Kragujevac Journal of Mathematics
Volume 47(4) (2023), Pages 599–612.

NUMERICAL SOLUTION OF LINEAR VOLTERRA INTEGRAL EQUATIONS USING NON-UNIFORM HAAR WAVELETS

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Abstract. In this paper, we presented a numerical method for solving linear Volterra integral equations (LVIE) which is based on the non-uniform Haar wavelets. By applying this method, the LVIE is reduced to a linear system of algebraic equations which can be solved by direct method. The min advantage of using non-uniform Haar wavelets is that the time of calculation can be adjusted arbitrarily. Also, we presented the error analysis of the proposed method. Furthermore, two examples are included for the demonstrating the convenient capabilities of the new method.

1. Introduction

Types of equations such as linear and nonlinear integral equations appear in many problems such as finance, mathematics, etc. [8–18]. In the recent years, a lot of numerical methods have been developed for solving these types of equations [1–7]. Since, obtaining the analytical solution of these equations is generally not possible, it is important to develop the numerical method to solve these equations. For this aim, we consider the linear Volterra integral equation of the second kind

$$Y(t) = K(t) + \lambda \int_0^t Y(s)P(s,t)ds, \tag{1.1}$$

where $Y(t)$ is the unknown function, $P(s,t)$ is the known function and the parameter λ is known. In the proposed method, first we transform Volterra integral equation (1.1) to a system of linear algebraic equations with the unknown non-uniform Haar

Key words and phrases. Non-uniform Haar wavelets, Volterra integral equations, grid points, function approximation.

2010 *Mathematics Subject Classification.* Primary: 65T60. Secondary: 45D05, 45L05, 65L60, 65R20, 41A30.

DOI 10.46793/KgJMat2304.599M

Received: November 26, 2019.

Accepted: October 06, 2020.

coefficients. Then, we obtain the unknown non-uniform Haar coefficients by solving this system by the known method. If we want to raise the exactness of the results, we must increase the number of the grid points. In the course of the solution, we have to invert some matrices, but by increasing the number of calculation points these matrices become nearly singular and therefore the inverse matrices cannot be evaluated with necessary accuracy. One possibility to find a way out of these difficulties is to make use of the non-uniform Haar method for which the length of the subintervals is unequal. This idea was proposed in [19]. Also, another advantage of proposed method is simple applicability and their efficiency. This article is organized as follows. In Section 2, we describe the non-uniform Haar wavelet and their property. The Heaviside step function is defined in Section 3. In the next section, we describe function approximation. In Section 5, a numerical method is proposed. In Section 6, we present the error analysis. In Section 7, some numerical examples are given. Finally in Section 8, conclusion is given.

2. Non-Uniform Haar Wavelets

Non-uniform Haar wavelets are characterized by two characteristics: the dilation parameter $j = 0, 1, \ldots, J$ (J is maximal level of resolution) and the translation parameter $k = 0, 1, \ldots, n-1$, where the integer $n = 2^j$. The number of wavelet is identified as $i = n + k + 1$. Also the maximal value is $i = 2N$, where $N = 2^J$.

For getting the grid points in interval $[0, L]$, where L is large enough constant, we consider the length of the c-th subinterval from this interval by $\Delta t_c = t_c - t_{c-1}$, $c = 1, 2, \ldots, 2N$. It is assume that $\Delta t_{c+1} = g\Delta t_c$, where $g > 1$ is a given constant. If we sum all the length of this subintervals, we find:

$$\Delta t_1(1 + g + g^2 + \cdots + g^{2N-1}) = L, \quad L = b - a,$$

or

$$\Delta t_1 = L\frac{g-1}{g^{2N}-1}.$$

Since

$$\tilde{t}(\ell) = \Delta t_1 \left(1 + g + \cdots + g^{\ell-1}\right) = \Delta t_1 \frac{g^\ell - 1}{g-1}, \quad \ell = 1, 2, \ldots, 2N, \tag{2.1}$$

so, we have obtained the grid points as

$$\tilde{t}(\ell) = L\frac{g^\ell - 1}{g^{2N} - 1}, \quad \ell = 1, 2, \ldots, 2N. \tag{2.2}$$

We divide the interval $[a, b]$ into $2N$ subinterval of unequal lengths with the division points $a = \tilde{x}(0) < \tilde{x}(1) < \cdots < \tilde{x}(2N) = b$. By using harmonize grid points, we describe the non-uniform Haar wavelet family based on [19] as follow:

$$H_i(t) = \begin{cases} 1, & \Upsilon_1(i) \le t \le \Upsilon_2(i), \\ -u_i, & \Upsilon_2(i) \le t \le \Upsilon_3(i), \\ 0, & \text{elsewhere}, \end{cases}$$

where

$$\Upsilon_1(i) = \tilde{t}(2k\theta), \quad \Upsilon_2(i) = \tilde{t}((2k+1)\theta), \quad \Upsilon_3(i) = \tilde{t}((2k+2)\theta), \quad \theta = \frac{N}{n},$$

and

$$u_i = \frac{\Upsilon_2(i) - \Upsilon_1(i)}{\Upsilon_3(i) - \Upsilon_2(i)}. \tag{2.3}$$

Clearly, these equations hold when $i > 2$. For the case $i = 1$ and $i = 2$, we have $\Upsilon_1(1) = a$, $\Upsilon_2(1) = \Upsilon_3(1) = b$, $\Upsilon_1(2) = a$, $\Upsilon_2(2) = \frac{\tilde{t}(2N)}{2}$, $\Upsilon_3(2) = b$.

For instance, by using the grid points defined in (2.2), if $g = 2$ the first eight bases non-uniform Haar functions are given by

$$H_1(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{elsewhere}, \end{cases} \quad H_2(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$H_3(t) = \begin{cases} 1, & 0 \le t \le \frac{3}{255}, \\ -\frac{1}{4}, & \frac{3}{255} \le t \le \frac{15}{255}, \\ 0, & \text{elsewhere}, \end{cases} \quad H_4(t) = \begin{cases} 1, & \frac{15}{255} \le t \le \frac{63}{255}, \\ -\frac{1}{4}, & \frac{63}{255} \le t \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$H_5(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{255}, \\ -\frac{1}{2}, & \frac{1}{255} \le t \le \frac{3}{255}, \\ 0, & \text{elsewhere}, \end{cases} \quad H_6(t) = \begin{cases} 1, & \frac{3}{255} \le t \le \frac{7}{255}, \\ -\frac{1}{2}, & \frac{7}{255} \le t \le \frac{15}{255}, \\ 0, & \text{elsewhere}, \end{cases}$$

$$H_7(t) = \begin{cases} 1, & \frac{15}{255} \le t \le \frac{31}{255}, \\ -\frac{1}{2}, & \frac{31}{255} \le t \le \frac{63}{255}, \\ 0, & \text{elsewhere}, \end{cases} \quad H_8(t) = \begin{cases} 1, & \frac{63}{255} \le t \le \frac{127}{255}, \\ -\frac{1}{2}, & \frac{127}{255} \le t \le 1, \\ 0, & \text{elsewhere}. \end{cases}$$

2.1. **Orthogonality property.** We know the Haar wavelet functions are piecewise orthogonal

$$\int_a^b H_r(t)H_s(t)dt = \begin{cases} (b-a)2^{-j}, & r = s, \\ 0, & r \ne s. \end{cases}$$

Similarly, the non-uniform Haar wavelet defined in $[a, b]$ are piecewise orthogonal,

$$\int_a^b H_i(t)H_j(t)dt = \begin{cases} (b-a)T_i, & i = j, \\ 0, & i \ne j, \end{cases} \tag{2.4}$$

with

$$T_i = u_i(\Upsilon_3(i) - \Upsilon_1(i)). \tag{2.5}$$

3. Definition of Heaviside Step Function

In this section, we recall definition of Heaviside step function. Clearly the Heaviside step function, hsf, is defined as follow:

$$hsf(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

with a useful property

$$hsf(t - \Upsilon_1(i))hsf(t - \Upsilon_2(i)) = hsf(t - \max\{\Upsilon_1(i), \Upsilon_2(i)\}), \quad \Upsilon_1(i), \Upsilon_2(i) \in \mathbb{R}.$$

Therefore, by using hsf, we can write $H_0(t) = hsf(t) - hsf(t-1)$,

$$H_\tau(t) = hsf(t - \Upsilon_1(i)) + (-u_i - 1)hsf(t - \Upsilon_2(i)) + u_i hsf(t - \Upsilon_3(i)), \tag{3.1}$$

where

$$\tau = 2^j + k, \quad j, k \in \mathbb{N} \cup \{0\},\ 0 \leq k < 2^j.$$

4. Function Approximation

For each integrable function $Y(t)$ in interval $[0,1]$, we can expand it by the non-uniform Haar wavelets as the following form:

$$Y(t) = q_1 H_1(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^{j-1}} q_{2^j+k+1} H_{2^j+k+1}(t), \quad t \in [0,1],$$

which coefficients q_i given by

$$q_1 = \frac{1}{T_1} \int_0^1 Y(t) H_1(t) dt, \quad q_i = \frac{1}{T_i} \int_0^1 Y(t) H_i(t) dt,$$

where $i = 2^j + k + 1$, $j \geq 0$, $0 \leq k < 2^j$ and $t \in [0,1]$. We can find coefficients q_i such that square error Γ

$$\Gamma = \int_0^1 \left(Y(t) - \sum_{i=1}^{2^j} q_i H_i(t) \right)^2 dt, \quad j \in \mathbb{N} \cup \{0\},$$

is minimized. By using (3.1) the non-uniform Haar coefficients can be rewritten as

$$q_i = \frac{1}{T_i} \left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} Y(t) dt - u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} Y(t) dt \right),$$

where $i = 2^j + k + 1$, $j, k = 0, 1, 2, 3, \dots$, and $0 \leq k < 2^j$. If $Y(t)$ is piecewise constant or can be approximated as piecewise constant during any subinterval, then $Y(t)$ will be terminated at n finite terms. This mean

$$Y(t) \simeq q_1 H_1(t) + \sum_{j=0}^{J} \sum_{k=0}^{2^{j-1}} q_{2^j+k+1} H_{2^j+k+1}(t),$$

rewriting above equation in the vector form, we have

$$Y(t) \simeq q^T H(t) = H^T(t) q(t), \quad t \in [0,1],$$

where the coefficients q^T and the non-uniform Haar functions vector $H(t)$ are defined as

$$q^T = [q_1, q_1, q_1, \dots, q_{2N}],$$
$$H(t) = [H_1(t), H_2(t), H_3(t), \dots, H_{2N}(t)]^T.$$

Therefore, any 2-variable function $p(s,t) \in L^2[0,1) \times L^2[0,1)$ can be expanded respect to the non-uniform Haar wavelets as

$$p(s,t) = H^T(s)pH(t) = H^T(t)p^T H(s),$$

where $H(s)$ and $H(t)$ are the non-uniform Haar wavelet vectors, and p is the $2N \times 2N$ Haar wavelet coefficients matrix which (i,j)-th element can be obtained as

$$\begin{aligned} p_{i,j} =& \frac{1}{T_i^2} \int_0^1 \int_0^1 p(s,t) H_i(s) H_j(t) dtds \\ =& \frac{1}{T_i^2} \left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} \int_{\Upsilon_1(j)}^{\Upsilon_2(j)} p(s,t)dtds - u_j \int_{\Upsilon_1(i)}^{\Upsilon_2(i)} \int_{\Upsilon_2(j)}^{\Upsilon_3(j)} p(s,t)dtds \right. \\ & \left. - u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} \int_{\Upsilon_1(j)}^{\Upsilon_2(j)} p(s,t)dtds + u_i u_j \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} \int_{\Upsilon_2(j)}^{\Upsilon_3(j)} p(s,t)dtds \right), \end{aligned}$$

where $i, j = 1, 2, 3, \dots, 2N$.

5. Description of the Proposed Method

In this section, we apply the operational matrix of integration of the non-uniform Harr wavelets for solving Volterra integral equations. For this purpose we approximate $Y(t)$, $K(t)$, $P(s,t)$ in (1.1) as follows:

$$\begin{aligned} Y(t) &\simeq Y^T H(t) = H^T(t) Y, \\ K(t) &\simeq K^T H(t) = H^T(t) K \end{aligned} \tag{5.1}$$

and

$$P(s,t) \simeq H^T(s) P H(t) = H^T(t) P^T H(s),$$

where Y and K are the non-uniform Haar wavelets coefficients vectors, and P is the non-uniform coefficient matrix. Substituting above approximations in (1.1), we have

$$\begin{aligned} Y^T H(t) &\simeq K^T H(t) + \int_0^t \left(Y^T H(s)\right) \left(H^T(s) P^T H(t)\right) ds \\ &= K^T H(t) + Y^T \left(\int_0^t H(s) H^T(s) ds \right) P^T H(t) \\ &= K^T H(t) + Y^T Q P^T H(t), \end{aligned}$$

where

$$Q = \int_0^t \left(H(s) H^T(s) \right) ds.$$

So, we can write

$$Y^T \left(I - Q P^T \right) \simeq K^T. \tag{5.2}$$

We know that, (5.2) is a linear system of equations for unknown vector Y. After solving this linear system and determining Y, we can approximate solution of Volterra integral equation (1.1) by substituting the obtained vector Y in (5.1).

6. Error Analysis

In this section, we prove the convergence and error analysis of the proposed method for solving Volterra integral equations. To do this, we need the following theorems.

Theorem 6.1. *Suppose that $q(t) \in L^2[0,1)$ is an arbitrary function with bounded first derivative $|q'(t)| \le M$ and we consider error function*

$$e_m(t) = q(t) - \sum_{i=0}^{m-1} q_i H_i(t),$$

where $i = 2^j + k + 1$, $m = 2^{J+1}$, $J > 0$ and

$$q_i = \frac{1}{T_i}\int_0^1 H_i(t)q(t)dt = \frac{1}{T_i}\left(\int_{\Upsilon_1(i)}^{\Upsilon_2(i)} q(t)dt - u_i \int_{\Upsilon_2(i)}^{\Upsilon_3(i)} q(t)dt\right).$$

Then, we have

$$\|e_m\|_2 = O\left(\frac{1}{m}\right).$$

Proof. Clearly, we have

$$\begin{aligned}
\|e_m\|_2^2 &= \int_0^1 \left(q(t) - \sum_{i=0}^{m-1} q_i H_i(t)\right)^2 dt \\
&= \int_0^1 \left(\sum_{i=m}^{\infty} q_i H_i(t)\right)^2 dt \\
&= \sum_{i=m}^{\infty} q_i^2 \int_0^1 H_i^2(t)dt.
\end{aligned}$$

By the mean value theorem for integrals, there are $\alpha_1 \in (\Upsilon_1(i), \Upsilon_2(i))$, $\alpha_2 \in (\Upsilon_2(i), \Upsilon_3(i))$, such that

$$\begin{aligned}
q_i &= \frac{1}{T_i}\left(q(\alpha_1)\left(\Upsilon_2(i) - \Upsilon_1(i)\right) - u_i\left(q(\alpha_2)\left(\Upsilon_3(i) - \Upsilon_2(i)\right)\right)\right) \\
&= \frac{1}{T_i}\left(q(\alpha_1)(\Upsilon_2(i) - \Upsilon_1(i)) - \frac{\Upsilon_2(i) - \Upsilon_1(i)}{\Upsilon_3(i) - \Upsilon_2(i)}(\Upsilon_3(i) - \Upsilon_2(i))q(\alpha_2)\right) \\
&= \frac{1}{T_i}\left((\Upsilon_2(i) - \Upsilon_1(i))(q(\alpha_1) - q(\alpha_2))\right) \\
&= \frac{1}{T_i}\left((\Upsilon_2(i) - \Upsilon_1(i))(\alpha_1 - \alpha_2)q'(\alpha)\right), \quad \alpha \in (\alpha_1, \alpha_2).
\end{aligned}$$

From (2.4) and definitions α_1, α_2, it follows that

$$\|e_m\|_2^2 = \sum_{i=m}^{\infty} \frac{1}{T_i^2} \Big((\Upsilon_2(i) - \Upsilon_1(i))^2 (\alpha_1 - \alpha_2)^2 (q'(\alpha))^2 \Big) T_i$$

$$\leq \sum_{i=m}^{\infty} \frac{1}{T_i} \Big((\Upsilon_2(i) - \Upsilon_1(i))^2 (\Upsilon_3(i) - \Upsilon_1(i))^2 M^2 \Big). \tag{6.1}$$

Now, by definitions of $\Upsilon_1(i)$, $\Upsilon_2(i)$, $\Upsilon_3(i)$ and (2.1), we have

$$\Upsilon_1(i) = \frac{2kN}{n} \Delta t_1,$$
$$\Upsilon_2(i) = \frac{(2k+1)N}{n} \Delta t_1$$

and

$$\Upsilon_3(i) = \frac{(2k+2)N}{n} \Delta t_1.$$

Therefore, we get

$$\Upsilon_2(i) - \Upsilon_1(i) = \frac{N \Delta t_1}{n}, \tag{6.2}$$

$$\Upsilon_3(i) - \Upsilon_1(i) = \frac{2N \Delta t_1}{n} \tag{6.3}$$

and

$$\Upsilon_3(i) - \Upsilon_2(i) = \frac{N \Delta t_1}{n}.$$

Since

$$\Upsilon_2(i) - \Upsilon_1(i) \leq \Upsilon_3(i) - \Upsilon_1(i),$$

we can write

$$\|e_m\|_2^2 \leq M^2 \sum_{i=m}^{\infty} \frac{1}{T_i} \Big((\Upsilon_3(i) - \Upsilon_1(i))^4 \Big). \tag{6.4}$$

Now, by using (2.3) and (2.5), we get

$$\frac{1}{T_i} = \frac{\Upsilon_3(i) - \Upsilon_2(i)}{(\Upsilon_2(i) - \Upsilon_1(i))(\Upsilon_3(i) - \Upsilon_1(i))}, \tag{6.5}$$

with

$$(\Upsilon_3(i) - \Upsilon_2(i)) \leq (\Upsilon_3(i) - \Upsilon_1(i)),$$

and applying (6.2) in (6.5), we can write

$$\frac{1}{T_i} \leq \frac{1}{(\Upsilon_2(i) - \Upsilon_1(i))} = \frac{1}{\frac{N(\Delta t_c)}{n}} = \frac{1}{\frac{N(\Delta t_c)}{2^j}}. \tag{6.6}$$

By using (6.3), (6.4) and (6.6) implies

$$\begin{aligned}\|e_m\|_2^2 &\leq M^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{2N^3(\Delta t_1)^3}{(2^j)^3} \\ &= 2M^2N^3(\Delta t_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^3} \times 2^j \\ &= 2M^2N^3(\Delta t_1)^3 \sum_{j=J+1}^{\infty} \frac{1}{(2^j)^2}.\end{aligned}$$

Applying series summation in above equation, we obtain

$$\|e_m\|_2^2 \leq \frac{8M^2N^3(\Delta t_1)^3}{3}\Big(\frac{1}{2^{J+1}}\Big)^2 = A\Big(\frac{1}{2^{J+1}}\Big)^2, \tag{6.7}$$

where

$$A = \frac{8M^2N^3(\Delta t_c)^3}{3}.$$

Since $m = 2^{J+1}$, we have

$$\|e_m\|_2 = O\left(\frac{1}{m}\right).$$

□

Theorem 6.2. *Suppose that $q(t,y) \in L^2[0,1)^2$ is a function with bounded partial derivative, $\left|\frac{\partial^2 q}{\partial t \partial y}\right| < E$, and we consider error function*

$$e_m(t,y) = q(t,y) - \sum_{i=0}^{m-1}\sum_{l=0}^{m-1} q_{i,l} H_i(t) H_l(y),$$

where $i = 2^{j_1} + k + 1$, $l = 2^{j_2} + k + 1$, $m = 2^{J+1}$, $J > 0$ and

$$q_{i,l} = \frac{1}{T_i^2} \int_0^1 \int_0^1 (H_i(t) H_l(y) q(t,y))\, dtdy.$$

Then, we have

$$\|e_m\|_2 = O\left(\frac{1}{m^2}\right).$$

Proof. By definition of error $e_m(t,y)$, we can write

$$\begin{aligned}\|e_m\|_2^2 &= \int_0^1 \int_0^1 \left(q(t,y) - \sum_{i=0}^{m-1}\sum_{l=0}^{m-1} q_{i,l} H_i(t) H_l(y)\right)^2 dtdy \\ &= \int_0^1 \int_0^1 \left(\sum_{i=m}^{\infty}\sum_{l=m}^{\infty} q_{i,l} H_i(t) H_l(y)\right)^2 dtdy \\ &= \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} \int_0^1 \int_0^1 \Big(q_{i,l}^2 H_i^2(t) H_l^2(y)\Big) dtdy.\end{aligned}$$

From the non-uniform wavelet definition, mean value theorem and Theorem 6.1, there are $\alpha, \alpha_1, \alpha_2$, also β, β_1 and β_2 such that

$$\begin{aligned}
q_{i,l} &= \frac{1}{T_i^2}\int_0^1 H_i(t)\Big(\int_0^1 H_l(y)q(t,y)dy\Big)dt\\
&= \frac{1}{T_i^2}\int_0^1 H_i(t)\left(\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\frac{\partial q(t,\beta)}{\partial y}\right)dt\\
&= \frac{1}{T_i^2}\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\int_0^1\left(H_i(t)\frac{\partial q(t,\beta)}{\partial y}\right)dt\\
&= \frac{1}{T_i^2}\left(\Upsilon_2(l)-\Upsilon_1(l)\right)(\beta_1-\beta_2)\left(\Upsilon_2(i)-\Upsilon_1(i)\right)(\alpha_1-\alpha_2)\frac{\partial^2 q(\alpha,\beta)}{\partial y\partial t}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\|e_m\|_2^2\\
=&\sum_{i=m}^{\infty}\sum_{l=m}^{\infty}\frac{1}{T_i^4}(\Upsilon_2(l)-\Upsilon_1(l))^2(\beta_1-\beta_2)^2(\Upsilon_2(i)-\Upsilon_1(i))^2(\alpha_1-\alpha_2)^2\left|\frac{\partial^2 q(\alpha,\beta)}{\partial y\partial t}\right|^2 T_i^2\\
\le&E^2\sum_{i=m}^{\infty}\sum_{l=m}^{\infty}\frac{1}{T_i^2}(\Upsilon_2(i)-\Upsilon_1(i))^2(\alpha_1-\alpha_2)^2(\Upsilon_2(l)-\Upsilon_1(l))^2(\beta_1-\beta_2)^2.
\end{aligned}$$

By using (6.7), we get

$$\|e_m\|_2^2 \le E^2A^2\times\frac{1}{m^4}=\frac{Z}{m^4},\quad Z=E^2A^2.$$

In the other words, we have

$$\|e_m\|_2 = O\left(\frac{1}{m^2}\right).$$

□

7. Numerical Examples

Here, we take two examples of linear Volterra integral equations to show the accuracy and efficiency of the non-uniform Haar wavelet method. We use MATLAB package to do all the computational work.

Example 7.1. We consider the following Volterra integral equation

$$Y(t)=K(t)+\int_0^t Y(s)(s-t)ds,$$

where, $K(t)=\frac{3}{4}e^{2t}+t^2+\frac{1}{2}t-\frac{7}{4}$, with the exact solution $Y(t)=e^{2t}-2$. Table 1 shows the comparison maximum absolute errors (MAE) of this example for $J=0,1,\ldots,6$. In Table 2, $\bar{X}_A$ is the exact solution, $\bar{X}_N$ is the numerical solution and $\bar{X}_E$ is the error of this method. Also, a comparison between the non-uniform Haar wavelet method (NHWM) and the uniform Haar wavelet method (UHWM) are shown in Table 3. Furthermore, CPU time is 303.203775 seconds.

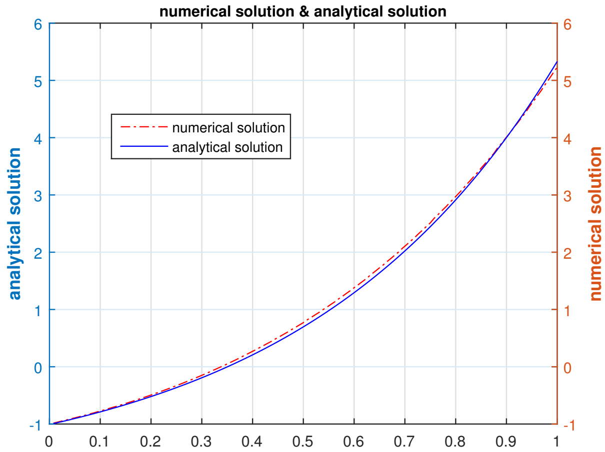

FIGURE 1. Analytical and numerical solution for $J = 6$.

TABLE 1. MAE of Example 1.

J	maximum absolute errors
0	0.4519
1	0.2784
2	0.1811
3	0.1341
4	0.1102
5	0.1005
6	0.0999

TABLE 2. The result of Example 1 for $J = 6$.

t	$\bar{X}_A$	$\bar{X}_N$	$\bar{X}_E$
0.0039	-0.9922	-0.9842	0.0080
0.0351	-0.9272	-0.9173	0.0099
0.0429	-0.9103	-0.8990	0.0114
0.0508	-0.8932	-0.8815	0.0117
0.1288	-0.7061	-0.6892	0.0183
0.1523	-0.6440	-0.6234	0.0207
0.8945	3.9831	3.9914	0.0083
0.9101	4.1731	4.1703	0.0002

TABLE 3. Comparison of maximum absolute errors for NHWM with UHWM.

J	Method	MAE
0	NHWM	0.4519
	UHWM	0.4383
2	NHWM	0.1811
	UHWM	0.1791
4	NHWM	0.1102
	UHWM	0.1167
6	NHWM	0.0999
	UHWM	0.1095

Example 7.2. We consider the following Volterra integral equation

$$Y(t) = K(t) + \int_0^t Y(s)(s^2 - t^2)ds,$$

where $K(t) = 2\cosh(t) - \sinh(t) - 2t\sinh(t) + t^2 + \frac{1}{3}t^3 - \frac{3}{2}$, with the exact solution $Y(t) = \frac{1}{2} - \sinh(t)$. Table 4 shows the comparison maximum absolute errors (MAE) of this example for $J = 0, 1, \ldots, 6$. In Table 5, $\bar{X}_A$ is the exact solution, $\bar{X}_N$ is the numerical solution and $\bar{X}_E$ is the error of this method. Also, a comparison between the non-uniform Haar wavelet method (NHWM) and the uniform Haar wavelet method (UHWM) are shown in Table 6. Furthermore, CPU time is 276.167482 seconds.

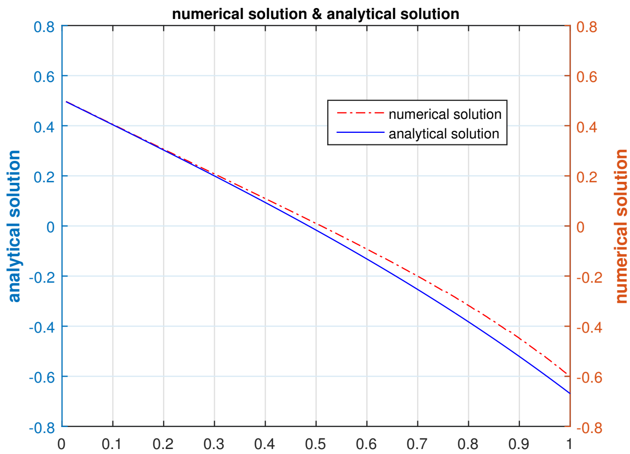

FIGURE 2. Analytical and numerical solution for $J = 6$.

Table 4. MAE of Example 2.

J	maximum absolute errors
0	0.0981
1	0.0928
2	0.0846
3	0.0794
4	0.0775
5	0.0749
6	0.0739

Table 5. The result of Example 2 for $J = 6$.

t	$\bar{X}_A$	$\bar{X}_N$	$\bar{X}_E$
0.0039	0.4961	0.4968	0.0006
0.0351	0.4649	0.4739	0.0009
0.1913	0.3075	0.3106	0.0035
0.3241	0.1702	0.1799	0.0097
0.4334	0.0529	0.0721	0.0193
0.4647	0.0184	0.0411	0.0227
0.5897	-0.1153	-0.0862	0.0377
0.6600	-0.2090	-0.1525	0.0487

Table 6. Comparison of maximum absolute errors for NHWM with UHWM.

J	Method	MAE
0	NHWM	0.0981
	UHWM	0.0874
2	NHWM	0.0846
	UHWM	0.0804
4	NHWM	0.0775
	UHWM	0.0793
6	NHWM	0.0739
	UHWM	0.0766

8. Conclusion

In this paper, a computational method based on the non-uniform Haar wavelets and their operational matrix of integration are proposed for solving Volterra integral equations. The main purpose of this method is reduce these equations to a linear system of algebraic equations. The convergence analysis of the proposed method is analyzed. For the future work, we can apply this method to solve Fredholm and Volterra-Fredholm integral equations, stochastic integral equations.

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