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ON ZERO FREE REGIONS FOR DERIVATIVES OF A POLYNOMIAL

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ABSTRACT. Let P_n denote the set of polynomials of the form

$$p(z) = (z-a)^m \prod_{k=1}^{n-m} (z - z_k),$$

with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$. For the polynomials of the form $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$, Brown [2] stated the problem "Find the best constant C_n such that $p'(z)$ does not vanish in $|z| < C_n$". He also conjectured in the same paper that $C_n = \frac{1}{n}$. This problem was solved by Aziz and Zarger [1]. In this paper, we obtain the results which generalizes the results of Aziz and Zarger.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = \prod_{k=1}^{n}(z - z_k)$ be a complex polynomial of degree n. The classical Gauss-Lucas theorem states that every critical point of a complex polynomial p of degree at least 2 lies in the convex hull of its zeros. This theorem has been further investigated and developed. About the location of critical point relative to each individual zero, a possible answer is given by the famous conjecture known in literature as Sendov's conjecture.

Conjecture 1 (Sendov's Conjecture). If all the zeros of a polynomial $p(z)$ lie in $|z| \leq 1$, then for any zero z_0 of p, the disc $|z - z_0| \leq 1$ contains at least one critical point of p.

This conjecture has attracted much attention. About 100 papers have been published related to this conjecture. This conjecture has so far been verified for general

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polynomials of degree less than or equal to 8. However the problem is still unproved in general.

In connection with this conjecture, Brown [2] observed that, if $p(z) = z(z-1)^{n-1}$, then $p'(\frac{1}{n}) = 0$ and posed the following problem.

"Let $p(z) = z\prod_{k=1}^{n-1}(z-z_k)$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$. Find the best constant C_n such that $p'(z)$ does not vanish in $|z| < C_n$".

However, Brown himself conjectured that $C_n = \frac{1}{n}$. This problem has been settled by Aziz and Zarger [1], in fact they proved the following.

Theorem 1.1. *If $p(z) = z\prod_{k=1}^{n-1}(z-z_k)$ is a polynomial of degree n, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in $|z| < \frac{1}{n}$.*

As a generalization of Theorem 1.1, N. A. Rather and F. Ahmad [3] have proved the following result.

Theorem 1.2. *Let $p(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k)$ with $|a| \leq 1$ be a polynomial of degree n with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in the region*

$$\left|z - \left(\frac{n-1}{n}\right)a\right| < \frac{1}{n}.$$

The result is best possible as is shown by the polynomial

$$p(z) = (z-a)(z-e^{i\alpha})^{n-1}, \quad 0 \leq \alpha < 2\pi.$$

N. A. Rather and F. Ahmad also proved the following result in the same paper.

Theorem 1.3. *Let $p(z) = (z-a)^m\prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree n with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then $p'(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros of $p'(z)$ lie in the region*

$$\left|z - \left(\frac{n-m}{n}\right)a\right| \geq \frac{m}{n}.$$

The result is best possible as is shown by the polynomial

$$p(z) = (z-a)^m(z-e^{i\alpha})^{n-m}, \quad 0 \leq \alpha < 2\pi.$$

Zarger and Manzoor [4] have extended Theorem 1.1 to the second derivative $p''(z)$ of a polynomial of the form $p(z) = z^m\prod_{k=1}^{n-m}(z-z_k)$, with $|z_k| \geq 1$ for $1 \leq k \leq n-m$. In fact they proved the following.

Theorem 1.4. *If $p(z) = z^m\prod_{k=1}^{n-m}(z-z_k)$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p''(z)$ does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$.*

Zarger and Manzoor [4] also obtained the following result for the polynomial $p^{(m)}(z)$, $m \geq 1$.

Theorem 1.5. *If $p(z) = z^m\prod_{k=1}^{n-m}(z-z_k)$ is a polynomial of degree n with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, does not vanish in $|z| < \frac{m!}{n(n-1)\cdots(n-m+1)}$.*

In this paper, we first prove the following theorem which generalize the result of Theorem 1.4.

Theorem 1.6. *Let* $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ *be a polynomial of degree* n *with* $|a| \le 1$, *and* $|z_k| \ge 1$ *for* $1 \le k \le n-m$, *then* $p''(z)$ *has* $(m-2)$ *fold zero at* $z = a$ *and remaining* $(n-m)$ *zeros lie in the region*

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

Proof. We can write

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, then by Theorem 1.3, the polynomial

$$p'(z) = (z-a)^{m-1} R(z),$$

where $R(z) = (z-a)Q'(z) + mQ(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{n-m}{n}\right) a \right| \ge \frac{m}{n}.$$

Now, consider the polynomial

$$S(z) = p'\left(\frac{m}{n}z + \frac{n-m}{n}a\right) \tag{1.1}$$

or

$$S(z) = \left(\frac{m}{n}\right)^{m-1} (z-a)^{m-1} R\left(\frac{m}{n}z + \frac{n-m}{n}a\right),$$

then $S(z)$ is a polynomial of degree $n-1$ with $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in $|z| \ge 1$.

Now, applying Theorem 1.3 to the polynomial $S(z)$, the derivative $S'(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{(n-1)-(m-1)}{n-1}\right) a \right| \ge \frac{m-1}{n-1},$$

which is equivalent to

$$\left| z - \left(\frac{n-m}{n-1}\right) a \right| \ge \frac{m-1}{n-1}.$$

Replacing z by $\frac{n}{m}z + \left(\frac{m-n}{m}\right) a$, in equation (1.1) and differentiating, we obtain

$$p''(z) = (z-a)^{m-2} T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$.

Applying above, we see $p''(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

This completes the proof. □

Remark 1.1. For $a = 0$, it reduces to Theorem 1.4.

Our next result generalizes Theorem 1.5 to the polynomial of the form $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$.

Theorem 1.7. *If $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree n with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, has all its zeros in the region*

$$\left|z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)}\right)a\right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

Proof. We can write

$$p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$$

or

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, $|z_k| \geq 1$, $1 \leq k \leq n-m$.

From the proof of Theorem 1.6, we can write

$$p''(z) = (z-a)^{m-2}T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$. Also, $p''(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \frac{n(n-1) - m(m-1)}{n(n-1)}a\right| \geq \frac{m(m-1)}{n(n-1)}.$$

Now, consider the polynomial

$$U(z) = p''\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1) - m(m-1)}{n(n-1)}a\right) \tag{1.2}$$

or

$$U(z) = \left(\frac{m(m-1)}{n(n-1)}\right)^{m-2}(z-a)^{m-2}T\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1) - m(m-1)}{n(n-1)}a\right).$$

Then $U(z)$ has $(m-2)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in $|z| \geq 1$.

Again, applying Theorem 1.3 to $U(z)$, which is a polynomial of degree $n-2$, the derivative $U'(z)$ has $(m-3)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(\frac{n-2-(m-2)}{n-2}\right)a\right| \geq \frac{m-2}{n-2},$$

which is equivalent to

$$\left|z - \left(\frac{n-m}{n-2}\right)a\right| \geq \frac{m-2}{n-2}.$$

Replacing z by $\frac{n(n-1)}{m(m-1)}z + \frac{m(m-1)-n(n-1)}{m(m-1)}a$, in (1.2) and differentiating, we obtain

$$p'''(z) = (z-a)^{m-3}V(z),$$

where $V(z) = (z-a)T'(z) + (m-2)T(z)$ has $(m-3)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie

$$\left| z - \left(1 - \frac{m(m-1)(m-2)}{n(n-1)(n-2)} \right) a \right| \geq \frac{m(m-1)(m-2)}{n(n-1)(n-2)}.$$

Proceeding similarly, for any positive integer $m = 1, 2, \ldots, n-1$, we see that the polynomial $p^{(m)}(z)$ has all its zeros in the region

$$\left| z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)} \right) a \right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

This completes the proof. □

Remark 1.2. For $a = 0$, it reduces to Theorem 1.5.

Remark 1.3. For $m = 1$, it reduces to Theorem 1.2.

Remark 1.4. For $a = 0$ and $m = 1$, it reduces to the result of Aziz and Zarger.

References

[1] A. Aziz and B. A. Zarger, *On the critical points of a polynomial,* Bull. Aust. Math. Soc. **57** (1998), 173–174. https://doi.org/10.1017/S000497270003152X
[2] J. E. Brown, *On the Ilief-Sendov conjecture,* Pacific J. Math. **135** (1988), 223–232.
[3] N. A. Rather and F. Ahmad, *On the critical points of a polynomial*, BIBECHANA **9** (2013), 28–32. https://doi.org/10.1017/S000497270003152X
[4] B. A. Zarger and A. W. Manzoor, *On zero free regions for derivative of a polynomial*, BIBECHANA **14** (2017), 48–53. https://doi.org/10.3126/bibechana.v14i0.15521

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