ON TWO PEXIDERIZED FUNCTIONAL EQUATIONS OF DAVISON TYPE

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Abstract. In this paper, we present the general solution of two Pexiderized functional equations of Davison type without assuming any regularity assumption on the unknown functions.

1. Introduction

In 1979, during the 17th International Symposium on Functional Equations (ISFE), Davison [2] introduced the following functional equation

\[ f(xy) + f(x + y) = f(xy + x) + f(y), \]

where the domain and range of \( f \) is a (commutative) field. At ISFE 17th Benz [1] determined the continuous solution of Davison functional equation. Indeed, he proved that if \( f : \mathbb{R} \rightarrow \mathbb{R} \), then every continuous solution of the equation (1.1) is of the form \( f(x) = ax + b \), where \( a \) and \( b \) are real constants. In 2000, Girgensohn and Lajkó [3] obtained the general solution of the Davison equation without any regularity assumption. They showed that the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfies the functional equation (1.1) for all \( x, y \in \mathbb{R} \) if and only if \( f \) is of the form \( f(x) = A(x) + b \), where \( A : \mathbb{R} \rightarrow \mathbb{R} \) is an additive function and \( b \) is an arbitrary real constant. For more on Davison functional equation (1.1) and its stability interested readers should referred to the book [5] and references therein. In [4] we studied the following functional

Key words and phrases. Additive mapping, functional equations of Davison type, Hosszu’s functional equation, Hyers-Ulam stability.

2010 Mathematics Subject Classification. Primary: 39B52. Secondary: 39B22.

DOI 10.46793/KgJMat2304.539N

Received: May 21, 2020.

Accepted: September 23, 2020.
Theorem 2.1. The functions \( f, g, h, k : \mathbb{R} \to \mathbb{X} \) satisfy the functional equation (1.7) for all \( x, y \in \mathbb{R} \) if and only if they have the form \( f(x) = A(x) + b_1, g(x) = A(x) + b_2, h(x) = A(x) + b_3 \) and \( k(x) = A(x) + b_4 \), where \( A : \mathbb{R} \to \mathbb{X} \) is additive and \( b_1, b_2, b_3, b_4 \in \mathbb{X} \) are constants with \( b_1 + b_2 = b_3 + b_4 \).

Proof. Sufficiency is obvious. Let \( f, g, h, k : \mathbb{R} \to \mathbb{X} \) satisfy (1.7). Substituting \( x = 0 \), \( y = 0 \) and \( y = 1 \), respectively, in (1.7), we get

\[
\begin{align*}
(2.1) & \quad f(y) + g(0) = h(0) + k(y), \\
(2.2) & \quad f(x) + g(0) = h(x) + k(0), \\
(2.3) & \quad f(x + 1) + g(-x) = h(0) + k(1).
\end{align*}
\]

If we use these equations in (1.7), we obtain

\[
(2.4) \quad f(x + y) - f(1 + xy) = f(x - xy) + f(y) + 2g(0) - 2h(0) - k(0) - k(1),
\]
for all \( x, y \in \mathbb{R} \). Letting \( x = 1 \) in (2.4), we obtain
\[
f(1 - y) = -f(y) - 2g(0) + 2h(0) + k(0) + k(1) \quad (y \in \mathbb{R}).
\]
Hence,
\[(2.5) \quad f(1 + xy) = -f(-xy) - 2g(0) + 2h(0) + k(0) + k(1) \quad (x, y \in \mathbb{R}).\]
It follows from (2.4) and (2.5) that
\[
f(x + y) + f(-xy) = f(x - xy) + f(y) \quad (x, y \in \mathbb{R}).
\]
Therefore \( f \) is of the form \( f(x) = A(x) + b_1 \), where \( A : \mathbb{R} \rightarrow X \) is additive and \( b_1 \in X \) is a constant (see [4, Theorem 3.1]). Now we obtain the asserted form of \( g, h \) and \( k \) by using (2.1), (2.2) and (2.3). The proof of the theorem is now complete. \( \square \)

**Lemma 2.1.** Let \( f : \mathbb{R} \rightarrow X \) be an odd function. Then \( f \) satisfies
\[(2.6) \quad f(x) + f(y) + f(y + 1) = f(x + y + xy) + f(y - xy + 1) \quad (x, y \in \mathbb{R})\]
if and only if \( f \) is additive.

**Proof.** Sufficiency is clear. Let \( f \) satisfy (2.6). Replacing \( y \) by \( y + 1 \) and \( y - 1 \), respectively, we get
\[
\begin{align*}
(2.7) \quad f(x) + f(y + 1) + f(y + 2) &= f(2x + y + xy + 1) + f(y - x - xy + 2), \\
(2.8) \quad f(x) + f(y - 1) + f(y) &= f(y + xy - 1) + f(x + y - xy),
\end{align*}
\]
for all \( x, y \in \mathbb{R} \). Interchanging \( x \) and \( y \) in (2.8), we see that
\[
(2.9) \quad f(y) + f(x - 1) + f(x) = f(x + xy - 1) + f(x + y - xy) \quad (x, y \in \mathbb{R}).
\]
Subtracting (2.9) from (2.8), we get
\[
(2.10) \quad f(y - 1) - f(x - 1) = f(y + xy - 1) - f(x + xy - 1) \quad (x, y \in \mathbb{R}).
\]
Replacing \( x \) by \( x + 1 \) and \( y \) by \( y + 1 \), respectively, in (2.10), we have
\[
(2.11) \quad f(y) - f(x) = f(2y + x + xy + 1) - f(2x + y + xy + 1) \quad (x, y \in \mathbb{R}).
\]
Adding the equations (2.7) and (2.11), we have
\[
(2.12) \quad f(y) + f(y + 1) + f(y + 2) = f(2y + x + xy + 1) + f(y - x - xy + 2),
\]
for all \( x, y \in \mathbb{R} \). Let \( u, v \in \mathbb{R} \) with \( u + v \neq -2 \). Setting \( x = \frac{v - u}{2 + u + v} \) and \( y = \frac{u + v}{2} \) in (2.12), we get
\[
(2.13) \quad f \left( \frac{u + v}{2} \right) + f \left( \frac{u + v}{2} + 1 \right) + f \left( \frac{u + v}{2} + 2 \right) = f \left( \frac{u + v}{2} + v + 1 \right) + f(u + 2).
\]
If \( u + v = -2 \), then (2.13) reduces to \( f(-1) + f(0) + f(1) = f(v) + f(-v) \), which holds automatically, since \( f \) is odd. Thus, (2.13) is true for all \( u, v \in \mathbb{R} \). Replacing \( v \) by \( v - u \) in (2.13), we have
\[
(2.14) \quad f \left( \frac{u}{2} \right) + f \left( \frac{u}{2} + 1 \right) + f \left( \frac{u}{2} + 2 \right) = f \left( \frac{3v - 2u}{2} + 1 \right) + f(u + 2).
\]
Replacing $u$ by $u-2$ and $v$ by $-\frac{2}{3}v$ in (2.14), we have
\[ f\left(\frac{-v}{3}\right) + f\left(\frac{-v}{3}+1\right) + f\left(\frac{-v}{3}+2\right) = f(3-(u+v)) + f(u). \]

This functional equation is a Pexider functional equation of the form
\[ (2.15) \quad F(x) = G(x+y) + H(y) \quad (x,y \in \mathbb{R}), \]
where
\[ F(t) := f\left(\frac{-t}{3}\right) + f\left(\frac{-t}{3}+1\right) + f\left(\frac{-t}{3}+2\right), \]
\[ G(t) := f(3-t), \]
\[ H(t) := f(t). \]

It is easy to show that (2.15) implies $H(x+y) = H(x) + H(y)$ for all $x,y \in \mathbb{R}$ since $G(x) = F(0) - H(x)$, $F(x) = F(0) - H(x)$ and $H(0) = 0$. Hence, $H$ is additive and thus $f$ is additive. The proof of the lemma is now complete. \hfill \Box

**Theorem 2.2.** The functions $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy (1.8) for all $x, y \in \mathbb{R}$ if and only if they have the form $f(x) = A(x) + b_1$, $g(x) = A(x) + b_2$, $h(x) = A(x) + b_3$, $k(x) = A(x) + b_4$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b_1, b_2, b_3, b_4 \in \mathbb{X}$ are constants with $2b_1 + 2b_2 = b_3 + b_4$.

**Proof.** Sufficiency is clear. Let $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy (1.8). Setting $x = 0$, $y = 0$ and $y = 1$, respectively, in (1.8), we get
\[ (2.16) \quad 2f(0) + 2g(y) = h(y) + k(y), \]
\[ (2.17) \quad 2f(x) + 2g(0) = h(x) + k(x), \]
\[ (2.18) \quad 2f(x) + 2g(1) = h(2x + 1) + k(1). \]

Therefore from (2.16) and (2.17), we obtain
\[ (2.19) \quad f(x) - f(0) = g(x) - g(0) \quad (x \in \mathbb{R}). \]

Replacing $y$ by $2y+1$ in (1.8), we have
\[ (2.20) \quad 2f(x) + 2g(2y+1) = h(2x + 2y + 2xy + 1) + k(2y - 2xy + 1) \quad (x, y \in \mathbb{R}). \]

Using (2.18) and (2.19) in (2.20), we get
\[ (2.21) \quad 2f(x) + 2f(2y+1) = 2f(x+y+xy) + k(2y - 2xy + 1) + 2f(0) - 2g(0) \]
\[ + 2g(1) - k(1), \]
for all $x, y \in \mathbb{R}$. It follows from (2.17) that
\[ 2f(2y - 2xy + 1) + 2g(0) = h(2y - 2xy + 1) + k(2y - 2xy + 1) \quad (x, y \in \mathbb{R}). \]

Using (2.18) in this equation, we have
\[ k(2y - 2xy + 1) = 2f(2y - 2xy + 1) - 2f(y - xy) \]
\[ + 2g(0) - 2g(1) + k(1) \quad (x, y \in \mathbb{R}). \]
Using this equation in (2.21), we get
\begin{equation}
(2.22) \quad f(x) + f(2y + 1) - f(0) = f(x + y + xy) + f(2y - 2xy + 1) - f(y - xy),
\end{equation}
for all \( x, y \in \mathbb{R} \). Letting \( x = -1 \) and replacing \( y \) by \( \frac{1}{2}y \) in (2.22), we have
\begin{equation}
(2.23) \quad f(2y + 1) = f(y + 1) + f(y) - f(0) \quad (y \in \mathbb{R}).
\end{equation}
Replacing \( y \) by \( y - xy \) in (2.23), we obtain
\begin{equation}
(2.24) \quad f(2y - 2xy + 1) = f(y - xy + 1) + f(y - xy) - f(0) \quad (y \in \mathbb{R}).
\end{equation}
Using this equation directly in the right-hand side of (2.22) and using (2.23) in the left-hand side of (2.22), we get
\begin{equation}
(2.25) \quad f(x) + f(y) - f(0) = f(x + y + xy) + f(2y - 2xy + 1) - f(y + 1),
\end{equation}
for all \( x, y \in \mathbb{R} \). Since the left-hand side of (2.25) is symmetric in \( x \) and \( y \), we get
\begin{equation}
(2.26) \quad f(x + 2y - 2xy) - f(2y) = f(2(x - xy) + 1) - f(x + 1).
\end{equation}
Using (2.23) in this equation, we have
\begin{equation}
(2.27) \quad f(x + 2y - 2xy) - f(2y) = f(x - xy + 1) + f(x - xy) - f(x + 1) - f(0),
\end{equation}
for all \( x, y \in \mathbb{R} \). Using (2.25) in (2.26), we have
\begin{equation}
(2.28) \quad f(2y) = f(1 + y) - f(1 - y) + f(0) \quad (y \in \mathbb{R}).
\end{equation}
Replacing \( y \) by \( -y \) in (2.28) and adding the obtained equation to (2.28), we get
\begin{equation}
(2.29) \quad f(2y) + f(-2y) = 2f(0) \quad (y \in \mathbb{R}).
\end{equation}
Hence, \( f - f(0) \) is odd. Since \( f \) satisfies (2.24), \( f - f(0) \) satisfies (2.6). Therefore, \( f - f(0) \) is additive by Lemma 2.1. Thus, \( f(x) = A(x) + b_1 \), where \( A : \mathbb{R} \to \mathbb{X} \) is an additive function and \( b_1 \in \mathbb{X} \) is a constant. Now, using (2.19), (2.18) and (2.17), we obtain the asserted form of \( g, h \) and \( k \). This finishes the proof of the theorem. \( \square \)

3. Open problems

In this section, we pose two open problems. Determine the general solution \((f, g, h, k)\) of the functional equations (1.7) and (1.8), respectively, where the domain and range of the unknown functions \( f, g, h, k \) are (commutative) fields. It should be noted that our arguments are not valid in Theorems 2.1 and 2.2 if the field characteristic (in domain) is equal to 2 or 3.
Acknowledgements. We wish to thank the anonymous reviewers whose comments helped us improve the presentation of the paper.

REFERENCES