable with parameter $\frac{\lambda}{2}$ and $\chi^2_{p+2k}$ is the central chi-square distribution with $p+2k$ degrees of freedom.

From the Definition 2.1, we deduce that if $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, then for $p \geq 3$ we have

$$E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{\sigma^2} E\left(\frac{1}{p-2+2K}\right), \tag{2.1}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

**Lemma 2.1** ([23])**.** *Let $X$ be a $N\left(\upsilon, \sigma^2\right)$ real random variable and let $f : \mathbb{R} \to \mathbb{R}$ be an indefinite integral of the Lebesgue measurable function, $f'$ essentially the derivative of $f$. Suppose also that $E\left|f'(X)\right| < +\infty$. Then*

$$E\left[\left(\frac{X-\upsilon}{\sigma}\right) f(X)\right] = E\left(f'(X)\right).$$

Now, let $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where $\sigma^2$ is unknown and estimated by $S^2$ ($S^2 \sim \sigma^2 \chi^2_n$). And let the balanced loss function defined as: for any estimator $\delta$ of $\theta$

$$L_\omega(\delta, \theta) = \omega \|\delta - \delta_0\|^2 + (1-\omega)\|\delta - \theta\|^2, \tag{2.2}$$

where $0 \leq \omega < 1$ and $\delta_0$ is the MLE. We associate to this balanced loss function the risk function defined by

$$R_\omega(\delta, \theta) = E(L_\omega(\delta, \theta)). \tag{2.3}$$

In this model, it is clear that the MLE is $\delta_0 = X$, its risk function is $(1-\omega)p\sigma^2$.

Indeed, $R_\omega(X, \theta) = \omega E(\|X - X\|^2) + (1-\omega)E(\|X - \theta\|^2)$, where $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, then $\frac{X-\theta}{\sigma} \sim N_p\left(0, I_p\right)$, thus $\frac{\|X-\theta\|^2}{\sigma^2} \sim \chi^2_p$. Hence, $E(\|X-\theta\|^2) = E(\sigma^2 \chi^2_p) = \sigma^2 p$.

It is well known that $\delta_0$ is minimax and inadmissible for $p \geq 3$, thus any estimator which dominates it is also minimax.

## 3. Minimaxity

### 3.1. **James-Stein estimator.** Consider the estimator

$$\delta_a = \left(1 - a\frac{S^2}{\|X\|^2}\right) X = X - a\frac{S^2}{\|X\|^2} X, \tag{3.1}$$

where $a$ is a real parameter.

**Proposition 3.1.** *Under the balanced loss function $L_\omega$, we have:*

$$R_\omega(\delta_a,\theta) = (1-\omega)p\sigma^2 + [a^2\sigma^2 n(n+2) - 2a(1-\omega)\sigma^2 n(p-2)]E\left(\frac{1}{p-2+2K}\right),$$

*where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$.*

*Proof.*

$$R_\omega(\delta_a,\theta) = \omega E(\|\delta_a - X\|^2) + (1-\omega)E(\|\delta_a-\theta\|^2).$$

From the independence between two random variables $S^2$ and $\|X\|^2$, we obtain

$$\begin{aligned}
E(\|\delta_a - X\|^2) &= E\left(\| - a\frac{S^2}{\|X\|^2}X\|^2\right)\\
&= a^2E(S^2)E\left(\frac{1}{\|X\|^2}\right)\\
&= a^2E((\sigma^2\chi^2_n)^2)E\left(\frac{1}{\|X\|^2}\right)\\
&= a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right),
\end{aligned}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$ and the last equality according to the formula (2.1) and the fact that $E((\chi^2_n)^2) = n(n+2)$. Now,

$$\begin{aligned}
E(\|\delta_a - \theta\|^2) =& E\left(\|X - a\frac{S^2}{\|X\|^2}X - \theta\|^2\right)\\
=& E(\|X-\theta\|^2) + a^2E(S^2)^2E\left(\frac{1}{\|X\|^2}\right)\\
&- 2aE(S^2)E\left(\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right).
\end{aligned}$$

As

$$E\left(\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right) = E\left[\sum_{i=1}^p\left(y_i - \frac{\theta_i}{\sigma}\right)\frac{y_i}{\|y\|^2}\right],$$

where for any $i = 1,\dots,p$, $y_i = \frac{x_i}{\sigma} \sim N\left(\frac{\theta_i}{\sigma},1\right)$ and by using Lemma 2.1, we get

$$\begin{aligned}
E\left[\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right] &= \sum_{i=1}^p E\left(\frac{\partial}{\partial y_i}\frac{1}{\sum_{j=1}^p y_j^2}y_i\right)\\
&= \sum_{i=1}^p E\left[\frac{1}{\|y\|^2} - \frac{2y_i^2}{\|y\|^4}\right]\\
&= (p-2)E\left(\frac{1}{\|y\|^2}\right)
\end{aligned}$$

$$= (p-2)E\left(\frac{1}{p-2+2K}\right),$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$ and the last equality comes from formula (2.1). Thus,

$$\begin{aligned}
R_\omega(\delta_a,\theta) =& \omega a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right)\\
&+(1-\omega)\left[p\sigma^2 + a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right)\right]\\
&-2a(1-\omega)\sigma^2 n(p-2)E\left(\frac{1}{p-2+2K}\right)\\
&=(1-\omega)p\sigma^2 + [a^2\sigma^2 n(n+2) - 2a(1-\omega)\sigma^2 n(p-2)]E\left(\frac{1}{p-2+2K}\right). \square
\end{aligned}$$

Using Proposition 3.1, we note that under the balanced loss function $L_\omega$, a sufficient condition so that $\delta_a$ dominating the MLE $X$ is

$$a \geq 0 \quad \text{and} \quad a(n+2) - 2(1-\omega)(p-2) \leq 0,$$

which is equivalent to

$$0 \leq a \leq \frac{2(1-\omega)(p-2)}{n+2}. \tag{3.2}$$

From Proposition 3.1 and the convexity of risk function $R_\omega(\delta_a,\theta)$ on $a$, one can easily show that the optimal value of $a$ that minimizes the risk function $R_\omega(\delta_a,\theta)$ is

$$\alpha = \frac{(1-\omega)(p-2)}{n+2}.$$

For $a = \alpha$, we obtain the James-Stein estimator

$$\delta_{JS} = \delta_\alpha = \left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X = \left(1 - \frac{(1-\omega)(p-2)}{n+2}\frac{S^2}{\|X\|^2}\right)X. \tag{3.3}$$

It follows from Proposition 3.1 that the risk function of $\delta_{JS}$ is given by

$$R_\omega(\delta_{JS},\theta) = (1-\omega)p\sigma^2 - (1-\omega)^2(p-2)^2\frac{n}{n+2}\sigma^2 E\left(\frac{1}{p-2+2K}\right), \tag{3.4}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$.

From the formula (3.4), it is easy to see that $R_\omega(\delta_{JS},\theta) \leq R_\omega(X,\theta)$, then the James-Stein estimator $\delta_{JS}$ dominates the MLE $X$, and thus it is minimax.

3.2. **Positive-Part of James-Stein estimator.** We consider the positive-part of James-Stein estimator defined by

$$\delta_{JS}^{+} = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{+} X = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right) X \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \leq 1}, \tag{3.5}$$

where $\left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{+} = \max\left(0, 1 - \alpha \frac{S^2}{\|X\|^2}\right)$. We recall that

$$\delta_{JS}^{-} = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{-} X = \left(\alpha \frac{S^2}{\|X\|^2} - 1\right) X \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}, \tag{3.6}$$

where $\mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}$ is the indicating function of the set $\left(\alpha \frac{S^2}{\|X\|^2} \geq 1\right)$.

We note that the positive-part of James-Stein estimator $\delta_{JS}^{+}$ has the form (3.1), corresponding to $a^{+} = \min\left\{\frac{(1-\omega)(p-2)}{n+2}, \frac{S^2}{\|X\|^2}\right\}$. Since $a^{+}$ satisfies the relation (3.2), $\delta_{JS}^{+}$ dominates the MLE $X$ under the balanced loss function $L_\omega$, thus $\delta_{JS}^{+}$ is minimax.

3.3. **Dominating the positive-part of James-Stein estimator to James-Stein estimator.** It is well known that the positive-part of James-Stein estimator dominates the James-Stein estimator for the standard case where $\omega = 0$ (see Baranchick [1]). In this part, we show that this property remains valid for any $0 < \omega < 1$.

**Theorem 3.1.** *Under the balanced loss function $L_\omega$, the positive-part of James-Stein estimator $\delta_{JS}^{+}$ dominates the James-Stein estimator $\delta_{JS}$.*

*Proof.*

$$R_\omega(\delta_{JS}^{+}, \theta) = \omega E(\|\delta_{JS}^{+} - X\|^2) + (1-\omega) E(\|\delta_{JS}^{+} - \theta\|^2)$$

and

$$R_\omega(\delta_{JS}, \theta) = \omega E(\|\delta_{JS} - X\|^2) + (1-\omega) E(\|\delta_{JS} - \theta\|^2).$$

Baranchick [1] has showed that $E(\|\delta_{JS}^{+} - \theta\|^2) \leq E(\|\delta_{JS} - \theta\|^2)$ for $p \geq 3$ and all $(\theta, \sigma) \in (\mathbb{R}^p \times \mathbb{R}^{+})$. Then $\delta_{JS}^{+}$ dominates $\delta_{JS}$ under the balanced loss function $L_\omega$, if and only if $E(\|\delta_{JS}^{+} - X\|^2) - E(\|\delta_{JS} - X\|^2) \leq 0$. Now,

$$\begin{aligned}
E(\|\delta_{JS}^{+} - X\|^2) =& E(\|\delta_{JS}^{+} - \delta_{JS} + \delta_{JS} - X\|^2) \\
=& E(\|\delta_{JS}^{+} - \delta_{JS}\|^2) + E(\|\delta_{JS} - X\|^2) + 2E[\langle \delta_{JS}^{+} - \delta_{JS}, \delta_{JS} - X\rangle] \\
=& E(\|\delta_{JS}^{-}\|^2) + E(\|\delta_{JS} - X\|^2) + 2E[\langle \delta_{JS}^{-}, \delta_{JS} - X\rangle] \\
=& E\left[\left\|\left(\alpha \frac{S^2}{\|X\|^2} - 1\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1} X\right\|^2\right] + E(\|\delta_{JS} - X\|^2) \\
&+ 2E\left[\left\langle \left(\alpha \frac{S^2}{\|X\|^2} - 1\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1} X, -\alpha \frac{S^2}{\|X\|^2} X\right\rangle\right] \\
=& E\left[\left(\alpha^2 \frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}\right] + E(\|\delta_{JS} - X\|^2)
\end{aligned}$$

$$- 2E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right].$$

Then

$$\begin{aligned}
&E(\|\delta_{JS}^{+} - X\|^2) - E(\|\delta_{JS} - X\|^2)\\
=&E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right] - 2E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]\\
=&E\left[\left(\|X\|^2 - \alpha^2\frac{S^4}{\|X\|^2}\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]\\
=&E\left[\left(\frac{1}{\|X\|^2}(\|X\|^2 - \alpha S^2)(\|X\|^2 + \alpha S^2)\right)\mathbb{I}_{(\|X\|^2-\alpha S^2)\leq 0}\right]\\
\leq&0.
\end{aligned}$$

□

## 4. Limits of Risks Ratios

4.1. **Bounds and limit of the risks ratio of James-Stein estimator.** In this part, we study the limit of risks ratio of the James-Stein estimator $\delta_{JS}$ to the MLE $X$, when the dimension $p$ tends to infinity and the sample size $n$ is fixed on one hand, and on the other hand when $p$ and $n$ tend simultaneously to infinity. The following lemma gives a lower and an upper bounds of the ratio $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$, which helps us to calculate the limit of risks ratio.

**Lemma 4.1.** *Assume the estimator $\delta_{JS}$ given in* (3.3). *Under the balanced loss function $L_\omega$, we have*

$$1 - \frac{n(1-\omega)(p-2)}{(n+2)(p+\frac{\|\theta\|^2}{\sigma^2})} \leq \frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} \leq 1 - \frac{n(1-\omega)(p-2)^2}{(n+2)p(p-2+\frac{\|\theta\|^2}{\sigma^2})}.$$

*Proof.* From Lemma 2.1 of Hamdaoui and Benmansour [12], we have

$$\frac{1}{p-2+\frac{\|\theta\|^2}{\sigma^2}} \leq E\left(\frac{1}{p-2+2K}\right) \leq \frac{p}{(p-2)(p+\frac{\|\theta\|^2}{\sigma^2})}.$$

Using the formula (3.4), we obtain the desired result. □

**Theorem 4.1.** *Assume the estimator $\delta_{JS}$ given in* (3.1), *if* $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$ $(c>0)$, *then*

i) $\lim_{p\to\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} = \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}$;

ii) $\lim\limits_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} = \frac{\omega+c}{1+c}$.

*Proof.* i) Using Lemma 4.1 and under the condition $\lim\limits_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, we have

$$\lim_{p\to\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} \leq 1-(1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{(p-2)^2}{p}\frac{\frac{1}{p}}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$

$$
\begin{aligned}
&= 1-(1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{(p-2)^2}{p^2}\frac{1}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]\\
&= 1-(1-\omega)\frac{n}{n+2}\frac{1}{1+c}\\
&= \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{p\to\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} &\geq 1-(1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{\frac{p-2}{p}}{\frac{p}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]\\
&= 1-(1-\omega)\frac{n}{n+2}\frac{1}{1+c}\\
&= \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}.
\end{aligned}
$$

ii) From Lemma 4.1 and under the condition $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2}=c$, we obtain

$$
\begin{aligned}
\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} &\leq 1-(1-\omega)\lim_{\substack{p\to\infty\\ n\to\infty}}\left[\frac{n}{n+2}\frac{(p-2)^2}{p}\frac{\frac{1}{p}}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]\\
&= 1-(1-\omega)\frac{1}{1+c}=\frac{\omega+c}{1+c}
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} &\geq 1-(1-\omega)\lim_{\substack{p\to\infty\\ n\to\infty}}\left[\frac{n}{n+2}\frac{\frac{p-2}{p}}{\frac{p}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]\\
&= 1-(1-\omega)\frac{1}{1+c}=\frac{\omega+c}{1+c}.
\end{aligned}
$$

□

*Remark* 4.1. As $0\leq\omega<1$, then

$$
\frac{1-\frac{n}{n+2}+c}{1+c}\leq\frac{(1-(1-\omega)\frac{n}{n+2}+c}{1+c}<1
$$

and $c/(1+c)\leq(\omega+c)/(1+c)<1$, thus for $p$ tends to infinity and $n$ is fixed, or for $p$ and $n$ tend simultaneously to infinity, the limit of risks ratio of James-Stein estimator $\delta_{JS}$ to the MLE $X$, is less than 1. Therefore, Theorem 4.1 show the stability of minimaxity property of James-Stein estimator $\delta_{JS}$ for the large values of $n$ and $p$.

4.2. **Bounds and limit of the risks ratio of the positive-part of James-Stein estimator.** The results for the positive-part of James-Stein estimator $\delta^+_{JS}$ are similar to those for the ordinary James-Stein estimator $\delta_{JS}$, although the calculations are a bit more difficult. In the following proposition, we give the explicit formula of the risk function of $\delta^+_{JS}$.

**Proposition 4.1.** *The risk function of the Positive-Part of James-Stein estimator $\delta^+_{JS}$ under the balanced loss function $L_\omega$, is*

$$\begin{aligned} R_\omega(\delta^+_{JS},\theta) =& R_\omega(\delta_{JS},\theta) \\ &+ E\left[\left(\|X\|^2 - \alpha^2\frac{S^4}{\|X\|^2} + 2(1-\omega)\sigma^2(p-2)\alpha\frac{S^2}{\|X\|^2} - p\sigma^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]. \end{aligned}$$

*Proof.*

$$\begin{aligned} R_\omega(\delta^+_{JS},\theta) =& \omega E(\|\delta^+_{JS} - X\|^2) + (1-\omega)E(\|\delta^+_{JS} - \theta\|^2) \\ =& \omega E(\|\delta^+_{JS} - \delta_{JS} + \delta_{JS} - X\|^2) + (1-\omega)E(\|\delta^+_{JS} - \delta_{JS} + \delta_{JS} - \theta\|^2) \\ =& \omega E[\|\delta^+_{JS} - \delta_{JS}\|^2 + \|\delta_{JS} - X\|^2 + 2\langle \delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] \\ &+ (1-\omega)E[\|\delta^+_{JS} - \delta_{JS}\|^2 + \|\delta_{JS} - \theta\|^2 + 2\langle \delta^+_{JS} - \delta_{JS}, \delta_{JS} - X + X - \theta\rangle] \\ =& [\omega E(\|\delta_{JS} - X\|^2) + (1-\omega)E(\|\delta_{JS} - \theta\|^2)] + E[\|\delta^+_{JS} - \delta_{JS}\|^2] \\ &+ 2E[\langle \delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle + 2(1-\omega)\langle \delta^+_{JS} - \delta_{JS}, X - \theta\rangle] \\ =& R_\omega(\delta_{JS},\theta) + E[\|\delta^+_{JS} - \delta_{JS}\|^2] + 2E[\langle \delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] \\ &+ 2(1-\omega)E[\langle \delta^+_{JS} - \delta_{JS}, X - \theta\rangle]. \end{aligned}$$

Now, we compute the expectations in the right side hand of the last equality.

$$\begin{aligned} E[\|\delta^+_{JS} - \delta_{JS}\|^2] &= E[\|\delta^-_{JS}\|^2] \\ &= E\left[\left\|\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X\right\|^2\right] \\ &= E\left[\left(\alpha^2\frac{S^4}{\|X\|^4} + 1 - 2\alpha\frac{S^2}{\|X\|^2}\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\|X\|^2\right] \\ &= E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right], \end{aligned} \tag{4.1}$$

$$\begin{aligned} E[\langle \delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] &= E[\langle \delta^-_{JS}, \delta_{JS} - X\rangle] \\ &= E\left[\left\langle\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X, -\alpha\frac{S^2}{\|X\|^2}X\right\rangle\right] \\ &= -E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right], \end{aligned} \tag{4.2}$$

and by using Lemma 2.1 of Shao and Strawdermen [21], we have

$$\begin{aligned} E[\langle \delta^+_{JS} - \delta_{JS}, X - \theta\rangle] &= E\left[\left\langle\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X, X - \theta\right\rangle\right] \\ &= \sigma^2 E\left[\left((p-2)\alpha\frac{S^2}{\|X\|^2} - p\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]. \end{aligned} \tag{4.3}$$

According to the formulas (4.1), (4.2) and (4.3) we get the desired result. □

In the Theorem 3.1 we showed that $R_\omega(\delta_{JS}^+,\theta) \leq R_\omega(\delta_{JS},\theta))$ for $p \geq 3$ and all $(\theta,\sigma) \in (\mathbb{R}^p \times \mathbb{R}^+)$, then the upper bound given in Lemma 4.1 plays the role of the upper bound of $R_\omega(\delta_{JS}^+,\theta)/R_\omega(X,\theta)$. Thus for calculate the limit of risks ratio $R_\omega(\delta_{JS}^+,\theta)/R_\omega(X,\theta)$, it suffices to determine a lower bound. The following proposition gives a lower bound of risks ratio $R_\omega(\delta_{JS}^+,\theta)/R_\omega(X,\theta)$.

**Proposition 4.2.** *For all $p \geq 3$, we have the following lower bound of the risks ratio* $\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}$

$$\begin{aligned}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)} \geq & \frac{R_\omega(\delta_{JS},\theta)}{(1-\omega)p\sigma^2} + \frac{p+\lambda}{(1-\omega)p}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p+4}^2(\lambda,du) \\ & -\frac{4}{p}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p-2}^2(\lambda,du) \\ & -\frac{(p-2)n}{(1-\omega)p(n+2)}\int_0^{+\infty}\mathbb{P}\left(\chi_{n+4}^2 \geq \frac{u}{\alpha}\right)\chi_{p-2}^2(\lambda,du). \end{aligned} \tag{4.4}$$

*Proof.* As $\frac{\|X\|^2}{\sigma^2} \sim \chi_p^2(\lambda)$ and $\frac{S^2}{\sigma^2} \sim \chi_n^2$, where $\lambda = \frac{\|\theta\|^2}{\sigma^2}$, we have

$$\begin{aligned}\sigma^2 E\left(\|X\|^2 \mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) =& \sigma^2 E\left(\chi_p^2(\lambda)\mathbb{I}_{\chi_n^2 \geq \frac{\chi_p^2(\lambda)}{\alpha}}\right) \\ =& \sigma^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}\chi_n^2(0,dt)\right)u\chi_p^2(\lambda,du) \\ =& \sigma^2 p\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p+2}^2(\lambda,du) \\ & +\sigma^2\lambda\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p+4}^2(\lambda,du).\end{aligned}$$

The last equality is obtained by using the formula (6.1) of Lemma 6.1 Appendix with $h(u) = \int_{\frac{u}{\alpha}}^{+\infty}\chi_n^2(0,dt)$. As the function $\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)$ is non increasing on $u$ and using the formula (6.2) of Lemma 6.2, we obtain

$$\sigma^2 E\left(\|X\|^2 \mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) \geq q\sigma^2(p+\lambda)\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p+4}^2(\lambda,du), \tag{4.5}$$

$$\begin{aligned}\sigma^2 E\left\{\left(2(p-2)\frac{\alpha S^2}{\|X\|^2} - 2p\right)\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right\} \geq & -4\sigma^2 E\left(\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) \\ =& -4\sigma^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}\chi_n^2(0,dt)\right)\chi_p^2(\lambda,du) \\ =& -4\sigma^2\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_p^2(\lambda,du) \\ \geq & -4\sigma^2\int_0^{+\infty}\mathbb{P}\left(\chi_n^2 \geq \frac{u}{\alpha}\right)\chi_{p-2}^2(\lambda,du).\end{aligned} \tag{4.6}$$

The last inequality comes from formula (6.1). Now,

$$\begin{aligned} E\left(-\frac{\alpha^2 S^4}{\|X\|^2}\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) &= -\sigma^2\alpha^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)\right)\frac{1}{u}\chi_p^2(\lambda,du) \\ &\geq -\frac{\sigma^2\alpha}{n+2}\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)\right)\chi_{p-2}^2(\lambda,du). \end{aligned}$$

The last inequality comes from formula (6.1), taking $h(u)=\frac{1}{u}\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)$. However, using formula (6.1) again, we get

$$\begin{aligned} \int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt) &= n\int_{\frac{u}{\alpha}}^{+\infty} t\chi_{n+2}^2(0,dt) \\ &= n(n+2)\int_{\frac{u}{\alpha}}^{+\infty}\chi_{n+4}^2(0,dt) \\ &= n(n+2)\mathbb{P}\left(\chi_{n+4}^2\geq\frac{u}{\alpha}\right), \end{aligned}$$

thus, we have

$$E\left(-\frac{\alpha^2 S^4}{\|X\|^2}\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right)\geq -\sigma^2\alpha n\int_0^{+\infty}\mathbb{P}\left(\chi_{n+4}^2\geq\frac{u}{\alpha}\right)\chi_{p-2}^2(\lambda,du), \tag{4.7}$$

combining to the formulas (4.5), (4.6) and (4.7), we get the desired result. □

**Theorem 4.2.** *Assume the estimator $\delta_{JS}^+$ given in* (3.5), *if* $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2}=c$ $(c>0)$, *then*

i) $\lim\limits_{p\to\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}=\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c}$;

ii) $\lim\limits_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}=\frac{\omega+c}{1+c}$.

*Proof.* In the one hand, from Theorem 3.1, we showed that $R_\omega(\delta_{JS}^+,\theta)\leq R_\omega(\delta_{JS},\theta)$ for $p\geq 3$ and all $(\theta,\sigma)\in(\mathbb{R}^p\times\mathbb{R}^+)$ and using Theorem 4.1, we have

$$\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}\leq\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c} \tag{4.8}$$

and

$$\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}\leq\frac{\omega+c}{1+c}. \tag{4.9}$$

In the other hand, when $p$ tends to infinity and $n$ is fixed, we have $\alpha=\frac{(1-\omega)(p-2)}{n+2}$ tending to $+\infty$. According to the Lebesque's Theorem by taking for example, the increasing sequel with $p$ $\left(f_p(u)=\int_{\frac{u}{\alpha}}^{+\infty}\chi_n^2(0,dt)=\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\right)$ and the fact that

$$\lim_{p\to+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)=\mathbb{P}\left(\chi_n^2\geq 0\right)=1,\quad\text{for all } n\geq 1,$$

we obtain

$$\lim_{p\to+\infty}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du)=1. \tag{4.10}$$

In the case where $p$ and $n$ tend simultaneously to infinity, we have

$$\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)=\mathbb{P}\left(\sum_{i=1}^n y_i^2\geq\frac{u(n+2)}{p-2}\right)=\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n y_i^2\geq\frac{u}{p-2}+\frac{2u}{n(p-2)}\right),$$

where $y_1,y_2,\dots,y_n$ are independent Gaussian random variables centered and reduced. Then by the strong law of large numbers, we have

$$\begin{aligned}\lim_{\substack{p\to\infty\\ n\to\infty}}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)&=\lim_{\substack{p\to\infty\\ n\to\infty}}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n y_i^2\geq\frac{u}{p-2}+\frac{2u}{n(p-2)}\right)\\&=\lim_{\substack{p\to\infty\\ n\to\infty}}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n y_i^2\geq 0\right)\\&=\mathbb{P}(1\geq 0)=1.\end{aligned}$$

Thus,

$$\lim_{\substack{p\to\infty\\ n\to\infty}}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du)=\int_0^{+\infty}\chi^2_{p+4}(\lambda,du)=1. \tag{4.11}$$

Using Proposition 4.2, formulas (4.10) and (4.11) and the condition

$$\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2}=\lim_{p\to\infty}\frac{\lambda}{p}=c,$$

leads to

$$\begin{aligned}\lim_{p\to+\infty}\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)}&\geq\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\lim_{p\to+\infty}\left[\frac{p+\lambda}{(1-\omega)p}-\frac{4}{p}-\frac{(p-2)n}{(1-\omega)p(n+2)}\right]\\&=\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\frac{1-\frac{n}{n+2}}{1-\omega}+\frac{c}{1-\omega}\\&\geq\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}\end{aligned}$$

and

$$\begin{aligned}\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)}&\geq\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\lim_{\substack{p\to\infty\\ n\to\infty}}\left[\frac{p+\lambda}{(1-\omega)p}-\frac{4}{p}-\frac{(p-2)n}{(1-\omega)p(n+2)}\right]\\&=\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\frac{c}{1-\omega}\\&\geq\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}.\end{aligned}$$

It follows from Theorem 4.1 that

$$\lim_{p\to+\infty}\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)}\geq\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c} \tag{4.12}$$

and

$$\lim_{\substack{p\to\infty \\ n\to\infty}} \frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)} \geq \frac{\omega+c}{1+c}. \tag{4.13}$$

Combining formulas (4.8), (4.9), (4.12) and (4.13) we get the desired result. □

## 5. Simulation Results

First, we illustrate graphically the risks ratios of the James-Stein estimator $\delta_{JS}$ and the positive-part of James-Stein estimator $\delta^+_{JS}$ to the MLE $X$ as a function of $\lambda = \|\theta\|^2/(2\sigma^2)$ for various values of $n$, $p$ and $\omega$. Secondly, we give the tables that show the values of risks ratios of the James-Stein estimator $\delta_{JS}$ and the positive-part of James-Stein estimator $\delta^+_{JS}$ to the MLE $X$ according to divers values of $\lambda = \|\theta\|^2/(2\sigma^2)$ but this time we fix $n$ and $p$ and vary $\omega$.

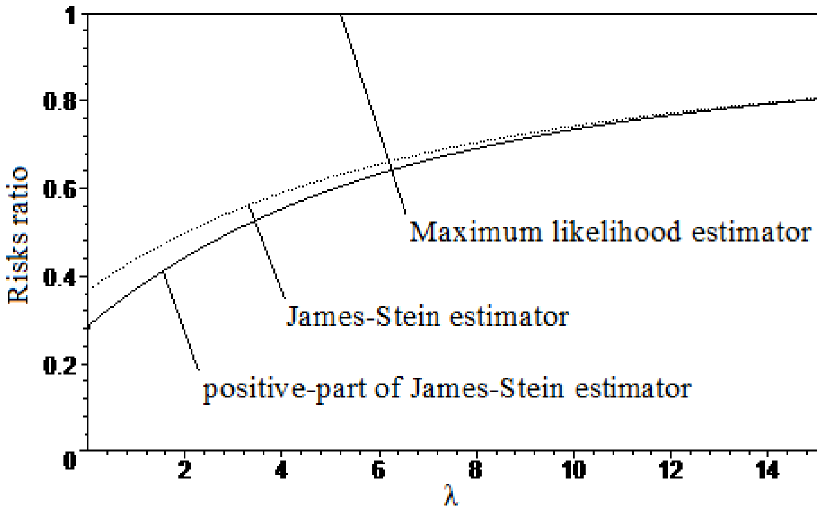


Figure 1. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n = 30,\ p = 8$ and $\omega = 0.1$

Figures 1–6 show that the risks ratios of the James-Stein estimator $\delta_{JS}$ and the positive-part of James-Stein estimator $\delta^+_{JS}$ to the MLE $X$ are less than 1, thus the estimators $\delta_{JS}$ and $\delta^+_{JS}$ dominate the MLE $X$ for large values of $n$ and $p$. We also observe that the gain increases if $\omega$ is near to 0 and decreases if $\omega$ is near to 1. Tables 1-6 illustrate this note.

In Table 1 and Table 2, we give the values of ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ for $n = 50$ and $p = 10$ and $n = 100$ and $p = 10$, respectively for divers values of $\lambda = (\|\theta\|^2)/(2\sigma^2)$ and $\omega$.

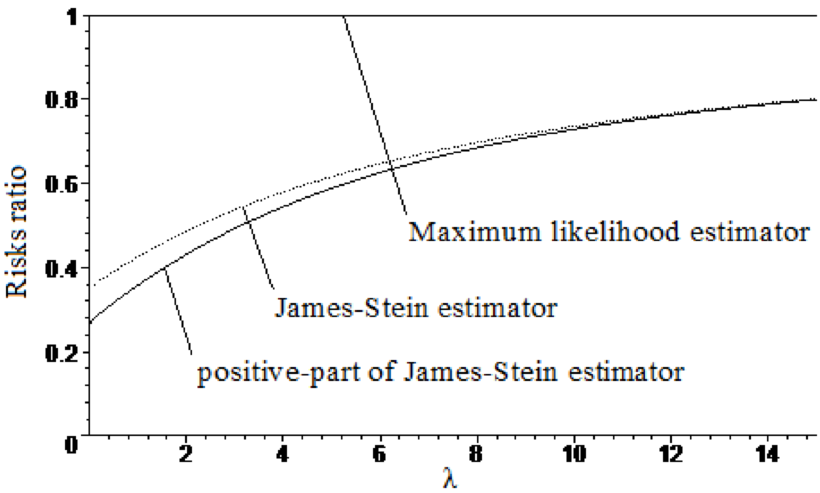


FIGURE 2. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n=50,\ p=8$ and $\omega=0.1$

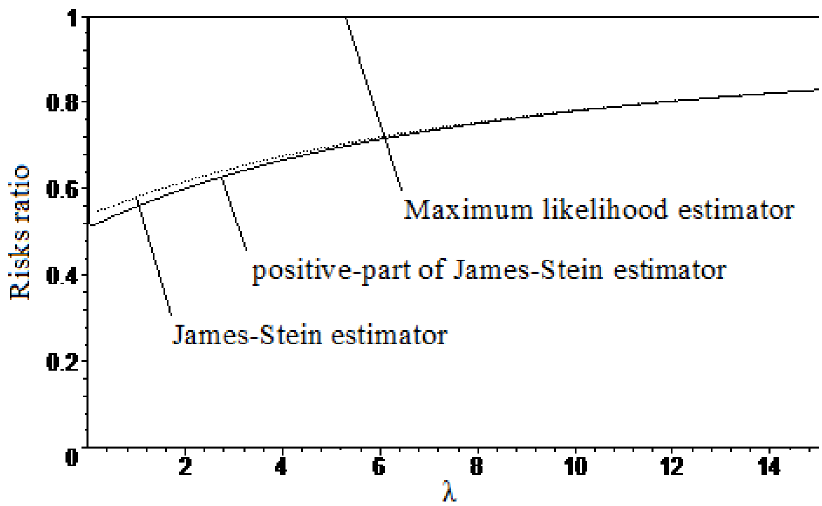


FIGURE 3. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n=50,\ p=10$ and $\omega=0.4$

From Tables 1–2, first, for any values of $\omega$ and $\lambda = \|\theta\|^2/(2\sigma^2)$, the ratio $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ is less than the ratio $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$, which shows that the positive-part of James-Stein estimator $\delta_{JS}^{+}$ dominates the James-Stein estimator $\delta_{JS}$. Secondly, on the one hand, if $\omega$ and $\lambda = \|\theta\|^2/(2\sigma^2)$ are small, the ratios are close to 0 than 1, and therefore the gain is very important. On the other hand, as much as $\omega$ goes to 1, the gain will be small and the risks ratios are almost equal. In the case

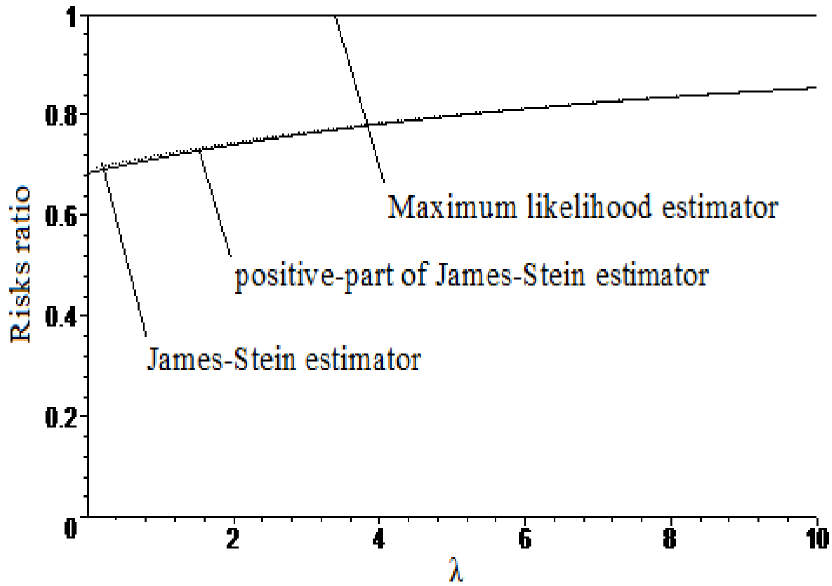


FIGURE 4. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n = 50,\ p = 10$ and $\omega = 0.6$

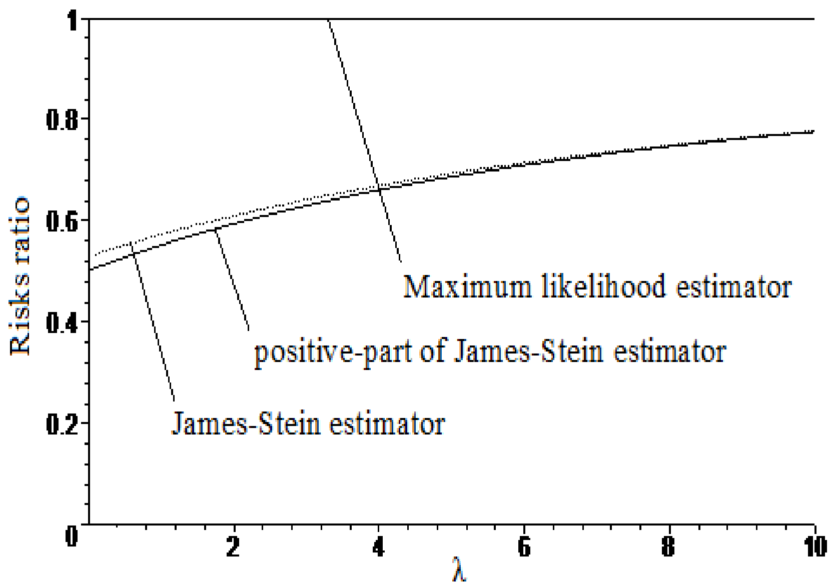


FIGURE 5. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n = 100,\ p = 10$ and $\omega = 0.4$

$\omega$ is near to 1 and $\lambda$ is large, the gain is almost equal to zero and the risks ratios are the same.

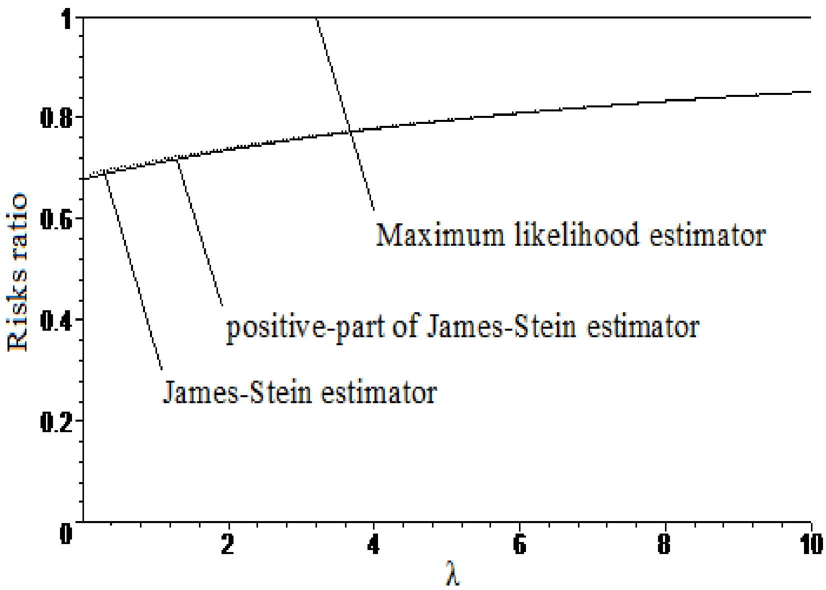


FIGURE 6. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n=100,\ p=10$ and $\omega=0.6$

## CONCLUSION

In this work, we established the minimaxity of the James-Stein estimator $\delta_{JS}$ and the positive-part of James-Stein estimator $\delta_{JS}^{+}$ of a multivariate normal mean distribution $X\sim N_p\left(\theta,\sigma^2 I_p\right)$ under the balanced loss function. If the limit of the ratio $\|\theta\|^2/p$ is a constant $c>0$, the risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ tend to the values less than 1, thus we ensured the stability of the minimaxity property of the James-Stein estimator $\delta_{JS}$ and the positive-part of James-Stein estimator $\delta_{JS}^{+}$ even if the dimension of the parameter spaces $p$ and the sample size $n$ tend to infinity. An extension of this work is to obtain the similar results in the case where the model has a symmetrical spherical distribution.

## 6. APPENDIX

**Lemma 6.1** (Bock [8]). *Let $X\sim N_p(\theta,I_p)$ where $X=(X_1,\dots,X_p)^\top$ and $\theta=(\theta_1,\dots,\theta_p)^\top$, then for any measurable function $h:[0,+\infty[\,\to\mathbb{R}$*

$$E\left(h\left(\|X\|^2\right)X_i^2\right)=E\left[h\left(\chi^2_{p+2}\left(\|\theta\|^2\right)\right)\right]+\theta_i^2E\left[h\left(\chi^2_{p+4}\left(\|\theta\|^2\right)\right)\right].$$

*Moreover,*

$$\begin{aligned}E\left(h\left(\|X\|^2\right)\|X\|^2\right)&=E\left[\chi^2_p\left(\|\theta\|^2\right)h\left(\chi^2_p\left(\|\theta\|^2\right)\right)\right]\\&=pE\left[h\left(\chi^2_{p+2}\left(\|\theta\|^2\right)\right)\right]+\|\theta\|^2E\left[h\left(\chi^2_{p+4}\left(\|\theta\|^2\right)\right)\right].\end{aligned}\tag{6.1}$$

**Lemma 6.2** ([3]). *Let $f$ is a real function. If for $p\geq 3$, $E_{\chi^2_p(\lambda)}\left[f(U)\right]$ exists, then*

TABLE 1. The values of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of $\lambda$ for $n=50$ and $p=10$

| $\lambda$ | risks ratio | $\omega=0.1$ | $\omega=0.2$ | $\omega=0.3$ | $\omega=0.4$ | $\omega=0.6$ | $\omega=0.8$ | $\omega=0.9$ |
|---|---|---|---|---|---|---|---|---|
| 0.4 | $\delta_{JS}$ | 0.3105 | 0.3871 | 0.4637 | 0.5403 | 0.6936 | 0.8468 | 0.9234 |
| | $\delta^+_{JS}$ | 0.2416 | 0.3671 | 0.4261 | 0.5015 | 0.6854 | 0.8462 | 0.9223 |
| 1 | $\delta_{JS}$ | 0.3715 | 0.4414 | 0.5112 | 0.5810 | 0.7207 | 0.8603 | 0.9302 |
| | $\delta^+_{JS}$ | 0.3124 | 0.3949 | 0.4775 | 0.5590 | 0,7148 | 0.8600 | 0.9302 |
| 2 | $\delta_{JS}$ | 0.4260 | 0.4898 | 0.5536 | 0.6173 | 0.7449 | 0.8724 | 0.9362 |
| | $\delta^+_{JS}$ | 0.3766 | 0.4522 | 0.5272 | 0.6007 | 0.7408 | 0.8722 | 0.9362 |
| 3 | $\delta_{JS}$ | 0,4728 | 0,5314 | 0,5900 | 0,6485 | 0,7657 | 0,8828 | 0,9414 |
| | $\delta^+_{JS}$ | 0,4321 | 0,5014 | 0,5696 | 0,6360 | 0,7628 | 0,8827 | 0,9414 |
| 5 | $\delta_{JS}$ | 0,5486 | 0,5987 | 0,6489 | 0,6991 | 0,7994 | 0,8997 | 0,9498 |
| | $\delta^+_{JS}$ | 0,5220 | 0,5802 | 0,6370 | 0,6922 | 0,7980 | 0,8996 | 0,9498 |
| 10 | $\delta_{JS}$ | 0,6719 | 0,7084 | 0,7448 | 0,7813 | 0,8542 | 0,9271 | 0,9635 |
| | $\delta^+_{JS}$ | 0,6640 | 0,7034 | 0,7420 | 0,7798 | 0,8540 | 0,9271 | 0,9635 |
| 15 | $\delta_{JS}$ | 0,7443 | 0,7727 | 0,8011 | 0,8296 | 0,8864 | 0,9432 | 0,9716 |
| | $\delta^+_{JS}$ | 0,7422 | 0,7715 | 0,8005 | 0,8293 | 0,8863 | 0,9432 | 0,9716 |
| 20 | $\delta_{JS}$ | 0,7912 | 0,8144 | 0,8376 | 0,8608 | 0,9072 | 0,9536 | 0,9768 |
| | $\delta^+_{JS}$ | 0,7907 | 0,8141 | 0,8375 | 0,8607 | 0,9072 | 0,9536 | 0,9768 |
| 25 | $\delta_{JS}$ | 0,8238 | 0,8434 | 0,8630 | 0,8825 | 0,9217 | 0,9608 | 0,9804 |
| | $\delta^+_{JS}$ | 0,8237 | 0,8433 | 0,8629 | 0,8825 | 0,9217 | 0,9608 | 0,9804 |
| 30 | $\delta_{JS}$ | 0,8477 | 0,8646 | 0,8816 | 0,8985 | 0,9323 | 0,9662 | 0,9831 |
| | $\delta^+_{JS}$ | 0,8477 | 0,8646 | 0,8815 | 0,8985 | 0,9323 | 0,9662 | 0,9831 |

a) *if $f$ is monotone non-increasing we have*

$$E_{\chi^2_{p+2}(\lambda)}\left[f(U)\right] \leq E_{\chi^2_p(\lambda)}\left[f(U)\right]; \tag{6.2}$$

b) *if $f$ is monotone non-decreasing we have*

$$E_{\chi^2_{p+2}(\lambda)}\left[f(U)\right] \geq E_{\chi^2_p(\lambda)}\left[f(U)\right]. \tag{6.3}$$

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TABLE 2. The values of risks ratios $R_\omega(\delta_{JS}, \theta)/R_\omega(X, \theta)$ and $R_\omega(\delta^+_{JS}, \theta)/R_\omega(X, \theta)$ as functions of $\lambda$ for $n = 100$ and $p = 10$

| $\lambda$ | risks ratio | $\omega = 0.1$ | $\omega = 0.2$ | $\omega = 0.3$ | $\omega = 0.4$ | $\omega = 0.6$ | $\omega = 0.8$ | $\omega = 0.9$ |
|---|---|---|---|---|---|---|---|---|
| 0.4 | $\delta_{JS}$ | 0.2970 | 0.3751 | 0.4532 | 0.5313 | 0.6876 | 0.8438 | 0.9219 |
| | $\delta^+_{JS}$ | 0.2092 | 0.3207 | 0.4025 | 0.5040 | 0.6800 | 0.8433 | 0.9219 |
| 1 | $\delta_{JS}$ | 0,3592 | 0,4304 | 0,5016 | 0,5728 | 0,7152 | 0,8576 | 0,9288 |
| | $\delta^+_{JS}$ | 0,3020 | 0,3857 | 0,4694 | 0,5519 | 0,7098 | 0,8573 | 0,9288 |
| 2 | $\delta_{JS}$ | 0,4147 | 0,4798 | 0,5448 | 0,6098 | 0,7399 | 0,8699 | 0,9350 |
| | $\delta^+_{JS}$ | 0,3671 | 0,4439 | 0,5198 | 0,5942 | 0,7361 | 0,8697 | 0,9350 |
| 3 | $\delta_{JS}$ | 0,4625 | 0,5222 | 0,5819 | 0,6416 | 0,7611 | 0,8805 | 0,9403 |
| | $\delta^+_{JS}$ | 0,4235 | 0,4937 | 0,5627 | 0,6300 | 0,7585 | 0,8804 | 0,9403 |
| 5 | $\delta_{JS}$ | 0,5397 | 0,5909 | 0,6420 | 0,6932 | 0,7954 | 0,8977 | 0,9489 |
| | $\delta^+_{JS}$ | 0,5146 | 0,5735 | 0,6309 | 0,6868 | 0,7942 | 0,8977 | 0,9489 |
| 10 | $\delta_{JS}$ | 0,6655 | 0,7027 | 0,7398 | 0,7770 | 0,8513 | 0,9257 | 0,9628 |
| | $\delta^+_{JS}$ | 0,6582 | 0,6982 | 0,7373 | 0,7757 | 0,8511 | 0,9257 | 0,9628 |
| 15 | $\delta_{JS}$ | 0,7393 | 0,7683 | 0,7972 | 0,8262 | 0,8841 | 0,9421 | 0,9710 |
| | $\delta^+_{JS}$ | 0,7375 | 0,7672 | 0,7967 | 0,8260 | 0,8841 | 0,9421 | 0,9710 |
| 20 | $\delta_{JS}$ | 0,7871 | 0,8108 | 0,8344 | 0,8581 | 0,9054 | 0,9527 | 0,9763 |
| | $\delta^+_{JS}$ | 0,7867 | 0,8105 | 0,8343 | 0,8580 | 0,9054 | 0,9527 | 0,9763 |
| 25 | $\delta_{JS}$ | 0,8203 | 0,8403 | 0,8603 | 0,8802 | 0,9202 | 0,9601 | 0,9800 |
| | $\delta^+_{JS}$ | 0,8203 | 0,8403 | 0,8603 | 0,8802 | 0,9202 | 0,9601 | 0,9800 |
| 30 | $\delta_{JS}$ | 0,8447 | 0,8620 | 0,8792 | 0,8965 | 0,9310 | 0,9655 | 0,9827 |
| | $\delta^+_{JS}$ | 0,8447 | 0,8620 | 0,8792 | 0,8965 | 0,9310 | 0,9655 | 0,9827 |

## REFERENCES


[1] A. J. Baranchik, *Multiple Regression and Estimation of the Mean of a Multivariate Normal Distribution*, Technical Report **51**(1), Stanford University, Stanford, 1964.

[2] A. Benkhaled and A. Hamdaoui, *General classes of shrinkage estimators for the multivariate normal mean with unknown variancee: Minimaxity and limit of risks ratios*, Kragujevac J. Math. **46**(2) (2020), 193–213.

[3] D. Benmansour and A. Hamdaoui, *Limit of the ratio of risks of James-Stein estimators with unknown variance*, Far East Journal of Theoretical Statistics **36**(1) (2011), 31–53.

[4] D. Benmansour and T. Mourid, *Etude d'une classe d'estimateurs avec rétrécisseur de la moyenne d'une loi gaussienne*, Ann. I.S.U.P. **51** (2007), 83–106.

[5] J. Berger, *Admissible minimax estimation of a multivariate normal mean with arbitrary quadratic loss*, Ann. Statist. **4**(1) (1976), 223–226. `https://doi.org/10.1214/aos/1176343356`

[6] J. Berger and R. Wolpert, *Estimating the mean function of a Gaussian process and the Stein effect*, J. Multivariate Anal. **13**(3) (1983), 401–424. `https://doi.org/10.1016/0047-259X(83)90018-0`

[7] P. K. Bhattacharya, *Estimating the mean of a multivariate normal population with general quadratic loss function*, The Annals of Mathematical Statistics **37**(6) (1966), 1819–1824. `https://doi.org/10.1214/aoms/1177699174`

[8] M. E. Bock, *Minimax estimators of the mean of a multivariate normal distribution*, Ann. Statist. **3**(1) (1975), 209–218. `https://doi.org/10.1214/aos/1176343009`

[9] G. Casella and J. T. Hwang, *Limit expressions for the risk of James-Stein estimators*, Canad. J. Statist. **10**(4) (1982), 305–309. `https://doi.org/10.2307/3556196`

[10] N. S. Farsipour and A. Asgharzadeh, *Estimation of a normal mean relative to balanced loss functions*, Statist. Papers **45**(2) (2004), 279–286. `https://doi.org/10.1007/BF02777228`

[11] A. Hamdaoui and A. Benkhaled and N. Mezouar, *Minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the bayesian case*, Stat. Optim. Inf. Comput. **8**(2) (2020), 507–520. `https://doi.org/10.19139/soic-2310-5070-735`

[12] A. Hamdaoui and D. Benmansour, *Asymptotic properties of risks ratios of shrinkage estimators*, Hacet. J. Math. Stat. **44**(5) (2015), 1181–1195. `https://doi.org/10.15672/HJMS.2014377624`

[13] A. Hamdaoui and N. Mezouar, *Risks ratios of shrinkage estimators for the multivariate normal mean*, Journal of Mathematics and Statistics **13**(2) (2017), 77–87. `https://doi.org/10.3844/jmssp.2017.77.87`

[14] A. Hamdaoui, N. Mezouar, D. Benmansour and D. Bouguenna, *Examples of shrinkage estimators of the mean, dominating the maximum likelihood estimator in larges dimension*, IOSR Journal of Mathematics **12**(3) (2016), 14–28. `https://doi.org/10.9790/5728-1203041428`

[15] B. E. Hansen, *The risk of James-Stein and lasso shrinkage*, Econometric Rev. **35** (2016), 1465–1470. `https://doi.org/10.1080/07474938.2015.1092799`

[16] W. James and C. Stein, *Estimation with quadratic loss*, in: *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, University of California Press, Berkeley, 1961, 361–379.

[17] T. F. Li and W. H. Kio, *Generalized James-Stein estimators*, Comm. Statist. Theory Methods **11**(20) (1982), 2249–2257. `https://doi.org/10.1080/03610928208828386`

[18] D. V. Lindley, *Discussion on Professor Stein's paper*, J. R. Stat. Soc. Ser. B. Stat. Methodol. **24** (1962), 285–287.

[19] K. Selahattin, S. Sadullah, M. R. Özkale and H. Güler, *More on the restricted ridge regression estimation*, J. Stat. Comput. Simul. **81**(11) (2011), 1433–1448. `https://doi.org/10.1080/00949655.2010.491480`

[20] K. Selahattin and D. Issam, *The optimal extended balanced loss function estimators*, J. Comput. Appl. Math. **345** (2019), 86–98. `https://doi.org/10.1016/j.cam.2018.06.021`

[21] P. Shao and W. E. Strawderman, *Improving on the James-Stein positive-part estimator of the multivariate normal mean vector for the case of common unknown variances*, Ann. Statist. **22**(3) (1994), 1517–1539.

[22] C. Stein, *Inadmissibilty of the usual estimator for the mean of a multivariate normal distribution*, in: *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, University of California Press, Berkeley, 1956, 197–206.

[23] C. Stein, *Estimation of the mean of a multivariate normal distribution*, Ann. Statist. **9**(6) (1981), 1135–1151. `https://doi.org/10.1214/aos/1176345632`

[24] X. Xie, S. C. Kou and L. Brown, *Optimal shrinkage estimators of mean parameters in family of distribution with quadratic variance*, Ann. Statist. **44**(2) (2016), 564–597. `https://doi.org/10.1214/15-AOS1377`

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# SOME RESULTS CONCERNED WITH HANKEL DETERMINANT

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Abstract. In this paper, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{K}(\alpha)$ class. Also, for the function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ defined in the unit disc such that $f \in \mathcal{K}(\alpha)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point $z_0$ with $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. That is, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$. The sharpness of this inequality is also proved.

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are analytic in $E = \{z : |z| < 1\}$. Also, $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha\right| < 1, \tag{1.1}$$

where $\alpha \in \mathbb{C}$. There are a lot of interesting studies regarding inequality (1.1) [16,17,24].

The certain analytic functions which is in the class of $\mathcal{K}(\alpha)$ on the unit disc $E$ are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{K}(\alpha)$ by applying Schwarz lemma. Schwarz lemma is a highly popular topic in electrical engineering. As exemplary applications, the use of positive real functions and boundary analysis of these functions for circuit synthesis can be given. Moreover, it is also possible to

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utilize Schwarz lemma for the analysis of transfer functions in control engineering and to design multi-notch filter structures in signal processing [14, 15].

Let $f \in \mathcal{A}$. The $q^{\text{th}}$ Hankel determinant of $f$ for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [23] as

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \quad c_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2.$$

Similarly, for $u = z - z_1$ and $f \in \mathcal{A}$, we have

$$D_s(m) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+s-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+s-1} & a_{m+s} & \cdots & a_{m+2s-2} \end{vmatrix}, \quad a_1 = 1.$$

From the Hankel determinant for $m = 1$ and $s = 2$, we have

$$D_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Here, the Hankel determinant $H_2(1) = c_3 - c_2^2$ and $D_2(1) = a_3 - a_2^2$ are well-known as Fekete-Szegö functional [22]. In [23], authors have obtained the upper bounds of the Hankel determinant $|c_2c_4 - c_3^2|$. Also, in [20], author have obtained the upper bounds the Hankel determinant $A_n^{(k)}$. Moreover, in [21], authors have given bounds for the Second Hankel determinant for class $\mathcal{M}_\alpha$. In [1], Schwarz lemma at the boundary has been examined for a class $\mathcal{K}$ of analytic functions, and the modulus of the second derivative has been estimated from below in terms of Hankel determinants $H_2(1)$.

We will obtain consideration for $f''(z)$ from below by using $H_2(1)$ and $D_2(1)$ determinants. In this consideration, the coefficients in Taylor expansion of $f(z)$ at $z = 0$ and $z = z_1$ points are used. The functions we use for our main results are as follows. The relationship between the Fekete-Szegö function, that is the Hankel determinant $H_2(1)$, and the second derivative of the function will be considered. In this consideration, the Taylor coefficients that form the analytic function $f(z)$ and the coefficients that form the Hankel determination will be correlated. In this correlation, Schwarz lemma and its results will be used.

Let $f \in \mathcal{K}(\alpha)$ and consider the following function

$$t(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha = 1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots .$$

It is an analytic function in $E$ and $t(0) = 1 - \alpha$. Consider the function

$$T(z) = \frac{R(z)}{\frac{z-z_1}{1-\overline{z_1}z}}, \quad R(z) = \frac{t(z) - t(0)}{1 - \overline{t(0)}t(z)}.$$

Here, $T(z)$ is an analytic function in $E$, $T(0) = 0$ and $|T(z)| < 1$ for $z \in E$.

Several studies on Schwarz lemma exist in literature as it has a wide applicability area. Some examples are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc which is also called as boundary version of Schwarz lemma. The classical Schwarz lemma implies the inequality

$$|f'(z_0)| \geq 1 \tag{1.2}$$

which is known as the Schwarz lemma on the boundary, and also as a part of the Lindelöf principle. The inequality (1.2) and its generalizations have important aplications in geometric theory of functions [2–7, 13–15, 15, 18]. Mercer [10] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [11]. In addition, he obtains an new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [12]. In [9], authors have given simple proofs of various versions of the Schwarz lemma for real-valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly HQR) mappings with the strip codomain. In [8], the authors have given different applications of the Schwarz lemma and the Jack lemma.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [19]). In addition, the second derivative of the function $f(z)$ will be considered from below. Therefore, here the existence of the second derivative gives the result of the Julia-Wolff lemma.

**Lemma 1.1** (Julia-Wolff lemma)**.** *Let $f$ be an analytic function in $E$, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function $f$ has an angular limit $f(z_0)$ at $z_0 \in \partial E$, $|f(z_0)| = 1$, then the angular derivative $f'(z_0)$ exists and $1 \leq |f'(z_0)| \leq \infty$.*

**Corollary 1.1.** *The analytic function $f$ has a finite angular derivative $f'(z_0)$ if and only if $f'$ has the finite angular limit $f'(z_0)$ at $z_0 \in \partial E$.*

## 2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{K}(\alpha)$ class. Also, for the function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ defined in the unit disc such that $f \in \mathcal{K}(\alpha)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point $z_0$ with $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. That is, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$. The sharpness of this inequality is also proved. Motivated by the results of the work presented in [2], the following result has been obtained.

**Theorem 2.1.** *Let $f \in \mathcal{K}(\alpha)$ and $\left(\frac{z_1}{f(z_1)}\right)^2 f'(z_1) = 1$ for $0 < |z_1| < 1$. Suppose that, for some $z_0 \in \partial E$, $f$ has an angular limit $f(z_0)$ at $z_0$, $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. Then we have the inequality*

(2.1)
$$
\begin{aligned}
|f''(z_0)| \geq & \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+2\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{\left(1-|1-\alpha|^2\right)|z_1|^2-|H_2(1)|}{\left(1-|1-\alpha|^2\right)|z_1|^2+|H_2(1)|}\right.\\
& \left.\times\left[1+\frac{A}{B}\cdot\frac{1-|z_1|^2}{|1-z_1|^2}\right]\right),
\end{aligned}
$$

*where*

$$
\begin{aligned}
A =& \left(1-|1-\alpha|^2\right)^2|z_1|^4+|D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)|-\left(1-|1-\alpha|^2\right)^2|D_2(1)||z_1|\\
&-\left(1-|1-\alpha|^2\right)|H_2(1)||z_1|,\\
B =& \left(1-|1-\alpha|^2\right)^2|z_1|^4+|D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)|+\left(1-|1-\alpha|^2\right)^2|D_2(1)||z_1|\\
&+\left(1-|1-\alpha|^2\right)|H_2(1)||z_1|.
\end{aligned}
$$

*This result is sharp for $\alpha \in \mathbb{R}$, with equality for each possible value of $|H_2(1)|$ and $|D_2(1)|$.*

*Proof.* Let

$$q(z) = \frac{z-z_1}{1-\overline{z_1}z}.$$

In addition, let $h : E \to E$ be an analytic and a point $z_1 \in E$ in order to satisfy

$$\left|\frac{h(z)-h(z_1)}{1-\overline{h(z_1)}h(z)}\right| \leq \left|\frac{z-z_1}{1-\overline{z_1}z}\right| = |q(z)|$$

and

$$|h(z)| \leq \frac{|h(z_1)|+|q(z)|}{1+|h(z_1)||q(z)|}, \tag{2.2}$$

by Schwarz-pick lemma [6]. If $p : E \to E$ is analytic function and $0 < |z_1| < 1$, letting

$$h(z) = \frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}$$

in (2.2), we obtain

$$\left|\frac{p(z)-p(0)}{z\left(1-\overline{p(0)}p(z)\right)}\right| \leq \frac{\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right|+|q(z)|}{1+\left|\frac{p(z_1)-p(0)}{z_1\left(1-\overline{p(0)}p(z_1)\right)}\right||q(z)|}$$

and

$$|p(z)| \leq \frac{|p(0)| + |z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}{1 + |p(0)|\,|z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}, \tag{2.3}$$

where

$$C = \frac{p(z_1) - p(0)}{z_1\left(1 - \overline{p(0)}p(z_1)\right)}.$$

Without loss of generality, we will assume that $z_0 = 1$. If we take

$$p(z) = \frac{R(z)}{z^2\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^2},$$

then

$$p(0) = \frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}, \quad p(z_1) = \frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}$$

and

$$C = \frac{\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2} + \frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}}{z_1\left(1 + \frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}\frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right)},$$

where $|C| \leq 1$. Let $|p(0)| = \beta$ and

$$\mathrm{T} = \frac{\left|\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}\right| + \left|\frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right|}{|z_1|\left(1 + \left|\frac{D_2(1)\left(1-|z_1|^2\right)^2}{\left(1-|1-\alpha|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-|1-\alpha|^2\right)z_1^2}\right|\right)}.$$

From (2.3), we get

$$|R(z)| \leq |z|^2 |q(z)|^2 \frac{\beta + |z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}$$

and

$$\frac{1-|R(z)|}{1-|z|} \geq \frac{1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|} - \beta\,|z|^2 |q(z)|^2 - |q(z)|^2 |z|^3 \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{(1-|z|)\left(1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}\right)}.$$

Let $\kappa(z) = 1 + \beta\,|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}$ and $\tau(z) = 1 + \mathrm{T}\,|q(z)|$. Therefore, we obtain

$$\begin{aligned} \frac{1-|R(z)|}{1-|z|} \geq & \frac{1}{\kappa(z)\tau(z)}\left\{\frac{1-|z|^3|q(z)|^3}{1-|z|} + \mathrm{T}\,|q(z)|\frac{1-|z|^3|q(z)|}{1-|z|}\right. \\ & \left. + \beta\,|z|\,|q(z)|\frac{1-|z|\,|q(z)|}{1-|z|} + \beta\,|z|\,\mathrm{T}\frac{1-|z|\,|q(z)|}{1-|z|}\right\}. \end{aligned} \tag{2.4}$$

Since

$$\lim_{z\to 1}\kappa(z)=\lim_{z\to 1}\left(1+\beta\left|z\right|\frac{\mathrm{T}+\left|q(z)\right|}{1+\mathrm{T}\left|q(z)\right|}\right)=1+\beta,$$
$$\lim_{z\to 1}\tau(z)=\lim_{z\to 1}\left(1+\mathrm{T}\left|q(z)\right|\right)=1+\mathrm{T},$$
$$\lim_{z\to 1}\frac{1-\left|z\right|^{i}\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^{j}}{1-\left|z\right|}=i+j\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2},$$

for nonnegative integers $i$ and $j$ and

$$1-\left|q(z)\right|^2=1-\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^2=\frac{\left(1-\left|z_1\right|^2\right)\left(1-\left|z\right|^2\right)}{\left|1-\overline{z_1}z\right|^2},$$

passing to the angular limit in (2.4) gives

$$\begin{aligned}\left|R'(1)\right|\geq&\frac{2}{(1+\beta)\left(1+\mathrm{T}\right)}\left(3+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}+\mathrm{T}\left[3+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]\right.\\&\left.+\beta\left[1+\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]+\beta\mathrm{T}\left[1+3\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right]\right)\\=&2+2\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}+\frac{1-\beta}{1+\beta}\left[1+\frac{1-\mathrm{T}}{1+\mathrm{T}}\frac{1-\left|z_1\right|^2}{\left|1-z_1\right|^2}\right].\end{aligned}$$

Moreover, since

$$\frac{1-\beta}{1+\beta}=\frac{1-\left|p(0)\right|}{1+\left|p(0)\right|}=\frac{1-\frac{\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2}}{1+\frac{\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2}}=\frac{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2-\left|H_2(1)\right|}{\left(1-\left|1-\alpha\right|^2\right)\left|z_1\right|^2+\left|H_2(1)\right|},$$

$$\frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{1-\frac{\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)}}{1+\frac{\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)}}$$

and

$$\frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)-\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|-\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}{\left|z_1\right|\left(1+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|\right)+\left|\frac{D_2(1)\left(1-\left|z_1\right|^2\right)^2}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|+\left|\frac{H_2(1)}{\left(1-\left|1-\alpha\right|^2\right)z_1^2}\right|}\cdot\frac{A_1}{B_1},$$

where

$$A_1=\left(1-\left|1-\alpha\right|^2\right)^2\left|z_1\right|^4+\left|D_2(1)\right|\left(1-\left|z_1\right|^2\right)^2\left|H_2(1)\right|-\left(1-\left|1-\alpha\right|^2\right)^2\left|D_2(1)\right|\left|z_1\right|$$

$$- \left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|\,,$$

$$B_1 = \left(1-|1-\alpha|^2\right)^2|z_1|^4 + |D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)| + \left(1-|1-\alpha|^2\right)^2|D_2(1)|\,|z_1| + \left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|\,,$$

we obtain

$$|R'(1)| \geq 2 + 2\frac{1-|z_1|^2}{|1-z_1|^2} + \frac{\left(1-|1-\alpha|^2\right)|z_1|^2 - |H_2(1)|}{\left(1-|1-\alpha|^2\right)|z_1|^2 + |H_2(1)|}\left[1 + \frac{A_2}{B_2}\cdot\frac{1-|z_1|^2}{|1-z_1|^2}\right],$$

where

$$A_2 = \left(1-|1-\alpha|^2\right)^2|z_1|^4 + |D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)| - \left(1-|1-\alpha|^2\right)^2|D_2(1)|\,|z_1| - \left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|\,,$$

$$B_2 = \left(1-|1-\alpha|^2\right)^2|z_1|^4 + |D_2(1)|\left(1-|z_1|^2\right)^2|H_2(1)| + \left(1-|1-\alpha|^2\right)^2|D_2(1)|\,|z_1| + \left(1-|1-\alpha|^2\right)|H_2(1)|\,|z_1|\,.$$

From definition of $R(z)$, we have

$$R'(z) = \frac{1-|t(0)|^2}{\left(1-\overline{t(0)}t(z)\right)^2}t'(z)$$

and

$$|R'(1)| = \frac{1-|1-\alpha|^2}{|\alpha|^2}\,|f''(1)|\,|1+\alpha|^2\,.$$

Thus, we obtain the inequality (2.1).

In order to show that the inequality (2.1) is sharp, choose arbitrary real numbers $z_1, x$ and $y$ such that $0 < x < \left(1-|1-\alpha|^2\right)|z_1|^2$, $0 < y < \frac{\left(1-|1-\alpha|^2\right)|z_1|^2}{\left(1-|z_1|^2\right)^2}$.

Let

$$\mathtt{K} = \frac{\frac{y}{z_1^2}\left(1-|z_1|^2\right)^2 - \frac{x}{z_1^2}}{z_1\left(1-\left(1-|z_1|^2\right)^2\frac{y}{z_1^2}\frac{x}{z_1^2}\right)}.$$

Let

$$R(z) = z^2\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^2\frac{\frac{x}{z_1^2} + z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1+\frac{x}{z_1^2}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}. \tag{2.5}$$

From (2.5), with the simple calculations, we obtain

$$\frac{R''(0)}{2!} = x, \quad \frac{R''(z_1)}{2!} = y$$

and

$$R'(1)=2+2\frac{1-|z_1|^2}{(1-z_1)^2}+\frac{z_1^2-x}{z_1^2+x}\left[1+\frac{z_1^4-y\left(1-|z_1|^2\right)^2x-y\left(1-|z_1|^2\right)^2z_1+xz_1}{z_1^4-y\left(1-|z_1|^2\right)^2x+y\left(1-|z_1|^2\right)^2z_1-xz_1}\right].$$

Choosing suitable signs of the numbers $z_1$, $x$ and $y$, we conslude from the last equality that the inequality (2.1) is sharp. □

## References


[1] T. Akyel, *Upper bound of Hankel determinant for a class of analytic functions*, Filomat (to appear).

[2] T. Akyel and B. N. Ornek, *Some remarks on Schwarz lemma at the boundary*, Filomat **31**(13) (2017), 4139–4151. https://doi.org/10.2298/FIL1713139A

[3] T. A. Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Var. Elliptic Equ. **58** (2013), 571–577. https://doi.org/10.1080/17476933.2012.718338

[4] H. P. Boas, *Julius and Julia: Mastering the art of the Schwarz lemma*, Amer. Math. Monthly **117** (2010), 770–785.

[5] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci. **122** (2004), 3623–3629. https://doi.org/10.1023/B:JOTH.0000035237.43977.39

[6] G. M. Golusin, *Geometric Theory of Functions of Complex Variable* (in Russian), 2nd Edition, American Math. Soc., Moscow, 1966.

[7] M. Mateljević, *Rigidity of holomorphic mappings & Schwarz and Jack lemma*, (to appear). https://doi.org/10.13140/RG.2.2.34140.90249

[8] M. Mateljević, N. Mutavdžić and B. N. Örnek, *Estimates for some classes of holomorphic functions in the unit disc*, Appl. Anal. Discrete Math. (to appear).

[9] M. Mateljević and M. Svetlik, *Hyperbolic metric on the strip and the Schwarz lemma for HQR mappings*, Appl. Anal. Discrete Math. **14** (2020), 150–168. https://doi.org/10.2298/AADM200104001M

[10] P. R. Mercer, *Sharpened versions of the Schwarz lemma*, J. Math. Anal. Appl. **205** (1997), 508–511. https://doi.org/10.1006/jmaa.1997.5217

[11] P. R. Mercer, *Boundary Schwarz inequalities arising from Rogosinski's lemma*, J. Class. Anal. **12** (2018), 93–97. https://doi.org/10.7153/jca-2018-12-08

[12] P. R. Mercer, *An improved Schwarz lemma at the boundary*, Open Math. **16** (2018), 1140–1144. https://doi.org/10.1515/math-2018-0096

[13] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. **128** (2000), 3513–3517. https://doi.org/10.1090/S0002-9939-00-05463-0

[14] B. N. Örnek and T. Düzenli, *Schwarz lemma for driving point impedance functions and its circuit applications*, International Journal of Circuit Theory and Applications **47** (2019), 813–824. https://doi.org/10.1002/cta.2616

[15] B. N. Örnek and T. Düzenli, *Boundary analysis for the derivative of driving point impedance functions*, IEEE Transactions on Circuits and Systems II: Express Briefs **65**(9) (2018), 1149–1153. https://doi.org/10.1109/TCSII.2018.2809539

[16] M. Obradović ana S. Ponnusamy, *On the class $U$*, in: *Proceedings of the 21st Annual Conference of the Jammu Mathematical Society and a National Seminar on Analysis and its Application*, Jamuu, 2011, 11–26.

[17] M. Obradović ana S. Ponnusamy, *Radius properties for subclasses of univalent functions*, Analysis (Munich) **25**(2005), 183–188. https://doi.org/10.1524/anly.2005.25.3.183

[18] B. N. Örnek, *A sharp Carathéodory's inequality on the right half plane*, J. Class. Anal. **14**(1) (2019), 39–48. https://doi.org/10.7153/jca-2019-14-04

[19] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.
[20] Ch. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika **14**(1967), 108–112. https://doi.org/10.1112/S002557930000807X
[21] J. Sokol and D. K. Thomas, *The second Hankel determinant for alpha-convex functions*, Lith. Math. J. (to appear). https://doi.org/10.1007/s10986-018-9397-0
[22] G. Szegö and M. Fekete, *Eine bemerkung uber ungerade schlichte funktionen*, J. Lond. Math. Soc. **2** (1933) 85–89. https://doi.org/10.1112/jlms/s1-8.2.85
[23] D. K. Thomas and J. W. Noonan, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc. **223** (1976), 337–346. https://doi.org/10.2307/1997533
[24] A. Vasudevarao and H. Yanagihara, *On the growth of analytic functions in the class $U(\alpha)$*, Comput. Methods Funct. Theory **13** (2013), 613–634. https://doi.org/10.1007/s40315-013-0045-8

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