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THREE SOLUTIONS FOR p-HAMILTONIAN SYSTEMS WITH IMPULSIVE EFFECTS

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ABSTRACT. In this paper, we give some new criteria that guarantee the existence of at least three weak solutions to a p-Hamiltonian boundary value problem generated by impulsive effects. To ensure the existence of these solutions, we use variational methods and critical point theory as our main tools.

1. INTRODUCTION.

In this research, we prove the existence of at least three weak solutions to the following second-order impulsive p-Hamiltonian system

(1.1)
$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \nabla G(t,u) + \nabla H(u), \quad \text{a.e. } t \in J, \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \quad i = 1,2,\ldots,N, j = 1,2,\ldots,m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Here, we assume that

- $N \geq 1$, $m \geq 2$, $p > 1$, $T > 0$ and $\lambda \in \mathbb{R}$;
- the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in $[0,T]$ and C^1 in $\mathbb{R}^N$;
- $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $G(\cdot, x)$ is continuous on $[0,T]$ for all $x \in \mathbb{R}^N$ and $G(t,\cdot)$ is C^1 on $\mathbb{R}^N$ for almost every $t \in [0,T]$;
- $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $J = [0,T] \setminus \{t_1, t_2, \ldots, t_m\}$, $u(t) = (u_1(t), \ldots, u_N(t))$ and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-)$ such that $u_i'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u_i'(t)$;

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- the functions $I_{ij} : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, m$, are continuous;
- $A(t) = (a_{ij}(t))_{N\times N}$ is an $N \times N$ continuous symmetric matrix and there is a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x, x) \geq \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0, T]$;
- $H : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function for which there is a constant $0 < L < \frac{\min\{1,\underline{\lambda}\}}{2p}$ such that $|H(x)| \leq L|x|^p$ for every $x \in \mathbb{R}^N$.

The study of the multiplicity of the solutions of Hamiltonian systems, as particular cases of dynamical systems, is mathematically important and interesting from a practical point of view. This is because these systems constitute a natural framework for the mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. Inspired by the monographs [16] and [21], the existence and multiplicity of weak solutions for Hamiltonian systems have been investigated by many authors using variational methods. See [6, 7, 9, 11, 12, 17–19, 27–31, 33] and the references therein for example.

On the other hand, impulsive effects describe some discontinuous processes and occur in many research fields such as SIR epidemic models, controllability and optimization, etc. (see [8, 20]). In the past few decades, a series of nonlinear functional methods were applied for dealing with the existence of solutions to boundary value problems for impulsive differential equations. These include the coincidence degree theory, the comparison principles and fixed point theorems.

In particular, in the recent years, the variational method has been used successfully in the investigation of the existence and multiplicity of solutions to boundary value problems for differential equations with impulsive effects. See [1, 2] and the references therein. For the background, theory and applications of impulsive differential equations, we refer the interested readers to [4, 13, 23]. Recently, a great deal of work has been done on the existence of multiple solutions for second-order impulsive p-Hamiltonian systems. We refer the interested reader to [10, 15, 25, 26, 32], in which second-order Hamiltonian systems with impulsive effects have been examined.

Our results are motivated by the recent papers [14] and [24]. In [14], Li et al. have studied the three periodic solutions for p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t, u) + \mu\nabla G(t, u), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.2}$$

Their technical approach was based on the two general three critical points theorems of Averna and Bonanno [3] and Ricceri [22]. In [24], Shang and Zhang obtained three solutions to the perturbed Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x, u) + g(x, u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

by using Theorem 2.1 below.

2. Preliminaries

In this article, we use the following theorem of Bonanno to prove the existence of three solutions for problem (1.1).

Theorem 2.1 ([5]). *Let X be a separable and reflexive real Banach space, and let $\phi, \psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals. Assume that ϕ is sequentially weakly lower semicontinuous and even, that ψ is sequentially weakly continuous and odd, and that, for some $a > 0$ and for each $\lambda \in [-a, a]$, the functional $\phi + \lambda\psi$ satisfies the Palais-Smale condition and*

$$\lim_{\|x\|\to+\infty} [\phi(x) + \lambda\psi(x)] = +\infty.$$

If there exists $k > 0$ such that

$$\inf_{x\in X} \phi(x) < \inf_{|\psi(x)|<k} \phi(x),$$

then, for every $b > 0$ there exists an open interval $\Lambda \subseteq [-b, b]$ and a positive real number σ such that for every $\lambda \in \Lambda$, the equation

$$\phi'(x) + \lambda\psi'(x) = 0,$$

admits at least three solutions in X whose norms are less than σ.

Here, we recall some basic concepts that will be used in what follows. Let

$$\begin{aligned} W_T^{1,p} =&\{u : [0, T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, u(0) = u(T), \\ &u' \in L^p([0, T], \mathbb{R}^N)\}, \end{aligned}$$

which is endowed with the norm

$$\|u\| = \left(\int_0^T |u'(t)|^p + \Big(A(t)|u(t)|^{p-2}u(t), u(t)\Big)\, dt\right)^{\frac{1}{p}}.$$

Observe that

$$\begin{aligned} (A(t)|x|^{p-2}x, x) &= |x|^{p-2} \sum_{i,j=1}^{N} a_{ij}(t)x_ix_j \\ &\le |x|^{p-2} \sum_{i,j=1}^{N} |a_{ij}(t)||x_i||x_j| \\ &\le \left(\sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty\right) |x|^p. \end{aligned}$$

Then, there exists a constant $\overline{\lambda} \le \sum_{i,j=1}^N \|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x, x) \le \overline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$. So,

$$\min\{1, \underline{\lambda}\}|||u|||^p \le \|u\|^p \le \max\{1, \overline{\lambda}\}|||u|||^p, \tag{2.1}$$

where

$$|||u||| = \left(\int_0^T |u(t)|^p dt + \int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}},$$

is the usual norm of $W_T^{1,p}$. Let

$$k_0 = \sup_{u\in W_T^{1,p}\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}, \quad \|u\|_\infty = \sup_{t\in[0,T]} |u(t)|, \tag{2.2}$$

where $|\cdot|$ is the usual norm of $\mathbb{R}^N$. Since $W_T^{1,p} \hookrightarrow C^0$ is compact, one has $k_0 < +\infty$ and for each $u \in W_T^{1,p}$ there exists $\xi \in [0,T]$ such that $|u(\xi)| = \min_{t\in[0,T]} |u(t)|$. Hence, by Hölder's inequality, one has

$$\begin{aligned}
|u(t)| &= \left|\int_\xi^t u'(s)ds + u(\xi)\right| \\
&\le \int_0^T |u'(s)|ds + \frac{1}{T}\int_0^T |u(\xi)|ds \\
&\le \int_0^T |u'(s)|ds + \frac{1}{T}\int_0^T |u(s)|ds \\
&\le T^{\frac{1}{q}}\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}} + T^{-\frac{1}{p}}\left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}} \\
&\le \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}} + \left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}}\right) \\
&\le \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\int_0^T |u'(s)|^p ds + \int_0^T |u(s)|^p ds\right)^{\frac{1}{p}} \\
&= \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}|||u|||,
\end{aligned}$$

for each $t \in [0,T]$ and $q = \frac{p}{p-1}$. So, by (2.1) and the above expression, we obtain

$$\|u\|_\infty \le \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}|||u||| \le \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}}\|u\|.$$

Then, from this and (2.2) it follows that

$$k_0 \le k = \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}}.$$

As usual, a weak solution to problem (1.1) is any $u \in W_T^{1,p}$ such that

$$\begin{aligned}
&\int_0^T \left[\left(|u'(t)|^{p-2}u'(t), v'(t)\right) + \left(A(t)|u(t)|^{p-2}u(t), v(t)\right)\right] dt - \int_0^T (\nabla G(t,u(t)), v(t))\, dt \\
&- \int_0^T (\nabla H(u(t)), v(t))\, dt + \sum_{j=1}^p\sum_{i=1}^N I_{ij}(u_i(t_j))v_i(t_j) - \lambda\int_0^T (\nabla F(t,u(t)), v(t))\, dt = 0,
\end{aligned}$$

for all $v \in W_T^{1,p}$.

3. The Main Results

Now, we present our main results.

Theorem 3.1. *Suppose that F, G, H and I_{ij} satisfy the following conditions.*

(H1) $H(\cdot)$ *is even.*

(H2) $G(t,\cdot)$ *is even and* $F(t,\cdot)$ *is odd for almost every* $t \in [0,T]$.

(H3) *The functions* I_{ij}, $i = 1,2,\dots,N,$ and $j = 1,2,\dots,m$, *are odd.*

(H4) $\lim_{|x|\to 0} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly for almost every* $t \in [0,T]$.

(H5) $\lim_{|x|\to +\infty} \frac{|\nabla G(t,x)|}{|x|^{p-1}} = 0$ *uniformly for almost every* $t \in [0,T]$.

(H6) *There exist constants* $c > 0$ *and* $1 \le q < p$ *such that*

$$|\nabla F(t,x)| \le c(1+|x|^{q-1}),$$

for all $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$.

(H7) *There is a constant* $B \ge 0$ *such that* $G(t,x) \ge 2r\frac{|x|^p}{p} - B$ *for all* $x \in \mathbb{R}^N$ *and almost every* $t \in [0,T]$. *Here,* $r = \sup\left\{\frac{1}{\int_0^T |u(t)|^p dt} : \ \|u\| = 1\right\}$.

(H8) *For any* $i \in \{1,2,\dots,N\}$ *and* $j \in \{1,2,\dots,m\}$ *there exist constants* $a_{ij} > 0$, $b_{ij} > 0$ *and* $\gamma_{ij} \in [0,1]$ *such that*

$$I_{ij}(y) \ge -a_{ij} - b_{ij}y^{\gamma_{ij}} \quad (y \ge 0) \quad \text{and} \quad I_{ij}(y) \le a_{ij} + b_{ij}(-y)^{\gamma_{ij}} \quad (y \le 0).$$

Then, for every $b > 0$ *there exist an open interval* $\Lambda \subseteq [-b,b]$ *and a positive real number* σ *such that for every* $\lambda \in \Lambda$, *problem* (1.1) *admits at least three solutions in* $W_T^{1,p}$ *whose norms are less than* σ.

Proof. Let $X = W_T^{1,p}$ be endowed with $\|\cdot\|$, and for each u in X let

$$\phi(u) = \frac{1}{p}\|u\|^p - \int_0^T G(t,u(t))dt - \int_0^T H(u(t))dt,$$

$$\psi(u) = \frac{1}{\lambda}\sum_{j=1}^m \sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(t)dt - \int_0^T F(t,u(t))dt.$$

Then, for every $u, v \in X$,

$$\begin{aligned}\phi'(u)(v) =& \int_0^T \left[\Big(|u'(t)|^{p-2}u'(t), v'(t)\Big) + \Big(A(t)|u(t)|^{p-2}u(t), v(t)\Big)\right] dt \\ &- \int_0^T (\nabla G(t,u(t)), v(t))\, dt - \int_0^T (\nabla H(u(t)), v(t))\, dt, \\ \psi'(u)(v) =& \frac{1}{\lambda}\sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_i(t_j))v_i(t_j) - \int_0^T (\nabla F(t,u(t)), v(t))\, dt.\end{aligned}$$

Since the critical points of the functional $\phi + \lambda\psi$ on X are exactly the weak solutions of problem (1.1), our aim is to apply Theorem 2.1 to ϕ and ψ. It is well-known that ϕ is a continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, ψ is continuously Gateaux differentiable and sequentially weakly continuous. Also, by (H1), (H2) and (H3), ϕ is even and ψ is odd. Owning to the assumption (H8), we have that

$$\begin{aligned}\int_0^z I_{ij}(t)dt &\geq -a_{ij}z - \frac{b_{ij}}{\gamma_{ij}+1}z^{\gamma_{ij}+1}\\ &= -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} \quad (z \geq 0),\end{aligned}$$

and

$$\begin{aligned}\int_z^0 I_{ij}(t)dt &\leq -a_{ij}z - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij}+1}z^{\gamma_{ij}+1}\\ &= a_{ij}|z| + \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} \quad (z < 0).\end{aligned}$$

Therefore, for every $i \in \{1,2,\ldots,N\}$, $j \in \{1,2,\ldots,m\}$ and $z \in \mathbb{R}$,

$$\int_0^z I_{ij}(t)dt \geq -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1}. \tag{3.1}$$

Thanks to (H4), given $\varepsilon > 0$ small enough, we may find a constant $C_\varepsilon > 0$ such that

$$|G(t,x)| \leq C_\varepsilon + \frac{\varepsilon}{p}|x|^p, \tag{3.2}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Also, taking (H6) into account, we get

$$|F(t,x)| \leq c|x| + \frac{c}{q}|x|^q, \tag{3.3}$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. Now by (3.1), (3.2) and (3.3), for all $u \in X$ and $\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned}\phi(u) + \lambda\psi(u) =& \frac{1}{p}\|u\|^p - \int_0^T G(t,u(t))dt - \int_0^T H(u(t))dt\\ &+ \sum_{j=1}^m\sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(t)dt - \lambda\int_0^T F(t,u(t))dt\\ \geq& \frac{1}{p}\|u\|^p - \int_0^T \left(C_\varepsilon + \frac{\varepsilon}{p}|u(t)|^p\right)dt - L\int_0^T |u(t)|^p dt\\ &- \lambda\int_0^T \left(c|u(t)| + \frac{c}{q}|u(t)|^q\right)dt\\ &- \sum_{j=1}^m\sum_{i=1}^N a_{ij}|u(t_j)| - \sum_{j=1}^m\sum_{i=1}^N \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1}\\ \geq& \frac{1}{p}\left(1 - \frac{2^{p-1}\varepsilon + Lp}{\min\{1,\underline{\lambda}\}}\right)\|u\|^p - \frac{1}{q}\left(\min\{1,\underline{\lambda}\}\right)^{-\frac{q}{p}} 2^{\frac{q(p-1)}{p}}\lambda c\|u\|^q\\ &- \left(\min\{1,\underline{\lambda}\}\right)^{-\frac{1}{p}} 2^{\frac{p-1}{p}}\lambda c\|u\| - C_\varepsilon T\end{aligned}$$

$$-\sum_{j=1}^{m}\sum_{i=1}^{N} a_{ij}|u(t_j)| - \sum_{j=1}^{m}\sum_{i=1}^{N} \frac{b_{ij}}{\gamma_{ij}+1}|u(t_j)|^{\gamma_{ij}+1}.$$

Since $p > q$ and ε is small enough,

$$\lim_{\|u\|\to+\infty} [\phi(u) + \lambda\psi(u)] = +\infty. \tag{3.4}$$

Now, we prove that $\varphi_\lambda = \phi + \lambda\psi$ satisfies the (P-S) condition. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a (P-S) sequence of φ_λ, that is, there exists $C > 0$ such that

$$\varphi_\lambda(u_n) \to C, \quad \varphi'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Assume that $\|u_n\| \to +\infty$. Then, (3.4) contradicts to the $\varphi_\lambda(u_n) \to C$, hence, $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_T^{1,p}$. We may assume that there exists $u_0 \in W_T^{1,p}$ satisfying $u_n \to u_0$ weakly in $W_T^{1,p}$, $u_n \to u_0$ in $L^p[0,T]$, $u_n(t) \to u_0(t)$ for almost every $t \in [0,T]$. Observe that

$$\begin{aligned}
\varphi'_\lambda(u_n)(u_n - u_0) = &\int_0^T \Big[\Big(|u'_n(t)|^{p-2}u'_n(t), u'_n(t) - u'_0(t)\Big) \\
&+ \Big(A(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t)\Big) \Big] dt \\
&- \int_0^T (\nabla G(t, u_n(t)), u_n(t) - u_0(t))\, dt \\
&- \int_0^T (\nabla H(u_n(t)), u_n(t) - u_0(t))\, dt \\
&- \int_0^T (\nabla F(t, u_n(t)), u_n(t) - u_0(t))\, dt \\
&+ \frac{1}{\lambda}\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}\left((u_n)_i(t_j)\right)\left((u_n)_i(t_j) - (u_0)_i(t_j)\right).
\end{aligned}$$

We already know that

$$\varphi'_\lambda(u_n)(u_n - u_0) \to 0 \quad \text{as} \quad n \to \infty.$$

Clearly,

$$\lim_{n\to\infty} \int_0^T (\nabla H(u_n(t)), u_n(t) - u_0(t))\, dt = 0.$$

By (H5), given $\varepsilon > 0$, we may find a constant $C_\varepsilon > 0$ such that

$$|\nabla G(t,x)| \le C_\varepsilon + \varepsilon|x|^{p-1},$$

for every $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. So,

$$\int_0^T (\nabla G(t, u_n(t)), u_n(t) - u_0(t))\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, by (H6)

$$\int_0^T (\nabla F(t, u_n(t)), u_n(t) - u_0(t))\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Also, $\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}\left((u_n)_i(t_j)\right)\left((u_n)_i(t_j)-(u_0)_i(t_j)\right)\to 0$ as $n\to\infty$. Therefore,

$$\int_0^T \Big[\Big(|u_n'(t)|^{p-2}u_n'(t), u_n'(t)-u_0'(t)\Big)+\Big(A(t)|u_n(t)|^{p-2}u_n(t), u_n(t)-u_0(t)\Big)\Big]\,dt\to 0,$$

as $n\to\infty$. This, together with the weak convergence of $u_n\to u_0$ in $W_T^{1,p}$, implies that $u_n\to u_0$ in $W_T^{1,p}$ as $n\to\infty$. Hence, φ_λ satisfies the (P-S) condition. Finally, we prove that $\inf_{u\in X}\phi(u)<\inf_{|\psi(u)|<k}\phi(u)$ for some $k>0$. To this end, we choose a nonnegative function $v\in W_T^{1,p}$ with $\|v\|=1$. By condition (H7), a simple calculation shows that

$$\begin{aligned}\phi(sv)&=\frac{1}{p}\|sv\|^p-\int_0^T G(t,sv(t))dt-\int_0^T H(sv(t))dt\\ &\le \frac{s^p}{p}\|v\|^p-2\frac{s^pr}{p}\int_0^T|v(t)|^pdt+BT+\frac{Ls^p}{\min\{1,\underline{\lambda}\}}\|v\|^p \qquad (3.5)\\ &\le\left(\frac{L}{\min\{1,\underline{\lambda}\}}-\frac{1}{p}\right)s^p+BT\to-\infty,\end{aligned}$$

as $s\to\infty$. Since $\frac{1}{2p}>\frac{L}{\min\{1,\underline{\lambda}\}}$, (3.5) implies that $\phi(sv)<0$ for $s>0$ large enough. So, we choose a large enough $s_0>0$, and let $u_1=s_0v$ such that $\phi(u_1)<0$. Thus, $\inf_{u\in X}\phi(u)<0$. From (H4), for every $\varepsilon>0$, there exists $\rho_0(\varepsilon)>0$ such that

$$|\nabla G(t,x)|\le\varepsilon|x|^{p-1},\quad \text{if } 0<|x|<\rho_0(\varepsilon).$$

Thus,

$$\int_0^T G(t,u(t))dt\le\int_0^T\frac{\varepsilon}{p}|u(t)|^pdt\le\frac{\varepsilon}{p\min\{1,\underline{\lambda}\}}\|u\|^p.$$

By choosing $\varepsilon=\frac{1}{2}\min\{1,\underline{\lambda}\}$, we get

$$\phi(u)\ge\left(\frac{1}{2p}-\frac{L}{\min\{1,\underline{\lambda}\}}\right)\|u\|^p>0.$$

Hence, there exists $k>0$ such that $\inf_{|\psi(u)|<k}\phi(u)=0$. So,

$$\inf_{u\in X}\phi(u)<\inf_{|\psi(u)|<k}\phi(u).$$

Now, all the assumptions of Theorem 2.1 are verified. Thus, for every $b>0$ there exists an open interval $\Lambda\subseteq[-b,b]$ and a positive real number σ such that for every $\lambda\in\Lambda$, problem (1.1) admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ. ∎

Theorem 3.2. *If F, G, H and I_{ij} satisfy the assumptions* (Hi) *for $i=1,2,3,4,5,6,8$ and* (H'7)*, which asserts that $\lim_{|x|\to 0}\frac{G(t,x)}{|x|^p}=+\infty$ for almost every $t\in[0,T]$, then for every $b>0$ there exist an open interval $\Lambda\subseteq[-b,b]$ and a positive real number σ such that for every $\lambda\in\Lambda$, problem* (1.1) *admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ.*

Proof. The proof is similar to that of Theorem 3.1. So we only give a sketch of it. By the proof of Theorem 3.1, the functionals ϕ, ψ are sequentially weakly lower semicontinuous and continuously Gateaux differentiable in $W_T^{1,p}$, ϕ is even and ψ is odd. For every $\lambda \in \mathbb{R}$, the functional $\phi + \lambda\psi$ satisfies the (P-S) condition and

$$\lim_{\|u\|\to+\infty} [\phi(u) + \lambda\psi(u)] = +\infty.$$

Owning to the assumption (H'7), we can find $\delta > 0$ such that, for every $M > 0$ one has $|G(t,x)| > M|x|^p$ for $0 < |x| \le \delta$ and almost every $t \in [0,T]$. We choose a nonzero nonnegative function $v \in C_0^\infty([0,T])$, put $M > \frac{3\|v\|^p}{2p\int_0^T |v(t)|^p dt}$ and take $\varepsilon > 0$ small enough. Then, we obtain

$$\begin{aligned}
\phi(\varepsilon v) &= \frac{1}{p}\|\varepsilon v\|^p - \int_0^T G(t,\varepsilon v(t))dt - \int_0^T H(\varepsilon v(t))dt \\
&\le \frac{1}{p}\varepsilon^p\|v\|^p - M\varepsilon^p \int_0^T |v(t)|^p dt + \frac{L\varepsilon^p}{\min\{1,\underline{\lambda}\}}\|v\|^p \\
&< \frac{3}{2p}\varepsilon^p\|v\|^p - M\varepsilon^p \int_0^T |v(t)|^p dt < 0.
\end{aligned}$$

So, we get

$$\inf_{u\in W_T^{1,p}} \phi(u) < 0.$$

By the proof of Theorem 3.1, we know that there exists $k > 0$ such that

$$\inf_{u\in X} \phi(u) < \inf_{|\psi(u)|<k} \phi(u).$$

Hence, our conclusion follows from Theorem 2.1. □

When $I_{ij} = G = H \equiv 0$, the problem (1.1) reduces to the following ordinary problem which has been considered in [14] by Li et al.

$$\left\{\begin{array}{ll} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u), & \text{a.e. } t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{array}\right. \tag{3.6}$$

By a reasoning just like that of Theorem 3.1, we obtain the following result.

Theorem 3.3. *If $F(t,\cdot)$ is odd for almost every $t \in [0,T]$ and there exist constants $c > 0$ and $1 \le q < p$ such that*

$$|\nabla F(t,x)| \le c(1 + |x|^{q-1}),$$

for all $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$, then for every $b > 0$ there exist an open interval $\Lambda \subseteq [-b,b]$ and a positive real number σ such that for every $\lambda \in \Lambda$, problem (3.6) *admits at least three solutions in $W_T^{1,p}$ whose norms are less than σ.*

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SKEW HURWITZ SERIES RINGS AND MODULES WITH BEACHY-BLIAR CONDITIONS

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Abstract. A ring R satisfies the right Beachy-Blair condition if for every faithful right ideal J of a ring R (that is, a right ideal J of a ring R is faithful if $r_R(J) = 0$) is co-faithful (that is, a right ideal J of a ring R is called co-faithful if there exists a finite subset $J_1 \subseteq J$ such that $r_R(J_1) = 0$). In this note, we prove two main results.

(a) Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.

(b) Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.

1. Introduction

Throughout this article, R and M_R denote an associative ring with identity and a unitary module, respectively. For any subset P of a ring R, $r_R(P)$ denotes the right annihilator of P in R. In fact, for any subset Y of a right R-module M_R, $r_R(Y)$ denotes the right annihilator of Y in M_R. In 1975, Beachy and Blair [4] discovered rings that satisfy the condition in which every faithful right ideal of R is co-faithful. On the other hand, Zelmanowitz [39] proved that any ring which satisfies the descending chain condition on right annihilators is right zip. The converse however does not

Key words and phrases. Ring with right Beachy-Blair condition, right R-module with right Beachy-Blair condition, skew Hurwitz series ring, skew Hurwitz series module, reduced ring, ω-compatible ring, ω-compatible module, ω-reduced module.

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hold. The term right zip was coined by Faith [10]. A ring R is right zip if the right annihilator $r_R(X)$ of a subset X of R is zero then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. Similarly, a left zip ring can be defined. A ring R is called zip if it is both right and left zip. From the above discussion it is clear that every right zip ring satisfies the right Beachy-Blair condition. Faith [10] also asked the following questions.

(a) Does R being a right zip ring imply $R[x]$ is right zip?
(b) Does R being a right zip imply $M_{n\times n}(R)$ is right zip?
(c) Does R being a right zip ring imply the group ring $R[G]$ is right zip when G is a finite group?

Cedó [7] answered all these questions negatively and positively question-2 for commutative rings. Above questions and their extensions have been studied by several authors, see [8, 10, 16, 18, 26, 29, 36, 37, 40], using some conditions. Motivated by above questions of Faith [10], any one can ask similar questions for rings with the Beachy-Blair condition. We have no idea of question-3 being answered so for. However, question-2 has been answered positively by Beachy and Blair [4]. In particular, they proved that a ring R satisfies the right Beachy-Blair condition if and only if $M_{n\times n}(R)$ satisfies the right Beachy-Blair condition. We now discuss question-1. Again, Beachy and Blair [4] have answered affirmatively in case of commutative rings. Desale and Varadarajan [9] attempted question-1 for non-commutative rings. In particular, they proved that if R is ω-reduced and satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x;\omega]]$ satisfies the right Beachy-Blair condition. Here, $\omega : R \to R$ is a ring automorphism of R. Recall that a ring R is called reduced if R has no nonzero nilpotent element. A reduced ring with condition $ab = 0$ if and only if $a\omega(b) = 0$ if and only if $\omega(a)b = 0$ (that is, ω-compatible) is called ω-reduced. Rodríguez- Jorge [34] continued the study of rings with the right Beachy-Blair condition. She gave a counterexample that a ring with the right Beachy-Blair condition need not be right zip. She also generalized the result of Desale and Varadarajan [9]. Moreover, they proved that if R satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x;\omega]]$ satisfies the right Beachy-Blair condition when R is strongly ω-skew Armendariz. A ring R is said to be strongly ω-skew Armendariz if for every $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and every $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\omega]]$, $f(x)g(x) = 0$ then $a_i\omega^i(b_j) = 0$ for all i, j, where $\omega : R \to R$ is a ring automorphism of R. This result of Rodríguez-Jorge [34] is also similar to the result of Cortes [8]. In [8], Cortes proved that if R is strongly ω-skew Armendariz then R is right zip if and only if the ω-twisted power series ring $R[[x;\omega]]$ is right zip. Recently, Ouyang et al. [28] generalized further the concept of rings with the right Beachy-Blair condition to a right R-module with the right Beachy-Blair condition. A right R-module satisfies the right Beachy-Blair condition if every faithful submodules of a right R-module is co-faithful. A right R-module M_R is called faithful if $r_R(M_R) = 0$. A right R-module M_R is called co-faithful if there exists a finite subset F of M_R such that $r_R(F) = 0$.

Moreover, Ouyang et al. [28] proved the relationship between a right R-module M_R with the right Beachy-Blair condition and its skew polynomial, skew monoid and skew generalized power series extensions. Recently, Sharma and Singh [36] studied the behavior of zip property of skew Hurwitz series rings and modules for non-commutative ring R. In this note, we prove two main results.

(a) Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.

(b) Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.

2. Construction of Skew Hurwitz Series Rings and Modules

Rings of formal power series have been interesting. These have important applications. One of these is differential algebra. Keigher [20] considered a variant of the ring of formal power series and studied some of its properties. In [21], he extended the study of this type of rings and introduced the ring of Hurwitz series over a commutative ring with identity. Moreover, he showed that the Hurwitz series ring HR is very closely connected to the base ring R itself if R is of positive characteristic. Recall the construction of Hurwitz series ring from [21]. The elements of the Hurwitz series HR are sequences of the form $a = (a_n) = (a_1, a_2, a_3, \ldots)$, where $a_n \in R$ for each $n \in \mathbb{N} \cup \{0\}$. Addition in HR is point-wise, while the multiplication of two elements (a_n) and (b_n) in HR is defined by $(a_n)(b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} C_k^n a_k b_{n-k}.$$

Here, C_k^n is a binomial symbol $\frac{n!}{k!(n-k)!}$ for all $n \geq k$, where $n, k \in \mathbb{N} \cup \{0\}$. This product is similar to the usual product of formal power series, except the binomial coefficients C_k^n. This type of product was considered first by Hurwitz [19], and then by Bochner and Martin [6], Fliess [12] and Taft [38] also. Inspired by the contribution of Hurwitz, Keigher [21] coined the term ring of Hurwitz series over commutative rings. After that, Hassenin [14] extended this construction to the skew Hurwitz series rings (HR, ω), where $\omega : R \to R$ is an automorphism of R. Here, the ring R is not necessarily commutative. Recall from [14], the elements of (HR, ω) are functions $f : \mathbb{N} \cup \{0\} \to R$. Addition in (HR, ω) is component wise. Multiplication is defined for every $f, g \in (HR, \omega)$, by

$$fg(p) = \sum_{k=0}^{p} C_k^p f(k)\omega^k(g(p-k)),$$

for all $p, k \in \mathbb{N} \cup \{0\}$.

It can be easily shown that (HR, ω) is a ring with identity h_1, defined by

$$h_1(n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$. It is clear that R is canonically embedded as a subring of (HR, ω) via $a \to h_a \in (HR, \omega)$, where

$$h_a(n) = \begin{cases} a, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

Further, Kamal [31, 32] gave the construction of the skew Hurwitz series ring by taking $\omega : R \to R$ to be an endomorphism of R and $\omega(1) = 1$ instead of $\omega : R \to R$ to be an automorphism of R. A number of authors, see for example [1, 14, 15, 31–33], have studied the properties of abstract ring structures of the skew Hurwitz series ring (HR, ω).

For any function $f \in (HR, \omega)$, $\operatorname{supp}(f) = \{n \in \mathbb{N} \cup \{0\} \mid f(n) \neq 0\}$ denote the support of f and $\pi(f)$ denote the minimal element of $\operatorname{supp}(f)$. For any nonempty subset X of R, we denote:

$$(HX, \omega) = \{f \in (HR, \omega) \mid f(n) \in X \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}.$$

In [33], Kamal generalized the construction of the skew Hurwitz series rings and proposed the concept of the skew Hurwitz series modules. He extended the properties of the simple and semisimple modules to the skew Hurwitz series module $HM_{(HR,\omega)}$. Let M_R be a right R-module and HM be the set of all maps $\phi : \mathbb{N} \cup \{0\} \to M$. With pointwise addition, $HM_{(HR,\omega)}$ is an abelian additive group. Moreover, $HM_{(HR,\omega)}$ becomes a module over the skew Hurwitz series ring (HR, ω), by the scalar multiplication for each $\phi \in HM_{(HR,\omega)}$ and each $g \in (HR, \omega)$ is defined by:

$$\phi g(p) = \sum_{k=0}^{p} C_k^p \phi(k) \omega^k(g(p-k)),$$

for each $p, k \in \mathbb{N} \cup \{0\}$.

For any $m \in M$ and any $n \in \mathbb{N} \cup \{0\}$, we define $h_m \in HM_{(HR,\omega)}$ by

$$h_m(p) = \begin{cases} m, & \text{if } p = 0, \\ 0, & \text{if } p \geq 1. \end{cases}$$

Then it is clear that $m \to h_m$ is a module embedding of M into $HM_{(HR,\omega)}$.

3. Skew Hurwitz Series Rings With The Right Beachy-Blair Condition

Beachy and Blair [4] proved that if R is commutative and satisfies the right Beachy-Blair condition, then $R[x]$ satisfies the right Beachy-Blair condition. Afterwards, Desale and Varadarajan [9] generalized above result. In particular, they proved that if R is ω-reduced and satisfies the right Beachy-Blair condition then the ω-twisted power series ring $R[[x; \omega]]$ satisfies the right Beachy-Blair condition. In [34], Rodríguez-Jorge gave the following example in which the Beachy-Blair condition passes to the power series ring $R[[x]]$.

Example 3.1 ([34, Example 3.1]). For any field $\mathbb{F}$, there exists a right zip $\mathbb{F}$-algebra R such that $R[[x]]$ is not zip but the power series ring $R[[x]]$ satisfies the right Beachy-Blair condition.

While in general it remains as an open problem whether or not the Beachy-Blair condition passes to the power series ring $R[[x]]$. Further, Rodríguez-Jorge [34] proved that if R is a strongly ω-Armendariz ring then the Beachy-Blair condition passes to the skew power series ring $R[[x;\omega]]$. This result of Rodríguez-Jorge [34] is a generalization of Desale and Varadarajan [9]. Motivated by this result, in this section, we prove that the right Beachy-Blair condition passes to the skew Hurwitz series ring (HR,ω) under certain conditions. To prove our main result of this section, we need some definitions and results.

Due to Krempa [23], a monomorphism ω of a ring R is said to be rigid if $a\omega(a)=0$ implies $a=0$, for $a\in R$. A ring R is called ω-rigid if there exists a rigid endomorphism ω of R. In [3], Annin said a ring R is ω-compatible if for each $a,b\in R$, $ab=0$ if and only if $a\omega(b)=0$. Hashemi and Moussavi [13] gave some examples of non-rigid ω-compatible rings and proved following lemma.

Lemma 3.1. *Let ω be an endomorphism of a ring R. Then*

(a) *if ω is compatible, then ω is injective;*

(b) *ω is compatible if and only if for all $a,b\in R$,*

$$\omega(a)b=0\Leftrightarrow ab=0;$$

(c) *the following conditions are equivalent:*

  (i) *ω is rigid;*

  (ii) *ω is compatible and R is reduced;*

  (iii) *for every $a\in R$, $\omega(a)a=0$ implies that $a=0$.*

In [1], Ahmadi et al. introduced the concept of skew Hurwitz series-wise Armendariz by considering R as a commutative ring and defined as follows.

Definition 3.1. Let R be a commutative ring and $\omega:R\to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series-wise Armendariz, if for every skew Hurwitz series $f,g\in(HR,\omega)$, $fg=0$ if and only if $f(n)g(m)=0$ for all n, m.

The concept of skew Hurwitz series-wise Armendariz in case of non-commutative ring was introduced by Sharma and Singh [36] and defined as follows.

Definition 3.2. Let R be a ring and $\omega:R\to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series-wise Armendariz, if for every skew Hurwitz series $f,g\in(HR,\omega)$, $fg=0$ implies $f(n)\omega^n g(m)=0$ for all n,m.

The following theorem shows that every reduced is skew Hurwitz series-wise Armendariz under some additional conditions.

Theorem 3.1. *Let R be a ring and ω be an automorphism of R. If R is reduced, ω-compatible and torsion-free as a $\mathbb{Z}$-module then R is skew Hurwitz series-wise Armendariz.*

Proof. Following the proof of Sharma and Singh [36, Theorem 3.5], we get the result. $\square$

Now, we prove our main result.

Theorem 3.2. *Let R be a ring which is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If R satisfies the right Beachy-Blair condition then the skew Hurwitz series ring (HR, ω) satisfies the right Beachy-Blair condition.*

Proof. Suppose R satisfies the right Beachy-Blair condition and U be a right ideal of (HR, ω) such that $r_{(HR,\omega)}(U) = 0$. Then the ideal generated by U is the two-sided ideal and $V = (HR, \omega)U$. Let $C_V = \cup_{f \in V} \{f(n) | f \in V, n \in \mathrm{supp}(f)\}$ which is a nonempty subset of R. Now, we show $r_R(C_V) = 0$. Let $a \in r_R(C_V)$, $f(n)a = 0$ for all $n \in \mathrm{supp}(f)$. Which gives $0 = f(n)a = f(n)h_a(0) = f(n)\omega^n(h_a(0))$ since R is ω-compatible and torsion-free as a $\mathbb{Z}$-module. It follows that $h_a \in r_{(HR,\omega)}(V)$. Thus, $h_a = 0$ which implies $a = 0$. Therefore, $r_R(C_V) = 0$.

Now, we show C_V is an ideal of R. Since ω is an automorphism of R, for any $r \in R$, $h_r, \omega^{-n}(h_r) \in (HR, \omega)$. Then $h_r f, f\omega^{-n}(h_r) \in V = (HR, \omega)U$ since V is an ideal of R. Thus, $h_r(0)f(n), f(n)\omega^n\omega^{-n}(h_r(0)) \in C_V$. It follows that $rf(n), f(n)r \in C_V$. Now, consider $a, b \in C_V$, then there exist $f, g \in V$ such that $f(n) = a$ and $g(n) = b$ for some $n \in \mathbb{N} \cup \{0\}$. Since V is an ideal, $f+g \in V$. Thus, $a+b = f(n)+g(n) = (f+g)(n) \in C_V$. Therefore, C_V is an ideal of R.

Since R satisfies the right Beachy-Blair condition so there exists a nonempty finite subset $Y = \{y_1, y_2, y_3, \ldots, y_s\}$ of C_V such that $r_R(Y) = 0$. Thus, for each $y_i \in Y_0$ there exists $f_i \in V$ such that $f_i(n) = y_i$ for some $n \in \mathbb{N}$, where $1 \leq i \leq s$. It follows that $V_1 = \{f_1, f_2, f_3, \ldots, f_s\}$ be a subset of V. Then $Y \subseteq C_{V_1}$ which implies that $r_R(C_{V_1}) = 0$. Now, we show $r_{(HR,\omega)}(V_1) = 0$. Let $g \in r_{(HR,\omega)}(V_1)$. Then $f_i g = 0$ for all $f_i \in V_1$. Since R is skew Hurwitz series-wise Armendariz, ω-compatible and torsion-free as a $\mathbb{Z}$-module so $f_i(n)g(m) = 0$ for every $n \in \mathrm{supp}(f_i)$ and $m \in \mathrm{supp}(g)$ from Theorem 3.1. Thus, $g(m) = 0$ which implies that $g = 0$. This proves that $r_{(HR,\omega)}(V_1) = 0$.

Since V_1 is a subset of V and V is an ideal of (HR, ω) generated by the right ideal U so

$$V_1 = \left\{ f_i = \sum_{j=1}^{m_s} g_i^j f_i^j \mid g_i^j \in (HR, \omega), f_i^j \in U, 1 \leq i \leq s, 1 \leq j \leq m_s \right\}.$$

Now consider $U_1 = \left\{ f_i^j \in U \mid 1 \leq i \leq s, 1 \leq j \leq m_s \right\}$ which is a finite subset of U. Thus, $r_{(HR,\omega)}(U_1) \subseteq r_{(HR,\omega)}(V_1) = 0$. Hence, (HR, ω) satisfies the right Beachy-Blair conditions. $\square$

As a direct consequence of the above theorem, we obtain the following corollary.

Corollary 3.1. *Let R be a reduced ring and be torsion-free as a $\mathbb{Z}$-module. If R satisfies the right Beachy-Blair condition then the Hurwitz series ring HR satisfies the right Beachy-Blair condition.*

Proof. Let ω be an identity automorphism of R, so $(HR, \omega) \cong HR$. Thus, from Theorem 3.2, we obtain the result. $\square$

4. Skew Hurwitz Series Modules With the right Beachy-Blair condition

In this section, we discuss the right R-module with the right Beachy-Blair condition to the skew Hurwitz series module $MH_{(HR,\omega)}$. In particular, we prove that a right R-module with the right Beachy-Blair condition passes to the skew Hurwitz series module $MH_{(HR,\omega)}$ under certain conditions.

Due to Annin [3], a right R-module M_R is called ω-compatible if for any $m \in M_R$ and $p \in R$, $mp = 0$ if and only if $m\omega(p) = 0$, where $\omega : R \to R$ is an endomorphism of R. It follows that, if M_R is ω-compatible, $mp = 0$ if and only if $m\omega^k(p) = 0$ for all k.

According to Lee and Zhou [24], a module M_R is called ω-reduced if for any $m \in M_R$ and $a \in R$, $ma = 0$ implies $mR \cap M_R a = 0$ and ω-compatible, where $\omega : R \to R$ is a ring of endomorphism of R with $\omega(1) = 1$. Henceforth, they also proved the following lemma.

Lemma 4.1. *The following are equivalent for a module M_R.*

(a) *M_R is ω-reduced.*

(b) *The following conditions holds: for any $m \in M_R$ and $a \in R$;*

  (i) *$ma = 0$ implies $mRa = mR\omega(a) = 0$;*

  (ii) *$ma\omega(a) = 0$ implies $ma = 0$;*

  (iii) *$ma^2 = 0$ implies $ma = 0$.*

In [24], Lee and Zhou introduced the concept of ω-Armendariz of power series type and defined as follows.

Definition 4.1. A right R-module M_R is said to be ω-Armendariz of power series type if the following conditions are satisfied.

(a) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\omega]]$ and $f(x) = \sum_{i=0}^{\infty} a_j x^j \in R[[x;\omega]]$, $m(x)f(x) = 0$ implies $m_i\omega^i(a_j) = 0$ for each $i, j \geq 0$.

(b) For any $m \in M_R$ and $a \in R$, $ma = 0$ if and only if $m\omega(a) = 0$.

Motivated by the above definition, Sharma and Singh [36] introduced the concept of ω-Armendariz of skew Hurwitz series type and defined as follows.

Definition 4.2. Let M_R be a right R-module and $\omega : R \to R$ be an endomorphism of R. A right R-module M_R is said to be ω-Armendariz of skew Hurwitz series type if the following conditions are satisfied.

(a) For every skew Hurwitz series $\phi \in MH_{(HR,\omega)}$ and $g \in (HR,\omega)$, $\phi g = 0$ implies $\phi(p)\omega^p g(q) = 0$ for all p, q.
(b) For any $m \in M_R$ and $a \in R$, $ma = 0$ if and only if $m\omega(a) = 0$.

Next theorem shows that every ω-reduced module is ω-Armendariz of skew Hurwitz series type under some additional conditions.

Theorem 4.1. *Let M_R be a right R-module and ω be an automorphism of R. If M_R be ω-reduced and torsion-free as a $\mathbb{Z}$-module then M_R is ω-Armendariz of skew Hurwitz series type.*

Proof. Following the proof of Sharma and Singh [36, Theorem 4.5], we obtain the result. □

Recently, Ouyang et al. [28] generalized further the concept of rings with the right Beachy-Blair condition to a right R-module with the right Beachy-Blair condition. They proved the relationship between a right R-module M_R with the right Beachy-Blair condition and its skew polynomial, skew monoid and skew generalized power series extensions. Moreover, they extended the well known results of Beachy and Blair [4], and Desale and Varadarajan [9]. In the following theorem, we prove that the right Beachy-Blair condition passes to the skew Hurwitz series module $HM_{(HR,\omega)}$.

Theorem 4.2. *Let M_R be a right R-module which is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module, and ω be an automorphism of R. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition.*

Proof. Suppose M_R satisfies the right Beachy-Blair condition and let I be a submodule of $HM_{(HR,\omega)}$ with $r_{(HR,\omega)}(I) = 0$. Then $N = (HR,\omega)I$ is a submodule of $HM_{(HR,\omega)}$ which is generated by I. Put $C_N = \cup_{\phi\in N} C_\phi$, where $C_\phi = \{\phi(n) | \phi \in N, n \in \mathrm{supp}(\phi)\}$ which is a nonempty subset of M_R. Now, we show $r_R(C_N) = 0$. Let $a \in r_R(C_N)$, $\phi(n)a = 0$ for all $n \in \mathrm{supp}(\phi)$ and all $\phi \in N$. Which gives $0 = \phi(n)a = \phi(n)h_a(0) = \phi(n)\omega^n(h_a(0))$ since M_R is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module. It follows that $\phi h_a = 0$. Thus, $h_a = 0$ which implies $a = 0$. Therefore, $r_R(C_N) = 0$.

Now, we show C_N is a submodule of M_R. Since ω is an automorphism of R so for any $r \in R$, $h_r, \omega^{-n}(h_r) \in (HR,\omega)$. Then $h_r\phi, \phi\omega^{-n}(h_r) \in N$ since N is a submodule of $HM_{(HR,\omega)}$. Thus, $h_r(0)\phi(n), \phi(n)\omega^n\omega^{-n}(h_r(0)) \in C_N$. Therefore, $r\phi(n), \phi(n)r \in C_N$ for all $n \in \mathrm{supp}(\phi)$. Let $a, b \in C_N$, then there exist $\phi_1, \phi_2 \in C_N$ such that $\phi_1(n) = a$ and $\phi_2(n) = b$ for some $n \in \mathbb{N} \cup \{0\}$. Since N is a submodule, $\phi_1 + \phi_2 \in N$. Thus, $a + b = \phi_1(n) + \phi_2(n) = (\phi_1 + \phi_2)(n) \in C_N$. Hence, C_N is a submodule of M_R.

Since M_R satisfies the right Beachy-Blair condition so there exists a nonempty finite subset $V = \{v_1, v_2, v_3, \ldots, v_s\}$ of C_N such that $r_R(V) = 0$. Thus, for each v_i there exists $\phi_i \in N$ such that $\phi_i(n) = v_i$ for some $n \in \mathbb{N}$, where $1 \leq i \leq s$. It follows that $N_1 = \{\phi_1, \phi_2, \phi_3, \ldots, \phi_s\}$ be a subset of N. Then $V \subseteq C_{N_1}$ which implies that

$r_R(C_{N_1}) = 0$. Now, we show $r_{(HR,\omega)}(N_1) = 0$. Let $g \in r_{(HR,\omega)}(N_1)$. Then $\phi_i g = 0$ for all $\phi_i \in N_1$. Since R is ω-Armendariz of skew Hurwitz series type and torsion-free as a $\mathbb{Z}$-module so $\phi_i(n)g(m) = 0$ for every $n \in \operatorname{supp}(\phi_i)$ and $m \in \operatorname{supp}(g)$ from Theorem 4.1. Thus, $g = 0$. This proves that $r_{(HR,\omega)}(N_1) = 0$.

Since N_1 is a subset of N and N is a submodule of $HM_{(HR,\omega)}$ generated by the submodule I so

$$N_1 = \left\{ \phi_i = \sum_{j=1}^{m_s} g_i^j \phi_i^j \mid g_i^j \in (HR,\omega), \phi_i^j \in I, 1 \le i \le s, 1 \le j \le m_s \right\}.$$

Now, consider $I_1 = \left\{ \phi_i^j \in I \mid 1 \le i \le s, 1 \le j \le m_s \right\}$ which is a finite subset of I. Thus, $r_{(HR,\omega)}(I_1) \subseteq r_{(HR,\omega)}(N_1) = 0$. Hence, the skew Hurwitz series module $HM_{(HR,\omega)}$ satisfies the right Beachy-Blair condition. □

Here, we obtain the following result as a special case of Theorem 4.2.

Corollary 4.1. *Let M_R be a right R-module, reduced and torsion-free as $\mathbb{Z}$-module. If M_R satisfies the right Beachy-Blair condition then the skew Hurwitz series module $HM_{(HR)}$ satisfies the right Beachy-Blair condition.*

Proof. Let ω be an identity automorphism of R, so $HM_{(HR,\omega)} \cong HM_{HR}$. Thus, from Theorem 4.2, we can get the proof. □

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QUASILINEAR PARABOLIC PROBLEM WITH $p(x)$-LAPLACIAN OPERATOR BY TOPOLOGICAL DEGREE

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Abstract. We prove the existence of a weak solution for the quasilinear parabolic initial boundary value problem associated to the equation

$$u_t - \Delta_{p(x)} u = h,$$

by using the Topological degree theory for operators of the form $L + S$, where L is a linear densely defined maximal monotone map and S is a bounded demicontinuous map of class (S_+) with respect to the domain of L.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Fixing a final time $T > 0$, we denote by Q the cylinder $\Omega\times]0,T[$ and $\Gamma = \partial\Omega\times]0,T[$ its lateral surface. We consider the following quasilinear parabolic initial-boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)} u = h, & \text{in } Q, \\ u(x,t) = 0, & \text{in } \Gamma, \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u := \operatorname{div}(|\nabla \mathrm{u}|^{\mathrm{p(x)}-2}\nabla \mathrm{u})$ is the $p(x)$-Laplacian applied on u, defined from

$$\mathcal{V} := \left\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \right\}$$

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introduced and discussed in [7, 22] (and that we think it is a reasonable framework to discuss our problem) to its dual $\mathcal{V}^*$. The variable exponent $p(\cdot) : \overline{\Omega} \to [1, +\infty[$ is a Log-Hölder continuous function only dependent on the space variable x (see definitions below). The right-hand side h is assumed to belong to $\mathcal{V}^*$ and u_0 lies in $L^2(\Omega)$.

The importance of investigating these problems lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids [19], the processes of filtration in complex media [4], image processing [10], mathematical biology [13], stratigraphy problems [15], and also elasticity [23].

Problems similar to the problem (1.1) are treated by several authors (see for instance [14, 22] and references therein) where the proved independently the existence of at least a weak solution for these problems.

In this work, we prove the existence of solutions for the quasilinear parabolic initial boundary value problem (1.1) using another approach: that of Topological degree theory.

The use of the theory of topological degrees is an efficient tool for solving some elliptical PDEs even in variable exponent spaces without resorting to variational methods (see [1–3]). This theory has recently been used also to solve some fractional differential equations (see [5, 18, 20, 21]). In this paper, we will use this approach to solve a parabolic problem in a space also with variable exponent.

The rest of this paper is organized as follows. In Section 2, we state some mathematical preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators and then the associated topological degree. We will prove the main results in Section 4.

2. Preliminaries

We first recall some basic properties of variable exponent Lebesgue and Sobolev spaces (see [8, 11, 12, 16, 17] for more details).

Let

$$p^- := \operatorname{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{esssup}_{x\in\Omega} p(x).$$

We will make use of the following assumption

$$1 < p^- \le p(x) \le p^+ < +\infty. \tag{2.1}$$

An interesting feature of generalized variable exponent Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(\cdot)$. However, when the exponent satisfies the following so-called log-Hölder condition

$$(\exists C > 0)\, |p(x) - p(y)| \log\left(e + \frac{1}{|x-y|}\right) \le C, \qquad \text{for all } x, y \in \overline{\Omega},$$

then $C_0^\infty(\overline{\Omega})$ is dense in $L^{p(\cdot)}(\Omega)$ (see [8, Theorem 3.7] and [11, Section 6.5.3]) and we have the Poincaré inequality (see [12, Theorem 8.2.4] and [16, Theorem 4.3])

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \qquad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where the constant $C > 0$ depends only on Ω and the function p.

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}$ given by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \qquad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

which equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact (see [17]).

We extend a variable exponent function $p : \overline{\Omega} \to [1, +\infty[$ to $\overline{Q} \to [1, +\infty[$ by setting $p(x,t) = p(x)$ for all $(x,t) \in \overline{Q}$.

As in [7], we consider the following functional space

$$\mathcal{V} := \Big\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \Big\},$$

which is a separable and reflexive Banach space endowed with the norm

$$|u|_{\mathcal{V}} := \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)}$$

or the equivalent norm

$$\|u\|_{\mathcal{V}} := \|\nabla u\|_{L^{p(\cdot)}(Q)}.$$

Note that, under the assumption (2.1), we have

$$\|u\|^{p^-}_{L^{p(\cdot)}(Q)} - 1 \le \int_Q |u|^{p(x)}\, dxdt \le \|u\|^{p^+}_{L^{p(\cdot)}(Q)} + 1. \tag{2.2}$$

Remark 2.1. ([7, Lemma 3.1]). $\mathcal{C}_0^\infty(Q)$ is dense in $\mathcal{V}$. Moreover we have the following continuous dense embedding

$$L^{p^+}(0,T; W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow_d \mathcal{V} \hookrightarrow_d L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)).$$

For the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T; W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow \mathcal{V}^* \hookrightarrow L^{(p^+)'}(0,T; W_0^{-1,p'(\cdot)}(\Omega)).$$

3. Classes of Mappings and Topological Degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle \cdot, \cdot \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

We consider a multi-values mapping T from X to 2^{X^*} (i.e., with values subsets of X^*). With each such map, we associate its graph

$$G(T) = \{(u,w) \in X \times X^* : w \in T(u)\}.$$

The multi-values mapping T is said to be *monotone* if for any pair of elements $(u_1, w_1), (u_2, w_2)$ in $G(T)$, we have the inequality

$$\langle w_1 - w_2, u_1 - u_2 \rangle \ge 0.$$

T is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(u_0, w_0) \in X \times X^*$ for which $\langle w_0 - w, u_0 - u\rangle \geq 0$, for all $(u, w) \in G(T)$, we have $(u_0, w_0) \in G(T)$.

We recall that a mapping $T : D(T) \subset X \to Y$ is *demicontinuous* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$. T is said to be *of class* (S_+) if for any $(u_n) \subset D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle Tu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

Let L be a linear maximal monotone map from $D(L) \subset X$ to X^* such that $D(L)$ is dense in X. For each open and bounded subset G on X, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_G :=&\{L + S : \bar{G} \cap D(L) \to X^* \mid S \text{ is bounded, demicontinuous map}\\
&\quad \text{of class } (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\},\\
\mathcal{H}_G :=&\{L + S(t) : \bar{G} \cap D(L) \to X^* \mid S(t) \text{ is a bounded homotopy of class}\\
&\quad (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}.
\end{aligned}$$

Note that the class $\mathcal{H}_G$ (class of *admissible homotopies*) includes all affine homotopies $L + (1 - t)S_1 + tS_2$ with $(L + S_i) \in \mathcal{F}_G$, $i = 1, 2$.

We introduce the topological degree for the class $\mathcal{F}_G$ due to Berkovits and Mustonen [6].

Theorem 3.1. *Let L be a linear maximal monotone densely defined map from $D(L) \subset X$ to X^*. There exists a topological degree function*

$$d : \{(F, G, h) : F \in \mathcal{F}_G, G \textit{ an open bounded subset in } X, h \notin F(\partial G \cap D(L))\} \to \mathbb{Z}$$

satisfying the following properties.

(a) (*Existence*) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G \cap D(L)$.*

(b) (*Additivity*) *If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F[(\bar{G} \setminus (G_1 \cup G_2)) \cap D(L)]$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

(c) (*Invariance under homotopies*) *If $F(t) \in \mathcal{H}_G$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for all $t \in [0, 1]$, where $h(t)$ is a continuous curve in X^*, then*

$$d(F(t), G, h(t)) = \textit{constant}, \quad \textit{for all } t \in [0, 1].$$

(d) (*Normalization*) *$L + J$ is a normalising map, where J is the duality mapping of X into X^*, that is,*

$$d(L + J, G, h) = 1, \quad \textit{whenever } h \in (L + J)(G \cap D(L)).$$

Theorem 3.2. *Let $L + S \in \mathcal{F}_X$ and $h \in X^*$. Assume that there exists $R > 0$ such that*

$$\langle Lu + Su - h, u\rangle > 0, \tag{3.1}$$

for all $u \in \partial B_R(0) \cap D(L)$. Then $(L + S)(D(L)) = X^*$.

Proof. Let $\varepsilon > 0$, $t \in [0,1]$ and

$$F_\varepsilon(t,u) = Lu + (1-t)Ju + t(Su + \varepsilon Ju - h).$$

Since $0 \in L(0)$ and by using the boundary condition (3.1), we see that

$$\begin{aligned}\langle F_\varepsilon(t,u), u\rangle &= \langle t(Lu + Su - h), u\rangle + \langle (1-t)Lu + (1-t+t\varepsilon)Ju, u\rangle \\ &\geq \langle (1-t)Lu + (1-t+t\varepsilon)Ju, u\rangle \\ &= (1-t)\langle Lu, u\rangle + (1-t+t\varepsilon)\langle Ju, u\rangle \\ &\geq (1-t+t\varepsilon)\|u\|^2 = (1-t+t\varepsilon)R^2 > 0.\end{aligned}$$

That is $0 \notin F_\varepsilon(t,u)$. Since J and $S + \varepsilon J$ are continuous, bounded and of type (S_+), $\{F_\varepsilon(t,\cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, by invariance under homotopy and normalisation, we obtain

$$d(F_\varepsilon(t,\cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

Hence, there exists $u_\varepsilon \in D(L)$ such that $0 \in F_\varepsilon(t,\cdot)$. Letting $\varepsilon \to 0^+$ and $t = 1$, we have $h \in Lu + Su$ for some $u \in D(L)$. Since $h \in X^*$ is arbitrary, we conclude that $(L+S)(D(L)) = X^*$. ☐

4. The Main Result

Lemma 4.1. *The operator $S := -\Delta_{p(x)}$ defined from $\mathcal{V}$ to $\mathcal{V}^*$ by*

$$\langle Su, v\rangle = \int_Q |\nabla u|^{p(x)-2}\nabla u \nabla v dxdt, \qquad \text{for all } u, v \in \mathcal{V},$$

is bounded, continuous and of class (S_+).

Proof. Let $t \in]0,T[$ and denote by A the operator defined from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$ by

$$\langle Au(x,t), v(x,t)\rangle = \int_\Omega |\nabla u(x,t)|^{p(x)-2}\nabla u(x,t)\nabla v(x,t)dx,$$

for all $u(\cdot,t), v(\cdot,t) \in W_0^{1,p(x)}(\Omega)$. Then

$$\langle Su, v\rangle = \int_0^T \langle Au(x,t), v(x,t)\rangle dt, \qquad \text{for all } u, v \in \mathcal{V}.$$

It is known by [9, Theorem 3.1] that A is bounded, continuous and of class (S_+); then it is the same for S. ☐

Our main result is the following existence theorem.

Theorem 4.1. *Let $h \in \mathcal{V}^*$ and $u_0 \in L^2(\Omega)$. There exists at least one weak solution $u \in D(L)$ of problem (1.1) in the following sense*

$$-\int_Q uv_t \, dxdt + \int_Q |\nabla u|^{p(x)-2}\nabla u\nabla v dxdt = \int_Q hvdxdt,$$

for all $v \in \mathcal{V}$.

Proof. Let L be the operator defined from $\mathcal{V} \supset D(L)$ to $\mathcal{V}^*$, where

$$D(L) = \{v \in \mathcal{V} : v' \in \mathcal{V}^*, v(0) = 0\},$$

by

$$\langle Lu, v\rangle = -\int_Q uv_t dxdt, \quad \text{for all } u \in D(L), v \in \mathcal{V}.$$

The operator L is generated by $\partial/\partial t$ via the relation

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \quad \text{for all } u \in D(L), v \in \mathcal{V}.$$

One can verify, as in [24] that L is a densely defined maximal monotone operator.

By the monotonicity of L ($\langle Lu, u\rangle \geq 0$ for all $u \in D(L)$) and by (2.2), we get

$$\langle Lu + Su, u\rangle \geq \langle Su, u\rangle = \int_Q |\nabla u|^{p(x)} dxdt \geq \|\nabla u\|^{p^-}_{L^{p(\cdot)}(Q)} - 1 = \|u\|^{p^-}_{\mathcal{V}} - 1,$$

for all $u \in \mathcal{V}$.

Since the right side of the above inequality approaches ∞ as $\|u\|_{\mathcal{V}} \to \infty$, then for each $h \in \mathcal{V}^*$ there exists $R = R(h)$ such that $\langle Lu + Su - h, u\rangle > 0$ for all $u \in B_R(0) \cap D(L)$. By applying Theorem 3.1, we conclude that the equation $Lu + Su = h$ is solvable in $D(L)$, that is, (1.1) admits at least one-weak solution. $\square$

Conclusion and future remark

So, we have used the theory of topological degree to solve a parabolic problem in a space with variable exponent. We hope in a future work to solve other parabolic problems by generalization of (1.1), by replacing, for example, the $p(x)$-Laplacian operator by a Leray-Lions type operator $-\operatorname{div} \mathrm{a}(\mathrm{x}, \mathrm{t}, \nabla \mathrm{u})$ under some suitable conditions.

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COMPUTING THE $\mathcal{H}_2$-NORM OF A FRACTIONAL-ORDER SYSTEM USING THE STATE-SPACE LINEAR MODEL

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Abstract. The main purpose of the present paper is to establish an alternative approach to compute the $\mathcal{H}_2$-norm for a fractional-order transfer function of the first kind based on Caputo fractional derivative. The key idea behind this new approach is the use of the concept of the parahermitian transfer matrices and the state-space realization. Numerical examples are presented to illustrate the new approach.

1. Introduction

In the last few years, many researchers pointed out that fractional derivatives revealed to be a more adequate tool for the description of properties of various real materials and in different fields [3, 8, 9, 11–13, 15]. Among these fields, dynamic systems appear since they can be described and modelled using fractional derivatives [3, 8, 11, 15].

One of the most important problems in modelling and control of dynamic systems is to compute the impulse response energy, known, also, as the $\mathcal{H}_2$-norm, for a fractional-order transfer function. The $\mathcal{H}_2$-norm often arises in control theory and can be used to measure the precision of a rational approximation of a fractional transfer function and inversely [1, 10, 14, 15]. More than that, the $\mathcal{H}_2$-norm is a useful measure for assessing the system's performance.

In the literature, several methods have been proposed to compute the $\mathcal{H}_2$-norm for the fractional transfer function, most of them use an analytic or an algebraic formulation [1, 10, 15]. However, in this paper, an alternative method is provided to

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calculate the $\mathcal{H}_2$-norm for the fractional transfer function of the first kind associated with the fractional-order linear system. The main concept of this new approach is the use of the state-space realization consisting of parameters that are extracted from the fractional-order transfer function and then a transformation of the parahermitian matrix which let it invariant. Finally, the general expression of the $\mathcal{H}_2$-norm is derived thanks to some concepts and some conditions set out.

The rest of the paper is organized as follows. In Section 2, mathematical concepts and the definition of the $\mathcal{H}_2$-norm are recalled. Section 3 describes the new approach for computing the $\mathcal{H}_2$-norm for the fractional transfer function of the first kind associated with a fractional linear system. In Section 4, some examples are presented to show the performance of the proposed method. Concluding remarks are drawn in the last section.

2. Preliminaries

The $\mathcal{H}_2$-norm of a rational transfer function matrix appears among other systems norms [2,7]. It is used in several contexts and domains [8,10,14,15], some of them use it to measure the intensity of the response to standard excitations. An added benefit is that different systems can be compared using the $\mathcal{H}_2$-norm. The definition of the $\mathcal{H}_2$-norm is presented in the following.

Let $\{a,\, b,\, c\}$ be a state-space representation of a linear system and let $G(s) = c(s-a)^{-1}b$ for some $s \in \mathbb{C}$ be its transfer function. The $\mathcal{H}_2$-norm of such system is defined as [15]

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)\, d\omega,$$

where $s = j\omega$, with $j = e^{j\frac{\pi}{2}}$ and $j^2 = -1$.

Note that, for $G(-j\omega) = G^*(j\omega)$, we get [15]

$$\int_{-\infty}^{0} G(j\omega)G^*(j\omega)d\omega = \int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

then

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{\pi}\int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega.$$

Let us recall that the main concept of this paper is to provide a numerical expression for computing the $\mathcal{H}_2$-norm of a rational transfer function matrix of the first kind using the state-space representation and a transformation of the parahermitian matrix which will be determined later under some conditions.

For this purpose, we will use the Schur complement [6], where the function G can be written through a block matrix as

$$S_G(s) = \left[\begin{array}{c|c} s-a & b \\ \hline -c & 0 \end{array}\right],$$

with respect to its right block entry.

The transfer function G is proper, thus, its conjugate transpose

$$G^*(s) = b(-s-a)^{-1}c,$$

is the Schur complement of the corresponding system matrix S_{G^*}

$$S_{G^*}(s) = \left[\begin{array}{c|c} -s-a & c \\ \hline -b & 0 \end{array}\right].$$

Using simple algebraic manipulation on the matrices S_G and S_{G^*} it follows

$$S_\phi(s) = \left[\begin{array}{cc|c} 0 & -s-a & c \\ s-a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

Note that the matrix S_ϕ is also known as the parahermitian matrix where its corresponding parahermitian transfer function is

$$\phi(s) = c(s-a)^{-1}b^2(-s-a)^{-1}c. \tag{2.1}$$

3. The $\mathcal{H}_2$-Norm of Fractional-Order Systems

A generalized fractional-order state-space model consisting of the parameters $\{a,\ b,\ c,\ \alpha\}$ can be represented as

$$\begin{cases} \mathbf{D}^\alpha x(t) = ax(t) + bu(t), \\ \quad y(t) = cx(t), \end{cases} \tag{3.1}$$

where $x, u, y \in \mathbb{R}^*$ are respectively the state, the input and the output and $a \in \mathbb{R}^*_-$ and $b,\ c \in \mathbb{R}^*$ with a null initial condition and $\mathbf{D}^\alpha$, where $n-1 \le \alpha < n$, for some $n \in \mathbb{N}^*$, is the α fractional-order derivation of the function x in the sense of the Caputo derivative, given by [4]

$$\mathbf{D}^\alpha x(t) = \frac{1}{\Gamma[n-\alpha]}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}, \quad n \in \mathbb{N}^*.$$

For almost $s \in \mathbb{C}$ we assume that the pencil $(1, a)$ is regular which is equivalent to $(s^\alpha - a) \neq 0$.

From the system (3.1), the transfer function G can be extracted. Indeed, by the means of the Laplace transform [12], the direct input-output relation of the system (3.1) is written as

$$Y(s) = c(s^\alpha - a)^{-1}b\,U(s), \quad \text{for all } s \in \mathbb{C}.$$

However,

$$Y(s) = G(s)\,U(s), \quad \text{for all } s \in \mathbb{C},$$

then, in time domain, the fractional transfer function associated with the system (3.1) is given by

$$G(s) = c(s^\alpha - a)^{-1}b, \quad \text{for all } s \in \mathbb{C},$$

which has the generalized state-space realization consisting on $\{a, b, c, \alpha\}$. Its $\mathcal{H}_2$-norm, in the frequency domain, is then obtained from the following definition [15]

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{\pi}\int_0^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

with $s = j\omega$, $s^\alpha = (j\omega)^\alpha$ and $\omega^\alpha = \tilde{\omega}$.

Then, for the fractional system (3.1) and in the frequency domain, the so-called parahemitian transfer function (formula (2.1)) becomes

$$\phi(\tilde{\omega}) = c(j^\alpha\tilde{\omega} - a)^{-1}b^2(\bar{j}^\alpha\tilde{\omega} - a)^{-1}c,$$

which is also the Schur complement of the so-called system matrix S_ϕ

$$S_\phi(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

It is well known that the parahermitian matrix can be transformed under row and column matrices transformations that leave the system state-space realization $\{a,\ b,\ c,\ \alpha\}$ invariant [6,16]. Therefore, the matrix $S_\phi(\tilde{\omega})$ can be transformed into the following matrix

$$S_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & f & p\,c \\ \hline -c & -c\,p & 0 \end{array}\right],$$

where

$$f = (j^\alpha\tilde{\omega} - a)p + p(\bar{j}^\alpha\tilde{\omega} - a) - b^2,$$

with p satisfying a condition given thereafter.

The existence of the value of

$$p = \frac{b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)},$$

solution of the equation $f = 0$ allows the matrix $S_{\tilde{\phi}}(\tilde{\omega})$ to become

$$S_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & 0 & \dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \\ \hline -c & -\dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} & 0 \end{array}\right].$$

In this case, the Schur complement of $S_{\tilde{\phi}}$ can be written as

$$\tilde{\phi}(\tilde{\omega}) = -\frac{c^2\,b^2}{2\,j\,a\,\sin\left(\frac{\alpha\pi}{2}\right)}\left[\frac{1}{\tilde{\omega} - a\,e^{-\frac{\alpha\pi}{2}j}} - \frac{1}{\tilde{\omega} - a\,e^{\frac{\alpha\pi}{2}j}}\right].$$

Thus, the $\mathcal{H}_2$-norm of the transfer function G is

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)d\omega$$

$$
\begin{aligned}
&= \frac{1}{\alpha\pi} \int_0^{+\infty} \tilde{\omega}^{\frac{1}{\alpha}-1}\, \tilde{\phi}(\tilde{\omega})\, d\tilde{\omega} \\
&= -\frac{c^2\, b^2}{2\,\pi\, j\, a\, \alpha\, \sin\left(\frac{\alpha\pi}{2}\right)} \left[\int_0^{+\infty} \left(\frac{1}{\tilde{\omega} - a\, e^{-\frac{\alpha\pi}{2} j}} - \frac{1}{\tilde{\omega} - a\, e^{\frac{\alpha\pi}{2} j}} \right) \tilde{\omega}^{\frac{1}{\alpha}-1}\, d\tilde{\omega} \right]. \qquad (3.2)
\end{aligned}
$$

The expression (3.2) can be readily computed for $\alpha = 1$. Nevertheless, for $\frac{1}{2} < \alpha < 2$ with $\alpha \neq 1$, we will use the Mellin integral transform [5] where the obtained result is presented in the following theorem.

Theorem 3.1. *Assuming that $\frac{1}{2} < \alpha < 2$ and $\alpha \neq 1$. Then the $\mathcal{H}_2$-norm of the fractional-order transfer function G, with generalized state-space realization $\{a, b, c, \alpha\}$, where $a \in \mathbb{R}_-^*$, $b, c \in \mathbb{R}^*$ and $(s^\alpha - a) \neq 0$ for almost $s \in \mathbb{C}$ is defined as*

$$\|G\|_{\mathcal{H}_2}^2 = -\frac{b^2\, c^2\, (-a)^{\frac{1}{\alpha}-2}\, \cot\left(\alpha\frac{\pi}{2}\right)}{\alpha\, \sin\left(\frac{\pi}{\alpha}\right)}.$$

The above theorem which is the main result of this paper present numerical formula for computing the $\mathcal{H}_2$-norm for a fractional-order transfer function of the first kind represented by the generalized state-space realization.

Remark 3.1. For $\alpha = 1$ we get

$$\|G\|_{\mathcal{H}_2}^2 = -\frac{c^2 b^2}{2a}.$$

Moreover, the same technique can be applied for any transfer function represented by a regular differential linear system written in a state-space as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \qquad (3.3)$$

where $x \in \mathbb{R}^q$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector and $A \in \mathbb{R}^{q\times q}$, $B \in \mathbb{R}^{q\times m}$, and $C \in \mathbb{R}^{p\times q}$ with a null initial condition.

In this case, the $\mathcal{H}_2$-norm of the transfer function $G(s) = C\,(sI - A)^{-1}\, B$, with generalized state-space realization $\{A, B, C\}$ and $\det(sI - A) \neq 0$ for almost $s \in \mathbb{C}$ is defined as

$$\|G\|_{\mathcal{H}_2}^2 = \operatorname{tr}\left(CPC^T\right).$$

The matrix P is a solution of the Lyapunov equation

$$AP + PA^T + BB^T = 0,$$

where A^T, B^T and C^T are respectively the transpose of the matrices A, B and C.

4. Numerical Examples

The algorithm has been tested for different examples, and compared to the existed methods in the state-of-art. The presented examples are taken from the references [10, 14] to validate our method. All examples have been performed using a Matlab code.

Example 4.1. Consider the system (3.3)

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The corresponding transfer function is

$$G(s) = \begin{bmatrix} \dfrac{-s^3 - 3s^2 - 13s + 5}{s^4 + 4s^3 + 36s^2 + 92s + 43} & \dfrac{-6s^2 - 22s - 16}{s^4 + 4s^3 + 36s^2 + 92s + 43} \\ \dfrac{-s^3 - 2s^2 - 31s - 48}{s^4 + 4s^3 + 36s^2 + 92s + 43} & \dfrac{6s + 16}{s^4 + 4s^3 + 36s^2 + 92s + 43} \end{bmatrix}.$$

Thus, using Remark 3.1, the $\mathcal{H}_2$-norm of the fractional transfer function G is

$$\|G\|_{\mathcal{H}_2} = 1.1751,$$

which is the same result when using the method presented in [14].

Example 4.2. Consider the transfer function $G(s) = \frac{k}{s^\alpha + \lambda}$ associated with the following system

$$\begin{cases} \mathbf{D}^\alpha\, x(t) = -\,\lambda\, x(t) + k\, u(t), \\ \quad\;\; y(t) = x(t), \end{cases}$$

with $\frac{1}{2} < \alpha < 2$, $\lambda \in \mathbb{R}_+^*$ and $k \in \mathbb{R}^*$.

The transfer function G satisfies the conditions of Theorem 3.1. Thus, the $\mathcal{H}_2$-norm of the transfer function G is given by

$$\|G\|^2_{\mathcal{H}_2} = \begin{cases} -\dfrac{k^2 \lambda^{\frac{1}{\alpha}-2} \cot\left(\alpha \frac{\pi}{2}\right)}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}, & \text{if } \frac{1}{2} < \alpha < 2 \text{ and } \alpha \neq 1, \\ \frac{k^2}{2\lambda}, & \text{if } \alpha = 1. \end{cases}$$

For $\alpha \in \left]\frac{1}{2}, 2\right[$, the obtained results are similar to the ones in [10]. For simplicity, if we take $k = 1$ and $\lambda = 2$, the comparison between both methods is plotted versus α in Figure 1.

5. Conclusion

In this paper, an efficient algorithm is proposed to compute the $\mathcal{H}_2$-norm for a fractional transfer function of the first kind associated with a fractional differential linear system. The approach consists of using the state-space realization and the parahermitian matrix and is based on the use of transformation matrices satisfying some conditions mentioned above. The extension of the proposed method for other types of systems, which are some of the most significant applications, will be discussed in a separate paper.

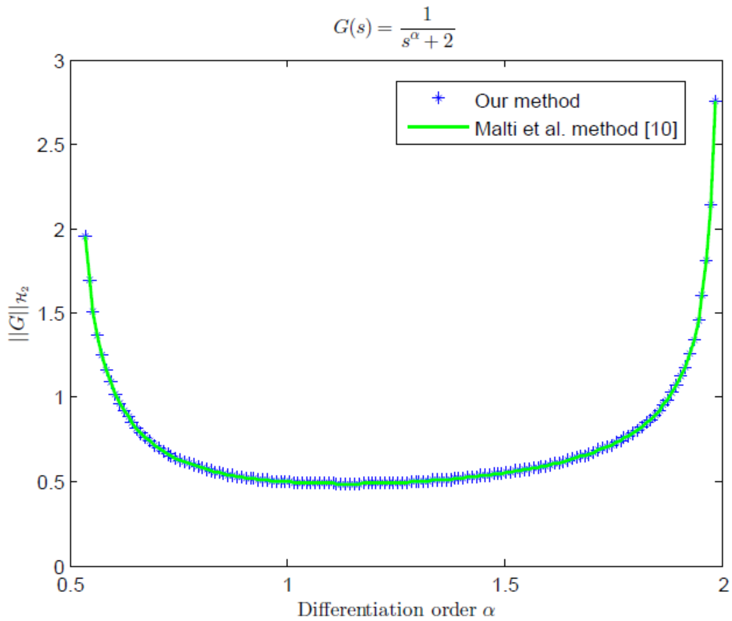

FIGURE 1. Comparison of the values of the $\mathcal{H}_2$-norm between our method and the method presented in [10] for $G(s) = \frac{1}{s^\alpha+2}$ and $\frac{1}{2} < \alpha < 2$.

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ON TWO PEXIDERIZED FUNCTIONAL EQUATIONS OF DAVISON TYPE

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Abstract. In this paper, we present the general solution of two Pexiderized functional equations of Davison type without assuming any regularity assumption on the unknown functions.

1. Introduction

In 1979, during the 17th International Symposium on Functional Equations (ISFE), Davison [2] introduced the following functional equation

$$f(xy) + f(x + y) = f(xy + x) + f(y), \tag{1.1}$$

where the domain and range of f is a (commutative) field. At ISFE 17th Benz [1] determined the continuous solution of Davison functional equation. Indeed, he proved that if $f : \mathbb{R} \to \mathbb{R}$, then every continuous solution of the equation (1.1) is of the form $f(x) = ax + b$, where a and b are real constants. In 2000, Girgensohn and Lajkó [3] obtained the general solution of the Davison equation without any regularity assumption. They showed that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.1) for all $x, y \in \mathbb{R}$ if and only if f is of the form $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and b is an arbitrary real constant. For more on Davison functional equation (1.1) and its stability interested readers should referred to the book [5] and references therein. In [4] we studied the following functional

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equations

$$f(x+y) = f(xy) + f(y) + f(x-xy), \tag{1.2}$$

$$f(x+y) = f(x-xy) + f(y+xy), \tag{1.3}$$

$$f(x+y) = f(x+y-xy) + f(xy), \tag{1.4}$$

$$f(x+y) = f(x-xy) + f(y) - f(-xy), \tag{1.5}$$

$$2f(x) + 2f(y) = f(x+y+xy) + f(x+y-xy), \tag{1.6}$$

without any regularity assumption on the unknown function f.

Let $\mathbb{X}$ be a nonempty set. The list $(\mathbb{X}, +, \cdot)$ is called a linear (or vector) space if $(\mathbb{X}, +)$ is an abelian group, and $\cdot$ is a mapping that assigns to each $(\lambda, x) \in \mathbb{R} \times \mathbb{X}$ an element $\lambda \cdot x$ of $\mathbb{X}$ (which will be denoted simply as λx) such that for all $\alpha, \lambda \in \mathbb{R}$ and $x, y \in \mathbb{X}$, we have (i) $\alpha(\lambda x) = (\alpha\lambda)x$; (ii) $(\alpha + \lambda)x = \alpha x + \lambda x$ and $\lambda(x+y) = \lambda x + \lambda y$; (iii) $1x = x$. A function $f : \mathbb{R} \to \mathbb{X}$, where $\mathbb{X}$ is a linear space, is said to be additive if and only if f satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\mathbb{X} = \mathbb{R}$, it is well known that every regular (measurable, continuous, integrable, or locally integrable) additive function is of the form $f(x) = ax$, where a is an arbitrary constant in $\mathbb{R}$.

The aim of the present paper is to present the general solutions (f, g, h, k) on the pexiderized functional equations

$$f(x+y) + g(-xy) = h(x-xy) + k(y) \tag{1.7}$$

and

$$2f(x) + 2g(y) = h(x+y+xy) + k(x+y-xy), \tag{1.8}$$

for all $x, y \in \mathbb{R}$ without assuming any regularity assumption of the unknown functions. This paper ends with two open problems related to the above functional equations.

2. General Solutions of (1.7) and (1.8) on $\mathbb{R}$

In this section $\mathbb{X}$ denotes a linear space.

Theorem 2.1. *The functions $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy the functional equation* (1.7) *for all $x, y \in \mathbb{R}$ if and only if they have the form $f(x) = A(x) + b_1$, $g(x) = A(x) + b_2$, $h(x) = A(x) + b_3$ and $k(x) = A(x) + b_4$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b_1, b_2, b_3, b_4 \in \mathbb{X}$ are constants with $b_1 + b_2 = b_3 + b_4$.*

Proof. Sufficiency is obvious. Let $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy (1.7). Substituting $x = 0$, $y = 0$ and $y = 1$, respectively, in (1.7), we get

$$f(y) + g(0) = h(0) + k(y), \tag{2.1}$$

$$f(x) + g(0) = h(x) + k(0), \tag{2.2}$$

$$f(x+1) + g(-x) = h(0) + k(1). \tag{2.3}$$

If we use these equations in (1.7), we obtain

$$f(x+y) - f(1+xy) = f(x-xy) + f(y) + 2g(0) - 2h(0) - k(0) - k(1), \tag{2.4}$$

for all $x, y \in \mathbb{R}$. Letting $x = 1$ in (2.4), we obtain

$$f(1-y) = -f(y) - 2g(0) + 2h(0) + k(0) + k(1) \quad (y \in \mathbb{R}).$$

Hence,

$$f(1+xy) = -f(-xy) - 2g(0) + 2h(0) + k(0) + k(1) \quad (x, y \in \mathbb{R}). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$f(x+y) + f(-xy) = f(x-xy) + f(y) \quad (x, y \in \mathbb{R}).$$

Therefore f is of the form $f(x) = A(x) + b_1$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b_1 \in \mathbb{X}$ is a constant (see [4, Theorem 3.1]). Now we obtain the asserted form of g, h and k by using (2.1), (2.2) and (2.3). The proof of the theorem is now complete. $\square$

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{X}$ be an odd function. Then f satisfies*

$$f(x) + f(y) + f(y+1) = f(x+y+xy) + f(y-xy+1) \quad (x, y \in \mathbb{R}) \tag{2.6}$$

if and only if f is additive.

Proof. Sufficiency is clear. Let f satisfy (2.6). Replacing y by $y+1$ and $y-1$, respectively, we get

$$f(x) + f(y+1) + f(y+2) = f(2x+y+xy+1) + f(y-x-xy+2), \tag{2.7}$$

$$f(x) + f(y-1) + f(y) = f(y+xy-1) + f(x+y-xy), \tag{2.8}$$

for all $x, y \in \mathbb{R}$. Interchanging x and y in (2.8), we see that

$$f(y) + f(x-1) + f(x) = f(x+xy-1) + f(x+y-xy) \quad (x, y \in \mathbb{R}). \tag{2.9}$$

Subtracting (2.9) from (2.8), we get

$$f(y-1) - f(x-1) = f(y+xy-1) - f(x+xy-1) \quad (x, y \in \mathbb{R}). \tag{2.10}$$

Replacing x by $x+1$ and y by $y+1$, respectively, in (2.10), we have

$$f(y) - f(x) = f(2y+x+xy+1) - f(2x+y+xy+1) \quad (x, y \in \mathbb{R}). \tag{2.11}$$

Adding the equations (2.7) and (2.11), we have

$$f(y) + f(y+1) + f(y+2) = f(2y+x+xy+1) + f(y-x-xy+2), \tag{2.12}$$

for all $x, y \in \mathbb{R}$. Let $u, v \in \mathbb{R}$ with $u + v \neq -2$. Setting $x = \frac{v-u}{2+u+v}$ and $y = \frac{u+v}{2}$ in (2.12), we get

$$f\Big(\frac{u+v}{2}\Big) + f\Big(\frac{u+v}{2}+1\Big) + f\Big(\frac{u+v}{2}+2\Big) = f\Big(\frac{u+v}{2}+v+1\Big) + f(u+2). \tag{2.13}$$

If $u + v = -2$, then (2.13) reduces to $f(-1) + f(0) + f(1) = f(v) + f(-v)$, which holds automatically, since f is odd. Thus, (2.13) is true for all $u, v \in \mathbb{R}$. Replacing v by $v - u$ in (2.13), we have

$$f\Big(\frac{v}{2}\Big) + f\Big(\frac{v}{2}+1\Big) + f\Big(\frac{v}{2}+2\Big) = f\Big(\frac{3v-2u}{2}+1\Big) + f(u+2). \tag{2.14}$$

Replacing u by $u-2$ and v by $-\frac{2}{3}v$ in (2.14), we have

$$f\Big(-\frac{v}{3}\Big)+f\Big(-\frac{v}{3}+1\Big)+f\Big(-\frac{v}{3}+2\Big)=f\left(3-(u+v)\right)+f(u).$$

This functional equation is a Pexider functional equation of the form

$$F(x)=G(x+y)+H(y)\quad (x,y\in\mathbb{R}), \tag{2.15}$$

where

$$\begin{aligned}F(t)&:=f\Big(-\frac{t}{3}\Big)+f\Big(-\frac{t}{3}+1\Big)+f\Big(-\frac{t}{3}+2\Big),\\ G(t)&:=f(3-t),\\ H(t)&:=f(t).\end{aligned}$$

It is easy to show that (2.15) implies $H(x+y)=H(x)+H(y)$ for all $x,y\in\mathbb{R}$ since $G(x)=F(0)-H(x)$, $F(x)=F(0)-H(x)$ and $H(0)=0$. Hence, H is additive and thus f is additive. The proof of the lemma is now complete. □

Theorem 2.2. *The functions $f,g,h,k:\mathbb{R}\to\mathbb{X}$ satisfy* (1.8) *for all $x,y\in\mathbb{R}$ if and only if they have the form $f(x)=A(x)+b_1$, $g(x)=A(x)+b_2$, $h(x)=A(x)+b_3$, $k(x)=A(x)+b_4$, where $A:\mathbb{R}\to\mathbb{X}$ is additive and $b_1,b_2,b_3,b_4\in\mathbb{X}$ are constants with $2b_1+2b_2=b_3+b_4$.*

Proof. Sufficiency is clear. Let $f,g,h,k:\mathbb{R}\to\mathbb{X}$ satisfy (1.8). Setting $x=0$, $y=0$ and $y=1$, respectively, in (1.8), we get

$$2f(0)+2g(y)=h(y)+k(y), \tag{2.16}$$

$$2f(x)+2g(0)=h(x)+k(x), \tag{2.17}$$

$$2f(x)+2g(1)=h(2x+1)+k(1). \tag{2.18}$$

Therefore from (2.16) and (2.17), we obtain

$$f(x)-f(0)=g(x)-g(0)\quad (x\in\mathbb{R}). \tag{2.19}$$

Replacing y by $2y+1$ in (1.8), we have

$$2f(x)+2g(2y+1)=h(2x+2y+2xy+1)+k(2y-2xy+1)\quad (x,y\in\mathbb{R}). \tag{2.20}$$

Using (2.18) and (2.19) in (2.20), we get

$$\begin{aligned}2f(x)+2f(2y+1)=&2f(x+y+xy)+k(2y-2xy+1)+2f(0)-2g(0)\\ &+2g(1)-k(1),\end{aligned} \tag{2.21}$$

for all $x,y\in\mathbb{R}$. It follows from (2.17) that

$$2f(2y-2xy+1)+2g(0)=h(2y-2xy+1)+k(2y-2xy+1)\quad (x,y\in\mathbb{R}).$$

Using (2.18) in this equation, we have

$$\begin{aligned}k(2y-2xy+1)=&2f(2y-2xy+1)-2f(y-xy)\\ &+2g(0)-2g(1)+k(1)\quad (x,y\in\mathbb{R}).\end{aligned}$$

Using this equation in (2.21), we get

$$f(x)+f(2y+1)-f(0)=f(x+y+xy)+f(2y-2xy+1)-f(y-xy), \tag{2.22}$$

for all $x,y\in\mathbb{R}$. Letting $x=-1$ and replacing y by $\frac{1}{2}y$ in (2.22), we have

$$f(2y+1)=f(y+1)+f(y)-f(0)\quad (y\in\mathbb{R}). \tag{2.23}$$

Replacing y by $y-xy$ in (2.23), we obtain

$$f(2y-2xy+1)=f(y-xy+1)+f(y-xy)-f(0)\quad (y\in\mathbb{R}).$$

Using this equation directly in the right-hand side of (2.22) and using (2.23) in the left-hand side of (2.22), we get

$$f(x)+f(y)-f(0)=f(x+y+xy)+f(y-xy+1)-f(y+1), \tag{2.24}$$

for all $x,y\in\mathbb{R}$. Since the left-hand side of (2.24) is symmetric in x and y, we get

$$f(y-xy+1)-f(y+1)=f(x-xy+1)-f(x+1), \tag{2.25}$$

for all $x,y\in\mathbb{R}$. Replacing y by $2y-1$ in (2.25), we get

$$f(x+2y-2xy)-f(2y)=f(2(x-xy)+1)-f(x+1).$$

Using (2.23) in this equation, we have

$$f(x+2y-2xy)-f(2y)=f(x-xy+1)+f(x-xy)-f(x+1)-f(0), \tag{2.26}$$

for all $x,y\in\mathbb{R}$. Using (2.25) in (2.26), we have

$$f(x+2y-2xy)-f(2y)=f(y-xy+1)+f(x-xy)-f(y+1)-f(0), \tag{2.27}$$

for all $x,y\in\mathbb{R}$. Setting $x=1$ in (2.27), we get

$$f(2y)=f(1+y)-f(1-y)+f(0)\quad (y\in\mathbb{R}). \tag{2.28}$$

Replacing y by $-y$ in (2.28) and adding the obtained equation to (2.28), we get

$$f(2y)+f(-2y)=2f(0)\quad (y\in\mathbb{R}).$$

Hence, $f-f(0)$ is odd. Since f satisfies (2.24), $f-f(0)$ satisfies (2.6). Therefore, $f-f(0)$ is additive by Lemma 2.1. Thus, $f(x)=A(x)+b_1$, where $A:\mathbb{R}\to\mathbb{X}$ is an additive function and $b_1\in\mathbb{X}$ is a constant. Now, using (2.19), (2.18) and (2.17), we obtain the asserted form of g, h and k. This finishes the proof of the theorem. □

3. Open problems

In this section, we pose two open problems. Determine the general solution (f,g,h,k) of the functional equations (1.7) and (1.8), respectively, where the domain and range of the unknown functions f,g,h,k are (commutative) fields. It should be noted that our arguments are not valid in Theorems 2.1 and 2.2 if the field characteristic (in domain) is equal to 2 or 3.

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GEOMETRIC PROPERTIES AND COMPACT OPERATOR ON FRACTIONAL RIESZ DIFFERENCE SPACE

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Abstract. In this article we introduce the Riesz difference sequence space $r_p^q\left(\Delta^{B\alpha}\right)$ of fractional order α, defined by the composition of fractional backward difference operator $\Delta^{B\alpha}$ given by $(\Delta^{B\alpha}v)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k-i}$ and the Riesz matrix R^q. We give some topological properties, obtain the Schauder basis and determine the α-, β- and γ- duals and investigate certain geometric properties of the space $r_p^q\left(\Delta^{B\alpha}\right)$. Finally, we characterize certain classes of compact operators on the space $r_p^q\left(\Delta^{B\alpha}\right)$ using Hausdorff measure of non-compactness.

1. Introduction

Throughout this article we shall use the symbol l^0 to denote the space of all real valued sequences. Let V and W be two sequence spaces and let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of real entries. In the rest of the paper, for ambiguity we shall write $A = (a_{nk})$ in place of $A = (a_{nk})_{n,k=0}^{\infty}$. We write A_n to denote the sequences in the nth row of the matrix A. We say that the matrix A defines a matrix mapping from V to W if for every sequence $v = (v_k)$, the A-transform of v, i.e., $Av = \{(Av)_n\} \in W$, where

$$(Av)_n = \sum_k a_{nk}v_k, \quad n \in \mathbb{N}. \tag{1.1}$$

Define the sequence space V_A by

$$V_A = \left\{v = (v_k) \in l^0 : Av \in V\right\}. \tag{1.2}$$

Key words and phrases. Riesz difference sequence space, difference operator $\Delta^{B\alpha}$, geometric properties, Hausdorff measure of non-compactness.

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Then the sequence space V_A is called the domain of the matrix A in the space V. Also, we use the notation (V, W) to represent the class of all matrices A from V to W. Thus $A \in (V, W)$ if and only if the series on the right hand side of the equality (1.1) converges for each $n \in \mathbb{N}$ and $v \in V$ such that $Av \in W$ for all $v \in V$. Besides, we denote the unit sphere and the closed unit ball of a set V by $S(V)$ and $B(V)$, respectively.

Throughout this paper s will denote the conjugate of p, that is $s = \frac{p}{p-1}$ for $1 < p < \infty$ or $s = \infty$ for $p = 1$ or $s = 1$ for $p = \infty$.

Definition 1.1. Let x be a real number such that $x \notin \{0, -1, -2, \ldots\}$. Then the gamma function of x is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.3}$$

Clearly, $\Gamma(x + 1) = x!$ for $x \in \mathbb{N}$. Also, $\Gamma(x + 1) = x\Gamma(x)$ for any real number $x \notin \{0, -1, -2, \ldots\}$.

The domains $c_0(\Delta^F)$, $c(\Delta^F)$ and $\ell_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and ℓ_∞ are introduced by Kızmaz [24]. Aftermore, the domain bv_p of the backward difference matrix Δ^B in the space ℓ_p have recently been investigated for $0 < p < 1$ by Altay and Başar [6], and for $1 \leq p \leq \infty$ by Başar and Altay [7]. Aftermore, several other authors [13, 15, 16, 18–21, 30, 31, 43] generalized the notion of difference operator Δ and studied difference sequence spaces of integer order. However, for a positive proper fraction α, Baliarsingh [10] (see also [9]) introduced generalized fractional forward and backward difference operators $\Delta^{F\alpha}$ and $\Delta^{B\alpha}$ defined by

$$(\Delta^{F\alpha}v)_k = \sum_i (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha - i + 1)} v_{k+i} \quad \text{and} \quad (\Delta^{B\alpha}v)_k = \sum_i (-1)^i \frac{\Gamma(\alpha + 1)}{i!\Gamma(\alpha - i + 1)} v_{k-i},$$

respectively. We give a short survey concerned with sequence spaces defined by fractional difference operator. Baliarsingh [10] introduced the difference sequence spaces $V(\Gamma, \Delta^\alpha, u)$ of fractional order α for $V = \{\ell_\infty, c, c_0\}$, where $u = (u_n)$ is a sequence satisfying certain conditions. Baliarsingh and Dutta [9] studied the difference sequence spaces $V(\Gamma, \Delta^\alpha, p)$ for $V = \{\ell_\infty, c, c_0\}$. Moreover, Altay and Başar [4] and Altay et al. [5] introduced the Euler sequence spaces e^r_0, e^r_c and e^r_∞, respectively. In [3], Polat and Başar introduced the spaces $e^r_0(\Delta^{Bm})$, $e^r_c(\Delta^{Bm})$ and $e^r_\infty(\Delta^{Bm})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e^r_0, e^r_c and e^r_∞, respectively. Kadak and Baliarsingh [22] studied Euler difference sequence spaces of fractional order $e^r_p(\Delta^{B\alpha})$, $e^r_0(\Delta^{B\alpha})$, $e^r_c(\Delta^{B\alpha})$ and $e^r_\infty(\Delta^{B\alpha})$ by introducing the Euler mean difference operator $E^r(\Delta^{B\alpha})$. Extending these spaces Meng and Mei [29] introduced binomial difference sequence spaces $b^{r,s}_0(\Delta^{B\alpha})$, $b^{r,s}_c(\Delta^{B\alpha})$ and $b^{r,s}_\infty(\Delta^{B\alpha})$ of fractional order. Yaying et al. [40] also studied the compactness related results on these spaces. Yaying and Hazarika [41] also examined the sequence space $b^{r,s}_p(\Delta^{B\alpha})$. Furthermore, Yaying [42] also studied paranormed Riesz difference sequence spaces $r^q_\infty(\Delta^{B\alpha})$, $r^q_0(\Delta^{B\alpha})$ and $r^q_c(\Delta^{B\alpha})$ of fractional order. Nayak, Et and Baliarsingh [35] examined the sequence spaces $V(u,v,\Delta^{B\alpha},p)$ derived by combining the weighted mean operator $G(u,v)$ and backward fractional difference operator $\Delta^{B\alpha}$. Özger [37] studied geometric properties and Hausdorff measure of non-compactness related results of certain sequence spaces defined by the fractional difference operators. More recently Baliarsingh and Kadak [11] investigated certain class of mappings and Hausdorff measure of non-compactness of certain generalised Euler difference sequence spaces of fractional order. Further, one may also refer [12] for a more generalized fractional difference operators.

Definition 1.2. Let (q_k) be a sequence of positive numbers and define $Q_n = \sum_{k=0}^n q_k$, $n \in \mathbb{N}$. Then the Riesz mean matrix $R^q = (r^q_{nk})$ is defined as

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Malkowsky [25] introduced the sequence spaces r^q_∞, r^q_c and r^q_0 as the set of all sequences whose R^q-transforms are in the spaces ℓ_∞, c and c_0, respectively. Altay and Başar [1] studied the sequence space $r^q(p)$ as

$$r^q(p) = \left\{ v = (v_k) \in l^0 : \sum_{n\in\mathbb{N}} \left| \frac{1}{Q_n} \sum_{k=0}^n q_k v_k \right|^{p_k} < \infty \right\},$$

where $p = (p_k)$ is a bounded sequence of positive real numbers. Altay and Başar [2] also studied the sequence spaces $r^q_\infty(p)$, $r^q_0(p)$ and $r^q_c(p)$ defined by

$$r^q_\infty(p) = \left\{ v = (v_k) \in l^0 : \sup_{n\in\mathbb{N}} \left| \frac{1}{Q_n} \sum_{k=0}^n q_k v_k \right|^{p_k} < \infty \right\},$$

$$r^q_0(p) = \left\{ v = (v_k) \in l^0 : \lim_{n\to\infty} \left| \frac{1}{Q_n} \sum_{k=0}^n q_k v_k \right|^{p_k} = 0 \right\} \quad \text{and}$$

$$r^q_c(p) = \left\{ v = (v_k) \in l^0 : \lim_{n\to\infty} \left| \frac{1}{Q_n} \sum_{k=0}^n q_k v_k - l \right|^{p_k} = 0, \text{ for some } l \in \mathbb{R} \right\}.$$

Since then several authors studied and examined Riesz sequence spaces. For more studies on Riesz sequence spaces, one may refer to [25,42] and the references mentioned therein.

2. Riesz Difference Operator of Fractional Order and Sequence Spaces

First we give the definitions of $R^q(\Delta^{B\alpha})$ and its inverse.

Definition 2.1 ([42])**.** The product matrix $R^q(\Delta^{B\alpha})$ of Riesz mean R^q and the backward difference operator $\Delta^{B\alpha}$ is defined as follows:

$$\left(R^q(\Delta^{B\alpha})\right)_{nk} = \begin{cases} \sum_{i=k}^n (-1)^{i-k} \frac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)} \cdot \frac{q_i}{Q_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Definition 2.2. ([42, Lemma 2.1]). The inverse of the product matrix $R^q(\Delta^{B\alpha})$ is given by:

$$\left(R^q(\Delta^{B\alpha})\right)^{-1}_{nk} = \begin{cases} (-1)^{n-k}\sum_{j=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)}\cdot\frac{Q_k}{q_j}, & 0 \le k < n, \\ \frac{Q_n}{q_n}, & k = n, \\ 0, & k > n. \end{cases}$$

We define the $R^q(\Delta^{B\alpha})$-transform of a sequence $v = (v_k)$ as follows:

$$u_n = \left(R^q(\Delta^{B\alpha})v\right)_n = \sum_{k=0}^{n-1}\left[\sum_{j=k}^{n}(-1)^{j-k}\frac{\Gamma(\alpha+1)}{(j-k)!\Gamma(\alpha-j+k+1)}\cdot\frac{q_j}{Q_n}\right]v_k + \frac{q_n}{Q_n}v_n, \tag{2.1}$$

where $n \in \mathbb{N}$. Now we introduce the Riesz difference sequence space $r^q_p(\Delta^{B\alpha})$ of fractional order α as follows:

$$r^q_p(\Delta^{B\alpha}) = \left\{v = (v_n) \in l^0 : R^q(\Delta^{B\alpha})v \in \ell_p\right\}, \quad \text{where } 1 \le p \le \infty.$$

The above sequence space can be expressed in the notation of (1.2) as follows:

$$r^q_p(\Delta^{B\alpha}) = (\ell_p)_{R^q(\Delta^{B\alpha})}, \quad 1 \le p \le \infty.$$

The sequence space $r^q_p(\Delta^{B\alpha})$ may be reduced to the following classes of sequence spaces in the special cases of α.

1. If $\alpha = 0$, then the sequence space $r^q_p(\Delta^{B\alpha})$ reduces to $r^q_p = (\ell_p)_{R^q}$ for $1 \le p \le \infty$.
2. If $\alpha = 1$, then the sequence space $r^q_p(\Delta^{B\alpha})$ reduces to $r^q_p(\Delta^B)$, where $(\Delta^B v)_k = v_k - v_{k-1}$ for all $k \in \mathbb{N}$.
3. If $\alpha = m \in \mathbb{N}$, then the sequence space $r^q_p(\Delta^{B\alpha})$ reduces to $r^q_p(\Delta^{Bm})$, where $(\Delta^{Bm}v)_k = \sum_{j=0}^{m}(-1)^j\binom{m}{j}v_{m-j}$ for all $k \in \mathbb{N}$.

We begin with the following theorem.

Theorem 2.1. *The sequence space $r^q_p(\Delta^{B\alpha})$ is a BK-space normed by*

$$\|v\|_{r^q_p(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \left(\sum_k\left|\left(R^q(\Delta^{B\alpha})v\right)_k\right|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{2.2}$$

and

$$\|v\|_{r^q_\infty(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}\left|\left(R^q(\Delta^{B\alpha})v\right)_k\right|. \tag{2.3}$$

Proof. The proof is a routine verification and hence omitted. □

Theorem 2.2. *The Riesz difference space $r^q_p(\Delta^{B\alpha})$ is linearly isomorphic to ℓ_p, where $1 \le p \le \infty$.*

Proof. We prove the result for the space $r^q_p(\Delta^{B\alpha})$, $1 \le p < \infty$. Define the mapping $T : r^q_p(\Delta^{B\alpha}) \to \ell_p$ by $v \mapsto u = Tv = R^q(\Delta^{(\alpha)})v$. It is easy to see that T is linear and injective. Let $u=(u_k)\in \ell_p$ and define the sequence $v=(v_k)$ by

$$v_k=\sum_{j=0}^{k-1}\left[\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\cdot\frac{Q_j}{q_i}u_j\right]+\frac{Q_k}{q_k}u_k,\quad k\in\mathbb{N}. \tag{2.4}$$

Then

$$\begin{aligned}\|v\|_{r_p^q(\Delta^{B\alpha})}&=\left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_p}=\left(\sum_k\left|\left(R^q(\Delta^{B\alpha})v\right)_k\right|^p\right)^{\frac{1}{p}}\\&=\left(\sum_k\left|\sum_{j=0}^{k-1}\left(\sum_{i=j}^{k}(-1)^{i-j}\frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)}\cdot\frac{q_i}{Q_k}\right)v_j+\frac{q_k}{Q_k}v_k\right|^p\right)^{\frac{1}{p}}\\&=\left(\sum_k\left|\sum_{j=0}^{k}\delta_{kj}u_j\right|^p\right)^{\frac{1}{p}}=\left(\sum_k|u_k|^p\right)^{\frac{1}{p}}=\|u\|_{\ell_p}<\infty,\end{aligned}$$

where

$$\delta_{kj}=\begin{cases}1, & \text{if } k=j,\\ 0, & \text{if } k\neq j.\end{cases}$$

Thus, $v\in r_p^q(\Delta^{B\alpha})$. Consequently, T is surjective and norm preserving. Thus, $r_p^q(\Delta^{B\alpha})\cong\ell_p$, $1\le p<\infty$. Similarly, we can show that $r_\infty^q(\Delta^{B\alpha})\cong\ell_\infty$. ☐

We now construct sequence of points in the space $r_p^q(\Delta^{B\alpha})$ which will form the Schauder basis for that space. First we recall the definition of Schauder basis for a normed space $(V,\|\cdot\|)$.

Definition 2.3. A sequence $v=(v_k)$ of a normed space $(V,\|\cdot\|)$ is called a Schauder basis of the space V if for every $\nu\in V$ there exists a unique sequence of scalars (c_k) such that

$$\lim_{n\to\infty}\left\|\nu-\sum_{k=0}^{n}c_kv_k\right\|=0.$$

We know by Theorem 2.2 that the mapping $T:r_p^q(\Delta^{B\alpha})\to\ell_p$ is an isomorphism. Hence it is evident that the inverse image of the usual basis $\{e^{(k)}\}_{k\in\mathbb{N}}$ of the space ℓ_p, $1\le p<\infty$, forms the basis of the new space $r_p^q(\Delta^{B\alpha})$. This immediately gives us the following theorem.

Theorem 2.3. *Let $1\le p<\infty$ and define the sequence $b^{(k)}(q)=(b_n^{(k)}(q))$ of the elements of the space $r_p^q(\Delta^{B\alpha})$ for every fixed $k\in\mathbb{N}$ by*

$$b_n^{(k)}(q)=\begin{cases}\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\cdot\frac{Q_j}{q_i}, & k<n,\\ \frac{Q_n}{q_n}, & k=n,\\ 0, & k>n.\end{cases} \tag{2.5}$$

Then the sequence $\{b^{(k)}(q)\}$ is basis for the space $r_p^q(\Delta^{B\alpha})$ and every $v \in r_p^q(\Delta^{B\alpha})$ has a unique representation of the form

$$v = \sum_k \lambda_k b^{(k)}(q), \tag{2.6}$$

where $\lambda_k = \Big(R^q(\Delta^{B\alpha})v\Big)_k$ for all $k \in \mathbb{N}$.

Corollary 2.1. *The sequence space $r_p^q(\Delta^{B\alpha})$ is separable for $1 \le p < \infty$.*

3. α-, β- and γ-Duals

In this section we obtain the α-, β- and γ-duals of $r_p^q(\Delta^{B\alpha})$. We note that the notation α used for α-dual has different meaning to that of the operator $\Delta^{B\alpha}$. First we recall the definitions of α-, β- and γ-duals of the space $V \subset l^0$.

Definition 3.1. The α-, β- and γ-duals of the subset $V \subset l^0$ are defined by

$$\begin{aligned} V^\alpha &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in \ell_1 \text{ for all } v \in V\},\\ V^\beta &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in cs \text{ for all } v \in V\},\\ V^\gamma &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in bs \text{ for all } v \in V\}, \end{aligned}$$

respectively.

Now, we quote certain lemmas given by Stielglitz and Tietz [38] which are necessary to establish our results. Throughout $\mathcal{N}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 3.1. *$A = (a_{nk}) \in (\ell_p, \ell_1)$ if and only if $\sup_{K \in \mathcal{N}} \sum_k \left| \sum_{n \in K} a_{nk} \right| < \infty,\ 1 < p \le \infty$.*

Lemma 3.2. *$A = (a_{nk}) \in (\ell_p, c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^s < \infty,\ 1 < p < \infty. \tag{3.2}$$

Lemma 3.3. *$A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (3.2) holds, with $1 < p \le \infty$.*

Lemma 3.4. *$A = (a_{nk}) \in (\ell_1, \ell_1)$ if and only if $\sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty$.*

Lemma 3.5. *$A = (a_{nk}) \in (\ell_1, c)$ if and only if (3.1) holds and*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty. \tag{3.3}$$

Lemma 3.6. *$A = (a_{nk}) \in (\ell_1, \ell_\infty)$ if and only if (3.2) holds.*

Theorem 3.1. *Define the sets $d_1(q)$ and $d_2(q)$ by*

$$d_1(q) = \left\{ t = (t_k) \in l^0 : \sup_{k \in \mathbb{N}} \sum_n |d_{nk}| < \infty \right\}$$

and

$$d_2(q) = \left\{ t = (t_k) \in l^0 : \sup_{K \in \mathcal{N}} \sum_k \left| \sum_{n \in K} d_{nk} \right|^q < \infty \right\},$$

where the matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum\limits_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \cdot \frac{Q_k}{q_k} t_n, & 0 \le k < n, \\ \frac{Q_n}{q_n} t_n, & k = n, \\ 0, & k > n. \end{cases}$$

Then $\left[r_1^q(\Delta^{B\alpha})\right]^\alpha = d_1(q)$ and $\left[r_p^q(\Delta^{B\alpha})\right]^\alpha = d_2(q)$ for $1 < p < \infty$.

Proof. Consider the sequence $t = (t_k) \in l^0$ and $v = (v_k)$ is as defined in (2.4), then we have

$$\begin{aligned} t_n v_n &= \sum_{j=0}^{n-1} \left[\sum_{i=j}^{j+1} (-1)^{n-j} \frac{\Gamma(-\alpha+1)}{(n-i)!\Gamma(-\alpha-n+i+1)} \cdot \frac{Q_j}{q_i} t_n u_j \right] + \frac{Q_n}{q_n} t_n u_n \\ &= (Du)_n, \quad \text{for each } n \in \mathbb{N}, \end{aligned} \tag{3.4}$$

Thus, we deduce from (3.4) that $tv = (t_k v_k) \in \ell_1$ whenever $v = (v_k) \in r_1^q(\Delta^{B\alpha})$ or $r_p^q(\Delta^{B\alpha})$ if and only if $Du \in \ell_1$ whenever $u = (u_k) \in \ell_1$ or ℓ_p. This yields us the fact that $t = (t_n) \in \left[r_1^q(\Delta^{B\alpha})\right]^\alpha$ or $\left[r_p^q(\Delta^{B\alpha})\right]^\alpha$ if and only if $D \in (\ell_1, \ell_1)$ or $D \in (\ell_p, \ell_1)$.

Thus, by using Lemma 3.1 and Lemma 3.4, we conclude that

$$\left[r_1^q(\Delta^{B\alpha})\right]^\alpha = d_1(q) \quad \text{and} \quad \left[r_p^q(\Delta^{B\alpha})\right]^\alpha = d_2(q).$$ □

Theorem 3.2. *Define the sets $d_3(q)$, $d_4(q)$ and $d_5(q)$ as follows:*

$$d_3(q) = \left\{ t = (t_k) \in l^0 : \sum_k \left| \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k \right|^q < \infty \right\},$$

$$d_4(q) = \left\{ t = (t_k) \in l^0 : \sup_{n,k} \left| \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k \right| < \infty \right\} \quad \textit{and}$$

$$d_5(q) = \left\{ t = (t_k) \in l^0 : \left\{ \frac{Q_k}{q_k} t_k \right\} \in \ell_\infty \right\},$$

where

$$\Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) = \frac{t_k}{q_k} + \sum_{j=k+1}^{n} (-1)^{j-k} t_j \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1) q_i}. \tag{3.5}$$

Then $\left[r_1^q(\Delta^{B\alpha})\right]^\beta = d_4(q) \cap d_5(q)$ and $\left[r_p^q(\Delta^{B\alpha})\right]^\beta = d_3(q) \cap d_5(q)$.

Proof. We give the proof for the space $r_p^q(\Delta^{B\alpha})$, $1 < p < \infty$, to avoid repetition of the similar statements. Let $t = (t_k) \in l^0$ and $v = (v_k)$ is as defined in (2.4). Consider the following equation

$$\begin{aligned}
\sum_{k=0}^{n} t_k v_k &= \sum_{k=0}^{n} t_k \left[\sum_{j=0}^{k-1} \left(\sum_{i=j}^{j+1} (-1)^{k-j} \frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)} \frac{Q_j}{q_i} u_j \right) + \frac{Q_k}{q_k} u_k \right] \\
&= \sum_{k=0}^{n-1} u_k Q_k \left[\frac{t_k}{q_k} + \sum_{j=k+1}^{n} (-1)^{j-k} t_j \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1) q_i} \right] + \frac{Q_n}{q_n} t_n u_n \qquad (3.6) \\
&= \sum_{k=0}^{n-1} u_k Q_k \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) + \frac{Q_n}{q_n} t_n u_n = (Cu)_n, \quad \text{for each } n \in \mathbb{N},
\end{aligned}$$

where $C = (c_{nk})$ is a matrix defined by

$$c_{nk} = \begin{cases} \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k, & 0 \leq k < n, \\ \frac{Q_n}{q_n} t_n, & k = n, \\ 0, & k > n, \end{cases}$$

and $\Delta^{B\alpha} \left(\frac{t_k}{q_k} \right)$ is as defined in (3.5). Clearly the columns of the matrix C are convergent, since

$$\lim_{n \to \infty} c_{nk} = \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k.$$

Thus, we deduce from (3.6) that $tv = (t_k v_k) \in cs$ whenever $v = (v_k) \in r_p^q(\Delta^{B\alpha})$ if and only if $Cu \in c$ whenever $u = (u_k) \in \ell_p$. This yields the fact that $t = (t_k) \in \left[r_p^q(\Delta^{B\alpha}) \right]^\beta$ if and only if $C \in (\ell_p, c)$. Thus by using Lemma 3.2 with (3.6), we get that

$$\sum_k \left| \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k \right|^q < \infty \quad \text{and} \quad \sup_k \left| \frac{Q_k}{q_k} t_k \right| < \infty.$$

Thus, $\left[r_p^q(\Delta^{B\alpha}) \right]^\beta = d_3(q) \cap d_5(q)$. □

Theorem 3.3. *Let* $1 < p < \infty$. *Then* $\left[r_p^q(\Delta^{B\alpha}) \right]^\gamma = d_3(q)$ *and* $\left[r_1^q(\Delta^{B\alpha}) \right]^\gamma = d_4(q)$.

Proof. The proof is analogous to the previous theorem except that Lemma 3.3 in case of $r_p^q(\Delta^{B\alpha})$ and Lemma 3.6 in case of $r_1^q(\Delta^{B\alpha})$ are employed instead of the Lemma 3.2. □

4. Certain Geometric Properties of the Space $r_p^q(\Delta^{B\alpha})$

In this section, we investigate certain geometric properties of the space $r_p^q(\Delta^{B\alpha})$. We first recall certain notions and definitions which are necessary to establish our results.

Definition 4.1. A point $w \in S(V)$ is an extreme point if for every $u, v \in S(V)$ the equality $2w = u + v$ implies $u = v$. A Banach space V is said to be rotund if every point of $S(V)$ is an extreme point.

Definition 4.2. A Banach space V is said to have Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 4.3. Let $1 < p < \infty$. A Banach space is said to have the Banack-Saks type p if every weakly null sequence has a subsequence (x_k) such that for some $K > 0$

$$\|x_k\| \leq K n^{\frac{1}{p}}, \quad \text{for all } n = 1, 2, 3, \ldots$$

Definition 4.4. Let V be a real vector space. A functional $\sigma : V \to [0, \infty)$ is called a modular if

(a) $\sigma(v) = 0$ if and only if $v = \theta$;
(b) $\sigma(\lambda v) = \sigma(v)$ for scalars $|\lambda| = 1$;
(c) $\sigma(\lambda u + \delta v) \leq \sigma(u) + \sigma(v)$ for all $u, v \in V$ and $\lambda, \delta > 0$ with $\lambda + \mu = 1$.

The modular σ is called convex if $\sigma(\lambda u + \delta v) \leq \lambda\sigma(u) + \delta\sigma(v)$ for $u, v \in V$ and $\lambda, \delta > 0$ with $\lambda + \delta = 1$.

We define the operator σ_p, $1 \leq p < \infty$, on $r_p^q(\Delta^{B\alpha})$ by

$$\sigma_p(v) = \sum_n \left| R^q(\Delta^{B\alpha}) \right|^p. \tag{4.1}$$

It is clear that $\sigma_p(v)$ is a convex modular on $r_p^q(\Delta^{B\alpha})$. Now we equip the sequence space $r_p^q(\Delta^{B\alpha})$ with the Luxemborg norm defined by

$$\|v\| = \inf \left\{ \kappa > 0 : \sigma_p \left(\frac{v}{\kappa} \right) \leq 1 \right\}.$$

Now, we give certain basic properties of the modular σ_p.

Proposition 4.1. *The modular σ_p on $r_p^q(\Delta^{B\alpha})$ satisfies the following statements.*

(a) *If $0 < k < 1$, then $k^p \sigma_p \left(\frac{v}{k} \right) \leq \sigma_p(v)$ and $\sigma_p(kv) \leq k\sigma_p(v)$.*
(b) *If $k > 1$, then $\sigma_p(v) \leq k^p \sigma_p \left(\frac{v}{k} \right)$.*
(c) *If $k \geq 1$, then $\sigma_p(v) \leq k\sigma_p(v) \leq \sigma_p(kv)$.*

Proposition 4.2. *The following statements hold for $v \in r_p^q(\Delta^{B\alpha})$.*

(a) *If $\|v\| < 1$, then $\sigma_p(v) \leq \|v\|$.*
(b) *If $\|v\| > 1$, then $\sigma_p(v) \geq \|v\|$.*
(c) *$\|v\| = 1$ if and only if $\sigma_p(v) = 1$.*
(d) *$\|v\| < 1$ if and only if $\sigma_p(v) < 1$.*
(e) *$\|v\| > 1$ if and only if $\sigma_p(v) > 1$.*
(f) *If $0 < k < 1$, $\|v\| > k$, then $\sigma_p(v) > k^p$.*
(g) *If $k \geq 1$, $\|v\| < k$, then $\sigma_p(v) < k^p$.*

Proof. The results can be established analogously to [44, Proposition 17, p.7] (also see [23, Proposition 3], [36, Proposition 6]). Hence, we omit details. ☐

Proposition 4.3. *Let (v_n) be a sequence in $r_p^q(\Delta^{B\alpha})$.*

(a) *If $\lim_{n\to\infty} \|x_n\| = 1$, then $\lim_{n\to\infty} \sigma_p(x_n) = 1$.*

(b) *If $\lim_{n\to\infty} \sigma_p(x_n) = 0$, then $\lim_{n\to\infty} \|x_n\| = 0$.*

Proof. The proof is analogous to the proof of the [36, Theorem 10, page 4]. So we omit details. ☐

Theorem 4.1. *The sequence space $r_p^q(\Delta^{B\alpha})$ is a Banach space with respect to the Luxemborg norm.*

Proof. It is enough to show that every Cauchy sequence in $r_p^q(\Delta^{B\alpha})$ is convergent in Luxemborg norm. Let $v^{(n)} = (v_j^{(n)})$ be a Cauchy sequence in $r_p^q(\Delta^{B\alpha})$ and $\varepsilon \in (0,1)$. Then there exists a positive integer n_0 such that $\left\|v^{(n)} - v^{(m)}\right\| < \varepsilon$ for all $m, n \geq n_0$. Using Part (a) of Proposition 4.2, we obtain

$$\sigma_p(v^{(n)} - v^{(m)}) < \left\|v^{(n)} - v^{(m)}\right\| < \varepsilon, \tag{4.2}$$

for all $n, m \geq n_0$. This gives

$$\sum_k \left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v^{(m)})\right)_k\right|^p < \varepsilon. \tag{4.3}$$

Thus, for each fixed k and for all $n, m \geq n_0$

$$\left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v^{(m)})\right)_k\right| = \left|\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k - \left(R^q(\Delta^{B\alpha})v^{(m)}\right)_k\right| < \varepsilon.$$

Hence, the sequence $\left\{\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k\right\}$ is Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k \in \mathbb{R}$ such that $\left\{\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k\right\} \to \left(R^q(\Delta^{B\alpha})v\right)_k$ as $n \to \infty$. Therefore as $n \to \infty$, using (4.3), we have

$$\sum_k \left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v)\right)_k\right|^p < \varepsilon, \quad \text{for all } n \geq n_0.$$

It remains to show that (v_k) is an element of $r_p^q(\Delta^{B\alpha})$. Since $\left\{\left(R^q(\Delta^{B\alpha})v^{(m)}\right)_k\right\} \to \left(R^q(\Delta^{B\alpha})v\right)_k$ as $m \to \infty$ we have

$$\lim_{m\to\infty} \sigma_p(v^{(n)} - v^{(m)}) = \sigma_p(v^{(n)} - v).$$

Thus, by using the inequality (4.2), we get that $\sigma_p(v^{(n)} - v) < \left\|v^{(n)} - v\right\| < \varepsilon$ for all $n \geq n_0$. This implies that $v^{(n)} \to v$ as $n \to \infty$. Thus, we have $v = v^{(n)} - (v^{(n)} - v) \in r_p^q(\Delta^{B\alpha})$.

Hence, the space $r_p^q(\Delta^{B\alpha})$ is complete under the Luxemborg norm. ☐

Theorem 4.2. *The sequence space $r_p^q(\Delta^{B\alpha})$ equipped with the Luxemborg norm is rotund if and only if $p > 1$.*

Proof. Let the space $r_p^q(\Delta^{B\alpha})$ be rotund and take $p=1$. Now consider the following sequences for a proper fraction α

$$u=\left(1,\alpha-\frac{q_0}{q_1},\frac{\alpha(\alpha+1)}{2!}-\alpha\frac{q_0}{q_1},\frac{\alpha(\alpha+1)(\alpha+2)}{3!}-\frac{\alpha(\alpha+1)}{2!}\cdot\frac{q_0}{q_1},\dots\right)$$

and

$$v=\left(0,\frac{Q_1}{q_1},\alpha\frac{Q_1}{q_1}-\frac{Q_1}{q_2},\frac{\alpha(\alpha+1)}{2!}\cdot\frac{Q_1}{q_1}-\alpha Q_1\frac{q_1}{q_2},\dots\right).$$

Then $u\neq v$ and it can be clearly seen that

$$\sigma_p(u)=\sigma_p(v)=\sigma_p\left(\frac{u+v}{2}\right)=1.$$

Then by Part (c) of Proposition 4.2, $u,v,\frac{u+v}{2}\in S\left[r_p^q(\Delta^{B\alpha})\right]$ which contradicts the fact that $r_p^q(\Delta^{B\alpha})$ is not rotund. Hence, $p>1$.

Conversely, let $w\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $u,v\in S\left[r_p^q(\Delta^{B\alpha})\right]$, $1<p<\infty$, be such that $w=\frac{u+v}{2}$. By the convexity of σ_p and using the property (c) of Proposition 4.2, we have

$$1=\sigma_p(w)\le\frac{1}{2}\left[\sigma_p(u)+\sigma_p(v)\right]\le\frac{1}{2}+\frac{1}{2}=1.$$

This implies that $\sigma_p(u)=\sigma_p(v)=1$ and $\sigma_p(w)=\frac{\sigma_p(u)+\sigma_p(v)}{2}$.

Thus from the definition of σ_p and from the above discussion, we get

$$\sum_n\left|\left(R^q(\Delta^{B\alpha})w\right)_n\right|^p=\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p.$$

Again $w=\frac{u+v}{2}$, we have

$$\sum_n\left|\left(R^q(\Delta^{B\alpha})\left(\frac{u+v}{2}\right)\right)_n\right|^p=\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p.$$

This implies that

$$\left|\left(R^q(\Delta^{B\alpha})\left(\frac{u+v}{2}\right)\right)_n\right|^p=\frac{1}{2}\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p. \tag{4.4}$$

From (4.4), it follows immediately that $u=v$. Thus the space $r_p^q(\Delta^{B\alpha})$ is rotund. □

Theorem 4.3. *The sequence space $R^q(\Delta^{B\alpha})$ has the Kadec-Klee property.*

Proof. Let $v\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $(v^{(n)})\subset r_p^q(\Delta^{B\alpha})$ such that $\left\|v^{(n)}\right\|\to 1$ and $v^{(n)}\to v$ weakly. Using Part (a) of Proposition 4.3, we get

$$\sigma_p(v^{(n)})\to 1\quad\text{as}\quad n\to\infty. \tag{4.5}$$

Also $v\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and using Part (c) of Proposition 4.2, we observe that

$$\sigma_p(v)=1. \tag{4.6}$$

Thus observing equations (4.5) and (4.6), we write

$$\sigma_p(v^{(n)})\to\sigma_p(v)\quad\text{as}\quad n\to\infty.$$

Since $v^{(n)} \to v$ weakly and the jth coordinate mapping $\pi_j : r_p^q(\Delta^{B\alpha}) \to \mathbb{R}$ defined by $\pi_j(v) = v_j$ is continuous imply that $v_k^{(n)} \to v_k$ as $n \to \infty$. Therefore, $v^{(n)} \to v$ as $n \to \infty$. This completes the proof. $\square$

Theorem 4.4. *The space* $r_p^q(\Delta^{B\alpha})$, $1 < p < \infty$, *has the Banach-Saks type* p.

Definition 4.5. The Gurarii's modulus of convexity for a normed linear space V is defined by

$$\beta_V(\varepsilon) = \inf\left\{1 - \inf_{0\leq\alpha\leq 1} \|\alpha v + (1-\alpha)u\| : v, u \in S(V), \|v - u\| = \varepsilon\right\},$$

where $0 < \varepsilon < 2$.

Theorem 4.5. *The Gurarii's modulus of convexity for the space* $r_p^q(\Delta^{B\alpha})$, $1 \leq p < \infty$, *is*

$$\beta_{r_p^q(\Delta^{B\alpha})} \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \quad \text{where } 0 \leq \varepsilon \leq 2.$$

Proof. Let $z \in r_p^q(\Delta^{B\alpha})$. Then

$$\|z\|_{r_p^q(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})z\right\|_{\ell_p} = \left(\sum_n \left|\left(R^q(\Delta^{B\alpha})z\right)_n\right|^p\right)^{\frac{1}{p}}.$$

Let $0 \leq \varepsilon \leq 2$ and we define the following two sequences:

$$u = \left(\left(\left[R^q(\Delta^{B\alpha})\right]^{-1}\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, \left[R^q(\Delta^{B\alpha})\right]^{-1}\left(\frac{\varepsilon}{2}\right), 0, 0, \dots\right)$$

and

$$v = \left(\left(\left[R^q(\Delta^{B\alpha})\right]^{-1}\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, \left[R^q(\Delta^{B\alpha})\right]^{-1}\left(\frac{-\varepsilon}{2}\right), 0, 0, \dots\right).$$

Then $\left\|R^q(\Delta^{B\alpha})u\right\|_{\ell_p} = \|u\|_{r_p^q(\Delta^{B\alpha})} = 1$ and $\left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \|v\|_{r_p^q(\Delta^{B\alpha})} = 1$. That is $u, v \in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $\left\|R^q(\Delta^{B\alpha})u - R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \|u - v\|_{r_p^q(\Delta^{B\alpha})} = \varepsilon$. Thus, for $0 \leq \alpha \leq 1$

$$\begin{aligned}\|\alpha u + (1-\alpha)v\|^p_{r_p^q(\Delta^{B\alpha})} &= \left\|\alpha R^q(\Delta^{B\alpha})u + (1-\alpha)R^q(\Delta^{B\alpha})v\right\|^p_{\ell_p} \\ &= 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|\left(\frac{\varepsilon}{2}\right)^p.\end{aligned}$$

Then $\inf_{0\leq\alpha\leq 1}\|\alpha u + (1-\alpha)v\|^p_{r_p^q(\Delta^{B\alpha})} = 1 - \left(\frac{\varepsilon}{2}\right)^p$. Therefore, for $p \geq 1$

$$\beta_{r_p^q(\Delta^{B\alpha})} \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$ $\square$

Corollary 4.1. (a) *For* $\varepsilon = 2$, $\beta_{r_p^q(\Delta^{B\alpha})} \leq 1$. *Hence,* $r_p^q(\Delta^{B\alpha})$ *is strictly convex.*

(b) *For* $0 < \varepsilon < 2$, $0 < \beta_{r_p^q(\Delta^{B\alpha})} < 1$. *Hence,* $r_p^q(\Delta^{B\alpha})$ *is uniformly convex.*

5. Hausdorff Measure of Non Compactness

In this section, we characterize certain classes of compact operators on the space $r_p^q(\Delta^{B\alpha})$ using Hausdorff measure of non-compactness. First we recall certain known definitions, results and notations that are essential for our investigation.

If V and W are Banach spaces then by $B(V, W)$, we denote the class of all bounded linear operators $L : V \to W$. $B(V, W)$ itself is a Banach space with the operator norm defined by $\|L\| = \sup_{v \in S(V)} \|L(v)\|$. We denote

$$\|a\|_V^* = \sup_{v \in S(V)} \left| \sum_k a_k v_k \right|, \tag{5.1}$$

for $a \in l^0$, provided that the series on the right hand side is finite which is the case whenever V is a BK space and $a \in V^\beta$ [39]. Also L is said to be compact if $D(V) = V$ for the domain of V and for every bounded sequence (v_n) in V, the sequence $(L(v_n))$ has a convergent subsequence in W. We denote the class of all such operators by $C(V, W)$.

The Hausdorff measure of noncompactness of a bounded set Q in a metric space V is defined by

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^n S(v_i, r_i),\ v_i \in V,\ r_i < \varepsilon,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N} \right\},$$

where $S(v_i, r_i)$ is the open ball centered at v_i and radius r_i for each $i = 1, 2, \ldots, n$. One may refer to [8, 11, 17, 27, 32, 34] for more details on compact operators and Hausdorff measure of non-compactness. We need following lemmas for our investigation.

Lemma 5.1. *$\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ and $\ell_\infty^\beta = \ell_1$, where $1 < p < \infty$. Further, if $V \in \{\ell_1, \ell_p, \ell_\infty\}$, then $\|a\|_V^* = \|a\|_{V^\beta}$ holds for all $a \in V^\beta$, where $\|\cdot\|_{V^\beta}$ is the natural norm on V^β.*

Lemma 5.2. ([39, Theorem 4.2.8]). *Let V and W be BK-spaces. Then we have $(V, W) \subset B(V, W)$, that is, every $A \in (V, W)$ defines a linear operator $L_A \in B(V, W)$, where $L_A(v) = A(v)$ for all $v \in V$.*

Lemma 5.3. ([28, Theorem 2.25, Corollary 2.26]). *Let V and W be Banach spaces and $L \in B(V, W)$. Then we have*

$$\|L\|_\chi = \chi(L(S(V))) = \chi(L(B(V))) \tag{5.2}$$

and

$$L \in C(V, W) \quad \text{if and only if} \quad \|L\|_\chi = 0. \tag{5.3}$$

Lemma 5.4. ([28, Theorem 1.23]). *Let $V \supset \varphi$ be a BK space. If $A \in (V, W)$ then $\|L_A\| = \|A\|_{(V,W)} = \sup_n \|A_n\|_V^* < \infty$.*

Lemma 5.5. ([28, Theorem 2.15]). *Let Q be a bounded subset of the normed space V, where V is ℓ_p, $1 \leq p < \infty$, or c_0. If $P_r : V \to V$ is the operator defined by $P_r(v_0, v_1, v_2 \ldots) = (v_0, v_1, v_2 \ldots, v_r, 0, 0, \ldots)$ for all $v = (v_k) \in V$, then*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{v\in Q}\|(I-P_r)(v)\|\right), \qquad \textit{where } I \textit{ is the identity operator on } V.$$

Lemma 5.6. ([33, Theorem 3.7]). *Let $V \supset \varphi$ be a BK-space. Then the following statements hold.*

(a) *If $A \in (V, c_0)$, then $\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|_V^*$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_V^* = 0$.*

(b) *If V has AK and $A \in (V, c)$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\|A_n - \alpha\|_V^* \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A_n - \alpha\|_V^*$$

and L_A is compact if and only if $\lim_{n\to\infty}\|A_n - \alpha\|_V^* = 0$, where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{n\to\infty} a_{nk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (V, \ell_\infty)$, then $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A_n\|_V^*$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_V^* = 0$.*

Lemma 5.7. ([33, Theorem 3.11]). *Let $V \supset \varphi$ be a BK-space. If $A \in (V, \ell_1)$, then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right) \leq \|L_A\|_\chi \leq 4\cdot\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right)$$

and L_A is compact if and only if $\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right) = 0$, where $\mathcal{N}_r$ is the subcollection of $\mathcal{N}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Lemma 5.8. ([33, Theorem 4.4, Corollary 4.5]). *Let $V \supset \varphi$ be a BK-space and let $\|A_n\|_{bs}^{[n]} = \left\|\sum_{m=0}^{n}A_m\right\|_V^*$. Then, the following statements hold.*

(a) *If $A \in (V, cs_0)$, then $\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|_{(V,bs)}^{[n]}$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_{(V,bs)}^{[n]} = 0$.*

(b) *If V has AK and $A \in (V, cs)$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\left\|\sum_{m=0}^{n}A_m - a\right\|_V^* \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left\|\sum_{m=0}^{n}A_m - a\right\|_V^*$$

and L_A is compact if and only if $\lim_{n\to\infty}\|\sum_{m=0}^{n}A_m - a\|_V^* = 0$, where $a = (a_k)$, with $a_k = \lim_{n\to\infty}\sum_{m=0}^{n}a_{mk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (V, bs)$, then $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A\|_{(V,bs)}^{[n]}$ and L_A is compact if and only if $\lim_{n\to\infty}\|A\|_{(V,bs)}^{[n]} = 0$.*

Define an associated matrix $F = (f_{nk})$ of the infinite matrix $A = (a_{nk})$ by

$$f_{nk} = \left(\frac{a_{nk}}{q_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)q_i} \right) Q_k, \tag{5.4}$$

for all $n, k \in \mathbb{N}$.

Lemma 5.9. *Let V be a sequence space and $A = (a_{nk})$ be an infinite matrix. If $A \in (r_p^q(\Delta^{B\alpha}), V)$, then $F \in (\ell_p, V)$ and $Av = Fu$ for all $v \in r_p^q(\Delta^{B\alpha})$, where A and F are related by* (5.4) *and* $1 \leq p \leq \infty$.

Theorem 5.1. *Let $1 < p < \infty$ and $s = \frac{p}{p-1}$. Then we have the following.*

(a) *If $A \in (r_p^q(\Delta^{B\alpha}), c_0)$, then* $\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}}$.

(b) *If $A \in (r_p^q(\Delta^{B\alpha}), c)$, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}},$$

where $f = (f_k)$ and $f_k = \lim_{n\to\infty} f_{nk}$ for each $k \in \mathbb{N}$.

(c) *If $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}}$.

(d) *If $A \in (r_p^q(\Delta^{B\alpha}), \ell_1)$, then*

$$\lim_{r\to\infty} \|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}), \ell_1)} \leq \|L_A\|_\chi \leq 4 \lim_{r\to\infty} \|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}), \ell_1)},$$

where $\|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}), \ell_1)} = \sup_{N \in \mathcal{N}_r} \left(\sum_k \left|\sum_{n\in N} f_{nk}\right|^s\right)^{\frac{1}{s}}$, $r \in \mathbb{N}$.

(e) *If $A \in (r_p^q(\Delta^{B\alpha}), cs_0)$, then* $\|L_A\|_\chi = \limsup\limits_{n\to\infty} \left(\sum_k \left|\sum_{m=0}^n f_{mk}\right|^s\right)^{\frac{1}{s}}$.

(f) *If $A \in (r_p^q(\Delta^{B\alpha}), cs)$, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^n f_{mk} - \tilde{f}_k \right|^s \right)^{\frac{1}{s}} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^n f_{mk} - \tilde{f}_k \right|^s \right)^{\frac{1}{s}},$$

where $\tilde{f} = (\tilde{f}_k)$ with $\tilde{f}_k = \lim_{n\to\infty} \left(\sum_{m=0}^n f_{mk}\right)$ for each $k \in \mathbb{N}$.

(g) *If $A \in (r_p^q(\Delta^{B\alpha}), bs)$, then* $0 \leq \|L_A\|_\chi \leq \limsup\limits_{n\to\infty} \left(\sum\limits_k \left| \sum\limits_{m=0}^n f_{mk} \right|^s \right)^{\frac{1}{s}}$.

Proof. (a) Using Lemma 5.1, one can notice that

$$\|A_n\|^*_{r_p^q(\Delta^{B\alpha})} = \|F_n\|^*_{\ell_p} = \|F_n\|_{\ell_s} = \left(\sum_k |f_{nk}|^s \right)^{\frac{1}{s}}, \quad \text{for } n \in \mathbb{N}.$$

Hence, using Lemma 5.6 (a), we get the desired result.

(b) We have

$$|F_n - f|^*_{\ell_p} = |F_n - f|_{\ell_s} = \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}}, \quad \text{for each } n \in \mathbb{N}.$$

Now, let $A \in (r_p^q(\Delta^{B\alpha}), c)$, then from Lemma 5.1, we have $F \in (\ell_p, c)$. Then we write, using Lemma 5.6 (b),

$$\frac{1}{2} \limsup_{n\to\infty} \|F_n - f\|^*_{\ell_p} \le \|L_A\|_\chi \le \limsup_{n\to\infty} \|F_n - f\|^*_{\ell_p}.$$

This implies

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}} \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}},$$

which is the desired result.

(c) The proof is similar to that of (a) and (b) except that we employ Lemma 5.6 (c) instead of Lemma 5.6 (a) or 5.6 (b).

(d) Clearly,

$$\left\|\sum_{n\in\mathbb{N}} F_n\right\|^*_{\ell_p} = \left\|\sum_{n\in\mathbb{N}} F_n\right\|_{\ell_s} = \left(\sum_k \left|\sum_{n\in\mathbb{N}} f_{nk}\right|^s\right)^{\frac{1}{s}}.$$

Let $A \in (r_p^q(\Delta^{B\alpha}), \ell_1)$. Then $F \in (\ell_p, \ell_1)$ by Lemma 5.9. Hence, using Lemma 5.7, we get

$$\lim_{r\to\infty} \left(\sup_{N\in\mathcal{N}_r} \left\|\sum_{n\in N} F_n\right\|^*_{\ell_p}\right) \le \|L_A\|_\chi \le 4 \cdot \lim_{r\to\infty} \left(\sup_{N\in\mathcal{N}_r} \left\|\sum_{n\in N} F_n\right\|^*_{\ell_p}\right).$$

This implies

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_k \|f_{nk}\|^s\right)^{\frac{1}{s}},$$

as desired.

(e) It is clear that

$$\left\|\sum_{m=0}^n A_m\right\|^*_{r_p^q(\Delta)} = \left\|\sum_{m=0}^n F_m\right\|^*_{\ell_p} = \left\|\sum_{m=0}^n F_m\right\|_{\ell_s} = \left(\sum_k \left|\sum_{m=0}^n f_{mk}\right|^s\right)^{\frac{1}{s}}.$$

Hence, by using Lemma 5.8 (a), we get the desired result.

(f) This is similar to the proof of part (e) with part (b) of Lemma 5.8 instead of part (a) of Lemma 5.8.

(g) This is similar to the proof of Part (e) with part (c) of Lemma 5.8 instead of Part (a) of Lemma 5.8. □

Now, we have the following corollaries.

Corollary 5.1. *Let* $1 < p < \infty$.

(a) *Let* $A \in (r_p^q(\Delta^{B\alpha}), c_0)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty} \left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}} = 0$.

(b) *Let* $A \in (r_p^q(\Delta^{B\alpha}), c)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}} = 0.$$

(c) *Let* $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}} = 0.$

(d) *Let* $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left(\sum_k\left|\sum_{n\in N} f_{nk}\right|^s\right)^{\frac{1}{s}}\right) = 0.$$

(e) *Let* $A \in (r_p^q(\Delta^{B\alpha}), cs_0)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk}\right|^s\right)^{\frac{1}{s}} = 0.$$

(f) *Let* $A \in (r_p^q(\Delta^{B\alpha}), cs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}\right|^s\right)^{\frac{1}{s}} = 0.$$

(g) *Let* $A \in (r_p^q(\Delta^{B\alpha}), bs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk}\right|^s\right)^{\frac{1}{s}} = 0.$$

Theorem 5.2. *The following statements hold.*

(a) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), c_0)$, *then* $\|L_A\|_\chi = \limsup\limits_{n\to\infty}\sum\limits_k |f_{nk}|$.

(b) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), c)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|\right) \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|\right),$$

where $f = (f_k)$ *and* $f_k = \lim\limits_{n\to\infty} f_{nk}$ *for each* $k \in \mathbb{N}$.

(c) *If* $A \in (r_\infty^q(\Delta^{(\alpha)}), \ell_\infty)$, *then* $0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}\sum_k |f_{nk}|$.

(d) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), \ell_1)$, *then*

$$\lim_{r\to\infty}\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)} \le \|L_A\|_\chi \le 4\lim_{r\to\infty}\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)},$$

where $\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)} = \sup_{N\in\mathcal{N}_r}\left(\sum_k\left|\sum_{n\in N} f_{nk}\right|\right)$, $r \in \mathbb{N}$.

(e) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), cs_0)$, *then* $\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^n f_{mk}\right|\right)$.

(f) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), cs)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right) \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right),$$

where $\tilde{f} = (\tilde{f}_k)$ *with* $\tilde{f}_k = \lim_{n\to\infty}\left(\sum_{m=0}^{n} f_{mk}\right)$ *for each* $k \in \mathbb{N}$.

(g) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), bs)$, *then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_k \left|\sum_{m=0}^{n} f_{mk}\right|\right).$

Proof. The proof is analogous to the proof of Theorem 5.1. □

Similarly, we have the following result.

Corollary 5.2. *The following statements hold.*

(a) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), c_0)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\sum_k |f_{nk}| = 0$.

(b) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), c)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\left(\sum_k |f_{nk} - f_k|\right) = 0$.

(c) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\sum_k |f_{nk}| = 0$.

(d) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), \ell_1)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left(\sum_k \left|\sum_{n\in N} f_{nk}\right|\right)\right) = 0.$$

(e) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), cs_0)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

(f) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), cs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k \left|\sum_{m=0}^{n} f_{mk} - \tilde{f}\right|\right) = 0.$$

(g) *Let* $A \in (r_\infty^q(\Delta^{B\alpha}), bs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

Theorem 5.3. *The following statements hold.*

(a) *If* $A \in (r_1^q(\Delta^{B\alpha)}), c_0)$, *then* $\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sup_k |f_{nk}|\right).$

(b) *If* $A \in (r_1^q(\Delta^{B\alpha}), c)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sup_k |f_{nk} - f_k|\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sup_k |f_{nk} - f_k|\right),$$

where $f = (f_k)$ *and* $f_k = \lim_{n\to\infty} f_{nk}$ *for each* $k \in \mathbb{N}$.

(c) *If* $A \in (r_1^q(\Delta^{B\alpha}), \ell_\infty)$, *then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sup_k |f_{nk}|\right).$

(d) *If* $A \in (r_1^q(\Delta^{B\alpha}), \ell_1)$, *then* $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |f_{nk}|\right).$

(e) *If* $A \in (r_1^q(\Delta^{B\alpha}), cs_0)$, *then* $\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right).$

(f) *If* $A \in (r_1^q(\Delta^{B\alpha}), cs)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right),$$

where $\tilde{f} = (\tilde{f}_k)$ *with* $\tilde{f}_k = \lim_{n\to\infty}\left(\sum_{m=0}^{n} f_{mk}\right)$ *for each* $k \in \mathbb{N}$.

(g) *If* $A \in (r_1^q(\Delta^{B\alpha}), bs)$, *then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right)$.

Proof. The proof is analogous to the proof of Theorem 5.1. □

Similarly, we have the following result.

Corollary 5.3. *The following statements hold.*

(a) *Let* $A \in (r_1^q(\Delta^{B\alpha}), c_0)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty} \left(\sup_k |f_{nk}|\right) = 0.$$

(b) *Let* $A \in (r_1^q(\Delta^{B\alpha}), c)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty} \left(\sup_k |f_{nk} - f_k|\right) = 0.$$

(c) *Let* $A \in (r_1^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty} \left(\sup_k |f_{nk}|\right) = 0.$$

(d) *Let* $A \in (r_1^q(\Delta^{B\alpha}), \ell_1)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty} \left(\sup_k \sum_{n=r}^{\infty} |f_{nk}|\right) = 0.$$

(e) *Let* $A \in (r_1^q(\Delta^{B\alpha}), cs_0)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

(f) *Let* $A \in (r_1^q(\Delta^{B\alpha}), cs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^{n} f_{mk} - \tilde{f}\right|\right) = 0.$$

(g) *Let* $A \in (r_1^q(\Delta^{B\alpha}), bs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

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SOME INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree n which has no zeros in $|z|<1$, then it was proved by Liman, Mohapatra and Shah [11] that

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|\\ \leq&\frac{n}{2}\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|+\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\max_{|z|=1}|P(z)|\\ &-\frac{n}{2}\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\min_{|z|=1}|P(z)|,\end{aligned}$$

for any β with $|\beta|\leq 1$ and $|z|=1$. In this paper we generalize the above inequality and our result also generalizes certain well known polynomial inequalities.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most n. If $P\in\mathcal{P}_n$, then according to Bernstein theorem [5], we have

$$\max_{|z|=1}|P'(z)|\leq n\max_{|z|=1}|P(z)|. \tag{1.1}$$

Bernstein proved it in 1912. Later, in 1930 he [6] revisited his inequality and proved the following result from which inequality (1.1) can be deduced for $Q(z)=Mz^n$, where $M=\max_{|z|=1}|P(z)|$.

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Theorem 1.1. *Let $P(z)$ and $Q(z)$ be two polynomials with degree of $P(z)$ not exceeding that of $Q(z)$. If $P(z)$ has all its zeros in $|z| \leq 1$ and*

$$|P(z)| \leq |Q(z)|, \quad \textit{for } |z| = 1,$$

then

$$|P'(z)| \leq |Q'(z)|, \quad \textit{for } |z| = 1 \tag{1.2}$$

More generally, it was proved by Malik and Vong [12] that for any β with $|\beta| \leq 1$, inequality (1.2) can be replaced by

$$\left| zP'(z) + \frac{n\beta}{2} P(z) \right| \leq \left| zQ'(z) + \frac{n\beta}{2} Q(z) \right|, \quad \text{for } |z| = 1. \tag{1.3}$$

By restricting the zeros of a polynomial, the maximum value may be smaller. Indeed, if $P \in \mathcal{P}_n$ has no zero inside the unit circle $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Inequality (1.4) was conjectured by Erdös and later proved verified by Lax [10]. This result was further improved by Aziz and Dawood [2] who, under the same hypothesis, proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.$$

Jain [8] generalized the inequality (1.4) and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $|z| = 1$,

$$\left| zP'(z) + \frac{n\beta}{2} P(z) \right| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \max_{|z|=1} |P(z)|. \tag{1.5}$$

As a refinement of (1.5), Deewan and Hans [7] proved the following.

Theorem 1.2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$*

$$\left| zP'(z) + \frac{n\beta}{2} P(z) \right| \leq \frac{n}{2} \left[\left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \max_{|z|=1} |P(z)| - \left\{ \left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right\} \min_{|z|=1} |P(z)| \right].$$

Let $D_\alpha P(z)$ be an operator that carries n^{th} degree polynomial $P(z)$ to the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z), \quad \alpha \in \mathbb{C},$$

of degree at most $(n-1)$. $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz was among the first to extend these results to polar derivatives. It is proved by Aziz [1] that for $P \in \mathcal{P}_n$ having no zeros in $|z| < 1$ and $|\alpha| \geq 1$,

$$|D_\alpha P(z)| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1) \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1.$$

As an extension of (1.1) for the polar derivative Aziz and Shah [4] proved the following.

Theorem 1.3. *If $P(z)$ is a polynomial of degree n, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$*

$$|D_\alpha P(z)| \leq n|\alpha z^{n-1}| \max_{|z|=1} |P(z)|, \quad \textit{for } |z| \geq 1.$$

Liman et al. [11] extended (1.3) to the polar derivative and proved the following result.

Theorem 1.4. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$,*

$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) P(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) Q(z) \right|, \quad \textit{for } |z| \geq 1.$$

2. Main Result

In this paper, we first prove the following result which is generalization of Theorem 1.4 and also obtain some compact generalization for polar derivative.

Theorem 2.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$,*

(2.1)
$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n} \right) P(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n} \right) Q(z) \right|, \quad \textit{for } |z| \geq k.$$

Remark 2.1. For $k = 1$, Theorem 2.1 reduces to the Theorem 1.4.

Dividing both sides of (2.1) by $|\alpha|$ and letting $|\alpha| \to \infty$ we get following generalization of (1.3).

Corollary 2.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then $\beta \in \mathbb{C}$ with $|\beta| \leq 1$*

$$\left| zP'(z) + \frac{n\beta}{1 + k^n} P(z) \right| \leq \left| zQ'(z) + \frac{n\beta}{1 + k^n} Q(z) \right|, \quad \textit{for } |z| \geq k.$$

By applying Theorem 2.1 to the polynomials $P(z)$ and $Q(z) = M\frac{z^n}{k^n}$, where $M = \max_{|z|=k} |P(z)|$, we get the following result.

Corollary 2.2. *If $P(z)$ is a polynomial of degree n, then for any α, β, with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| \geq k$*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \leq n\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|M.$$

By applying Theorem 2.1 to the polynomials $P(z)$ and $Q(z) = m\frac{z^n}{k^n}$, where $m = \min_{|z|=k}|P(z)|$, we get the following result.

Corollary 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for any α, β with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| \geq k$*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \geq n\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m.$$

Theorem 2.2. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$*

$$|zD_\alpha P(z)| + n\left(\frac{|\alpha|-k}{1+k^n}\right)|Q(z)| \leq |zD_\alpha Q(z)| + n\left(\frac{|\alpha|-k}{1+k^n}\right)|P(z)|. \tag{2.2}$$

Dividing both sides of (2.2) by α and letting $|\alpha| \to \infty$, we get the following result.

Corollary 2.4. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then for $|z| = 1$*

$$\left|\frac{P'(z)}{n}\right| + \left|\frac{Q(z)}{1+k^n}\right| \leq \left|\frac{Q'(z)}{n}\right| + \left|\frac{P(z)}{1+k^n}\right|.$$

Theorem 2.3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$ then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for $|z| \geq k$*

$$\begin{aligned}
&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \\
\leq &\frac{n}{2}\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| + k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)| \\
&-\frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| - \left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}m,
\end{aligned} \tag{2.3}$$

where $m = \min_{|z|=k}|P(z)|$.

Dividing both sides of (2.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following generalization of a result due to Dewan and Hans [7].

Corollary 2.5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and for $|z| \geq k$*

$$\begin{aligned}\left|zP(z) + \frac{n\beta}{1+k^n}P(z)\right| \leq & \frac{n}{2}\left\{|z|^n\left|1+\frac{\beta}{1+k^n}\right| + k^n\left|\frac{\beta}{1+k^n}\right|\right\}\max_{|z|=1}|P(z)| \\ & - \frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|1+\frac{\beta}{1+k^n}\right| - \left|\frac{\beta}{1+k^n}\right|\right\}m,\end{aligned}$$

where $m = \min_{|z|=k}|P(z)|$.

3. Lemma

For the proofs of these theorems we need the following lemmas. The first lemma which we need is due to Laguerre (see [9, page 38]).

Lemma 3.1. *If all the zeros of an n^{th} degree polynomial $P(z)$ lie in a circular region C and w is any zero of $D_\alpha P(z)$, then at most one of the points w and α may lie outside C.*

Lemma 3.2. *Let A and B be any two complex numbers, then the following holds.*

(i) *If $|A| \geq |B|$ and $B \neq 0$, then $A \neq vB$ for all complex numbers v with $|v| < 1$.*
(ii) *Conversely, if $A \neq vB$ for all complex number v with $|v| < 1$, then $|A| \geq |B|$.*

Lemma 3.2 is due to Xin Li [13].

Lemma 3.3. *If $P(z)$ is a polynomial of degree n, then for $k \geq 1$*

$$\max_{|z|=k}|P(z)| \leq k^n \max_{|z|=1}|P(z)|.$$

Lemma 3.3 is simple consequence of maximum modulus theorem.

Lemma 3.4. *If the polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$*

$$n(|\alpha| - k)|P(z)| \leq (1+k^n)|D_\alpha P(z)|.$$

Lemma 3.4 is due to Aziz and Rather [3].

Lemma 3.5. *If $P(z)$ is a polynomial of degree n, then for any α with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = k$*

$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| + \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| \\ \leq & n\left\{|z|^n\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| + k^n\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|,\end{aligned}$$

where $q(z) = \left(\frac{z}{k}\right)^n \overline{P\left(\frac{k^2}{\overline{z}}\right)}$.

Proof. Let $M = \max_{|z|=k} |P(z)|$. An application of *Rouche's Theorem* shows that all the zeros of the polynomial $G(z) = k^n P(z) + \lambda M z^n$ lie in $|z| < k$, $k \geq 1$ for every λ with $|\lambda| > 1$. If $H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\bar{z}}\right)} = k^n Q(z) + \overline{\lambda} M k^n$, then $|G(z)| = |H(z)|$ for $|z| = k$ and hence for any γ with $|\gamma| < 1$ the polynomial $\gamma H(z) + G(z)$ has all its zeros in $|z| < k$, $k \geq 1$. By applying Lemma 3.4, we have for any α with $|\alpha| \geq k$

$$(1 + k^n)|z(\gamma D_\alpha H(z) + D_\alpha G(z))| \geq n(|\alpha| - k)|\gamma H(z) + G(z)|.$$

Since $\gamma H(z) + G(z) \neq 0$ for $|z| \geq k$, $k \geq 1$, so the right hand side is non zero. Thus, by using (i) of Lemma 3.2 we have for all β satisfying $|\beta| < 1$ and for $|z| \geq k$

$$T(z) = \beta n(|\alpha| - k)\gamma H(z) + G(z) + (1 + k^n) z(\gamma D_\alpha H(z) + D_\alpha G(z)) \neq 0,$$

or, equivalently, for $|z| \geq k$

$$T(z) = \gamma(1 + k^n) z D_\alpha H(z) + n\beta(|\alpha| - k)H(z) + (1 + k^n) z D_\alpha G(z) + n\beta(|\alpha| - k)G(z) \neq 0.$$

Using (ii) of Lemma 3.2 we have for $|\gamma| < 1$ and for $|z| \geq k$

$$|(1 + k^n) z D_\alpha H(z) + n\beta(|\alpha| - k)H(z)| \leq |(1 + k^n) z D_\alpha G(z) + n\beta(|\alpha| - k)G(z)|. \tag{3.1}$$

Now by putting $G(z) = k^n P(z) + \lambda M z^n$ and $H(z) = k^n Q(z) + \overline{\lambda} M k^n$ in (3.1) we get

$$\begin{aligned} &\left| z D_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| - n|\lambda| \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M \\ \leq &\left| z D_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) + n\lambda \frac{z^n}{k^n} \left(\alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right)\right) M \right|. \end{aligned} \tag{3.2}$$

By Corollary 2.2, it is possible to choose the argument of λ such that

$$\left| z D_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right| = n \frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M. \tag{3.3}$$

Using (3.3) in (3.2) and letting $|\lambda| \to 1$ we get for $|\alpha| > k$ and $|\beta| < 1$

$$\begin{aligned} &\left| z D_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| - n \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M \\ \leq & n \frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M - \left| z D_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right|. \end{aligned}$$

That is

$$\begin{aligned} &\left| z D_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right| + \left| z D_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| \\ \leq & n \left\{ \frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| + \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| \right\} M. \end{aligned} \tag{3.4}$$

Using Lemma 3.3 in (3.4) we get

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|+\left|zD_\alpha q(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|\\ \leq& n\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|.\end{aligned}$$

That proves Lemma 3.5 completely. □

4. Proof of Theorems

Proof of Theorem 2.1. By *Rouche's Theorem*, the polynomial $\lambda P(z)-Q(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$ for $|\lambda|<1$. Therefore, for $r>1$, all the zeros of $\lambda P(rz)-Q(rz)$ lie in $|z|\leq \frac{k}{r}<k$. By applying Lemma 3.4 to the polynomial $\lambda P(rz)-Q(rz)$, we have for $|z|=1$

$$n(|\alpha|-k)|\lambda P(rz)-Q(rz)|\leq (1+k^n)|z(\lambda D_\alpha P(rz)-D_\alpha Q(rz))|.$$

As in the proof of Lemma 3.1, we have for $|\beta|<1$ and for $|z|\geq k$

$$(1+k^n)z\{\lambda D_\alpha P(rz)-D_\alpha Q(rz)\}+n\beta(|\alpha|-k)\{\lambda P(rz)-Q(rz)\}\neq 0.$$

This implies for $|z|\geq k$

(4.1)
$$|(1+k^n)zD_\alpha P(rz)+n\beta(|\alpha|-k)P(rz)|\leq |(1+k^n)zD_\alpha Q(rz)+n\beta(|\alpha|-k)Q(rz)|.$$

Now making $r\to 1$ and using the continuity for $|\beta|$ in (4.1), the theorem follows. □

Proof of Theorem 2.2. Since all the zeros of $Q(z)$ lie in $|z|\leq k$, $k\geq 1$, we have for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$

$$(1+k^n)|zD_\alpha Q(z)|\geq n(|\alpha|-k)|Q(z)|.$$

This gives for every β with $|\beta|\leq 1$

$$(1+k^n)|zD_\alpha Q(z)|-n|\beta|(|\alpha|-k)|Q(z)|\geq 0.$$

Therefore, it is possible to choose the argument of β in the right hand side of Theorem 2.1 such that

$$|(1+k^n)zD_\alpha Q(z)-n\beta(|\alpha|-k)Q(z)|=|zD_\alpha Q(z)|-n|\beta|\left(\frac{|\alpha|-k}{1+k^n}\right)|Q(z)|. \tag{4.2}$$

Using (4.2) in Theorem 2.1 and letting $|\beta|\to 1$, we get the desired result. □

Proof of Theorem 2.3. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z|\leq k$, $k\geq 1$. If $q(z)=\left(\frac{z}{k}\right)^n\overline{P\left(\frac{k^2}{\bar z}\right)}$, then $q(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$

and $|P(z)| = |q(z)|$ for $|z| = k$. Hence, by Theorem 2.1, we have for all α, β satisfying $|\alpha| \geq k$, $|\beta| \leq 1$

(4.3)
$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \leq \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|, \quad \text{for } |z| \geq k.$$

Let $m = \min_{|z|=k}|P(z)|$. If $P(z)$ has a zero in $|z| = k$, then $m = 0$ and result follows by combining Lemma 3.5 with (4.3). Therefore, we suppose that all the zeros of $P(z)$ lie in $|z| > k$ and so $m > 0$. We have $|\gamma m| < |P(z)|$ on $|z| = k$ for any γ with $|\gamma| < 1$. By *Rouche's Theorem* the polynomial $F(z) = P(z) + \gamma m$ has no zeros in $|z| < k$. Therefore, the polynomial $G(z) = \left(\frac{z}{k}\right)^n \overline{F\left(\frac{k^2}{\bar{z}}\right)} = q(z) - \bar{\gamma} m \frac{z^n}{k^n}$ will have all its zeros in $|z| \leq k$. Also $|F(z)| = |G(z)|$ for $|z| = k$. On applying Theorem 2.1, we get for any β, α with $|\beta| \leq 1$, $|\alpha| \geq k$

$$\left|zD_\alpha F(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)F(z)\right| \leq \left|zD_\alpha G(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)G(z)\right|, \quad \text{for } |z| \geq k.$$

Equivalently,

(4.4)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - n|\gamma|\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| m\\ \leq&\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z) - n\bar{\gamma}\frac{z^n}{k^n}\left(\alpha + \left(\frac{|\alpha|-k}{1+k^n}\right)\right)m\right|.\end{aligned}$$

Since $q(z)$ has all its zeros in $|z| \leq k$ and $\min_{|z=k|}|p(z)| = \min_{|z=k|}|q(z)| = m$, therefore, by Corollary 2.3, we have

(4.5)
$$\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| \geq n\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| m.$$

Therefore, we can write (4.4) in view of (4.5) as

(4.6)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - n\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| m\\ \leq&\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| - n\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| m.\end{aligned}$$

Letting $|\gamma| \to 1$, we get from inequality (4.6)

(4.7)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|\\ \leq& - n\left\{\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| - \left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\} m.\end{aligned}$$

Now, by Lemma 3.5, we have

$$\begin{aligned}
&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|+\left|zD_\alpha q(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| \\
&\leq n\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|.
\end{aligned}\tag{4.8}$$

Inequalities (4.7) and (4.8) together lead to

$$\begin{aligned}
&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \\
\leq&\frac{n}{2}\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)| \\
&-\frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|-\left|z+\beta\frac{(|\alpha|-k)}{1+k^n}\right|\right\}m.
\end{aligned}$$

That proves Theorem 2.3 completely. □

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PICTURE FUZZY SUBSPACE OF A CRISP VECTOR SPACE

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Abstract. In this paper, the notion of picture fuzzy subspace of a crisp vector space is established and some related properties are explored on the basis of some basic operations (intersection, Cartesian product, union, (θ, ϕ, ψ)-cut etc.) on picture fuzzy sets. Direct sum of two picture fuzzy subspaces is initiated here over the direct sum of two crisp vector spaces. Also, the concepts of picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors are introduced and some corresponding results are presented. Isomorphism between two picture fuzzy subspaces is developed here as an extension of isomorphism between two fuzzy subspaces.

1. Introduction

Vector space and subspace of a vector space are two pioneer concepts in the field of algebra. Rosenfeld [13] applied the notion of fuzzy set to group theory and established the idea of fuzzy group after the initiation of fuzzy set by Zadeh [15]. After that many researchers worked on different topics of algebra in the environment of fuzzy set. The concept of fuzzy subspace was initiated by Katsaras and Liu [10]. Vector space in fuzzy sense under triangular norm was studied by Das [5]. Kumar [11] enriched the idea of Das. The concept of picture fuzzy set, a generalization of the concepts of fuzzy set and intuitionistic fuzzy set, was introduced by Cuong [4]. With the advancement of time, different kinds of research works under picture fuzzy environment were performed by several researchers [6–9, 12, 14].

Key words and phrases. Picture fuzzy subspace, direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation, picture fuzzy linearly independent set of vectors.

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In this paper, we will introduce the notion of picture fuzzy subspace of a crisp vector space and study some basic results related to it on the basis of some basic operations on picture fuzzy sets. Also, we will establish the concepts of direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors and explore some properties connected to these.

2. Preliminaries

In the current section, we will call again some basic concepts about fuzzy set (FS), fuzzy subspace (FSS) of a crisp vector space (crisp VS), intuitionistic fuzzy set (IFS), picture fuzzy set (PFS) and some basic operations on picture fuzzy sets (PFSs).

Definition 2.1 ([15])**.** Let A be the set of universe. Then a FS P over A is defined as $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of membership of a in P.

Togethering the concepts of FS and subspace of a crisp VS, Kumar defined FSS of a crisp VS as follows.

Definition 2.2 ([11])**.** Let V be a crisp VS over the field F and $P = \{(a, \mu_P(a)) : a \in V\}$ be a FS in V. Then P is said to be FSS of V if the below stated conditions are meet.

(i) $\mu_P(a_1 - a_2) \geqslant \mu_P(a_1) \wedge \mu_P(a_2)$.

(ii) $\mu_P(ra_1) \geqslant \mu_P(a)$ for all $a_1, a_2 \in V$ and for all $r \in F$.

The measure of non-membership was missing in FS. Including this type of uncertainty, Atanassov [1] defined IFS.

Definition 2.3 ([1])**.** Let A be the set of universe. An IFS P over A is defined as $P = \{(a, \mu_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of membership of a in P and $v_P(a) \in [0,1]$ is the measure of non-membership of a in P with the condition $0 \leqslant \mu_P(a) + v_P(a) \leqslant 1$ for all $a \in A$.

It can be noted that $s_P(a) = 1 - (\mu_P(a) + v_P(a))$ is the measure of suspicion of a in P, which excludes the measure of membership and the measure of non-membership.

Including more possible types of uncertainty, Cuong [4] defined PFS generalizing the concepts of FS and IFS.

Definition 2.4 ([4])**.** Let A be the set of universe. Then a PFS P over the universe A is defined as $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of positive membership of a in P, $\eta_P(a) \in [0,1]$ is the measure of neutral membership of a in P and $v_P(a) \in [0,1]$ is the measure of negative membership of a in P with the condition $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$ for all $a \in A$. For all $a \in A$, $1 - (\mu_P(a) + \eta_P(a) + v_P(a))$ is the measure of denial membership a in P.

The basic operations on PFSs consist of equality, union and intersection are defined below.

Definition 2.5 ([4])**.** Let $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), v_Q(a)) : a \in A\}$ be two PFSs over the universe A. Then

(i) $P \subseteq Q$ if and only if $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$, $v_P(a) \geqslant v_Q(a)$ for all $a \in A$;

(ii) $P = Q$ if and only if $\mu_P(a) = \mu_Q(a), \eta_P(a) = \eta_Q(a), v_P(a) = v_Q(a)$ for all $a \in A$;

(iii) $P \cup Q = \{(a, \max(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \min(v_P(a), v_Q(a))) : a \in A\}$;

(iv) $P \cap Q = \{(a, \min(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \max(v_P(a), v_Q(a))) : a \in A\}$.

Definition 2.6. Let $P = \{(a, \mu_P, \eta_P, v_P) : a \in A\}$ be a PFS over the universe A. Then (θ, ϕ, ψ)-cut of P is the crisp set in A denoted by $C_{\theta,\phi,\psi}(P)$ and is defined as $C_{\theta,\phi,\psi}(P) = \{a \in A : \mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi, v_P(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0, 1]$ with the condition $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Definition 2.7. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a surjective mapping and $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), v_P(a_1)) : a_1 \in A_1\}$ be a PFS in A_1. Then image of P under the map h is the PFS $h(P) = \{(a_2, \mu_{h(P)}(a_2), \eta_{h(P)}(a_2), v_{h(P)}(a_2)) : a_2 \in A_2\}$, where

$$\mu_{h(P)}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} \mu_P(a_1), \quad \eta_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \eta_P(a_1)$$

and

$$v_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} v_P(a_1),$$

for all $a_2 \in A_2$.

Definition 2.8. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a mapping and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), v_Q(a_2)) : a_2 \in A_2\}$ be a PFS in A_2. Then inverse image of Q under the map h is the PFS $h^{-1}(Q) = \{(a_1, \mu_{h^{-1}(Q)}(a_1), \eta_{h^{-1}(Q)}(a_1), v_{h^{-1}(Q)}(a_1)) : a_1 \in A_1\}$, where $\mu_{h^{-1}(Q)}(a_1) = \mu_Q(h(a_1))$, $\eta_{h^{-1}(Q)}(a_1) = \eta_Q(h(a_1))$ and $v_{h^{-1}(Q)}(a_1) = v_Q(h(a_1))$ for all $a_1 \in A_1$.

Definition 2.9. Let $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), v_P(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), v_Q(a_2)) : a_2 \in A_2\}$ be two PFSs over A_1 and A_2 respectively, where A_1, A_2 be two sets of universe. Then the Cartesian product of P and Q is the PFS $P \times Q$=$\{((a, b), \mu_{P\times Q}((a, b)), \eta_{P\times Q}((a, b)), v_{P\times Q}((a, b))) : (a, b) \in A_1 \times A_2\}$, where $\mu_{P\times Q}((a, b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a, b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_{P\times Q}((a, b)) = v_P(a) \vee v_Q(b)$ for all $(a, b) \in A_1 \times A_2$.

Throughout the paper, we write PFS $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ as $P = (\mu_P, \eta_P, v_P)$.

Now, it is time to introduce picture fuzzy subspace (PFSS) of a crisp vector space (crisp VS).

3. Picture Fuzzy Subspace

In the current section, the concept of PFSS is initiated and some basic results on PFSS are explored on the basis of intersection, union, Cartesian product and (θ, ϕ, ψ)-cut on PFSs. Also, some properties of PFSS under image and inverse image of PFS are studied when the map is a linear map in crisp sense.

Now, let us define PFSS of a crisp VS.

Definition 3.1. Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFS in V. Then P is said to be a PFSS of V if

(i) $\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a-b) \leqslant v_P(a) \vee v_P(b)$;

(ii) $\mu_P(ra) \geqslant \mu_P(a)$, $\eta_P(ra) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra) \leqslant v_P(a)$ for all $a, b \in V$ and for all $r \in F$.

Proposition 3.1. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Also, let ρ be the null vector in V. Then*

(i) $\mu_P(\rho) \geqslant \mu_P(a)$, $\eta_P(\rho) \geqslant \eta_P(a)$, $v_P(\rho) \leqslant v_P(a)$;

(ii) $\mu_P(ra) = \mu_P(a)$, $\eta_P(ra) = \eta_P(a)$ and $v_P(ra) = v_P(a)$ for all $a \in V$ and for any non-zero $r \in F$.

Proof. (i) Since P is a PFSS of V, therefore

$$\mu_P(\rho) = \mu_P(a-a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a),$$
$$\eta_P(\rho) = \eta_P(a-a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a),$$
$$v_P(\rho) = v_P(a-a) \leqslant v_P(a) \vee v_P(a) = v_P(a).$$

Thus, it is obtained that $\mu_P(\rho) \geqslant \mu_P(a)$, $\eta_P(\rho) \geqslant \eta_P(a)$ and $v_P(\rho) \leqslant v_P(a)$ for all $a \in V$.

(ii) Since P is a PFSS of V therefore

$$\mu_P(ra) \geqslant \mu_P(a), \quad \eta_P(ra) \geqslant \eta_P(a) \quad \text{and} \quad v_P(ra) \leqslant v_P(a),$$

for all $a \in V$ and for all $r \in F$. Let r be a non-zero scalar. Then

$$\mu_P(a) = \mu_P(r^{-1}(ra)) \geqslant \mu_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$
$$\eta_P(a) = \eta_P(r^{-1}(ra)) \geqslant \eta_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$
$$v_P(a) = v_P(r^{-1}(ra)) \leqslant v_P(ra) \quad [\text{because } P \text{ is a PFSS of } V],$$

for all $a \in V$. Consequently, $\mu_P(ra) = \mu_P(a)$, $\eta_P(ra) = \eta_P(a)$ and $v_P(ra) = v_P(a)$ for all $a \in V$ and for any non-zero $r \in F$. ☐

Proposition 3.2. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFS in V. Then P is a PFSS of V if and only if $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$.*

Proof. Let us suppose that P is a PFSS of V. Therefore,

$$\mu_P(ra+sb) = \mu_P(ra-(-sb)) \geqslant \mu_P(ra) \wedge \mu_P((-s)b) \geqslant \mu_P(a) \wedge \mu_P(b),$$

$$\eta_P(ra+sb) = \eta_P(ra-(-sb)) \geqslant \eta_P(ra) \wedge \eta_P((-s)b) \geqslant \eta_P(a) \wedge \eta_P(b),$$
$$v_P(ra+sb) = v_P(ra-(-sb)) \leqslant v_P(ra) \vee v_P((-s)b) \leqslant v_P(a) \vee v_P(b),$$

for all $a, b \in V$ and for all $r, s \in F$. Thus, it is obtained that $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$.

Conversely, let $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(ra+sb) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$ and for all $r, s \in F$. Let us suppose that ρ be the null vector in V.

Now, setting $r = 1$ and $s = -1$, it is obtained that $\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a-b) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in V$. Now, setting $a = b$, it is obtained that

$$\begin{aligned}
&\mu_P(a-a) \geqslant \mu_P(a) \wedge \mu_P(a) \quad \text{i.e. } \mu_P(\rho) \geqslant \mu_P(a),\\
&\eta_P(a-a) \geqslant \eta_P(a) \wedge \eta_P(a) \quad \text{i.e. } \eta_P(\rho) \geqslant \eta_P(a),\\
&v_P(a-a) \leqslant v_P(a) \vee v_P(a) \quad \text{i.e. } v_P(\rho) \leqslant v_P(a), \quad \text{for all } a \in V.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mu_P(ra) = \mu_P(ra+s\rho) \geqslant \mu_P(a) \wedge \mu_P(\rho) = \mu_P(a),\\
&\eta_P(ra) = \eta_P(ra+s\rho) \geqslant \eta_P(a) \wedge \eta_P(\rho) = \eta_P(a),\\
&v_P(ra) = v_P(ra+s\rho) \leqslant v_P(a) \vee v_P(\rho) = v_P(a), \quad \text{for all } a \in V \text{ and for all } r \in F.
\end{aligned}$$

Consequently, P is a PFSS of V. ☐

Example 3.1. Let us consider a crisp VS $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$ and a PFS $P = (\mu_P, \eta_P, v_P)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.3, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.15, & \text{otherwise,} \end{cases}$$

$$v_P(a) = \begin{cases} 0.15, & \text{when } a \in \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\},\\ 0.45, & \text{otherwise.} \end{cases}$$

Clearly, P is a PFSS of V.

Proposition 3.3. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. If for $a, b \in V$, $\mu_P(a-b) = \mu_P(\rho)$, $\eta_P(a-b) = \eta_P(\rho)$ and $v_P(a-b) = v_P(\rho)$ then $\mu_P(a) = \mu_P(b)$, $\eta_P(a) = \eta_P(b)$ and $v_P(a) = v_P(b)$, where ρ be the null vector in V.*

Proof. Here it is observed that

$$\begin{aligned}
\mu_P(a) = \mu_P((a-b)+b) &\geqslant \mu_P(a-b) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \mu_P(\rho) \wedge \mu_P(b)
\end{aligned}$$

$$\begin{aligned}
&= \mu_P(b) \quad \text{[by Proposition 3.1]},\\
\eta_P(a) = \eta_P((a-b)+b) &\geqslant \eta_P(a-b) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(\rho) \wedge \eta_P(b)\\
&= \eta_P(b) \quad \text{[by Proposition 3.1]},\\
\upsilon_P(a) = \upsilon_P((a-b)+b) &\leqslant \upsilon_P(a-b) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \upsilon_P(\rho) \vee \upsilon_P(b)\\
&= \upsilon_P(b) \quad \text{[by Proposition 3.1]}.
\end{aligned}$$

Thus, $\mu_P(a) \geqslant \mu_P(b)$, $\eta_P(a) \geqslant \eta_P(b)$ and $\upsilon_P(a) \leqslant \upsilon_P(b)$.

Also,

$$\begin{aligned}
\mu_P(b) = \mu_P(a-(a-b))) &= \mu_P(a+(-1)(a-b))\\
&\geqslant \mu_P(a) \wedge \mu_P(a-b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \mu_P(a) \wedge \mu_P(\rho)\\
&= \mu_P(a) \quad \text{[by Proposition 3.1]},\\
\eta_P(b) = \eta_P(a-(a-b)) &= \eta_P(a+(-1)(a-b))\\
&\geqslant \eta_P(a) \wedge \eta_P(a-b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(a) \wedge \eta_P(\rho)\\
&= \eta_P(a) \quad \text{[by Proposition 3.1]},\\
\upsilon_P(b) = \upsilon_P(a-(a-b)) &= \upsilon_P(a+(-1)(a-b))\\
&\leqslant \upsilon_P(a) \vee \upsilon_P(a-b) \quad \text{[becuase } P \text{ is a PFSS of } V]\\
&= \upsilon_P(a) \vee \upsilon_P(\rho)\\
&= \upsilon_P(a) \quad \text{[by Proposition 3.1]}.
\end{aligned}$$

Thus, $\mu_P(b) \geqslant \mu_P(a)$, $\eta_P(b) \geqslant \eta_P(a)$ and $\upsilon_P(b) \leqslant \upsilon_P(a)$.

Consequently, it is obtained that $\mu_P(a) = \mu_P(b)$, $\eta_P(a) = \eta_P(b)$ and $\upsilon_P(a) = \upsilon_P(b)$. ☐

Proposition 3.4. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSS of V. If for $a, b \in V$, $\mu_P(a) < \mu_P(b)$, $\eta_P(a) < \eta_P(b)$ and $\upsilon_P(a) > \upsilon_P(b)$ hold then $\mu_P(a-b) = \mu_P(a) = \mu_P(b-a)$, $\eta_P(a-b) = \eta_P(a) = \eta_P(b-a)$ and $\upsilon_P(a-b) = \upsilon_P(a) = \upsilon_P(b-a)$.*

Proof. It is observed that

$$\begin{aligned}
\mu_P(a-b) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \mu_P(a) \quad [\text{as } \mu_P(a) < \mu_P(b)],\\
\eta_P(a-b) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V]\\
&= \eta_P(a) \quad [\text{as } \eta_P(a) < \eta_P(b)],\\
\upsilon_P(a-b) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSS of } V]
\end{aligned}$$

$$= v_P(a) \quad [\text{as } v_P(a) > v_P(b)].$$

Thus, it is obtained that $\mu_P(a-b) \geqslant \mu_P(a)$, $\eta_P(a-b) \geqslant \eta_P(a)$ and $v_P(a-b) \leqslant v_P(a)$. Also,

$$\begin{aligned}
\mu_P(a) &\geqslant \mu_P(a-b) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= \mu_P(a-b) \quad \text{or} \quad \mu_P(b), \\
\eta_P(a) &\geqslant \eta_P(a-b) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= \eta_P(a-b) \quad \text{or} \quad \eta_P(b), \\
v_P(a) &\leqslant v_P(a-b) \vee v_P(b) \quad [\text{because } P \text{ is a PFSS of } V] \\
&= v_P(a-b) \quad \text{or} \quad v_P(b).
\end{aligned}$$

If $\mu_P(a) \geqslant \mu_P(b)$, $\eta_P(a) \geqslant \eta_P(b)$ and $v_P(a) \leqslant v_P(b)$ then they contradict the given conditions $\mu_P(a) < \mu_P(b)$, $\eta_P(a) < \eta_P(b)$ and $v_P(a) > v_P(b)$. So, it follows that $\mu_P(a) \geqslant \mu_P(a-b)$, $\eta_P(a) \geqslant \eta_P(a-b)$ and $v_P(a) \leqslant v_P(a-b)$.

Consequently, it is obtained that $\mu_P(a) = \mu_P(a-b)$, $\eta_P(a) = \eta_P(a-b)$ and $v_P(a) = v_P(a-b)$.

Moreover, it is clear that $\mu_P(a-b) = \mu_P(-(b-a)) = \mu_P(b-a)$, $\eta_P(a-b) = \eta_P(-(b-a)) = \eta_P(b-a)$ and $v_P(a-b) = v_P(-(b-a)) = v_P(b-a)$ [by Proposition 3.1].

Consequently, $\mu_P(a-b) = \mu_P(b-a) = \mu_P(a)$, $\eta_P(a-b) = \eta_P(b-a) = \eta_P(a)$ and $v_P(a-b) = v_P(b-a) = v_P(a)$. $\square$

Proposition 3.5. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSS of V. Then $P \cap Q$ is a PFSS of V.*

Proof. Let $P \cap Q = R = (\mu_R, \eta_R, v_R)$, where $\mu_R(a) = \mu_P(a) \wedge \mu_Q(a)$, $\eta_R(a) = \eta_P(a) \wedge \eta_Q(a)$ and $v_R(a) = v_P(a) \vee v_Q(a)$ for all $a \in V$.

Now,

$$\begin{aligned}
\mu_R(ra+sb) &= \mu_P(ra+sb) \wedge \mu_Q(ra+sb) \\
&\geqslant (\mu_P(a) \wedge \mu_P(b)) \wedge (\mu_Q(a) \wedge \mu_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (\mu_P(a) \wedge \mu_Q(a)) \wedge (\mu_P(b) \wedge \mu_Q(b)) \\
&= \mu_R(a) \wedge \mu_R(b), \\
\eta_R(ra+sb) &= \eta_P(ra+sb) \wedge \eta_Q(ra+sb) \\
&\geqslant (\eta_P(a) \wedge \eta_P(b)) \wedge (\eta_Q(a) \wedge \eta_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (\eta_P(a) \wedge \eta_Q(a)) \wedge (\eta_P(b) \wedge \eta_Q(b)) \\
&= \eta_R(a) \wedge \eta_R(b), \\
v_R(ra+sb) &= v_P(ra+sb) \vee v_Q(ra+sb) \\
&\leqslant (v_P(a) \vee v_P(b)) \vee (v_Q(a) \vee v_Q(b)) \quad [\text{because } P, Q \text{ are PFSSs of } V] \\
&= (v_P(a) \vee v_Q(a)) \vee (v_P(b) \vee v_Q(b))
\end{aligned}$$

$$= v_R(a) \vee v_R(b), \quad \text{for all } a, b \in V \text{ and all } r, s \in F.$$

Consequently, $R = P \cap Q$ is a PFSS of V.

Thus, we have proved that the intersection of two PFSSs is a PFSS. But union of two PFSSs is not necessarily a PFSS. This can be proved by two examples. If P, Q be two PFSSs of a crisp VS V over the field F then Example 3.2 shows that $P \cup Q$ is not a PFSS of V while Example 3.3 shows that $P \cup Q$ is a PFSS of V. □

Example 3.2. Let us consider a crisp VS $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two PFSs $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.45, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.15, & \text{otherwise,} \end{cases}$$

$$v_P(a) = \begin{cases} 0.1, & \text{when } a = (k, 0) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.4, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.4, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$v_Q(a) = \begin{cases} 0.2, & \text{when } a = (0, k) \text{ for some } k \neq 0 \text{ or } a = (0, 0), \\ 0.35, & \text{otherwise.} \end{cases}$$

Thus, $P \cup Q$ is given by

$$\mu_{P \cup Q}(a) = \begin{cases} 0.45, & \text{when } a = (0, 0), \\ 0.45, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.4, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P \cup Q}(a) = \begin{cases} 0.25, & \text{when } a = (0, 0), \\ 0.2, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.15, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.15, & \text{otherwise,} \end{cases}$$

$$v_{P \cup Q}(a) = \begin{cases} 0.1, & \text{when } a = (0, 0), \\ 0.1, & \text{when } a = (k, 0) \text{ for some } k \neq 0, \\ 0.2, & \text{when } a = (0, k) \text{ for some } k \neq 0, \\ 0.35, & \text{otherwise.} \end{cases}$$

It is observed that

$$0.2 = \mu_{P \cup Q}((2, 2)) \ngeq \mu_{P \cup Q}((2, 0)) \wedge \mu_{P \cup Q}(0, 2) = 0.45 \wedge 0.4 = 0.4,$$

$$0.35 = v_{P \cup Q}((2, 2)) \nleq v_P((2, 0)) \vee v_P((0, 2)) = 0.1 \vee 0.2 = 0.2,$$

but

$$0.15 = \eta_{P\cup Q}((2,2)) \geqslant \eta_{P\cup Q}((2,0)) \wedge \eta_{P\cup Q}(0,2) = 0.2 \wedge 0.15 = 0.15.$$

Hence, $P \cup Q$ is not a PFSS of V.

Example 3.3. Let us consider a crisp VS $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two PFSs $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ in V defined below.

$$\mu_P(a) = \begin{cases} 0.3, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.15, & \text{when } a = (0,0), \\ 0.4, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.2, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Thus, $P \cup Q$ is given by

$$\mu_{P\cup Q}(a) = \begin{cases} 0.3, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a = (0,0), \\ 0.1, & \text{otherwise,} \end{cases}$$

$$\upsilon_{P\cup Q}(a) = \begin{cases} 0.15, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Here, $P \cup Q$ is a PFSS of V.

Proposition 3.6. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V. Then $P \cup Q$ is a PFSS of V if either $P \subseteq Q$ or $Q \subseteq P$.*

Proof. Case 1. Let $P \subseteq Q$. Then $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$ and $\upsilon_P(a) \geqslant \upsilon_Q(a)$ for all $a \in A$. Then $\mu_{P\cup Q}(a) = \mu_P(a) \vee \mu_Q(a) = \mu_Q(a)$, $\eta_{P\cup Q}(a) = \eta_P(a) \wedge \eta_Q(a) = \eta_P(a)$ and $\upsilon_{P\cup Q}(a) = \upsilon_P(a) \wedge \upsilon_Q(a) = \upsilon_Q(a)$ for all $a \in V$.

Now,

$$\begin{aligned} \mu_{P\cup Q}(ra+sb) &= \mu_Q(ra+sb) \\ &\geqslant \mu_Q(a) \wedge \mu_Q(b) \quad \text{[because } Q \text{ is a PFSS of } V] \end{aligned}$$

$$
\begin{aligned}
&= \mu_{P\cup Q}(a) \wedge \mu_{P\cup Q}(b),\\
\eta_{P\cup Q}(ra+sb) &= \eta_P(ra+sb)\\
&\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \eta_{P\cup Q}(a) \wedge \eta_{P\cup Q}(b),\\
\upsilon_{P\cup Q}(ra+sb) &= \upsilon_Q(ra+sb)\\
&\leqslant \upsilon_Q(a) \vee \upsilon_Q(b) \quad [\text{because } Q \text{ is a PFSS of } V]\\
&= \upsilon_{P\cup Q}(a) \vee \mu_{P\cup Q}(b) \text{ for all } a,b \in V.
\end{aligned}
$$

Thus, $P \cup Q$ is a PFSS of V whenever $P \subseteq Q$.

Case 2. Let $Q \subseteq P$. Then $\mu_Q(a) \leqslant \mu_P(a)$, $\eta_Q(a) \leqslant \eta_P(a)$ and $\upsilon_Q(a) \geqslant \upsilon_P(a)$ for all $a \in V$. Then $\mu_{P\cup Q}(a) = \mu_P(a) \vee \mu_Q(a) = \mu_P(a)$, $\eta_{P\cup Q}(a) = \eta_P(a) \wedge \eta_Q(a) = \eta_Q(a)$ and $\upsilon_{P\cup Q}(a) = \upsilon_P(a) \vee \upsilon_Q(a) = \upsilon_P(a)$ for all $a \in V$. Proceeding in the similar way like case 1, it is obtained that $P \cup Q$ is a PFSS of V whenever $Q \subseteq P$. □

Proposition 3.7. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V. Then $P \times Q$ is a PFSS of $V \times V$.*

Proof. Let $P \times Q = (\mu_{P\times Q}, \eta_{P\times Q}, \upsilon_{P\times Q})$, where $\mu_{P\times Q}((a,b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a,b)) = \eta_P(a) \vee \eta_Q(b)$ and $\upsilon_{P\times Q}((a,b)) = \upsilon_P(a) \vee \upsilon_Q(b)$ for all $(a,b) \in V \times V$. Now,

$$
\begin{aligned}
\mu_{P\times Q}(r(a,b)+s(c,d)) &= \mu_P(ra+sc) \wedge \mu_Q(rb+sd)\\
&\geqslant (\mu_P(a) \wedge \mu_P(c)) \wedge (\mu_Q(b) \wedge \mu_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\mu_P(a) \wedge \mu_Q(b)) \wedge (\mu_P(c) \wedge \mu_Q(d))\\
&= \mu_{P\times Q}((a,b)) \wedge \mu_{P\times Q}((c,d)),\\
\eta_{P\times Q}(r(a,b)+s(c,d)) &= \eta_P(ra+sc) \wedge \eta_Q(rb+sd)\\
&\geqslant (\eta_P(a) \wedge \eta_P(c)) \wedge (\eta_Q(b) \wedge \eta_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\eta_P(a) \wedge \eta_Q(b)) \wedge (\eta_P(c) \wedge \eta_Q(d))\\
&= \eta_{P\times Q}((a,b)) \wedge \eta_{P\times Q}((c,d)),\\
\upsilon_{P\times Q}(r(a,b)+s(c,d)) &= \upsilon_P(ra+sc) \wedge \upsilon_Q(rb+sd)\\
&\leqslant (\upsilon_P(a) \vee \upsilon_P(c)) \vee (\upsilon_Q(b) \vee \upsilon_Q(d))\\
&\quad [\text{because } P, Q \text{ are PFSSs of } V]\\
&= (\upsilon_P(a) \vee \upsilon_Q(b)) \vee (\upsilon_P(c) \vee \upsilon_Q(d))\\
&= \upsilon_{P\times Q}((a,b)) \vee \upsilon_{P\times Q}((c,d)),
\end{aligned}
$$

for all $(a,b),(c,d) \in V \times V$ and for all $r,s \in F$. Consequently, $P \times Q$ is a PFSS of $V \times V$. □

Proposition 3.8. *Let V_1 and V_2 be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V_1 and V_2 respectively. Also, let ρ_1 and ρ_2 be two null vectors in V_1 and V_2 respectively. Then $\mu_{P\times Q}((\rho_1, \rho_2)) \geqslant \mu_{P\times Q}((a, b))$, $\eta_{P\times Q}((\rho_1, \rho_2)) \geqslant \eta_P((a, b))$ and $v_{P\times Q}((\rho_1, \rho_2)) \leqslant v_P((a, b))$ for all $(a, b) \in V_1 \times V_2$.*

Proof. Here, it is observed that

$$\begin{aligned}
\mu_{P\times Q}((\rho_1, \rho_2)) &= \mu_P(\rho_1) \wedge \mu_Q(\rho_2) \\
&\geqslant \mu_P(a) \wedge \mu_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2 \\
&= \mu_{P\times Q}((a, b)), \quad \text{for all } (a, b) \in V_1 \times V_2, \\
\eta_{P\times Q}((\rho_1, \rho_2)) &= \eta_P(\rho_1) \wedge \eta_Q(\rho_2) \\
&\geqslant \eta_P(a) \wedge \eta_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2 \\
&= \eta_{P\times Q}((a, b)), \quad \text{for all } (a, b) \in V_1 \times V_2, \\
\text{and } v_{P\times Q}((\rho_1, \rho_2)) &= v_P(\rho_1) \wedge \eta_Q(\rho_2) \\
&\leqslant v_P(a) \vee v_Q(b), \quad \text{for all } a \in V_1 \text{ and for all } b \in V_2 \\
&= v_{P\times Q}((a, b)), \quad \text{for all } (a, b) \in V_1 \times V_2.
\end{aligned}$$

Consequently, it is obtained that $\mu_{P\times Q}((\rho_1, \rho_2)) \geqslant \mu_{P\times Q}((a, b))$, $\eta_{P\times Q}((\rho_1, \rho_2)) \geqslant \eta_{P\times Q}((a, b))$ and $v_{P\times Q}((\rho_1, \rho_2)) \leqslant v_{P\times Q}((a, b))$ for all $(a, b) \in V_1 \times V_2$. □

Proposition 3.9. *Let V_1 and V_2 be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V_1 and V_2 respectively. Then one of the below stated conditions must hold.*

(i) $\mu_P(a) \leqslant \mu_Q(\rho_2)$, $\eta_P(a) \leqslant \eta_Q(\rho_2)$ and $v_P(a) \geqslant v_Q(\rho_2)$.

(ii) $\mu_Q(b) \leqslant \mu_P(\rho_1)$, $\eta_Q(b) \leqslant \eta_P(\rho_1)$ and $v_Q(b) \geqslant v_P(\rho_1)$ for all $a \in V_1$ and for all $b \in V_2$, where ρ_1 and ρ_2 be two null vectors in V_1 and V_2, respectively.

Proof. Let none of the stated conditions be hold. Then there exist $a \in V_1$ and $b \in V_2$ such that $\mu_P(a) > \mu_Q(\rho_2)$, $\eta_P(a) > \eta_Q(\rho_2)$, $v_P(a) < v_Q(\rho_2)$ and $\mu_Q(b) > \mu_P(\rho_1)$, $\eta_Q(b) > \eta_P(\rho_1)$, $v_Q(b) < v_P(\rho_1)$. Now,

$$\begin{aligned}
\mu_{P\times Q}((a, b)) &= \mu_P(a) \wedge \mu_Q(b) > \mu_Q(\rho_2) \wedge \mu_P(\rho_1) = \mu_{P\times Q}((\rho_1, \rho_2)), \\
\eta_{P\times Q}((a, b)) &= \eta_P(a) \wedge \eta_Q(b) > \eta_Q(\rho_2) \wedge \eta_P(\rho_1) = \eta_{P\times Q}((\rho_1, \rho_2)), \\
v_{P\times Q}((a, b)) &= v_P(a) \vee v_Q(b) < v_Q(\rho_2) \vee v_P(\rho_1) = v_{P\times Q}((\rho_1, \rho_2)).
\end{aligned}$$

Thus, it is obtained that $\mu_{P\times Q}((a, b)) > \mu_P((\rho_1, \rho_2))$, $\eta_{P\times Q}((a, b)) > \eta_{P\times Q}((\rho_1, \rho_2))$ and $v_P((a, b)) < v_{P\times Q}((\rho_1, \rho_2))$. But it is known from Proposition 3.8 that $\mu_{P\times Q}((a, b)) \leqslant \mu_P((\rho_1, \rho_2))$, $\eta_{P\times Q}((a, b)) \leqslant \eta_{P\times Q}((\rho_1, \rho_2))$ and $v_{P\times Q}((a, b)) \geqslant v_{P\times Q}((\rho_1, \rho_2))$. Hence, one of the stated conditions must hold. □

Proposition 3.10. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Then $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V, provided that $\mu_P(\rho) \geqslant \theta$, $\eta_P(\rho) \leqslant \phi$ and $v_P(\rho) \leqslant \psi$, where ρ be the null vector in V.*

Proof. Clearly, $C_{\theta,\phi,\psi}(P)$ is non-empty. Let $a, b \in C_{\theta,\phi,\psi}(P)$ and $r, s \in F$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$, $\upsilon_P(a) \leqslant \psi$ and $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$, $\upsilon_P(b) \leqslant \psi$. Now,

$$\begin{aligned}
\mu_P(ra+sb) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\geqslant \theta \wedge \theta\\
&= \theta,\\
\eta_P(ra+sb) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\geqslant \phi \wedge \phi\\
&= \phi,\\
\upsilon_P(ra+sb) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSS of } V\text{]}\\
&\leqslant \psi \vee \psi\\
&= \psi.
\end{aligned}$$

Thus, $a, b \in C_{\theta,\phi,\psi}(P)$ and $r, s \in F$ imply $ra+sb \in C_{\theta,\phi,\psi}(P)$. Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V. □

Proposition 3.11. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFS in V. Then P is a PFSS of V if all (θ, ϕ, ψ)-cuts of P are crisp subspaces of V.*

Proof. Let $a, b \in V$. Take, $\mu_P(a)\wedge\mu_P(b) = \theta$, $\eta_P(a)\wedge\eta_P(b) = \phi$ and $\upsilon_P(a)\vee\upsilon_P(b) = \psi$. Clearly, $\theta \in [0,1]$, $\phi \in [0,1]$ and $\psi \in [0,1]$ with $0 \leqslant \theta+\phi+\psi \leqslant 1$.

Now,

$$\begin{aligned}
\mu_P(a) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(a) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
\upsilon_P(a) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) = \psi.
\end{aligned}$$

Thus, $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $\upsilon_P(a) \leqslant \psi$. So, $a \in C_{\theta,\phi,\psi}(P)$. Also,

$$\begin{aligned}
\mu_P(b) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(b) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
\upsilon_P(b) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) = \psi.
\end{aligned}$$

Thus, $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$ and $\upsilon_P(b) \leqslant \psi$. So, $b \in C_{\theta,\phi,\psi}(P)$.

Since $C_{\theta,\phi,\psi}(P)$ is a crisp subspace of V therefore $ra+sb \in C_{\theta,\phi,\psi}(P)$ for all $r, s \in F$.

As a result,

$$\begin{aligned}
\mu_P(ra+sb) &\geqslant \theta = \mu_P(a) \wedge \mu_P(b),\\
\eta_P(ra+sb) &\geqslant \phi = \eta_P(a) \wedge \eta_P(b),\\
\upsilon_P(ra+sb) &\leqslant \psi = \upsilon_P(a) \vee \upsilon_P(b), \quad \text{for all } r, s \in F.
\end{aligned}$$

Since a, b are arbitrary elements of V therefore $\mu_P(ra+sb) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(ra+sb) \geqslant \eta_P(a)\wedge\eta_P(b)$ and $\upsilon_P(ra+sb) \leqslant \upsilon_P(a)\vee\upsilon_P(b)$ for all $a, b \in V$ and for all $r, s \in F$. Thus, P is a PFSS of V. □

Proposition 3.12. *Let V and W be two crisp VSs over the field F and Q be a PFSS of W. Then for a linear transformation (LT) $h : V \to W$, $h^{-1}(Q)$ is a PFSS of V.*

Proof. Let $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, v_{h^{-1}(Q)})$. Then $\mu_{h^{-1}(Q)}(a) = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $v_{h^{-1}(Q)}(a) = v_Q(h(a))$ for all $a \in V$.

Now,

$$\begin{aligned}
\mu_{h^{-1}(Q)}(ra+sb) &= \mu_Q(h(ra+sb)) \\
&= \mu_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\geqslant \mu_Q(h(a)) \wedge \mu_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= \mu_{h^{-1}(Q)}(a) \wedge \mu_{h^{-1}(Q)}(b), \\
\eta_{h^{-1}(Q)}(ra+sb) &= \eta_Q(h(ra+sb)) \\
&= \eta_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\geqslant \eta_Q(h(a)) \wedge \eta_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= \eta_{h^{-1}(Q)}(a) \wedge \eta_{h^{-1}(Q)}(b), \\
v_{h^{-1}(Q)}(ra+sb) &= v_Q(h(ra+sb)) \\
&= v_Q(rh(a)+sh(b)) \quad [\text{because } h \text{ is a crisp LT from } V \text{ to } W] \\
&\leqslant v_Q(h(a)) \vee v_Q(h(b)) \quad [\text{because } Q \text{ is a PFSS of } W] \\
&= v_{h^{-1}(Q)}(a) \vee v_{h^{-1}(Q)}(b), \quad \text{for all } a,b \in V \text{ and for all } r,s \in F.
\end{aligned}$$

Consequently, $h^{-1}(Q)$ is a PFSS of V. □

Proposition 3.13. *Let V and W be two VSs over the same field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Then for a bijective LT $h : V \to W$, $h(P)$ is a PFSS of W.*

Proof. Let $h(P) = (\mu_{h(P)}, \eta_{h(P)}, v_{h(P)})$. Then

$$\begin{aligned}
\mu_{h(P)}(q) &= \bigvee_{p \in h^{-1}(q)} \mu_P(p), \\
\eta_{h(P)}(q) &= \bigwedge_{p \in h^{-1}(q)} \eta_P(p), \\
v_{h(P)}(q) &= \bigwedge_{p \in h^{-1}(q)} v_P(p).
\end{aligned}$$

Since h is bijective therefore $h^{-1}(q)$ must be a singleton set. So, for $q \in W$, there exists an unique $p \in V$ such that $p = h^{-1}(q)$, i.e., $h(p) = q$. Thus, in this case, $\mu_{h(P)}(q) = \mu_{h(P)}(h(p)) = \mu_P(p)$, $\eta_{h(P)}(q) = \eta_{h(P)}(h(p)) = \eta_P(p)$ and $v_{h(P)}(q) = v_{h(P)}(h(p)) = v_P(p)$. Now,

$$\begin{aligned}
\mu_{h(P)}(rc+sd) &= \mu_{h(P)}(rh(a)+sh(b)) \\
&\quad [\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a,b \in V] \\
&= \mu_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}] \\
&= \mu_P(ra+sb)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \mu_P(a) \wedge \mu_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \mu_{h(P)}(h(a)) \wedge \mu_{h(P)}(h(b))\\
&= \mu_{h(P)}(c) \wedge \mu_{h(P)}(d),\\
\eta_{h(P)}(rc+sd) &= \eta_{h(P)}(rh(a)+sh(b))\\
&[\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in V]\\
&= \eta_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}]\\
&= \eta_P(ra+sb)\\
&\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \eta_{h(P)}(h(a)) \wedge \eta_{h(P)}(h(b))\\
&= \eta_{h(P)}(c) \wedge \eta_{h(P)}(d),\\
\upsilon_{h(P)}(rc+sd) &= \upsilon_{h(P)}(rh(a)+sh(b))\\
&[\text{where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in V]\\
&= \upsilon_{h(P)}(h(ra+sb)) \quad [\text{because } h \text{ is a crisp LT}]\\
&= \upsilon_P(ra+sb)\\
&\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad [\text{because } P \text{ is a PFSS of } V]\\
&= \upsilon_{h(P)}(h(a)) \vee \upsilon_{h(P)}(h(b))\\
&= \upsilon_{h(P)}(c) \vee \upsilon_{h(P)}(d), \quad \text{for all } r, s \in F.
\end{aligned}
$$

Since, c, d are arbitrary elements of W therefore $\mu_{h(P)}(rc+sd) \geqslant \mu_{h(P)}(c) \wedge \mu_{h(P)}(d)$, $\eta_{h(P)}(rc+sd) \geqslant \eta_{h(P)}(c) \wedge \eta_{h(P)}(d)$ and $\upsilon_{h(P)}(rc+sd) \leqslant \upsilon_{h(P)}(c) \vee \upsilon_{h(P)}(d)$ for all $c, d \in W$ and for all $r, s \in F$. Consequently, $h(P)$ is a PFSS of W. □

4. Direct Sum of two Picture Fuzzy Subspaces

The current section introduces direct sum of two PFSSs over the direct sum of two crisp VSs and investigates some important results connected to it.

Definition 4.1. Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V and W respectively. Then direct sum of P and Q is defined as the PFS $P \oplus Q = (\mu_{P\oplus Q}, \eta_{P\oplus Q}, \upsilon_{P\oplus Q})$ over the set of universe $V \oplus W$, where

$$
\begin{aligned}
&\mu_{P\oplus Q}(c) = \mu_P(a) \wedge \mu_Q(b),\\
&\eta_{P\oplus Q}(c) = \eta_P(a) \wedge \eta_Q(b),\\
&\upsilon_{P\oplus Q}(c) = \upsilon_P(a) \vee \upsilon_Q(b), \quad \text{for any } c \in V \oplus W,
\end{aligned}
$$

with $c = a + b$, where $a \in V$ and $b \in W$.

Example 4.1. Let us consider two crisp VSs $V_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$ and $V_2 = \{(0, a_2) : a_2 \in \mathbb{R}\}$ over the field $F = \mathbb{R}$. Also, let us suppose two PFSSs P_1 and P_2 of V_1 and

V_2 respectively defined below.

$$\mu_P(a) = \begin{cases} 0.55, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.45, & \text{when } a = (0,0), \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.05, & \text{when } a = (0,0), \\ 0.37, & \text{otherwise,} \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.4, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.1, & \text{when } a = (0,0), \\ 0.3, & \text{otherwise.} \end{cases}$$

Thus, for any $a \in V_1 \oplus V_2$, $P \oplus Q$ is defined as follows.

$$\mu_{P\oplus Q}(a) = \begin{cases} 0.35, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\eta_{P\oplus Q}(a) = \begin{cases} 0.4, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.2, & \text{otherwise,} \end{cases}$$

$$\upsilon_{P\oplus Q}(a) = \begin{cases} 0.1, & \text{when } a = (0,0), \\ 0.3, & \text{when } a \in V_2 - \{(0,0)\}, \\ 0.37, & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ be two PFSSs of V and W respectively. Then $P \oplus Q$ is a PFSS of $V \oplus W$.*

Proof. Let $c_1, c_2 \in V \oplus W$ with $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$ where $a_1, a_2 \in V$ and $b_1, b_2 \in W$. Let $r, s \in F$. Now,

$$\begin{aligned} \mu_{P\oplus Q}(rc_1 + sc_2) &= \mu_{P\oplus Q}(r(a_1 + b_1) + s(a_2 + b_2)) \\ &= \mu_{P\oplus Q}((ra_1 + sa_2) + (rb_1 + sb_2)) \\ &= \mu_P(ra_1 + sa_2) \wedge \mu_Q(rb_1 + sb_2) \\ &\geqslant (\mu_P(a_1) \wedge \mu_P(a_2)) \wedge (\mu_Q(b_1) \wedge \mu_Q(b_2)) \\ &\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W] \\ &= (\mu_P(a_1) \wedge \mu_Q(b_1)) \wedge (\mu_P(a_2) \wedge \mu_Q(b_2)) \\ &= \mu_{P\oplus Q}(c_1) \wedge \mu_{P\oplus Q}(c_2), \end{aligned}$$

$$\begin{aligned}
\eta_{P\oplus Q}(rc_1+sc_2) &= \eta_{P\oplus Q}(r(a_1+b_1)+s(a_2+b_2)) \\
&= \eta_{P\oplus Q}((ra_1+sa_2)+(rb_1+sb_2)) \\
&= \eta_P(ra_1+sa_2)\wedge \eta_Q(rb_1+sb_2) \\
&\geqslant (\eta_P(a_1)\wedge \eta_P(a_2))\wedge(\eta_Q(b_1)\wedge \eta_Q(b_2)) \\
&\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W] \\
&= (\eta_P(a_1)\wedge \eta_Q(b_1))\wedge(\eta_P(a_2)\wedge \eta_Q(b_2)) \\
&= \eta_{P\oplus Q}(c_1)\wedge \eta_{P\oplus Q}(c_2), \\
v_{P\oplus Q}(rc_1+sc_2) &= v_{P\oplus Q}(r(a_1+b_1)+s(a_2+b_2)) \\
&= v_{P\oplus Q}((ra_1+sa_2)+(rb_1+sb_2)) \\
&= v_P(ra_1+sa_2)\vee v_Q(rb_1+sb_2) \\
&\leqslant (v_P(a_1)\vee v_P(a_2))\vee(v_Q(b_1)\vee v_Q(b_2)) \\
&\quad [\text{because } P \text{ is a PFSS of } V \text{ and } Q \text{ is a PFSS of } W] \\
&= (v_P(a_1)\vee v_Q(b_1))\vee(v_P(a_2)\vee v_Q(b_2)) \\
&= v_{P\oplus Q}(c_1)\vee v_{P\oplus Q}(c_2).
\end{aligned}$$

Since c_1, c_2 are arbitrary elements of $V\oplus W$ and r, s are arbitrary scalars of F therefore $\mu_{P\oplus Q}(rc_1+sc_2)\geqslant \mu_{P\oplus Q}(c_1)\wedge \mu_{P\oplus Q}(c_2)$, $\eta_{P\oplus Q}(rc_1+sa_2)\geqslant \eta_{P\oplus Q}(c_1)\wedge \eta_{P\oplus Q}(c_2)$ and $v_{P\oplus Q}(rc_1+rc_2)\leqslant v_{P\oplus Q}(c_1)\vee v_{P\oplus Q}(c_2)$ for all $c_1, c_2\in V\oplus W$ and for all $r, s\in F$. Consequently, $P\oplus Q$ is a PFSS of $V\oplus W$. ☐

Proposition 4.2. *Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V and W respectively. Then $C_{\theta,\phi,\psi}(P\oplus Q) = C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$.*

Proof. Let $c\in C_{\theta,\phi,\psi}(P\oplus Q)$. Then clearly, $c\in V\oplus W$. Say, $c = a+b$ with $a\in V$ and $b\in W$. Then

$$\begin{aligned}
\mu_{P\oplus Q}(c) &= \mu_P(a)\wedge \mu_Q(b)\geqslant \theta \Rightarrow \mu_P(a)\geqslant \theta \quad \text{and} \quad \mu_Q(b)\geqslant \theta, \\
\eta_{P\oplus Q}(c) &= \eta_P(a)\wedge \eta_Q(b)\geqslant \phi \Rightarrow \eta_P(a)\geqslant \phi \quad \text{and} \quad \eta_Q(b)\geqslant \phi, \\
v_{P\oplus Q}(c) &= v_P(a)\vee v_Q(b)\leqslant \psi \Rightarrow v_P(a)\leqslant \psi \quad \text{and} \quad \eta_Q(b)\leqslant \psi.
\end{aligned}$$

Thus, $a\in C_{\theta,\phi,\psi}(P)$ and $b\in C_{\theta,\phi,\psi}(Q)$. So, $c = a+b\in C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$. Consequently, $C_{\theta,\phi,\psi}(P\oplus Q)\subseteq C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$.

Conversely, let $c\in C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)$. Then there exists $a\in C_{\theta,\phi,\psi}(P)$ and $b\in C_{\theta,\phi,\psi}(Q)$ with $c = a+b$. Then $\mu_P(a)\geqslant \theta$, $\eta_P(a)\geqslant \phi$, $v_P(a)\leqslant \psi$ and $\mu_Q(b)\geqslant \theta$, $\eta_Q(b)\geqslant \phi$, $v_Q(b)\leqslant \psi$. Thus, $\mu_P(a)\wedge\mu_Q(b)\geqslant \theta$, $\eta_P(a)\wedge\eta_Q(b)\geqslant \phi$ and $v_P(a)\vee v_Q(b)\leqslant \psi$. Now, $a\in C_{\theta,\phi,\psi}(P)\Rightarrow a\in V$ and $b\in C_{\theta,\phi,\psi}(Q)\Rightarrow b\in W$. As a result, $c = a+b\in V\oplus W$. It follows that $\mu_{P\oplus Q}(c)\geqslant \theta$, $\eta_{P\oplus Q}(c)\geqslant \phi$ and $v_{P\oplus Q}(c)\leqslant \psi$. Therefore, $c\in C_{\theta,\phi,\psi}(P\oplus Q)$. Thus, it is obtained that $C_{\theta,\phi,\psi}(P)\oplus C_{\theta,\phi,\psi}(Q)\subseteq C_{\theta,\phi,\psi}(P\oplus Q)$.

Consequently, we get $C_{\theta,\phi,\psi}(P\oplus Q) = C_{\theta,\phi,\psi}\oplus C_{\theta,\phi,\psi}(Q)$. ☐

5. Isomorphism between two Picture Fuzzy Subspaces

Isomorphism is a pioneer concept in crisp sense. In this section, the notion of isomorphism is introduced between two PFSSs. An important result is established here through a proposition.

Definition 5.1. Let V and W be two crisp VSs over the same field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V and W respectively. Then P is said to be isomorphic to Q if there exists an isomorphism $H : V \to W$ such that $\mu_Q(H(a)) = \mu_P(a)$, $\eta_Q(H(a)) = \eta_P(a)$ and $v_Q(H(a)) = \mu_P(a)$ for all $a \in V$.

Proposition 5.1. *Let V and W be two crisp VSs over the field F and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSSs of V and W respectively. Let P is isomorphic to Q. Then $C_{\theta,\phi,\psi}(P)$ is isomorphic to $C_{\theta,\phi,\psi}(Q)$, provided that*

(i) $\mu_P(\rho_1) \geqslant \theta$, $\eta_P(\rho_1) \geqslant \phi$ and $v_P(\rho_1) \leqslant \psi$;

(ii) $\mu_Q(\rho_2) \geqslant \theta$, $\eta_Q(\rho_2) \geqslant \phi$ and $v_Q(\rho_2) \leqslant \psi$,

where ρ_1, ρ_2 be the null vectors in V and W, respectively.

Proof. Since P and Q are isomorphic therefore there exists an isomorphism $H : V \to W$ such that $\mu_Q(H(a)) = \mu_P(a)$, $\eta_Q(H(a)) = \eta_P(a)$ and $v_Q(H(a)) = v_P(a)$. Now, let us define $h : C_{\theta,\phi,\psi}(P) \to C_{\theta,\phi,\psi}(Q)$ such that $h(a) = H(a)$ for every $a \in C_{\theta,\phi,\psi}(P)$. Let $a \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $v_P(a) \leqslant \psi$. Since P is isomorphic to Q therefore it follows that $\mu_Q(H(a)) \geqslant \theta$, $\eta_Q(H(a)) \geqslant \phi$ and $v_Q(H(a)) \leqslant \psi$. Thus, for $a \in C_{\theta,\phi,\psi}(P)$, it is obtained that $H(a) \in C_{\theta,\phi,\psi}(Q)$. So, h is well defined.

Since H is an isomorphism therefore H is one-one and onto. Due to injectivity of H, $\ker H = \{a \in V : H(a) = \rho_2\} = \{\rho_1\}$. It follows that $\{a \in C_{\theta,\phi,\psi}(P) : H(a) = \rho_2\} = \{\rho_1\}$ because $C_{\theta,\phi,\psi}(P) \subseteq V$. So, $\ker h$=$\{\rho_1\}$. Thus, h is one-one.

Let us suppose $b \in C_{\theta,\phi,\psi}(Q)$. Then $\mu_Q(b) \geqslant \theta$, $\eta_Q(b) \geqslant \phi$ and $v_Q(b) \leqslant \psi$. Since H is an isomorphism therefore there exists $a \in V$ such that $H(a) = b$. So, we can write $\mu_Q(H(a)) \geqslant \theta$, $\eta_Q(H(a)) \geqslant \phi$ and $v_Q(H(a)) \leqslant \psi$. Since P is isomorphic to Q therefore $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $v_P(a) \leqslant \psi$. So, $a \in C_{\theta,\phi,\psi}(P)$. Thus, for each $b \in C_{\theta,\phi,\psi}(Q)$ there exists a pre-image $a \in C_{\theta,\phi,\psi}(P)$. So, h is onto.

Consequently, $C_{\theta,\phi,\psi}(P)$ is isomorphic to $C_{\theta,\phi,\psi}(Q)$. □

6. Picture Fuzzy Linear Transformation

In the current section, the notion of picture fuzzy linear transformation (PFLT) is initiated with a suitable example and some corresponding properties are studied.

Definition 6.1. Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Then a map $T : V \to V$ is said to be PFLT on V if

(i) T is a linear map in crisp sense;

(ii) $\mu_P(T(a)) \geqslant \mu_P(a)$, $\eta_P(T(a)) \geqslant \eta_P(a)$ and $v_P(T(a)) \leqslant v_P(a)$ for all $a \in V$.

Example 6.1. Consider the Example 3.1. Define a map $T : V \to V$ by $T((a_1, a_2, a_3)) = (a_1 + a_2, a_2 + a_3, 0)$. Clearly, T is a linear map in crisp sense. For any $(a_1, a_2, a_3) \in V$, it is observed that

$$\begin{aligned}\mu_P(T(a_1,a_2,a_3)) &= \mu_P((a_1+a_2,a_2+a_3,0)) = 0.3 \geqslant \mu_P((a_1,a_2,a_3)),\\ \eta_P(T(a_1,a_2,a_3)) &= \eta_P((a_1+a_2,a_2+a_3,0)) = 0.35 \geqslant \eta_P((a_1,a_2,a_3)),\\ v_P(T(a_1,a_2,a_3)) &= v_P((a_1+a_2,a_2+a_3,0)) = 0.15 \leqslant v_P((a_1,a_2,a_3)).\end{aligned}$$

Thus, T is a PFLT on V.

Proposition 6.1. *Let V be a crisp VS over the field F and P be a PFSS of V. If T_1 and T_2 are two PFLTs on V then so is $T_1 + T_2$.*

Proof. Let $a \in V$. Now,

$$\begin{aligned}\mu_P((T_1+T_2)(a)) &= \mu_P(T_1(a)+T_2(a))\\ &\geqslant \mu_P(T_1(a)) \wedge \mu_P(T_2(a)) \quad [\text{because } P \text{ is a PFSS of } V]\\ &\geqslant \mu_P(a) \wedge \mu_P(a) \quad [\text{because } T \text{ is a PFLT on } V]\\ &= \mu_P(a),\\ \eta_P((T_1+T_2)(a)) &= \eta_P(T_1(a)+T_2(a))\\ &\geqslant \eta_P(T_1(a)) \wedge \eta_P(T_2(a)) \quad [\text{because } P \text{ is a PFSS of } V]\\ &\geqslant \eta_P(a) \wedge \eta_P(a) \quad [\text{because } T \text{ is a PFLT on } V]\\ &= \eta_P(a),\\ \text{and } v_P((T_1+T_2)(a)) &= v_P(T_1(a)+T_2(a))\\ &\leqslant v_P(T_1(a)) \vee v_P(T_2(a)) \quad [\text{because } P \text{ is a PFSS of } V]\\ &\leqslant v_P(a) \vee v_P(a) \quad [\text{because } T \text{ is a PFLT on } V]\\ &= v_P(a).\end{aligned}$$

Since a is an arbitrary element of V therefore $\mu_P((T_1+T_2)(a)) \geqslant \mu_P(a)$, $\eta_P((T_1+T_2)(a)) \geqslant \eta_P(a)$ and $v_P((T_1+T_2)(a)) \leqslant v_P(a)$ for all $a \in V$. Consequently, $T_1 + T_2$ is a PFLT on V. ☐

Proposition 6.2. *Let V be a crisp VS over the field F and P be a PFSS of V. If T is a PFLT on V then so is kT for some scalar $k \in F$.*

Proof. Let $a \in V$. Now,

$$\begin{aligned}\mu_P((kT)(a)) &= \mu_P(kT(a))\\ &\geqslant \mu_P(T(a)) \quad [\text{because } P \text{ is a PFSS of } V]\\ &\geqslant \mu_P(a) \quad [\text{because } T \text{ is a PFLT on } V],\\ \eta_P((kT)(a)) &= \eta_P(kT(a))\\ &\geqslant \eta_P(T(a)) \quad [\text{because } P \text{ is a PFSS of } V]\\ &\geqslant \eta_P(a) \quad [\text{because } T \text{ is a PFLT on } V],\\ \text{and } v_P((kT)(a)) &= v_P(kT(a))\end{aligned}$$

$$\leqslant v_P(T(a)) \quad \text{[because } P \text{ is a PFSS of } V]$$
$$\leqslant v_P(a) \quad \text{[because } T \text{ is a PFLT on } V].$$

Since a is an arbitrary element of V therefore $\mu_P((kT)(a)) \geqslant \mu_P(a)$, $\eta_P((kT)(a)) \geqslant \eta_P(a)$ and $v_P((kT)(a)) \leqslant v_P(a)$ for all $a \in V$ and for some scalar $k \in F$. Consequently, kT is a PFLT on V. □

7. Linear Independency of a Finite Set of Vectors in Picture Fuzzy Sense

The current section introduces the concept of picture fuzzy linearly independent (PFLI) set of vectors with suitable example. An important result related to it is highlighted through a proposition.

Definition 7.1. Let V be a crisp VS over the field F and $P = (\mu_p, \eta_P, v_P)$ be a PFSS of V. A finite set of vectors $\{a_1, a_2, a_3, \dots, a_n\}$ in V is said to be PFLI in V with respect to PFSS P if

(i) $\{a_1, a_2, a_3, \dots, an\}$ is linearly independent set of vectors in V;

(ii)

$$\mu_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = \mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_na_n),$$
$$\eta_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = \eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_na_n),$$
$$v_P(c_1a_1 + c_2a_2 + \cdots + c_na_n) = v_P(c_1a_1) \vee v_P(c_2a_2) \vee \cdots \vee v_P(c_na_n),$$

where $c_i \in F$ for $i = 1, 2, \dots, n$.

Proposition 7.1. *Let V be a crisp VS over the field F and $P = (\mu_P, \eta_P, v_P)$ be a PFSS of V. Also, let M be a finite set of vectors in V which is PFLI in V with respect to PFSS P. Then any subset of M is PFLI in V with respect to PFSS P.*

Proof. Let $M = \{a_1, a_2, \dots, a_n\}$. Now, let $\{a_1, a_2, a_3, \dots, a_r\}$ be any subset of M, where $r \leqslant n$. It is known that $\{a_1, a_2, a_3, \dots, a_r\}$ is linearly independent in V. Let ρ be the null vector in V. Now, $c_1a_1 + c_2a_2 + \cdots + c_ra_r = c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n$, where $c_{r+1} = c_{r+2} = \cdots = c_n = 0$. Now,

$$\begin{aligned}
&\mu_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\mu_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \mu_P(c_ra_r) \wedge \mu_P(c_{r+1}a_{r+1}) \wedge \cdots \wedge \mu_P(c_na_n)\\
&\text{[since } \{a_1, a_2, \dots, a_n\} \text{ is PFLI set of vectors in } V \text{ with respect to PFSS } P]\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_ra_r) \wedge \mu_P(\rho)\\
=&\mu_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \mu_P(c_ra_r) \quad \text{[by Proposition 3.1]},\\
&\eta_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\eta_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\eta_P(c_1a_1) \wedge \mu_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r) \wedge \eta_P(c_{r+1}a_{r+1}) \wedge \cdots \wedge \eta_P(c_na_n)
\end{aligned}$$

[since $\{a_1, a_2, \ldots, a_n\}$ is PFLI set of vectors in V with respect to PFSS P]

$=\eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r) \wedge \eta_P(\rho)$

$=\eta_P(c_1a_1) \wedge \eta_P(c_2a_2) \wedge \cdots \wedge \eta_P(c_ra_r)$ [by Proposition 3.1]

and

$$
\begin{aligned}
&\upsilon_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r)\\
=&\upsilon_P(c_1a_1 + c_2a_2 + \cdots + c_ra_r + c_{r+1}a_{r+1} + \cdots + c_na_n)\\
=&\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r) \vee \upsilon_P(c_{r+1}a_{r+1}) \vee \cdots \vee \upsilon_P(c_na_n)
\end{aligned}
$$

[since $\{a_1, a_2, \ldots, a_n\}$ is PFLI set of vectors in V with respect to PFSS P]

$=\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r) \vee \upsilon_P(\rho)$

$=\upsilon_P(c_1a_1) \vee \upsilon_P(c_2a_2) \vee \cdots \vee \upsilon_P(c_ra_r)$ [by Proposition 3.1],

for $c_1, c_2, c_3, \ldots, c_n \in F$. Thus, $\{a_1, a_2, a_3, \ldots, a_r\}$ is PFLI in V with respect to PFSS P. Therefore, any subset of M is PFLI in V with respect to PFSS P. □

Example 7.1. Let us consider a crisp VS $V = \mathbb{R}^2$ and a PFSS P of V as follows:

$$\mu_P(a) = \begin{cases} 0.57, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\}, \\ 0.23, & \text{otherwise,} \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.32, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\}, \\ 0.17, & \text{otherwise,} \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0.11, & \text{when } a \in \{(k,0) : k \in \mathbb{R}\}, \\ 0.37, & \text{otherwise.} \end{cases}$$

It is clear that $\{(a_1, 0), (0, a_2)\}$ is a PFLI set of vectors in V with respect to PFSS P for $a_1 \neq 0$ and $a_2 \neq 0$.

8. Conclusion

In this paper, the notion of PFSS of a crisp VS is introduced. Some basic properties of PFSS in context of some basic operations on PFSs are studied. Here, we have shown that the intersection of two PFSSs is a PFSS. Similar type of result is also true for Cartesian product of two PFSSs, but not necessarily true in case of union which is highlighted with two suitable examples. Also, it is shown that (θ, ϕ, ψ)-cut of a PFSS is a crisp subspace. A result on (θ, ϕ, ψ)-cut of a PFS is established here which gives a condition under which a PFS will be a PFSS. The idea of direct sum of two PFSSs over the direct sum of two crisp VSs is established and related properties are studied. The concept of isomorphism between two PFSSs is initiated here and related result is investigated. Also, the notions of PFLT and PFLI set of vectors are introduced here. It is proved that the sum of two PFLTs is a PFLT and scalar multiplication with PFLT is a PFLT. Also, it is proved that any subset of a PFLI set of vectors is PFLI. We expect that our works will help the researchers to go through more advanced level of works on PFSS and it will also encourage them to explore the idea of subspace in the environment of some other kinds of sets.

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NUMERICAL SOLUTION OF LINEAR VOLTERRA INTEGRAL EQUATIONS USING NON-UNIFORM HAAR WAVELETS

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Abstract. In this paper, we presented a numerical method for solving linear Volterra integral equations (LVIE) which is based on the non-uniform Haar wavelets. By applying this method, the LVIE is reduced to a linear system of algebraic equations which can be solved by direct method. The min advantage of using non-uniform Haar wavelets is that the time of calculation can be adjusted arbitrarily. Also, we presented the error analysis of the proposed method. Furthermore, two examples are included for the demonstrating the convenient capabilities of the new method.

1. Introduction

Types of equations such as linear and nonlinear integral equations appear in many problems such as finance, mathematics, etc. [8–18]. In the recent years, a lot of numerical methods have been developed for solving these types of equations [1–7]. Since, obtaining the analytical solution of these equations is generally not possible, it is important to develop the numerical method to solve these equations. For this aim, we consider the linear Volterra integral equation of the second kind

$$Y(t) = K(t) + \lambda \int_0^t Y(s)P(s,t)ds, \tag{1.1}$$

where $Y(t)$ is the unknown function, $P(s,t)$ is the known function and the parameter λ is known. In the proposed method, first we transform Volterra integral equation (1.1) to a system of linear algebraic equations with the unknown non-uniform Haar

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coefficients. Then, we obtain the unknown non-uniform Haar coefficients by solving this system by the known method. If we want to raise the exactness of the results, we must increase the number of the grid points. In the course of the solution, we have to invert some matrices, but by increasing the number of calculation points these matrices become nearly singular and therefore the inverse matrices cannot be evaluated with necessary accuracy. One possibility to find a way out of these difficulties is to make use of the non-uniform Haar method for which the length of the subintervals is unequal. This idea was proposed in [19]. Also, another advantage of proposed method is simple applicability and their efficiency. This article is organized as follows. In Section 2, we describe the non-uniform Haar wavelet and their property. The Heaviside step function is defined in Section 3. In the next section, we describe function approximation. In Section 5, a numerical method is proposed. In Section 6, we present the error analysis. In Section 7, some numerical examples are given. Finally in Section 8, conclusion is given.

2. Non-Uniform Haar Wavelets

Non-uniform Haar wavelets are characterized by two characteristics: the dilation parameter $j = 0, 1, \ldots, J$ (J is maximal level of resolution) and the translation parameter $k = 0, 1, \ldots, n-1$, where the integer $n = 2^j$. The number of wavelet is identified as $i = n + k + 1$. Also the maximal value is $i = 2N$, where $N = 2^J$.

For getting the grid points in interval $[0, L]$, where L is large enough constant, we consider the length of the c-th subinterval from this interval by $\Delta t_c = t_c - t_{c-1}$, $c = 1, 2, \ldots, 2N$. It is assume that $\Delta t_{c+1} = g\Delta t_c$, where $g > 1$ is a given constant. If we sum all the length of this subintervals, we find:

$$\Delta t_1(1 + g + g^2 + \cdots + g^{2N-1}) = L, \quad L = b - a,$$

or

$$\Delta t_1 = L\frac{g-1}{g^{2N}-1}.$$

Since

$$\tilde{t}(\ell) = \Delta t_1 \left(1 + g + \cdots + g^{\ell-1}\right) = \Delta t_1 \frac{g^\ell - 1}{g-1}, \quad \ell = 1, 2, \ldots, 2N, \tag{2.1}$$

so, we have obtained the grid points as

$$\tilde{t}(\ell) = L\frac{g^\ell - 1}{g^{2N}-1}, \quad \ell = 1, 2, \ldots, 2N. \tag{2.2}$$

We divide the interval $[a, b]$ into $2N$ subinterval of unequal lengths with the division points $a = \tilde{x}(0) < \tilde{x}(1) < \cdots < \tilde{x}(2N) = b$. By using harmonize grid points, we describe the non-uniform Haar wavelet family based on [19] as follow:

$$H_i(t) = \begin{cases} 1, & \Upsilon_1(i) \le t \le \Upsilon_2(i), \\ -u_i, & \Upsilon_2(i) \le t \le \Upsilon_3(i), \\ 0, & \text{elsewhere,} \end{cases}$$

where

$$\Upsilon_1(i) = \tilde{t}(2k\theta), \quad \Upsilon_2(i) = \tilde{t}((2k+1)\theta), \quad \Upsilon_3(i) = \tilde{t}((2k+2)\theta), \quad \theta = \frac{N}{n},$$

and

$$u_i = \frac{\Upsilon_2(i) - \Upsilon_1(i)}{\Upsilon_3(i) - \Upsilon_2(i)}. \tag{2.3}$$

Clearly, these equations hold when $i > 2$. For the case $i = 1$ and $i = 2$, we have $\Upsilon_1(1) = a$, $\Upsilon_2(1) = \Upsilon_3(1) = b$, $\Upsilon_1(2) = a$, $\Upsilon_2(2) = \frac{\tilde{t}(2N)}{2}$, $\Upsilon_3(2) = b$.

For instance, by using the grid points defined in (2.2), if $g = 2$ the first eight bases non-uniform Haar functions are given by

$$H_1(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{elsewhere,} \end{cases} \qquad H_2(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$H_3(t) = \begin{cases} 1, & 0 \le t \le \frac{3}{255}, \\ -\frac{1}{4}, & \frac{3}{255} \le t \le \frac{15}{255}, \\ 0, & \text{elsewhere,} \end{cases} \qquad H_4(t) = \begin{cases} 1, & \frac{15}{255} \le t \le \frac{63}{255}, \\ -\frac{1}{4}, & \frac{63}{255} \le t \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$H_5(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{255}, \\ -\frac{1}{2}, & \frac{1}{255} \le t \le \frac{3}{255}, \\ 0, & \text{elsewhere,} \end{cases} \qquad H_6(t) = \begin{cases} 1, & \frac{3}{255} \le t \le \frac{7}{255}, \\ -\frac{1}{2}, & \frac{7}{255} \le t \le \frac{15}{255}, \\ 0, & \text{elsewhere,} \end{cases}$$

$$H_7(t) = \begin{cases} 1, & \frac{15}{255} \le t \le \frac{31}{255}, \\ -\frac{1}{2}, & \frac{31}{255} \le t \le \frac{63}{255}, \\ 0, & \text{elsewhere,} \end{cases} \qquad H_8(t) = \begin{cases} 1, & \frac{63}{255} \le t \le \frac{127}{255}, \\ -\frac{1}{2}, & \frac{127}{255} \le t \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

2.1. **Orthogonality property.** We know the Haar wavelet functions are piecewise orthogonal

$$\int_a^b H_r(t)H_s(t)dt = \begin{cases} (b-a)2^{-j}, & r = s, \\ 0, & r \ne s. \end{cases}$$

Similarly, the non-uniform Haar wavelet defined in $[a, b]$ are piecewise orthogonal,

$$\int_a^b H_i(t)H_j(t)dt = \begin{cases} (b-a)T_i, & i = j, \\ 0, & i \ne j, \end{cases} \tag{2.4}$$

with

$$T_i = u_i(\Upsilon_3(i) - \Upsilon_1(i)). \tag{2.5}$$

3. Definition of Heaviside Step Function

In this section, we recall definition of Heaviside step function. Clearly the Heaviside step function, hsf, is defined as follow:

$$hsf(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

with a useful property

$$hsf(t - \Upsilon_1(i))hsf(t - \Upsilon_2(i)) = hsf(t - \max\{\Upsilon_1(i), \Upsilon_2(i)\}), \quad \Upsilon_1(i), \Upsilon_2(i) \in \mathbb{R}.$$

Therefore, by using hsf, we can write $H_0(t) = hsf(t) - hsf(t-1)$,

$$H_\tau(t) = hsf(t - \Upsilon_1(i)) + (-u_i - 1)hsf(t - \Upsilon_2(i)) + u_i hsf(t - \Upsilon_3(i)), \tag{3.1}$$

where

$$\tau = 2^j + k, \quad j, k \in \mathbb{N} \cup \{0\},\ 0 \leq k < 2^j.$$

4. Function Approximation

For each integrable function $Y(t)$ in interval $[0,1]$, we can expand it by the non-uniform Haar wavelets as the following form:

$$Y(t) = q_1 H_1(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^{j-1}} q_{2^j+k+1} H_{2^j+k+1}(t), \quad t \in [0,1],$$

which coefficients q_i given by

$$q_1 = \frac{1}{T_1} \int_0^1 Y(t) H_1(t) dt, \quad q_i = \frac{1}{T_i} \int_0^1 Y(t) H_i(t) dt,$$

where $i = 2^j + k + 1$, $j \geq 0$, $0 \leq k < 2^j$ and $t \in [0,1]$. We can find coefficients q_i such that square error Γ

$$\Gamma = \int_0^1 \left(Y(t) - \sum_{i=1}^{2^j} q_i H_i(t) \right)^2 dt, \quad j \in \mathbb{N} \cup \{0\},$$

is minimized. By using (3.1) the non-uniform Haar coefficients can be rewritten as

$$q_i = \frac{1}{T_i} \left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} Y(t) dt - u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} Y(t) dt \right),$$

where $i = 2^j + k + 1$, $j, k = 0, 1, 2, 3, \ldots$, and $0 \leq k < 2^j$. If $Y(t)$ is piecewise constant or can be approximated as piecewise constant during any subinterval, then $Y(t)$ will be terminated at n finite terms. This mean

$$Y(t) \simeq q_1 H_1(t) + \sum_{j=0}^{J} \sum_{k=0}^{2^{j-1}} q_{2^j+k+1} H_{2^j+k+1}(t),$$

rewriting above equation in the vector form, we have

$$Y(t) \simeq q^T H(t) = H^T(t) q(t), \quad t \in [0,1],$$

where the coefficients q^T and the non-uniform Haar functions vector $H(t)$ are defined as

$$\begin{aligned} q^T &= [q_1, q_1, q_1, \ldots, q_{2N}] \,, \\ H(t) &= [H_1(t), H_2(t), H_3(t), \ldots, H_{2N}(t)]^T. \end{aligned}$$

Therefore, any 2-variable function $p(s,t) \in L^2[0,1) \times L^2[0,1)$ can be expanded respect to the non-uniform Haar wavelets as

$$p(s,t) = H^T(s)pH(t) = H^T(t)p^T H(s),$$

where $H(s)$ and $H(t)$ are the non-uniform Haar wavelet vectors, and p is the $2N \times 2N$ Haar wavelet coefficients matrix which (i,j)-th element can be obtained as

$$\begin{aligned} p_{i,j} =& \frac{1}{T_i^2} \int_0^1 \int_0^1 p(s,t) H_i(s) H_j(t) dtds \\ =& \frac{1}{T_i^2} \left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} \int_{\Upsilon_1(j)}^{\Upsilon_2(j)} p(s,t) dtds - u_j \int_{\Upsilon_1(i)}^{\Upsilon_2(i)} \int_{\Upsilon_2(j)}^{\Upsilon_3(j)} p(s,t) dtds \right. \\ & \left. -u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} \int_{\Upsilon_1(j)}^{\Upsilon_2(j)} p(s,t) dtds + u_i u_j \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} \int_{\Upsilon_2(j)}^{\Upsilon_3(j)} p(s,t) dtds \right), \end{aligned}$$

where $i, j = 1, 2, 3, \ldots, 2N$.

5. Description of the Proposed Method

In this section, we apply the operational matrix of integration of the non-uniform Harr wavelets for solving Volterra integral equations. For this purpose we approximate $Y(t)$, $K(t)$, $P(s,t)$ in (1.1) as follows:

$$\begin{aligned} Y(t) &\simeq Y^T H(t) = H^T(t) Y, \\ K(t) &\simeq K^T H(t) = H^T(t) K \end{aligned} \tag{5.1}$$

and

$$P(s,t) \simeq H^T(s) P H(t) = H^T(t) P^T H(s),$$

where Y and K are the non-uniform Haar wavelets coefficients vectors, and P is the non-uniform coefficient matrix. Substituting above approximations in (1.1), we have

$$\begin{aligned} Y^T H(t) &\simeq K^T H(t) + \int_0^t \left(Y^T H(s) \right) \left(H^T(s) P^T H(t) \right) ds \\ &= K^T H(t) + Y^T \left(\int_0^t H(s) H^T(s) ds \right) P^T H(t) \\ &= K^T H(t) + Y^T Q P^T H(t), \end{aligned}$$

where

$$Q = \int_0^t \left(H(s) H^T(s) \right) ds.$$

So, we can write

$$Y^T \left(I - Q P^T \right) \simeq K^T. \tag{5.2}$$

We know that, (5.2) is a linear system of equations for unknown vector Y. After solving this linear system and determining Y, we can approximate solution of Volterra integral equation (1.1) by substituting the obtained vector Y in (5.1).

6. Error Analysis

In this section, we prove the convergence and error analysis of the proposed method for solving Volterra integral equations. To do this, we need the following theorems.

Theorem 6.1. *Suppose that $q(t) \in L^2[0,1)$ is an arbitrary function with bounded first derivative $|q'(t)| \leq M$ and we consider error function*

$$e_m(t) = q(t) - \sum_{i=0}^{m-1} q_i H_i(t),$$

where $i = 2^j + k + 1$, $m = 2^{J+1}$, $J > 0$ and

$$q_i = \frac{1}{T_i}\int_0^1 H_i(t)q(t)dt = \frac{1}{T_i}\left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} q(t)dt - u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} q(t)dt\right).$$

Then, we have

$$\|e_m\|_2 = O\left(\frac{1}{m}\right).$$

Proof. Clearly, we have

$$\begin{aligned}
\|e_m\|_2^2 &= \int_0^1 \left(q(t) - \sum_{i=0}^{m-1} q_i H_i(t)\right)^2 dt\\
&= \int_0^1 \left(\sum_{i=m}^{\infty} q_i H_i(t)\right)^2 dt\\
&= \sum_{i=m}^{\infty} q_i^2 \int_0^1 H_i^2(t)dt.
\end{aligned}$$

By the mean value theorem for integrals, there are $\alpha_1 \in (\Upsilon_1(i), \Upsilon_2(i))$, $\alpha_2 \in (\Upsilon_2(i), \Upsilon_3(i))$, such that

$$\begin{aligned}
q_i &= \frac{1}{T_i}\left(q(\alpha_1)\left(\Upsilon_2(i) - \Upsilon_1(i)\right) - u_i\left(q(\alpha_2)\left(\Upsilon_3(i) - \Upsilon_2(i)\right)\right)\right)\\
&= \frac{1}{T_i}\left(q(\alpha_1)(\Upsilon_2(i) - \Upsilon_1(i)) - \frac{\Upsilon_2(i) - \Upsilon_1(i)}{\Upsilon_3(i) - \Upsilon_2(i)}(\Upsilon_3(i) - \Upsilon_2(i))q(\alpha_2)\right)\\
&= \frac{1}{T_i}\left((\Upsilon_2(i) - \Upsilon_1(i))(q(\alpha_1) - q(\alpha_2))\right)\\
&= \frac{1}{T_i}\left((\Upsilon_2(i) - \Upsilon_1(i))(\alpha_1 - \alpha_2)q'(\alpha)\right), \quad \alpha \in (\alpha_1, \alpha_2).
\end{aligned}$$

From (2.4) and definitions α_1, α_2, it follows that

$$\begin{aligned}\|e_m\|_2^2 &= \sum_{i=m}^{\infty} \frac{1}{T_i^2}\Big((\Upsilon_2(i)-\Upsilon_1(i))^2(\alpha_1-\alpha_2)^2(q'(\alpha))^2\Big)T_i \\ &\le \sum_{i=m}^{\infty} \frac{1}{T_i}\Big((\Upsilon_2(i)-\Upsilon_1(i))^2(\Upsilon_3(i)-\Upsilon_1(i))^2M^2\Big). \end{aligned} \tag{6.1}$$

Now, by definitions of $\Upsilon_1(i)$, $\Upsilon_2(i)$, $\Upsilon_3(i)$ and (2.1), we have

$$\begin{aligned}\Upsilon_1(i) =& \frac{2kN}{n}\Delta t_1, \\ \Upsilon_2(i) =& \frac{(2k+1)N}{n}\Delta t_1\end{aligned}$$

and

$$\Upsilon_3(i) = \frac{(2k+2)N}{n}\Delta t_1.$$

Therefore, we get

$$\Upsilon_2(i)-\Upsilon_1(i) = \frac{N\Delta t_1}{n}, \tag{6.2}$$

$$\Upsilon_3(i)-\Upsilon_1(i) = \frac{2N\Delta t_1}{n} \tag{6.3}$$

and

$$\Upsilon_3(i)-\Upsilon_2(i) = \frac{N\Delta t_1}{n}.$$

Since

$$\Upsilon_2(i)-\Upsilon_1(i) \le \Upsilon_3(i)-\Upsilon_1(i),$$

we can write

$$\|e_m\|_2^2 \le M^2\sum_{i=m}^{\infty}\frac{1}{T_i}\Big((\Upsilon_3(i)-\Upsilon_1(i))^4\Big). \tag{6.4}$$

Now, by using (2.3) and (2.5), we get

$$\frac{1}{T_i} = \frac{\Upsilon_3(i)-\Upsilon_2(i)}{(\Upsilon_2(i)-\Upsilon_1(i))(\Upsilon_3(i)-\Upsilon_1(i))}, \tag{6.5}$$

with

$$(\Upsilon_3(i)-\Upsilon_2(i)) \le (\Upsilon_3(i)-\Upsilon_1(i)),$$

and applying (6.2) in (6.5), we can write

$$\frac{1}{T_i} \le \frac{1}{(\Upsilon_2(i)-\Upsilon_1(i))} = \frac{1}{\frac{N(\Delta t_c)}{n}} = \frac{1}{\frac{N(\Delta t_c)}{2^j}}. \tag{6.6}$$

By using (6.3), (6.4) and (6.6) implies

$$\begin{aligned}\|e_m\|_2^2 &\leq M^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{2N^3(\Delta t_1)^3}{(2^j)^3} \\ &= 2M^2N^3(\Delta t_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^3} \times 2^j \\ &= 2M^2N^3(\Delta t_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^2}.\end{aligned}$$

Applying series summation in above equation, we obtain

$$\|e_m\|_2^2 \leq \frac{8M^2N^3(\Delta t_1)^3}{3}\left(\frac{1}{2^{J+1}}\right)^2 = A\left(\frac{1}{2^{J+1}}\right)^2, \tag{6.7}$$

where

$$A = \frac{8M^2N^3(\Delta t_c)^3}{3}.$$

Since $m = 2^{J+1}$, we have

$$\|e_m\|_2 = O\left(\frac{1}{m}\right). \qquad \square$$

Theorem 6.2. *Suppose that $q(t,y) \in L^2[0,1)^2$ is a function with bounded partial derivative, $\left|\frac{\partial^2 q}{\partial t \partial y}\right| < E$, and we consider error function*

$$e_m(t,y) = q(t,y) - \sum_{i=0}^{m-1} \sum_{l=0}^{m-1} q_{i,l} H_i(t) H_l(y),$$

where $i = 2^{j_1} + k + 1$, $l = 2^{j_2} + k + 1$, $m = 2^{J+1}$, $J > 0$ and

$$q_{i,l} = \frac{1}{T_i^2} \int_0^1 \int_0^1 \left(H_i(t) H_l(y) q(t,y)\right) dtdy.$$

Then, we have

$$\|e_m\|_2 = O\left(\frac{1}{m^2}\right).$$

Proof. By definition of error $e_m(t,y)$, we can write

$$\begin{aligned}\|e_m\|_2^2 &= \int_0^1 \int_0^1 \left(q(t,y) - \sum_{i=0}^{m-1} \sum_{l=0}^{m-1} q_{i,l} H_i(t) H_l(y)\right)^2 dtdy \\ &= \int_0^1 \int_0^1 \left(\sum_{i=m}^{\infty} \sum_{l=m}^{\infty} q_{i,l} H_i(t) H_l(y)\right)^2 dtdy \\ &= \sum_{i=m}^{\infty} \sum_{l=m}^{\infty} \int_0^1 \int_0^1 \left(q_{i,l}^2 H_i^2(t) H_l^2(y)\right) dtdy.\end{aligned}$$

From the non-uniform wavelet definition, mean value theorem and Theorem 6.1, there are $\alpha, \alpha_1, \alpha_2$, also β, β_1 and β_2 such that

$$
\begin{aligned}
q_{i,l} &= \frac{1}{T_i^2}\int_0^1 H_i(t)\left(\int_0^1 H_l(y)q(t,y)dy\right)dt \\
&= \frac{1}{T_i^2}\int_0^1 H_i(t)\left(\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\frac{\partial q(t,\beta)}{\partial y}\right)dt \\
&= \frac{1}{T_i^2}\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\int_0^1\left(H_i(t)\frac{\partial q(t,\beta)}{\partial y}\right)dt \\
&= \frac{1}{T_i^2}\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\left(\Upsilon_2(i)-\Upsilon_1(i)\right)(\alpha_1-\alpha_2)\frac{\partial^2 q(\alpha,\beta)}{\partial y\partial t}.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\|e_m\|_2^2 \\
=&\sum_{i=m}^{\infty}\sum_{l=m}^{\infty}\frac{1}{T_i^4}(\Upsilon_2(l)-\Upsilon_1(l))^2(\beta_1-\beta_2)^2(\Upsilon_2(i)-\Upsilon_1(i))^2(\alpha_1-\alpha_2)^2\left|\frac{\partial^2 q(\alpha,\beta)}{\partial y\partial t}\right|^2 T_i^2 \\
\leq& E^2\sum_{i=m}^{\infty}\sum_{l=m}^{\infty}\frac{1}{T_i^2}(\Upsilon_2(i)-\Upsilon_1(i))^2(\alpha_1-\alpha_2)^2(\Upsilon_2(l)-\Upsilon_1(l))^2(\beta_1-\beta_2)^2.
\end{aligned}
$$

By using (6.7), we get

$$\|e_m\|_2^2 \leq E^2A^2 \times \frac{1}{m^4} = \frac{Z}{m^4}, \quad Z = E^2A^2.$$

In the other words, we have

$$\|e_m\|_2 = O\left(\frac{1}{m^2}\right).$$

□

7. Numerical Examples

Here, we take two examples of linear Volterra integral equations to show the accuracy and efficiency of the non-uniform Haar wavelet method. We use MATLAB package to do all the computational work.

Example 7.1. We consider the following Volterra integral equation

$$Y(t) = K(t) + \int_0^t Y(s)(s-t)ds,$$

where, $K(t) = \frac{3}{4}e^{2t} + t^2 + \frac{1}{2}t - \frac{7}{4}$, with the exact solution $Y(t) = e^{2t} - 2$. Table 1 shows the comparison maximum absolute errors (MAE) of this example for $J = 0, 1, \ldots, 6$. In Table 2, $\bar{X}_A$ is the exact solution, $\bar{X}_N$ is the numerical solution and $\bar{X}_E$ is the error of this method. Also, a comparison between the non-uniform Haar wavelet method (NHWM) and the uniform Haar wavelet method (UHWM) are shown in Table 3. Furthermore, CPU time is 303.203775 seconds.

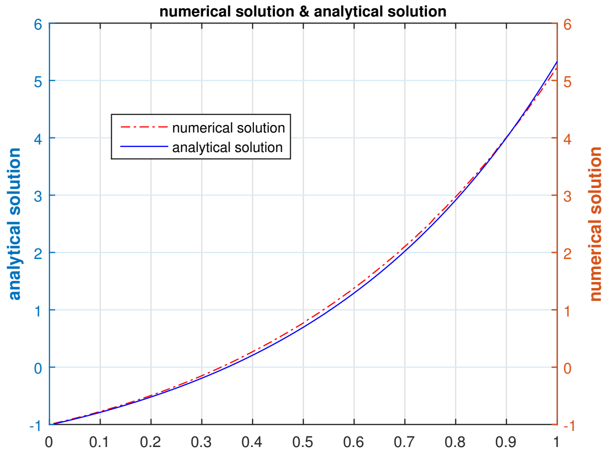

FIGURE 1. Analytical and numerical solution for $J = 6$.

TABLE 1. MAE of Example 1.

J	maximum absolute errors
0	0.4519
1	0.2784
2	0.1811
3	0.1341
4	0.1102
5	0.1005
6	0.0999

TABLE 2. The result of Example 1 for $J = 6$.

t	$\bar{X}_A$	$\bar{X}_N$	$\bar{X}_E$
0.0039	-0.9922	-0.9842	0.0080
0.0351	-0.9272	-0.9173	0.0099
0.0429	-0.9103	-0.8990	0.0114
0.0508	-0.8932	-0.8815	0.0117
0.1288	-0.7061	-0.6892	0.0183
0.1523	-0.6440	-0.6234	0.0207
0.8945	3.9831	3.9914	0.0083
0.9101	4.1731	4.1703	0.0002

TABLE 3. Comparison of maximum absolute errors for NHWM with UHWM.

J	Method	MAE
0	NHWM	0.4519
	UHWM	0.4383
2	NHWM	0.1811
	UHWM	0.1791
4	NHWM	0.1102
	UHWM	0.1167
6	NHWM	0.0999
	UHWM	0.1095

Example 7.2. We consider the following Volterra integral equation

$$Y(t) = K(t) + \int_0^t Y(s)(s^2 - t^2)ds,$$

where $K(t) = 2\cosh(t) - \sinh(t) - 2t\sinh(t) + t^2 + \frac{1}{3}t^3 - \frac{3}{2}$, with the exact solution $Y(t) = \frac{1}{2} - \sinh(t)$. Table 4 shows the comparison maximum absolute errors (MAE) of this example for $J = 0, 1, \ldots, 6$. In Table 5, $\bar{X}_A$ is the exact solution, $\bar{X}_N$ is the numerical solution and $\bar{X}_E$ is the error of this method. Also, a comparison between the non-uniform Haar wavelet method (NHWM) and the uniform Haar wavelet method (UHWM) are shown in Table 6. Furthermore, CPU time is 276.167482 seconds.

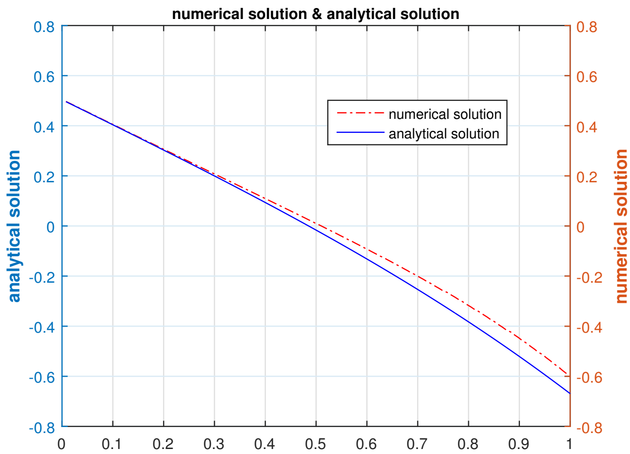

FIGURE 2. Analytical and numerical solution for $J = 6$.

Table 4. MAE of Example 2.

J	maximum absolute errors
0	0.0981
1	0.0928
2	0.0846
3	0.0794
4	0.0775
5	0.0749
6	0.0739

Table 5. The result of Example 2 for $J = 6$.

t	$\bar{X}_A$	$\bar{X}_N$	$\bar{X}_E$
0.0039	0.4961	0.4968	0.0006
0.0351	0.4649	0.4739	0.0009
0.1913	0.3075	0.3106	0.0035
0.3241	0.1702	0.1799	0.0097
0.4334	0.0529	0.0721	0.0193
0.4647	0.0184	0.0411	0.0227
0.5897	-0.1153	-0.0862	0.0377
0.6600	-0.2090	-0.1525	0.0487

Table 6. Comparison of maximum absolute errors for NHWM with UHWM.

J	Method	MAE
0	NHWM	0.0981
	UHWM	0.0874
2	NHWM	0.0846
	UHWM	0.0804
4	NHWM	0.0775
	UHWM	0.0793
6	NHWM	0.0739
	UHWM	0.0766

8. Conclusion

In this paper, a computational method based on the non-uniform Haar wavelets and their operational matrix of integration are proposed for solving Volterra integral equations. The main purpose of this method is reduce these equations to a linear system of algebraic equations. The convergence analysis of the proposed method is analyzed. For the future work, we can apply this method to solve Fredholm and Volterra-Fredholm integral equations, stochastic integral equations.

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GENERALIZATION OF CERTAIN INEQUALITIES CONCERNING THE POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. In this paper, we prove some more general results concerning the maximum modulus of the polar derivative of a polynomial. A variety of interesting results follow as special cases from our results.

1. INTRODUCTION

Let $\mathbb{P}_n$ denote the space of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and let $P'(z)$ be its derivative then

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} \left|P(z)\right|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein (for reference see [3]) and is best possible with equality holds for $P(z) = \lambda z^n$, where λ is a complex number. Where as concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have (for reference see [15]),

$$\max_{|z|=R} \left|P(z)\right| \leq R^n \max_{|z|=1} \left|P(z)\right|, \quad R \geq 1. \tag{1.2}$$

Inequality (1.2) holds for $P(z) = \lambda z^n$, where λ is a complex number.

If we restrict ourselves to the class of polynomials $P \in \mathbb{P}_n$, with $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1} \left|P(z)\right| \tag{1.3}$$

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and

$$\max_{|z|=R\geq 1}\left|P(z)\right| \leq \frac{R^n+1}{2}\max_{|z|=1}\left|P(z)\right|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [10], where as inequality (1.4) was proved by Ankeny and Rivlin [1].

Inequality (1.1) can be seen as a special case of the following inequality which is also due to Bernstein [3].

Theorem 1.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \leq |F(z)|$ for $|z| = 1$, then for $|z| \geq 1$, we have*

$$\left|f'(z)\right| \leq \left|F'(z)\right|. \tag{1.5}$$

Equality holds in (1.5) *for $f(z) = e^{i\eta}F(z)$, $\eta \in \mathbb{R}$.*

Inequality (1.1) can be obtained from inequality (1.5) by taking $F(z) = Mz^n$, where $M = \max_{|z|=1}|f(z)|$. In the same way, inequality (1.2) follows from the following result which is a special case of Bernstein-Walsh lemma [14], Corollary 12.1.3.

Theorem 1.2. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \leq |F(z)|$ for $|z| = 1$, then*

$$\left|f(z)\right| < \left|F(z)\right|, \quad \text{for } |z| > 1,$$

unless $f(z) = e^{i\eta}F(z)$ for some $\eta \in \mathbb{R}$.

In 2011, Govil et al. [4] proved a more general result which provides a compact generalization of inequalities (1.1), (1.2), (1.3) and (1.4) and includes Theorem 1.1 and Theorem 1.2 as special cases. In fact, they proved that if $f(z)$ and $F(z)$ are as in Theorem 1.1, then for any β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\left|f(Rz) - \beta f(rz)\right| \leq \left|F(Rz) - \beta F(rz)\right|, \quad \text{for } |z| \geq 1. \tag{1.6}$$

Further, as a generalization of (1.6), Liman et al. [8] in the same year 2011 and under the same hypothesis as in Theorem 1.1, proved that

$$\begin{aligned}&\left|f(Rz) - \beta f(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}f(rz)\right|\\ \leq&\left|F(Rz) - \beta F(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}F(rz)\right|,\end{aligned} \tag{1.7}$$

for every $\beta, \gamma \in \mathbb{C}$ with $|\beta| \leq 1$, $|\gamma| \leq 1$ and $R > r \geq 1$.

Jain [6] proved a result concerning the minimum modulus of polynomials by showing that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \leq 1$, then for every β with $|\beta| \leq 1$ and $R \geq 1$,

$$\min_{|z|=1}\left|f(Rz) + \beta\left(\frac{R+1}{2}\right)^n f(z)\right| \geq \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|\min_{|z|=1}|f(z)|. \tag{1.8}$$

Mezerji et al. [13] besides proving some other results also obtained a generalization of (1.8) by proving that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then for every $|\beta| \leq 1$ and $R \geq 1$

$$\min_{|z|=1}\left|f(Rz) + \beta\Big(\frac{R+k}{1+k}\Big)^n f(z)\right| \geq \frac{1}{k^n}\left|R^n + \beta\Big(\frac{R+k}{1+k}\Big)^n\right| \min_{|z|=1}|f(z)|. \tag{1.9}$$

Recently, Kumar [7] found that there is a room for the generalization of the condition $R \geq 1$ in (1.8) and (1.9) to $R \geq r > 0$ and proved that if $f \in \mathbb{P}_n$ and $f(z)$ has all its zeros in $|z| \leq k$, $k > 0$, then for every β with $|\beta| \leq 1$, $|z| \geq 1$ and $R \geq r$, $Rr \geq k^2$,

$$\min_{|z|=1}\left|f(Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^n f(rz)\right| \geq \frac{1}{k^n}\left|R^n + \beta r^n\Big(\frac{R+k}{r+k}\Big)^n\right| \min_{|z|=k}|f(z)|. \tag{1.10}$$

For $f \in \mathbb{P}_n$, let $D_\alpha f(z)$ denote the polar derivative of $f(z)$ of degree n with respect to α (see [11]) then

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z).$$

The polynomial $D_\alpha f(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\frac{D_\alpha f(z)}{\alpha} := f'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

The latest development and research can be found in the papers by Jiraphorn Somsuwan and Meneeruk Nakprasit [16] and Abdullah Mir [12].

Recently, Liman et al. [9] besides proving some other results also proved the following generalization of (1.6) and (1.7) to the polar derivative $D_\alpha f(z)$ of a polynomial $f(z)$ with respect to α, $|\alpha| \geq 1$.

Theorem 1.3. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. If $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have*

$$\begin{aligned}&\bigg|z\Big[(n-m)\Big\{f(Rz) - \beta f(rz)\Big\} + D_\alpha f(Rz) - \beta D_\alpha f(rz)\Big]\\ &+ \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) - \beta f(rz)\Big\}\bigg|\\ &\leq\bigg|z\Big\{D_\alpha F(Rz) - \beta D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) - \beta F(rz)\Big\}\bigg|.\end{aligned} \tag{1.11}$$

Equality holds in (1.11) *for $f(z) = e^{i\eta}F(z)$, $\eta \in \mathbb{R}$.*

2. Main Results

The main aim of this paper is to obtain some more general results for the maximum modulus of the polar derivative of a polynomial under certain constraints on the zeros and on the functions considered. We first prove the following generalization of inequalities (1.6) and (1.7) and of Theorem 1.3.

Theorem 2.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq k, k > 0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\Big|f(z)\Big| \leq |F(z)|, \quad \text{for } |z| = k. \tag{2.1}$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1$, $|\beta| \leq 1$, $|\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r$, $rR \geq k^2$ and $|z| \geq 1$, we have

$$\begin{aligned}&\Bigg|z\Bigg[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Bigg]\\&\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Bigg|\\ \leq&\Bigg|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Bigg|,\end{aligned} \tag{2.2}$$

where

$$\psi = \psi_k(R, r, \beta, \gamma) = \gamma\Bigg\{\Big(\frac{R+k}{r+k}\Big)^n - |\beta|\Bigg\}. \tag{2.3}$$

The result is sharp and equality in (2.2) holds for $f(z) = e^{i\eta}F(z)$, η is real and $F(z)$ has all its zeros in $|z| \leq k$.

We now present and discuss some consequences of Theorem 2.1. Suppose $f \in \mathbb{P}_n$ and $f(z) \neq 0$ in $|z| < k$, the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$ and $Q(z)$ has all its zeros in $|z| \leq \frac{1}{k}$. Note that

$$|Q(z)| = \frac{1}{k^n}|f(k^2z)|, \quad \text{for } |z| = \frac{1}{k}.$$

Applying Theorem 2.1 with $F(z)$ replaced by $k^nQ(z)$, we get the following corollary.

Corollary 2.1. *If $f \in \mathbb{P}_n$ and $f(z) \neq 0$ in $|z| < k$, $k > 0$, then for every $|\alpha| \geq 1$, $|\beta| \leq 1$, $|\gamma| \leq 1$ and $|\lambda| < 1$, we have for $R > r$, $rR \geq \frac{1}{k^2}$ and $|z| \geq 1$,*

$$\begin{aligned}&\Bigg|z\Big\{D_\alpha f(Rk^2z)+\phi D_\alpha f(rk^2z)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rk^2z)+\phi f(rk^2z)\Big\}\Bigg|\\ \leq& k^n\Bigg|z\Big\{D_\alpha Q(Rz)+\phi D_\alpha Q(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{Q(Rz)+\phi Q(rz)\Big\}\Bigg|,\end{aligned} \tag{2.4}$$

$Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$ and

$$\phi = \phi_k(R, r, \beta, \gamma) = \gamma\Bigg\{\Big(\frac{Rk+1}{rk+1}\Big)^n - |\beta|\Bigg\}. \tag{2.5}$$

Equality holds in (2.4) for $f(z) = e^{i\eta}Q(z), \eta \in \mathbb{R}$.

Remark 2.1. For $k=1$ and $\gamma=0$, Corollary 2.1 in particular yields a result of Liman et al. [9, Corollary 1.4]. Taking $\beta=\lambda=0$ in Corollary 2.1 we get the following result.

Corollary 2.2. *If $f\in\mathbb{P}_n$ and $f(z)\neq 0$ in $|z|<k$, $k>0$, then for every $|\alpha|\geq 1$, $|\gamma|\leq 1$, we have for $R>r$, $rR\geq \frac{1}{k^2}$ and $|z|\geq 1$,*

$$\begin{aligned}&\left|D_\alpha f(Rk^2z)+\gamma\Big(\frac{Rk+1}{rk+1}\Big)^n D_\alpha f(rk^2z)\right|\\ \text{(2.6)}\qquad &\leq k^n\left|D_\alpha Q(Rz)+\gamma\Big(\frac{Rk+1}{rk+1}\Big)D_\alpha Q(rz)\right|,\end{aligned}$$

$Q(z)=z^n\overline{f(\frac{1}{\bar z})}$.

Inequality (2.6) should be compared with a result recently proved by Kumar [7, Lemma 2.2], where $f(z)$ is replaced by $D_\alpha f(z)$, $|\alpha|\geq 1$.

Remark 2.2. For $r=1$, Corollary 2.2 gives the polar derivative analog of a result due to Mezerji et al. ([13], Lemma 4). If we take $\beta=0$ in Theorem 2.1 we get the following.

Corollary 2.3. *Let $F\in\mathbb{P}_n$, having all zeros in $|z|\leq k$, $k>0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\Big|f(z)\Big|\leq\Big|F(z)\Big|,\quad \text{for } |z|=k.$$

If $\alpha,\gamma,\lambda\in\mathbb{C}$ be such that $|\alpha|\geq 1$, $|\gamma|\leq 1$ and $|\lambda|<1$, then for $R>r$, $rR\geq k^2$ and $|z|\geq 1$, we have

$$\begin{aligned}&\Bigg|z\Bigg[(n-m)\Big\{f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n f(rz)\Big\}+D_\alpha f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n D_\alpha f(rz)\Bigg]\\ &\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n f(rz)\Big\}\Bigg|\\ &\leq\Bigg|z\Big\{D_\alpha F(Rz)+\gamma\Big(\frac{R+k}{r+k}\Big)^n D_\alpha F(rz)\Big\}\\ \text{(2.7)}\quad &\quad+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\Big(\frac{R+k}{r+k}\Big)^n F(rz)\Big\}\Bigg|.\end{aligned}$$

Equality holds in (2.7) for $f(z)=e^{i\eta}F(z)$, $\eta\in\mathbb{R}$ and $F(z)$ has all its zeros in $|z|\leq k$.

If we apply Theorem 2.1 to polynomials $f(z)$ and $\frac{z^n}{k^n}\min_{|z|=k}|f(z)|$, we get the following result.

Corollary 2.4. *If $f\in\mathbb{P}_n$ and $f(z)$ has all its zeros in $|z|\leq k, k>0$, then for every $\alpha,\beta,\gamma,\lambda\in\mathbb{C}$ such that $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$ and $|\lambda|<1$, we have for $R>r$, $rR\geq k^2$ and $|z|\geq 1$,*

$$\Big|z\Big\{D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Big|$$

$$\geq \frac{n|z|^n}{k^n}\left|\alpha(R^{n-1}+\psi r^{n-1})+\frac{\lambda}{2}(|\alpha|-1)(R^n+\psi r^n)\right|\min_{|z|=k}|f(z)|, \tag{2.8}$$

where ψ is defined by the equation (2.3). *Equality holds in* (2.8) *for $f(z)=az^n$, $a\neq 0$.*

Taking $\lambda=0$ in Corollary 2.4 we get the following result.

Corollary 2.5. *If $f\in\mathbb{P}_n$ and $f(z)$ has all its zeros in $|z|\leq k$, $k>0$, then for every $\alpha,\beta,\gamma,\in\mathbb{C}$ such that $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$ and for $R>r$, $rR\geq k^2$, we have*

$$\min_{|z|=1}\left|D_\alpha f(Rz)+\psi D_\alpha f(rz)\right|\geq\frac{n|\alpha|}{k^n}\left|R^{n-1}+\psi r^{n-1}\right|\min_{|z|=k}|f(z)|, \tag{2.9}$$

ψ is defined by the equation (2.3). *Equality holds in* (2.8) *for $f(z)=az^n$, $a\neq 0$.*

Remark 2.3. For $\beta=0$, the above inequality (2.9) gives the polar derivative analog of (1.10).

Theorem 2.2. *Let $F\in\mathbb{P}_n$, having all its zeros in $|z|\leq k$, $k>0$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\left|f(z)\right|\leq|F(z)|,\quad \text{for } |z|=k.$$

If $\alpha,\beta,\gamma,\in\mathbb{C}$ be such that $|\alpha|\geq 1$, $|\beta|\leq 1$ and $|\gamma|\leq 1$, then for $R>r$, $rR\geq k^2$ and $|z|\geq 1$, we have

$$\begin{aligned}&\left|z\left[(n-m)\left\{f(Rz)+\psi f(rz)\right\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\right]\right|\\&+\frac{n}{2}(|\alpha|-1)\left|F(Rz)+\psi F(rz)\right|\\&\leq\left|z\left\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\right\}\right|+\frac{n}{2}(|\alpha|-1)\left|f(Rz)+\psi f(rz)\right|,\end{aligned} \tag{2.10}$$

where ψ is defined by the equation (2.3). *Equality holds in* (2.10) *for $f(z)=e^{i\eta}F(z)$, $\eta\in\mathbb{R}$ and $F(z)$ has all its zeros in $|z|\leq k$.*

Remark 2.4. $\gamma=0$ and $k=1$, Theorem 2.2 gives in particular a result of Liman et al. [9, Theorem 2]. From Theorem 2.2 we have the following.

Corollary 2.6. *If $f\in\mathbb{P}_n$, and $f(z)$ does not vanish in $|z|<k$, $k>0$, then for every $\alpha,\beta,\gamma\in\mathbb{C}$ with $|\alpha|\geq 1$, $|\beta|\leq 1$, $|\gamma|\leq 1$, we have for $R>r$, $rR\geq\frac{1}{k^2}$ and $|z|\geq 1$,*

$$\begin{aligned}&\left|z\left\{D_\alpha f(Rk^2z)+\phi D_\alpha f(rk^2z)\right\}\right|+\frac{n}{2}(|\alpha|-1)k^n\left|Q(Rz)+\phi Q(rz)\right|\\&\leq k^n\left|z\left\{D_\alpha Q(Rz)+\phi D_\alpha Q(rz)\right\}\right|+\frac{n}{2}(|\alpha|-1)\left|f(Rk^2z)+\phi f(rk^2z)\right|,\end{aligned} \tag{2.11}$$

where $Q(z)=z^n\overline{f(\frac{1}{\bar z})}$ and ϕ is defined by the equation (2.5).

Remark 2.5. We recover a result of Liman et al. [9, Corollary 2.3] from Corollary 2.5 when we take $\gamma=0$ and $k=1$.

3. Lemmas

We need the following lemmas to prove our theorems. The first lemma is due to Aziz and Zargar [2].

Lemma 3.1. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \le k$, $k \ge 0$, then for every $R > r$, $rR \ge k^2$*
$$\Big|f(Rz)\Big| > \left(\frac{R+k}{r+k}\right)^n \Big|f(rz)\Big|, \quad \text{for } |z| = 1.$$

Lemma 3.2. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$, then for every α with $|\alpha| \ge 1$,*
$$2\Big|zD_\alpha f(z)\Big| \ge n(|\alpha|-1)\Big|f(z)\Big|, \quad \text{for } |z| = 1.$$

The above lemma is due to Shah [17].

Lemma 3.3. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \le k$, then for $|\alpha| \ge k$, the polar derivative*
$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z),$$
of $f(z)$ at the point α also has all its zeros in $|z| \le k$.

The above lemma is due to Laguerre [11, page 49].

4. Proof of the Theorems

Proof of Theorem 2.1. By hypothesis, $F(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$ and $f(z)$ is a polynomial of degree at most n such that

$$\Big|f(z)\Big| \le |F(z)|, \quad \text{for } |z| = k, \tag{4.1}$$

therefore, if $F(z)$ has a zero of multiplicity ν at $z = ke^{i\theta_0}$, then $f(z)$ must also have a zero of multiplicity at least ν at $z = ke^{i\theta_0}$. We assume that $\frac{f(z)}{F(z)}$ is not a constant, otherwise, the inequality (2.2) is obvious. It follows by the maximum modulus principle that
$$\Big|f(z)\Big| < |F(z)|, \quad \text{for } |z| > k.$$
Suppose $F(z)$ has m zeros on $|z| = k$, where $0 \le m < n$, so that we can write
$$F(z) = F_1(z)F_2(z),$$
where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree $n - m$ whose all zeros lie in $|z| < k$. This gives with the help of (4.1) that
$$f(z) = P_1(z)F_1(z),$$
where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (4.1), we get
$$|P_1(z)| \le |F_2(z)|, \quad \text{for } |z| = k,$$

and $F_2(z) \neq 0$ for $|z| = k$. Therefore, for a given complex number δ with $|\delta| > 1$, it follows from Rouche's theorem that the polynomial $P_1(z) - \delta F_2(z)$ of degree $n - m \geq 1$ has all its zeros in $|z| < k$. Hence, the polynomial

$$P(z) = F_1(z)(P_1(z) - \delta F_2(z)) = f(z) - \delta F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$P(z) = (z - \eta e^{i\gamma})H(z),$$

where $\eta < k$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leq k$. Applying Lemma 3.1 to $H(z)$, we obtain for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} \left|P(Re^{i\theta})\right| &= \left|Re^{i\theta} - \eta e^{i\gamma}\right|\left|H(Re^{i\theta})\right| \\ &> \left|Re^{i\theta} - \eta e^{i\gamma}\right|\left(\frac{R+k}{r+k}\right)^{n-1}\left|H(re^{i\theta})\right| \\ &= \left(\frac{R+k}{r+k}\right)^{n-1}\frac{\left|Re^{i\theta} - \eta e^{i\gamma}\right|}{\left|re^{i\theta} - \eta e^{i\gamma}\right|}\left|re^{i\theta} - \eta e^{i\gamma}\right|\left|H(re^{i\theta})\right|. \end{aligned} \tag{4.2}$$

Now for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right|^2 &= \frac{R^2 + \eta^2 - 2R\eta\cos(\theta - \gamma)}{r^2 + \eta^2 - 2r\eta\cos(\theta - \gamma)} \\ &\geq \left(\frac{R+\eta}{r+\eta}\right)^2, \quad \text{for } R > r \text{ and } rR \geq k^2 \\ &> \left(\frac{R+k}{r+k}\right)^2, \quad \text{since } \eta < k. \end{aligned}$$

This implies

$$\left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right| > \frac{R+k}{r+k},$$

which on using in (4.2) gives for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\left|P(Re^{i\theta})\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(re^{i\theta})\right|.$$

Equivalently,

$$\left|P(Rz)\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(rz)\right|, \tag{4.3}$$

for $R > r$, $rR \geq k^2$ and $|z| = 1$. This implies for every $|\beta| \leq 1$, $R > r$, $rR \geq k^2$ and $|z| = 1$,

$$\left|P(Rz) - \beta P(rz)\right| \geq \left|P(Rz)\right| - |\beta|\left|P(rz)\right| > \left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}\left|P(rz)\right|. \tag{4.4}$$

Again, since $r < R$, it follows that $\left(\frac{r+k}{R+k}\right)^n < 1$, inequality (4.3) implies that

$$|P(rz)| < |P(Rz)|, \quad \text{for } |z| = 1.$$

Also, all the zeros of $P(Rz)$ lie in $|z| \leq \frac{k}{R}$ and $R^2 > rR \geq k^2$, we have $\frac{k}{R} < 1$. A direct application of Rouche's theorem shows that the polynomial $P(Rz) - \beta f(rz)$ has all its zeros in $|z| < 1$, for every $|\beta| \leq 1$. Applying Rouche's theorem again, it follows from (4.4) that for every $|\gamma| \leq 1$, $|\beta| \leq 1$, $R > r$, $rR \geq k^2$, all the zeros of the polynomial

$$g(z) := P(Rz) - \beta P(rz) + \gamma\left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}P(rz) = P(Rz) + \psi P(rz) \tag{4.5}$$

lie in $|z| < 1$. Using Lemma 3.2 we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $|z| = 1$

$$2\Big|zD_\alpha g(z)\Big| \geq n(|\alpha| - 1)\Big|g(z)\Big|.$$

Hence, for any complex number λ with $|\lambda| < 1$, we have for $|z| = 1$,

$$2\Big|zD_\alpha g(z)\Big| > n|\lambda|(|\alpha| - 1)\Big|g(z)\Big|.$$

Therefore, it follows by Lemma 3.3 that all the zeros of

$$\begin{aligned} W(z) :=& 2zD_\alpha g(z) + n\lambda(|\alpha| - 1)g(z) \\ =& 2zD_\alpha P(Rz) + 2z\psi D_\alpha P(rz) + n\lambda\Big(|\alpha| - 1\Big)\Big(P(Rz) + \psi P(rz)\Big) \end{aligned} \tag{4.6}$$

lie in $|z| < 1$.

Replacing $P(z)$ by $f(z) - \delta F(z)$ and using definition of polar derivative give

$$\begin{aligned} W(z) =& 2z\left[n\Big\{f(Rz) - \delta F(Rz)\Big\} + (\alpha - Rz)\Big\{f(Rz) - \delta F(Rz)\Big\}'\right] \\ &+ 2z\psi\left[n\Big\{f(rz) - \delta F(rz)\Big\} + (\alpha - rz)\Big\{f(rz) - \delta F(rz)\Big\}'\right] \\ &+ n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\}, \end{aligned}$$

which on simplification gives

$$\begin{aligned} W(z) =& 2z\bigg[(n - m)f(Rz) + mf(Rz) + (\alpha - Rz)\Big(f(Rz)\Big)' \\ &- \delta\Big\{nF(rz) + (\alpha - rz)\Big(F(Rz)\Big)'\Big\}\bigg] \\ &+ 2z\psi\bigg[(n - m)f(rz) + mf(rz) + (\alpha - rz)\Big(f(rz)\Big)' \\ &- \delta\Big\{nF(rz) + (\alpha - rz)\Big(F(rz)\Big)'\Big\}\bigg] \\ &+ n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\} \end{aligned}$$

$$
\begin{aligned}
=&2z\Big\{\Big(n-m)f(Rz)+D_\alpha f(Rz)-\delta D_\alpha F(Rz)\Big\}\\
&+2z\psi\Big\{\Big(n-m)f(rz)+D_\alpha f(rz)-\delta D_\alpha F(rz)\Big\}\\
&+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\}\\
=&2z\Big\{(n-m)f(Rz)+\psi(n-m)f(rz)+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\}\\
&+n\lambda\psi(|\alpha|-1)f(Rz)+n\lambda\psi(|\alpha|-1)f(rz)\\
&-\delta\Big\{2zD_\alpha F(Rz)+2z\psi D_\alpha F(rz)\\
&+n\lambda(|\alpha|-1)F(Rz)+n\lambda\psi(|\alpha|-1)F(rz)\Big\}.
\end{aligned}
\tag{4.7}
$$

Since by (4.6), $W(z)$ has all its zeros in $|z|<1$, therefore, by (4.7), we get for $|z|\geq 1$

$$
\begin{aligned}
&\Bigg|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]\\
&+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Bigg|\\
\leq&\Bigg|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Bigg|.
\end{aligned}
\tag{4.8}
$$

To see that the inequality (4.8) holds, note that if the inequality (4.8) is not true, then there is a point $z=z_0$ with $|z_0|\geq 1$, such that

$$
\begin{aligned}
&\Bigg|z_0\Big[(n-m)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]\\
&+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}\Bigg|\\
>&\Bigg|z_0\Big\{D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\Bigg|.
\end{aligned}
\tag{4.9}
$$

Now, because by hypothesis all the zeros of $F(z)$ lie in $|z|\leq k$, the polynomial $F(Rz)$ has all its zeros in $|z|\leq\frac{k}{R}<1$, and therefore, if we use Rouche's theorem and Lemmas 3.1 and 3.3 and argument similar to the above we will get that all the zeros of

$$z\Big(D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}$$

lie in $|z|<1$ for every $|\alpha|\geq 1$, $|\lambda|<1$ and $R>r$, $rR\geq k^2$, that is,

$$z\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\neq 0,$$

for every z_0 with $|z_0| \geq 1$. Therefore, if we take

$$\delta = \frac{z_0\Big[(n-m)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]}{z_0\Big(D_\alpha F(Rz_0)+\psi F(rz_0)D_\alpha\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}}$$
$$+\frac{\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz_0)+\psi f(rz_0)\Big\}}{z_0\Big(D_\alpha F(Rz_0)+\psi F(rz_0)D_\alpha\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}},$$

then δ is a well-defined real or complex number, and in view of (4.9) we also have $|\delta| > 1$. Hence, with the choice of δ, we get from (4.7) that $W(z_0) = 0$ for some z_0, satisfying $|z_0| \geq 1$, which is clearly a contradiction to the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus for every $R > r$, $rR \geq k^2$, $|\alpha| \geq 1$, $|\lambda| < 1$ and $|z| \geq 1$, inequality (4.8) holds and this completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since all the zeros of $F(z)$ lie in $|z| \leq k$, $k > 0$, for $R > r$, $rR \geq k^2$, $|\beta| \leq 1$, $|\gamma| \leq 1$, it follows as in the proof of Theorem 2.1, that all the zeros of

$$h(z) := F(Rz) - \beta F(rz) + \gamma\Big\{\Big(\frac{R+k}{r+k}\Big)^n - |\beta|\Big\}F(rz) = F(Rz) + \psi F(rz)$$

lie in $|z| < 1$. Hence, by Lemma 3.2 we get for $|\alpha| \geq 1$,

$$2\Big|zD_\alpha h(z)\Big| \geq n(|\alpha|-1)\Big|h(z)\Big|, \quad \text{for } |z| \geq 1.$$

This gives for every λ with $|\lambda| < 1$

$$\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big| \geq 0, \tag{4.10}$$

for $|z| \geq 1$. Therefore, it is possible to choose the argument of λ in the right hand side of (4.8) such that for $|z| \geq 1$

$$\begin{aligned}&\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Big|\\ &=\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big|.\end{aligned} \tag{4.11}$$

Hence, from (4.8), we get by using (4.11) for $|z| \geq 1$

$$\begin{aligned}&\Big|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]\Big|\\ &\quad-\frac{n|\lambda|}{2}(|\alpha|-1)\Big|f(Rz)+\psi f(rz)\Big|\\ &\leq\Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}\Big| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz)+\psi F(rz)\Big|.\end{aligned} \tag{4.12}$$

Letting $|\lambda| \to 1$ in (4.12), we immediately get (2.10) and this proves Theorem 2.2 completely. $\square$

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COMPACTNESS ESTIMATE FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON A Q-PSEUDOCONVEX DOMAIN IN A STEIN MANIFOLD

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Abstract. We consider a smoothly bounded q-pseudoconvex domain Ω in an n-dimensional Stein manifold X and suppose that the boundary $b\Omega$ of Ω satisfies $(q-P)$ property, which is the natural variant of the classical P property. Then, one prove the compactness estimate for the $\overline{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev k-space. Applications to the boundary global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev k-space are given. Moreover, we prove the boundary global regularity of the $\overline{\partial}$-operator on Ω.

1. Introduction and main results

The existence and regularity properties of the solutions of the system of Cauchy-Riemann equations $\bar{\partial} f = g$ on strongly pseudo-convex domains have been a central theme in the theory of several complex variables for many years. Classically many different approaches have been used: a) Vanishing of the $\bar{\partial}$-cohomology group, b) The abstract L^2-theory of the $\bar{\partial}$-Neumann problem, and c) The construction of rather explicit integral solution operators for $\bar{\partial}$, in analogy to the Cauchy transform in C^1. The first approach used by Grauert−Riemenschneider [6]. Saber [15], used this method and studied the solvability of the $\overline{\partial}$-problem with C^∞ regularity up to the boundary on a strictly q-convex domain of an n-dimensional Kähler manifold X. The second approach was first used by Kohn [11] in studying the boundary regularity of the $\overline{\partial}$-equation when Ω is pseudoconvex with C^∞ boundary. For solvability with regularity up to the boundary in a pseudoconvexity domain without corners, one refer to Kohn

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[12]. Zampieri [18] introduced a new type of notion of q-pseudoconvexity in $\mathbb{C}^n$. Under this condition he proved local boundary regularity for any degree $\geq q$. Other results in this direction belong to Heungju [10], Baracco-Zampieri [1] and Saber [16]. Thus the method of L^2 a priori estimates for the weighted $\overline{\partial}$-Neumann operator has yielded many important results on the local and global boundary regularity of the $\overline{\partial}$-problem. The integral formula approach was pioneered by Henkin [8] and Grauert-Lieb [7] for strictly pseudoconvex domains. They obtained uniform and Hölder estimates for the solution of $\overline{\partial}$ on such domains. For the related results for $\overline{\partial}$ on the pseudoconcave domains in $\mathbb{P}^n$, see Henkin-Iordan [9].

In this paper, he compactness estimate proved in Khanh and Zampieri [17] is extended E-valued forms. Such compactness estimates immediately lead to very important qualitative properties of the $\overline{\partial}$-operator, such as smoothless of solutions and closed range. The main theorem generalizes Khanh and Zampieri [17] result to forms with values in a vector bundle. The proof starting with the known estimate on scalar differential forms and then obtains a similar estimate locally on bundle-valued forms using a local frame. Then, by using a partition of unity, we globalize this estimate at the cost of the constants. Consequently, we study the boundary regularity of the $\overline{\partial}$-equation, $\overline{\partial}u = f$, for forms in a vector bundle on bounded q-pseudoconvex domain Ω in a Stein manifold X of dimension n. Moreover, some standard consequences of compactness are deduced.

2. $(q-P)$ PROPERTY

Let Ω be a bounded domain of $\mathbb{C}^n$ with C^1-boundary $b\Omega$ and ρ its a C^1-defining function. An (r,s)-form on Ω is given by

$$f = \sum_{\substack{|I|=r\\|J|=s}}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ are multiindices and $dz^I = dz_1 \wedge \cdots \wedge dz_r$, $d\bar{z}^J = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_s$. Here, the coefficients $f_{I,J}$ are functions (belonging to various function classes) on Ω. Then for two (r,s)-forms

$$f = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

$$g = \sum_{I,J}{}' g_{I,J} dz^I \wedge d\bar{z}^J.$$

One defines the inner product and the norm as

$$(f,g) = \sum_{I,J}{}' f_{I,J}\, \overline{g_{I,J}},$$

$$|f| = (f,f).$$

The notation $\sum'$ means the summation over strictly increasing multiindices. This definition is independent of the choice of the orthonormal basis. Denote by $C^\infty_{r,s}(\overline{\Omega})$

the space of complex-valued differential forms of class C^∞ and of type (r,s) on Ω that are smooth up to the boundary and $\mathcal{D}_{r,s}(U)$ denotes the elements in $C^\infty_{r,s}(\overline{\Omega})$ that are compactly supported in $U\cap\overline{\Omega}$. $L^2_{r,s}(\Omega)$ consists of the (r,s)-forms u satisfies

$$\|u\|^2 = \sum_{\substack{|I|=r\\|J|=s}}{}' |u_{I,J}|^2 dV < \infty.$$

Let

$$\overline{\partial} : L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega)$$

be the maximal closed extension and

$$\overline{\partial}^* : L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega)$$

its Hilbert space adjoint. The Laplace-Beltrami operator $\Box_{r,s}$ is defined as

$$\Box_{r,s} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathrm{dom}\ \Box_{r,s} \to L^2_{r,s}(\Omega).$$

Let

$$\mathcal{H}^{r,s} = \{\varphi \in \mathrm{dom}\,\overline{\partial} \cap \mathrm{dom}\,\overline{\partial}^\star : \overline{\partial}\varphi = 0 \text{ and } \mathrm{dom}\,\overline{\partial}^\star\varphi = 0\}.$$

One defines the $\overline{\partial}$-Neumann operator

$$N : L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega),$$

as the inverse of the restriction of $\Box_{r,s}$ to $(\mathcal{H}^{r,s})^\perp$. For nonnegative integer k, one defines the Sobolev k-space

$$W^k_{r,s}(\Omega) = \{f \in L^2_{r,s}(\Omega) : \|f\|_k < +\infty\},$$

where the Sobolev norm of order k is defined as

$$\|f\|^2_{W^k} = \int_\Omega \sum_{|\alpha|\le k} |D^\alpha f|^2 dV,$$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_{2n}}\right)^{\alpha_{2n}}, \quad \text{for } \alpha = (\alpha_1,\dots,\alpha_{2n}),\ |\alpha| = \sum \alpha_j,$$

and $x_1,\dots,x_{2n}$ are real coordinates for Ω. Detailed information on Sobolev spaces may be found for example in [4], [5]. Let p be a point in the boundary of Ω. Then one can choose a neighborhood U of p and a local coordinate system $(x_1,\dots,x_{2n-1},\rho) \in \mathbb{R}^{2n-1}\times\mathbb{R}$, satisfies the last coordinate is a local defining function of the boundary. Call $(U,(x,\rho))$ a special boundary chart. Denote the dual variable of x by ξ, and define

$$\langle x,\xi\rangle = \sum_{j=1}^{2n-1} x_j\xi_j.$$

The tangential Fourier transform for $f \in \mathcal{D}(\overline{\Omega}\cap U)$ is given in this special boundary chart by

$$\tilde{f}(\xi,\rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i\langle x,\xi\rangle} f(x,\rho)dx,$$

where $dx = dx_1 \cdots dx_{2n-1}$. For each $k \geq 0$, the standard tangential Bessel potential operator Λ^k of order k (see e.g., Chen-Shaw [4], Section 5.2) is defined as

$$(\Lambda^k f)(x,\rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i\langle x,\xi\rangle}\,(1+|\xi|^2)^{\frac{k}{2}}\,\tilde{f}(\xi,\rho)d\xi.$$

The tangential L^2-Sobolev norm of f of order k is defined as

$$|||f|||^2_{W^k_{r,s}(\Omega)} = \|\Lambda^k f\|^2 = \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} (1+|\xi|^2)^k\,|\tilde{f}(\xi,\rho)|^2 d\xi\,d\rho.$$

Clearly, for $k > 0$, this norm is weaker than the full L^2-Sobolev norm of order k, since it just measures derivatives in the tangential directions.

Let $T^{\mathbb{C}}b\Omega$ be the complex tangent bundle to the boundary, $L_{b\Omega} = (\rho_{ij})|_{T^{\mathbb{C}}b\Omega}$ the Levi form of $b\Omega$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ are the ordered eigenvalues of $L_{b\Omega}$. For every positive number M and if $\varphi^M \in C^\infty(\overline{\Omega} \cap V)$, one denote by

$$\lambda_1^{\varphi^M} \leq \lambda_2^{\varphi^M} \leq \cdots \leq \lambda_{n-1}^{\varphi^M}$$

the ordered eigenvalues of the Levi form (φ^M_{ij}). Choose an orthonormal basis of $(1,0)$-forms $\omega_1, \omega_2, \ldots, \omega_n = \partial\rho$ and the dual basis of $(1,0)$-vector fields $L_1, L_2, \ldots, L_n$; note that $T^{1,0}\partial\Omega = \mathrm{Span}\{L_1, L_2, \ldots, L_{n-1}\}$. We denote by ρ_j and ρ_{jj} the coefficients of $\partial\rho$ and $\partial\overline{\partial}\rho$ in this basis. Following Khanh and Zampieri [17] in Section 2, we have the following definitions.

Definition 2.1. $b\Omega$ is called q-pseudoconvex in a neighborhood V of z_0 if there exist a bundle $\Xi \subset T^{1,0}b\Omega$ of rank $q_0 < q$ with smooth coefficients in V, say the bundle of the first q_0 vector fields $L_1, \ldots, L_{q_0}$ of our basis of $T^{1,0}b\Omega$, satisfies

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} \rho_{jj} \geq 0 \quad \text{ on } b\Omega \cap V. \tag{2.1}$$

Since $\sum_{j=1}^{q_0} \rho_{jj}$ is the trace of the restricted form $(\rho_{jj})|_\Xi$, then Definition 2.1 depends only on the choice of the bundle Ξ, not of its basis. Condition (2.1) is equivalent to

$$\sideset{}{'}\sum_{\substack{|I|=r\\|K|=s-1}} \sum_{i,j=1}^{n-1} \rho_{ij} u_{IiK} \overline{u}_{IjK} - \sum_{j=1}^{q_0} \rho_{jj}|u|^2 \geq 0, \tag{2.2}$$

for any (r,s) form u with $s \geq q$. It is in this form that (2.1) will be applied. In some case, it is better to consider instead of (2.2), the variant

$$\sideset{}{'}\sum_{\substack{|I|=r\\|K|=s-1}} \sum_{i,j=1}^{n-1} \rho_{ij} u_{IiK} \overline{u}_{IjK} - \sideset{}{'}\sum_{\substack{|I|=r\\|K|=s-1}} \sum_{j=1}^{q_0} \rho_{jj}|u_{IjK}|^2 \geq 0. \tag{2.3}$$

It is obvious that if $L_{b\Omega}|_{\Xi}$ is assumed to be diagonal, instead of less than or equalo to 0, then the left side of (2.3) equals

$$\sum_{\substack{|I|=r\\|K|=s-1}}{}' \sum_{i,j=q_0+1}^{n-1} \rho_{ij} u_{IiK}\overline{u}_{IjK}.$$

Thus, if $\urcorner$ is the Levi-orthogonal complement of Ξ, then (2.3) is equivalent to $L_{b\Omega}|_{\urcorner} \geq 0$. The condition in the definition below generalizes to domains which are not necessarily pseudoconvex, the celebrated P property by Catlin [3].

Definition 2.2. $b\Omega$ is said to has the $(q-P)$ property in V if for every positive number M there exists a function $\varphi^M \in C^\infty(\overline{\Omega}\cap V)$ with

(i) $|\varphi^M| \leq 1$ on Ω;

(ii) $\sum_{j=1}^q \lambda_j^{\varphi^M} - \sum_{j=1}^{q_0} \varphi^M_{jj} \geq c\sum_{j=1}^{q_0} |\varphi^M_j|^2$ on $\overline{\Omega}\cap V$;

(iii) $\sum_{j=1}^q \lambda_j^{\varphi^M} - \sum_{j=1}^{q_0} \varphi^M_{jj} \geq M$ on $b\Omega\cap V$,

where the constant $c>0$ does not depend on M. (The point here is that (ii) holds in the whole $\overline{\Omega}$, (iii) only on $b\Omega$.)

There are obvious variants of (ii) and (iii) adapted to (2.3). Condition (iii) is a modification of (ii) in Definition 2 of [14]. The bigger flexibility of our condition consists in allowing subtraction of φ^M_{jj} for $j=1,\dots,q_0$. We say that a compact subset $F\subset b\Omega\cap V$ satisfies $(q-P)$ if and only if (iii) holds for any $z\in F$.

Theorem 2.1 ([13])**.** *Let X be a complex manifold of complex dimension n with a Hermitian metric g and Ω be a bounded domain of X. Let $\Omega \Subset X$ be an submanifold with smooth boundary. Suppose the compactness estimate* (3.1) *holds on Ω. Suppose further that the $\bar\partial$-closed (r,s)-form α is in $W^k(\Omega)$ and $\alpha \perp \mathcal{H}^{r,s}$, there exists a constant C_k so that the canonical solution u of $\bar\partial u=\alpha$, with $u\perp \ker\bar\partial$ satisfies*

$$\|u\|_{W^k} \leq C_k(\|\alpha\|_{W^k} + \|u\|).$$

Since $C^\infty(\overline{\Omega}) = \cap_{k=0}^\infty W^k(\Omega)$, it follows that if $\alpha\in C^\infty_{r,s}(\overline{\Omega})$, then $u\in C^\infty_{r,s-1}(\overline{\Omega})$.

3. Solvability of $\bar\partial$ in $\mathbb{C}^n$

Following Khanh and Zampieri [17], one obtains the following theorem.

Theorem 3.1. *Let Ω be a smoothly bounded q-pseudoconvex domain in $\mathbb{C}^n$, and suppose that $b\Omega$ satisfies property $(q-P)$ in a neighborhood V of z_0. Then for every $\epsilon>0$ there exists a function $C_\varepsilon \in \mathcal{D}(\Omega)$ satisfying*

$$\|u\|^2 \leq \varepsilon\, Q(u,u) + C_\varepsilon \|u\|^2_{W^{-1}_{r,s}(\Omega)}, \tag{3.1}$$

for $u\in \mathcal{D}_{r,s}(\overline{\Omega}\cap V)\cap \operatorname{dom}\bar\partial^*$ and for any $s\geq q$. Here

$$Q(u,u) = \|\bar\partial u\|^2 + \|\bar\partial^\star u\|^2 + \|u\|^2,$$

and $\|u\|_{W^{-1}_{r,s}(\Omega)}$ is the Sobolev norm of order -1.

The same statement holds if q-pseudoconvexity is understood in the sense of the variant (2.3) and if $(q-P)$ property has the corresponding variants.

Definition 3.1. We will refer to (3.1) as a compactness estimate.

Theorem 3.2. *Let Ω be the same as in Theorem 3.1. Then, for $f \in C^{\infty}_{r,s}(\overline{\Omega})$, $q \leq s \leq n-2$, satisfying $\overline{\partial} f = 0$, there exists $u \in C^{\infty}_{r,s-1}(\overline{\Omega})$, satisfies $\overline{\partial} u = f$.*

Proof. The proof follows from the estimate (3.1) and Theorem 2.1. □

Property $(q-P)$ is related to the $\overline{\partial}$-Neumann problem by the following theorem.

Remark 3.1. It is easy to observe that (3.1) implies for $u \in \mathrm{dom}\square_{r,s}$:

$$\|u\|^2 \leq \varepsilon \ \|\square_{r,s} u\|^2 + C_\varepsilon \|u\|^2_{W^{-1}_{r,s}(\Omega)} .$$

We now discuss the global regularity for $N_{r,s}$. From the estimate (3.1) one can derive a priori estimates for $N_{r,s}$ in the Sobolev k-space.

Theorem 3.3. *Let Ω be the same as in Theorem 3.1. A compactness estimate implies boundedness of the $\overline{\partial}$-Neumann operator $N_{r,s}$ in $W^k_{r,s}(\Omega)$ for any $k > 0$.*

Proof. By a standard fact of elliptic regularization, one sees that the global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ holds if

$$\|u\|_{W^k_{r,s}(\Omega)} \lesssim \|\square_{r,s} u\|_{W^k_{r,s}(\Omega)}, \tag{3.2}$$

for any $u \in C^{\infty}_{r,s}(\overline{\Omega}) \cap \mathrm{dom}\,\square_{r,s}$. Hence,

$$\|u\|^2_{W^k_{r,s}(\Omega)} \lesssim \|\square_{r,s} u\|_{W^{k-2}_{r,s}(\Omega)} + \|\Lambda^{k-1} D u\|^2, \tag{3.3}$$

where Λ is the tangential differential operator of order k. By Theorem 3.1, the estimate (3.1) implies that

$$\left\|D\Lambda^{-1} u\right\|^2 \lesssim Q(u,u) + C \|u\|^2_{W^{-1}_{r,s}(\Omega)} . \tag{3.4}$$

In fact, it follows by the non-characteristic with respect to the boundary of $\bar{L}_n$; the operator D can be understood as D_r or Λ.

Now we estimate the last term of (3.3), we have

$$\begin{aligned}
\|\Lambda^{k-1} D u\|^2 \lesssim & \|D\Lambda^{-1}\Lambda^k u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & Q(\Lambda^k u, \Lambda^k u) + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & < \Lambda^k \square_{r,s} u, \Lambda^k u > + \|[\overline{\partial}, \Lambda^k] u\|^2 + \|[\overline{\partial}^*, \Lambda^k] u\|^2 \\
& + \|[\overline{\partial}^*, [\overline{\partial}, \Lambda^k] u\|^2 + \|[\overline{\partial}, [\overline{\partial}^*, \Lambda^k] u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & \|\Lambda^k \square_{r,s} u\|^2 + \|\Lambda^{k-1} D u\|^2 + \|\Lambda^{k-2} D^2 u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & \|\square_{r,s} u\|^2_{W^k_{r,s}(\Omega)} + \|\Lambda^{k-1} D u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)},
\end{aligned}$$

where the second inequality follows by (3.4). Then the term $\|\Lambda^{k-1}Du\|^2$ can be absorbed by the left-hand side term. By induction method, we obtain the estimate (3.2). $\square$

Proposition 3.1. *Let Ω be the same as in Theorem 3.1. Then the following are equivalent.*

(i) *The validity of global compactness estimates.*

(ii) *The embedding of the space $\operatorname{dom}\overline{\partial}\cap\operatorname{dom}\overline{\partial}^*$, provided with the graph norm*

$$\|u\|+\|\overline{\partial}u\|+\|\overline{\partial}^*u\|,$$

into $L^2_{r,s}(\Omega)$ is compact.

(iii) *The $\overline{\partial}$-Neumann operators*

$$N_{r,s}:L^2_{r,s}(\Omega,E)\to L^2_{r,s}(\Omega,E),$$

for $q\le s\le n-1$ are compact from $L^2_{r,s}(\Omega)$ to itself.

(iv) *The canonical solution operators to $\overline{\partial}$ given by*

$$\overline{\partial}^*N_{r,s}:L^2_{r,s}(\Omega)\to L^2_{r,s-1}(\Omega),$$
$$N_{r,s+1}\overline{\partial}^*:L^2_{r,s+1}(\Omega)\to L^2_{r,s}(\Omega),$$

are compact.

Proof. The equivalence of (ii) and (i) is a result of Lemma 1.1 in [13]. The general L^2-theory and the fact that $L^2_{r,s}(\Omega)$ embeds compactly into $W^{-1}_{r,s}(\Omega)$ shows that (iii) is equivalent to (ii) and (i). Finally, the equivalence of (iii) and (iv) follows from the formula

$$N_{r,s}=(\overline{\partial}^*N_{r,s})^*\overline{\partial}^*N_{r,s}+\overline{\partial}^*N_{r,s+1}(\overline{\partial}^*N_{r,s+1})^*,$$

(see [4], page 55). We refer the reader to [14] for similar calculations. $\square$

4. Solvability of $\overline{\partial}$ in Stein manifold

Let X be complex manifold of complex dimension n with a Hermitian metric g and Ω be a bounded domain of X. Let $\pi:E\to X$ be a vector bundle, of rank p, over X with Hermitian metric h. Let $\{U_j\}$, $j\in J$, be an open covering of X by charts with coordinates mappings $z_j:U_j\to\mathbb{C}^n$ satisfies $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j)=U_j\times\mathbb{C}$, and $(z_j^1,z_j^2,\dots,z_j^n)$ be local coordinates on U_j. Let $\{\zeta_j\}_{j\in J}$ be a partition of unity subordinate to the holomorphic atlas (U_j,z_j), of X. We denote by T_zX the tangent bundle of X at $z\in X$. An E-valued differential (r,s)-form u on X is given locally by a column vector $u=(u^1,u^2,\dots,u^p)$, where u^a, $1\le a\le p$, are $\mathbb{C}$-valued differential forms of type (r,s) on X. The spaces $C^\infty_{r,s}(X,E)$, $\mathcal{D}_{r,s}(X,E)$, $C^\infty_{r,s}(\overline{\Omega},E)$, $\mathcal{D}_{r,s}(\Omega,E)$ and $W^k_{r,s}(\Omega,E)$ are defined as in Section 2 but for E-valued forms. Let $L^2_{r,s}(\Omega,E)$ be the Hilbert space of E-valued differential forms u on Ω, of type (r,s), satisfies

$$\|u\|_\Omega=\sum_j\sum_{a=1}^p\|u_j^a\|_{U_j\cap\Omega}<\infty,$$

where $\|u_j^a\|_{U_j\cap\Omega}$ is defined in (2.1). Let $\overline{\partial} : L^2_{r,s}(\Omega, E) \to L^2_{r,s+1}(\Omega, E)$ be the maximal closed extension of the original $\overline{\partial}$ and $\overline{\partial}^* : L^2_{r,s}(\Omega, E) \to L^2_{r,s-1}(\Omega, E)$ its Hilbert space adjoint. For $k \in \mathbb{R}$, we define a $W^k(X, E)$-norm by the following:

$$\|u\|^2_{k(X)} := \sum_j \|\zeta_j u_j\|^2_{k(W_j)}, \tag{4.1}$$

where $W_j = z_j(U_j)$ and $\sum_j \|\zeta_j u_j\|^2_{k(W_j)}$ is defined as in the Euclidean case.

Theorem 4.1. *Let Ω be a smoothly bounded q-pseudoconvex domain in an of n-dimensional Stein manifold X, $n \geq 3$, and suppose that $b\Omega$ satisfies property $(q - P)$ in a neighborhood V of z_o. Let E be a vector bundle, of rank p, on X. Then, for $f \in C^\infty_{r,s}(\overline{\Omega}, E)$, $q \leq s \leq n-2$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X, there exists $u \in C^\infty_{r,s-1}(\overline{\Omega}, E)$, satisfies $\overline{\partial} u = f$ in the distribution sense in X.*

Proof. Let $\{U_j\}_{j=1}^N$ be a finite covering of $b\Omega$ by a local patching. Let $e_1, e_2, \ldots, e_p$ be an orthonormal basis on $E_z = \pi^{-1}(z)$, for every $z \in U_j$, $j \in J$. Thus, every E-valued differential (r, s)-form u on X can be written locally, on U_j, as

$$u(z) = \sum_{a=1}^p u^a(z)\, e_a(z),$$

where u^a are the components of the restriction of u on U_j. Since $b\Omega$ is compact, there exists a finite number of elements of the covering $\{U_j\}$, say, U_j, $j = 1, 2, \ldots, m$ satisfies $\bigcup_{\nu=1}^m U_{j_\nu}$ cover $b\Omega$. Let $\{\zeta_j\}_{j=0}^m$ be a partition of the unity satisfies $\zeta_0 \in \mathcal{D}_{r,s}(\Omega)$, $\zeta_j \in \mathcal{D}_{r,s}(U_j)$, $j = 1, 2, \ldots, m$, and

$$\sum_{j=0}^m \zeta_j^2 = 1 \quad \text{on } \overline{\Omega},$$

where $\{U_j\}_{j=1,\ldots,m}$ is a covering of $b\Omega$. Let U be a small neighborhood of a given boundary point $\xi_0 \in b\Omega$ satisfies $U \Subset V \Subset U_{j_\nu}$, for a certain $j_\nu \in I$. If $u \in \mathcal{D}_{r,s}(\Omega, E)$, $0 \leq r \leq n$, $q \leq s \leq n-2$, on applying the compactness estimate of Khanh and Zampieri [17] to each u^a and adding for $a = 1, \ldots, p$, one gets compactness estimate for $u|_{\Omega\cap U}$

$$\begin{aligned}\|\zeta_0 u\|^2 &\lesssim \epsilon Q(\zeta_0 u, \zeta_0 u) + C_\epsilon \|\zeta_0 u\|^2_{W^{-1}_{r,s}(\Omega)}\\ &\lesssim \epsilon Q(u, u) + C_\epsilon \|u\|_{W^{-1}_{r,s}(\Omega)}.\end{aligned}$$

Similarly, for $j = 1, \ldots, m$, we obtain compactness estimate for $u|_{\Omega\cap U_j}$

$$\begin{aligned}\|\zeta_j u\|^2 &\lesssim \epsilon Q(\zeta_j u, \zeta_j u) + C_\epsilon \|\zeta_j u\|^2_{W^{-1}_{r,s}(\Omega)}\\ &\lesssim \epsilon Q(u, u) + C_\epsilon \|u\|^2_{W^{-1}_{r,s}(\Omega)}.\end{aligned}$$

Summing up over j, we obtain

$$\|u\|^2 \leq \varepsilon\, Q(u, u) + C_\varepsilon \, \|u\|^2_{W^{-1}_{r,s}(\Omega)}\,. \tag{4.2}$$

Thus the proof follows by using Theorem 2.1 and the compactness estimate (4.2). $\square$

Theorem 4.2. *Denote by* Ω, E *and* X *as in Theorem* 4.1. *A compactness estimate* (4.2) *implies boundedness of the* $\overline{\partial}$*-Neumann operator* $N_{r,s}$ *in* $W^k_{r,s}(\Omega, E)$ *for any* $k > 0$.

Proof. By a standard fact of elliptic regularization, one sees that the boundary global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ holds if

$$\|u\|_{W^k} \lesssim \|\Box_{r,s} u\|_{W^k},$$

for any $u \in C^\infty_{r,s}(\overline{\Omega}, E) \cap \operatorname{dom} \Box_{r,s}$ and for any positive integer k. As in the proof of Theorem 4.1, let U be a small neighborhood of a given boundary point $\xi_0 \in b\Omega$ satisfies $U \Subset V \Subset U_{j_\nu}$, for a certain $j_\nu \in I$. If $u \in \mathcal{D}_{r,s}(\Omega, E)$, $0 \le r \le n$, $q \le s \le n-2$, on applying the estimate (3.2) to each u^a and adding for $a = 1, \dots, p$, one gets compactness estimate for $u|_{\Omega\cap U}$

$$\|\zeta_0 u\|_{W^k} \lesssim \|\Box_{r,s}\zeta_0 u\||_{W^k}.$$

Similarly, for $j = 1, \dots, m$, one gets compactness estimate for $u|_{\Omega\cap U_j}$

$$\|\zeta_j u\||_{W^k} \lesssim \|\Box_{r,s}\zeta_j u\||_{W^k} \lesssim \|\Box_{r,s} u\||_{W^k}.$$

Summing up over j, we obtain

$$\|u\||_{W^k} \lesssim \|\Box_{r,s} u\||_{W^k}.$$

Thus the proof follows. $\square$

As in Proposition 3.1, one can prove the following proposition.

Proposition 4.1. *Denote by* Ω, E *and* X *as in Theorem* 4.1. *Then the following are equivalent.*

(i) *The compactness estimates are valid.*

(ii) *The embedding of* $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^*$, *with the graph norm*

$$\|u\| + \|\overline{\partial} u\| + \|\overline{\partial}^* u\|$$

into $L^2_{r,s}(\Omega, E)$ *is compact.*

(iii) *The* $\overline{\partial}$*-Neumann operator*

$$N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),$$

for $q \le s \le n-1$ *is compact from* $L^2_{r,s}(\Omega, E)$ *to itself.*

(iv) *The canonical solution operators to* $\overline{\partial}$ *are given by*

$$\overline{\partial}^* N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s-1}(\Omega, E),$$
$$N_{r,s+1}\overline{\partial}^* : L^2_{r,s+1}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),$$

are compact.

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FITTED OPERATOR FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. This study presents a fitted operator numerical method for solving singularly perturbed boundary value problems with integral boundary condition. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, a model problem is considered for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, h. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$ where the classical numerical methods fails to give good result and it also improves the results of the methods existing in the literature.

1. Introduction

Boundary value problems with integral boundary conditions are an important class of problems which arise in various fields of applications such as electro-chemistry, thermo-elasticity, heat conduction, underground water flow and population dynamics, see, for example [12, 17, 19]. In fact, boundary value problems involving integral boundary conditions have received considerable attention in recent years [7, 9, 11] and [13]. For a discussion of existence and uniqueness results and for applications of problems with integral boundary conditions one can refer, [4–8], [10, 11] and the references therein. In [1, 2, 9, 11, 13, 16] it has been considered some approximating or numerical treatment aspects of this kind of problems. However, the methods or algorithms developed so far mainly concerned with the regular cases (i.e., when

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the boundary layers are absent). Boundary value problems with integral boundary conditions in which the leading derivative term is multiplied by a small parameter ε are called singularly perturbed problems with integral boundary conditions. The solutions of such types of problems manifest boundary layer phenomena where the solution changed abruptly. As a result, numerical analysis of singular perturbation cases has been far from trivial because of the boundary layer behavior of the solution. The solutions of the problems with boundary layer undergo rapid changes within very thin layers near the boundary or inside the problem domain [3], [13–15], [18] and hence classical numerical methods for solving such problems are unstable and fail to give good results when the perturbation parameter is small (i.e., for $h \geq \varepsilon$) [18]. Therefore, it is important to develop a numerical method that gives good results for small values of the perturbation parameter where others fails to give good result and convergent independent of the values of the perturbation parameter and the mesh sizes. Hence, this paper proposed a fitted operator numerical method that is simple, stable and uniformly convergent.

2. Statement of the Problem

Consider the following singularly perturbed problem with integral boundary condition

$$\varepsilon y''(x) + a(x)y'(x) = f(x), \quad 0 < x < l, \tag{2.1}$$

with the given conditions

$$y'(0) = \frac{\mu_0}{\varepsilon}, \tag{2.2}$$

$$\int_0^l b(x)y(x)dx = \mu_1, \tag{2.3}$$

where $0 < \varepsilon \ll 1$ is a positive parameter, $0 < a \leq a(x)$, $f(x)$, $b(x)$ are sufficiently smooth functions in the $[0, l]$ and μ_i $(i = 0, 1)$ are given constants. The function $y(x)$ has in general a boundary layer of thickness $O(\varepsilon)$ near $x = 0$.

In this paper, we analyze a fitted finite-difference scheme on uniform mesh for the numerical solution of the problem (2.1)–(2.3). Uniform convergence is proved in the discrete maximum norm. Finally, we formulate the algorithm for solving the discrete problem and give the illustrative numerical results.

3. Properties of Continuous Solution

The differential operator for the problem under consideration is given by

$$L_\varepsilon \equiv \varepsilon \frac{d^2}{dx^2} + \frac{d}{dx},$$

and it satisfies the following minimum principle for boundary value problems (BVPs). The following lemmas [15] are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 3.1 (Continuous minimum principle)**.** *Assume that $v(x)$ is any sufficiently smooth function satisfying $v(0) \geq 0$ and $v(l) \geq 0$. Then $Lv(x) \leq 0$, for all $x \in \Omega = (0, l)$ implies that $v(x) > 0$, for all $x \in \Omega = [0, l]$.*

Proof. Let x^* be such that $v(x^*) = \min_{x\in[0,l]} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0, l\}$. Therefore, $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Moreover, $Lv(x^*) = \varepsilon v''(x^*) + a(x^*)v'(x^*) \geq 0$, which is a contradiction. It follows that $v(x^*) > 0$ and thus $v(x) \geq 0$, for all $x \in [0, l]$. □

The uniqueness of the solution is implied by this minimum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of (2.1)–(2.3) is bounded.

Lemma 3.2. *If y is the solution of the boundary value problem* (2.1)–(2.3) *and $y \in C^2(\Omega)$ then*

$$\|y(x)\| \leq \|f\| + \max\{|y_0|, |y_l|\},$$

where $k = 0, 1, 2, 3$ and $x \in [0, l]$.

Proof. We handle first the case when $k = 0$. Consider the barrier functions defined by

$$\psi^{\pm}(x) = [(l - x)\|f\| + \max\{|y_0|, |y_l|\}] \pm y(x),$$

when $x = 0$, we have

$$\psi^{\pm}(0) = \|f\|l + \max\{|y_0|, |y_l|\} \pm y(0) \geq 0, \quad \text{since} \quad \max\{|y_0|, |y_l|\}] \geq y(0).$$

When $x = l$, we have

$$\psi^{\pm}(l) = (l - l)\|f\| + \max\{|y_0|, |y_l|\} \pm y(l) \geq 0, \quad \text{since} \quad \max\{|y_0|, |y_l|\}] \geq y(l).$$

Now,

$$L_\varepsilon\psi^{\pm}(x) = \varepsilon(\psi^{\pm}(x))'' + (\psi^{\pm}(x))' = \|f\| + \max\{|y_0|, |y_l|\} \pm Ly(x) \leq 0.$$

Applying the minimum principle, we conclude that $\psi^{\pm}(x) \geq 0$, and therefore

$$\|y(x)\| \leq \|f\| + \max\{|y_0|, |y_l|\}.$$

□

The following lemma shows the bound for the derivatives of the solution.

Lemma 3.3. *Let y_ε be the solution of the continuous problem (P_ε). Then, for $k = 0, 1, 2, 3$,*

$$|y_\varepsilon^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right), \quad \textit{for all } x \in [0, l].$$

Proof. The homogeneous differential equation of (2.1) is

$$\varepsilon y''(x) + a(x)y'(x) = 0. \tag{3.1}$$

The characteristic equation of (3.1) becomes

$$\varepsilon m^2 + am = 0 \Rightarrow m = 0 \quad \text{or} \quad m = \frac{-a}{\varepsilon}.$$

The asymptotic solution of (3.1) is given by

$$u(x) = A + B\exp\left(\frac{-a}{\varepsilon}x\right),$$

where A and B are arbitrary constants.

To get the k^{th} derivative of the asymptotic solution of the homogeneous part of (3.1)

$$\begin{aligned}
u'(x) =& C\varepsilon^{-1}\exp\left(\frac{-a}{\varepsilon}x\right),\\
u''(x) =& C\varepsilon^{-2}\exp\left(\frac{-a}{\varepsilon}x\right),\\
u'''(x) =& C\varepsilon^{-3}\exp\left(\frac{-a}{\varepsilon}x\right).
\end{aligned}$$

In general, for $k = 1, 2, 3$

$$u^{(k)}(x) = C\varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right).$$

The reduced problem obtained from (2.1) takes the $a(x)v_0'(x) = f(x)$, where $v_0(0) = y_0$ and has the solution

$$\begin{aligned}
v_0(x) =& y_0 + \int_0^x \frac{f(t)}{a(t)}dt \le |y_0| + \int_0^x \left|\frac{f(t)}{a(t)}\right| dt\\
\le& C + \left|\frac{f(\zeta)}{a(\zeta)}\right| \int_0^x dt \le C + \left|\frac{f(\zeta)}{a(\zeta)}\right| x, \quad x \in (0, l),\\
\le& C,
\end{aligned}$$

from the assumptions on a and f, it is clear that for $k = 0, 1, 2, 3$

$$|v_0^{(k)}(x)| \le C, \quad \text{for all } x \in [0, l].$$

So, from the relation $y_\varepsilon = v_0 + u$ we have $y_\varepsilon^{(k)} = v_0^{(k)} + u^{(k)}$ and from the relation of triangular inequality

$$|y_\varepsilon^{(k)}| \le |v_0^{(k)}| + |u^{(k)}| \le C + C\varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right) \le C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right).$$

Therefore, it is well accepted that the solution of (2.1) has a boundary layer near $x = 0$ and its derivatives satisfy

$$|y_\varepsilon^{(k)}(x)| \le C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right), \quad \text{for all } x \in [0, l]. \qquad \square$$

4. Formulation of the Method

Consider the homogeneous differential equation with constant coefficient $\varepsilon y''(x) + ay'(x) = 0$ whose solution is given by

$$y(x) = A + B\exp\left(\frac{-a}{\varepsilon}x\right), \tag{4.1}$$

where A and B are constants which will be determined depending on the given conditions. Now, dividing the interval $[0, l]$ into N equal parts with constant mesh length $h = \frac{l}{N}$, we obtain $x_i = x_0 + ih$, for $i = 1, 2, \ldots, N$, where $x_0 = 0$, $x_N = l$.

To demonstrate the procedure, we consider (2.1), at discrete nodes x_i

$$\varepsilon y_i''(x) + a_i(x)y_i'(x) = f_i. \tag{4.2}$$

Approximating (4.2) by central difference approximations, we obtain:

$$\varepsilon\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + a_i\frac{y_{i+1} - y_{i-1}}{2h} = f_i. \tag{4.3}$$

Under the assumption that f_i is bounded, introducing the fitting parameter σ onto the higher order difference approximation of (4.3), multiply both sides by h and evaluating its limit gives

$$\sigma = -\frac{\rho a \lim\limits_{h\longrightarrow 0}(y_{i+1} - y_{i-1})}{2\lim\limits_{h\longrightarrow 0}(y_{i+1} - 2y_i + y_{i-1})}, \tag{4.4}$$

where $\rho = \frac{h}{\varepsilon}$.
Evaluating (4.1) at each nodal point x_i, we obtain

$$\begin{cases} \lim\limits_{h\to 0} y_i = A + B\exp(-ai\rho), \\ \lim\limits_{h\to 0} y_{i+1} = A + B\exp(-ai\rho)\exp(-a\rho), \\ \lim\limits_{h\to 0} y_{i-1} = A + B\exp(-ai\rho)\exp(a\rho), \end{cases} \tag{4.5}$$

$$\sigma = -\frac{\rho a \lim\limits_{h\to 0}(y_{i+1} - y_{i-1})}{2\lim\limits_{h\to 0}(y_{i+1} - 2y_i + y_{i-1})} = \frac{a\rho}{2}\coth\left(\frac{a\rho}{2}\right). \tag{4.6}$$

Hence, from (4.3) and (4.6), we get

$$\frac{\varepsilon\sigma}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{a_i}{2h}(y_{i+1} - y_{i-1}) = f_i.$$

This can be rewritten as three term recurrence relation

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{4.7}$$

where

$$\begin{cases} E_i = \dfrac{\varepsilon\sigma}{h^2} - \dfrac{a_i}{2h}, \\ F_i = \dfrac{-2\varepsilon\sigma}{h^2}, \\ G_i = \dfrac{\varepsilon\sigma}{h^2} + \dfrac{a_i}{2h}, \\ H_i = f_i. \end{cases}$$

Since the problem has no Dirichlet boundary conditions, we apply the following two cases, to obtain two equations at each end point.

For $i = 0$, (4.7) becomes

$$E_0 y_{-1} + F_0 y_0 + G_0 y_1 = H_0. \tag{4.8}$$

Here, in (4.8) the term is out of the domain, so that using (2.2) we have

$$y'(0) = \frac{y_1 - y_{-1}}{2h} \Rightarrow y_{-1} = y_1 - 2hy'(0). \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$F_0 y_0 + (E_0 + G_0) y_1 = H_0 + 2hE_0 y'(0). \tag{4.10}$$

For $i = N$ (Simpson's rule) suppose $b(x)y(x)$ is a function defined on the interval $[0, l]$ and let x_i be a uniform partition of with step length h. The composite Simpson's rule approximates the integral of $b(x)y(x)$ by

(4.11)

$$\int_0^l b(x)y(x)dx = \frac{h}{3}\left[b(0)y(0) + b(l)y(l) + 2\sum_{i=1}^{N-1} b(x_{2i})y(x_{2i}) + 4\sum_{i=1}^{N} b(x_{2i-1})y(x_{2i-1}) \right].$$

Using the integral boundary condition given in condition in (2.3), (4.11) can be written as

$$\frac{h}{3}\left[b(0)y(0) + b(l)y(l) + 2\sum_{i=1}^{N-1} b(x_{2i})y(x_{2i}) + 4\sum_{i=1}^{N} b(x_{2i-1})y(x_{2i-1}) \right] = \mu_1. \tag{4.12}$$

Therefore, the problem in (2.1) with the given boundary conditions (2.2) and (2.3), can be solved using the schemes in (4.7), (4.10) and (4.12) which gives $N \times N$ system of algebraic equations.

5. Uniform Convergence Analysis

In this section, we need to show the discrete scheme in (4.7), (4.10) and (4.12) satisfy the discrete minimum principle, uniform stability estimates, and uniform convergence.

Lemma 5.1 (Discrete Minimum Principle)**.** *Let v_i be any mesh function that satisfies $v_0 \geq 0$, $v_N \geq 0$ and $L_\varepsilon v_i \leq 0$, $i = 1, 2, 3, \ldots, N-1$, then $v_i \geq 0$ for $i = 0, 1, 2, \ldots, N$.*

Proof. The proof is by contradiction. Let j be such that $v_j = \min_i v_i$ and suppose that $v_j \leq 0$. Clearly, $j \notin \{0, N\}$, $v_{j+1} - v_j \geq 0$ and $v_j - v_{j-1} \leq 0$.

Therefore,

$$\begin{aligned}
L_\varepsilon v_j =& \varepsilon \left[\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} \right] + a_j \left[\frac{v_{j+1} - v_{j-1}}{2h} \right] \\
=& \frac{\varepsilon}{h^2}[v_{j+1} - 2v_j + v_{j-1}] + \frac{a_j}{2h}[v_{j+1} - v_{j-1}] \\
=& \frac{\varepsilon}{h^2}[(v_{j+1} - v_j) - (v_j - v_{j-1})] + \frac{a_j}{2h}[(v_{j+1} - v_j) + (v_j - v_{j-1})] \\
\geq& 0,
\end{aligned}$$

where the strict inequality holds if $v_{j+1} - v_j > 0$. This is a contradiction and therefore $v_j \geq 0$. Since j is arbitrary, we have $v_i \geq 0$, $i = 0, 1, 2, \ldots, N$. From the discrete minimum principle we obtain an ε-uniform stability property for the operator L_ε^N. $\square$

Lemma 5.2 (Uniform stability estimate). *If ϕ_j is any mesh function such that*

$$\phi_0 = \phi_N = 0,$$

then

$$|\phi_j| \le \frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|, \quad j = 0, 1, 2, \dots, N.$$

Proof. As in [21], we introduce two mesh functions ψ_j^+, ψ_j^- defined by

$$\psi_j^{\pm} = \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_j.$$

It follows that

$$\begin{aligned}\psi^{\pm}(0) &= \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_0\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i| \pm \phi_0\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i|\\ &\ge 0\end{aligned}$$

and

$$\begin{aligned}\psi^{\pm}(N) &= \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_N\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i| \pm \phi_N\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i|\\ &\ge 0,\end{aligned}$$

and for all $j = 1, 2, \dots, N-1$,

$$L_\varepsilon^N \psi_j^{\pm} = \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm L_\varepsilon^N \phi_j \le 0.$$

From discrete minimum principle, if $\psi_0 \ge 0, \psi_N \ge 0$ and $L_\varepsilon^N \psi_j \le 0$, for all $0 < j < N$, then $\psi_j^{\pm} \ge 0$, $0 \le j \le N$. □

We provide above the discrete operator L_ε^N satisfy the minimum principle. Next we analyze the uniform convergence analysis.

Theorem 5.1 (Uniform Convergence). *The numerical solution y^h of (P_ε^h) and the exact solution y of (P_ε) satisfying ε-uniform error estimates, if there exist a positive integer N_0 and positive numbers C and P, all independent of N and ε, such that for all $N \ge N_0$, $|y^h - y|_\Omega^h \le Ch^2$.*

Proof. Consider the convection-diffusion problem of a linear singularly perturbed two-point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x) = f(x), \quad x \in \Omega = (0, l). \tag{5.1}$$

Now, introducing a variable fitting factor (4.6), $\sigma_i = \frac{a_i\rho_i}{2}\coth(\frac{a_i\rho_i}{2})$, in our scheme, we obtain

$$\frac{\sigma_i}{\rho_i}(y_{i+1} - 2y_i + y_{i-1}) + \left(\frac{y_{i+1} - y_{i-1}}{2}\right) = hf_i, \quad \rho_i = \frac{h}{\varepsilon}. \tag{5.2}$$

Multiply both sides of (5.2) by $2\rho_i$ and rearranging, we get

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i, \tag{5.3}$$

where

$$\begin{cases} E_i = 2\sigma_i - \rho_i, \\ F_i = 4\sigma_i, \\ G_i = 2\sigma_i + \rho_i, \\ H_i = -2\rho_i h f_i. \end{cases}$$

Consider the given problem on two distinct meshes with step sizes h and $k = \frac{h}{2}$ which implies the following relations. For the mesh size h

$$\rho_1 = \frac{h}{\varepsilon}, \quad E_1 = 2\sigma_1 - \rho_1, \quad F_1 = 4\sigma_1, \quad G_1 = 2\sigma_1 + \rho_1, \quad \sigma_1 = \frac{\rho_1}{2}\coth\left(\frac{\rho_1}{2}\right).$$

For the mesh size k,

$$\rho_2 = \frac{k}{\varepsilon} = \frac{\rho_1}{2}, \quad E_2 = 2\sigma_2 - \rho_2, \quad F_2 = 4\sigma_2, \quad G_2 = 2\sigma_2 + \rho_2, \quad \sigma_2 = \frac{\rho_1}{4}\coth\left(\frac{\rho_1}{4}\right).$$

For the operator we have

$$L_\varepsilon^h y_{ih}^h = -Ey_{i-1} + Fy_i - Gy_{i+1} = H_i. \tag{5.4}$$

Now, consider the given problem on two mesh sizes h and k of (5.4) as

$$L_\varepsilon^h y_{ih}^h = - E_1 y_{i-1} + F_1 y_i - G_1 y_{i+1} = H_{ih}, \tag{5.5}$$

$$L_\varepsilon^h y_{2ih}^{\frac{h}{2}} = - E_2 y_{2i-2} + F_2 y_{2i} - G_2 y_{2i+2} = H_{2ih}, \tag{5.6}$$

where

$$\begin{aligned} E_1 &= 2\sigma_1 - \rho_1, \quad F_1 = 4\sigma_1, \quad G_1 = 2\sigma_1 + \rho_1, \\ E_2 &= 2\sigma_2 - \rho_2, \quad F_2 = 4\sigma_2, \quad G_2 = 2\sigma_2 + \rho_2. \end{aligned}$$

Similarly, using the second mesh size k, we have

$$L_\varepsilon^{\frac{h}{2}} y_{2i\frac{h}{2}}^{\frac{h}{2}} = - E_2 y_{2i-1} + F_2 y_{2i} - G_2 y_{2i+1} = H_{2i\frac{h}{2}}, \tag{5.7}$$

$$L_\varepsilon^{\frac{h}{2}} y_{(2i+1)\frac{h}{2}}^{\frac{h}{2}} = - E_2 y_{2i} + F_2 y_{2i+1} - G_2 y_{2i+2} = H_{(2i+1)\frac{h}{2}}, \tag{5.8}$$

$$L_\varepsilon^{\frac{h}{2}} y_{(2i-1)\frac{h}{2}}^{\frac{h}{2}} = - E_2 y_{2i-2} + F_2 y_{2i-1} - G_2 y_{2i} = H_{(2i-1)\frac{h}{2}}. \tag{5.9}$$

To eliminate y_{2i+1} using (5.7) and (5.8), we have

$$-G_2^2 y_{2i+2} - F_2 E_2 y_{2i-1} + (F_2^2 - G_2 E_2) y_{2i} = F_2 H_{2ik} + G_2 H_{(2i+1)k}.$$

Thus, we have the values of y_{2i+2} as

$$y_{2i+2} = \frac{-F_2E_2}{G_2^2}y_{2i-1} + \frac{(F_2^2 - G_2E_2)}{G_2^2}y_{2i} - \frac{F_2}{G_2^2}H_{2ik} - \frac{1}{G_2}H_{(2i+1)k}. \tag{5.10}$$

Also,to eliminate y_{2i-1} using (5.7) and (5.9),we have

$$-E_2^2y_{2i-2} + (F_2^2 - E_2G_2)y_{2i} - F_2G_2y_{2i+1} = F_2H_{2ik} + E_2H_{(2i-1)K}.$$

Thus, we have the value of y_{2i-2} as

$$y_{2i-2} = \frac{(F_2^2 - E_2G_2)}{E_2^2}y_{2i} - \frac{-F_2G_2}{E_2^2}y_{2i+1} - \frac{F_2}{E_2^2}H_{2ik} - \frac{1}{E_2}H_{(2i-1)k}. \tag{5.11}$$

Substituting both values of y_{2i+2} and y_{2i-2} from (5.10) and (5.11) into (5.6)

$$\begin{aligned}L_\varepsilon^h y_{2ih}^{\frac{h}{2}} =& - E_2y_{2i-2} + F_2y_{2i} - G_2y_{2i+2}\\ =& - E2\left\{\frac{(F_2^2 - E_2G_2)}{E_2^2}y_{2i} - \frac{F_2G_2}{E_2^2}y_{2i+1} - \frac{F_2}{E_2^2}H_{2ik} - \frac{1}{E_2}H_{(2i-1)k}\right\}\\ &+ F_2y_{2i} - G_2\left\{\frac{-F_2E_2}{G_2^2}y_{2i-1} + \frac{(F_2^2 - G_2E_2)}{G_2^2}y_{2i} - \frac{F_2}{G_2^2}H_{2i} - \frac{1}{G_2}H_{(2i+1)}\right\},\end{aligned}$$

(5.12)

$$\begin{aligned}L_\varepsilon^h y_{2ih}^{\frac{h}{2}} =&\left\{F_2 - \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2 - E_2G_2}{G_2}\right\}y_{2i} + \frac{F_2E_2}{G_2}y_{2i-1}\\ &+ \frac{F_2G_2}{E_2}y_{2i+1} + \left\{\frac{F_2}{E_2} + \frac{F_2}{G_2}\right\}H_{2i} + H_{2i-1} + H_{2i+1}.\end{aligned}$$

Using (5.5) and (5.12)

$$\begin{aligned}&\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right|\\ =&\left|L_\varepsilon^h(y_i^h - y_{2i}^{\frac{h}{2}})(ih)\right| \qquad (5.13)\\ =&\left| - E_1y_{i-1} + F_1y_1 - G_1y_{i+1} - \left\{F_2 - \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2 - E_2G_2}{G_2}\right\}y_{2i}\right.\\ &\left. - \frac{F_2E_2}{G_2}y_{2i-1} - \frac{F_2G_2}{E_2}y_{2i+1} + \left\{\frac{F_2}{E_2} + \frac{F_2}{G_2}\right\}H_{2i} - (H_{2i-1} + H_{2i+1})\right|,\end{aligned}$$

(5.14)

$$\begin{aligned}\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| =&\left| - \left\{F_2 - \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2 - E_2G_2}{G_2}\right\}y_{2i} - \frac{F_2E_2}{G_2}y_{2i-1}\right.\\ &\left. - \frac{F_2G_2}{E_2}y_{2i+1} - \left\{ - 1 + \frac{F_2}{E_2} + \frac{F_2}{G_2}\right\}H_i - \left(H_{i-\frac{h}{2}} + H_{i+\frac{h}{2}}\right)\right|.\end{aligned}$$

Using Taylor series expansion up to third term, we have the following

$$
(5.15)\qquad \begin{cases}
y_{2i+h} = y_{i+\frac{h}{2}} = y_i + \dfrac{h}{2}y_i' + \dfrac{h^2}{8}y_i'' + \dfrac{h^3}{48}y_i''' + O(h^4),\\
H_{i+\frac{h}{2}} = H_i + \dfrac{h}{2}H_i' + \dfrac{h^2}{8}H_i'' + \dfrac{h^3}{48}H_i''' + O(h^4),\\
y_{2i-h} = y_{i-\frac{h}{2}} = y_i - \dfrac{h}{2}y_i' + \dfrac{h^2}{8}y_i'' - \dfrac{h^3}{48}y_i''' + O(h^4),\\
H_{i-\frac{h}{2}} = H_i - \dfrac{h}{2}H_i' + \dfrac{h^2}{8}H_i'' - \dfrac{h^3}{48}H_i''' + O(h^4).
\end{cases}
$$

Now, substituting the expanded parts of (5.15) into (5.14), we get

$$
\begin{aligned}
\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| =& \Bigg|\left\{ - F_2 + \frac{F_2^2 - E_2G_2}{E_2} + \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} y_i\\
&+ \left\{1 - \frac{F_2}{E_2} - \frac{F_2}{G_2} - 2\right\} H_i + \frac{h}{2}\left\{\frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} y_i'\\
&+ \frac{h^2}{8}\left\{ - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} y_i''\\
&- \frac{h^2}{4}H_i' + \frac{h^3}{48}\left\{ - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} y_i'''\Bigg|.
\end{aligned}
$$

For simplicity, let re-write the above equation as

$$
\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| \le |A|y_i + |B|H_i + \frac{h}{2}|D|y_i' + \frac{h^2}{8}|M|y_i'' + K + \frac{h^3}{48}|N|y_i''',
$$

where

$$
\begin{cases}
A = -F_2 + \dfrac{F_2^2 - E_2G_2}{E_2} + \dfrac{F_2^2 - E_2G_2}{E_2} - \dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2},\\
B = -1 - \dfrac{F_2}{E_2} - \dfrac{F_2}{G_2},\\
D = \dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2},\\
M = -\dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2},\\
K = -\dfrac{h^2}{4}H_i',\\
N = -\dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2}.
\end{cases}
$$

Now, when we evaluate the limit of each variables separatly using L'Hospital's rule

$$\lim_{\rho_1\to 0}|A| = \lim_{\rho_1\to 0}\left\{-F_2 + \frac{F_2^2 - E_2G_2}{E_2} + \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} = 0,$$

$$\lim_{\rho_1\to 0}|B|H_i = \left(\lim_{\rho_1\to 0}\left\{-1 - \frac{F_2}{E_2} - \frac{F_2}{G_2}\right\}\right)\left(\lim_{\rho_1\to 0} -2\rho_1 hf\right) = 0,$$

$$\lim_{\rho_1\to 0}|D| = \lim_{\rho_1\to 0}\left\{\frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} \Rightarrow \lim_{\rho_1\to 0}\frac{h}{2}|D| = 0,$$

$$\lim_{\rho_1\to 0}|K| = \frac{h^2}{4}\lim_{\rho_1\to 0} H' \le C_1h^2,$$

$$\lim_{\rho_1\to 0}|M| = \lim_{\rho_1\to 0}\left\{-\frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} = -8 \Rightarrow \frac{h^2}{8}|M|y_i'' \le C_2h^2,$$

$$\lim_{\rho_1\to 0}|N| = \lim_{\rho_1\to 0}\left\{\frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2}\right\} = 0 \Rightarrow \frac{h^3}{48}|N|y_i''' = 0.$$

Therefore,

$$\begin{aligned}\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| &\le |A|y_i + |B|H_i + \frac{h}{2}|D|y_i' + \frac{h^2}{8}|M|y_i'' + K + \frac{h^3}{48}|N|y_i''' \\ &\le 0 + 0 + 0 + C_1h^2 + C_2h^2 + 0 \\ &\le (C_1 + C_2)h^2 \\ &\le Ch^2.\end{aligned}$$

□

Lemma 5.3. *For all $0 < h < h_0$ and for all $\varepsilon > 0$, assume that L^h is stable with stability constant C and that*

$$\max\left\{\left|\left(y^h - y^{\frac{h}{2}}\right)(0)\right|, \left|\left(y^h - y^{\frac{h}{2}}\right)(l)\right|\right\} + C\left|L^h\left(y^h - y^{\frac{h}{2}}\right)\right| \le C_2h^p,$$

then

$$\left|\left(y^h - y^{\frac{h}{2}}\right)(x_i)\right| \le C_2h^p,$$

where C_2 is independent of ε.

Since $\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| \le Ch^2$, we conclude that $\max_{1\le j\le N-1}|y(x_j) - Y(x_j)| \le Ch^2$.

6. Numerical Example and Results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in (2.1)–(2.3).

Example 6.1. Consider the model singularly perturbed boundary value problem

$$\varepsilon y''(x) + y'(x) = 1, \quad 0 < x < 1,$$

subject to the boundary conditions

$$y'(0) = \frac{1}{\varepsilon} \quad \text{and} \quad \int_0^1 y(x)dx = \frac{1}{2}.$$

Having $y_j \equiv y_j^h$ (the approximated solution obtained via fitted operator finite difference method) for different values of h and ε, the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^h) are evaluated using the double mesh principle [15] for fitted operator finite difference methods using formula

$$E_\varepsilon^h := \max_{0\leq j\leq n} |y_j^h - y_{2i}^{2h}|.$$

Further, we will tabulate the ε-uniform error

$$E^N = \max_{0<\varepsilon\leq 1} E_\varepsilon^h.$$

The numerical rate of convergence are computed using the formula [15]

$$r_\varepsilon^h := \frac{\log(E_\varepsilon^h) - \log(E_\varepsilon^{\frac{h}{2}})}{\log(2)}$$

and the ε-uniform rate of convergence is computed using

$$R^N = \frac{\log(E^h) - \log(E^{\frac{h}{2}})}{\log(2)}.$$

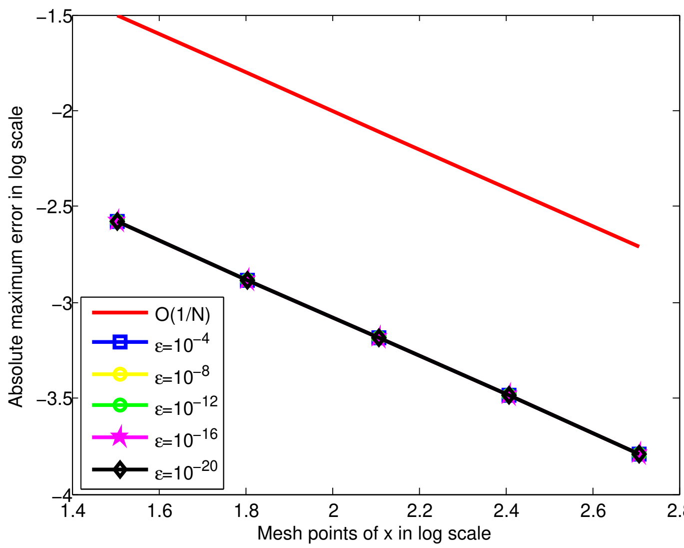

FIGURE 1. ε-uniform convergence with fitted operator in Log-Log scale

TABLE 1. Maximum absolute errors for different values of ε and mesh size, h with fitted parameter (WFP) and without fitted parameter (WOFP) for Example 6.1

ε	64	128	256	512	1024
WFP					
10^{-4}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-8}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-12}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-16}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-20}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
E^N	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
WOFP					
10^{-4}	8.2229e+03	1.8281e+03	4.1007e+02	8.7013e+01	1.7867e+01
10^{-8}	9.1177e+11	2.2841e+11	5.7162e+10	1.4297e+10	3.5745e+09
10^{-12}	9.1178e+19	2.2842e+19	5.7162e+18	1.4298e+18	3.5754e+17
10^{-16}	9.1083e+27	2.2845e+27	5.7140e+26	1.4301e+26	3.5756e+25
10^{-20}	5.7341e+37	7.1295e+36	8.8165e+35	1.0782e+35	2.2303e+34
E^N	5.7341e+37	7.1295e+36	8.8165e+35	1.0782e+35	2.2303e+34

TABLE 2. Maximum absolute errors and rate of convergence of Example 6.1 for different ε and mesh size h

ε	64	128	256	512	1024
WFP					
10^{-4}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-8}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-12}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-16}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-20}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
E^N	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
R^N	1.7267	1.5726	1.4023	1.2526	

7. DISCUSSION AND CONCLUSION

This study introduces fitted operator numerical method for solving singularly perturbed boundary value problems with integral boundary conditions. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh by introducing the fitting operator

TABLE 3. ε-uniform Maximum absolute errors and ε-uniform rate of convergence for Example 6.1

ε	N=64	N=128	N=256	N=512
Present method				
E^N	0.0026454	0.0013159	0.00065359	0.00032616
R^N	1.0074	1.0096	1.0028	
Method in[20]				
E^N	0.0273271	0.0155869	0.00852830	0.00032616
R^N	0.81	0.87	0.97	

in to the higher order finite difference approximation used to replace the derivatives in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, a model problem/example is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see tables 1-3) and compared with the results of the previously developed numerical methods existing in the literature (Table 3). Further, the ε-uniform convergence of the method is shown by the log-log plot of the ε-uniform error (Figure 1). In a concise manner, the present method approximates the exact solution very well for reasonable value of the mesh size, $h \geq \varepsilon$, where existing classical numerical methods fails to give good results. Moreover, the method is convergent independent of the perturbation parameter ε and mesh size h and it improves the results of the methods developed so far for solving the problem under consideration.

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