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SOME INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree n which has no zeros in $|z| < 1$, then it was proved by Liman, Mohapatra and Shah [11] that

$$\begin{aligned} &\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \\ \leq &\frac{n}{2}\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\max_{|z|=1}|P(z)| \\ &-\frac{n}{2}\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right| - \left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\min_{|z|=1}|P(z)|, \end{aligned}$$

for any β with $|\beta| \leq 1$ and $|z| = 1$. In this paper we generalize the above inequality and our result also generalizes certain well known polynomial inequalities.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most n. If $P \in \mathcal{P}_n$, then according to Bernstein theorem [5], we have

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

Bernstein proved it in 1912. Later, in 1930 he [6] revisited his inequality and proved the following result from which inequality (1.1) can be deduced for $Q(z) = Mz^n$, where $M = \max_{|z|=1}|P(z)|$.

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Theorem 1.1. *Let $P(z)$ and $Q(z)$ be two polynomials with degree of $P(z)$ not exceeding that of $Q(z)$. If $P(z)$ has all its zeros in $|z| \leq 1$ and*

$$|P(z)| \leq |Q(z)|, \quad \textit{for } |z| = 1,$$

then

$$|P'(z)| \leq |Q'(z)|, \quad \textit{for } |z| = 1 \tag{1.2}$$

More generally, it was proved by Malik and Vong [12] that for any β with $|\beta| \leq 1$, inequality (1.2) can be replaced by

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \left|zQ'(z) + \frac{n\beta}{2}Q(z)\right|, \quad \text{for } |z| = 1. \tag{1.3}$$

By restricting the zeros of a polynomial, the maximum value may be smaller. Indeed, if $P \in \mathcal{P}_n$ has no zero inside the unit circle $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Inequality (1.4) was conjectured by Erdös and later proved verified by Lax [10]. This result was further improved by Aziz and Dawood [2] who, under the same hypothesis, proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.$$

Jain [8] generalized the inequality (1.4) and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $|z| = 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2} \left\{ \left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right| \right\} \max_{|z|=1} |P(z)|. \tag{1.5}$$

As a refinement of (1.5), Deewan and Hans [7] proved the following.

Theorem 1.2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$*

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2} \left[\left\{ \left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right| \right\} \max_{|z|=1} |P(z)| - \left\{ \left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right| \right\} \min_{|z|=1} |P(z)| \right].$$

Let $D_\alpha P(z)$ be an operator that carries n^{th} degree polynomial $P(z)$ to the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z), \quad \alpha \in \mathbb{C},$$

of degree at most $(n-1)$. $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz was among the first to extend these results to polar derivatives. It is proved by Aziz [1] that for $P \in \mathcal{P}_n$ having no zeros in $|z| < 1$ and $|\alpha| \geq 1$,

$$|D_\alpha P(z)| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1) \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1.$$

As an extension of (1.1) for the polar derivative Aziz and Shah [4] proved the following.

Theorem 1.3. *If $P(z)$ is a polynomial of degree n, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$*

$$|D_\alpha P(z)| \leq n|\alpha z^{n-1}| \max_{|z|=1} |P(z)|, \quad \textit{for } |z| \geq 1.$$

Liman et al. [11] extended (1.3) to the polar derivative and proved the following result.

Theorem 1.4. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$,*

$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) P(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) Q(z) \right|, \quad \textit{for } |z| \geq 1.$$

2. Main Result

In this paper, we first prove the following result which is generalization of Theorem 1.4 and also obtain some compact generalization for polar derivative.

Theorem 2.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$,*

(2.1)
$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n} \right) P(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n} \right) Q(z) \right|, \quad \textit{for } |z| \geq k.$$

Remark 2.1. For $k = 1$, Theorem 2.1 reduces to the Theorem 1.4.

Dividing both sides of (2.1) by $|\alpha|$ and letting $|\alpha| \to \infty$ we get following generalization of (1.3).

Corollary 2.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then $\beta \in \mathbb{C}$ with $|\beta| \leq 1$*

$$\left| zP'(z) + \frac{n\beta}{1 + k^n} P(z) \right| \leq \left| zQ'(z) + \frac{n\beta}{1 + k^n} Q(z) \right|, \quad \textit{for } |z| \geq k.$$

By applying Theorem 2.1 to the polynomials $P(z)$ and $Q(z) = M\frac{z^n}{k^n}$, where $M = \max_{|z|=k} |P(z)|$, we get the following result.

Corollary 2.2. *If $P(z)$ is a polynomial of degree n, then for any α, β, with $|\alpha|\geq k$, $|\beta|\leq 1$ and $|z|\geq k$*

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|\leq n\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|M.$$

By applying Theorem 2.1 to the polynomials $P(z)$ and $Q(z)=m\frac{z^n}{k^n}$, where $m=\min_{|z|=k}|P(z)|$, we get the following result.

Corollary 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then for any α, β with $|\alpha|\geq k$, $|\beta|\leq 1$ and $|z|\geq k$*

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|\geq n\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m.$$

Theorem 2.2. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z|\leq k$, $k\geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)|\leq|Q(z)|$ for $|z|=k$, then for all $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$*

$$|zD_\alpha P(z)|+n\left(\frac{|\alpha|-k}{1+k^n}\right)|Q(z)|\leq|zD_\alpha Q(z)|+n\left(\frac{|\alpha|-k}{1+k^n}\right)|P(z)|. \tag{2.2}$$

Dividing both sides of (2.2) by α and letting $|\alpha|\to\infty$, we get the following result.

Corollary 2.4. *Let $Q(z)$ be a polynomial of degree n having all its zeros $|z|\leq k$, $k\geq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)|\leq|Q(z)|$ for $|z|=k$, then for $|z|=1$*

$$\left|\frac{P'(z)}{n}\right|+\left|\frac{Q(z)}{1+k^n}\right|\leq\left|\frac{Q'(z)}{n}\right|+\left|\frac{P(z)}{1+k^n}\right|.$$

Theorem 2.3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<k$, $k\geq 1$ then for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\geq k$, $|\beta|\leq 1$ and for $|z|\geq k$*

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|\\ \leq&\frac{n}{2}\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|\\ &-\frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|-\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}m,\end{aligned} \tag{2.3}$$

where $m=\min_{|z|=k}|P(z)|$.

Dividing both sides of (2.3) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following generalization of a result due to Dewan and Hans [7].

Corollary 2.5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and for $|z| \geq k$*

$$\left|zP(z) + \frac{n\beta}{1+k^n}P(z)\right| \leq \frac{n}{2}\left\{|z|^n\left|1+\frac{\beta}{1+k^n}\right| + k^n\left|\frac{\beta}{1+k^n}\right|\right\}\max_{|z|=1}|P(z)| - \frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|1+\frac{\beta}{1+k^n}\right| - \left|\frac{\beta}{1+k^n}\right|\right\}m,$$

where $m = \min_{|z|=k}|P(z)|$.

3. Lemma

For the proofs of these theorems we need the following lemmas. The first lemma which we need is due to Laguerre (see [9, page 38]).

Lemma 3.1. *If all the zeros of an n^{th} degree polynomial $P(z)$ lie in a circular region C and w is any zero of $D_\alpha P(z)$, then at most one of the points w and α may lie outside C.*

Lemma 3.2. *Let A and B be any two complex numbers, then the following holds.*

(i) *If $|A| \geq |B|$ and $B \neq 0$, then $A \neq vB$ for all complex numbers v with $|v| < 1$.*
(ii) *Conversely, if $A \neq vB$ for all complex number v with $|v| < 1$, then $|A| \geq |B|$.*

Lemma 3.2 is due to Xin Li [13].

Lemma 3.3. *If $P(z)$ is a polynomial of degree n, then for $k \geq 1$*

$$\max_{|z|=k}|P(z)| \leq k^n \max_{|z|=1}|P(z)|.$$

Lemma 3.3 is simple consequence of maximum modulus theorem.

Lemma 3.4. *If the polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$*

$$n(|\alpha| - k)|P(z)| \leq (1 + k^n)|D_\alpha P(z)|.$$

Lemma 3.4 is due to Aziz and Rather [3].

Lemma 3.5. *If $P(z)$ is a polynomial of degree n, then for any α with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = k$*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| + \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| \leq n\left\{|z|^n\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| + k^n\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|,$$

where $q(z) = \left(\frac{z}{k}\right)^n \overline{P\left(\frac{k^2}{\bar{z}}\right)}$.

Proof. Let $M = \max_{|z|=k} |P(z)|$. An application of *Rouche's Theorem* shows that all the zeros of the polynomial $G(z) = k^n P(z) + \lambda M z^n$ lie in $|z| < k$, $k \geq 1$ for every λ with $|\lambda| > 1$. If $H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\overline{z}}\right)} = k^n Q(z) + \overline{\lambda} M k^n$, then $|G(z)| = |H(z)|$ for $|z| = k$ and hence for any γ with $|\gamma| < 1$ the polynomial $\gamma H(z) + G(z)$ has all its zeros in $|z| < k$, $k \geq 1$. By applying Lemma 3.4, we have for any α with $|\alpha| \geq k$

$$(1 + k^n)|z(\gamma D_\alpha H(z) + D_\alpha G(z))| \geq n(|\alpha| - k)|\gamma H(z) + G(z)|.$$

Since $\gamma H(z) + G(z) \neq 0$ for $|z| \geq k$, $k \geq 1$, so the right hand side is non zero. Thus, by using (i) of Lemma 3.2 we have for all β satisfying $|\beta| < 1$ and for $|z| \geq k$

$$T(z) = \beta n(|\alpha| - k)\gamma H(z) + G(z) + (1 + k^n)z(\gamma D_\alpha H(z) + D_\alpha G(z)) \neq 0,$$

or, equivalently, for $|z| \geq k$

$$T(z) = \gamma(1 + k^n)zD_\alpha H(z) + n\beta(|\alpha| - k)H(z) + (1 + k^n)zD_\alpha G(z) + n\beta(|\alpha| - k)G(z) \neq 0.$$

Using (ii) of Lemma 3.2 we have for $|\gamma| < 1$ and for $|z| \geq k$

$$|(1 + k^n)zD_\alpha H(z) + n\beta(|\alpha| - k)H(z)| \leq |(1 + k^n)zD_\alpha G(z) + n\beta(|\alpha| - k)G(z)|. \tag{3.1}$$

Now by putting $G(z) = k^n P(z) + \lambda M z^n$ and $H(z) = k^n Q(z) + \overline{\lambda} M k^n$ in (3.1) we get

$$\begin{aligned} &\left| zD_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| - n|\lambda| \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M \\ \leq &\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) + n\lambda \frac{z^n}{k^n} \left(\alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right)\right) M \right|. \end{aligned} \tag{3.2}$$

By Corollary 2.2, it is possible to choose the argument of λ such that

$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right| = n\frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M. \tag{3.3}$$

Using (3.3) in (3.2) and letting $|\lambda| \to 1$ we get for $|\alpha| > k$ and $|\beta| < 1$

$$\begin{aligned} &\left| zD_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| - n \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M \\ \leq &n\frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| M - \left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right|. \end{aligned}$$

That is

$$\begin{aligned} &\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z) \right| + \left| zD_\alpha q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) q(z) \right| \\ \leq &n \left\{ \frac{|z|^n}{k^n} \left| \alpha + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| + \left| z + \beta \left(\frac{|\alpha| - k}{1 + k^n}\right) \right| \right\} M. \end{aligned} \tag{3.4}$$

Using Lemma 3.3 in (3.4) we get

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|+\left|zD_\alpha q(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|\\ \leq& n\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|.\end{aligned}$$

That proves Lemma 3.5 completely. □

4. Proof of Theorems

Proof of Theorem 2.1. By *Rouche's Theorem*, the polynomial $\lambda P(z)-Q(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$ for $|\lambda|<1$. Therefore, for $r>1$, all the zeros of $\lambda P(rz)-Q(rz)$ lie in $|z|\leq \frac{k}{r}<k$. By applying Lemma 3.4 to the polynomial $\lambda P(rz)-Q(rz)$, we have for $|z|=1$

$$n(|\alpha|-k)|\lambda P(rz)-Q(rz)|\leq(1+k^n)|z(\lambda D_\alpha P(rz)-D_\alpha Q(rz))|.$$

As in the proof of Lemma 3.1, we have for $|\beta|<1$ and for $|z|\geq k$

$$(1+k^n)z\{\lambda D_\alpha P(rz)-D_\alpha Q(rz)\}+n\beta(|\alpha|-k)\{\lambda P(rz)-Q(rz)\}\neq 0.$$

This implies for $|z|\geq k$

$$|(1+k^n)zD_\alpha P(rz)+n\beta(|\alpha|-k)P(rz)|\leq|(1+k^n)zD_\alpha Q(rz)+n\beta(|\alpha|-k)Q(rz)|. \tag{4.1}$$

Now making $r\to 1$ and using the continuity for $|\beta|$ in (4.1), the theorem follows. □

Proof of Theorem 2.2. Since all the zeros of $Q(z)$ lie in $|z|\leq k$, $k\geq 1$, we have for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$

$$(1+k^n)|zD_\alpha Q(z)|\geq n(|\alpha|-k)|Q(z)|.$$

This gives for every β with $|\beta|\leq 1$

$$(1+k^n)|zD_\alpha Q(z)|-n|\beta|(|\alpha|-k)|Q(z)|\geq 0.$$

Therefore, it is possible to choose the argument of β in the right hand side of Theorem 2.1 such that

$$|(1+k^n)zD_\alpha Q(z)-n\beta(|\alpha|-k)Q(z)|=|zD_\alpha Q(z)|-n|\beta|\left(\frac{|\alpha|-k}{1+k^n}\right)|Q(z)|. \tag{4.2}$$

Using (4.2) in Theorem 2.1 and letting $|\beta|\to 1$, we get the desired result. □

Proof of Theorem 2.3. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z|\leq k$, $k\geq 1$. If $q(z)=\left(\frac{z}{k}\right)^n\overline{P\left(\frac{k^2}{\bar z}\right)}$, then $q(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$

and $|P(z)| = |q(z)|$ for $|z| = k$. Hence, by Theorem 2.1, we have for all α, β satisfying $|\alpha| \geq k$, $|\beta| \leq 1$

(4.3)
$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| \leq \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|, \quad \text{for } |z| \geq k.$$

Let $m = \min_{|z|=k}|P(z)|$. If $P(z)$ has a zero in $|z| = k$, then $m = 0$ and result follows by combining Lemma 3.5 with (4.3). Therefore, we suppose that all the zeros of $P(z)$ lie in $|z| > k$ and so $m > 0$. We have $|\gamma m| < |P(z)|$ on $|z| = k$ for any γ with $|\gamma| < 1$. By *Rouche's Theorem* the polynomial $F(z) = P(z) + \gamma m$ has no zeros in $|z| < k$. Therefore, the polynomial $G(z) = \left(\frac{z}{k}\right)^n \overline{F\left(\frac{k^2}{\bar{z}}\right)} = q(z) - \bar{\gamma}m\frac{z^n}{k^n}$ will have all its zeros in $|z| \leq k$. Also $|F(z)| = |G(z)|$ for $|z| = k$. On applying Theorem 2.1, we get for any β, α with $|\beta| \leq 1$, $|\alpha| \geq k$

$$\left|zD_\alpha F(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)F(z)\right| \leq \left|zD_\alpha G(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)G(z)\right|, \quad \text{for } |z| \geq k.$$

Equivalently,

(4.4)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - n|\gamma|\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m\\ \leq&\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z) - n\bar{\gamma}\frac{z^n}{k^n}\left(\alpha + \left(\frac{|\alpha|-k}{1+k^n}\right)\right)m\right|.\end{aligned}$$

Since $q(z)$ has all its zeros in $|z| \leq k$ and $\min_{|z=k|}|p(z)| = \min_{|z=k|}|q(z)| = m$, therefore, by Corollary 2.3, we have

(4.5)
$$\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| \geq n\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m.$$

Therefore, we can write (4.4) in view of (4.5) as

(4.6)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - n\left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m\\ \leq&\left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right| - n\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|m.\end{aligned}$$

Letting $|\gamma| \to 1$, we get from inequality (4.6)

(4.7)
$$\begin{aligned}&\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right| - \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|\\ \leq& -n\left\{\frac{|z|^n}{k^n}\left|\alpha + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right| - \left|z + \beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}m.\end{aligned}$$

Now, by Lemma 3.5, we have

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|+\left|zD_\alpha q(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)q(z)\right|\\ \le& n\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|.\end{aligned}\tag{4.8}$$

Inequalities (4.7) and (4.8) together lead to

$$\begin{aligned}&\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-k}{1+k^n}\right)P(z)\right|\\ \le&\frac{n}{2}\left\{|z|^n\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|+k^n\left|z+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|\right\}\max_{|z|=1}|P(z)|\\ &-\frac{n}{2}\left\{\frac{|z|^n}{k^n}\left|\alpha+\beta\left(\frac{|\alpha|-k}{1+k^n}\right)\right|-\left|z+\beta\frac{(|\alpha|-k)}{1+k^n}\right|\right\}m.\end{aligned}$$

That proves Theorem 2.3 completely. □

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