Kragujevac Journal of Mathematics
Volume 47(4) (2023), Pages 627–636.

COMPACTNESS ESTIMATE FOR THE $\overline{\partial}$-NEUMANN PROBLEM ON A Q-PSEUDOCONVEX DOMAIN IN A STEIN MANIFOLD

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Abstract. We consider a smoothly bounded q-pseudoconvex domain Ω in an n-dimensional Stein manifold X and suppose that the boundary $b\Omega$ of Ω satisfies $(q-P)$ property, which is the natural variant of the classical P property. Then, one prove the compactness estimate for the $\overline{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev k-space. Applications to the boundary global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev k-space are given. Moreover, we prove the boundary global regularity of the $\overline{\partial}$-operator on Ω.

1. Introduction and main results

The existence and regularity properties of the solutions of the system of Cauchy-Riemann equations $\bar{\partial} f = g$ on strongly pseudo-convex domains have been a central theme in the theory of several complex variables for many years. Classically many different approaches have been used: a) Vanishing of the $\bar{\partial}$-cohomology group, b) The abstract L^2-theory of the $\bar{\partial}$-Neumann problem, and c) The construction of rather explicit integral solution operators for $\bar{\partial}$, in analogy to the Cauchy transform in C^1. The first approach used by Grauert−Riemenschneider [6]. Saber [15], used this method and studied the solvability of the $\overline{\partial}$-problem with C^∞ regularity up to the boundary on a strictly q-convex domain of an n-dimensional Kähler manifold X. The second approach was first used by Kohn [11] in studying the boundary regularity of the $\overline{\partial}$-equation when Ω is pseudoconvex with C^∞ boundary. For solvability with regularity up to the boundary in a pseudoconvexity domain without corners, one refer to Kohn

Key words and phrases. Stein manifold, q-pseudoconvex domain, compactness estimate, $\overline{\partial}$-operator, $\overline{\partial}$-Neumann operator.
2010 *Mathematics Subject Classification.* Primary: 32F10. Secondary: 32W05.
DOI 10.46793/KgJMat2304.627S
Received: April 27, 2020.
Accepted: October 12, 2020.

[12]. Zampieri [18] introduced a new type of notion of q-pseudoconvexity in $\mathbb{C}^n$. Under this condition he proved local boundary regularity for any degree $\geq q$. Other results in this direction belong to Heungju [10], Baracco-Zampieri [1] and Saber [16]. Thus the method of L^2 a priori estimates for the weighted $\overline{\partial}$-Neumann operator has yielded many important results on the local and global boundary regularity of the $\overline{\partial}$-problem. The integral formula approach was pioneered by Henkin [8] and Grauert-Lieb [7] for strictly pseudoconvex domains. They obtained uniform and Hölder estimates for the solution of $\overline{\partial}$ on such domains. For the related results for $\overline{\partial}$ on the pseudoconcave domains in $\mathbb{P}^n$, see Henkin-Iordan [9].

In this paper, he compactness estimate proved in Khanh and Zampieri [17] is extended E-valued forms. Such compactness estimates immediately lead to very important qualitative properties of the $\overline{\partial}$-operator, such as smoothless of solutions and closed range. The main theorem generalizes Khanh and Zampieri [17] result to forms with values in a vector bundle. The proof starting with the known estimate on scalar differential forms and then obtains a similar estimate locally on bundle-valued forms using a local frame. Then, by using a partition of unity, we globalize this estimate at the cost of the constants. Consequently, we study the boundary regularity of the $\overline{\partial}$-equation, $\overline{\partial}u = f$, for forms in a vector bundle on bounded q-pseudoconvex domain Ω in a Stein manifold X of dimension n. Moreover, some standard consequences of compactness are deduced.

2. $(q - P)$ PROPERTY

Let Ω be a bounded domain of $\mathbb{C}^n$ with C^1-boundary $b\Omega$ and ρ its a C^1-defining function. An (r, s)-form on Ω is given by

$$f = \sum_{\substack{|I|=r \\ |J|=s}}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ are multiindices and $dz^I = dz_1 \wedge \cdots \wedge dz_r$, $d\bar{z}^J = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_s$. Here, the coefficients $f_{I,J}$ are functions (belonging to various function classes) on Ω. Then for two (r, s)-forms

$$f = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

$$g = \sum_{I,J}{}' g_{I,J} dz^I \wedge d\bar{z}^J.$$

One defines the inner product and the norm as

$$(f, g) = \sum_{I,J}{}' f_{I,J} \, \overline{g_{I,J}},$$

$$|f| = (f, f).$$

The notation $\sum'$ means the summation over strictly increasing multiindices. This definition is independent of the choice of the orthonormal basis. Denote by $C^\infty_{r,s}(\overline{\Omega})$

the space of complex-valued differential forms of class C^∞ and of type (r,s) on Ω that are smooth up to the boundary and $\mathcal{D}_{r,s}(U)$ denotes the elements in $C^\infty_{r,s}(\overline{\Omega})$ that are compactly supported in $U \cap \overline{\Omega}$. $L^2_{r,s}(\Omega)$ consists of the (r,s)-forms u satisfies

$$\|u\|^2 = \sum_{\substack{|I|=r \\ |J|=s}}{}' |u_{I,J}|^2 dV < \infty.$$

Let

$$\overline{\partial} : L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega)$$

be the maximal closed extension and

$$\overline{\partial}^* : L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega)$$

its Hilbert space adjoint. The Laplace-Beltrami operator $\Box_{r,s}$ is defined as

$$\Box_{r,s} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \text{dom } \Box_{r,s} \to L^2_{r,s}(\Omega).$$

Let

$$\mathcal{H}^{r,s} = \{\varphi \in \text{dom}\,\overline{\partial} \cap \text{dom}\,\overline{\partial}^\star : \overline{\partial}\varphi = 0 \text{ and } \text{dom}\,\overline{\partial}^\star\varphi = 0\}.$$

One defines the $\overline{\partial}$-Neumann operator

$$N : L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega),$$

as the inverse of the restriction of $\Box_{r,s}$ to $(\mathcal{H}^{r,s})^\perp$. For nonnegative integer k, one defines the Sobolev k-space

$$W^k_{r,s}(\Omega) = \{f \in L^2_{r,s}(\Omega) : \|f\|_k < +\infty\},$$

where the Sobolev norm of order k is defined as

$$\|f\|^2_{W^k} = \int_\Omega \sum_{|\alpha| \le k} |D^\alpha f|^2 dV,$$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_{2n}}\right)^{\alpha_{2n}}, \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_{2n}),\ |\alpha| = \sum \alpha_j,$$

and $x_1, \ldots, x_{2n}$ are real coordinates for Ω. Detailed information on Sobolev spaces may be found for example in [4], [5]. Let p be a point in the boundary of Ω. Then one can choose a neighborhood U of p and a local coordinate system $(x_1, \ldots, x_{2n-1}, \rho) \in \mathbb{R}^{2n-1} \times \mathbb{R}$, satisfies the last coordinate is a local defining function of the boundary. Call $(U, (x, \rho))$ a special boundary chart. Denote the dual variable of x by ξ, and define

$$\langle x, \xi \rangle = \sum_{j=1}^{2n-1} x_j \xi_j.$$

The tangential Fourier transform for $f \in \mathcal{D}(\overline{\Omega} \cap U)$ is given in this special boundary chart by

$$\tilde{f}(\xi, \rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i \langle x, \xi \rangle} f(x, \rho) dx,$$

where $dx = dx_1 \cdots dx_{2n-1}$. For each $k \geq 0$, the standard tangential Bessel potential operator Λ^k of order k (see e.g., Chen-Shaw [4], Section 5.2) is defined as

$$(\Lambda^k f)(x,\rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i\langle x,\xi\rangle} (1+|\xi|^2)^{\frac{k}{2}} \tilde{f}(\xi,\rho) d\xi.$$

The tangential L^2-Sobolev norm of f of order k is defined as

$$|||f|||^2_{W^k_{r,s}(\Omega)} = \|\Lambda^k f\|^2 = \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} (1+|\xi|^2)^k \, |\tilde{f}(\xi,\rho)|^2 d\xi \, d\rho.$$

Clearly, for $k > 0$, this norm is weaker than the full L^2-Sobolev norm of order k, since it just measures derivatives in the tangential directions.

Let $T^{\mathbb{C}} b\Omega$ be the complex tangent bundle to the boundary, $L_{b\Omega} = (\rho_{ij})|_{T^{\mathbb{C}} b\Omega}$ the Levi form of $b\Omega$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ are the ordered eigenvalues of $L_{b\Omega}$. For every positive number M and if $\varphi^M \in C^\infty(\overline{\Omega} \cap V)$, one denote by

$$\lambda_1^{\varphi^M} \leq \lambda_2^{\varphi^M} \leq \cdots \leq \lambda_{n-1}^{\varphi^M}$$

the ordered eigenvalues of the Levi form (φ^M_{ij}). Choose an orthonormal basis of $(1,0)$-forms $\omega_1, \omega_2, \ldots, \omega_n = \partial\rho$ and the dual basis of $(1,0)$-vector fields $L_1, L_2, \ldots, L_n$; note that $T^{1,0}\partial\Omega = \text{Span}\{L_1, L_2, \ldots, L_{n-1}\}$. We denote by ρ_j and ρ_{jj} the coefficients of $\partial\rho$ and $\partial\overline{\partial}\rho$ in this basis. Following Khanh and Zampieri [17] in Section 2, we have the following definitions.

Definition 2.1. $b\Omega$ is called q-pseudoconvex in a neighborhood V of z_0 if there exist a bundle $\Xi \subset T^{1,0}b\Omega$ of rank $q_0 < q$ with smooth coefficients in V, say the bundle of the first q_0 vector fields $L_1, \ldots, L_{q_0}$ of our basis of $T^{1,0}b\Omega$, satisfies

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} \rho_{jj} \geq 0 \quad \text{on } b\Omega \cap V. \tag{2.1}$$

Since $\sum_{j=1}^{q_0} \rho_{jj}$ is the trace of the restricted form $(\rho_{jj})|_\Xi$, then Definition 2.1 depends only on the choice of the bundle Ξ, not of its basis. Condition (2.1) is equivalent to

$$\sum_{\substack{|I|=r\\|K|=s-1}}{}' \sum_{i,j=1}^{n-1} \rho_{ij} u_{IiK} \overline{u}_{IjK} - \sum_{j=1}^{q_0} \rho_{jj} |u|^2 \geq 0, \tag{2.2}$$

for any (r,s) form u with $s \geq q$. It is in this form that (2.1) will be applied. In some case, it is better to consider instead of (2.2), the variant

$$\sum_{\substack{|I|=r\\|K|=s-1}}{}' \sum_{i,j=1}^{n-1} \rho_{ij} u_{IiK} \overline{u}_{IjK} - \sum_{\substack{|I|=r\\|K|=s-1}}{}' \sum_{j=1}^{q_0} \rho_{jj} |u_{IjK}|^2 \geq 0. \tag{2.3}$$

It is obvious that if $L_{b\Omega}|_{\Xi}$ is assumed to be diagonal, instead of less than or equalo to 0, then the left side of (2.3) equals

$$\sum_{\substack{|I|=r\\|K|=s-1}}{}' \sum_{i,j=q_0+1}^{n-1} \rho_{ij} u_{IiK}\overline{u}_{IjK}.$$

Thus, if $\daleth$ is the Levi-orthogonal complement of Ξ, then (2.3) is equivalent to $L_{b\Omega}|_{\daleth} \geq 0$. The condition in the definition below generalizes to domains which are not necessarily pseudoconvex, the celebrated P property by Catlin [3].

Definition 2.2. $b\Omega$ is said to has the $(q-P)$ property in V if for every positive number M there exists a function $\varphi^M \in C^\infty(\overline{\Omega}\cap V)$ with

(i) $|\varphi^M| \leq 1$ on Ω;

(ii) $\sum_{j=1}^{q} \lambda_j^{\varphi^M} - \sum_{j=1}^{q_0} \varphi^M_{jj} \geq c\sum_{j=1}^{q_0} |\varphi_j^M|^2$ on $\overline{\Omega}\cap V$;

(iii) $\sum_{j=1}^{q} \lambda_j^{\varphi^M} - \sum_{j=1}^{q_0} \varphi^M_{jj} \geq M$ on $b\Omega\cap V$,

where the constant $c>0$ does not depend on M. (The point here is that (ii) holds in the whole $\overline{\Omega}$, (iii) only on $b\Omega$.)

There are obvious variants of (ii) and (iii) adapted to (2.3). Condition (iii) is a modification of (ii) in Definition 2 of [14]. The bigger flexibility of our condition consists in allowing subtraction of φ^M_{jj} for $j=1,\dots,q_0$. We say that a compact subset $F\subset b\Omega\cap V$ satisfies $(q-P)$ if and only if (iii) holds for any $z\in F$.

Theorem 2.1 ([13])**.** *Let X be a complex manifold of complex dimension n with a Hermitian metric g and Ω be a bounded domain of X. Let $\Omega \Subset X$ be an submanifold with smooth boundary. Suppose the compactness estimate* (3.1) *holds on Ω. Suppose further that the $\bar\partial$-closed (r,s)-form α is in $W^k(\Omega)$ and $\alpha \perp \mathcal{H}^{r,s}$, there exists a constant C_k so that the canonical solution u of $\bar\partial u=\alpha$, with $u\perp \ker\bar\partial$ satisfies*

$$\|u\|_{W^k} \leq C_k(\|\alpha\|_{W^k} + \|u\|).$$

Since $C^\infty(\overline{\Omega}) = \cap_{k=0}^\infty W^k(\Omega)$, it follows that if $\alpha\in C^\infty_{r,s}(\overline{\Omega})$, then $u\in C^\infty_{r,s-1}(\overline{\Omega})$.

3. Solvability of $\bar\partial$ in $\mathbb{C}^n$

Following Khanh and Zampieri [17], one obtains the following theorem.

Theorem 3.1. *Let Ω be a smoothly bounded q-pseudoconvex domain in $\mathbb{C}^n$, and suppose that $b\Omega$ satisfies property $(q-P)$ in a neighborhood V of z_0. Then for every $\epsilon>0$ there exists a function $C_\varepsilon \in \mathcal{D}(\Omega)$ satisfying*

$$\|u\|^2 \leq \varepsilon\, Q(u,u) + C_\varepsilon \|u\|^2_{W^{-1}_{r,s}(\Omega)}, \tag{3.1}$$

for $u\in \mathcal{D}_{r,s}(\overline{\Omega}\cap V)\cap \operatorname{dom}\bar\partial^*$ and for any $s\geq q$. Here

$$Q(u,u) = \|\bar\partial u\|^2 + \|\bar\partial^\star u\|^2 + \|u\|^2,$$

and $\|u\|_{W^{-1}_{r,s}(\Omega)}$ is the Sobolev norm of order -1.

The same statement holds if q-pseudoconvexity is understood in the sense of the variant (2.3) and if $(q-P)$ property has the corresponding variants.

Definition 3.1. We will refer to (3.1) as a compactness estimate.

Theorem 3.2. *Let* Ω *be the same as in Theorem* 3.1. *Then, for* $f \in C^{\infty}_{r,s}(\overline{\Omega})$, $q \le s \le n-2$, *satisfying* $\overline{\partial} f = 0$, *there exists* $u \in C^{\infty}_{r,s-1}(\overline{\Omega})$, *satisfies* $\overline{\partial} u = f$.

Proof. The proof follows from the estimate (3.1) and Theorem 2.1. ☐

Property $(q-P)$ is related to the $\overline{\partial}$-Neumann problem by the following theorem.

Remark 3.1. It is easy to observe that (3.1) implies for $u \in \mathrm{dom}\square_{r,s}$:

$$\|u\|^2 \le \varepsilon \,\|\square_{r,s} u\|^2 + C_\varepsilon \,\|u\|^2_{W^{-1}_{r,s}(\Omega)}.$$

We now discuss the global regularity for $N_{r,s}$. From the estimate (3.1) one can derive a priori estimates for $N_{r,s}$ in the Sobolev k-space.

Theorem 3.3. *Let* Ω *be the same as in Theorem* 3.1. *A compactness estimate implies boundedness of the* $\overline{\partial}$*-Neumann operator* $N_{r,s}$ *in* $W^k_{r,s}(\Omega)$ *for any* $k > 0$.

Proof. By a standard fact of elliptic regularization, one sees that the global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ holds if

$$\|u\|_{W^k_{r,s}(\Omega)} \lesssim \|\square_{r,s} u\|_{W^k_{r,s}(\Omega)}, \tag{3.2}$$

for any $u \in C^{\infty}_{r,s}(\overline{\Omega}) \cap \mathrm{dom}\,\square_{r,s}$. Hence,

$$\|u\|^2_{W^k_{r,s}(\Omega)} \lesssim \|\square_{r,s} u\|_{W^{k-2}_{r,s}(\Omega)} + \|\Lambda^{k-1} D u\|^2, \tag{3.3}$$

where Λ is the tangential differential operator of order k. By Theorem 3.1, the estimate (3.1) implies that

$$\left\|D\Lambda^{-1} u\right\|^2 \lesssim Q(u,u) + C\,\|u\|^2_{W^{-1}_{r,s}(\Omega)}. \tag{3.4}$$

In fact, it follows by the non-characteristic with respect to the boundary of $\bar{L}_n$; the operator D can be understood as D_r or Λ.

Now we estimate the last term of (3.3), we have

$$\begin{aligned}
\|\Lambda^{k-1} D u\|^2 \lesssim & \|D\Lambda^{-1}\Lambda^k u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & Q(\Lambda^k u, \Lambda^k u) + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & < \Lambda^k \square_{r,s} u, \Lambda^k u > + \|[\overline{\partial}, \Lambda^k] u\|^2 + \|[\overline{\partial}^*, \Lambda^k] u\|^2 \\
& + \|[\overline{\partial}^*, [\overline{\partial}, \Lambda^k] u\|^2 + \|[\overline{\partial}, [\overline{\partial}^*, \Lambda^k] u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & \|\Lambda^k \square_{r,s} u\|^2 + \|\Lambda^{k-1} D u\|^2 + \|\Lambda^{k-2} D^2 u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)} \\
\lesssim & \|\square_{r,s} u\|^2_{W^k_{r,s}(\Omega)} + \|\Lambda^{k-1} D u\|^2 + C\|u\|^2_{W^{k-1}_{r,s}(\Omega)},
\end{aligned}$$

where the second inequality follows by (3.4). Then the term $\|\Lambda^{k-1}Du\|^2$ can be absorbed by the left-hand side term. By induction method, we obtain the estimate (3.2). □

Proposition 3.1. *Let Ω be the same as in Theorem 3.1. Then the following are equivalent.*

(i) *The validity of global compactness estimates.*

(ii) *The embedding of the space* $\operatorname{dom}\overline{\partial}\cap\operatorname{dom}\overline{\partial}^*$*, provided with the graph norm*

$$\|u\|+\|\overline{\partial}u\|+\|\overline{\partial}^*u\|,$$

into $L^2_{r,s}(\Omega)$ is compact.

(iii) *The $\overline{\partial}$-Neumann operators*

$$N_{r,s}:L^2_{r,s}(\Omega,E)\to L^2_{r,s}(\Omega,E),$$

for $q\le s\le n-1$ are compact from $L^2_{r,s}(\Omega)$ to itself.

(iv) *The canonical solution operators to $\overline{\partial}$ given by*

$$\overline{\partial}^*N_{r,s}:L^2_{r,s}(\Omega)\to L^2_{r,s-1}(\Omega),$$
$$N_{r,s+1}\overline{\partial}^*:L^2_{r,s+1}(\Omega)\to L^2_{r,s}(\Omega),$$

are compact.

Proof. The equivalence of (ii) and (i) is a result of Lemma 1.1 in [13]. The general L^2-theory and the fact that $L^2_{r,s}(\Omega)$ embeds compactly into $W^{-1}_{r,s}(\Omega)$ shows that (iii) is equivalent to (ii) and (i). Finally, the equivalence of (iii) and (iv) follows from the formula

$$N_{r,s}=(\overline{\partial}^*N_{r,s})^*\overline{\partial}^*N_{r,s}+\overline{\partial}^*N_{r,s+1}(\overline{\partial}^*N_{r,s+1})^*,$$

(see [4], page 55). We refer the reader to [14] for similar calculations. □

4. Solvability of $\overline{\partial}$ in Stein manifold

Let X be complex manifold of complex dimension n with a Hermitian metric g and Ω be a bounded domain of X. Let $\pi:E\to X$ be a vector bundle, of rank p, over X with Hermitian metric h. Let $\{U_j\}$, $j\in J$, be an open covering of X by charts with coordinates mappings $z_j:U_j\to\mathbb{C}^n$ satisfies $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j)=U_j\times\mathbb{C}$, and $(z_j^1,z_j^2,\dots,z_j^n)$ be local coordinates on U_j. Let $\{\zeta_j\}_{j\in J}$ be a partition of unity subordinate to the holomorphic atlas (U_j,z_j), of X. We denote by T_zX the tangent bundle of X at $z\in X$. An E-valued differential (r,s)-form u on X is given locally by a column vector $u=(u^1,u^2,\dots,u^p)$, where u^a, $1\le a\le p$, are $\mathbb{C}$-valued differential forms of type (r,s) on X. The spaces $C^\infty_{r,s}(X,E)$, $\mathcal{D}_{r,s}(X,E)$, $C^\infty_{r,s}(\overline{\Omega},E)$, $\mathcal{D}_{r,s}(\Omega,E)$ and $W^k_{r,s}(\Omega,E)$ are defined as in Section 2 but for E-valued forms. Let $L^2_{r,s}(\Omega,E)$ be the Hilbert space of E-valued differential forms u on Ω, of type (r,s), satisfies

$$\|u\|_\Omega=\sum_j\sum_{a=1}^{p}\|u_j^a\|_{U_j\cap\Omega}<\infty,$$

where $\|u_j^a\|_{U_j\cap\Omega}$ is defined in (2.1). Let $\overline{\partial} : L^2_{r,s}(\Omega, E) \to L^2_{r,s+1}(\Omega, E)$ be the maximal closed extension of the original $\overline{\partial}$ and $\overline{\partial}^* : L^2_{r,s}(\Omega, E) \to L^2_{r,s-1}(\Omega, E)$ its Hilbert space adjoint. For $k \in \mathbb{R}$, we define a $W^k(X, E)$-norm by the following:

$$\|u\|^2_{k(X)} := \sum_j \|\zeta_j u_j\|^2_{k(W_j)}, \tag{4.1}$$

where $W_j = z_j(U_j)$ and $\sum_j \|\zeta_j u_j\|^2_{k(W_j)}$ is defined as in the Euclidean case.

Theorem 4.1. *Let Ω be a smoothly bounded q-pseudoconvex domain in an of n-dimensional Stein manifold X, $n \geq 3$, and suppose that $b\Omega$ satisfies property $(q - P)$ in a neighborhood V of z_o. Let E be a vector bundle, of rank p, on X. Then, for $f \in C^\infty_{r,s}(\overline{\Omega}, E)$, $q \leq s \leq n - 2$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X, there exists $u \in C^\infty_{r,s-1}(\overline{\Omega}, E)$, satisfies $\overline{\partial} u = f$ in the distribution sense in X.*

Proof. Let $\{U_j\}_{j=1}^N$ be a finite covering of $b\Omega$ by a local patching. Let $e_1, e_2, \ldots, e_p$ be an orthonormal basis on $E_z = \pi^{-1}(z)$, for every $z \in U_j$, $j \in J$. Thus, every E-valued differential (r, s)-form u on X can be written locally, on U_j, as

$$u(z) = \sum_{a=1}^p u^a(z)\, e_a(z),$$

where u^a are the components of the restriction of u on U_j. Since $b\Omega$ is compact, there exists a finite number of elements of the covering $\{U_j\}$, say, U_j, $j = 1, 2, \ldots, m$ satisfies $\bigcup_{\nu=1}^m U_{j_\nu}$ cover $b\Omega$. Let $\{\zeta_j\}_{j=0}^m$ be a partition of the unity satisfies $\zeta_0 \in \mathcal{D}_{r,s}(\Omega)$, $\zeta_j \in \mathcal{D}_{r,s}(U_j)$, $j = 1, 2, \ldots, m$, and

$$\sum_{j=0}^m \zeta_j^2 = 1 \quad \text{on } \overline{\Omega},$$

where $\{U_j\}_{j=1,\ldots,m}$ is a covering of $b\Omega$. Let U be a small neighborhood of a given boundary point $\xi_0 \in b\Omega$ satisfies $U \Subset V \Subset U_{j_\nu}$, for a certain $j_\nu \in I$. If $u \in \mathcal{D}_{r,s}(\Omega, E)$, $0 \leq r \leq n$, $q \leq s \leq n - 2$, on applying the compactness estimate of Khanh and Zampieri [17] to each u^a and adding for $a = 1, \ldots, p$, one gets compactness estimate for $u|_{\Omega\cap U}$

$$\begin{aligned}\|\zeta_0 u\|^2 &\lesssim \epsilon Q(\zeta_0 u, \zeta_0 u) + C_\epsilon \|\zeta_0 u\|^2_{W^{-1}_{r,s}(\Omega)}\\ &\lesssim \epsilon Q(u, u) + C_\epsilon \|u\|_{W^{-1}_{r,s}(\Omega)}.\end{aligned}$$

Similarly, for $j = 1, \ldots, m$, we obtain compactness estimate for $u|_{\Omega\cap U_j}$

$$\begin{aligned}\|\zeta_j u\|^2 &\lesssim \epsilon Q(\zeta_j u, \zeta_j u) + C_\epsilon \|\zeta_j u\|^2_{W^{-1}_{r,s}(\Omega)}\\ &\lesssim \epsilon Q(u, u) + C_\epsilon \|u\|^2_{W^{-1}_{r,s}(\Omega)}.\end{aligned}$$

Summing up over j, we obtain

$$\|u\|^2 \leq \varepsilon\, Q(u, u) + C_\varepsilon\, \|u\|^2_{W^{-1}_{r,s}(\Omega)}\,. \tag{4.2}$$

Thus the proof follows by using Theorem 2.1 and the compactness estimate (4.2). □

Theorem 4.2. *Denote by* Ω*,* E *and* X *as in Theorem* 4.1*. A compactness estimate* (4.2) *implies boundedness of the* $\overline{\partial}$*-Neumann operator* $N_{r,s}$ *in* $W^k_{r,s}(\Omega, E)$ *for any* $k > 0$*.*

Proof. By a standard fact of elliptic regularization, one sees that the boundary global regularity for the $\overline{\partial}$-Neumann operator $N_{r,s}$ holds if

$$\|u\|_{W^k} \lesssim \|\Box_{r,s}u\|_{W^k},$$

for any $u \in C^\infty_{r,s}(\overline{\Omega}, E) \cap \operatorname{dom} \Box_{r,s}$ and for any positive integer k. As in the proof of Theorem 4.1, let U be a small neighborhood of a given boundary point $\xi_0 \in b\Omega$ satisfies $U \Subset V \Subset U_{j_\nu}$, for a certain $j_\nu \in I$. If $u \in \mathcal{D}_{r,s}(\Omega, E)$, $0 \le r \le n$, $q \le s \le n-2$, on applying the estimate (3.2) to each u^a and adding for $a = 1, \dots, p$, one gets compactness estimate for $u|_{\Omega \cap U}$

$$\|\zeta_0 u\|_{W^k} \lesssim \|\Box_{r,s}\zeta_0 u\||_{W^k}.$$

Similarly, for $j = 1, \dots, m$, one gets compactness estimate for $u|_{\Omega \cap U_j}$

$$\|\zeta_j u\||_{W^k} \lesssim \|\Box_{r,s}\zeta_j u\||_{W^k} \lesssim \|\Box_{r,s}u\||_{W^k}.$$

Summing up over j, we obtain

$$\|u\||_{W^k} \lesssim \|\Box_{r,s}u\||_{W^k}.$$

Thus the proof follows. □

As in Proposition 3.1, one can prove the following proposition.

Proposition 4.1. *Denote by* Ω*,* E *and* X *as in Theorem* 4.1*. Then the following are equivalent.*

(i) *The compactness estimates are valid.*

(ii) *The embedding of* $\operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$*, with the graph norm*

$$\|u\| + \|\overline{\partial}u\| + \|\overline{\partial}^* u\|$$

into $L^2_{r,s}(\Omega, E)$ *is compact.*

(iii) *The* $\overline{\partial}$*-Neumann operator*

$$N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),$$

for $q \le s \le n-1$ *is compact from* $L^2_{r,s}(\Omega, E)$ *to itself.*

(iv) *The canonical solution operators to* $\overline{\partial}$ *are given by*

$$\overline{\partial}^* N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s-1}(\Omega, E),$$
$$N_{r,s+1}\overline{\partial}^* : L^2_{r,s+1}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),$$

are compact.

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