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ON THE SEMIGROUP OF BI-IDEALS OF AN ORDERED SEMIGROUP

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Abstract. The purpose of this paper is to characterize an ordered semigroup S in terms of the properties of the associated semigroup $\mathcal{B}(S)$ of all bi-ideals of S. We show that an ordered semigroup S is a Clifford ordered semigroup if and only if $\mathcal{B}(S)$ is a semilattice. The semigroup $\mathcal{B}(S)$ is a normal band if and only if the ordered semigroup S is both regular and intra regular. For each subvariety $\mathcal{V}$ of bands, we characterize the ordered semigroup S such that $\mathcal{B}(S) \in \mathcal{V}$.

1. Introduction and Preliminaries

The passage from semigroup without order to ordered semigroup is not straightforward. Regular rings and semigroups have been influenced many authors to study the order structure on regular semigroups as well as to introduce a natural notion of regularity which arises out of a combination of the partial order and binary operation on an ordered semigroup. Bhuniya and Hansda [1] presented a natural analogy between these two regularities. Thus it is quite obvious to explore a natural analogy between the subclasses of these two regularities.

An ordered semigroup $(S, \cdot, \leq)$ is a partially ordered set $(S, \leq)$ and at the same time a semigroup $(S, \cdot)$ such that for all a, b and $x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. Let $(S, \cdot, \leq)$ be an ordered semigroup, $(\emptyset \neq) A \subseteq S$ is called a subsemigroup of S if for every $a, b \in A$, $ab \in A$. Every subsemigroup A of S with the relation $\leq_A$ on A defined by $\leq_A = \leq \cap \{(a, b) \in A \times A\}$ is an ordered semigroup (called an ordered

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subsemigroup of S). Clearly, $\leq_A = \leq \cap A \times A$. For an ordered semigroup S and $H \subseteq S$, denote $(H] := \{t \in H : t \leq h \text{ for some } h \in H\}$.

Let I be a non-empty subset of an ordered semigroup S. I is a left(right) ideal of S, if $SI \subseteq I(IS \subseteq I)$ and $(I] = I$. We call I is an ideal of S if it is both a left and a right ideal of S. We denote the set of all left and right ideals of S by $\mathcal{L}(S)$ and $\mathcal{R}(S)$ respectively. Following Kehayopulu and Tsingelis [9], a subsemigroup B of S is called a bi-ideal of S if $BSB \subseteq B$ and $(B] = B$. We denote the set of all bi-ideals of S by $\mathcal{B}(S)$. The principal left ideal, right ideal, ideal and bi-ideal generated by $a \in S$ are denoted by $L(a), R(a), I(a)$ and $B(a)$ respectively and defined by $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$, $B(a) = (a \cup a^2 \cup aSa]$.

Characterizations of a semigroup (without order) S by the set of all bi-ideals of S, were beautifully presented by S. Lajos [11]. Here our approach allows one to characterize an ordered semigroup S by the set $\mathcal{B}(S)$ of all bi-ideals of S as a semigroup without order. We show that product of two bi-ideals in an ordered semigroup S is again a bi-ideal of S. Thus, $\mathcal{B}(S)$ is closed under this product. The main object of this paper is to study the semigroup $\mathcal{B}(S)$ of all bi-ideals of S whenever S is in different important subclasses of the regular ordered semigroups.

Kehayopulu [6] defined Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S in the following way: for $a, b \in S$ $a\mathcal{L}b$ if $L(a) = L(b)$; $a\mathcal{R}b$ if $R(a) = R(b)$; $a\mathcal{J}b$ if $I(a) = I(b)$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These four are equivalence relations on S. An ordered semigroup S is said to be *regular* if for every $a \in S$, $a \in (aSa]$ and is intra-regular if for every $a \in S$, $a \in (Sa^2S]$. An ordered semigroup S is group like ordered semigroup [1] if for all $a, b \in S$ there are $x, y \in S$ such that $a \leq xb$ and $a \leq by$. A regular ordered semigroup S is called a left group like ordered semigroup [1] if for all $a, b \in S$ there is $x \in S$ such that $a \leq xb$. Right group like ordered semigroup defined dually. Class of *Clifford* [4] as well as *left Clifford* [4] ordered semigroups are subclasses of class of regular ordered semigroups. A regular ordered semigroup S is called a *Clifford* (*left Clifford*) [4] ordered semigroup if for all $a, b \in S$ there is $x \in S$ such that $ab \leq bxa$ ($ab \leq xa$). Following results have been given for the sake of convenience of general readers.

Theorem 1.1. *Let S be an ordered semigroup. Then following conditions hold in S.*

(1) *If S is regular, then $B = (BSB]$ for every bi-ideal B of S (see [8]).*
(2) *If S is regular, then a nonempty subset B of S is a bi-ideal of S if and only if $B = (RL]$ for some right ideal R and left ideal L of S (see [5]).*

Theorem 1.2 ([1])**.** *An ordered semigroup S is a group like ordered semigroup if and only if it is both left group like and right group like ordered semigroup.*

For the sake of convenience of general readers we give some definitions and results from semigroup theory. By a band F we mean a semigroup $(F, \cdot)$ with the property $a^2 = a$ for every $a \in F$. A band $(F, \cdot)$ is called rectangular if for every $a, b \in F$ $aba = a$. A left(right) zero band is a band $(F, \cdot)$ with the property $ab = a$ $(ba = a)$ for every $a, b \in F$. A band $(F, \cdot)$ is said to be left (right) normal band if for every $a, b, c \in F$,

$abc = acb$ ($abc = bac$) and F is said to be normal if $abca = acba$. A commutative band is called a semilattice. A semigroup in which every finitely generated subsemigroup is finite called locally finite. A locally finite semigroup S is called locally testable [3] if for every idempotent f of S, fSf is a semilattice.

2. Semigroup of Bi-Ideals in Regular Ordered Semigroups

First we define a product of two bi-ideals of an ordered semigroup S. Let $(S, \cdot, \leq)$ be an ordered semigroup and $P(S)$ be the set of all subsets of S. We define a binary operation $*$ on S as follows: For $A, B \in P(S)$, $A * B = (AB]$, where AB=$\{ab : a \in A, b \in B\}$. It is easy to check that $(P(S), *)$ forms semigroup. Throughout the paper $A * A$ will be denoted by A^2, for every bi-ideal A of S. It is also noted that A^2 is not AA rather $A^2 = (AA]$. Followed by above, it is a routine task to verify that $\mathcal{L}(S)$, $\mathcal{R}(S)$ and $\mathcal{B}(S)$ are semigroups with respect to $*$.

In the following proposition we show that regularity of an ordered semigroup is equivalent to the regularity of the semigroup $B(S)$.

Proposition 2.1. *Let S be an ordered semigroup. Then S is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideals is regular.*

Proof. First assume that $\mathcal{B}(S)$ is a regular semigroup. Let $a \in S$. Then $B(a) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is regular, there is $C \in \mathcal{B}(S)$ such that $B(a) = B(a) * C * B(a) = (B(a)CB(a)]$. Since $a \in B(a)$, there are $b \in B(a)$, $x \in C$ and $c \in B(a)$ such that $a \leq bxc$. Also, for $b, c \in B(a)$ there are $s_1, s_2 \in S$ such that $b \leq a$ or $b \leq as_1a$ and $c \leq a$ or $c \leq as_2a$. Thus, in either case $a \leq bxc$ gives that $a \in (aSa]$ and therefore S is a regular ordered semigroup.

The converse follows directly from Theorem 1.1. ☐

Theorem 2.1. *Let S be a regular ordered semigroup. Then $\mathcal{R}(S)(\mathcal{L}(S))$ is a band and $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

Proof. Let $R \in \mathcal{R}(S)$ and $a \in R$. Since S is regular there exist $x \in S$ such that $a \leq axa$. Also $ax \in R$ which gives that $a \in (RR] = R*R = R^2$ and so $R \subseteq R^2$. Thus, $R^2 = R$. Hence, $\mathcal{R}(S)$ is a band. Similarly, $\mathcal{L}(S)$ is a band.

Choose $R \in \mathcal{R}(S)$ and $L \in \mathcal{L}(S)$. Let $B = R * L$. Then $B = (RL]$ and B is a subsemigroup of S. Now $BSB = (RL]S(RL] \subseteq (RLSRL] \subseteq (RL] = B$, by Theorem 1.1. This shows that $B \in \mathcal{B}(S)$ and so $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$. Next choose $D \in \mathcal{B}(S)$. Now $D \in \mathcal{B}(S) \subseteq \mathcal{R}(S)\mathcal{L}(S)$. Thus, $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$. Hence, the theorem is proved. ☐

Theorem 2.2. *An ordered semigroup S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

Proof. Suppose S is both regular and intra-regular ordered semigroup. Let $B \in \mathcal{B}(S)$ and $a \in B$. Then $a \leq axa \leq axaxa$ for some $x \in S$. Since S is intra-regular there are $s_1, s_2 \in S$ such that $a \leq s_1a^2s_2$ which implies that $a \leq axs_1a^2s_2xa \leq (axs_1a)(as_2xa)$.

Since $axs_1a \in BSB \subseteq B$, $axs_1a^2s_2xa \in B^2$ so that $a \in (BB] = B * B = B^2$. Also, $B^2 \subseteq B$ and thus $B^2 = B$.

Conversely, assume that $\mathcal{B}(S)$ is a band. Let $a \in S$. Then $B(a) \in B(S)$ and so $a \in B(a) = B(a)^2 = B(a) * B(a) = (B(a)B(a)]$. Thus, $a \leq bc$ for some $b, c \in B(a)$. This gives that $b \leq a$ or $b \leq asa$ for some $s \in S^1$ and $c \leq a$ or $c \leq ata$ for some $t \in S^1$. Then $a \leq bc$ implies that either $a \leq a^2$ or $a \in (aSa^2Sa]$ which gives that a is both regular and intra-regular. Thus, S is both regular and intra-regular. □

Lemma 2.1. *Let S is a both regular and intra-regular ordered semigroup. Then*

(1) *for every $B, C, D \in \mathcal{B}(S)$, $((BCB](BDB]] = (BCB] \cap (BDB]$;*

(2) *$\mathcal{B}(S)$ is locally testable semigroup.*

Proof. (1) We have, $((BCB](BDB]] \subseteq ((BCB](B]] \subseteq ((BCB]] \subseteq (BCB]$. Similarly, $((BCB](BDB]] \subseteq (BDB]$. Thus, $((BCB](BDB]] \subseteq (BCB] \cap (BDB]$. Now let $u \in (BCB] \cap (BDB]$. Then there are $b \in B$, $c \in C$, $d \in D$ such that $u \leq bcb$ and $u \leq bdb$. Since S is both regular and intra-regular, then there are $x, t, s \in S$ such that $u \leq uxu, b \leq btb$ and $b \leq s_1b^2s_2$ this implies $u \leq bcbxbdb \leq bcbtbxbdb \leq bcbts_1b^2s_2xbdb \leq (bcbts_1b)(bs_2xbdb)$. So, $u \in ((BCB](BDB]]$. Hence, $(BCB] \cap (BDB] \subseteq ((BCB](BDB]]$. Thus, $((BCB](BDB]] = (BCB] \cap (BDB]$.

(2) Consider $B \in \mathcal{B}(S)$. Then $B\mathcal{B}(S)B$ is a subsemigroup of $\mathcal{B}(S)$ and so a band. Now for every $C, D \in \mathcal{B}(S)$, $(BCB] * (BDB] = ((BCB](BDB]] = (BCB] \cap (BDB] = (BDB] \cap (BCB] = ((BDB](BCB]] = (BDB] * (BCB]$ shows that $B\mathcal{B}(S)B$ is a semilattice. Thus, $\mathcal{B}(S)$ is locally testable. □

Nambooripad [3] proved that a regular semigroup S is locally testable if and only if for every $f \in E(S)$, fSf is a semilattice. Also, following Zalcstein [12] a locally testable semigroup is a band if and only if it is a normal band.

Corollary 2.1. *Let S be an ordered semigroup. If S is both regular and intra-regular then $\mathcal{B}(S)$ is a band if and only if $\mathcal{B}(S)$ is a normal band.*

This follows from Theorem 2.2, Lemma 2.1 and Theorem 5 of [12].

Theorem 2.3. *Let S be an ordered semigroup. Then $\mathcal{B}(S)$ is a rectangular band if and only if S is regular and simple.*

Proof. First suppose that $\mathcal{B}(S)$ is a rectangular band. Let $a, b \in S$. Then $B(a), B(b) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is rectangular band, we have $B(a) = B(a) * B(b) * B(a)$ and $B(b) = B(b) * B(a) * B(b)$. Also, by Theorem 2.2, S is regular. Since $a \in B(a) = B(a) * B(b) * B(a) = (B(a)B(b)B(a)]$, there are $w, z \in B(a)$, $u \in B(b)$ such that $a \leq zuw$. Since $w, z \in B(a)$, $z \leq as_1a$ and $w \leq as_2a$ for some $s_1, s_2 \in S$. Also, for $u \in B(b)$ there is $s_3 \in S$ such that $u \leq bs_3b$. Thus, $a \leq (as_1abs_3)b(as_2a)$, i.e., $a \leq xby$ for some $x, y \in S$. Hence, S is simple.

Conversely, let S is a regular and simple ordered semigroup. Consider, $a \in S$. Now by given condition we have $a \in (Sa^2S]$ so that S is intra-regular. So by Theorem 2.2, $\mathcal{B}(S)$ is a band. Next let $A, B \in \mathcal{B}(S)$. We show that $A = A * B * A$. For this let $a \in A$ and $b \in B$. Since $a, aba \in S$ and $a\mathcal{J}b$ so $a \leq y_1abay_2$ for some $y_1, y_2 \in S$. The regularity of S yields that $a \leq axa \leq axaxa$ for some $x \in S$. Then $a \leq (axy_1a)b(ay_2xa)$ so that $a \in ((ASA)B(ASA)] \subseteq (ABA] = A * B * A$ that is, $A \subseteq A * B * A$. Again $A * B * A \subseteq (ASA] = A$. Thus, $A = A * B * A$ hence $\mathcal{B}(S)$ is a rectangular band. □

Theorem 2.4. *Let S be an ordered semigroup. Then $\mathcal{B}(S)$ is a left (right) zero band if and only if S is a left (right) group like ordered semigroup.*

Proof. Let $\mathcal{B}(S)$ is a left zero band. Then by Proposition 2.2, S is regular. Let $a, b \in S$. Then $B(a), B(b) \in \mathcal{B}(S)$. Since $\mathcal{B}(S)$ is a left zero band, $B(a) = B(a) * B(b)$, so $a \in (B(a)B(b)]$. Then there are $z \in B(a)$ and $w \in B(b)$ such that $a \leq zw$. Also, $w \leq bsb$ for some $s \in S$. Therefore, $a \leq (zbs)b$ and hence S is a left group like ordered semigroup.

Conversely, let S be a left group like ordered semigroup. Let $B, C \in \mathcal{B}(S)$. Let $u \in B * C$, then there are $b \in B$ and $c \in C$ such that $u \leq bc$. Since S is a left group like ordered semigroup we have $c \leq tb$ for some $t \in S$. Then for $c \leq tb$ together with $u \leq bc \leq btb$ gives $u \in B$. Thus, $B * C \subseteq B$. Now for any $d \in B$, $d \leq dtd$ for some $t \in S$. Since d, $dc \in S$, $d \leq t_1dc$ for some $t_1 \in S$. So, $d \leq dtt_1dc$. Clearly $d \in BSB \subseteq B$ so that $d \in (BC] = B * C$. Hence, $B = B * C$ and so B is a left zero band. □

Thus, it is very logical step to study the set of all bi-ideals $\mathcal{B}(S)$ for a group like ordered semigroup S.

Theorem 2.5. *Let S be an ordered semigroup. Then $\mathcal{B}(S)$ is both left zero and right zero band if and only if S is a group like ordered semigroup.*

Proof. This is similar to the proof of the Theorem 2.4. □

We now focus on the characterization of *Clifford* and *left Clifford* ordered semigroup S by the semigroup $\mathcal{B}(S)$.

Theorem 2.6. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is a Clifford ordered semigroup;*
(2) *$B_1 * B_2 = B_1 \cap B_2$ for all $B_1, B_2 \in \mathcal{B}(S)$;*
(3) *$(\mathcal{B}(S), *)$ is a semilattice.*

Proof. (1) ⇒ (2) First suppose that S is a Clifford ordered semigroup. Let $B_1, B_2 \in \mathcal{B}(S)$ and $u \in B_1 * B_2$. Then $u \leq b_1b_2$ for $b_1 \in B_1$ and $b_2 \in B_2$. Since S is regular there is $x \in S$ such that $u \leq uxu \leq b_1b_2xb_1b_2$. Since S is Clifford, there is $x_1 \in S$ such that $b_1b_2 \leq b_2x_1b_1$, so that $u \leq b_1b_2xb_2x_1b_1$. This implies $u \in B_1$. Similarly $u \in B_2$. Hence, $B_1 * B_2 \subseteq B_1 \cap B_2$. Next let $b \in B_1 \cap B_2$. Since S is regular, there is $y \in S$ such that $b \leq byb \leq bybyb$. Since S is Clifford, $yb \leq bzy$ for some $z \in S$. Thus,

$b \leq bbzy^2b$. Since $b \in B_2$ and B_2 is a bi-ideal of S it yields that $bzy^2b \in B_2SB_2 \subseteq B_2$. Also, $b \in B_1$ so that $b \in (B_1B_2] = B_1 * B_2$. Hence, $B_1 * B_2 = B_1 \cap B_2$.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Assume that $(B(S), *)$ is a semilattice. Then S is a regular ordered semigroup (by Theorem 2.2). Consider $a, b \in S$. Then $ab \in B(a) * B(b) = B(b) * B(a)$ implies that $ab \leq vu$ for some $u \in B(a)$ and $v \in B(b)$. Since S is regular, there are $s, t \in S$ such that $u \leq asa$ and $v \leq btb$. Thus, $ab \leq btbasa = bza$ where $z = tbas \in S$. Hence, S is a Clifford ordered semigroup. □

Theorem 2.7. *Let S be an ordered semigroup. Then $\mathcal{B}(S)$ is a left normal band if and only if S is a left Clifford ordered semigroup.*

Proof. First suppose that S is a left Clifford ordered semigroup. Let A, B and $C \in \mathcal{B}(S)$ and $x \in A * B * C$. Then $x \in (ABC]$ so $x \leq abc$ for some $a \in A$, $b \in B$ and $c \in C$. Since S is regular, there is $s \in S$ such that $x \leq xsx$ so that $x \leq abcsabc$. Since S is a left Clifford ordered semigroup, it follows $bc \leq s_1b$ for some $s_1 \in S$, so $x \leq abc(sas_1)b \leq abs_2cb$ for $s_2 \in S$. Since S is regular there is $t \in S$ such that $a \leq ata$ implies $x \leq atabs_2cb$. Also there are $s_3, s_4 \in S$, $x \leq ats_3as_2cb \leq ats_3s_4acb$ implies $x \in A * C * B$. Therefore, $A * B * C \subseteq A * C * B$. Similarly it can be shown that $A * C * B \subseteq A * B * C$. Hence, $A * B * C = A * C * B$ and so $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then S is regular, by Theorem 2.2. Let $a, b \in S$. Then there is $x \in S$ such that $ab \leq abxab$ which implies $ab \in (B(abx)B(a)B(b)] = (B(abx)B(b)B(a)]$, since $\mathcal{B}(S)$ is a left normal band. Then $ab \leq uvw$, where $u \in B(abx), v \in B(b), w \in B(a)$. Again, $w \leq asa$ for some $s \in S$. Now $ab \leq uvw \leq (uvas)a \leq s_1a$, where $s_1 = uvas \in S$. Thus, S is left Clifford ordered semigroup. □

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