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$L^\infty-$ASYMPTOTIC BEHAVIOR OF A FINITE ELEMENT METHOD FOR A SYSTEM OF PARABOLIC QUASI-VARIATIONAL INEQUALITIES WITH NONLINEAR SOURCE TERMS

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Abstract. This paper is an extension and a generalization of the previous results, cf. [3,6,8,11]. It is devoted to studying the finite element approximation of the non coercive system of parabolic quasi-variational inequalities related to the management of energy production problem. Specifically, we prove optimal L^∞-asymptotic behavior of the system of evolutionary quasi-variational inequalities with nonlinear source terms using the finite element spatial approximation and the subsolutions method.

1. Introduction

This paper is concerned with the semi-implicit time scheme combined with a finite element spatial approximation for a system of parabolic quasi-variational inequalities with nonlinear source terms: Find $(u^1,\dots,u^J)\in\left(L^2\left((0,T),H_0^1(\Omega)\right)\right)^J$ satisfying

$$
\begin{cases}
\dfrac{\partial u^i}{\partial t}+A^iu^i\le f^i\left(u^i\right) \text{ in } \Phi,\\
u^i\le Mu^i,\quad i=1,\dots,J,\\
\left(\dfrac{\partial u^i}{\partial t}+A^iu^i-f^i\left(u^i\right)\right)\left(u^i-Mu^i\right)=0 \text{ in } \Phi,\\
u^i(x,0)=u_0^i \text{ in } \Omega,\quad u^i=0 \text{ on } \Sigma.
\end{cases}
\tag{1.1}
$$

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Here A^i denotes uniformly second order elliptic operators on a bounded convex domain Ω in $\mathbb{R}^J$, $J \geq 1$ with smooth boundary $\partial\Omega$ and Φ set in $\mathbb{R}^J \times \mathbb{R}$ defined as $\Phi = \Omega \times [0, T]$, with $T < +\infty$, $\Sigma = \partial\Omega \times [0, T]$.

$f^i\left(u^i\right)$ are J nonlinear and Lipschitz functions with Lipschitz constant $\alpha < \beta$ and satisfying the following condition

(1.2)
$$f^i \in \left(L^2\left((0,T), L^\infty(\Omega)\right) \cap C^1\left((0,T), H^{-1}(\Omega)\right)\right)^J, \quad f^i > 0, \text{ also is increasing.}$$

This system arises from the management of energy production problems (see [4] and the references therein). In the case studied here, Mu^i represents a "cost function" and the prototype encountered is

$$Mu^i(x) = \mathbf{k} + \inf_{\mu \neq i} u^\mu, \quad \text{where } \mathbf{k} > 0 \text{ and } \mu > 0, \tag{1.3}$$

and we know by [25] on page 243 that M satisfies some proprieties as M is a concave operator, i.e.,

$$M\left(\delta u + (1-\delta) v\right) \geq \delta M(u) + (1-\delta) M(v), \quad \text{for all } u, v \in C(\Omega),$$

and it also satisfies

$$M(u + \eta) = M(u) + \eta, \quad \text{for all } \eta \in \mathbb{R},$$

where $\mathbf{k}$ represents the switching cost. It is positive when the unit is "turned on" and equal to zero when the unit is "turned off".

Many results on error estimates for the classical obstacle problems, system of stationary and evolutionary quasi-variational and variational inequalities have been achieved in this norm, (cf., e.g., [1–3, 5, 9, 18, 20, 22]).

Moreover, in [11] Boulaaras, Bencheikh and Haiour established quasi-optimal L^∞-asymptotic behavior of the system of parabolic quasi-variational inequality related the management of energy production problems with mixed boundary condition using a discrete algorithm based on a θ-scheme combined with a finite element spatial approximation, that is, for $\theta \geq \frac{1}{2}$

$$\| U_h^n - U^\infty \|_\infty \leq C\left[h^2 \mid \log h \mid^3 + \left(\frac{1}{1+\theta\Delta t}\right)^n\right],$$

and for $0 \leq \theta < \frac{1}{2}$

$$\|U_h^n - U^\infty\|_\infty \leq C\left[h^2 \mid \log h \mid^3 + \left(\frac{2}{2+\beta\theta\left(1-2\theta\right)\rho\left(A^i\right)}\right)^n\right],$$

where $\rho\left(A^i\right)$ is the spectral radios of the elliptic operator A^i and U_h^n, the discrete solution of the system of QVIs calculated at the moment-end $T = n\Delta t$ for an index of the time discretization $k = 1, \ldots, n$, and U^∞, the asymptotic continuous solution of the system of QVIs.

Also, in [8] Boulaaras, Haiour proved quasi-optimal L^∞-asymptotic behavior of the evolutionary Hamilton-Jacobi-Bellman equations using the semi-implicit scheme with respect to the t-variable combined with a finite element spatial approximation where $k = \Delta t$, that is

$$\|U_h^n - U^\infty\|_\infty \leq C^* \left[h^2 |\log h|^3 + \left(\frac{1+kc}{1+k\beta}\right)^n\right],$$

where U_h^n, the discrete solution of the evolutionary Hamilton-Jacobi-Bellman equations calculated at the moment-end $T = n\Delta t$ for an index of the time discretization $k = 1, \ldots, n$ and U^∞, the asymptotic continuous solution of the evolutionary Hamilton-Jacobi-Bellman equations.

In [14] Boulbrachene, Cortey Dumont established optimal L^∞-error estimate of a finite element approximation of the Hamilton-Jacobi-Bellman (HJB) equations using the discrete regularity introduced by Cortey Dumont in [20], that is

$$\|u - u_h\|_\infty \leq Ch^2 |\log h|^2,$$

where u, the continuous solution of the Hamilton-Jacobi-Bellman (HJB) equations, and u_h, the discrete solution of the Hamilton-Jacobi-Bellman (HJB) equations.

In a recent work in [7] Bencheikh, Boulaaras and Haiour also established optimal L^∞-asymptotic behavior for a system of parabolic quasi-variational inequalities related to stochastic control problems using the regularization of the obstacles appearing in the discrete system of QVIs "the discrete regularity", they have the following estimation

$$\|U_h(T,\cdot) - U^\infty(\cdot)\|_\infty \leq C\left[h^2 |\log h|^2 + \left(\frac{1}{1+\theta\Delta t}\right)^N\right],$$

where $U_h(T,\cdot)$, the discrete solution of the system of parabolic quasi-variational inequalities related to stochastic control problems calculated at the moment-end $T = N\Delta t$ for an index of the time discretization $k = 1, \ldots, N$, and $U^\infty(\cdot)$, the asymptotic continuous solution of the system of parabolic quasi-variational inequalities related to stochastic control problems.

In this paper we propose a new proof to get the optimal L^∞-asymptotic behavior of the system of parabolic QVIs with nonlinear source terms without going through the discrete regularity of the obstacles appearing in the discrete system of QVIs and we improve the convergence order in works of Boulaaras, Haiour [8, 9] and Boulaaras, Bencheikh and Haiour [11] for the system of parabolic quasi-variational inequalities.

The subsolutions method (see [14,17,21]) characterizes the continuous solution (resp. the discrete solution) as the least upper bound of the set of continuous subsolutions (resp. the discrete subsolution) will also be crucial to determine the convergence order.

The approximation method developed in this paper stands on the construction a sequence of continuous subsolution denoted $\beta^k = (\beta^{1,k}, \ldots, \beta^{J,k})$ such that

$$\beta^{i,k} \leq u^{i,k} \quad \text{and} \quad \left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2, \quad \text{for all } k \geq 1,\ i = 1, 2, \ldots, J,$$

and the construction of a sequence of discrete subsolution $\alpha_h^k = (\alpha_h^{1,k}, \dots, \alpha_h^{J,k})$ such that

$$\alpha_h^{i,k} \le u_h^{i,k} \quad \text{and} \quad \left\|\alpha_h^{i,k} - u^{i,k}\right\|_\infty \le Ch^2 |\ln h|^2, \quad \text{for all } k \ge 1,\ i = 1, 2, \dots, J,$$

to obtain

$$\max_{1\le i\le J} \left\|u^{i,k} - u_h^{i,k}\right\|_\infty \le Ch^2 |\ln h|^2, \quad \text{for all } k \ge 1.$$

In this situation, we establish the optimal L^∞-asymptotic behavior of the system of parabolic QVIs, that is

$$\| U_h^N - U^\infty \|_\infty = \max_{1\le i\le J} \| u_h^{i,N} - u^{i,\infty} \|_\infty \le C \left[h^2 \ \ln h|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N \right].$$

The paper is organized as follows. In Section 2, we consider system of continuous quasi-variational inequalities and we give some related qualitative properties. In Section 3, we characterize the discrete solution as a fixed point of a contraction. In Section 4, we introduce two auxiliary problems which allow us to define sequences of continuous and discrete subsolutions. In Section 5, we present the main result of the paper.

2. The Continuous Problem

2.1. **Notations, Assumptions.** Let $a_{jp}^i(x)$, $a_p^i(x)$, $a_0^i(x)$ in $L^\infty(\Omega) \cap C^2(\overline{\Omega})$, $x \in \bar{\Omega}$, $j, p = 1, \dots, S$, are sufficiently smooth coefficients and satisfying the following conditions:

$$\sum_{j,p=1}^S a_{jp}^i(x)\zeta_j\zeta_p \geqq \gamma|\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^S,\ \gamma > 0,\ x \in \overline{\Omega},$$

and

$$a_{jp}^i = a_{pj}^i, \quad a_0^i(x) \geqslant \beta > 0, \quad \beta \text{ is a constant.} \tag{2.1}$$

We define the second order differential operators A^i:

$$A^i = -\sum_{j,p=1}^S a_{jp}^i(x) \frac{\partial^2}{\partial x_j \partial x_p} + \sum_{p=1}^S a_p^i(x) \frac{\partial}{\partial x_p} + a_0^i(x),$$

and the associated variational forms for any $u, v \in H_0^1(\Omega)$

$$a^i(u,v) = \int_\Omega \left(\sum_{j,p=1}^S a_{jp}^i(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_p} + \sum_{p=1}^S a_p^i(x) \frac{\partial u}{\partial x_p} v + a_0^i(x) uv \right) dx.$$

We shall also need the following notations

$$\|W\|_\infty = \max_{1\le i\le J} \left\|w^i\right\|_\infty, \quad \text{for all } W = \left(w^1, w^2, \dots, w^J\right) \in \prod_{i=1}^J L^\infty(\Omega),$$

where $\|\cdot\|_\infty$ denotes the well-known L^∞-norm, $(\cdot,\cdot)$ be the inner product in $L^2(\Omega)$.

2.2. **The system of continuous parabolic quasi-variational inequalities.** The problem (1.1) can be approximated by the following system of continuous parabolic quasi-variational inequalities: Find $U = (u^1, u^2, \ldots, u^J) \in (L^2((0,T), H_0^1(\Omega)))^J$ solution for:

$$\begin{cases} \left(\dfrac{\partial u^i}{\partial t}, v^i - u^i\right) + a^i\left(u^i, v^i - u^i\right) \geqq \left(f^i\left(u^i\right), v^i - u^i\right), \\ u^i \leq Mu^i, \quad v^i \leq Mu^i, \quad 1 \leq i \leq J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

Now, we apply the semi-implicit scheme of the system to the continuous parabolic quasi-variational inequalities (2.2). Therefore, we seek a sequence of elements $u^{i,k} \in (H_0^1(\Omega))^J$, $1 \leq i \leq J$, which approaches $u^i(t_k)$, $t_k = k\Delta t$, with initial data $u^{i,0}$. Thus, we have $k = 1, \ldots, N$,

$$\begin{cases} \left(\dfrac{u^{i,k} - u^{i,k-1}}{\Delta t}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k}\right), v^i - u^{i,k}\right), \\ u^{i,k} \leq Mu^{i,k}, \quad v^i \leq Mu^{i,k}, \quad 1 \leq i \leq J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.3}$$

2.2.1. *Existence and uniqueness of continuous solution of the system of parabolic QVIs.* Let us recall just the main steps leading to the existence of a unique solution to system (2.3). For more details, we refer the reader to [4].

A fixed point mapping associated with the continuous problem.

Let $\mathbb{H}^+ = (L_+^\infty(\Omega))^J = \{\ V = (v^1, \ldots, v^J) \text{ such that } v^i \in L_+^\infty(\Omega)\}$, where $L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$.

We introduce the following mapping:

$$\begin{aligned} T &: \mathbb{H}^+ \to (L^\infty(\Omega))^J, \\ W &\to TW = \zeta^k = (\zeta^{1,k}, \ldots, \zeta^{J,k}), \end{aligned} \tag{2.4}$$

we note $\zeta^{i,k} = \partial(F^{i,k}(w^i), Mw^i) \in (H_0^1(\Omega))^J$ for all $i = 1, \ldots, J$, the solution of the following problem:

$$\begin{cases} b^i\left(\zeta^{i,k}, v^i - \zeta^{i,k}\right) \geqq \left(f^{i,k}\left(w^i\right) + \lambda w^i, v^i - \zeta^{i,k}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \zeta^{i,k} \leq Mw^i, \quad v^i \leq Mw^i, \end{cases}$$

where $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$.

An iterative continuous algorithm.

Let us also define the vector $U^0 = (u^{1,0}, \ldots, u^{J,0})$, where $u^{i,0}$ is the solution to the continuous equation:

$$b^i\left(u^{i,0}, v^i\right) = \left(g^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J,$$

where $g^{i,0}$ is a linear and a regular function.

Now we give the following continuous algorithm

$$u^{i,k} = Tu^{i,k-1}, \quad k = 1, \dots, N,\ i = 1, \dots, J, \tag{2.5}$$

or

$$U^k = TU^{k-1},$$

where $U^k = \left(u^{1,k}, \dots, u^{J,k}\right)$ is the solution of the problem (2.3).

Remark 2.1. We denote

$$\mathbb{C} = \left\{W \in \mathbb{H}^+ \mid 0 \le W \le U^0\right\},$$

where $U^0 = U_0 = \left(u_0^1, \dots, u_0^J\right)$, $\mathbb{H}^+ = (L_+^\infty(\Omega))^J$. Since $f^{i,k}(\cdot) \ge 0$, combining comparison results in variational inequalities with simple induction, we obtain $U^k = \left(u^{1,k}, \dots, u^{J,k}\right) \ge 0$ for all $k = 1, \dots, N$ and $TW \ge 0$.

Similarly as in [12], the mapping T is monotone increasing for the stationary free boundary problem with nonlinear source term. Then it can be easily verified that

$$U^2 = TU^1 \le TU^0 = U^1 \le U^0,$$

thus, inductively,

$$U^{k+1} = TU^k \le U^k \le \dots \le U^0, \quad \text{for all } k = 1, \dots, N,$$

and also it can be seen that the sequence U^k stays in $\mathbb{C}$.

According to assumption (1.2), f is increasing, for $k = 1, \dots, N$, $i = 1, \dots, J$, and using the Remark 2.1, we have

$$f\left(U^k\right) \le f\left(U^{k-1}\right)$$

or

$$f\left(u^{i,k}\right) \le f\left(u^{i,k-1}\right),$$

which implies

$$\begin{cases} \left(\dfrac{u^{i,k}}{\Delta t}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k-1}\right) + \dfrac{u^{i,k-1}}{\Delta t}, v^i - u^{i,k}\right), \\ u^{i,k} \le Mu^{i,k}, \quad v^i \le Mu^{i,k}, \quad 1 \le i \le J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.6}$$

Then, the problem (2.6) can be reformulated into the following coercive continuous system of elliptic quasi-variational inequalities (EQVIs)

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) \geqq \left(f^{i,k}\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v^i - u^{i,k}\right), \quad u^{i,k} \in \left(H_0^1(\Omega)\right)^J, \\ u^{i,k} \le Mu^{i,k}, \quad v^i \le Mu^{i,k}, \quad 1 \le i \le J, \\ u^i(x,0) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.7}$$

where

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) = \lambda\left(u^{i,k}, v^i - u^{i,k}\right) + a^i\left(u^{i,k}, v^i - u^{i,k}\right), \quad u^{i,k} \in \left(H_0^1(\Omega)\right)^J, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{N}{T}, \quad k = 1, \ldots, N. \end{cases}$$

Then the bilinear form $b(\cdot,\cdot)$ is strongly coercive see [26]. There exist two constants $\lambda > 0$ and $\gamma > 0$ such that:

$$b^i(v, v) = a^i(v, v) + \lambda \|v\|^2_{L^2(\Omega)} \geqq \gamma \|v\|^2_{H_0^1(\Omega)}, \quad \text{for all } v \in H_0^1(\Omega).$$

Let $\mathbb{C} = \{W \in \mathbb{H}^+ \mid 0 \le W \le U^0\}$, where $U^0 = U_0 = \left(u_0^1, \ldots, u_0^J\right)$ and $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$, $\tilde{F}^{i,k}(\tilde{w}^i) = f^{i,k}(\tilde{w}^i) + \lambda \tilde{w}^i \in (L^\infty(\Omega))^J$ be the corresponding right-hand sides to the continuous PQVIs and $\mathbf{k}$ and $\tilde{\mathbf{k}}$ be two parameters that are defined in (1.2) and (1.3).

A monotonicity property

Proposition 2.1 ([16,20])**.** *If* $F^{i,k}(w^i) \le F^{i,k}(\tilde{w}^i)$ *and* $\mathbf{k} \le \tilde{\mathbf{k}}$, *then*

$$u^{i,k} = \partial\left(F^{i,k}\left(w^i\right), \mathbf{k}\right) \le \tilde{u}^{i,k} = \partial\left(F^{i,k}\left(\tilde{w}^i\right), \tilde{\mathbf{k}}\right).$$

Proposition 2.2 ([8,12])**.** *Under the previous assumption and notations* (1.2), (2.1), (2.4)*, the mapping T is a contraction in $\mathbb{H}^+$ with contraction constant $\frac{\alpha+\lambda}{\beta+\lambda}$. Therefore, T admits a unique fixed point which coincides with the continuous solution of the system of parabolic QVIs* (2.7)*.*

Proposition 2.3 ([8])**.** *Under the conditions of Proposition* 2.2 *and notations* (1.2), (2.1), (2.4)*, we have the following estimate of geometric convergence*

$$\| U^k - U^\infty \|_\infty = \max_{1\le i\le J} \| u^{i,k} - u^{i,\infty} \|_\infty \le \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^k \| U^\infty - U^0 \|_\infty,$$

where U^∞ is an asymptotic continuous solution of the following system of QVIs

$$\begin{cases} b^i\left(u^{i,\infty}, v^i - u^{i,\infty}\right) \ge \left(f^i\left(u^{i,\infty}\right) + \lambda u^{i,\infty}, v^i - u^{i,\infty}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ u^{i,\infty} \le M u^{i,\infty}, \quad i = 1, \ldots, J. \end{cases}$$

Lipschitz dependence with respect to the right-hand sides and the parameter k

Proposition 2.4 ([14,21])**.** *Under the conditions of Proposition* 2.1*. Then we have:*

$$\max_{1\le i\le J} \| u^{i,k} - \tilde{u}^{i,k} \|_\infty \le C \max_{1\le i\le J} \left(\left|\mathbf{k} - \tilde{\mathbf{k}}\right| + \| F^{i,k} - \tilde{F}^{i,k} \|_\infty\right).$$

Characterization of the solution of the system (2.7) **as the envelope of continuous subsolutions**

Definition 2.1 ([4])**.** $Z = (z^1, \dots, z^J) \in (H_0^1(\Omega))^J$ is said to be a continuous subsolution for the system of quasi-variational inequalities (2.7) if

$$\begin{cases} b^i\left(z^{i,k}, v^i\right) \leq \left(f^{i,k}\left(z^{i,k-1}\right) + \lambda z^{i,k-1}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \, v^i \geqq 0, \\ z^{i,k} \leq M z^{i,k}, \quad i = 1, \dots, J, \, k = 1, \dots, N. \end{cases}$$

Let $\mathbb{Y}$ denote the set of such continuous subsolutions.

Theorem 2.1 ([4,21])**.** *The solution of the system* (2.7) *is the maximum element of the set* $\mathbb{Y}$.

3. The Discrete Problem

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements, $h > 0$ is the mesh size. We assume the family τ_h is regular and quasi-uniform. We consider φ_l, $l = 1, 2, \dots, m(h)$, are the nodal basis functions defined by $\varphi_l\left(M_s\right) = \delta_{ls}$ where M_s, $s = 1, \dots, m\left(h\right)$, is a vertex of the considered triangulation and r_h is the usual interpolation operator.

Let $\mathbb{V}_h$ denote the standard piecewise linear finite element space

$$\begin{aligned} \mathbb{V}_h = &\{u^i \in (L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\bar{\Omega})\right))^J \mid u^i \mid_{k_i} \in P_1, \\ &k_i \in \tau_h^i \text{ and } u^i\left(\cdot, 0\right) = u_0^i \text{ in } \Omega, u^i = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

P_1 denotes the space of polynomials with degree no more than 1 and $\mathbb{B}^i$, $1 \leq i \leq J$, denote the finite element matrices defined by

$$(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s), \quad 1 \leq l, s \leq m(h).$$

The Discrete Maximum Principle Assumption (dmp) (cf. [19]). We assume that the matrices $(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s) = a^i(\varphi_l, \varphi_s) + \lambda(\varphi_l, \varphi_s)$ are M-matrices.

Under the **dmp**, we shall achieve a similar study to that devoted to the continuous problem.

We discretize in space the problem (2.2), i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $\mathbb{V}_h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we seek a sequence of elements $u_h^{i,k} \in \left(\mathbb{V}_h\right)^J$, $1 \leq i \leq J$, which approaches $u_h^i\left(t_k\right)$, $t_k = k\Delta t$ with initial data $u^{i,0}$. Thus, we have $k = 1, \dots, N$,

$$\begin{cases} \left(\dfrac{u_h^{i,k} - u_h^{i,k-1}}{\Delta t}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right), v_h^i - u_h^{i,k}\right), \\ u_h^{i,k} \leq r_h M u_h^{i,k}, \quad v_h^i \leq r_h M u_h^{i,k}, \quad 1 \leq i \leq J. \end{cases} \tag{3.1}$$

Then we can write (3.1) as follows:

$$\begin{cases} \left(\dfrac{u_h^{i,k}}{\Delta t}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right) + \dfrac{u_h^{i,k-1}}{\Delta t}, v_h^i - u_h^{i,k}\right). \\ u_h^{i,k} \le r_h M u_h^{i,k}, \quad v_h^i \le r_h M u_h^{i,k}, \quad 1 \le i \le J. \end{cases} \tag{3.2}$$

The problem (3.2) can be reformulated into the following coercive system of discrete elliptic quasi-variational inequalities:

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \geqq \left(f^{i,k}\left(u_h^{i,k-1}\right) + \lambda u_h^{i,k-1}, v_h^i - u_h^{i,k}\right), \quad \text{for all } v_h^i \in (\mathbb{V}_h)^J, \\ u_h^{i,k} \le r_h M u_h^{i,k}, \quad v_h^i \le r_h M u_h^{i,k}, \end{cases} \tag{3.3}$$

such that

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) = \lambda\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right), \quad u_h^{i,k} \in (\mathbb{V}_h)^J, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{N}{T}, \quad k = 1, \dots, N. \end{cases}$$

3.0.1. *Existence and uniqueness for discrete solution of the system of PQVI.* As in the continuous problem, we shall characterize the discrete solution of system of PQVI as the unique fixed point of a contraction.

A fixed point mapping associated with discrete problem

We introduce the following mapping:

$$T_h : \mathbb{H}^+ \to (\mathbb{V}_h)^J, \tag{3.4}$$

$$W \to T_h W = \zeta_h^k = (\zeta_h^{1,k}, \dots, \zeta_h^{J,k}),$$

we keep the precedent notation, i.e., $\zeta_h^{i,k} = \partial_h(F^{i,k}(w^i), r_h M w^i) \in (\mathbb{V}_h)^J$ for all $i = 1, \dots, J$, the solution to the following problem:

$$\begin{cases} b^i\left(\zeta_h^{i,k}, v_h^i - \zeta_h^{i,k}\right) \geqq \left(f^{i,k}\left(w^i\right) + \lambda w^i, v_h^i - \zeta_h^{i,k}\right), \quad \text{for all } v_h^i \in (\mathbb{V}_h)^J, \\ \zeta_h^{i,k} \le r_h M w^i, \quad v_h^i \le r_h M w^i, \end{cases}$$

where $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$.

An iterative discrete algorithm

Let us also define the vector $U_h^0 = (u_h^{1,0}, \dots, u_h^{J,0})$, where $u_h^{i,0}$ is the solution of the continuous equation:

$$b^i\left(u_h^{i,0}, v^i\right) = \left(g^{i,0}, v^i\right), \quad \text{for all } v^i \in (\mathbb{V}_h)^J,$$

where $g^{i,0}$ is a linear and a regular function.

Now we give the following discrete algorithm

$$u_h^{i,k} = T_h u_h^{i,k-1}, \quad k = 1, \dots, N,\ i = 1, \dots, J, \tag{3.5}$$

or

$$U_h^k = T_h U_h^{k-1},$$

where $U_h^k = \left(u_h^{1,k}, \ldots, u_h^{J,k}\right)$ is the solution of the problem (3.3).

We denote $\mathbb{C}_h = \{W \in \mathbb{H}^+ \mid 0 \leq W \leq U_h^0\}$, where $U_h^0 = \left(u_{h0}^1, \ldots, u_{h0}^J\right)$ and $F^{i,k}(w^i) = f^{i,k}(w^i) + \lambda w^i$ $\tilde{F}^{i,k}(\tilde{w}^i) = f^{i,k}(\tilde{w}^i) + \lambda \tilde{w}^i \in (L^\infty(\Omega))^J$ are the corresponding right-hand sides to the discrete PQVIs and $\mathbf{k}$ and $\tilde{\mathbf{k}}$ be two parameters.

As in the continuous case, we give some related qualitative properties of the discrete solution of the system of parabolic QVIs (3.3).

A monotonicity property

Proposition 3.1 ([16, 20])**.** *If $F^{i,k}(w^i) \leq F^{i,k}(\tilde{w}^i)$ and $\mathbf{k} \leq \widetilde{\mathbf{k}}$, then*

$$u_h^{i,k} = \partial_h\left(F^{i,k}\left(w^i\right), \mathbf{k}\right) \leq \tilde{u}_h^{i,k} = \partial_h\left(F^{i,k}\left(\tilde{w}^i\right), \tilde{\mathbf{k}}\right).$$

Proposition 3.2 ([8, 12])**.** *Under the previous assumption, notations* (1.2), (2.1), (3.4) *and the **dmp**, the mapping T_h is a contraction in $\mathbb{H}^+$ with contraction constant $\frac{\alpha+\lambda}{\beta+\lambda}$. Therefore, T_h admits a unique fixed point which coincides with the discrete solution of the system of parabolic QVIs* (3.3)*.*

Proposition 3.3 ([8])**.** *Under the conditions of Proposition* 3.2 *and notations* (1.2), (2.1), (3.4)*, we have the following estimate of geometric convergence*

$$\| U_h^k - U_h^\infty \|_\infty = \max_{1\leq i\leq J} \| u_h^{i,k} - u_h^{i,\infty} \|_\infty \leq \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^k \| U_h^\infty - U_h^0 \|_\infty,$$

where U_h^∞ is an asymptotic discrete solution of the following system of QVIs

$$\begin{cases} b^i(u_h^{i,\infty}, v_h^i - u_h^{i,\infty}) \geq \left(f^i\left(u_h^{i,\infty}\right) + \lambda u_h^{i,\infty}, v_h^i - u_h^{i,\infty}\right), & \text{for all } v_h^i \in (\mathbb{V}_h)^J, \\ u_h^{i,\infty} \leq r_h M u_h^{i,\infty}, \quad i = 1, \ldots, J. \end{cases}$$

Lipschitz dependence with respect to the right-hand sides and the parameter k

Proposition 3.4 ([14, 21])**.** *Under the **dmp** and the Proposition* 3.1*, we have:*

$$\max_{1\leq i\leq J} \| u_h^{i,k} - \tilde{u}_h^{i,k} \|_\infty \leq C \max_{1\leq i\leq J}\left(\left|\mathbf{k} - \tilde{\mathbf{k}}\right| + \| F^{i,k} - \tilde{F}^{i,k} \|_\infty\right).$$

Characterization of the solution of system (3.3) **as the envelope of discrete subsolutions**

Definition 3.1 ([4])**.** $Z_h = (z_h^1, \ldots, z_h^J) \in (\mathbb{V}_h)^J$ is said to be a discrete subsolution for the system of quasi-variational inequalities (3.3) if

$$\begin{cases} b^i\left(z_h^{i,k}, \varphi_l\right) \leq \left(f^{i,k}\left(z_h^{i,k-1}\right) + \lambda z_h^{i,k-1}, \varphi_l\right), \quad \text{for all } v_h^i \in (\mathbb{V}_h)^J, \varphi_l \geq 0, \\ l = 1, \ldots, m(h), \\ z_h^{i,k} \leq r_h M z_h^{i,k}, \quad i = 1, \ldots, J,\ k = 1, \ldots, N. \end{cases}$$

Let $\mathbb{Y}_h$ denote the set of such discrete subsolutions.

Theorem 3.1 ([4, 21])**.** *The discrete solution of the system* (3.3) *is the maximum element of the set $\mathbb{Y}_h$.*

4. L^∞-Error Estimates

In this section, we first introduce the following two auxiliary systems of variational inequalities and next we prove a fundamental lemma of the subsolutions method.

4.1. **Two auxiliary sequences of system of variational inequalities.** We define the sequence $\left\{\bar{U}^k\right\}_{k\geq 1} = \left(\bar{u}^{1,k}, \ldots, \bar{u}^{J,k}\right)$ such that $\bar{U}^k$ solves the continuous system of V.I.

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i - \bar{u}^{i,k}\right) \geqq \left(f^i\left(u_h^{i,k-1}\right) + \lambda u_h^{i,k-1}, v^i - \bar{u}^{i,k}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k} \leq M u_h^{i,k-1}, \quad v^i \leq M u_h^{i,k-1}, \end{cases}$$

where $U_h^{k-1} = \left(u_h^{1,k-1}, \ldots, u_h^{J,k-1}\right)$ is defined in (3.5), and the sequence $\left\{\bar{U}_h^k\right\}_{k\geq 1} = \left(\bar{u}_h^{1,k}, \ldots, \bar{u}_h^{J,k}\right)$ is such that $\bar{U}_h^k$ solves the discrete system of V.I.

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, v_h^i - \bar{u}_h^{i,k}\right) \geqq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v_h^i - \bar{u}_h^{i,k}\right), \quad \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J, \\ \bar{u}_h^{i,k} \leq r_h M u^{i,k-1}, \quad v_h^i \leq r_h M u^{i,k-1}, \end{cases}$$

where $U^{k-1} = \left(u^{1,k-1}, \ldots, u^{J,k-1}\right)$ is defined in (2.5).

Lemma 4.1 ([20, 21])**.** *There exists a constant C independent of h and k such that*

$$\max_{1\leq i\leq J} \| \bar{u}^{i,k} - u_h^{i,k} \|_\infty \leq Ch^2 |\ln h|^2$$

and

$$\max_{1\leq i\leq J} \| \bar{u}_h^{i,k} - u^{i,k} \|_\infty \leq Ch^2 |\ln h|^2.$$

4.2. **Optimal L^∞-error estimates.** Now, we obtain the optimal L^∞-error estimate between the k-th continuous iterates $u^{i,k}$ and k-th discrete iterates $u_h^{i,k}$ defined in (2.7) and (3.3), respectively.

In this theorem, we exploit the idea of Boulbrachene in [13] given for variational inequalities with noncoercive operators, where we have adapted it to a system of QVIs related to the management of energy production problem.

Theorem 4.1.

$$\left\|U^k - U_h^k\right\|_\infty = \max_{1\leq i\leq J} \left\|u^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2.$$

The following lemma plays crucial role in proving the Theorem 4.1.

Lemma 4.2. *There exists a sequence of continuous subsolutions $\left(\beta^k\right)_{k\geq 1} = \left(\beta^{1,k}, \ldots, \beta^{J,k}\right)$, such that*

$$\beta^{i,k} \leq u^{i,k}, \quad 1 \leq k \leq N,\ 1 \leq i \leq J,$$

and

$$\left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2,$$

and a sequence of discrete subsolutions $\left(\alpha_h^k\right)_{k\geq 1} = \left(\alpha_h^{1,k}, \ldots, \alpha_h^{J,k}\right)$, *such that*

$$\alpha_h^{i,k} \leq u_h^{i,k}, \quad 1 \leq k \leq N,\, 1 \leq i \leq J,$$

and

$$\left\|\alpha_h^{i,k} - u^{i,k}\right\|_\infty \leq Ch^2 |\ln h|^2.$$

Proof. Let $\bar{U}^1$ be continuous solution of the system of V.I.

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i - \bar{u}^{i,1}\right) \geqq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0}, v^i - \bar{u}^{i,1}\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf_{\mu\neq i} u_h^{\mu,0}, \quad v^i \leq \mathbf{k} + \inf_{\mu\neq i} u_h^{\mu,0}. \end{cases}$$

Then, as $\bar{U}^1 = (\bar{u}^{i,1})_{1\leq i\leq J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i\right) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf_{\mu\neq i} u_h^{\mu,0}, \quad v^i \leq \mathbf{k} + \inf_{\mu\neq i} u_h^{\mu,0}, \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}^{i,1}, v^i\right) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda u_h^{i,0} - \lambda u^{i,0} + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \inf_{\mu\neq i} u_h^{\mu,0} - \inf_{\mu\neq i} u^{\mu,0} + \inf_{\mu\neq i} u^{\mu,0}. \end{cases}$$

We have

$$\left\|u_h^{i,0} - u^{i,0}\right\|_\infty \leq Ch^2 \left|\ln h\right|^{\frac{3}{2}} \quad \text{(see [23])}, \tag{4.1}$$

then

$$\begin{cases} b^i(\bar{u}^{i,1}, v^i) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda \left\|u_h^{i,0} - u^{i,0}\right\|_\infty + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + \left\|\inf_{\mu\neq i} u_h^{\mu,0} - \inf_{\mu\neq i} u^{\mu,0}\right\|_\infty + \inf_{\mu\neq i} u^{\mu,0}, \end{cases}$$

and using (4.1), we get

$$\begin{cases} b^i(\bar{u}^{i,1}, v^i) \leq \left(f^i\left(u_h^{i,0}\right) + \lambda Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0}, v^i\right), \quad \text{for all } v^i \in \left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,1} \leq \mathbf{k} + Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \inf_{\mu\neq i} u^{\mu,0}. \end{cases}$$

As $\bar{U}^1 = (\bar{u}^{i,1})_{1\leq i\leq J}$ is a subsolution for the system of V.I., where the solution is $\tilde{U}^1 = (\tilde{u}^{i,1})_{1\leq i\leq J} = \partial\left(f^i\left(u_h^{i,0}\right) + \lambda Ch^2 \left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0}, \mathbf{k} + Ch^2 \left|\ln h\right|^{\frac{3}{2}}\right)$.

Let $U^1 = (u^{i,1})_{1\leq i\leq J} = \partial\left(f^i\left(u_h^{i,0}\right) + \lambda u^{i,0}, \mathbf{k}\right)$ using the Proposition 2.4, we get

$$\begin{aligned}\left\|\tilde{u}^{i,1} - u^{i,1}\right\|_\infty \leq & C\left(\left\|f^i\left(u_h^{i,0}\right) + \lambda Ch^2\left|\ln h\right|^{\frac{3}{2}} + \lambda u^{i,0} - f^i\left(u_h^{i,0}\right) - \lambda u^{i,0}\right\|_\infty\right.\\ & \left.+\left\|\mathbf{k} + Ch^2\left|\ln h\right|^{\frac{3}{2}} - \mathbf{k}\right\|_\infty\right)\\ \leq & C\left(\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}} + Ch^2\left|\ln h\right|^{\frac{3}{2}}\right)\\ \leq & Ch^2\left|\ln h\right|^{\frac{3}{2}},\end{aligned}$$

and using the Theorem 2.1, we have

$$\bar{u}^{i,1} \leq \tilde{u}^{i,1} \leq u^{i,1} + Ch^2\left|\ln h\right|^{\frac{3}{2}}.$$

Now taking $\beta^{i,1} = \bar{u}^{i,1} - Ch^2\left|\ln h\right|^{\frac{3}{2}}$, we have

$$\beta^{i,1} \leq u^{i,1}, \tag{4.2}$$

and

$$\begin{aligned}\left\|\beta^{i,1} - u_h^{i,1}\right\|_\infty &\leq \left\|\bar{u}^{i,1} - Ch^2\left|\ln h\right|^{\frac{3}{2}} - u_h^{i,1}\right\|_\infty\\ &\leq \left\|\bar{u}^{i,1} - u_h^{i,1}\right\|_\infty + Ch^2\left|\ln h\right|^{\frac{3}{2}}\\ &\leq Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^{\frac{3}{2}}\\ &\leq Ch^2\left|\ln h\right|^2.\end{aligned} \tag{4.3}$$

Let $\bar{U}_h^1$ be the discrete solution of the system of V.I.

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, v_h^i - \bar{u}_h^{i,1}\right) \geqq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0}, v_h^i - \bar{u}_h^{i,1}\right), & \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J,\\ \bar{u}_h^{i,1} \leq r_h\left(k + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \quad v_h^i \leq r_h\left(k + \inf\limits_{\mu\neq i} u^{\mu,0}\right). \end{cases}$$

Then, as $\bar{U}_h^1 = \left(\bar{u}_h^{i,1}\right)_{1\leq i\leq J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \quad v_h^i \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right), \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0} - \lambda u_h^{i,0} + \lambda u_h^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq r_h\left(\mathbf{k} + \inf\limits_{\mu\neq i} u^{\mu,0}\right). \end{cases}$$

Then

$$\begin{cases} b^i\left(\bar{u}_h^{i,1}, \varphi_s\right) \leq \left(f^i\left(u^{i,0}\right) + \lambda u^{i,0} - \lambda u_h^{i,0} + \lambda u_h^{i,0}, \varphi_s\right), & \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,1} \leq \mathbf{k} + r_h\left(\inf\limits_{\mu\neq i} u^{\mu,0}\right) - r_h\left(\inf\limits_{\mu\neq i} u_h^{\mu,0}\right) + r_h\left(\inf\limits_{\mu\neq i} u_h^{\mu,0}\right), \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}_h^{i,1},\varphi_s\right) \le \left(f^i\left(u^{i,0}\right)+\lambda\left\|u^{i,0}-u_h^{i,0}\right\|_\infty+\lambda u_h^{i,0},\varphi_s\right), \quad \text{for all } \varphi_s,\ s=1,\dots,m(h), \\ \bar{u}_h^{i,1} \le \mathbf{k}+\left\|r_h\left(\inf_{\mu\neq i} u^{\mu,0}\right)-r_h\left(\inf_{\mu\neq i} u_h^{\mu,0}\right)\right\|_\infty+r_h\left(\inf_{\mu\neq i} u_h^{\mu,0}\right), \end{cases}$$

using (4.1) , we get

$$\begin{cases} b^i\left(\bar{u}_h^{i,1},\varphi_s\right) \le \left(f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0},\varphi_s\right), \quad \text{for all } \varphi_s,\ s=1,\dots,m(h), \\ \bar{u}_h^{i,1} \le \mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}+r_h\left(\inf_{\mu\neq i} u_h^{\mu,0}\right). \end{cases}$$

As $\bar{U}_h^1=\left(\bar{u}_h^{i,1}\right)_{1\le i\le J}$ is a subsolution for the system of V.I., where the solution is $\tilde{U}_h^1=\left(\tilde{u}_h^{i,1}\right)_{1\le i\le J}=\partial_h\left(f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0},\mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}\right)$.

Let $U_h^1=\left(u_h^{i,1}\right)_{1\le i\le J}=\partial_h\left(f^i\left(u^{i,0}\right)+\lambda u_h^{i,0},\mathbf{k}\right)$. Using Proposition 3.4, we have

$$\begin{aligned} \left\|\tilde{u}_h^{i,1}-u_h^{i,1}\right\|_\infty \le & C\left(\left\|f^i\left(u^{i,0}\right)+\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+\lambda u_h^{i,0}-f^i\left(u^{i,0}\right)-\lambda u_h^{i,0}\right\|_\infty\right. \\ & \left.+\left\|\mathbf{k}+Ch^2\left|\ln h\right|^{\frac{3}{2}}-\mathbf{k}\right\|_\infty\right) \\ \le & C\left(\lambda Ch^2\left|\ln h\right|^{\frac{3}{2}}+Ch^2\left|\ln h\right|^{\frac{3}{2}}\right) \\ \le & Ch^2\left|\ln h\right|^{\frac{3}{2}}, \end{aligned}$$

and using Theorem 3.1, we get

$$\bar{u}_h^{i,1} \le \tilde{u}_h^{i,1} \le u_h^{i,1}+Ch^2\left|\ln h\right|^{\frac{3}{2}}.$$

Now taking $\alpha_h^{i,1}=\bar{u}_h^{i,1}-Ch^2\left|\ln h\right|^{\frac{3}{2}}$, we get

$$\alpha_h^{i,1} \le u_h^{i,1} \tag{4.4}$$

and

$$\begin{aligned} \left\|\alpha_h^{i,1}-u^{i,1}\right\|_\infty & \le \left\|\bar{u}_h^{i,1}-Ch^2\left|\ln h\right|^{\frac{3}{2}}-u^{i,1}\right\|_\infty \\ & \le \left\|\bar{u}_h^{i,1}-u^{i,1}\right\|_\infty+Ch^2\left|\ln h\right|^{\frac{3}{2}} \\ & \le Ch^2\left|\ln h\right|^2+Ch^2\left|\ln h\right|^{\frac{3}{2}} \\ & \le Ch^2\left|\ln h\right|^2. \end{aligned} \tag{4.5}$$

Then, according to (4.2), (4.3) and (4.4), (4.5), we get

$$\begin{aligned} u^{i,1} \le & \alpha_h^{i,1}+Ch^2\left|\ln h\right|^2 \le u_h^{i,1}+Ch^2\left|\ln h\right|^2, \\ u_h^{i,1} \le & \beta^{i,1}+Ch^2\left|\ln h\right|^2 \le u^{i,1}+Ch^2\left|\ln h\right|^2. \end{aligned}$$

Thus,

$$\left\|u^{i,1}-u_h^{i,1}\right\|_\infty \le Ch^2\left|\ln h\right|^2.$$

Therefore,

$$\max_{1\leq i\leq J}\left\|u^{i,1}-u_h^{i,1}\right\|_\infty \leq Ch^2\left|\ln h\right|^2.$$

For k we assume that

$$\left\|u^{i,k-1}-u_h^{i,k-1}\right\|_\infty \leq Ch^2\left|\ln h\right|^2 \tag{4.6}$$

and we prove that

$$\left\|u^{i,k}-u_h^{i,k}\right\|_\infty \leq Ch^2\left|\ln h\right|^2.$$

For that, consider the following system of continuous V.I.

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i-\bar{u}^{i,k}\right) \geqq \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}, v^i-u^{i,k}\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k}\leq \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}, \quad v^i\leq \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

Then, as $\bar{U}^k=\left(\bar{u}^{i,k}\right)_{1\leq i\leq J}$ be a solution to a system of V.I. it is also a subsolution i.e.,

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i\right) \leq \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}, v^i\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k}\leq \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}, \quad v^i\leq \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

Then

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i\right) \leq \left(f^i\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1}-\lambda u^{i,k-1}+\lambda u^{i,k-1}, v^i\right), \\ \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k}\leq \mathbf{k}+\inf\limits_{\mu\neq i} u_h^{\mu,k-1}-\inf\limits_{\mu\neq i} u^{\mu,k-1}+\inf\limits_{\mu\neq i} u^{\mu,k-1}, \end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i\right) \leq \left(f^i\left(u_h^{i,k-1}\right)+\lambda\left\|u_h^{i,k-1}-u^{i,k-1}\right\|_\infty+\lambda u^{i,k-1}, v^i\right), \\ \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k}\leq \mathbf{k}+\left\|\inf\limits_{\mu\neq i} u_h^{\mu,k-1}-\inf\limits_{\mu\neq i} u^{\mu,k-1}\right\|_\infty+\inf\limits_{\mu\neq i} u^{\mu,k-1}. \end{cases}$$

Using (4.6), we get

$$\begin{cases} b^i\left(\bar{u}^{i,k}, v^i\right) \leq \left(f^i\left(u_h^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u^{i,k-1}, v^i\right), \quad \text{for all } v^i\in\left(H_0^1(\Omega)\right)^J, \\ \bar{u}^{i,k}\leq \mathbf{k}+Ch^2\left|\ln h\right|^2+\inf\limits_{\mu\neq i} u^{\mu,k-1}. \end{cases}$$

Let $\bar{U}^k=\left(\bar{u}^{i,k}\right)_{1\leq i\leq J}$ be a subsolution for the system of V.I. whose solution is $\tilde{U}^k=\left(\tilde{u}^{i,k}\right)_{1\leq i\leq J}=\partial\left(f^i\left(u_h^{i,k-1}\right)+\lambda Ch^2\left|\ln h\right|^2+\lambda u^{i,k-1}, \mathbf{k}+Ch^2\left|\ln h\right|^2\right)$.

Then, as $U^k = \left(u^{i,k}\right)_{1\le i\le J} = \partial\left(f^i\left(u_h^{i,k-1}\right) + \lambda u^{i,k-1}, \mathbf{k}\right)$ making use of Proposition 2.4, we have

$$\begin{aligned}\left\|\tilde{u}^{i,k} - u^{i,k}\right\|_\infty \le & C\left(\left\|f^i\left(u_h^{i,k-1}\right) + \lambda Ch^2\left|\ln h\right|^2 - f^i\left(u_h^{i,k-1}\right)\right\|_\infty\right.\\ & \left. + \left\|\mathbf{k} + Ch^2\left|\ln h\right|^2 - \mathbf{k}\right\|_\infty\right)\\ \le & C(\lambda Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2)\\ \le & Ch^2\left|\ln h\right|^2,\end{aligned}$$

and, using Theorem 2.1, we have

$$\bar{u}^{i,k} \le \tilde{u}^{i,k} \le u^{i,k} + Ch^2\left|\ln h\right|^2.$$

Now putting $\beta^{i,k} = \bar{u}^{i,k} - Ch^2\left|\ln h\right|^2$, we get

$$\beta^{i,k} \le u^{i,k} \tag{4.7}$$

and

$$\begin{aligned}\left\|\beta^{i,k} - u_h^{i,k}\right\|_\infty \le & \left\|\bar{u}^{i,k} - Ch^2\left|\ln h\right|^2 - u_h^{i,k}\right\|_\infty\\ \le & \left\|\bar{u}^{i,k} - u_h^{i,k}\right\|_\infty + Ch^2\left|\ln h\right|^2\\ \le & Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2\\ \le & Ch^2\left|\ln h\right|^2.\end{aligned} \tag{4.8}$$

Let $\bar{U}_h^k$ be the discrete solution of the following system of V.I.

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, v_h^i - \bar{u}_h^{i,k}\right) \geqq \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, v_h^i - \bar{u}_h^{i,k}\right), \quad \text{for all } v_h^i \in \left(\mathbb{V}_h\right)^J,\\ \bar{u}_h^{i,k} \le r_h\left(\mathbf{k} + \inf\limits_{\mu\ne i} u^{\mu,k-1}\right), \quad v \le r_h\left(\mathbf{k} + \inf\limits_{\mu\ne i} u^{\mu,k-1}\right).\end{cases}$$

Then, as $\bar{U}_h^k = \left(\bar{u}_h^{i,k}\right)_{1\le i\le J}$ is a solution to a system of V.I. it is also a subsolution, i.e.,

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \le \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1}, \varphi_s\right), \quad \text{for all } \varphi_s,\ s = 1, \ldots, m(h),\\ \bar{u}_h^{i,k} \le r_h\left(\mathbf{k} + \inf\limits_{\mu\ne i} u^{\mu,k-1}\right), \quad v \le r_h\left(\mathbf{k} + \inf\limits_{\mu\ne i} u^{\mu,k-1}\right).\end{cases}$$

Then we have

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \le \left(f^i\left(u^{i,k-1}\right) + \lambda u^{i,k-1} - \lambda u_h^{i,k-1} + \lambda u_h^{i,k-1}, \varphi_s\right), \quad \text{for all } \varphi_s,\\ \bar{u}_h^{i,k} \le \mathbf{k} + r_h \inf\limits_{\mu\ne i} u^{\mu,k-1} - r_h \inf\limits_{\mu\ne i} u_h^{\mu,k-1} + r_h \inf\limits_{\mu\ne i} u_h^{\mu,k-1},\end{cases}$$

and

$$\begin{cases} b^i\left(\bar{u}_h^{i,k}, \varphi_s\right) \le \left(f^i\left(u^{i,k-1}\right) + \lambda\left\|u^{i,k-1} - u_h^{i,k-1}\right\|_\infty + \lambda u_h^{i,k-1}, \varphi_s\right), \quad \text{for all } \varphi_s,\\ \bar{u}_h^{i,k} \le \mathbf{k} + \left\|r_h \inf\limits_{\mu\ne i} u^{\mu,k-1} - r_h \inf\limits_{\mu\ne i} u_h^{\mu,k-1}\right\|_\infty + r_h \inf\limits_{\mu\ne i} u_h^{\mu,k-1}.\end{cases}$$

Using (4.6), we obtain

$$\begin{cases} b^i\left(\bar{u}_h^{i,k},\varphi_s\right) \leq \left(f^i\left(u^{i,k-1}\right) + \lambda Ch^2\left|\ln h\right|^2 + \lambda u_h^{i,k-1},\varphi_s\right), & \text{for all } \varphi_s,\\ \bar{u}_h^{i,k} \leq \mathbf{k} + Ch^2\left|\ln h\right|^2 + r_h \inf_{\mu\neq i} u_h^{\mu,k-1}. \end{cases}$$

So, $\bar{U}_h^k = \left(\bar{u}_h^{i,k}\right)_{1\leq i\leq J}$ is a subsolution for the system of V.I. whose solution is $\tilde{U}_h^k = \left(\tilde{u}_h^{i,k}\right)_{1\leq i\leq J} = \partial_h\left(f^i\left(u^{i,k-1}\right) + \lambda Ch^2\left|\ln h\right|^2 + \lambda u_h^{i,k-1}, \mathbf{k} + Ch^2\left|\ln h\right|^2\right)$. Then, as $U_h^k = \left(u_h^{i,k}\right)_{1\leq i\leq J} = \partial_h\left(f^i\left(u^{i,k-1}\right) + \lambda u_h^{i,k-1}, \mathbf{k}\right)$ making use of Proposition 3.4, we have

$$\begin{aligned} \left\|\tilde{u}_h^{i,k} - u_h^{i,k}\right\|_\infty \leq & C\left(\left\|f^i\left(u^{i,k-1}\right) + \lambda Ch^2\left|\ln h\right|^2 - f^i\left(u^{i,k-1}\right)\right\|_\infty \right.\\ & \left. + \left\|\mathbf{k} + Ch^2\left|\ln h\right|^2 - \mathbf{k}\right\|_\infty\right)\\ \leq & C(\lambda Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2)\\ \leq & Ch^2\left|\ln h\right|^2, \end{aligned}$$

and, using Theorem 3.1, we have

$$\bar{u}_h^{i,k} \leq \tilde{u}_h^{i,k} \leq u_h^{i,k} + Ch^2\left|\ln h\right|^2.$$

Now, putting $\alpha_h^{i,k} = \bar{u}_h^{i,k} - Ch^2\left|\ln h\right|^2$, we have

$$\alpha_h^{i,k} \leq u_h^{i,k} \tag{4.9}$$

and

$$\begin{aligned} \left\|\alpha_h^{i,k} - u^{i,k}\right\|_\infty \leq & \left\|\bar{u}_h^{i,k} - Ch^2\left|\ln h\right|^2 - u^{i,k}\right\|_\infty\\ \leq & \left\|\bar{u}_h^{i,k} - u^{i,k}\right\|_\infty + Ch^2\left|\ln h\right|^2\\ \leq & Ch^2\left|\ln h\right|^2 + Ch^2\left|\ln h\right|^2\\ \leq & Ch^2\left|\ln h\right|^2. \end{aligned} \tag{4.10}$$

Then, combining (4.7), (4.8) and (4.9), (4.10), we get

$$\begin{aligned} u^{i,k} \leq & \alpha_h^{i,k} + Ch^2\left|\ln h\right|^2 \leq u_h^{i,k} + Ch^2\left|\ln h\right|^2,\\ u_h^{i,k} \leq & \beta^{i,k} + Ch^2\left|\ln h\right|^2 \leq u^{i,k} + Ch^2\left|\ln h\right|^2. \end{aligned}$$

Thus,

$$\left\|u^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2\left|\ln h\right|^2.$$

Therefore,

$$\left\|U^k - U_h^k\right\|_\infty = \max_{1\leq i\leq J}\left\|u^{i,k} - u_h^{i,k}\right\|_\infty \leq Ch^2\left|\ln h\right|^2. \qquad \square$$

5. Asymptotic Behavior in L^∞-Norm

This section is devoted to the proof of the main result of the present paper, where we prove the optimal L^∞-asymptotic behavior for the system of parabolic quasi-variational inequalities with nonlinear source terms. More precisely, we evaluate the variation in L^∞ between U_h^N, the discrete solution calculated at the moment $T = N\Delta t$ and U^∞, the stationary continuous solution of the system of QVIs.

Theorem 5.1. *Under the results of the Proposition* 2.3 *and Theorem* 4.1*, we have*

$$\left\| U_h^N - U^\infty \right\|_\infty \leq C \left[h^2 \left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t} \right)^N \right], \tag{5.1}$$

where C is a constant independent of h and N, $\beta > 0$ is constant and $\alpha < \beta$ Lipschitz constant.

Proof. We have

$$u_h^{i,k} = u_h^i\left(t, x\right), \quad \text{for } t \in \left](k-1)\, t, kt\right[.$$

Thus,

$$u_h^{i,N} = u_h^i\left(T, x\right),$$

then

$$\begin{aligned} \left\| u_h^{i,N} - u^{i,\infty} \right\|_\infty &= \left\| u_h^{i,N} - u^{i,N} + u^{i,N} - u^{i,\infty} \right\|_\infty \\ &\leq \left\| u_h^{i,N} - u^{i,N} \right\|_\infty + \left\| u^{i,N} - u^{i,\infty} \right\|_\infty . \end{aligned}$$

Using Theorem 4.1 and Proposition 2.3, we get,

$$\left\| u_h^{i,N} - u^{i,\infty} \right\|_\infty \leq C \left[h^2 \left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t} \right)^N \right],$$

which yields the following estimate:

$$\left\| U_h^N - U^\infty \right\|_\infty = \max_{1\leq i\leq J} \left\| u_h^{i,N} - u^{i,\infty} \right\|_\infty \leq C \left[h^2 \left|\ln h\right|^2 + \left(\frac{1+\alpha\Delta t}{1+\beta\Delta t} \right)^N \right]. \qquad \square$$

Remark 5.1. In the previous estimate (5.1), $\left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N$ tends to 0 when $N \to +\infty$. Then, we obtain the optimal L^∞-error estimate for the system of elliptic quasi-variational inequalities related to management of energy production problems (cf. [16]):

$$\left\| U_h^\infty - U^\infty \right\|_\infty \leq C h^2 \left|\ln h\right|^2 .$$

If we replace Mu^i in (1.3) by $Mu = \mathbf{k} + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} \left(u + \xi\right)$ and $f\left(u\right)$ by f, the problem (2.2) reduces to the parabolic quasi-variational inequalities related to impulse control problem with linear source term (cf. [10]). Find $u \in K(u)$

$$\left(\frac{\partial u}{\partial t}, v - u \right) + a\left(u, v-u\right) \geq \left(f, v-u\right), \quad \text{for all } v \in K(u),$$

with

$$K(u)=\left\{u\in L^2\left(0,T;H_0^1\left(\Omega\right)\right)\mid u\leq Mu,\ u\left(0,x\right)=u_0 \text{ in } \Omega\right\}.$$

In this case, the error estimate given in (5.1) becomes

$$\left\|u_h^N-u^\infty\right\|_\infty\leq C\left[h^2\left|\ln h\right|^2+\left(\frac{1}{1+\beta\Delta t}\right)^N\right].$$

If we replace Mu^i in (1.3) by $Mu^i=l+u^{i+1}$, where $Mu^i=l+u^{i+1}$ represents the obstacle of Hamilton Jacobi Bellman equation, the problem (2.2) reduces to the system of evolutionary Hamilton Jacobi Bellman (HJB) equation with nonlinear source terms (cf [8]): Find a victor $U=\left(u^1,\ldots,u^J\right)\in\left(L^2\left(0,T;H_0^1\left(\Omega\right)\right)\right)^J$ such that

$$\begin{cases}\left(\dfrac{\partial u^i}{\partial t},v^i-u^i\right)+a^i\left(u^i,v^i-u^i\right)\geqq\left(f^i\left(u^i\right),v^i-u^i\right),\\ u^i\leq l+u^{i+1},\quad v^i\leq l+u^{i+1},\quad u^{J+1}=u^1,\quad 1\leq i\leq J,\\ u^i\left(x,0\right)=u_0^i \text{ in } \Omega,\quad u^i=0 \text{ on } \partial\Omega.\end{cases}$$

In this case, we get the following error estimate:

$$\max_{1\leq i\leq J}\left\|u_h^{i,N}-u^{i,\infty}\right\|_\infty\leq C\left[h^2\left|\ln h\right|^2+\left(\frac{1+\alpha\Delta t}{1+\beta\Delta t}\right)^N\right].$$

Conclusion 1. We have introduced a new approach and we have obtained the optimal L^∞-asymptotic behavior for the finite element approximation of the system of parabolic quasi-variational inequalities with nonlinear source terms. This method stands on the Bensoussan-Lions algorithm and the concept of subsolutions. A future work will consolidate our theoretical results by numerical simulation, where efficient numerical monotone algorithms will be treated.

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