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INEQUALITIES AMONG TOPOLOGICAL DESCRIPTORS

ZEHUI SHAO[1], HUIQIN JIANG[2], AND ZAHID RAZA[3]

Abstract. A topological index is a type of molecular descriptor that is calculated based on the molecular graph of a chemical compound. Topological indices are used for example in the development of QSAR QSPR in which the biological activity or other properties of molecules are correlated with their chemical structure. In this paper, we establish several inequalities among the molecular descriptors such as the generalized version of the first Zagreb index, the Randić index, the ABC index, AZI index, and the redefined first, second and third Zagreb indices.

1. Introduction

Graph theory is an important tool to study properties of chemical molecules. In chemical graph theory, the vertices of the graph correspond to the atoms of molecules and the edges correspond to chemical bonds, and such a molecular graph is established to define topological indices which are used to study, or predict its structural features [13]. These topological indices are very important in the the quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR) studies [10]. They can reflect many phisico-chemical properties such as the stability of linear and branched alkanes, strain energy of cycloalkanes [4], heat of formation for heptanes and octanes [7], and the bioactivity of chemical compounds [11].

In this paper, we only consider undirected simple graphs. Let G be a graph, we denote $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. The complete graph, path, cycle and star on n vertices are denoted by K_n, P_n, C_n and S_n (or $S_{1,n-1}$), respectively. A graph is an (n,m)-graph if it has order n and size m. We denote

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TABLE 1. The most common molecular descriptors defined in last two decays

Atomic Bond Connectivity Index	$ABC(G)$	$\sum\limits_{uv\in E(G)} \left(\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}\right)$
Augmented Zagreb Index	$AZI(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{d(u)d(v)}{d(u)+d(v)-2}\right)^3$
Randic Connectivity Index	$R(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{1}{\sqrt{d(u)d(v)}}\right)$
Sum-Connectivity Index	$X(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{1}{\sqrt{d(u)+d(v)}}\right)$
First Zagreb Index	$Zg_1(G)$	$\sum\limits_{uv\in E(G)} (d(u)+d(v))$
Second Zagreb Index	$Zg_2(G)$	$\sum\limits_{uv\in E(G)} (d(u)d(v))$
Generalized Zagreb Index	$M_\alpha(G)$	$\sum\limits_{uv\in E(G)} (d(u)^{\alpha-1}+d(v)^{\alpha-1})$
Modified Zagreb Index	$M_2^*(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{1}{d(u)d(v)}\right)$
Redefined First Zagreb Index	$ReZg_1(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{d(u)+d(v)}{d(u)d(v)}\right)$
Redefined Second Zagreb Index	$ReZg_2(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{d(u)d(v)}{d(u)+d(v)}\right)$
Redefined Third Zagreb Index	$ReZg_3(G)$	$\sum\limits_{uv\in E(G)} (d(u)+d(v))\, d(u)d(v)$
Forgotten Index	$F(G)$	$\sum\limits_{v\in V(G)} (d(v))^3$
Harmonic Index	$H(G)$	$\sum\limits_{v\in V(G)} \left(\frac{2}{d(u)+d(v)}\right)$
Geometric-Arithmetic Index	$GA(G)$	$\sum\limits_{uv\in E(G)} \left(\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\right)$

by $d(v)$ the degree of a vertex v of a graph G. We denote by $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ) the maximum and minimum degree of G, respectively. A graph is *regular* if $\Delta = \delta$. A bipartite graph is *biregular* if each vertex in the same part has the same degree, e.g., $K_{2,3}$ is a biregular graph. For positive integers s, t, we denote by $E_{s,t}$ the set of edges with two end vertices with degree s and t, respectively. That is $E_{s,t} = \{uv : d(u) = s, d(v) = t\}$. The most common molecular descriptors defined in last two decays are defined in Table 1.

When a new topological index is proposed in chemical graph theory, one of the important problems is to find lower and upper bounds for this index on a class of graphs such as trees or general graphs or graph operations [3, 6]. Motivated with the importance of bounds and inequalities, this paper continue to study the inequalities among topological indices.

2. Related Work on Inequalities Among Topological Indices

In recent years, there has been an increasing amount of literature on inequality of topological indices [1, 19–22]. Various relations of different topological indices have been extensively researched, and we summarize main known results as follows.

Theorem 2.1. *Let G be a connected graph having $n \geq 3$ vertices. Then*

$$\left(\frac{1536}{343}\right) X(G) \leq AZI(G) \leq \left(\frac{\sqrt{(n-1)^{13}}}{\sqrt{32}(n-2)^3}\right) X(G),$$

with left equality if and only if $G \cong S_{1,8}$ and right equality if and only if $G \cong K_n$.

Theorem 2.2 ([1])**.** *Let G be a connected graph having $n \geq 3$ vertices. Then*

$$\begin{aligned}
\left(\frac{343\sqrt{7}}{216}\right) R(G) \leq & AZI(G) \leq \left(\frac{\sqrt{(n-1)^7}}{8(n-2)^3}\right) R(G), \\
\left(\frac{375}{64}\right) H(G) \leq & AZI(G) \leq \left(\frac{\sqrt{(n-1)^7}}{8(n-2)^3}\right) H(G), \\
\left(\frac{n-1}{n-2}\right)^{\frac{7}{2}} ABC(G) \leq & AZI(G) \leq \left(\frac{(n-1)^2}{2(n-2)}\right)^{\frac{7}{2}} ABC(G), \\
8GA(G) \leq & AZI(G) \leq \left(\frac{\sqrt{(n-1)^6}}{8(n-2)^3}\right) GA(G), \quad \delta \geq 2, \\
4M_2^*(G) \leq & AZI(G) \leq \left(\frac{\sqrt{(n-1)^4}}{2(n-2)}\right) M_2^*(G).
\end{aligned}$$

The left equality in the above inequalities holds if and only if $G \cong S_{1,7}$, $G \cong S_{1,5}$, $G \cong S_{1,n-1}$, $G \cong C_n$, $G \cong P_3$, respectively and right equality in all the aforementioned inequalities holds if and only if $G \cong K_n$.

Theorem 2.3 ([20])**.** *Let G be a simple connected graph on $n \geq 3$ vertices with minimum degree $\delta \geq s$ and maximum degree $\Delta \leq t$, where $1 \leq s \leq t \leq n-1$ and $t \geq 2$. Then*

(i) $\left(\frac{\sqrt{2t-2}}{t}\right) GA(G) \leq ABC(G)$, *with equality if and only if G is a t-regular graph;*

(ii) $ABC(G) \leq \left(\frac{\sqrt{2s-2}}{s}\right) GA(G)$, *with equality if and only if G is a s-regular graph and $ABC(G) \leq \left(\frac{(s+t)\sqrt{s+t-2}}{st}\right) GA(G)$ if $t \geq 2s - 3 + \sqrt{5s^2 - 14s + 9}$, with equality if and only if one vertex has degree s and the other vertex has degree t for every edge of G.*

Theorem 2.4 ([1]). *Let G be a connected graph having $n \geq 2$ vertices. Then*

$$\begin{aligned} M_2^*(G) \leq & R(G) \leq (n-1)M_2^*(G), \\ \frac{M_2^*(G)}{\sqrt{2}} \leq & X(G) \leq \left(\frac{(n-1)^{\frac{3}{2}}}{\sqrt{2}}\right) M_2^*(G), \\ M_2^*(G) \leq & H(G) \leq (n-1)M_2^*(G), \\ M_2^*(G) \leq & GA(G) \leq (n-1)^2 M_2^*(G), \\ \sqrt{2}M_2^*(G) \leq & ABC(G) \leq (n-1)\sqrt{2(n-2)}M_2^*(G). \end{aligned}$$

The left equality in the first four inequalities and in the fifth inequality is attained if and only if $G \cong P_2$ and $G \cong P_3$, respectively. The right equality in all inequalities is attained if and only if $G \cong K_n$.

Theorem 2.5 ([19]). *Let G be a connected graph having $n \geq 3$ vertices. Then*
(i) $\left(\sqrt{\frac{3}{2}}\right) X(G) \leq ABC(G)$, *with equality if and only if $G \cong P_3$;*
(ii) $ABC(G) \leq \sqrt{2}X(G)$, *if $n = 3$, the equality holds if and only if $G \cong K_3$, $ABC(G) \leq \left(\sqrt{\frac{8}{3}}\right) X(G)$, if $n = 4$, the equality holds if and only if $G \cong K_4$ or $G \cong S_4$, $ABC(G) \leq \left(\sqrt{\frac{n(n-2)}{n-1}}\right) X(G)$, if $n \geq 5$, the equality holds if and only if $G \cong S_n$.*

Theorem 2.6 ([19]). *Let G be a connected graph having $n \geq 3$ vertices. Then*
(i) $\left(\frac{3\sqrt{2}}{4}\right) H(G) \leq ABC(G)$, *with equality if and only if $G \cong P_3$;*
(ii) $ABC(G) \leq \sqrt{2n-4}H(G)$, *if $3 \leq n \leq 6$, with equality if and only if $G \cong K_n$, $ABC(G) \leq \left(\frac{n}{2}\sqrt{\frac{n-2}{n-1}}\right) H(G)$, if $n \geq 7$, with equality if and only if $G \cong S_n$.*

2.1. **Other inequalities and some inequality chains.**

Proposition 2.1 ([22]). *Let G be a graph. Then $X(G) \geq \left(\frac{1}{\sqrt{2}}\right) R(G)$ with equality if and only if all non-isolated vertices have degree one. Moreover, if G has no components on two vertices, then $X(G) \geq \left(\sqrt{\frac{2}{3}}\right) R(G)$ with equality if and only if all non-trivial components of G are paths on three vertices, and if no pendant vertices, then $X(G) \geq R(G)$ with equality if and only if all non-isolated vertices have degree two.*

Proposition 2.2 ([22]). *Let G be a graph with m edges. Then $X(G) \leq \sqrt{\frac{mR(G)}{2}}$ with equality if and only if G is regular.*

Theorem 2.7 ([19]). *Let G be a graph with n vertices. Then $\left(\frac{2\sqrt{n-1}}{n}\right) R(G) \leq H(G) \leq R(G)$. The lower bound is attained if and only if $G \cong S_n$, and the upper bound is attained if and only if all connected components of G are regular.*

Theorem 2.8 ([20]). *Let G be a connected graph with $\delta \geq 2$. Then*

$$H(G) \leq R(G) \leq X(G) < ABC(G),$$

where the first inequality holds as equality if and only if G is a regular graph, and the second inequality holds as equality if and only if G is a cycle.

3. Results and Discussion

For a graph G, we say G has Property A if for each edge uv we have $d(u)+d(v)=k$ for some k, and G has Property B if for each edge uv we have $d(u)d(v)=k$ for some k, and G has Property C if for each edge uv we have $\frac{d(u)d(v)}{d(u)+d(v)}=k$ for some k.

Definition 3.1. We define the four classes of graphs as follows.

- Let $\mathcal{G}_1$ be the set of graphs without isolated vertices with property A.
- Let $\mathcal{G}_2$ be the set of graphs without isolated vertices with property B.
- Let $\mathcal{G}_3$ be the set of graphs without isolated vertices with properties A and B.
- Let $\mathcal{G}_4$ be the set of graphs without isolated vertices with property C.

Lemma 3.1. *If G is a graph without isolated vertices we have $d(u_1)d(v_1)=d(u_2)d(v_2)$ and $d(u_1)+d(v_1)=d(u_2)+d(v_2)$ for any pair of edges u_1v_1 and u_2v_2, then G has properties A and B. Equivalently, G is either regular or a biregular.*

Proof. Since $d(u_1)d(v_1)=d(u_2)d(v_2)$ and $d(u_1)+d(v_1)=d(u_2)+d(v_2)$ for any pair of edges u_1v_1 and u_2v_2, we have $d(u)d(v)=k_1$ and $d(u)+d(v)=k_2$ for some k_1 and k_2. Then we have $uv\in E_{s,t}$ where s and t are the roots of the equation $x^2-k_1x+k_2=0$. If $s=t$, then we have G is regular. If $s\neq t$, then the degree of each neighbor of each vertex with degree s is t and thus G is biregular. $\square$

The following lemma is the well known power mean inequality [2].

Lemma 3.2. *Let $x_1,x_2,\ldots,x_n>0$ and $p>q>0$. Then*

$$\sqrt[p]{\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}}\geq\sqrt[q]{\frac{x_1^q+x_2^q+\cdots+x_n^q}{n}},$$

with equality if and only if $x_i=x_j$ for each $i\neq j$.

The Cauchy-Schwarz inequality is arguably one of the most widely used inequalities in mathematics (see [18]).

Lemma 3.3. *Let $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ be two real sequences. Then $\left(\sum\limits_{i=1}^{n}a_i^2\right)\left(\sum\limits_{i=1}^{n}b_i^2\right)\geq\left(\sum\limits_{i=1}^{n}a_ib_i\right)^2$, with equality if and only if there exists a constant c such that $a_i=cb_i$ for all $i=1,2,\ldots,n$.*

The following is the well known Jensen's inequality (see [14]).

Lemma 3.4. *Let $(x_1,x_2,\ldots,x_n)$ be a real sequence and $f(x)$ be a continuous real function.*

i) *If $f''(x)>0$, then*

$$\frac{\sum\limits_{i=1}^{n}f(x_i)}{n}\geq f\left(\frac{\sum\limits_{i=1}^{n}x_i}{n}\right),$$

with equality if and only if $x_i=x_j$ for each $i\neq j$.

ii) *If $f''(x) < 0$, then*

$$\frac{\sum_{i=1}^{n} f(x_i)}{n} \leq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right),$$

with equality if and only if $x_i = x_j$ for each $i \neq j$.

By Lemma 3.2, the following result is immediate.

Theorem 3.1. *Let G be an (n,m)-graph without isolated vertices, $k = k_1 + k_2$ with $k_1 > 0$ and $k_2 > 0$. Then*

$$\frac{M_k(G)}{n} \geq \left(\frac{M_{k_1}(G)}{n}\right)\left(\frac{M_{k_2}(G)}{n}\right).$$

Proof. Let $x_i = d(v_i)$ for $i = 1, 2, \ldots, n$. Since G has no isolated vertex, so, we have $x_i \geq 1$. Note that $k > k_1$, by Lemma 3.2, we have

$$\left(\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}\right)^{\frac{1}{k}} \geq \left(\frac{x_1^{k_1} + x_2^{k_1} + \cdots + x_n^{k_1}}{n}\right)^{\frac{1}{k_1}}.$$

That is

$$\left(\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}\right)^{\frac{k_1}{k}} \geq \frac{x_1^{k_1} + x_2^{k_1} + \cdots + x_n^{k_1}}{n}. \tag{3.1}$$

Similarly, we have

$$\left(\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}\right)^{\frac{k_2}{k}} \geq \frac{x_1^{k_2} + x_2^{k_2} + \cdots + x_n^{k_2}}{n}. \tag{3.2}$$

By (3.1) $\times$ (3.2), we get

$$\frac{x_1^k + x_2^k + \cdots + x_n^k}{n} \geq \left(\frac{x_1^{k_1} + x_2^{k_1} + \cdots + x_n^{k_1}}{n}\right)\left(\frac{x_1^{k_2} + x_2^{k_2} + \cdots + x_n^{k_2}}{n}\right).$$

That is

$$\frac{M_k(G)}{n} \geq \left(\frac{M_{k_1}(G)}{n}\right)\left(\frac{M_{k_2}(G)}{n}\right).$$

□

Corollary 3.1. *Let G be an (n,m)-graph without isolated vertices and $k \geq 2$. Then*

$$M_k(G) \geq n\left(\frac{M_{k-1}(G)}{n}\right)^{\frac{k}{k-1}}.$$

Proof. By Theorem 3.1 with $k_1 = 1$ and $k_2 = k - 1$, the result holds. □

Remark 3.1. By Corollary 3.1, we have $F(G) \geq n\left(\frac{M_1(G)}{n}\right)^{\frac{3}{2}} \geq \frac{8m^3}{n^2}$.

Theorem 3.2 ([12]). *Suppose a_i and b_i, $1 \leq i \leq n$, are positive real numbers. Then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b),$$

where a, b, A and B are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$, $b \leq b_i \leq B$. Further, $\alpha(n) = n\lceil \frac{n}{2} \rceil (1 - \frac{1}{n}\lceil \frac{n}{2} \rceil)$.

By Theorem 3.2, we have the following.

Theorem 3.3. *Let G be an (n,m)-graph without isolated vertices, $k = k_1 + k_2$ with $k_1 > 0$ and $k_2 > 0$. Then*

$$M_k(G) \leq \left(\frac{M_{k_1}(G)M_{k_2}(G) + \alpha(n)(\Delta^{k_1} - \delta^{k_1})(\Delta^{k_2} - \delta^{k_2})}{n} \right).$$

Proof. Let $a_i = d(v_i)^{k_1}$, $b_i = d(v_i)^{k_2}$, $A = \Delta^{k_1}$, $B = \Delta^{k_2}$, $a = \delta^{k_1}$ and $b = \delta^{k_2}$, then by Theorem 3.2, we have

$$nM_k(G) - M_{k_1}(G)M_{k_2}(G) \leq \alpha(n)(\Delta^{k_1} - \delta^{k_1})(\Delta^{k_2} - \delta^{k_2}). \tag{3.3}$$

So, we have

$$M_k(G) \leq \left(\frac{M_{k_1}(G)M_{k_2}(G) + \alpha(n)(\Delta^{k_1} - \delta^{k_1})(\Delta^{k_2} - \delta^{k_2})}{n} \right). \qquad \square$$

Theorem 3.4. *Let G be an (n,m)-graph without isolated vertices. Then $R(G)^2 Zg_2(G) \geq m^3$, with equality if and only if $G \in \mathcal{G}_2$.*

Proof. Let $f(x) = \frac{1}{\sqrt{x}}$, then we have $f''(x) = \frac{3}{4x^{\frac{5}{2}}} > 0$ if $x > 0$. By Lemma 3.4, we have

$$\frac{\sum\limits_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}}{m} \geq \frac{1}{\sqrt{\frac{\sum\limits_{uv \in E(G)} d(u)d(v)}{m}}}.$$

That is

$$\frac{R(G)}{m} \geq \frac{1}{\sqrt{\frac{Zg_2(G)}{m}}}.$$

Consequently, we have $R(G)^2 Zg_2(G) \geq m^3$. The equality holds if and only if

$$\frac{1}{\sqrt{d(u_1)d(v_1)}} = \frac{1}{\sqrt{d(u_2)d(v_2)}},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$. Since

$$\frac{1}{\sqrt{d(u_1)d(v_1)}} = \frac{1}{\sqrt{d(u_2)d(v_2)}} \Leftrightarrow d(u_1)d(v_1) = d(u_2)d(v_2),$$

we have the equality holds if and only if $G \in \mathcal{G}_2$. $\square$

Theorem 3.5. *Let G be an (n,m)-graph without isolated vertices. Then $X(G)^2 Zg_1(G) \geq m^3$ with equality if and only if $G \in \mathcal{G}_1$.*

Proof. Let $f(x) = \frac{1}{\sqrt{x}}$. Then $f''(x) = \frac{3}{4x^{\frac{5}{2}}} > 0$ for $x > 0$. By Lemma 3.4, we have

$$\frac{\sum\limits_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)}}}{m} \geq \frac{1}{\sqrt{\frac{\sum\limits_{uv\in E(G)} (d(u)+d(v))}{m}}}.$$

That is

$$\frac{X(G)}{m} \geq \frac{1}{\sqrt{\frac{Zg_1(G)}{m}}}.$$

Consequently, we have $X(G)^2 Zg_1(G) \geq m^3$. The equality holds if and only if

$$\frac{1}{\sqrt{d(u_1)+d(v_1)}} = \frac{1}{\sqrt{d(u_2)+d(v_2)}},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$. Since

$$\frac{1}{\sqrt{d(u_1)+d(v_1)}} = \frac{1}{\sqrt{d(u_2)+d(v_2)}} \Leftrightarrow d(u_1)+d(v_1) = d(u_2)+d(v_2),$$

we have the equality holds if and only if $G \in \mathcal{G}_1$. ☐

Theorem 3.6. *Let G be an (n,m)-graph without isolated vertices. Then $ABC(G)^2 + 2R(G)^2 \leq mn$ with equality holds if and only if G is regular or biregular.*

Proof. Let $f(x) = \sqrt{x}$, we have $f''(x) < 0$ if $x > 0$. By Lemma 3.4,

$$\begin{aligned}
\frac{ABC(G)}{m} &= \frac{\sum\limits_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}}{m} \\
&\leq \sqrt{\frac{\sum\limits_{uv\in E(G)} \frac{d(u)+d(v)-2}{d(u)d(v)}}{m}} \\
&= \sqrt{\frac{\sum\limits_{uv\in E(G)} (\frac{1}{d(u)} + \frac{1}{d(u)}) - 2\sum\limits_{uv\in E(G)} \frac{1}{d(u)d(v)}}{m}} \\
&= \sqrt{\frac{n - 2\sum\limits_{uv\in E(G)} \frac{1}{d(u)d(v)}}{m}}. \qquad (3.4)
\end{aligned}$$

The equality holds if and only if

$$\frac{d(u_1)+d(v_1)-2}{d(u_1)d(v_1)} = \frac{d(u_2)+d(v_2)-2}{d(u_2)d(v_2)},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$.

On the other hand, by Lemma 3.3 we have

$$R(G)^2 = \left(\sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}} \right)^2 \leq m \left(\sum_{uv\in E(G)} \frac{1}{d(u)d(v)} \right). \tag{3.5}$$

The equality holds if and only if

$$\frac{1}{\sqrt{d(u_1)d(v_1)}} = \frac{1}{\sqrt{d(u_2)d(v_2)}},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$.

From (3.4) and (3.5) above, we have

$$\frac{ABC(G)}{m} \leq \sqrt{\frac{n - 2 \sum\limits_{uv\in E(G)} \frac{1}{d(u)d(v)}}{m}} \leq \sqrt{\frac{n - 2\frac{R(G)^2}{m}}{m}}.$$

Then, we get $ABC(G)^2 + 2R(G)^2 \leq mn$. So, we have the equality holds if and only if $G \in \mathcal{G}_3$. Then by Lemma 3.1, we have G is regular or biregular. □

Theorem 3.7. *Let G be an (n, m)-graph without isolated vertices. Then*

$$AZI(G)(mn - 2R(G)^2)^3 \geq m^7$$

and the equality holds if and only if G is regular or biregular.

Proof. Let $f(x) = \frac{1}{x^3}$, we have $f''(x) = 12x^{-5} > 0$ if $x > 0$. By Lemma 3.4, we have

$$\begin{aligned}
\frac{AZI(G)}{m} &= \frac{\left(\sum\limits_{uv\in E(G)} \frac{d(u)d(v)}{d(u)+d(v)-2}\right)^3}{m} \\
&\geq \frac{1}{\left(\frac{\sum\limits_{uv\in E} \frac{d(u)+d(v)-2}{d(u)d(v)}}{m}\right)^3} \\
&= \frac{1}{\left(\frac{\sum\limits_{uv\in E(G)} (\frac{1}{d(u)} + \frac{1}{d(u)}) - 2 \sum\limits_{uv\in E(G)} \frac{1}{d(u)d(v)}}{m}\right)^3} \\
&= \frac{1}{\left(\frac{n-2 \sum\limits_{uv\in E(G)} \frac{1}{d(u)d(v)}}{m}\right)^3}. \qquad (3.6)
\end{aligned}$$

The equality holds if and only if

$$\frac{d(u_1) + d(v_1) - 2}{d(u_1)d(v_1)} = \frac{d(u_2) + d(v_2) - 2}{d(u_2)d(v_2)},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$.

On the other hand, by Lemma 3.3 we have

$$R(G)^2 = \left(\sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}} \right)^2 \leq m \left(\sum_{uv \in E(G)} \frac{1}{d(u)d(v)} \right). \tag{3.7}$$

The equality holds if and only if

$$\frac{1}{\sqrt{d(u_1)d(v_1)}} = \frac{1}{\sqrt{d(u_2)d(v_2)}},$$

for any two distinct edges $u_1v_1, u_2v_2 \in E(G)$.

From (3.6) and (3.7) above, we have

$$\frac{AZI(G)}{m} \geq \frac{1}{\left(\frac{n-2 \sum_{uv \in E(G)} \frac{1}{d(u)d(v)}}{m} \right)^3} \geq \frac{1}{\left(\frac{n-2 \sum_{uv \in E(G)} \frac{1}{d(u)d(v)}}{m} \right)^3}.$$

Then we get $AZI(G)(mn - 2R(G)^2)^3 \geq m^7$. Thus, the equality holds if and only if we have $G \in \mathcal{G}_3$. Then by Lemma 3.1, we have G is regular or biregular. □

Theorem 3.8. *Let G be an (n,m)-graph without isolated vertices. Then*

i) *$ReZg_2(G) \geq \frac{m^2}{n}$ and the equality holds if and only if $G \in \mathcal{G}_4$;*

ii) *$ReZg_2(G)ReZg_3(G) \geq Zg_2(G)^2$ and the equality holds if and only if $G \in \mathcal{G}_1$;*

iii) *$ReZg_1(G)ReZg_3(G) \geq Zg_1(G)^2$ and the equality holds if and only if $G \in \mathcal{G}_2$.*

Proof. i) By Lemma 3.3, we have

$$ReZg_1(G)ReZg_2(G) = \left(\sum_{uv \in E(G)} \frac{d(u)+d(v)}{d(u)d(v)} \right) \left(\sum_{uv \in E(G)} \frac{d(u)d(v)}{d(u)+d(v)} \right) \geq m^2.$$

Since

$$ReZG_1(G)ReZG_2(G) = \sum_{uv \in E(G)} \frac{d(u)+d(v)}{d(u)d(v)} = \sum_{uv \in E(G)} (\frac{1}{d(u)} + \frac{1}{d(u)}) = n,$$

we have $ReZG_2(G) \geq \frac{m^2}{n}$. By Lemma 3.3, we have the equality holds if and only if $G \in \mathcal{G}_4$.

ii) By Lemma 3.3, we have

$$\begin{aligned} ReZg_2(G)ReZg_3(G) &= \left(\sum_{uv \in E(G)} \frac{d(u)d(v)}{d(u)+d(v)} \right) \left(\sum_{uv \in E(G)} d(u)d(v)(d(u)+d(v)) \right) \\ &\geq \left(\sum_{uv \in E(G)} d(u)d(v) \right)^2 \\ &= Zg_2(G)^2. \end{aligned}$$

By Lemma 3.3, we have the equality holds if and only if $G \in \mathcal{G}_1$.

iii) By Lemma 3.3, we have

$$\begin{aligned} ReZg_1(G)ReZg_3(G) &= \left(\sum_{uv\in E(G)} \frac{d(u)+d(v)}{d(u)d(v)}\right)\left(\sum_{uv\in E(G)} d(u)d(v)(d(u)+d(v))\right) \\ &\geq \left(\sum_{uv\in E(G)} (d(u)+d(v))\right)^2 \\ &= Zg_1(G)^2. \end{aligned}$$

By Lemma 3.3, we have the equality holds if and only if $G \in \mathcal{G}_2$. □

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References

[1] A. Ali, A. A. Bhatti and Z. Raza, *Further inequalities between vertex-degree-based topological indices*, Int. J. Appl. Comput. Math. **3** (2016) 1921–1930. https://doi.org/10.1007/s40819-016-0213-4

[2] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their Inequalities*, Reidel, Dordrecht, 1988.

[3] K. C. Das and S. A. Mojallal, *Upper bounds for the energy of graphs*, MATCH Commun. Math. Comput. Chem. **70** (2013), 657–662. https://doi.org/10.15672/HJMS.20164513097

[4] E. Estrada, L. Torres, L. Rodriguez and I. Gutman, *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian Journal of Chemistry - Section A **37** (1998), 849–855. http://nopr.niscair.res.in/handle/123456789/40308

[5] S. Fajtlowicz, *On conjectures of graffiti II*, Congr. Numer. **60** (1987), 187–197.

[6] G. H. Fath-Tabar, B. Vaez-Zadeh, A. R. Ashrafi and A. Graovac, *Some inequalities for the atom-bond connectivity index of graph operations*, Discrete Appl. Math. **159** (2011), 1323–1330. https://doi.org/10.1016/j.dam.2011.04.019

[7] B. Furtula, A. Graovac and D. Vukičević, *Augmented Zagreb index*, J. Math. Chem. **48** (2010), 370–380. https://doi.org/10.1007/s10910-010-9677-3

[8] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z

[9] I. Gutman, A. Ghalavand, T. Dehgjan-Zadeh and A. R. Ashrafi, *Graphs with smallest forgotten index*, Iranian Journal of Mathematical Chemistry **8**(3) (2017), 259–273. https://doi.org/10.22052/ijmc.2017.43258

[10] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals: Total π-electron energy of alternant hydrocarbons*, Chemical Physics Letters **17** (1972), 535–538. https://doi.org/10.1016/0009-2614(72)85099-1

[11] S. M. Hosamani, *Computing Sanskruti index of certain nanostructures*, J. Appl. Math. Comput. **54** (2017), 425–433. https://doi.org/10.1007/s12190-016-1016-9

[12] I. Ž. Milovanovć, E. I. Milovanovć and A. Zakić, *A short note on graph energy*, MATCH Commun. Math. Comput. Chem. **72**(2014), 179–182.

[13] S. Nikolić, G. Kovačević, A. Miličević and N. Trinajstić, *The Zagreb indices* 30 *years after*, Croatica Chemica Acta **76** (2003), 113–124.

[14] J. Pecarič and S. S. Dragomir, *A refinements of Jensen inequality and applications*, Stud. Univ. Babes-Bolyai Math. **24** (1989), 15–19.

[15] M. Randić, *The connectivity index* 25 *years later*, Journal of Molecular Graphics and Modelling **20** (2001), 19–35. https://doi.org/10.1016/S1093-3263(01)00098-5

[16] R. S. Ranjini, V. Lokesha and A. Usha, *Relation between phenylene and hexagonal squeeze using harmonic index*, Int. J. Graph Theory **1** (2013), 116–121.

[17] J. M. Rodríguez and J. M. Sigarreta, *On the geometric-arithmetic index*, MATCH Commun. Math. Comput. Chem. **74** (2015), 103–120.

[18] J. M. Steele, *The Cauchy-Schwarz Master Class*, Cambridge University Press, New Yersey, 2004.

[19] L. Zhong and K. Xu, *Inequalities between vertex-degree-based topological indices*, MATCH Commun. Math. Comput. Chem. **71** (2014), 627–642.

[20] L. Zhong and Q. Cui, *On a relation between the atom-bond connectivity and the first geometric-arithmetic indices*, Discrete Appl. Math. **185** (2015), 249–253. https://doi.org/10.1016/j.dam.2014.11.027

[21] B. Zhou and N. Trinajstić, *On a novel connectivity index*, J. Math. Chem. **46** (2009), 1252–1270. https://doi.org/10.1007/s10910-008-9515-z

[22] B. Zhou and N. Trinajstić, *Relations between the product and sum-connectivity indices*, Croatica Chemica Acta **85** (2012), 363–365. https://doi.org/10.5562/cca2052

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A NEW APPROACH FOR SOLVING A NEW CLASS OF NONLINEAR OPTIMAL CONTROL PROBLEMS GENERATED BY ATANGANA-BALEANU-CAPUTO VARIABLE ORDER FRACTIONAL DERIVATIVE AND FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

F. GHOMANJANI[1]

ABSTRACT. In the sequel, the numerical solution of a new class of nonlinear optimal control problems (OCPs) generated by Atangana-Baleanu-Caputo (ABC) variable order (V-O) fractional derivative (FD) and fractional Volterra-Fredholm integro-differential equations (FVFIDEs) is found by Bezier curve method (BCM). The main idea behind this work is the use of the BCM. In this technique, the solution is found in the form of a rapid convergent series. Using this method, it is possible to obtain BCM solution of the general form of multipoint boundary value problems. To shown the efficiency of the developed method, numerical results are stated as the main results in this study.

1. INTRODUCTION

OCPs is one of the main topics refered to the V-O fractional. These problems are related to the V-O fractional operators in their cost functional and dynamical system. Recently, many numerical methods have been stated such as in [1] and [2], the Bernstein functions for nonlinear V-O fractional OCPs is stated. In [3], generalized polynomials is studied for a kind of V-O fractional 2D OCPs. B-splines (where Bezier form is a special case of B-splines), due to numerical stability and arbitrary order of accuracy, have become popular tools for solving differential equations. The use of Bezier curves for solving V-O fractional OCPs (2.1) and FVFIDEs is a novel idea.

Key words and phrases. variable order, Bezier curve, nonlinear optimal control problems (NOCPs), Volterra-Fredholm integro-differential equations.

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Additionally some papers spent the Bezier curves. In [4] and [5], the authors utilized the Bezier curves for solving delay differential equation (DDE) and optimal control of switched systems numerically. In [6], the authors proposed the utilization of Bezier curves on some linear optimal control systems with pantograph delays. Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, the Bezier control points strategy is utilized (see [7]). Some other uses of the Bezier functions are found in (see [8]). The organization of this study is classified as follows. Problem statement is introduced in Section 2. Also solving ABC V-O FD based on the Bezier curves is stated in Section 3. Convergence analysis is stated in Section 4. A numerical example is solved in Section 5, then a remark is stated about FVFIDEs. Solving FVFIDEs based on Bezier curves is presented in Section 6. Section 7 will give a problem statement for FVFIDEs. Numerical applications for FVFIDEs are presented in Section 8. Finally, Section 9 will give a conclusion briefly.

2. Problem Statement

In this paper, the following definition is considered.

Definition 2.1. Let $\alpha : [0, \tau_{max}] \to (0, 1)$ be a continuous function and $x \in C^1[0, \tau_{max}]$. The V-O FD of order $\alpha(\tau)$ in the ABC sense of $x(\tau)$ is defined as follows (see [9]):

$$ {}_{0}^{ABC}D_t^{\alpha(\tau)} x(\tau) = \frac{C(\alpha(\tau))}{1-\alpha(\tau)} \int_0^\tau x'(s) E_{\alpha(\tau)} \left(\frac{-\alpha(\tau)(\tau - s)^{\alpha(\tau)}}{1-\alpha(\tau)} \right) ds, \tag{2.1}$$

where

$$\begin{aligned} E_\lambda(\tau) =& \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma(j\lambda + 1)}, \quad \lambda \in R^+,\, \tau \in R, \\ C(\alpha(\tau)) =& 1 - \alpha(\tau) + \frac{\alpha(\tau)}{\Gamma(\alpha(\tau))}, \\ {}_{0}^{ABC}D_t^{\alpha(\tau)} c =& 0, \quad \text{for any constant } c. \end{aligned}$$

So, we focus on the following problem

$$\min J = \int_0^{\tau_{\max}} L(\tau, x_1(\tau), x_2(\tau), \dots, x_m(\tau), u(\tau)) d\tau,$$

such that

$$\begin{aligned} & {}_{0}^{ABC}D_t^{\alpha_i(\tau)}\, x(\tau) = G_i(\tau, x_1(\tau), x_2(\tau), \dots, x_m(\tau), u(\tau)), \\ & \alpha_i(\tau) \in (0, 1), \quad i = 1, 2, \dots, m,\, m \in \mathbb{N}, \\ & x_i(0) = x_{i,0}, \quad i = 0, 1, \dots, m, \quad \tau_{max} \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where L, G_i for $i = 1, 2, \dots, m$, are continuous operators, α_i for $i = 1, 2, \dots, m$, is a continuous function on $[0, \tau_{max}]$ and $x_{i,0}$ is a given real constant.

3. Solving ABC V-O FD Based on the Bezeir Curves

Our aim is utilizing Bezier curves to approximate the solutions $x(\tau)$ and $u(\tau)$ where $x(\tau)$ and $u(\tau)$ are given below. Define the Bezier polynomials of degree n over the interval $[\tau_0, \tau_f]$ as follows:

$$x(\tau) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{\tau - \tau_0}{h}\right), \quad \tau_f = 1, \quad \tau_0 = 0, \quad u(\tau) = \sum_{r=0}^{n} b_r B_{r,n}\left(\frac{\tau - \tau_0}{h}\right), \tag{3.1}$$

where $h = \tau_f - \tau_0$ and

$$B_{r,n}\left(\frac{\tau - \tau_0}{h}\right) := \binom{n}{r}\frac{1}{h^n}(\tau_f - \tau)^{n-r}(\tau - \tau_0)^r$$

is the Bernstein polynomial of degree n over the interval $[\tau_0, \tau_f]$ and a_r, b_r, $r = 0, 1, \dots, n$, and they are unknown control points. Also, we have

$$\begin{aligned}
\frac{dB_{r,n}(\tau)}{d\tau} =& n\Big(B_{r-1,n-1}(\tau) - B_{r,n-1}(\tau)\Big),\\
\frac{dx(\tau)}{d\tau} =& \sum_{r=0}^{n-1} na_r B_{r-1,n-1}(\tau) - \sum_{r=0}^{n-1} na_r B_{r,n-1}(\tau)\\
=& \sum_{r=0}^{n-1} na_{r+1} B_{r,n-1}(\tau) - \sum_{r=0}^{n-1} na_r B_{r,n-1}(t)\\
=& \sum_{r=0}^{n-1} B_{r,n-1}(\tau) n(a_{r+1} - a_r),
\end{aligned}$$

then

$$\begin{aligned}
{}_0^{ABC}D_t^{\alpha(\tau)}x(\tau) =& \frac{C(\alpha(\tau))}{1-\alpha(\tau)}\int_0^\tau \left(\sum_{r=0}^{n} a_r B_{r,n}(\tau)\right)' E_{\alpha(\tau)}\left(\frac{-\alpha(\tau)(\tau - s)^{\alpha(\tau)}}{1-\alpha(\tau)}\right) ds\\
=& \frac{C(\alpha(\tau))}{1-\alpha(\tau)}\int_0^\tau \sum_{i=0}^{n-1} B_{i,n-1}(\tau) n\left(a_{i+1} - a_i\right) E_{\alpha(\tau)}\left(\frac{-\alpha(\tau)(\tau - s)^{\alpha(\tau)}}{1-\alpha(\tau)}\right) ds.
\end{aligned} \tag{3.2}$$

By substituting $x(\tau)$ and $u(\tau)$ and (3.2) in (2.2), we obtain a simplified problem then we can solve this problem by Maple 16. Our goal is to solve the following optimization problem over the interval $[\tau_0, \tau_f]$ to find the entries of the vectors a_r, b_r, for $r = 0, 1, \dots, n$.

4. Convergence Analysis

In this section, we can suppose the following problem

$$\min J = \int_0^{\tau_{\max}} x^T(\tau)P(\tau)(\tau) + u^T(\tau)Q(\tau)u(\tau)\ d\tau, \tag{4.1}$$

such that

$$\begin{aligned}
&{}_{0}^{ABC}D_{t}^{\alpha(\tau)}\, x(\tau) - A(\tau)x(\tau) - B(\tau)u(\tau) = F(\tau),\\
&\alpha_i(\tau) \in (0,1), \quad i = 1,2,\ldots,m,\ m \in \mathbb{N},\\
&x(0) = x_0 = a, \quad u(0) = u_0 = b, \quad a, b \in \mathbb{R}, \quad \tau_{max} = 1,
\end{aligned}$$

where $P(\tau)$ and $Q(\tau)$ are given non-negative functions for $\tau \in [0,1]$.

Lemma 4.1. For a polynomial in Bezier form

$$x(\tau) = \sum_{i=0}^{n_2} a_{i,n_2} B_{i,n_2}(\tau),$$

where a_{i,n_2+m_1} is the Bezier coefficient of $x(\tau)$ after being degree-elevated to degree $n_2 + m_1$. Now, we have

$$\frac{\sum_{i=0}^{n_2} a_{i,n_2}^2}{n_2+1} \geq \frac{\sum_{i=0}^{n_2+1} a_{i,n_2+1}^2}{n_2+2} \geq \cdots \geq \frac{\sum_{i=0}^{n_2+m_1} a_{i,n_2+m_1}^2}{n_2+m_1+1}.$$

Proof. See [10]. □

Theorem 4.1. If the problem (4.1) has a unique C^1 continuous solution $\bar{x}$, C^0 continuous control solution $\bar{u}$, then the approximate solution obtained by the control-point-based method converges to the exact solution $(\bar{x}, \bar{u})$ as the degree of the approximate solution tends to infinity.

Proof. Given an arbitrary small positive number $\epsilon > 0$, by the Weierstrass Theorem, one can find polynomials $Q_{1,N_1}(\tau)$ and $Q_{2,N_2}(\tau)$ of degree N_1 and N_2 such that (see [11])

$$\begin{aligned}
\|Q_{1,N_1}(\tau) - \bar{x}(\tau)\|_\infty &\leq \frac{\epsilon}{16\|A(\tau)\|_\infty},\\
\|Q_{2,N_2}(\tau) - \bar{u}(\tau)\|_\infty &\leq \frac{\epsilon}{16\|B(\tau)\|_\infty},\\
\|\,{}_{0}^{ABC}D_{t}^{\alpha(\tau)}\, Q_{1,N_1}(\tau) - {}_{0}^{ABC}D_{t}^{\alpha(\tau)}\bar{x}(\tau)\|_\infty &\leq \frac{\epsilon}{16},
\end{aligned}$$

where $\|\cdot\|_\infty$ stands for the L_∞-norm over $[0,1]$. Now, we have

$$\begin{aligned}
\|a - Q_{1,N_1}(0)\|_\infty &\leq \frac{\epsilon}{16},\\
\|b - Q_{2,N_2}(0)\|_\infty &\leq \frac{\epsilon}{16}.
\end{aligned} \tag{4.2}$$

In general, $Q_{1,N_1}(\tau)$ and $Q_{2,N_2}(\tau)$ do not satisfy the boundary conditions. After a small perturbation with linear and constant polynomials β and γ for $Q_{1,N_1}(\tau)$, $Q_{2,N_2}(\tau)$ we can obtain polynomials $P_{1,N_1}(\tau) = Q_{1,N_1}(\tau) + \beta$, $P_{2,N_1}(\tau) = Q_{2,N_2}(\tau) + \gamma$ such that $P_{1,N_1}(\tau)$ satisfy the boundary conditions $P_{1,N_1}(0) = a$, $P_{2,N_2}(0) = b$. Thus

$Q_{1,N_1}(0) + \beta = a$, $Q_{2,N_2}(0) + \gamma = b$ by utilizing (4.2), one have

$$\begin{aligned} \|a - Q_{1,N_1}(0)\|_\infty =& \|\beta\|_\infty \le \frac{\epsilon}{16}, \\ \|b - Q_{2,N_2}(0)\|_\infty =& \|\gamma\|_\infty \le \frac{\epsilon}{16}. \end{aligned}$$

Now, we have

$$\begin{aligned} \|P_{1,N_1}(\tau) - \bar{x}(\tau)\|_\infty =& \|Q_{1,N_1}(\tau) + \beta - \bar{x}(\tau)\|_\infty \\ \le& \|Q_{1,N_1}(\tau) - \bar{x}(\tau)\|_\infty + \|\beta\|_\infty \le \frac{2\epsilon}{16}, \\ \| {}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau) - {}_0^{ABC}D_t^\alpha \bar{x}(\tau) \|_\infty =& \| {}_0^{ABC}D_t^\alpha Q_{1,N_1}(\tau) - {}_0^{ABC}D_t^\alpha \bar{x}(\tau) \|_\infty < \frac{\epsilon}{16}. \end{aligned}$$

Now, let define

$$\begin{aligned} LP_N(x) = L\left(P_{1,N_1}(\tau), P_{2,N_2}(\tau), {}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau)\right) =& {}_0^{ABC}D_t^\alpha \, P_{1,N_1}(x) \\ & - A(\tau)x(\tau) - B(\tau)u(\tau) = F(\tau), \end{aligned}$$

for every $\tau \in [0,1]$. Thus, for $N \ge N_1$, one may find an upper bound for the following residual:

$$\begin{aligned} \|LP_N(x) - F(\tau)\|_\infty =& \|L\left(P_{1,N_1}(\tau), P_{2,N_2}(\tau), {}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau)\right) - F(\tau)\|_\infty \\ \le& \| {}_0^{ABC}D_t^\alpha \, P_{1,N_1}(\tau) - {}_0^{ABC}D_t^\alpha \bar{x}(\tau)\|_\infty \\ & + \|A(\tau)\|_\infty \|P_{1,N_1}(\tau) - \bar{x}(\tau)\|_\infty + \|B(\tau)\|_\infty \|P_{2,N_2}(\tau) - \bar{u}(\tau)\|_\infty \\ \le& \frac{\epsilon}{16} + \|A(\tau)\|_\infty \frac{\epsilon}{16\|A(\tau)\|_\infty} + \|B(\tau)\|_\infty \frac{\epsilon}{16\|B(\tau)\|_\infty} \le \epsilon. \end{aligned}$$

Since the residual $R(P_N) := LP_N(x) - F(x)$ is a polynomial, we can represent it by a Bezier form. Thus we have

$$R(P_N) := \sum_{i=0}^{m} d_{i,m} B_{i,m}(x). \tag{4.3}$$

Then from Lemma 1 in [10], there exists an integer M, $M \ge N$, such that when $m > M$, we have

$$\left| \frac{1}{m+1} \sum_{i=0}^{m} d_{i,m}^2 - \int_0^1 (R(P_N))^2 dx \right| < \epsilon,$$

which gives

$$\frac{1}{m+1} \sum_{i=0}^{m} d_{i,m}^2 < \epsilon + \int_0^1 (R(P_N))^2 \; dt \le \epsilon. \tag{4.4}$$

Suppose $x(\tau)$ and $u(\tau)$ are approximated solutions of (4.1) obtained by the control-point-based method of degree k, $k \geq m \geq M$. Let

$$\begin{aligned} R(x(\tau), u(\tau), {}_0^{ABC}D_t^{\alpha} x(\tau)) =& L(x(\tau), u(\tau), D^{\alpha} x(\tau)) - F(\tau) \\ =& \sum_{i=0}^{k} c_{i,k} B_{i,k}(x), \quad k \geq m \geq M,\, x \in [0,1]. \end{aligned}$$

Define the following norm for difference approximated solution $(x(\tau), u(\tau))$ and exact solution $(\bar{x}(\tau), \bar{u}(\tau))$:

$$\begin{aligned} \|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| :=& \int_0^1 |\, {}_0^{ABC}D_t^{\alpha}\, x(\tau) - {}_0^{ABC}D_t^{\alpha} \bar{x}(\tau)| d\tau \\ &+ \int_0^1 |x(\tau) - \bar{x}(\tau)| d\tau + \int_0^1 |u(0) - \bar{u}(0)| d\tau. \end{aligned} \tag{4.5}$$

It is easy to show that

$$\begin{aligned} & \|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| \qquad (4.6) \\ =& C(|R(x(\tau), u(\tau), {}_0^{ABC}D_t^{\alpha} x(\tau) - \bar{x}(\tau), \bar{u}(\tau), {}_0^{ABC}D_t^{\alpha} \bar{x}(\tau))| \\ & + |x(0) - \bar{x}(0)| + |u(0) - \bar{u}(0)|) \\ =& \int_0^1 \sum_{i=0}^{k} (c_{i,k} B_{i,k}(t))^2 dx \leq \frac{C}{k+1} \sum_{i=0}^{k} c_{i,k}^2. \qquad (4.7) \end{aligned}$$

Last inequality in (4.6) is obtained from Lemma 1 in [10] which C is a constant positive number. Now from Lemma 4.1 and (4.3), one can easily show that:

$$\begin{aligned} \|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| \leq& \frac{C}{k+1} \sum_{i=0}^{k} c_{i,k}^2 \\ \leq& \frac{C}{k+1} \sum_{i=0}^{k} d_{i,k}^2 \leq \cdots \leq \frac{C}{m+1} \sum_{i=0}^{m} d_{i,m}^2 \\ \leq& C(\epsilon) = \epsilon_1, \quad m \geq M, \end{aligned} \tag{4.8}$$

where last inequality in (4.8) is coming from (4.4). This completes the proof. □

5. Numerical Application

Now, a numerical example of ABC V-O FD is stated to illustrate the BCM. All results are obtained by utilizing Maple 16.

Example 5.1. The following ABC V-O FD is considered (see [9])

$$\begin{aligned} & \min J[u] = \frac{1}{2} \int_0^1 x^2(\tau) + u^2(\tau) d\tau, \\ & {}_0^{ABC}D_t^{\alpha(\tau)}\, x(\tau) = u(\tau) - x(\tau), \\ & x(0) = 1, \end{aligned}$$

where the exact solution with $\alpha(\tau) = 1$ is followed as:

$$\begin{aligned} x_{exact}(\tau) =& \cosh(\sqrt{2}\tau) + \nu \sinh(\sqrt{2}\tau), \\ u_{exact}(\tau) =& (1+\sqrt{2}\nu)\cosh(\sqrt{2}\tau) + (\sqrt{2}+\nu)\sinh(\sqrt{2}\tau), \\ \nu =& -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} = -0.98, \\ J_{exact} =& 0.192909. \end{aligned}$$

By BCM with $\alpha = 0.95 + 0.04\sin(\tau)$, we obtain the following solution:

$$\begin{aligned} x_{approx}(\tau) =& -1.383424070\tau + 0.9779939940\tau^2 - 0.3971018840\tau^3 \\ & + 0.08450149506\tau^4 + 0.9999999998, \\ u_{approx}(\tau) =& -2.365740323 - 0.06762181093\tau + 2.572114069\tau^2 - 8.284294539\tau^3 \\ & + 3.832909652\tau^4, \\ J_{approx} =& 0.1929636321. \end{aligned}$$

The graphs of approximated and exact solution x and u are plotted in Figs. 1, 2 (with $n = 4$). Table 1 shows comparison of the values of $x(\tau)$ and $u(\tau)$ for proposed method, Chebyshev cardinal function method [9], and exact solution (with $\alpha = 0.95 + 0.04\sin(\tau)$).

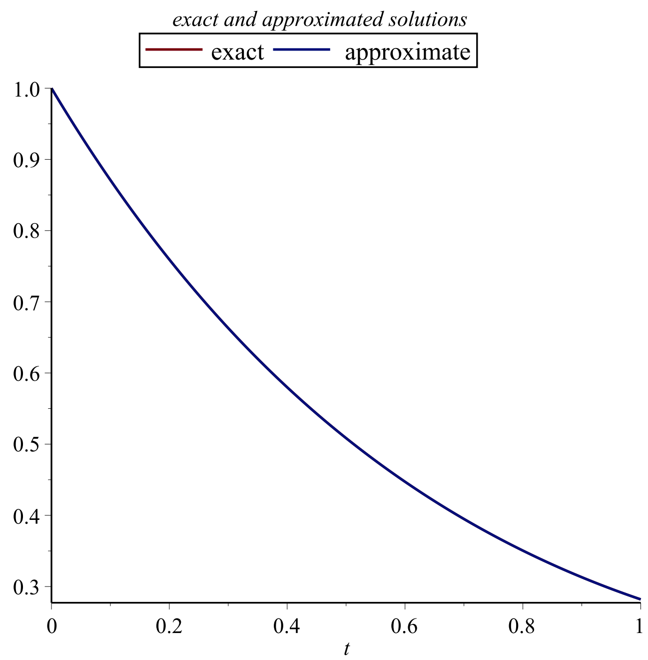

FIGURE 1. The graphs of approximated and exact solution x ($n = 4$) for Example 5.1

Table 1. Comparison of the value of $x(\tau)$ and $u(\tau)$ for proposed method, Chebyshev cardinal function method, and exact solution for Example 5.1

τ	$x(\tau)$ in proposed method	$x(\tau)$ in Chebyshev	$x(\tau)$ in exact	$u(\tau)$ in proposed method	$u(\tau)$ in Chebyshev	$u(\tau)$ in exact
0.2	0.7593933324	0.711981	0.759393	−0.2768738382	−0.305356	−0.276873
0.4	0.5798581284	0.549930	0.579944	−0.1854511779	−0.210872	−0.190227
0.6	0.4472007827	0.430800	0.447200	−0.1141705380	−0.129592	−0.118900
0.8	0.3504725480	0.343133	0.350472	−0.05714883910	−0.061000	−0.057148
1.0	0.2819695347	0.280370	0.281969	0.0	0.012271	0.0

Remark 5.1. In science, some problems such as earthquake engineering, biomedical engineering, can be modeled by fractional integro-differential equations (FIDEs). For analyzing these systems, it is required to obtain the solution of FIDEs. Finding the solution of them is not easy. For solving FIDEs, various techniques are suggested such as, Adomian decomposition method (ADM) [12, 13], Laplace decomposition method (LDM) [14], Taylor expansion method (TEM) [15], Spectral collocation method (SCM) [16]. In this paper, the following FVFIDEs are considered:

$$\begin{aligned} \mu_2(x)y'' + \mu_1(x)y' + \mu_\alpha(x)D_x^\alpha y + \mu_0(x)y =& g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt, \\ &0 < \alpha \le 1,\ \lambda_1, \lambda_2 \in \mathbb{R}, \end{aligned} \tag{5.1}$$

where D_x^α is the Caputo sense fractional derivative. Here, the given functions g, μ_i, for $i = 1, 2, 3$, μ_α, K_1 and K_2 are supposed to be sufficiently smooth. Such equations

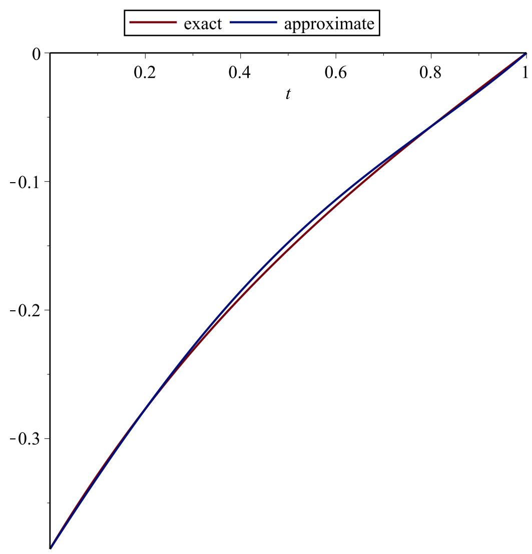

Figure 2. The graphs of approximated and exact solution u $(n = 4)$ for Example 5.1

arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory (see [17]). In this work, one may utilize Bezier curves technique for solving FVFIDEs.

6. Solving FVFIDEs Based on Bezier Curves

Several definitions of a fractional derivative of order $\alpha > 0$ existed.

Definition 6.1. The Caputo's fractional derivative of order α is stated in [18]

$$(D^{\alpha}y)(x) = \frac{1}{\Gamma(n_1-\alpha)}\int_0^x (x-s)^{-\alpha-1+n_1}y^{(n_1)}(s)ds, \quad n_1-1 \le \alpha \le n_1,\, n_1 \in \mathbb{N},$$

where $\alpha > 0$ and n_1 is the smallest integer greater than α.

Definition 6.2. The Riemann-Liouville fractional integer operator of order α is presented in [18]

$$I^{\alpha}y(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}y(s)ds, & \alpha > 0, \\ y(x), & \alpha = 0. \end{cases}$$

6.1. **Function approximation.** Utilizing Bezier curves, this technique is to approximate the solutions $y(x)$ where $y(x)$ is given in (6.2). We define the Bezier polynomials of degree n that approximate over the interval $x \in [x_0, x_f]$ as follows:

$$y \approx P^n y = \sum_{i=0}^{n} c_i B_{i,n}\left(\frac{x-x_0}{h}\right) = C^T B(x),$$

where $h = x_f - x_0$,

$$\begin{aligned} C^T &= [c_0, c_1, \dots, c_n]^T, \\ B^T(x) &= [B_{0,n}(x), B_{1,n}(x), \dots, B_{n,n}(x)]^T, \\ B_{i,n}\left(\frac{x-x_0}{h}\right) &= \binom{n}{i}\frac{1}{h^n}(x_f - x)^{n-i}(x-x_0)^i, \end{aligned} \tag{6.1}$$

is the Bernstein polynomial with degree n for $x \in [x_0, x_f]$, and c_r is the control point [6]. Our technique is utilizing Bezier curves to approximate the solution $y(x)$ in Eq. (5.1). Define the Bezier polynomials of degree n over the interval $[x_0, x_f] = [0, 1]$ as follows:

$$y_n(x) \simeq \sum_{i=0}^{n} c_i B_{i,n}(x), \quad 0 \le x \le 1, \tag{6.2}$$

where

$$B_{i,n}(x) = \binom{n}{i}(1-x)^{n-i}x^i, \quad i = 0, 1, \dots, n.$$

7. Problem Statement for FVFIDEs

From (5.1), one may have

$$\begin{aligned}\mu_\alpha(x)D_x^\alpha y =& g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y), \quad 0<\alpha\le 1.\end{aligned}$$

By utilizing Bezier curve method, one may have

$$y_n(x) \simeq \sum_{i=0}^{n} c_i B_{i,n}(x), \quad 0 \le x \le 1.$$

Therefore,

$$\begin{aligned}\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)' =& n\sum_{i=0}^{n-1}(x)(c_{i+1}-c_i),\\ \left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)'' =& n(n-1)\sum_{i=0}^{n-2} B_{i,n-2}(x)(c_{i+2}-2c_{i+1}+c_i),\\ \frac{\partial}{\partial c_i}\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)'' =& \frac{\partial}{\partial c_i}\left(\sum_{i=2}^{n} B_{i+2,n}(x)(n+2)(n+1)c_i\right.\\ & \left.-2\sum_{i=1}^{n-1} B_{i+1,n-1}(x)n(n+1)c_i + \sum_{i=0}^{n-2} B_{i,n-2}(x)n(n+1)c_i\right)\\ =& \left(\sum_{i=2}^{n} B_{i+2,n}(x)(n+2)(n+1) - 2\sum_{i=1}^{n-1} B_{i+1,n-1}(x)n(n+1)\right.\\ & \left.+\sum_{i=0}^{n-2} B_{i,n-2}(x)n(n+1)\right).\end{aligned}$$

Now

$$\begin{aligned}D^\alpha\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right) =& g(x) + \lambda_1\int_a^x K_1(x,t)\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)dt\\ &+ \lambda_2\int_a^b K_2(x,t)\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)dt\\ &-\left(\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)\right).\end{aligned}$$

One may define

$$\begin{aligned}R(x,c_0,c_1,\ldots,n) =& \sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \left(g(x) + \lambda_1\int_a^x K_1(x,t)y(t)dt\right.\\ &\left.+\lambda_2\int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y)\right).\end{aligned}$$

Also define the following equation

$$S(x, c_0, c_1, \ldots, c_n) = \int_0^1 R(x, c_0, c_1, \ldots, n)^2 w_1(x)dx, \quad w_1(x) = 1,$$

now, we have

$$\begin{aligned} S(x, c_0, c_1, \ldots, c_n) = \int_0^1 \Bigg(\sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \Big(g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ + \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y)\Big)\Bigg)^2 dx. \end{aligned} \tag{7.1}$$

Then

$$\frac{\partial S}{\partial c_i} = 0, \quad 0 \leq i \leq n. \tag{7.2}$$

Using (7.1) and (7.2), we have

$$\begin{aligned} &\int_0^1 \Bigg(\sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \Big(g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y)\Big)\Bigg)^2 dx \times \Big(D^\alpha B_{i,n}(x) \\ &-\lambda_1 \int_a^x K_1(x,t)B_{i,n}(t)dt - \lambda_2 \int_a^b K_2(x,t)B_{i,n}(t)dt - \mu_2(x)\left(B_{i,n}(x)\right)''\Big) = 0. \end{aligned} \tag{7.3}$$

By (7.3), one can obtain a system of $n+1$ linear equations with $n+1$ unknown coefficients c_i. Also by utilizing many subroutine algorithm for solving this linear equations, one can find the unknown coefficients c_i, $i = 0, 1, \ldots, n$.

8. Numerical Applications for FVFIDEs

In this section, we present some numerical examples to illustrate the proposed method.

Example 8.1. Consider the following problem (see [19])

$$\begin{aligned} &y''(x) + D_x^\alpha y(x) - g(x) + 2\int_0^x K_1(x,t)y(t)dt - \int_0^1 K_2(x,t)y(t)dt = 0, \\ &y(0) = 0, \quad y(1) = 0, \\ &g(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15}, \\ &K_1(x,t) = x - t, \quad K_2(x,t) = x^2 - t, \\ &y_{exact}(x) = x^3(x-1), \end{aligned}$$

using the described technique, one may have

$$y_{approx}(x) = -5.551115125 \times 10^{-15} x(1-x)^4 - 1.110223025 \times 10^{-14} x^2(1-x)^3 - 1.000000000 x^3(1-x)^2 - 1.000000000 x^4(1-x),$$

where the absolute error of the proposed method is zero (see Table 2). One may note that Alkan and Hatipoglu [19] obtained the absolute error around 10^{-3}, with $N = 32$. The graphs of approximated solution and exact solution $y(x)$ are plotted in Figure 3.

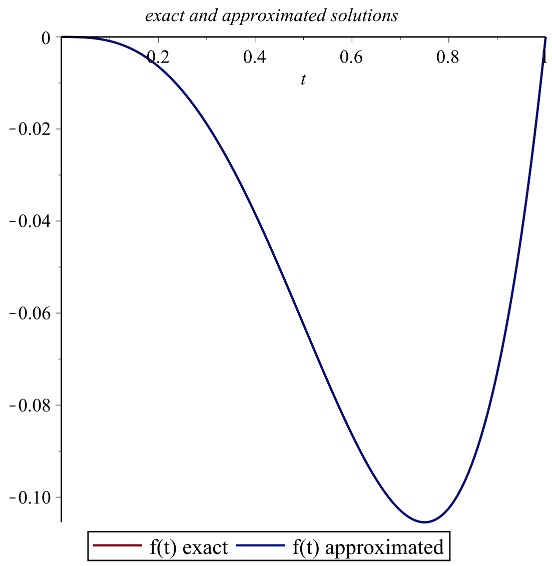

FIGURE 3. The graphs of approximated and exact solution $y(x)$ for Example 8.1

TABLE 2. Exact, estimated values and absolute error of $y(x)$ for Example 8.1

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	0.0009000000000	0.0009000000000	0.0
0.2	0.006400000000	0.006400000000	0.0
0.3	0.01890000000	0.01890000000	0.0
0.4	0.03840000000	0.03840000000	0.0
0.5	0.06250000000	0.06250000000	0.0
0.6	0.08640000000	0.08640000000	0.0
0.7	0.1029000000	0.1029000000	0.0
0.8	0.1024000000	0.1024000000	0.0
0.9	0.07290000000	0.07290000000	0.0
1.0	0.000000000000	0.000000000000	0.0

Example 8.2. Consider the following LFIDE (see [19])

$$y''(x) + \frac{1}{x}D_x^{0.5}y(x) + \frac{1}{x^2} - g(x) - \int_0^x K_1(x,t)y(t)dt - \int_0^1 K_2(x,t)y(t)dt = 0,$$
$$y(0) = 0, \quad y(1) = 0,$$
$$g(x) = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.0674\cos(x) + 5.95385\sin(x),$$
$$K_1(x,t) = \sin(x-t), \quad K_2(x,t) = \cos(x-t),$$
$$y_{exact}(x) = x^2(1-x),$$

using the described technique, one may have

$$y_{approx}(x) = x^2(1-x)^3 + 2x^3(1-x)^2 + x^4(1-x),$$

where the absolute error is zero (see Table 3). One may note that Alkan and Hatipoglu [19] obtained the absolute error around 10^{-7}, with $N = 64$. The graphs of approximated and exact solution $y(x)$ are plotted in Fig. 4.

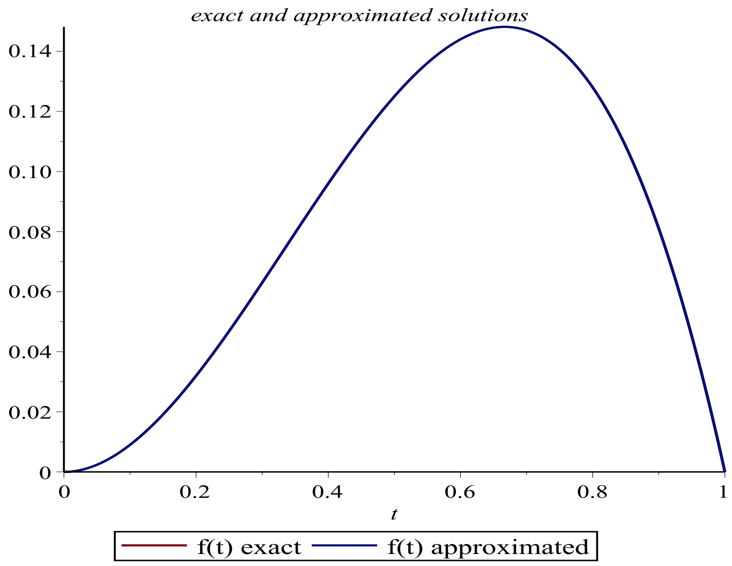

FIGURE 4. The graphs of approximated and exact solution $y(x)$ for Example 8.2

Example 8.3. Consider the following fractional integro-differential equation (see [17])

$$D^\alpha y(x) - x(1+e^x) - 3e^x - y(x) + \int_0^x y(t)dt = 0, \quad \alpha = 4,$$
$$y(0) = 1, \quad y''(0) = 2, \quad y(1) = 1 + e, \quad y''(1) = 3e,$$
$$y_{exact}(x) = 1 + xe^x,$$

using the described technique, one may have

$$y_{approx}(x) = -(1-x)^5 + 5.999923400x(1-x)^4 + 14.99969361x^2(1-x)^3 + 19.50425782x^3(1-x)^2 + 13.15738369x^4(1-x) + x^5(1+e),$$

where the absolute error is less 10^{-5} (see Table 4). The graphs of approximated and exact solution $y(x)$ are plotted in Figure 5.

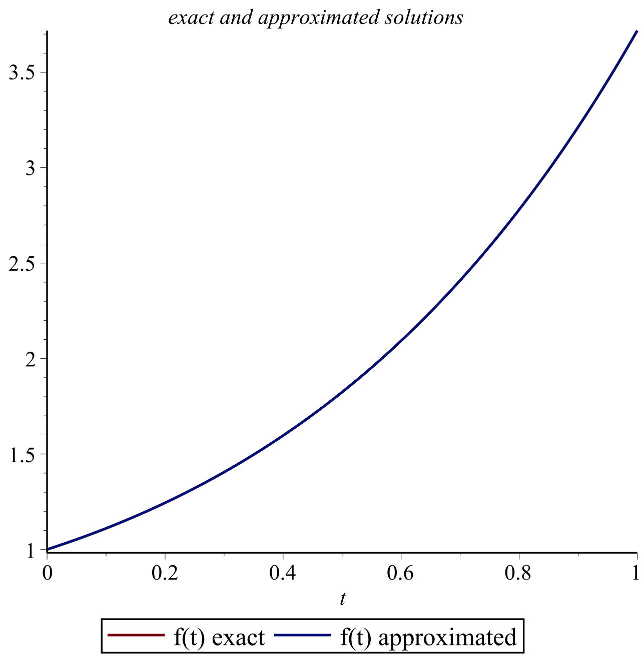

FIGURE 5. The graphs of approximated and exact solution $y(x)$ for Example 8.3

TABLE 3. Exact, estimated values and absolute error of $y(x)$ for Example 8.2

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	0.0009000000000	0.0009000000000	0.0
0.2	0.03200000000	0.03200000000	0.0
0.3	0.06300000000	0.06300000000	0.0
0.4	0.09600000000	0.09600000000	0.0
0.5	0.1250000000	0.1250000000	0.0
0.6	0.1440000000	0.1440000000	0.0
0.7	0.1470000000	0.1470000000	0.0
0.8	0.1280000000	0.1280000000	0.0
0.9	0.08100000000	0.08100000000	0.0
1.0	0.000000000000	0.000000000000	0.0

TABLE 4. Exact, estimated values and absolute error of $y(x)$ for Example 8.3

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	1.110517092	1.110512538	$0.4554981720\times10^{-5}$
0.2	1.244280552	1.244280551	$3.150000000\times10^{-10}$
0.3	1.404957642	1.404964146	$0.6503142000\times10^{-5}$
0.4	1.596729879	1.596736160	$0.6280720000\times10^{-5}$
0.5	1.824360636	1.824360635	$3.200000000\times10^{-10}$
0.6	2.093271280	2.093271279	$1.000000000\times10^{-10}$
0.7	2.409626895	2.409649925	$0.2303010000\times10^{-4}$
0.8	2.780432742	2.780504994	$0.7225290000\times10^{-4}$
0.9	3.213642800	3.213749965	$0.1071650000\times10^{-3}$
1.0	3.718281828	3.718281828	0.0

Example 8.4. We consider the following fourth-order, nonlinear fractional integro-differential equation

$$\begin{aligned}&D^{\alpha}y(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4,\\ &y(0) = 1, \quad y(1) = e, \quad y''(0) = 1, \quad y''(1) = e,\\ &y_{exact}(x) = e^x,\end{aligned}$$

using this method with $n = 6$, we have

$$\begin{aligned}y_{approx}(x) =& 1 + 0.002209653x^6 + 0.00723447x^5 + 0.50002963x^2 + 0.1664541100x^3\\ &+ 0.04235543000x^4 + 0.9999985750x.\end{aligned}$$

From Table 5, we see that, the results obtained with the present method are in good agreement with the results of Momani and Noor [17] and Legendre method [17].

TABLE 5. Comparison of the value of $y(x)$ for Momani and Noor [17], Legendre method [17], stated method, exact value and absolute error of our method for Example 8.4 with $\alpha = 4$

x	Momani and Noor	Legendre	stated method	exact value	our absolute error
0.1	1.10516012	1.10517092	1.105170918	1.105170918	$2.818280000\times10^{-10}$
0.2	1.22138187	1.22140276	1.221402758	1.221402758	$3.700000000\times10^{-11}$
0.3	1.34982923	1.34985881	1.349858768	1.349858808	3.954700000×10^{-8}
0.4	1.49178854	1.49182470	1.491824662	1.491824698	3.563000000×10^{-8}
0.5	1.64868133	1.64872127	1.648721271	1.648721271	$4.400000000\times10^{-10}$
0.6	1.82207855	1.82211880	1.822118800	1.822118800	0.0
0.7	2.01371621	2.01375271	2.013752666	2.013752707	4.120000000×10^{-8}
0.8	2.22551265	2.22554093	2.225540928	2.225540928	$5.000000000\times10^{-10}$
0.9	2.45958740	2.45960311	2.459603313	2.459603111	2.020000000×10^{-7}
1.0	2.71828183	2.71828183	2.718281828	2.718281828	0.0

9. CONCLUSIONS

This paper deals with the approximate solution of ABC V-O fractional derivative and FVFIDEs via BCM. The solution obtained using the suggested method is in very good agreement with the already existing ones and state that this approach can solve the problem effectively. The stated technique reduces the CPU time and the computer memory comparing with existing methods (see some examples). Although the stated technique is very easy to utilize and the obtained results are satisfactory.

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References

[1] H. Hassani and Z. Avazzadeh, *Transcendental bernstein series for solving nonlinear variable order fractional optimal control problems*, Appl. Math. Comput. **366** (2019), Paper ID 124563. https://doi.org/10.1016/j.amc.2019.124563

[2] H. Hassani, Z. Avazzadeh and J. A. T. Machado, *Solving two-dimensional variable-order fractional optimal control problems with transcendental Bernstein series*, Journal of Computational and Nonlinear Dynamics **14**(6) (2019), Paper ID 061001. https://doi.org/10.1115/1.4042997

[3] F. Mohammadi and H. Hassani, *Numerical solution of two-dimensional variable-order fractional optimal control problem by generalized polynomial basis*, J. Optim. Theory Appl. **10**(2) (2019), 536–555. https://doi.org/10.1007/s10957-018-1389-z

[4] F. Ghomanjani and M. H. Farahi, *The Bezier control points method for solving delay differential equation*, Intelligent Control and Automation **3**(2) (2012), 188–196.

[5] F. Ghomanjani and M. H. Farahi, *Optimal control of switched systems based on bezier control points*, International Journal of Intelligent Systems Technologies and Applications **4**(7) (2012), 16–22.

[6] F. Ghomanjani, M. H. Farahi and A. V. Kamyad, *Numerical solution of some linear optimal control systems with pantograph delays*, IMA J. Math. Control Inform. (2015), 225–243. https://doi.org/10.1093/imamci/dnt037

[7] F. Ghomanjani and E. Khorram, *Approximate solution for quadratic Riccati differential equation*, Journal of Taibah University for Science **11**(2) (2017), 246–250. https://doi.org/10.1016/j.jtusci.2015.04.001

[8] F. Ghomanjani, *A new approach for solving fractional differential-algebraic equations*, Journal of Taibah University for Science **11**(6) (2017). https://doi.org/10.1016/j.jtusci.2017.03.006

[9] M. H. Heydari, *Chebyshev cardinal functions for a new class of nonlinear optimal control problems generated by Atangana-Baleanu-Caputo variable-order fractional derivative*, Chaos Solitons Fractals **130** (2020), Paper ID 109401. https://doi.org/10.1016/j.chaos.2019.109401

[10] J. Zheng, T. W. Sederberg and R. W. Johnson, *Least squares methods for solving differential equations using Bezier control points*, Appl. Numer. Math. **48** (2004), 237–252. https://doi.org/10.1016/j.apnum.2002.01.001

[11] F. Ghomanjani, *A new approach for solving fractional differential-algebraic equations*, Journal of Taibah University for Science **11**(6) (2017), 1158–1164. https://doi.org/10.1016/j.jtusci.2017.03.006

[12] R. C. Mittal and R. Nigam, *Solution of fractional integro-differential equations by Adomian decomposition method*, International Journal of Applied Mathematics and Mechanics **4**(2) (2008), 87–94.

[13] S. Momani and M. A. Noor, *Numerical methods for fourth-order fractional integro-differential equations*, Appl. Math. Comput. **182**(1) (2006), 754–760. https://doi.org/10.1016/j.amc.2006.04.041

[14] C. Yang and J. Hou, *Numerical solution of Volterra Integro-differential equations of fractional order by Laplace decomposition method*, World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering **7**(5) (2013), 863–867. https://doi.org/10.5281/zenodo.1087866

[15] L. Huang, X. F. Li, Y. Zhao and X. Y. Duan, *Approximate solution of fractional integro-differential equations by Taylor expansion method*, Comput. Math. Appl. **62**(3) (2011), 1127–1134. https://doi.org/10.1016/j.camwa.2011.03.037

[16] X. Ma and C. Huang, *Spectral collocation method for linear fractional integro-differential equations*, Appl. Math. Model. **38**(4) (2014), 1434–1448. https://doi.org/10.1016/j.apm.2013.08.013

[17] A. Saadatmandi and M. Dehghan, *A Legendre collocation method for fractional integro-differential equations*, J. Vib. Control **17**(13) (2011), 2050–2058. https://doi.org/10.1177/1077546310395977

[18] S. Yuzbasi, *Numerical solution of the Bagley-Torvik equation by the Bessel collocation method*, Math. Methods Appl. Sci. **36** (2013), 300–312. https://doi.org/10.1002/mma.2588

[19] S. Alkan and V. F. Hatipoglu, *Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order*, Tbilisi Math. J. **10**(2) (2017), 1–13. https://doi.org/10.1515/tmj-2017-0021

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APPROXIMATION BY AN EXPONENTIAL-TYPE COMPLEX OPERATORS

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Abstract. In the present paper, we discuss the approximation properties of a complex exponential kind operator. Upper estimate, Voronovskaya-type formula and exact estimate are obtained.

1. Introduction

In the year 1978, Ismail [10] and Ismail and May [11] introduced and studied some exponential type operators. A type of the operators constructed in [11, (3.11)] is the following sequence

$$Q_n(f,x)=\int_0^\infty W(n,x,t)f(t)dt,\quad x\in(0,\infty),\, n\in\mathbb{N}, \tag{1.1}$$

where the kernel is given by

$$W(n,x,t)=\left(\frac{n}{2\pi}\right)^{1/2}\exp\left(n/x\right)t^{-3/2}\exp\left(-\frac{nt}{2x^2}-\frac{n}{2t}\right).$$

The kernel of these operators satisfies the partial differential equation

$$\frac{\partial}{\partial x}W(n,x,t)=\frac{n(t-x)}{x^3}W(n,x,t). \tag{1.2}$$

Due to its complicated behavior in integration, these operators were not previously much studied by researchers. Recently in case of real variables these operators were studied by Gupta [8], who established some direct results. The asymptotic formula for certain exponential type operators are discussed in [1].

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Also, in the recent years, the study of approximation by complex operators on compact disks is an active area of research, see for instance [2–4, 6, 7, 9] and [12] etc.

In this paper, we study the approximation properties of the complex variant in (1.1), obtained by replacing x with z in the formula (1.1). Section 2 contains some auxiliary results used in the next sections. Section 3 deals with upper estimate, while in Section 4, we study a Voronovskaya-type result and the exact estimate in approximation.

2. Auxiliary Results

The proofs of our main results require three additional lemmas, as follows.

Lemma 2.1. *If we denote $T_{n,m}(x) = Q_n(e_m, x)$, $e_m(t) = t^m$, then using Mapple, we find that $T_{n,0}(x) = 1$ and there holds the following recurrence relation:*

$$nT_{n,m+1}(x) = x^3[T_{n,m}(x)]' + nxT_{n,m}(x), \quad n, m \in \mathbb{N}.$$

In particular

$$\begin{aligned}
T_{n,0}(x) =&1,\\
T_{n,1}(x) =&x,\\
T_{n,2}(x) =&x^2 + \frac{x^3}{n},\\
T_{n,3}(x) =&x^3 + \frac{3x^4}{n} + \frac{3x^5}{n^2},\\
T_{n,4}(x) =&x^4 + \frac{6x^5}{n} + \frac{15x^6}{n^2},\\
T_{n,5}(x) =&x^5 + \frac{10x^6}{n} + \frac{45x^7}{n^2} + \frac{105x^8}{n^3} + \frac{105x^9}{n^4},\\
T_{n,6}(x) =&x^6 + \frac{15x^7}{n} + \frac{105x^8}{n^2} + \frac{420x^9}{n^3} + \frac{945x^{10}}{n^4} + \frac{945x^{11}}{n^5}.
\end{aligned}$$

Proof. By definition

$$T_{n,m}(x) \quad = \quad \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/x) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2x^2} - \frac{n}{2t}\right) t^m dt.$$

Thus, differentiating w.r.t x both the sides and using (1.2), we have

$$\begin{aligned}
x^3[T_{n,m}(x)]' =& \int_0^\infty x^3[W(n,x,t)]' t^m dt\\
=& \int_0^\infty n(t-x)W(n,x,t)t^m dt\\
=& nT_{n,m+1}(x) - nxT_{n,m}(x).
\end{aligned}$$

This completes the proof of lemma, other consequences follow from the recurrence relation. $\square$

Lemma 2.2. *Suppose that* $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *is an entire function satisfying the condition* $|c_k| \leq M\frac{A^k}{k!}$, $k = 0, 1, \ldots,$ *with* $M > 0$ *and* $A \in (0, 1/2)$ (*which implies that* f *is of exponential growth since* $|f(z)| \leq M \exp(A|z|)$ *for all* $z \in \mathbb{C}$). *Then* $Q_n(f, z)$ *is well defined for any* $n \in \mathbb{N}$ *and any* $z \in \mathbb{C}$ *satisfying*

$$\mathrm{Re}\,(z^2) > 0 \quad \textit{and} \quad \frac{|z|^2}{\mathrm{Re}\,(z^2)} < \frac{1}{2A}. \tag{2.1}$$

Proof. Since $|\exp(z)| = \exp(\mathrm{Re}\,(z))$, $\mathrm{Re}\,(1/z) = \mathrm{Re}\,(z)/|z|$ and $\mathrm{Re}\,(1/z^2) = \mathrm{Re}\,(z^2)/|z|^2$, we get

$$\begin{aligned}
&|Q_n(f, z)| \\
\leq& M \left(\frac{n}{2\pi}\right)^{1/2} |e(n/z)| \int_0^{\infty} t^{-3/2} \exp(-n/(2t) + At)| \exp(-nt/(2z^2))| dt \\
=& M \exp(n\mathrm{Re}\,(z)/|z|) \int_0^{\infty} t^{-3/2} \exp(-n/(2t)) \exp(-t[n\mathrm{Re}\,(z^2)/(2|z|^2) - A]) dt.
\end{aligned}$$

By the hypothesis on z, we easily seen that $n\mathrm{Re}\,(z^2)/(2|z|^2) - A > 0$ for all $n \geq 1$. Therefore, for fixed z as in the hypothesis and denoting $n\mathrm{Re}\,(z^2)/(2|z|^2) - A$ with $C > 0$, we have to deal with the existence of the integral

$$I := \int_0^{\infty} t^{-3/2} \exp(-n/(2t)) \exp(-Ct) dt.$$

Changing the variable $t = \frac{1}{v}$, we easily obtain

$$I = \int_0^{\infty} v^{-1/2} \exp(-nv/2) \exp(-C/v) dv < \infty.$$

Indeed, for $K > 0$ an arbitrary fixed constant, we have

$$\begin{aligned}
I =& \int_0^{K} v^{-1/2} \exp(-nv/2) \exp(-C/v) dv + \int_K^{\infty} v^{-1/2} \exp(-nv/2) \exp(-C/v) dv \\
:=& I_1 + I_2,
\end{aligned}$$

where

$$I_1 \leq \int_0^{K} \exp(-nv/2) v^{-1/2} \frac{v}{C} dv \leq \frac{1}{C} \int_0^{K} v^{1/2} \exp(-nv/2) dv < \infty$$

and $I_2 \leq \frac{1}{\sqrt{K}} \int_K^{\infty} e(-nv/2) dv < \infty$. □

Lemma 2.3. *Suppose that* f *is an entire function, i.e.,* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $z \in \mathbb{C}$ *such that there exist* $M > 0$ *and* $A \in (0, 1)$, *with the property* $|c_k| \leq M\frac{A^k}{k!}$ *for all* $k = 0, 1, \ldots$ (*which implies* $|f(z)| \leq M \exp(A|z|)$ *for all* $z \in \mathbb{C}$).

Then for all $n \in \mathbb{N}$ *and* z *satisfying* (2.1), *we have*

$$Q_n(f, z) = \sum_{k=0}^{\infty} c_k Q_n(e_k, z).$$

Proof. Since we can write

$$Q_n(f;z) = \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/z) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right) \left(\sum_{k=0}^\infty c_k t^k\right) dt,$$

if above the integral would commute with the infinite sum, then we would obtain

$$\begin{aligned} Q_n(f,z) =& \sum_{k=0}^\infty c_k \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/z) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right) t^k dt \\ =& \sum_{k=0}^\infty c_k Q_n(e_k, z). \end{aligned}$$

It is well-known by the Fubini type result that a sufficient condition for the commutativity is that

$$\int_0^\infty t^{-3/2} \left|\exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right)\right| \left(\sum_{k=0}^\infty |c_k| t^k\right) dt < \infty.$$

Applied to our case, for $n \in \mathbb{N}$ and z satisfying (2.1), we get

$$\begin{aligned} &\int_0^\infty t^{-3/2} \left|\exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right)\right| \left(\sum_{k=0}^\infty |c_k| t^k\right) dt \\ \leq& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right) \left(\sum_{k=0}^\infty \frac{A^k t^k}{k!}\right) dt \\ =& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right) e^{At} dt \\ =& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2) + At\right) dt < \infty, \end{aligned}$$

by the proof of Lemma 2.2. □

Remark 2.1. It is easy to see that from geometric point of view, the conditions on z in (2.1) means that z belongs to two symmetric cones with respect to origin (but without containing the origin) containing the x axis, which are included in the two symmetric cones with respect to origin between the first and second bisectrix, containing the x axis. Indeed, since $|z|^2 = x^2 + y^2$ and $\mathrm{Re}\,(z^2) = x^2 - y^2$, simple calculations show that the condition (2.1) satisfied by $z = x + iy$ can easily be written under the form

$$\sqrt{\left(1 + \frac{1}{2A}\right)}|y| < \sqrt{\left(\frac{1}{2A} - 1\right)}|x|,$$

that is

$$\frac{|y|}{|x|} < \frac{\sqrt{1/(2A) - 1}}{\sqrt{1/(2A) + 1}} < 1.$$

3. Upper Estimate

The first main result concerns an upper estimate in approximation by $Q_n(f,z)$.

Theorem 3.1. *Suppose that f is an entire function, i.e., $f(z)=\sum_{k=0}^{\infty}c_k z^k$ for all $z\in\mathbb{C}$ such that there exist $M>0$ and $A\in(0,1/2)$, with the property $|c_k|\le M\frac{A^k}{k!}$, for all $k=0,1,\dots$ (which implies $|f(z)|\le Me^{A|z|}$ for all $z\in\mathbb{C}$). Consider $1\le r<\frac{1}{A}$.*

Then for all $n\ge r^2$, $|z|\le r$ and z satisfying (2.1)*, the following estimate hold:*

$$|Q_n(f,z)-f(z)|\le \frac{C_{r,M,A}}{n},$$

where $C_{r,M,A}=Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k<\infty$.

Proof. By Lemma 2.1 written with x replaced by z, we easily obtain

$$n[T_{n,m+1}(z)-z^{m+1}]=z^3[T_{n,m}(z)-z^m]'+nz[T_{m,n}(z)-z^m]+mz^{m+2}.$$

Applying the Bernstein's inequality on $|z|\le r$ to the polynomial of degree m, $T_{n,m}(z)-z^m$, we get $\|[T_{n,m}(z)-z^m]'\|_r\le\frac{m}{r}\|T_{n,m}(z)-z^m\|_r$, where $\|P\|_r=\sup_{|z|\le r}|P(z)|$. Then, denoting $e_m=z^m$, from the above recurrence we immediately obtain

$$\|T_{n,m+1}-e_{m+1}\|_r\le\left(r+\frac{mr^2}{n}\right)\|T_{m,n}-e_m\|_r+\frac{mr^{m+2}}{n}.$$

In what follows we prove by mathematical induction with respect to m that for $n\ge r^2$, this recurrence implies

$$||T_{n,m}-e_m||_r\le\frac{(m+1)!}{n}r^{m+1},\quad \text{for all } m\ge 0.$$

Indeed for $m=0$ and $m=1$ it is trivial, as the left-hand side is zero. Suppose that it is valid for m, the above recurrence relation implies that

$$||T_{n,m+1}-e_{m+1}||_r\le\left(r+\frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1}+\frac{m}{n}r^{m+2}.$$

It remains to prove that

$$\left(r+\frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1}+\frac{m}{n}r^{m+2}\le\frac{(m+2)!}{n}r^{m+2},$$

or after simplifications, equivalently to

$$\left(r+\frac{r^2m}{n}\right)(m+1)!+rm\le(m+2)!r,$$

for all $m\in\mathbb{N}$ and $r\ge 1$.

Since $n\ge r^2$, we get

$$\left(r+\frac{r^2m}{n}\right)(m+1)!+rm\le(r+m)\,(m+1)!+rm,$$

it is good enough if we prove that

$$(r+m)\,(m+1)!+rm\leq (m+2)!r.$$

But this last inequality is obviously equivalent with

$$m(m+1)!+rm\leq rm(m+1)!+r(m+1)!,$$

which is clearly valid for all $m\geq 1$ (and fixed $r\geq 1$).

Finally, taking into account Lemma 2.3, for all $n\geq r^2$, we obtain

$$\begin{aligned}|Q_n(f,z)-f(z)|&\leq \sum_{k=0}^{\infty}|c_k|\cdot|Q_n(e_k,z)-e_k(z)|\\ &\leq \frac{M}{n}\cdot\sum_{k=2}^{\infty}\frac{A^k}{k!}\cdot(k+1)!r^{k+1}=\frac{C_{r,M,A}}{n},\end{aligned}$$

where $C_{r,M,A}=Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k<\infty$. □

Remark 3.1. The smaller A is, the larger is the portion of the symmetrical cones where the estimation in Theorem 3.1 takes place. This happens because of the intersection between the symmetrical cones and the disk $\{|z|\leq r\}$ with $1\leq r<\frac{1}{A}$, where if $A\searrow 0$ then $r\nearrow\infty$.

4. Voronovskaya Type Formula and Exact Estimate

The following estimate is a Voronovskaja-kind quantitative result.

Theorem 4.1. *Suppose that f is an entire function, i.e., $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{C}$ such that there exist $M>0$ and $A\in(0,1/2)$, with the property $|c_k|\leq M\frac{A^k}{k!}$, for all $k=0,1,\dots$ (which implies $|f(z)|\leq M\exp(A|z|)$ for all $z\in\mathbb{C}$). Consider $1\leq r<\frac{1}{A}$.*

Then for all $n\geq r^2$, $|z|\leq r$ and z satisfying (2.1), *the following estimate holds:*

$$\left|Q_n(f,z)-f(z)-\frac{z^3f''(z)}{2n}\right|\leq\frac{E_{r,M,A}(f)}{n^2},$$

where

$$E_{r,M,A}(f)=3Mr^2\sum_{k=2}^{\infty}(k+1)^2(Ar)^k<\infty.$$

Proof. Everywhere in the proof consider z and n as in hypothesis.

By the proof of Lemma 2.3, we can write $Q_n(f,z)=\sum_{k=0}^{\infty}c_k\,Q_n(e_k,z)$. Also, since

$$\frac{z^3f''(z)}{2n}=\frac{z^3}{2n}\sum_{k=2}^{\infty}c_kk(k-1)z^{k-2}=\frac{1}{2n}\sum_{k=2}^{\infty}c_k\,k(k-1)\,z^{k+1},$$

we get

$$\left|Q_n(f,z)-f(z)-\frac{z^3\,f''(z)}{2n}\right|\leq\sum_{k=2}^{\infty}|c_k|\left|T_{n,k}(z)-e_k(z)-\frac{k(k-1)z^{k+1}}{2n}\right|.$$

By Lemma 2.1, we have

$$T_{n,k}(z) = \frac{z^3}{n}\, T'_{n,k-1}(z) + z\, T_{n,k-1}(z).$$

If we denote

$$J_{n,k}(z) = T_{n,k}(z) - e_k(z) - \frac{k(k-1)z^{k+1}}{2n},$$

then it is obvious that $J_{n,k}(z)$ is a polynomial of degree less than or equal to $k+2$ and by simple computation and the use of above recurrence relation, we are led to

$$J_{n,k}(z) \quad = \quad \frac{z^3}{n} J'_{n,k-1}(z) + zJ_{n,k-1}(z) + X_{n,k}(z),$$

where after simple computation, we have

$$X_{n,k}(z) \quad = \quad \frac{k(k-1)(k-2)z^{k+2}}{2n^2}.$$

Using the estimate in the proof of Theorem 3.1, we have

$$|T_{n,k}(z) - e_k(z)| \leq \frac{(k+1)!}{n} \cdot r^{k+1}.$$

It follows

$$|J_{n,k}(z)| \leq \frac{r^3}{n}\, |J'_{n,k-1}(z)| + r\, |J_{n,k-1}(z)| + |X_{n,k}(z)|,$$

where

$$|X_{n,k}(z)| \leq \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.$$

Now we shall find the estimation of $|J'_{n,k-1}(z)|$. Taking into account the fact that $J_{n,k-1}(z)$ is a polynomial of degree $\leq k+1$, we have

$$\begin{aligned}
|J'_{n,k-1}(z)| \leq & \frac{k}{r}||J_{n,k-1}(z)||_r \\
\leq & \frac{k}{r}\left[||T_{n,k-1}(z) - e_{k-1}(z)||_r + \frac{(k-1)(k-2)r^k}{2n}\right] \\
\leq & \frac{(k+1)!}{n} \cdot r^{k-1} + \frac{k(k-1)(k-2)r^{k-1}}{2n}.
\end{aligned}$$

Thus,

$$\frac{r^3}{n}|J'_{n,k-1}(z)| \leq \frac{1}{n}\left[\frac{(k+1)!}{n}r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n}\right]$$

and

$$\begin{aligned}
|J_{n,k}(z)| \leq & r|J_{n,k-1}(z)| + \frac{1}{n}\left[\frac{(k+1)!}{n}r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n}\right] \\
& + \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.
\end{aligned}$$

This immediately implies

$$|J_{n,k}(z)| \leq r|J_{n,k-1}(z)| + \frac{3}{n^2}(k+1)!r^{k+2}.$$

By writing this inequality for $k = 1, 2, 3, \ldots$, we easily obtain step by step the following

$$|J_{n,k}(z)| \leq \frac{3}{n^2} r^{k+2} \left[\sum_{j=1}^{k+1} j! \right] \leq \frac{3}{n^2} r^{k+2} (k+1)!(k+1).$$

In conclusion,

$$\begin{aligned} \left| Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n} \right| \leq & \frac{3}{n^2} \cdot \sum_{k=2}^{\infty} |c_k| r^{k+2} \cdot (k+1)!(k+1) \\ \leq & \frac{3Mr^2}{n^2} \cdot \sum_{k=2}^{\infty} (k+1)^2 (Ar)^k. \end{aligned}$$

This completes the proof of theorem. □

Using the above Voronovskaja's theorem, we obtain the following lower order in approximation.

Theorem 4.2. *Under the hypothesis in Theorem* 4.1, *if f is not a polynomial of degree ≤ 1, then for all $n \geq r^2$ we have*

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{K_{r,M,A}(f)}{n},$$

where $\|F\|_r^* = \sup\{|F(z)| : |z| \leq r$ and z satisfies (2.1)$\}$ *and $K_{r,M,A}(f)$ is a constant which depends only on f, M, A and r.*

Proof. For all $n \geq r^2$, $|z| \leq r$ and z satisfying (2.1), we have

$$Q_n(f,z) - f(z) = \frac{1}{n} \left[0.5\, z^3 f''(z) + \frac{1}{n} \left\{ n^2 \left(Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n} \right) \right\} \right].$$

Also, we have

$$||F + G||_r^* \geq |||F||_r^* - ||G||_r^*| \geq ||F||_r^* - ||G||_r^*.$$

It follows

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{1}{n} \left[||\, 0.5\, e_3\, f''\, ||_r^* - \frac{1}{n} \left\{ n^2 \left\| Q_n(f,\cdot) - f - \frac{e_3\, f''}{2n} \right\|_r^* \right\} \right].$$

Taking into account that by hypothesis, f is not a polynomial of degree ≤ 1, we get $||0.5\, e_3\, f''||_r^* > 0$. Indeed, supposing the contrary it follows that $z^3 f''(z) = 0$, which by the fact that f is entire function, clearly implies $f''(z) = 0$, i.e., f is a polynomial of degree ≤ 1, a contradiction with the hypothesis.

Now by Theorem 4.1, we have

$$n^2 \left\| Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n} \right\|_r^* \leq E_{r,M,A}(f).$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$, we have

$$|| \, 0.5 \, e_3 \, f'' \, ||_r^* - \frac{1}{n} \left\{ n^2 \left\| Q_n(f,z) - f(z) - \frac{0.5 \, z^3 \, f''(z)}{n} \right\|_r^* \right\} \geq \frac{1}{2} \, || \, 0.5 e_3 \, f'' \, ||_r^* \, ,$$

which immediately implies

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{1}{2n} || \, 0.5 \, e_3 \, f'' \, ||_r^*, \quad \text{for all } n \geq n_0.$$

For $n \in \{1, 2, \ldots, n_0 - 1\}$ we obviously have

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{M_{r,n}(f)}{n},$$

with $M_{r,n}(f) = n||Q_n(f,\cdot) - f||_r^* > 0$. Indeed, if we would have $||Q_n(f,\cdot) - f||_r^* = 0$, then would follow $Q_n(f,z) = f(z)$ for all $|z| \leq r$, z satisfying (2.1), which is valid only for f a polynomial of degree ≤ 1, contradicting the hypothesis on f. Hence, we obtain $||Q_n(f,\cdot) - f||_r^* \geq \frac{K_{r,M,A}(f)}{n}$ for all n, where

$$K_{r,M,A}(f) = \min \left\{ M_{r,1}(f), M_{r,2}(f), \ldots, M_{r,n_0-1}(f), \, \frac{1}{2} \, || \, 0.5 \, e_3 \, f'' \, ||_r^* \right\},$$

which completes the proof. □

Combining Theorem 3.1 with Theorem 4.2, we immediately get the following exact estimate.

Corollary 4.1. *Under the hypothesis in Theorem* 4.1*, if f is not a polynomial of degree ≤ 1, then we have*

$$||Q_n(f,\cdot) - f||_r^* \sim \frac{1}{n}, \quad n \in \mathbb{N},$$

where the symbol $\sim$ represents the well-known equivalence between the orders of approximation.

Remark 4.1. Particular cases of the exponential-type operators studied in the real case in [11], are the Bernstein polynomials, the operators of Szász, of Post-Widder, of Gauss-Weierstrass, of Baskakov, to mention only a few. In the complex variable case, only the approximation properties of the operators of Bernstein, Szász, Baskakov and Post-Widder were already studied, see, e.g., [5, 7, 9]. It remains as open question to use the method in this paper for other complex exponential-type operators, too.

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References

[1] T. Acar, *Asymptotic formulas for generalized Szász-Mirakyan operators*, Appl. Math. Comput. **263** (2015), 233–239. https://doi.org/10.1016/j.amc.2015.04.060

[2] R. P. Agarwal and V. Gupta, *On q-analogue of a complex summation-integral type operators in compact disks*, J. Inequal. Appl. **2012**(1) (2012), Article ID 111. https://doi.org/10.1186/1029-242X-2012-111

[3] S. G. Gal and V. Gupta, *Quantitative estimates for a new complex Durrmeyer operator in compact disks*, Appl. Math. Comput. **218**(6) (2011), 2944–2951. https://doi.org/10.1016/j.amc.2011.08.044

[4] S. G. Gal, V. Gupta and N. I. Mahmudov, *Approximation by a Durrmeyer-type operator in compact disks*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **58**(2) (2012), 65–87. https://doi.org/10.1007/s11565-011-0124-6

[5] S. G. Gal and V. Gupta, *Approximation by a complex Post-Widder type operator*, Anal. Theory Appl. **34**(4) (2018), 297–305. https://doi.org/10.4208/ata.OA-2018-0003

[6] S. G. Gal, *Approximation by Complex Bernstein and Convolution Type Operators*, World Scientific, 2009. https://doi.org/10.1142/7426

[7] S. G. Gal, *Overconvergence in Complex Approximation*, Springer, New York, 2013. https://doi.org/10.1007/978-1-4614-7098-4

[8] V. Gupta, *Approximation with certain exponential operators*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114**(2) (2020). https://doi.org/10.1007/s13398-020-00792-9

[9] V. Gupta and R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-02765-4

[10] M. Ismail, *Polynomials of binomial type and approximation theory*, J. Approx. Theory **23**(1978), 177–186. https://doi.org/10.1016/0021-9045(78)90105-3

[11] M. Ismail and C. P. May, *On a family of approximation operators*, J. Math. Anal. Appl. **63** (1978), 446–462. https://doi.org/10.1016/0022-247X(78)90090-2

[12] A. S. Kumar, P. N. Agrawal and T. Acar, *Quantitative estimates for a new complex q-Durrmeyer type operators on compact disks*, UPB Scientific Bulletin, Series A **80**(1) (2018), 191–210.

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ON THE GENERALIZATION OF FRACTIONAL KINETIC EQUATION COMPRISING INCOMPLETE H-FUNCTION

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Abstract. In the present work, a novel and even more generalized fractional kinetic equation has been formulated in terms of polynomial weighted incomplete H-function, incomplete Fox-Wright function and incomplete generalized hypergeometric function, considering the importance of the fractional kinetic equations arising in the various science and engineering problems. All the derived findings are of natural type and can produce a variety of fractional kinetic equations and their solutions.

1. Introduction and Mathematical Preliminaries

In order to explain the memory effects of complicated systems, the fact that fractional derivatives add a convolution integral with a power-law memory kernel or exponential-law memory kernel strengthens the value of fractional differential equations. It can also be shown that in the last few decades, very fascinating and revolutionary applications of fractional calculus operators have been developed in physics, chemistry, biology, engineering, finance and other fields of research. Some of the applications include: diffusion processes, mechanics of materials, combinatorics, inequalities, signal processing, image processing, advection and dispersion of solutes in porous or fractured media, modeling of viscoelastic materials under external forces, bioengineering, relaxation and reaction kinetics of polymers, random walks, mathematical finance, modeling of combustion, control theory, heat propagation, modeling

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of viscoelastic materials, in biological systems and many more. The recent work [1–4], and references therein, can be referred to for further information.

Because of the importance in astrophysics, control systems, and mathematical physics, the research on the fractional kinetic equations and their solution has attracted interest from many researchers [5–12]. The fractional kinetic equation has indeed been commonly used to test various physical phenomena regulating the diffusion in porous media, reactions and relaxation mechanisms in complex structures. As a consequence, a significant number of research papers (see [13–20]) focused on the solution of these equations including generalized Mittag-Leffler function, Bessel's function, Struve function, G-function, H-function, and Aleph-function have recently been written. In this new fractional generalization of the kinetic equation, use of the incomplete special functions gives a different dimension to this study. The equation involves a family of polynomials, incomplete H-function, incomplete Fox-Wright function and incomplete generalized hypergeometric function. For these fractional kinetic equations, the Laplace transformation technique is used to derive the solution. Special cases are also illustrated in brief.

Haubold and Mathai [16] set the fractional differential equation within the rate of change of reaction, $\mathcal{N} = \mathcal{N}(\mathfrak{t})$, the rate of destruction, $\delta(\mathcal{N}_\mathfrak{t})$, and the rate of growth, $p(\mathcal{N}_\mathfrak{t})$, as follows:

$$\frac{d\mathcal{N}}{d\mathfrak{t}} = -\delta(\mathcal{N}_\mathfrak{t}) + p(\mathcal{N}_\mathfrak{t}), \tag{1.1}$$

where $\mathcal{N}_\mathfrak{t}$ is given by $\mathcal{N}_\mathfrak{t}(\mathfrak{t}^*) = \mathcal{N}(\mathfrak{t} - \mathfrak{t}^*)$, $\mathfrak{t}^* > 0$.

In addition, Haubold and Mathai [16] gave the limiting case of (1.1) when $\mathcal{N}(\mathfrak{t})$ in the quantity of spatial fluctuations or homogeneities is ignored and given as

$$\frac{d\mathcal{N}_j}{d\mathfrak{t}} = -c_j\,\mathcal{N}_j(\mathfrak{t}), \tag{1.2}$$

where $\mathcal{N}_j(\mathfrak{t} = 0) = \mathcal{N}_0$ is the amount of density of species j at time $\mathfrak{t} = 0$, $c_j > 0$. If the index j is dropped and the typical kinetic equation (1.2) is integrated, we receive

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0 = -c\,{}_0D_\mathfrak{t}^{-1}\mathcal{N}(\mathfrak{t}), \tag{1.3}$$

where ${}_0D_\mathfrak{t}^{-1}$ is the specialized case of the Riemann-Liouville fractional integral operator ${}_0D_\mathfrak{t}^{-\nu}$ lay it out as

$${}_0D_\mathfrak{t}^{-\nu}f(\mathfrak{t}) = \frac{1}{\Gamma(\nu)}\int_0^\mathfrak{t}(\mathfrak{t} - u)^{\nu-1}f(u)du, \quad \mathfrak{t} > 0,\ \mathrm{Re}\,(\nu) > 0. \tag{1.4}$$

Haubold and Mathai [16] gave the fractional thought to the classical kinetic equation by considering fractional derivative rather than the total derivative in (1.2)

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0 = -c^\nu\,{}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}). \tag{1.5}$$

Then the solution for $\mathcal{N}(\mathfrak{t})$ is a Mittag-Leffler function $\mathrm{E}_\nu(\cdot)$

$$\mathcal{N}(\mathfrak{t}) = \mathcal{N}_0\sum_{r=0}^{\infty}\frac{(-1)^r(c\,\mathfrak{t})^{\nu r}}{\Gamma(\nu r + 1)} = \mathcal{N}_0\,\mathrm{E}_\nu(-c^\nu\mathfrak{t}^\nu). \tag{1.6}$$

In addition, Saxena and Kalla [17] thought about the subsequent fractional kinetic equation

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0 f(\mathfrak{t}) = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{1.7}$$

where $f(\mathfrak{t}) \in \mathcal{L}(0,\infty)$.

The Laplace transformation of the Riemann-Liouville fractional integration of $f(t)$ given in the equation (1.4) is specified as

$$\mathrm{L}\left[{}_0D_{\mathfrak{t}}^{-\nu} f(\mathfrak{t});\omega\right] = \omega^{-\nu}\, F(\omega), \quad \mathfrak{t} > 0,\ \mathrm{Re}\,(\nu) > 0,\ \mathrm{Re}\,(\omega) > 0, \tag{1.8}$$

where $F(\omega)$ is the Laplace transform of the function $f(\mathfrak{t})$ and given by

$$F(\omega) = \mathrm{L}[f(\mathfrak{t});\omega] = \int_0^{\infty} e^{-\omega\mathfrak{t}} f(\mathfrak{t})\, d\mathfrak{t}, \quad \mathfrak{t} > 0,\ \mathrm{Re}\,(\omega) > 0. \tag{1.9}$$

On the other hand, the familiar lower and upper incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, respectively are defined as (see [21]):

$$\gamma(\xi,x) = \int_0^{x} u^{\xi-1}\, e^{-u}\, du, \quad \mathrm{Re}\,(\xi) > 0,\ x \geq 0, \tag{1.10}$$

and

$$\Gamma(\xi,x) = \int_x^{\infty} u^{\xi-1}\, e^{-u}\, du, \quad x \geq 0,\ \mathrm{Re}\,(\xi) > 0 \text{ if } x = 0. \tag{1.11}$$

These functions fulfill the following relation:

$$\gamma(\xi,x) + \Gamma(\xi,x) = \Gamma(\xi), \quad \mathrm{Re}\,(\xi) > 0. \tag{1.12}$$

By the use of above defined incomplete gamma functions, Srivastava et al. [21] defined the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ given as

$$\begin{aligned} {}_p\gamma_q\left[\begin{matrix}(a_1,x), & a_2,\dots, & a_p; & \\ & b_1,\dots, & b_q; & \end{matrix}\; z\right] &= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} \sum_{\ell=0}^{\infty} \frac{\gamma(a_1+\ell,x)\prod_{j=2}^{p}\Gamma(a_j+\ell)}{\prod_{j=1}^{q}\Gamma(b_j+\ell)} \cdot \frac{z^{\ell}}{\ell!} \\ &= \sum_{\ell=0}^{\infty} \frac{(a_1,x)_{\ell}\,(a_2)_{\ell}\cdots(a_p)_{\ell}}{(b_1)_{\ell}\cdots(b_q)_{\ell}} \cdot \frac{z^{\ell}}{\ell!} \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} {}_p\Gamma_q\left[\begin{matrix}(a_1,x), & a_2,\dots, & a_p; & \\ & b_1,\dots, & b_q; & \end{matrix}\; z\right] &= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} \sum_{\ell=0}^{\infty} \frac{\Gamma(a_1+\ell,x)\prod_{j=2}^{p}\Gamma(a_j+\ell)}{\prod_{j=1}^{q}\Gamma(b_j+\ell)} \cdot \frac{z^{\ell}}{\ell!} \\ &= \sum_{\ell=0}^{\infty} \frac{[a_1,x]_{\ell}\,(a_2)_{\ell}\cdots(a_p)_{\ell}}{(b_1)_{\ell}\cdots(b_q)_{\ell}} \cdot \frac{z^{\ell}}{\ell!}, \end{aligned} \tag{1.14}$$

where $(a,x)_{\ell}$ and $[a,x]_{\ell}$ are incomplete Pochhammer symbols defined below and $(a)_{\ell}$ is Pochhammer symbol

$$(a,x)_{\ell} = \frac{\gamma(a+\ell,x)}{\Gamma(a)}, \quad a,\ell \in \mathbb{C},\ x \geq 0, \tag{1.15}$$

and

$$[a,x]_\ell = \frac{\Gamma(a+\ell,x)}{\Gamma(a)}, \quad a,\ell \in \mathbb{C},\ x \geq 0. \tag{1.16}$$

The existence and convergence conditions of the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ are set out in [21].

The incomplete Fox-Wright functions, ${}_p\Psi_q^{(\gamma)}$ and ${}_p\Psi_q^{(\Gamma)}$, are the generalization of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, and defined as follows (see [22]):

(1.17)
$${}_p\Psi_q^{(\gamma)}\left[\begin{array}{rc}(a_1,A_1,x),(a_j,A_j)_{2,p}\,; & \\ (b_j,B_j)_{1,q}\,; & z\end{array}\right] = \sum_{\ell=0}^{\infty} \frac{\gamma(a_1+A_1\ell,x)\prod_{j=2}^{p}\Gamma(a_j+A_j\ell)}{\prod_{j=1}^{q}\Gamma(b_j+B_j\ell)} \cdot \frac{z^\ell}{\ell!}$$

and

(1.18)
$${}_p\Psi_q^{(\Gamma)}\left[\begin{array}{rc}(a_1,A_1,x),(a_j,A_j)_{2,p}\,; & \\ (b_j,B_j)_{1,q}\,; & z\end{array}\right] = \sum_{\ell=0}^{\infty} \frac{\Gamma(a_1+A_1\ell,x)\prod_{j=2}^{p}\Gamma(a_j+A_j\ell)}{\prod_{j=1}^{q}\Gamma(b_j+B_j\ell)} \cdot \frac{z^\ell}{\ell!},$$

where $A_j,\, B_j \in \mathbb{R}^+$, $a_j,\, b_j \in \mathbb{C}$ and series converges absolutely for all $z \in \mathbb{C}$ when $\Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\ > 0$.

The incomplete Fox-Wright functions, ${}_p\Psi_q^{(\gamma)}$ and ${}_p\Psi_q^{(\Gamma)}$ satisfy the following decomposition formula

$${}_p\Psi_q^{(\gamma)}(z) + {}_p\Psi_q^{(\Gamma)}(z) = {}_p\Psi_q(z), \tag{1.19}$$

where ${}_p\Psi_q(z)$ is Fox-Wright function.

Inspired by the applications of ${}_p\gamma_q$ and ${}_p\Gamma_q$ functions (defined above) and their representation as Mellin-Barnes contour integrals, Srivastava et al. [22] presented and researched the incomplete H-functions as follows:

$$\gamma_{u,\,v}^{r,\,s}(z) = \gamma_{u,\,v}^{r,\,s}\left[z \,\middle|\, \begin{array}{c}(a_1,A_1,x),(a_j,A_j)_{2,u} \\ (b_j,B_j)_{1,v}\end{array}\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} g(\xi,x)\ z^{-\xi}\ d\xi \tag{1.20}$$

and

$$\Gamma_{u,\,v}^{r,\,s}(z) = \Gamma_{u,\,v}^{r,\,s}\left[z \,\middle|\, \begin{array}{c}(a_1,A_1,x),(a_j,A_j)_{2,u} \\ (b_j,B_j)_{1,v}\end{array}\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} G(\xi,x)\ z^{-\xi}\ d\xi, \tag{1.21}$$

where

$$g(\xi,x) = \frac{\gamma(1-a_1-A_1\xi,x)\ \prod\limits_{j=1}^{r}\Gamma(b_j+B_j\xi)\ \prod\limits_{j=2}^{s}\Gamma(1-a_j-A_j\xi)}{\prod\limits_{j=r+1}^{v}\Gamma(1-b_j-B_j\xi)\ \prod\limits_{j=s+1}^{u}\Gamma(a_j+A_j\xi)} \tag{1.22}$$

and

$$G(\xi,x)=\frac{\Gamma(1-a_1-A_1\xi,x)\prod_{j=1}^{r}\Gamma(b_j+B_j\xi)\prod_{j=2}^{s}\Gamma(1-a_j-A_j\xi)}{\prod_{j=r+1}^{v}\Gamma(1-b_j-B_j\xi)\prod_{j=s+1}^{u}\Gamma(a_j+A_j\xi)}, \tag{1.23}$$

with the set of conditions setout in [22].

These incomplete H-functions fulfill the following relation (known as decomposition formula):

$$\gamma_{p,q}^{m,n}(z)+\Gamma_{p,q}^{m,n}(z)=H_{p,q}^{m,n}(z). \tag{1.24}$$

The general class of polynomials of index n, $n=0,1,2,\dots$, was defined by Srivastava [23] as:

$$S_n^m[x]=\sum_{s=0}^{[n/m]}\frac{(-n)_{m\,s}}{s!}A_{n,\,s}\,x^s, \tag{1.25}$$

where m is positive integer and $A_{n,\,s}\in\mathbb{R}$ (or $\mathbb{C}$) are arbitrary positive constants. The notations $(-n)_m$ and $[\,\cdot\,]$, respectively represent the Pochhammer symbol and the greatest integer function. Srivastava's polynomials give a number of known polynomials as its special cases on suitably specializing the coefficients $A_{n,\,s}$.

Throughout this paper we assume that the incomplete H-functions, incomplete Fox-Wright functions and incomplete expanded hypergeometric functions exist under the same sets of conditions setout in [21, 22].

2. Solution of Generalized Fractional Kinetic Equations

Theorem 2.1. *Assume that* $\zeta,\eta,\mathfrak{a},\mathfrak{b}>0$ *and* $\mu>0$*, then the solution of*

$$\mathcal{N}(\mathfrak{t})-\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\,S_n^m[\mathfrak{a}\,\mathfrak{t}^{\zeta}]\,\Gamma_{u,\,v}^{r,\,s}[\mathfrak{b}\,\mathfrak{t}^{\eta}]=-c^{\nu}\,{}_0D_{\mathfrak{t}}^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{2.1}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t})=&\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\sum_{i=0}^{\infty}(-c^{\nu}\,\mathfrak{t}^{\nu})^i\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,\,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^{\zeta})^k\\ &\times\Gamma_{u+1,\,v+1}^{r,\,s+1}\left[\mathfrak{b}\,\mathfrak{t}^{\eta}\,\middle|\,\begin{matrix}(a_1,A_1,x),(1-\mu-\zeta k,\eta),(a_j,A_j)_{2,u}\\ (b_j,B_j)_{1,v}\,,(1-\mu-\zeta k-\nu i,\eta)\end{matrix}\right].\end{aligned} \tag{2.2}$$

Proof. To prove the result, Laplace transform method has been used. Taking the Laplace transform of (2.1) and using (1.21), (1.25) and (1.8), after little simplification, we obtain

(2.3)

$$[1+c^{\nu}\omega^{-\nu}]\mathcal{N}(\omega)=\mathcal{N}_0\sum_{k=0}^{[n/m]}\frac{(-n)_{m\,k}\,A_{n,\,k}\,\mathfrak{a}^k}{k!}\cdot\frac{1}{2\pi i}\int_{\mathfrak{L}}G(\xi,x)\,\mathfrak{b}^{\xi}\,\frac{\Gamma(\mu+\zeta k-\eta\xi)}{\omega^{\mu+\zeta k-\eta\xi}}\,d\xi,$$

where $\mathcal{N}(\omega) = \mathrm{L}\{\mathcal{N}(\mathfrak{t});\omega\}$ and $G(\xi, x)$ is defined in (1.23). Since $(1+x)^{-1} = \sum_{r=0}^{\infty}(-1)^r x^r$, therefore (2.3) implies that

$$\mathcal{N}(\omega) = \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}\, A_{n,k}\, \mathfrak{a}^k}{k!} \cdot \frac{1}{2\pi i}\int_{\mathfrak{L}} G(\xi,x)\, \mathfrak{b}^{\xi}\, \Gamma(\mu+\zeta k-\eta\xi)\, d\xi \times \sum_{i=0}^{\infty}(-c^{\nu})^i\, \omega^{-(\mu+\zeta k-\eta\xi+\nu i)}. \tag{2.4}$$

Now, take the inverse Laplace transform of (2.4), we obtain

$$\mathcal{N}(\mathfrak{t}) = \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}\, A_{n,k}\, \mathfrak{a}^k}{k!} \cdot \frac{1}{2\pi i}\int_{\mathfrak{L}} G(\xi,x)\, \mathfrak{b}^{\xi}\, \Gamma(\mu+\zeta k-\eta\xi)\, d\xi \times \sum_{i=0}^{\infty}(-c^{\nu})^i\, \frac{\mathfrak{t}^{(\mu+\zeta k-\eta\xi+\nu i-1)}}{\Gamma(\mu+\zeta k-\eta\xi+\nu i)}. \tag{2.5}$$

Finally, using (1.21) therein, we get the required result (2.2). ☐

Theorem 2.2. *Assume that $\zeta, \eta, \mathfrak{a}, \mathfrak{b} > 0$ and $\mu > 0$, then the solution of*

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0\, \mathfrak{t}^{\mu-1}\, S_n^m[\mathfrak{a}\,\mathfrak{t}^{\zeta}]\, \gamma_{u,v}^{r,s}[\mathfrak{b}\,\mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_t^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{2.6}$$

is provided as

$$\mathcal{N}(\mathfrak{t}) = \mathcal{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty}(-c^{\nu}\,\mathfrak{t}^{\nu})^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^{\zeta})^k \times \gamma_{u+1,v+1}^{r,s+1}\left[\mathfrak{b}\,\mathfrak{t}^{\eta}\,\middle|\, \begin{matrix} (a_1,A_1,x),(1-\mu-\zeta k,\eta),(a_j,A_j)_{2,u} \\ (b_j,B_j)_{1,v}\,,(1-\mu-\zeta k-\nu i,\eta) \end{matrix}\right]. \tag{2.7}$$

Proof. The proof is the immediate consequence of the definitions (1.20), (1.25) and parallel to the Theorem 2.1. Hence, we skip the proof. ☐

The incomplete H-functions and the incomplete Fox-Wright functions are connected with the following relations (see, [22, (6.3) and (6.4)]):

$$\Gamma_{p,q+1}^{1,p}\left[-z\,\middle|\, \begin{matrix} (1-a_1,A_1,x),(1-a_j,A_j)_{2,p} \\ (0,1),(1-b_j,B_j)_{1,q} \end{matrix}\right] = {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix} (a_1,A_1,x),(a_j,A_j)_{2,p}\,; \\ (b_j,B_j)_{1,q}\,; \end{matrix}\; z\right] \tag{2.8}$$

and

$$\gamma_{p,q+1}^{1,p}\left[-z\,\middle|\, \begin{matrix} (1-a_1,A_1,x),(1-a_j,A_j)_{2,p} \\ (0,1),(1-b_j,B_j)_{1,q} \end{matrix}\right] = {}_p\Psi_q^{(\gamma)}\left[\begin{matrix} (a_1,A_1,x),(a_j,A_j)_{2,p}\,; \\ (b_j,B_j)_{1,q}\,; \end{matrix}\; z\right]. \tag{2.9}$$

If we make the substitution $\mathfrak{b} = -\mathfrak{b}$, $r = 1$, $s = p$, $v = q+1$, $a_j \mapsto (1-a_j)$, $j = 1,\dots,p$, $b_j \mapsto (1-b_j)$ $j = 1,\dots,q$, and multiply by $\Gamma(\xi)$ (i.e., put $b = 0$ and $B = 1$) in (2.1), (2.2), (2.6) and (2.7), use of the equations (2.8) and (2.9) respectively leads to the following corollaries.

Corollary 2.1. *Assume that $\zeta, \eta, \mathfrak{a}, \mathfrak{b} > 0$ and $\mu > 0$, then the solution of*

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0\, \mathfrak{t}^{\mu-1}\, S_n^m[\mathfrak{a}\,\mathfrak{t}^\zeta]\, {}_p\Psi_q^{(\Gamma)}[\mathfrak{b}\,\mathfrak{t}^\eta] = -c^\nu\, {}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{2.10}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t}) =& \mathcal{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty}(-c^\nu\, \mathfrak{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^\zeta)^k \\ &\times\, {}_{p+1}\Psi_{q+1}^{(\Gamma)}\left[\begin{array}{c} (a_1, A_1, x), (\mu+\zeta k, \eta), (a_j, A_j)_{2,p}\,; \\ (bj, B_j)_{1,q},\, (\mu+\zeta k+\nu i, \eta)\,; \end{array}\ \mathfrak{b}\,\mathfrak{t}^\eta\right].\end{aligned} \tag{2.11}$$

Corollary 2.2. *Assume that $\zeta, \eta, \mathfrak{a}, \mathfrak{b} > 0$ and $\mu > 0$, then the solution of*

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0\, \mathfrak{t}^{\mu-1}\, S_n^m[\mathfrak{a}\,\mathfrak{t}^\zeta]\, {}_p\Psi_q^{(\gamma)}[\mathfrak{b}\,\mathfrak{t}^\eta] = -c^\nu\, {}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{2.12}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t}) =& \mathcal{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty}(-c^\nu\, \mathfrak{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^\zeta)^k \\ &\times\, {}_{p+1}\Psi_{q+1}^{(\gamma)}\left[\begin{array}{c} (a_1, A_1, x), (\mu+\zeta k, \eta), (a_j, A_j)_{2,p}\,; \\ (bj, B_j)_{1,q},\, (\mu+\zeta k+\nu i, \eta)\,; \end{array}\ \mathfrak{b}\,\mathfrak{t}^\eta\right].\end{aligned} \tag{2.13}$$

Remark 2.1. If we set $x = 0$, $p = 1$, $q = 2$, $a_1 = 1$, $A_1 = 1$, $b_1 = l + 1 + \frac{b}{2}$, $b_2 = \frac{3}{2}$, $B_1 = B_2 = 1$, $\eta = 2$, $\mathfrak{b} = -\frac{c}{4}$, $\mu = l + 2$, $c^\nu = d^\nu$ and $S_n^m[\mathfrak{a}\,t^\zeta] = \frac{1}{2^{l+1}}$ (i.e., $m = 1$, $\zeta = 0$, $\mathfrak{a} = \frac{1}{2}$, $A_{n,k} = \frac{k!}{(-n)_k}$ for $k = l + 1$ and $A_{n,k} = 0$, otherwise) into (2.10) and (2.11), then the resulting equations would correspond to the kinetic equation and its solution involving generalized Struve function given by Nisar et al. [24, page 168, (14) and (15)].

The incomplete Fox-Wright functions are related to the incomplete generalized hypergeometric functions, ${}_p\Gamma_q$ and ${}_p\gamma_q$ (see [21]). In consequence of (2.8) and (2.9), the incomplete H-functions are related to the incomplete generalized hypergeometric functions as below:

$${}_p\Psi_q^{(\Gamma)}\left[\begin{array}{c} (a_1, 1, x), (a_j, 1)_{2,p}\,; \\ (b_j, 1)_{1,q}\,; \end{array}\ z\right] = \mathcal{C}_q^p\, {}_p\Gamma_q\left[\begin{array}{c} (a_1, x), (a_j)_{2,p}\,; \\ (b_j)_{1,q}\,; \end{array}\ z\right] \tag{2.14}$$

and

$${}_p\Psi_q^{(\gamma)}\left[\begin{array}{c} (a_1, 1, x), (a_j, 1)_{2,p}\,; \\ (b_j, 1)_{1,q}\,; \end{array}\ z\right] = \mathcal{C}_q^p\, {}_p\gamma_q\left[\begin{array}{c} (a_1, x), (a_j)_{2,p}\,; \\ (b_j)_{1,q}\,; \end{array}\ z\right], \tag{2.15}$$

where $\mathcal{C}_q^p$ is defined by

$$\mathcal{C}_q^p = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)}. \tag{2.16}$$

Thus,

$$\Gamma_{p,q+1}^{1,p}\left[-z\ \middle|\ \begin{array}{c} (1-a_1, 1, x), (1-a_j, 1)_{2,p} \\ (0,1), (1-b_j, 1)_{1,q} \end{array}\right] = \mathcal{C}_q^p\, {}_p\Gamma_q\left[\begin{array}{c} (a_1, x), (a_j)_{2,p}\,; \\ (b_j)_{1,q}\,; \end{array}\ z\right] \tag{2.17}$$

and

$$\gamma_{p,\,q+1}^{1,\,p}\left[-z\,\middle|\,\begin{array}{r}(1-a_1,1,x),(1-a_j,1)_{2,\,p}\\(0,1),(1-b_j,1)_{1,\,q}\end{array}\right]=\mathcal{C}_q^p\;{}_p\gamma_q\left[\begin{array}{r}(a_1,x),(a_j)_{2,\,p}\,;\\(b_j)_{1,\,q}\,;\end{array}\;z\right].\tag{2.18}$$

If we substitute $\mathfrak{b}=-\mathfrak{b}$, $r=1$, $s=p$, $v=q+1$, $a_j\mapsto(1-a_j)$, $j=1,\dots,p$, $b_j\mapsto(1-b_j)$, $j=1,\dots,q$, $A_j=1$, $j=1,\dots,p$, $B_j=1$, $j=2,\dots,q$, and multiply by $\Gamma(\xi)$ (i.e., put $b=0$ and $B=1$) in (2.1), (2.2), (2.6) and (2.7). Then the use of (2.17) and (2.18), respectively, leads to the following corollaries.

Corollary 2.3. *Assume that $\zeta,\eta,\mathfrak{a},\mathfrak{b}>0$ and $\mu>0$, then the solution of*

$$\mathcal{N}(\mathfrak{t})-\mathcal{N}_0\,\mathcal{C}_q^p\,\mathfrak{t}^{\mu-1}\,S_n^m[\mathfrak{a}\,\mathfrak{t}^\zeta]\,{}_p\Gamma_q[\mathfrak{b}\,\mathfrak{t}^\eta]=-c^\nu\,{}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}),\tag{2.19}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t})=&\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\sum_{i=0}^{\infty}(-c^\nu\,\mathfrak{t}^\nu)^i\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^\zeta)^k\\&\times\mathcal{C}_{q+1}^{p+1}\;{}_{p+1}\Gamma_{q+1}\left[\begin{array}{r}(a_1,x),\mu+\zeta k,a_2,a_3,...,a_p\,;\\\mu+\zeta k+\nu i,b_1,b_2,...,b_q\,;\end{array}\;\mathfrak{b}\,\mathfrak{t}^\eta\right],\end{aligned}\tag{2.20}$$

where $\mathcal{C}_q^p$ is defined in (2.16) and $\mathcal{C}_{q+1}^{p+1}=\mathcal{C}_q^p\frac{\Gamma(\mu+\zeta k)}{\Gamma(\mu+\zeta k+\nu i)}$.

Corollary 2.4. *Assume that $\zeta,\eta,\mathfrak{a},\mathfrak{b}>0$ and $\mu>0$, then the solution of*

$$\mathcal{N}(\mathfrak{t})-\mathcal{N}_0\,\mathcal{C}_q^p\,\mathfrak{t}^{\mu-1}\,S_n^m[\mathfrak{a}\,\mathfrak{t}^\zeta]\,{}_p\gamma_q[\mathfrak{b}\,\mathfrak{t}^\eta]=-c^\nu\,{}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}),\tag{2.21}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t})=&\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\sum_{i=0}^{\infty}(-c^\nu\,\mathfrak{t}^\nu)^i\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,k}}{k!}(\mathfrak{a}\,\mathfrak{t}^\zeta)^k\\&\times\mathcal{C}_{q+1}^{p+1}\;{}_{p+1}\gamma_{q+1}\left[\begin{array}{r}(a_1,x),\mu+\zeta k,a_2,a_3,\dots,a_p\,;\\\mu+\zeta k+\nu i,b_1,b_2,\dots,b_q\,;\end{array}\;\mathfrak{b}\,\mathfrak{t}^\eta\right],\end{aligned}\tag{2.22}$$

where $\mathcal{C}_q^p$ is defined in (2.16) and $\mathcal{C}_{q+1}^{p+1}=\mathcal{C}_q^p\frac{\Gamma(\mu+\zeta k)}{\Gamma(\mu+\zeta k+\nu i)}$.

3. Applications

In this section, some consequences and applications of the above results are considered. Specific special cases of the derived findings can be developed by suitably specializing the coefficient $A_{n,s}$ to obtain a large number of spectrum of the known polynomials. To illustrate that we consider the following examples.

Example 3.1. Show that the solution of

$$\mathcal{N}(\mathfrak{t})-\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\,\Gamma_{u,\,v}^{r,\,s}[\mathfrak{b}\,\mathfrak{t}^\eta]=-c^\nu\,{}_0D_\mathfrak{t}^{-\nu}\mathcal{N}(\mathfrak{t}),\tag{3.1}$$

is provided as

$$\mathcal{N}(\mathfrak{t})=\mathcal{N}_0\,\mathfrak{t}^{\mu-1}\sum_{i=0}^{\infty}(-c^\nu\,\mathfrak{t}^\nu)^i\,\Gamma_{u+1,\,v+1}^{r,\,s+1}\left[\mathfrak{b}\,\mathfrak{t}^\eta\,\middle|\,\begin{array}{c}(a_1,A_1,x),(1-\mu,\eta),(a_j,A_j)_{2,u}\\(b_j,B_j)_{1,v}\,,(1-\mu-\nu i,\eta)\end{array}\right],\tag{3.2}$$

$$\mathfrak{N}(\mathfrak{t}) - \mathfrak{N}_0\, \mathfrak{t}^{\mu-1}\, {}_p\Psi_q^{(\Gamma)}[\mathfrak{b}\, \mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathfrak{N}(\mathfrak{t}), \tag{3.3}$$

is provided as

$$\mathfrak{N}(\mathfrak{t}) = \mathfrak{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^{\nu}\, \mathfrak{t}^{\nu})^i\, {}_{p+1}\Psi_{q+1}^{(\Gamma)} \left[\begin{array}{c} (a_1, A_1, x), (\mu, \eta), (a_j, A_j)_{2,\, p}\,; \\ (bj, B_j)_{1,\, q},\ (\mu + \nu i, \eta)\,; \end{array} \ \mathfrak{b}\, \mathfrak{t}^{\eta} \right], \tag{3.4}$$

$$\mathfrak{N}(\mathfrak{t}) - \mathfrak{N}_0\, \mathfrak{C}_q^p\, \mathfrak{t}^{\mu-1}\, {}_p\Gamma_q[\mathfrak{b}\, \mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathfrak{N}(\mathfrak{t}), \tag{3.5}$$

is provided as

$$\mathfrak{N}(\mathfrak{t}) = \mathfrak{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^{\nu}\, \mathfrak{t}^{\nu})^i\, \mathfrak{C}'^{p+1}_{q+1}\ {}_{p+1}\Gamma_{q+1} \left[\begin{array}{c} (a_1, x), \mu, a_2, a_3, \ldots, a_p\,; \\ \mu + \nu i, b_1, b_2, \ldots, b_q\,; \end{array} \ \mathfrak{b}\, \mathfrak{t}^{\eta} \right], \tag{3.6}$$

with $\mathfrak{C}'^{p+1}_{q+1} = \mathfrak{C}_q^p \frac{\Gamma(\mu)}{\Gamma(\mu+\nu i)}$.

Solution. Here, setting $m = 1$, $\mathfrak{a} = 1$, $\zeta = 0$ and $A_{n,\, s} = \frac{s!}{(-n)_{ms}}$ for $s = 0$ and $A_{n,\, s} = 0$ for $s \neq 0$ (i.e., $S_n^m[\mathfrak{a}\, t^{\zeta}] = 1$) in (2.1), (2.10) and (2.19). The assertions (3.1), (3.3) and (3.5) of the example follow from the Theorem 2.1, Corollary 2.1 and Corollary 2.3, respectively.

Remark 3.1. It is important to note that for $x = 0$, the kinetic equation and its solution given by (3.1) and (3.2) respectively, would give the corresponding results given earlier by Choi and Kumar [13].

Example 3.2. Show that the solution of

$$\mathfrak{N}(\mathfrak{t}) - \mathfrak{N}_0\, \mathfrak{t}^{\mu+\frac{n}{2}-1}\, \mathrm{H}_n\left(\frac{1}{2\sqrt{\mathfrak{t}}}\right) \Gamma_{u,\, v}^{r,\, s}[\mathfrak{b}\, \mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathfrak{N}(\mathfrak{t}), \tag{3.7}$$

is provided as

$$\begin{aligned} \mathfrak{N}(\mathfrak{t}) =& \mathfrak{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^{\nu}\, \mathfrak{t}^{\nu})^i \sum_{k=0}^{[n/2]} \frac{(-1)^k\, \mathfrak{t}^k}{k!\,(n-2k)!} \\ & \times\ \Gamma_{u+1,\, v+1}^{r,\, s+1}\left[\mathfrak{b}\, \mathfrak{t}^{\eta} \,\middle|\, \begin{array}{c} (a_1, A_1, x), (1-\mu-k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}\,, (1-\mu-k-\nu i, \eta) \end{array} \right], \end{aligned} \tag{3.8}$$

$$\mathfrak{N}(\mathfrak{t}) - \mathfrak{N}_0\, \mathfrak{t}^{\mu+\frac{n}{2}-1}\, \mathrm{H}_n\left(\frac{1}{2\sqrt{\mathfrak{t}}}\right) {}_p\Psi_q^{(\Gamma)}[\mathfrak{b}\, \mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathfrak{N}(\mathfrak{t}), \tag{3.9}$$

is provided as

$$\begin{aligned} \mathfrak{N}(\mathfrak{t}) =& \mathfrak{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^{\nu}\, \mathfrak{t}^{\nu})^i \sum_{k=0}^{[n/2]} \frac{(-1)^k\, \mathfrak{t}^k}{k!\,(n-2k)!} \\ & \times\ {}_{p+1}\Psi_{q+1}^{(\Gamma)} \left[\begin{array}{c} (a_1, A_1, x), (\mu+k, \eta), (a_j, A_j)_{2,\, p}\,; \\ (bj, B_j)_{1,\, q},\ (\mu+k+\nu i, \eta)\,; \end{array} \ \mathfrak{b}\, \mathfrak{t}^{\eta} \right], \end{aligned} \tag{3.10}$$

$$\mathcal{N}(\mathfrak{t}) - \mathcal{N}_0\, \mathcal{C}_q^p\, \mathfrak{t}^{\mu+\frac{n}{2}-1}\, \mathrm{H}_n\left(\frac{1}{2\sqrt{t}}\right)\, {}_p\Gamma_q[\mathfrak{b}\,\mathfrak{t}^{\eta}] = -c^{\nu}\, {}_0D_{\mathfrak{t}}^{-\nu}\mathcal{N}(\mathfrak{t}), \tag{3.11}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathfrak{t}) =& \mathcal{N}_0\, \mathfrak{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^{\nu}\, \mathfrak{t}^{\nu})^i \sum_{k=0}^{[n/2]} \frac{(-1)^k\, \mathfrak{t}^k}{k!\,(n-2k)!} \\ & \times \mathcal{C}'^{p+1}_{q+1}\, {}_{p+1}\Gamma_{q+1}\left[\begin{array}{c} (a_1,x), \mu+k, a_2, a_3, ..., a_p\,; \\ \mu+k+\nu i, b_1, b_2, ..., b_q\,; \end{array} \mathfrak{b}\,\mathfrak{t}^{\eta}\right],\end{aligned} \tag{3.12}$$

with $\mathcal{C}'^{p+1}_{q+1} = \mathcal{C}_q^p \frac{\Gamma(\mu+k)}{\Gamma(\mu+k+\nu i)}$.

Solution. Set $m=2$, $\mathfrak{a}=1$, $\zeta=1$ and $A_{n,s}=(-1)^s$ (i.e., $S_n^2[\mathfrak{t}] = \mathfrak{t}^{n/2}\mathrm{H}_n\left(\frac{1}{2\sqrt{\mathfrak{t}}}\right)$, where $\mathrm{H}_n(\mathfrak{t})$ is Hermite polynomial) in (2.1), (2.10) and (2.19). Thus, assertions (3.7), (3.9) and (3.11) of the example follow from the Theorem 2.1, Corollary 2.1 and Corollary 2.3, respectively.

Remark 3.2. As an application of the results (2.6), (2.12) and (2.21), a number of consequent results can be derived.

4. Concluding remarks

Our attempt in this paper is to propose a new fractional generalization of the standard kinetic equation and to use the integral transformation approach to analyze its solution. A study of several interesting fractional kinetic equations and their solutions has been made, which include a family of polynomials and the incomplete H-function, incomplete Fox-Wright function and incomplete generalized hypergeometric function. The main results contained in the Theorem 2.1, Theorem 2.2 and their corollaries are of general nature. Analogously, various fractional kinetic equations and their solutions available in literature (see, [15–20]) can be obtained as special cases of the main results. Through addition, a number of recognized polynomials are produced by the polynomials family as their specific cases on a properly specialized connected sequence $A_{n,s}$. As a consequence, by providing appropriate basic values to the arbitrary sequences and the constraints, the main results can be used to generate a set of kinetic equations and their potential solutions. We intend to continue this study of the more generalized kinetic equations and their proposed solutions in the future work.

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References

[1] H. Habenom, D. L. Suthar, D. Baleanu and S. D. Purohit, *A numerical simulation on the effect of vaccination and treatments for the fractional hepatitis B model*, Journal of Computational and Nonlinear Dynamics **16**(1) (2021), Paper ID 011004. https://doi.org/10.1115/1.4048475

[2] Kritika, R. Agarwal and S. D. Purohit, *Mathematical model for anomalous subdiffusion using conformable operator*, Chaos Solitons & Fractals **140** (2020), Paper ID 110199.https://doi.org/10.1016/j.chaos.2020.110199

[3] R. Agarwal, M. P. Yadav, D. Baleanu and S. D. Purohit, *Existence and uniqueness of miscible flow equation through porous media with a nonsingular fractional derivative*, AIMS Mathematics **5**(2) (2020), 1062–1073. https://doi.org/10.3934/math.2020074

[4] R. Agarwal, S. D. Purohit and Kritika, *A mathematical fractional model with non-singular kernel for thrombin-receptor activation in calcium signaling*, Math. Methods Appl. Sci. **42** (2019), 7160–7171.https://doi.org/10.1002/mma.5822

[5] M. Chand, J. C. Prajapati and E. Bonyah, *Fractional integrals and solution of fractional kinetic equations involving k-Mittag-Leffler function*, Trans. A. Razmadze Math. Inst. **171**(2) (2017), 144–166. https://doi.org/10.1016/j.trmi.2017.03.003

[6] R. K. Gupta, A. Atangana, B. S. Shaktawat and S. D. Purohit, *On the solution of generalized fractional kinetic equations involving generalized M-series*, Caspian Journal of Applied Mathematics, Ecology and Economics **7**(1) (2019), 88–98.

[7] H. Habenom, D. L. Suthar and M. Gebeyehu, *Application of Laplace transform on fractional kinetic equation pertaining to the generalized Galue type Struve function*, Adv. Math. Phys. **2019** (2019), Article ID 5074039, 8 pages. https://doi.org/10.1155/2019/5074039

[8] O. Khan, N. Khan, D. Baleanu and K. S. Nisar, *Computable solution of fractional kinetic equations using Mathieu-type series*, Adv. Difference Equ. **2019**(234) (2019). https://doi.org/10.1186/s13662-019-2167-4

[9] D. Kumar, J. Choi and H. M. Srivastava, *Solution of a general family of fractional kinetic equations associated with the Mittag-Leffler function*, Nonlinear Funct. Anal. Appl. **23**(3) (2018), 455–471.

[10] S. D. Purohit and F. Ucar, *An application of q-Sumudu transform for fractional q-kinetic equation*, Turkish J. Math. **42** (2018), 726–734. https://doi:10.3906/mat-1703-7

[11] D. L. Suthar, D. Kumar and H. Habenom, *Solutions of fractional kinetic equation associated with the generalized multiindex Bessel function via Laplace-transform*, Differ. Equ. Dyn. Syst. **2019** (2019). https://doi.org/10.1007/s12591-019-00504-9.

[12] D. L. Suthar, S. D. Purohit and S. Araci, *Solution of fractional kinetic equations associated with the (p, q)-Mathieu-type series*, Discrete Dyn. Nat. Soc. **2020** (2020), Article ID 8645161, 7 pages. https://doi.org/10.1155/2020/8645161

[13] J. Choi and D. Kumar, *Solutions of generalized fractional kinetic equations involving Aleph functions*, Math. Commun. **20** (2015), 113–123.

[14] A. Chouhan, S. D. Purohit and S. Srivastava, *An alternative method for solving generalized differential equations of fractional order*, Kragujevac J. Math. **37**(2) (2013), 299–306.

[15] D. Kumar, S. D. Purohit, A. Secer and A. Atangana, *On generalized fractional kinetic equations involving generalized Bessel function of the first kind*, Math. Probl. Eng. **2015** (2015), Article ID 289387, 7 pages. https://doi.org/10.1155/2015/289387

[16] H. J. Haubold and A. M. Mathai, *The fractional kinetic equation and thermonuclear functions*, Astrophys. Space Sci. **327** (2000), 53–63. https://doi.org/10.1023/A:1002695807970

[17] R. K. Saxena and S. L. Kalla, *On the solutions of certain fractional kinetic equations*, Appl. Math. Comput. **199** (2008), 504–511. https://doi.org/10.1016/j.amc.2007.10.005

[18] R. K. Saxena, A. M. Mathai and H. J. Haubold, *On fractional kinetic equations*, Astrophys. Space Sci. **282** (2002), 281–287. https://doi.org/10.1023/A:1021175108964

[19] R. K. Saxena, A. M. Mathai and H. J. Haubold, *On generalized fractional kinetic equations*, Phys. A **344** (2004), 657–664. https://doi.org/10.1016/j.physa.2004.06.048

[20] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Functions: Theory and Applications*, Springer, New York, 2010. https://doi.org/10.1007/978-1-4419-0916-9

[21] H. M. Srivastava, M. A. Chaudhary and R. P. Agarwal, *The incomplete Pochhammer symbols and their applications to hypergeometric and related functions*, Integral Transforms Spec. Funct. **23** (2012), 659–683. https://doi.org/10.1080/10652469.2011.623350

[22] H. M. Srivastava, R. K. Saxena and R. K. Parmar, *Some families of the incomplete H-functions and the incomplete $\overline{H}$-functions and associated integral transforms and operators of fractional calculus with applications*, Russ. J. Math. Phys. **25**(1) (2018), 116–138. https://doi.org/10.1134/S1061920818010119

[23] H. M. Srivastava, *A contour integral involving Fox's H-function*, Indian J. Math. **14** (1972), 1–6. https://doi.org/10.12691/ajams-5-1-3.

[24] K. S. Nisar, S. D. Purohit and S. R. Mondal, *Generalized fractional kinetic equations involving generalized Struve function of the first kind*, Journal of King Saud University - Science **28**(2) (2016), 167–171. https://doi.org/10.1016/j.jksus.2015.08.005

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SYMMETRIES, NOETHER'S THEOREM, CONSERVATION LAWS AND NUMERICAL SIMULATION FOR SPACE-SPACE-FRACTIONAL GENERALIZED POISSON EQUATION

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Abstract. In the present paper Lie theory of differential equations is expanded for finding symmetry geometric vector fields of Poisson equation. Similarity variables extracted from symmetries are applied in order to find reduced forms of the considered equation by using Erdélyi-Kober operator. Conservation laws of the space-space-fractional generalized Poisson equation with the Riemann-Liouville derivative are investigated via Noether's method. By means of the concept of non-linear self-adjointness, Noether's operators, formal Lagrangians and conserved vectors are computed. A collocation technique is also applied to give a numerical simulation of the problem.

1. Introduction

Theory of fractional order differential equations (FDEs) due to the non-local property of fractional derivatives are used to describe many phenomena and various fields of physics and other sciences for example fluid mechanics, physics, chemistry, biology, engineering, control, signal and image processing, dynamic systems, biology, environmental science, materials, economic, etc. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. Sun et al. are given a comprehensive package for the tangible examples of fractional calculus in the nature [24].

Key words and phrases. Riemann-Liouville derivative, Lie point symmetry, Erdelyi-Kober operator, conservation laws, Jacobi polynomial.

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Notwithstanding several definitions for fractional calculus, such as Sonin–Letnikov derivative, Liouville derivative, Caputo derivative, Hadamard derivative, Riesz–Miller derivative, Che–Machado derivative, Caputo–Katugampola derivative, Hilfer–Katugampola derivative, Pichaghchi derivative, etc., have been presented by different researchers [5,15,19,25]. Also Lin et al. established another important integral identity for once differentiable function involving Riemann–Liouville fractional integrals which will be used to derive some new Riemann–Liouville fractional Hermite–Hadamard inequalities via r-convex function and geometric-arithmetically s-convex function respectively [13].

The concept of symmetry of FDEs is similar to symmetry of PDEs. Symmetries of a fractional equation are transformations that map any solution to another solution of the equation but unlike PDEs that are considered in many references, symmetries of time-fractional differential equations (TFDEs) have been investigated somewhat and symmetries of space-time-fractional differential equations (STFDE) and space-space-fractional differential equations (SSFDE) have been attended seldom [8–10, 17, 23].

One of the applications of symmetries is to calculate conservations laws of a given system [12, 20]. Conservation laws are fundamental laws of science especially physics that keep a certain quantity that are not variable in time during of processes and they can be used to reduce dimension of equations. To calculate conservation laws by symmetries we should utilize Noether's theorem, Euler-Lagrange operator and formal Lagrangian [1, 7, 11, 14, 16].

In mathematics, Poisson's equation is a PDE of elliptic type with broad utility in mechanical engineering and theoretical physics. It arises, for instance, to describe the potential field caused by a given charge or mass density distribution; with the potential field known, one can then calculate gravitational or electrostatic field. It is a generalization of Laplace's equation, which is also frequently seen in physics. The equation is named after the French mathematician, geometer, and physicist Simeon Denis Poisson.

In this paper the generalized fractional order of the Poisson equation, the space-space-fractional equation of the form,

$$\mathcal{D}_x^{\beta}(u) + \mathcal{D}_y^{\alpha}(u) = F(u) \tag{1.1}$$

is considered where $1 < \alpha, \beta < 1$ [22].

If $\alpha = \beta = 2$, (1.1) becomes to the second order elliptic PDE $u_{xx} + u_{yy} = F(u)$ which is named Poisson equation. The generalized form of (1.1) by considering $1 < \alpha, \beta < 2$ can be written as

$$\mathcal{D}_x^{\beta}(u) + \mathcal{D}_y^{\alpha}(u) = F(u, u_x). \tag{1.2}$$

(1.2) has some special cases. For example if y replace by t and u_{xx} replaces by $-u_{xx}$ it converts to Klein-Gordon equation. Also, if we replace $F(u)$ by $\sin u$ the equation reduces to sine-Gordon equation [22].

The rest of the paper is organized as follow. In Section 2, the method of finding symmetry operators is introduced and this section is concluded by reduction of the equation via the similarity variables obtaining from symmetries. Section 3 establishes the Noether's operators and the associated conservation laws for the fractional Poisson equation. Finally, in section 4 a collocation method based on Jacobi polynomials is suggested to obtain the numerical solution of the problem. It in noteworthy that Pintarelli et al. give a useful method for finding the numerical solutions of the both linear and non-linear Poisson equation [18].

2. Symmetry Analysis of Space-Time-Fractional Poisson Equation

This section is devoted to Lie group analysis of the (1.2).

2.1. Lie symmetry method for (1.2)**.** Consider an SSFDE of the form

$$\mathcal{M} = \left\{\partial_y^\alpha u + \partial_x^\beta u - F(x, y, u, u_x, u_{xx}, \dots) = 0\right\}, \tag{2.1}$$

for an arbitrary function $F(x, y, u, u_x, u_{xx}, \dots)$. To obtain symmetries of (2.1) two different cases are considered.

Case 1. Let us substitute $F(x, y, u, u_x, u_{xx}, \dots) = -uu_x$ into (1.2). Then (2.2) is obtained:

$$\Delta := \mathcal{D}_y^\alpha(u) + \mathcal{D}_x^\beta(u) + uu_x = 0. \tag{2.2}$$

First, we should investigate one-parameter Lie group of infinitesimal transformation with a small group parameter $\varepsilon \ll 1$ such as:

$$x \mapsto x + \varepsilon\xi(x, y, u), \quad y \mapsto y + \varepsilon\rho(x, y, u), \quad u \mapsto u + \varepsilon\eta(x, y, u). \tag{2.3}$$

The transformation (2.3) takes

$$X = \xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} \tag{2.4}$$

as an associated infinitesimal generator. If Δ admitted X as a symmetry, the invariance condition implies that

$$\mathrm{Pr}^{(\alpha,\beta,1)}X(\Delta)|_{\Delta=0} = 0, \tag{2.5}$$

where $\mathrm{Pr}^{(\alpha,\beta,1)}X$ denotes the prolongation of (2.4). The extended form of Eq. (2.5) yields the following appearance:

$$\mathrm{Pr}^{(\alpha,\beta,1)}X = \xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta_y^\alpha\frac{\partial}{\partial u_y^\alpha} + \eta_x^\beta\frac{\partial}{\partial u_x^\beta} + \eta^x\frac{\partial}{\partial u_x},$$

which determines the point symmetries of (2.2). Expanding the invariance condition (2.5) yields:

$$\mathrm{Pr}^{(\alpha,\beta,1)}X(F)|_{(F=0)} = \eta^{(\alpha,y)} + \eta^{(\beta,x)} + \eta^x u + \eta u_x = 0, \tag{2.6}$$

where

$$\begin{aligned} \eta^{(\beta,x)} =& \mathfrak{D}_x^\beta(\eta)+\rho\mathfrak{D}_x^\beta(u_y)-\mathfrak{D}_x^\beta(\rho u_y)+\mathfrak{D}_x^\beta(uD_x(\xi))-\mathfrak{D}_x^{\beta+1}(\xi u)+\xi\mathfrak{D}_x^{\beta+1}(u),\\ \eta^{(\alpha,y)} =& \mathfrak{D}_y^\alpha(\eta)+\xi\mathfrak{D}_y^\alpha(u_x)-\mathfrak{D}_y^\alpha(\xi u_x)+\mathfrak{D}_y^\alpha(uD_y(\rho))-\mathfrak{D}_y^{\alpha+1}(\rho u)+\rho\mathfrak{D}_y^{\alpha+1}(u),\\ \eta^x =& D_x(\eta)-u_yD_x(\rho)-u_xD_x(\xi), \end{aligned} \tag{2.7}$$

are the prolongation's coefficients.

The operators D_y and D_x mention the total derivatives dependent to y and x, respectively. Also operator $\mathfrak{D}_x^\beta, \mathfrak{D}_y^\alpha$ are the total space fractional derivative with respect to x, y. These operators are not similar to ordinary operators. Because there are differents between PDEs and FDEs for using Leibniz rule, non-commutation and Laplace transform, see [21] for more details. By inserting (2.7) to (2.6), the following solution is obtained:

$$\xi = -C_2\alpha x + C_1, \quad \rho = -C_2\beta y + C_3, \quad \eta = C_2\beta\alpha u - C_2\alpha u,$$

where C_i $(i = 1, 2, 3)$ are arbitrary constants.

Because of the maintaining the structure of SSFDE in the lower limit of the integral in Riemann-Liouville derivative respect to x or y, we should work under the assumption that $\rho(x, y, u)\Big|_{y=0} = 0$ and $\xi(x, y, u)\Big|_{x=0} = 0$. According to Lie symmetry theory we have the following Lie algebra for (2.2), with arbitrary $\alpha, \beta \in (1, 2)$,

$$X_1 = -\alpha x\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} + (u\beta\alpha - u\alpha)\frac{\partial}{\partial u}. \tag{2.8}$$

The condition $\rho(x, y, u)\Big|_{y=0} = \xi(x, y, u)\Big|_{x=0} = 0$ does not involve any assumption about PDEs then $X_2 = \frac{\partial}{\partial x}$ and $X_3 = \frac{\partial}{\partial y}$ are more symmetries for (2.2) by $\alpha = \beta = 2$.

Case 2. Similarly if $F(x, y, u, u_x, u_{xx}, \dots) = u$, with $\alpha, \beta \in (1, 2)$, the equation

$$F = \mathfrak{D}_y^\alpha(u) + \mathfrak{D}_x^\beta(u) - u = 0,$$

has the following symmetries:

$$X_1 = 2\alpha x\frac{\partial}{\partial x} + 2\beta y\frac{\partial}{\partial y} + \alpha(\beta-1)u\frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}.$$

2.2. **Reductions.** In this section we will acquire reduction of (2.2) by the obtained symmetries of section 2 and then we will obtain a space-fractional order differential equation (SFODE). For Case 1, pursuant to the infinitesimals (2.8), we can write the similarity variables those are found by solving the corresponding characteristic equations in the form

$$\frac{dy}{-\beta y} = \frac{du}{\alpha(\beta-1)u} = \frac{dx}{-\alpha x}.$$

Solving the above differential equation, one can get the similarity variables $xy^{-\frac{\alpha}{\beta}}$ and $uy^{\frac{\alpha(\beta-1)}{\beta}}$. As regards we arrive an answer that has the form $u(x, y) = y^{-\frac{\alpha(\beta-1)}{\beta}}H(z)$

where the function z is given by $xy^{-\frac{\alpha}{\beta}}$. Then we can write left-hand-sided of the Riemann-Liouville fractional derivative $\mathcal{D}_y^{\alpha}(u)$ in the form

(2.9)

$$\mathcal{D}_y^{\alpha}u(x,y)=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_0^y\frac{H\left(xs^{-\frac{\alpha}{\beta}}\right)s^{-\frac{\alpha(\beta-1)}{\beta}}ds}{(y-s)^{\alpha-n+1}},\quad n-1<\alpha<n,\ n=2,3,\ldots$$

The assumptions $\omega=\frac{y}{s}$ ($s=\frac{y}{\omega}$ and $ds=\frac{-y}{\omega^2}d\omega$) is considered in the sequel. Then (2.9) can be written as:

$$\mathcal{D}_y^{\alpha}u(y,x)=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_1^{\infty}\frac{y^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}}}{\omega^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}+1}}(\omega-1)^{n-\alpha-1}H(z\omega^{\frac{\alpha}{\beta}})d\omega.$$

Thus, we obtain

$$\mathcal{D}_y^{\alpha}u(y,x)=\frac{\partial^n}{\partial y^n}\left[y^{n-\alpha-\frac{\alpha(\beta-1)}{\beta}}\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1-\frac{\alpha(\beta-1)}{\beta},n-\alpha}H\right)(z)\right],\tag{2.10}$$

where

$$\left(K_{\delta}^{\zeta,\alpha}g\right)(z)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\zeta+\alpha)}g(zu^{\frac{1}{\delta}})du, & \alpha>0,\\ g(z), & \alpha=0,\end{cases}$$

$$n=\begin{cases}[n]+1, & \alpha\notin\mathbb{N},\\ \alpha, & \alpha\in\mathbb{N}.\end{cases}$$

By inserting $z=xy^{-\frac{\alpha}{\beta}}$ into (2.10) and by considerinmg $y\frac{d}{dy}\Psi(z)=-\frac{\alpha}{\beta}z\frac{d}{dz}\Psi(z)$, it concludes that

$$\mathcal{D}_y^{\alpha}u(y,x)=y^{-\alpha-\frac{\alpha(\beta-1)}{\beta}}\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha-\frac{\alpha(\beta-1)}{\beta},\alpha}H\right)(z),$$

where $\mathcal{P}_{\beta}^{\zeta,\alpha}$ is the left-hand-sided of Erdélyi-Kober fractional differential operator [5].

The left-hand-side of the Riemann-Liouville fractional derivative $\mathcal{D}_x^{\beta}(u)$ can be written the same as the $\mathcal{D}_y^{\alpha}(u)$, but with a few differences. First of all, we will need the assumptions $\omega=\frac{x}{s}$,$x=\omega s$ and $ds=\frac{-x}{\omega^2}d\omega$. So we get

$$\mathcal{D}_x^{\beta}u(x,y)=y^{-\frac{\alpha(\beta-1)}{\beta}}x^{-\beta}\left(\mathcal{P}_{-1}^{1-\beta,\beta}H\right)(z).$$

Thus, (2.2) can be reduced to the following FPDE where it is written in terms of Erdelyi-Kober fractional differential operator,

$$\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha-\frac{\alpha(\beta-1)}{\beta},\alpha}H\right)(z)+z^{-\beta}\left(\mathcal{P}_{-1}^{1-\beta,\beta}H\right)(z)=-H(z)H'(z)y^{\frac{\alpha(\beta-1)}{\beta}}.$$

3. Conservation Laws

In this section conservation laws are computed via the modified version of Noether's theorem based on non-linear self-adjointness concept [9].

3.1. **Non-linear self-adjointness of the fractional Poisson equation.** There are two methods during the conservation laws calculations, first method is stablished by a usual Lagrangian and the second method is constructed by a formal Lagrangian.

The formal Lagrangian for the (1.2) can be written as follows:

$$\mathcal{L} = vF(x, y, u, \mathcal{D}_y^\alpha u, \mathcal{D}_x^\beta u, u_x) = v\mathcal{D}_y^\alpha(u) + v\mathcal{D}_x^\beta(u) + vuu_x, \quad v = v(x, y),$$

where v is new dependent variable. Indeed, by multiplying a new dependent variable in the equation that is equaled to zero we can find the formal Lagrangian equation.

The Euler-Lagrange operator with respect to u for a finite space interval $x \in [x, 0]$ and $y \in [0, y]$ is defined by:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\mathcal{D}_y^\alpha)^* \frac{\partial}{\partial \mathcal{D}_y^\alpha u} + (\mathcal{D}_x^\beta)^* \frac{\partial}{\partial \mathcal{D}_x^\alpha u} + \sum_{m=1}^{\infty} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1,\ldots,i_m}},$$

where $(\mathcal{D}_j^\mu)^*$ is adjoint operator for Riemman-Liouville derivative $(\mathcal{D}_j^\mu)$ such that $j = x, y$ and $\mu = \alpha, \beta$, that is defined by (see [14]):

$$({}_0\mathcal{D}_x^\beta)^* = (-1)^n \, {}_xJ_X^{n-\beta}(\mathcal{D}_x^n) \equiv {}_x^C\mathcal{D}_X^\beta, \tag{3.1}$$

where ${}_xJ_X^{n-\beta}$ is the right-sided fractional integral in Riemann-Liouville derivative, and ${}_x\mathcal{D}_X^\beta$ and ${}_x^C\mathcal{D}_X^\beta$ are the right-sided Riemann-Liouville and Caputo fractional derivative of order β. By applying operator $\frac{\delta}{\delta u}$ and formal Lagrangian we obtain the adjoint equation for (2.2) as

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = (\mathcal{D}_y^\alpha)^* v + (\mathcal{D}_x^\beta)^* v - v_x u. \tag{3.2}$$

The (2.2) is non-linearly self-adjoint if the adjoint (3.2) holds for all solution u of the initial (2.2) upon the substitution $v = \Phi(x, y, u)$, where $v = \Phi(x, y, u)$ satisfies the condition $\Phi(x, y, u) \neq 0$. It means that the following equation holds:

$$F^*\Big|_{v=\Phi(x,y,u)} = \lambda F, \tag{3.3}$$

where the coefficients λ is indefinite function, which is obtained during calculations.

For the first position we consider v in general term $v = \Phi(x, y, u)$ and its necessary derivative is $v_x = \Phi_x + \Phi_u u_x$. By inserting elements v and v_x, we shall write the expression (3.3) as

$$-\Phi_x u - \Phi_u u_x u + (\mathcal{D}_y^\alpha)^* v + (\mathcal{D}_x^\beta)^* v = \lambda((\mathcal{D}_y^\alpha u) + (\mathcal{D}_x^\beta u) + u_x u). \tag{3.4}$$

Then expansion of (3.4) and comparing of the coefficients for $1, u_x$ one can verify that $\lambda = \Phi_u$ and $\Phi_x = 0$. Hence, (2.2) is non-linearly self-adjoint by considering $v = a_1$, where a_1 is a constant.

For the second position, without loss of generality we can certainly assume $v = \varphi(y)\chi(x)$ then $v_x = \varphi(y)\chi'(x)$. Inserting v and v_x into (3.3) the (2.4) yields

$$-\varphi\chi' + (\mathcal{D}_y^\alpha)^* \varphi\chi + (\mathcal{D}_x^\beta)^* \varphi\chi = \lambda((\mathcal{D}_y^\alpha u) + (\mathcal{D}_x^\beta u) + u_x u)\Big|_{(2.2))} = 0.$$

According to (3.1), the above expression is written as:

$$-\varphi\chi' + \chi(x){}_y^c\mathcal{D}_Y^\alpha(X)\Big(\varphi(y)\Big) + \varphi(y){}_x^c\mathcal{D}_X^\beta\Big(\chi(x)\Big) = 0. \tag{3.5}$$

The expanded form of (3.5) concludes that $\chi(x) = a_2$ and $\varphi(y) = y$.

3.2. **Basic definitions for constructing conservation laws.** The main difficulty in carrying out conservation laws is that we can't define usual Lagrangian for many equations. The generalized Poisson (2.2) hasn't got the usual Lagrangian. Accordingly under the above results, Noether's operators for the Riemann-Liouville based on the formal Lagrangian are given by

(3.6)
$$C^y = \mathcal{N}^y = \sum_{k=0}^{n-1}(-1)^k\mathcal{D}_y^{\alpha-1-k}(W)\mathcal{D}_y^k\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_y^\alpha u)}\right) \times (-1)^n J\left(W, \mathcal{D}_y^n\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_y^\alpha u)}\right)\right),$$

(3.7)
$$C^x = \mathcal{N}^x = \sum_{k=0}^{m-1}(-1)^k\mathcal{D}_x^{\beta-1-k}(W)\mathcal{D}_x^k\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_x^\beta u)}\right) \times (-1)^m J_1\left(W, \mathcal{D}_x^m\left(\frac{\partial\mathcal{L}}{\partial(\mathcal{D}_x^\beta u)}\right)\right),$$

where J and J_1 are defined as:

$$J(f,g) = \frac{\displaystyle\int_0^y\int_y^Y \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}d\mu d\tau}{\Gamma(n-\alpha)},$$

$$J_1(f,g) = \frac{\displaystyle\int_0^x\int_x^X \frac{f(\tau,y)g(\mu,y)}{(\mu-\tau)^{\beta+1-m}}d\mu d\tau}{\Gamma(m-\beta)},$$

and W is the characteristic of Lie's symmetry generator defined by [4,16]:

$$W = \eta - \xi u_x - \rho u_y.$$

The formal Lagrangian for (2.2) after substitution acceptable $v = a_1$ is defined as $\mathcal{L} = a_1\Big(\mathcal{D}_y^\alpha(u) + \mathcal{D}_x^\beta(u) + uu_x\Big)$. In this case, using (3.6), (3.7) and considering $W = \alpha(\beta-1)u + \alpha x u_x + \beta y u_y$, one can get components of conserved vectors:

$$C^y = a_1\alpha(\beta-1)\mathcal{D}_y^{\alpha-1}(u) + a_1\alpha x\mathcal{D}_y^{\alpha-1}(u_x) + a_1\beta\mathcal{D}_y^{\alpha-1}(yu_y),$$
$$C^x = a_1\alpha(\beta-1)\mathcal{D}_x^{\beta-1}(u) + a_1\alpha\mathcal{D}_x^{\beta-1}(xu_x) + a_1\beta y\mathcal{D}_x^{\beta-1}(u_y).$$

The formal Lagrangian for (2.2) after replacement $v = a_2 y$ is given by $\mathcal{L} = a_2 y\Big(\mathcal{D}_y^\alpha(u) + \mathcal{D}_x^\beta(u) + uu_x\Big)$ and then we can formulate main results as following:

$$\begin{aligned} C^y =& a_2\alpha(\beta-1)y\mathcal{D}_y^{\alpha-1}(u) + a_2\alpha yx\mathcal{D}_y^{\alpha-1}(u_x) + a_2\beta y\mathcal{D}_y^{\alpha-1}(yu_y) \\ &- a_2\alpha(\beta-1)\mathcal{D}_y^{\alpha-2}(u) - a_2\alpha x\mathcal{D}_y^{\alpha-2}(u_x) - a_2\beta\mathcal{D}_y^{\alpha-2}(yu_y), \\ C^x =& a_2\alpha(\beta-1)\mathcal{D}_x^{\beta-1}(u) + a_2\alpha y\mathcal{D}_x^{\beta-1}(xu_x) + a_2\beta y^2\mathcal{D}_x^{\beta-1}(u_y). \end{aligned}$$

4. A Numerical Simulation

In this section, we will propose a numerical solution for the equation

$$D_x^{\beta}(u) + D_y^{\alpha}(u) = f(u, u_x). \tag{4.1}$$

To do this, we consider a specific example with $f(u, u_x) = u(x, y)$ and the boundary conditions as follows

$$u(x, 0) = \frac{\sin x}{100}, \quad x \in [0, 1], \tag{4.2}$$

$$u(0, y) = \frac{\sin y}{100}, \quad y \in [0, 1]. \tag{4.3}$$

Since that is not the focus of this work, we briefly describe an efficient numerical method to perform it here. It should be noted that in some research works [2, 3, 6], the authors used an operational matrix of fractional differentiation to solve the problems of this type. However, in this part, we use a collocation method based on Jacobi polynomials and compute them and their fractional derivative by some suitable commands in `MAPLE` software. The well-known Jacobi polynomials are defined on the interval $[-1, 1]$ and can be generated with the aid of the following recurrence formula [2]

$$\begin{aligned} J_i^{(a,b)}(t) =& \frac{(a+b+2i-1)\{a^2 - b^2 + t(a+b+2i)(a+b+2i-2)\}}{2i(a+b+i)(a+b+2i-2)} J_{i-1}^{(a,b)} \\ &- \frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)} J_{i-2}^{(a,b)}, \quad i = 2, 3, \dots, \end{aligned}$$

where

$$J_0^{(a,b)}(t) = 1 \quad \text{and} \quad J_1^{(a,b)}(t) = \frac{a+b+2}{2} t + \frac{a-b}{2}.$$

In order to use these polynomials on the interval $[0, 1]$ we defined the so-called shifted Jacobi polynomials by introducing the change of variable $t = 2x - 1$. Let the shifted Jacobi polynomials $J_{1,i}^{(a,b)}(2x-1)$ be denoted by $J_{1,i}^{(a,b)}(x)$. Then $J_{1,i}^{(a,b)}(x)$ can be generated from

$$\begin{aligned} J_{1,i}^{(a,b)}(x) =& \frac{(a+b+2i-1)\{a^2 - b^2 + (2x-1)(a+b+2i)(a+b+2i-2)\}}{2i(a+b+i)(a+b+2i-2)} J_{1,i-1}^{(a,b)} \\ &- \frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)} J_{1,i-2}^{(a,b)}, \quad i = 2, 3, \dots, \end{aligned}$$

where

$$J_{1,0}^{(a,b)}(x) = 1 \quad \text{and} \quad J_{1,1}^{(a,b)}(x) = \frac{a+b+2}{2}(2x-1) + \frac{a-b}{2}.$$

The analytic form of the shifted Jacobi polynomials $J_{1,i}^{(a,b)}(x)$ of degree i is given by

$$J_{1,i}^{(a,b)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+b+1)\Gamma(i+k+a+b+1)}{\Gamma(k+b+1+1)\Gamma(i+a+b+1)(i-k)!k!} x^k,$$

where

$$J_{1,i}^{(a,b)}(0) = (-1)^i \frac{\Gamma(i+b+1)}{\Gamma(b+1)i!} \quad \text{and} \quad J_{1,i}^{(a,b)}(1) = \frac{\Gamma(i+a+1)}{\Gamma(a+1)i!}.$$

The choice $a = b = 0$ yields the Legendre polynomials, while choosing $a = b = -\frac{1}{2}$ gives Chebyshev polynomials. We now assume that, the solutions of (4.1)–(4.3) can be approximated by the shifted Jacobi polynomials as follows

$$u(x,y) \simeq u_N(x,y) = \sum_{m=0}^{N} \sum_{n=0}^{N} u_{mn} J_{1,m}^{(a,b)}(x) J_{1,n}^{(a,b)}(y),$$

where u_{mn}, $m = 0, 1, \ldots, N$, and $n = 0, 1, \ldots, N$, are unknown coefficients to be determined and $J_{1,m}^{(a,b)}(x)$ and $J_{1,n}^{(a,b)}(y)$ are the shifted Jacobi polynomials. In our numerical simulation we consider $a = b = 0$.

The shifted Jacobi polynomials $J_{1,i}^{(a,b)}(x)$ can be written in the `MAPLE` software in the form

$$J_{1,i}^{(a,b)}(x) = \text{Jacobi}P(i, a, b, 2x - 1).$$

The fractional integral of order $\alpha > 0$ of function f can also be determined using the following command

$${}_0I_x^{\alpha} f(x) = \text{fracdiff}(f(x), x, -\alpha).$$

Then the Rieman–Liouville fractional derivative of order $1 \leq \alpha \leq 2$ of function f is normally written as

$$D_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^2 I_x^{2-\alpha} f(x) = \text{diff}(\text{fracdiff}(f(x), x, \alpha - 2), x, x).$$

Since there exist $(N+1)^2$ unknown coefficients u_{mn}, $m = 0, 1, \ldots, N$, and $n = 0, 1, \ldots, N$, we should construct system of $(N+1)^2$ algebraic equations. For this reason, we define the well-known Chebyshev Gauss Lobatto (CGL) collocation points as

$$\eta_i = \frac{1}{2}\left(1 - \cos\left(\frac{(i-1)\pi}{N}\right)\right), \quad 1 \leq i \leq N+1,$$

$$\tau_j = \frac{1}{2}\left(1 - \cos\left(\frac{(j-1)\pi}{N}\right)\right), \quad 1 \leq j \leq N+1.$$

We discretize now equation (4.1) using the CGL points as

$$D_x^{\beta} u_N(\eta_i, \tau_j) + D_y^{\alpha} u_N(\eta_i, \tau_j) = u_N(\eta_i, \tau_j), \quad 2 \leq i \leq N+1,\ 2 \leq j \leq N+1,$$

and the boundary conditions (4.2)–(4.3) as follows

$$u_N(\eta_i, 0) = \frac{\sin \eta_i}{100}, \quad 1 \leq i \leq N+1,$$

$$u_N(0, \tau_j) = \frac{\sin \tau_j}{100}, \quad 2 \leq j \leq N+1.$$

In this case, the considered equations are collocated and then transformed into the associated systems of $(N+1)^2$ algebraic equations and $(N+1)^2$ unknowns which can be solved through an iterative method in Maple software by $fsolve$ command. We can also check the accuracy of our proposed numerical approach. To do this, we replace the $u_N(x,y), D_x^\beta u_N(x,y)$ and $D_y^\alpha u_N(x,y)$ in equations (4.1)–(4.3). Then (4.1)–(4.3) can be satisfied approximately. In other words we define the absolute error as

$$E = \left| D_x^\beta u_N(x,y) + D_y^\alpha u_N(x,y) - u_N(x,y) \right| \simeq 0.$$

Diagrams of the solutions of the system (4.1)–(4.3) using the suggested numerical method are shown in Figure 1. Figure 2, gives some numerical results obtained by this method for $u(x,y)$. The absolute error E is also depicted in Figure 3.

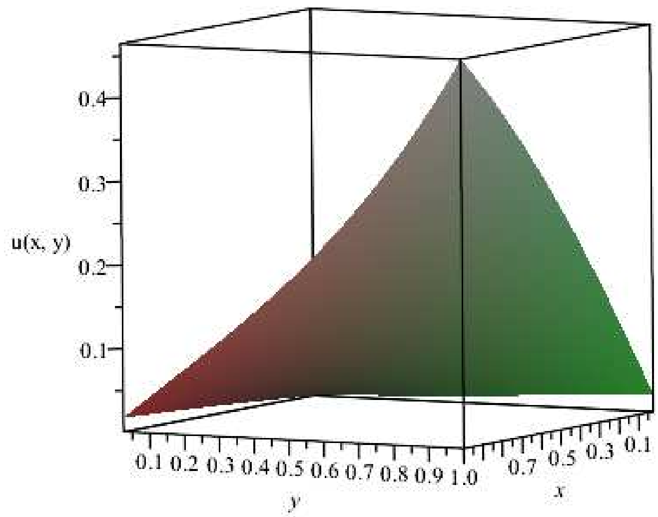

FIGURE 1. Numerical solution $u_N(x,y)$ for $\beta = 1.9$ and $\alpha = 2$ with $N = 5$.

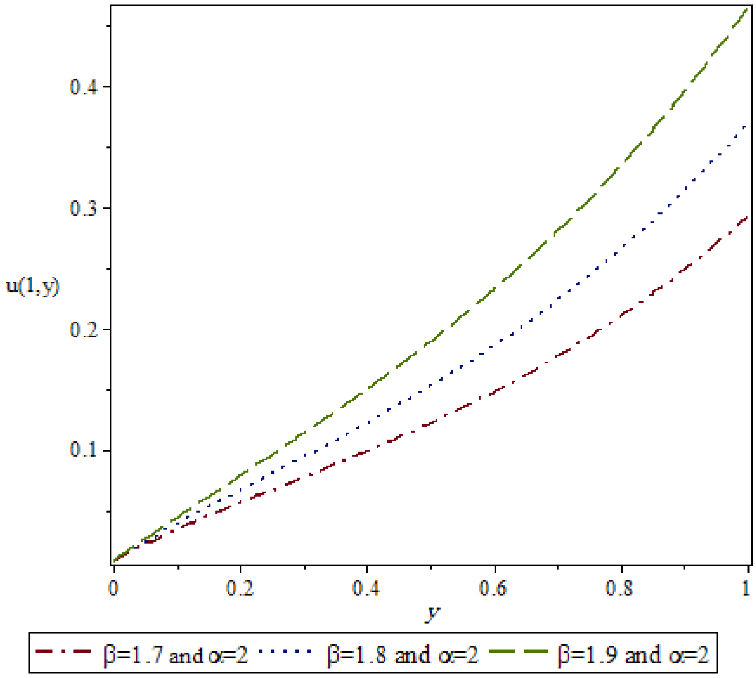

FIGURE 2. Numerical solutions of the $u_N(x, y)$ for $\beta = 1.9, 1.8, 1.7$ and $\alpha = 2$ with $x = 1$ and $N = 5$.

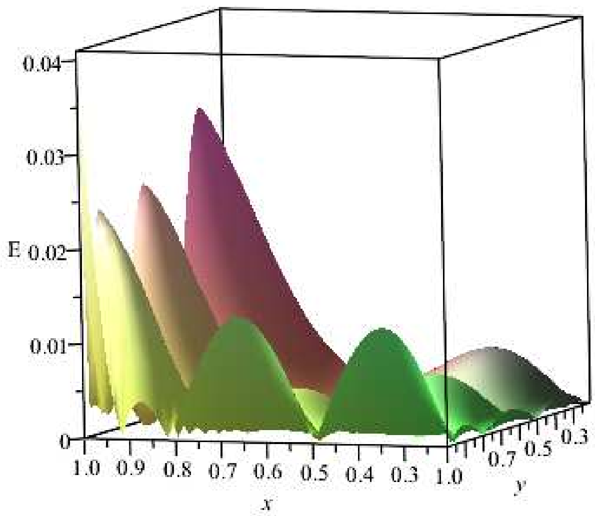

FIGURE 3. Absolute error E of the presented method for $N = 5$.

TABLE 1. Some numerical results of $u_N(x, y)$ with $N = 5$ and different values of α and β.

(x, y)	$\alpha = 2,\ \beta = 1.9$	$\alpha = 2,\ \beta = 1.8$	$\alpha = 2,\ \beta = 1.7$
(1, 0.1)	0.04400193	0.03886688	0.03410741
(1, 0.2)	0.07856733	0.06686172	0.05642837
(1, 0.3)	0.11351537	0.09422739	0.07751767
(1, 0.4)	0.15008387	0.12244699	0.09901679
(1, 0.5)	0.18939548	0.15273855	0.12217625
(1, 0.6)	0.23250915	0.18613527	0.14796318
(1, 0.7)	0.28047157	0.22356557	0.17716893
(1, 0.8)	0.33436854	0.26593330	0.21051668
(1, 0.9)	0.39537657	0.31419779	0.24876897
(1, 1)	0.46481412	0.36945403	0.29283537

REFERENCES

[1] T. M. Atanacković, S. Konjik, S. Pilipović and S. Simić, *Variational problems with fractional derivatives: Invariance conditions and Nothers theorem*, Nonlinear Anal. **71**(5–6) (2009), 1504–1517. https://doi.org/10.1016/j.na.2008.12.043

[2] A. H. Bhrawy, E. H. Doha, D. Baleanu and R. M. Hafez, *A highly accurate Jacobi collocation algorithm for systems of high-order linear differential-difference equations with mixed initial conditions*, Math. Methods Appl. Sci. **38**(14) (2015), 3022–3032. https://doi.org/10.1002/mma.3277

[3] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, *A new Jacobi operational matrix: An application for solving fractional differential equations*, Appl. Math. Model. **36**(10) (2012), 4931–4943. https://doi.org/10.1016/j.apm.2011.12.031

[4] G. W. Bluman, A. F. Cheviakov and C. Anco, *Application of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2000.

[5] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics **301**, John Wiley & Sons, New York, 1994.

[6] Y. Chen and H. L. An, *Numerical solutions of coupled Burgers equations with time and space fractional derivatives*, Appl. Math. Comput. **200**(1) (2008), 87–95. https://doi.org/10.1016/j.amc.2007.10.050

[7] G. S. F. Frederico and D. F. M. Torres, *A formulation of Noether's theorem for fractional problems of the calculus of variations*, J. Math. Anal. Appl. **334**(2) (2007), 834–846. https://doi.org/10.1016/j.jmaa.2007.01.013

[8] R. K. Gazizov, A. A. Kasatkin and S. Y. Lukashchuk, *Symmetry properties of fractional diffusion equations*, Physica Scripta **136** (2009), 014–016. https://doi.org/10.1088/0031-8949/2009/T136/014016

[9] R. K. Gazizov, N. H. Ibragimov and S. Y. Lukashchuk, *Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations*, Commun. Nonlinear Sci. Numer. Simul. **23**(1–3) (2015), 153–163. https://doi.org/10.1016/j.cnsns.2014.11.010

[10] Q. Huang and R. Zhdanov, *Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann-Liouville derivative*, Phys. A **409** (2014), 110–118. https://doi.org/10.1016/j.physa.2014.04.043

[11] E. Lashkarian, S. R. Hejazi, N. Habibi and A. Motamednezhad, *Symmetry properties, conservation laws, reduction and numerical approximations of time-fractional cylindrical-Burgers*

equation, Commun. Nonlinear Sci. Numer. Simul. **67** (2019), 176–191. https://doi.org/10.1016/j.cnsns.2018.06.025

[12] E. Lashkarian, S. R. Hejazi and E. Dastranj, *Conservation laws of* $(3+\alpha)$*-dimensional time-fractional diffusion equation*, Comput. Math. Appl. **75**(3) (2018), 740–754. https://doi.org/10.1016/j.camwa.2017.10.001

[13] Z. Lin and J. R. Wang, *New Riemann–Liouville fractional Hermite-Hadamard inequalities via two kinds of convex functions*, Journal of Interdisciplinary Mathematics **20**(2) (2017), 357–382. https://doi.org/10.1080/09720502.2014.914281

[14] S. Y. Lukashchuk, *Conservation laws for time-fractional subdiffusion and diffusion-wave equations*, Nonlinear Dyn. **80**(1–2) (2014). https://doi.org/10.1007/s11071-015-1906-7

[15] F. Mainardy, *Fractional Calculus and Waves in Linear Viscoelasticity, An Introduction to Mathematical Models*, Imperial College Press, Singapore, 2010.

[16] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics **107**, Springer, New York, 1993.

[17] A. Ouhadan and E. H. Elkinani, *Exact solutions of time fractional Kolmogorov equation by using Lie symmetry analysis*, J. Fract. Calc. Appl. **5**(1) (2014), 97–104.

[18] M. B. Pintarelli and F. Vericat, *On the numerical solution of the linear and nonlinear Poisson equations seen as bi-dimensional inverse moment problems*, Journal of Interdisciplinary Mathematics **19**(5–6) (2016), 927–944. https://doi.org/10.1080/09720502.2014.916845

[19] J. Sabatier, O. P. Agrawal and J. A. T Machado, *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[20] E. Saberi and S. R. Hejazi, *A comparison of conservation laws of the Boussinesq system*, Kragujevac J. Math. **43**(2) (2019), 173–200.

[21] S. Samko, A. A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach science, Yverdon, Switzerland, 1993.

[22] I. Shingareva and C. Lizarraga-Celaya, *Solving Nonlinear Partial Differential Equations with Maple and Mathematica*, Springer Wien, NewYork, 2011.

[23] K. Singla and R. K. Gupta, *Space-time fractional nonlinear partial differential equations: symmetry analysis and conservation laws*, Nonlinear Dyn. **89** (2017), 321–331. https://doi.org/10.1007/s11071-017-3456-7

[24] H. G. Sun, Y. Zhang, D. Baleanu, W. Chen and Y. Q. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul. **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

[25] Y. Zhou, J. Wang and L. Zhang, *Basic Theory of Fractional Differential Equations*, World Scientific, London, 2016.

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A PRODUCT FORMULA AND CERTAIN q-LAPLACE TYPE TRANSFORMS FOR THE q-HUMBERT FUNCTIONS

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ABSTRACT. The present work deals with the mathematical investigation of the product formulas and several q-Laplace type integral transforms of certain q-Humbert functions. In our investigation, the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform of certain q^2-Humbert functions are considered. Several useful special cases have been deduced as applications of main results.

1. INTRODUCTION AND PRELIMINARIES

Integral transforms have been widely used in many areas of science and engineering and therefore so much work has been done on the theory and applications of integral transforms. The integral transform method is a persuasive way to solve numerous differential equations. Thus, in the literature there are lots of works on several integral transforms such as Laplace, Fourier, Mellin, Hankel. Two of the most frequently used formulas in the area of integral transforms are the classical Laplace and Sumudu transform and their corresponding q-analogues, see for example [1–6, 20, 21]. The Laplace transform is the most popular and extensively used in applied mathematics. Yürekli and Sadek [24] introduced a new type of Laplace transform, known as the $\mathcal{L}_2$-transform. These transforms were studied in more details by Yürekli [22,23]. After that Uçar and Albayrak [19] have investigated the q-analogue of this $\mathcal{L}_2$-transforms, which are called the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform and are defined as follows [19]:

$$ {}_qL_2\{f(t);s\} = \frac{1}{[2]_q} \cdot \frac{(q^2;q^2)_\infty}{s^2} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} f\left(q^n s^{-1}\right) \tag{1.1}$$

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and

$$ {}_q\mathcal{L}_2\{f(t);s\} = \frac{1}{[2]_q} \cdot \frac{1}{(-s^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} q^{2n} \left(-s^2;q^2\right)_n f(q^n), \tag{1.2}$$

respectively.

In order to better understand the work, some notations and preliminaries of the quantum theory are recollected. For any real number b, the q-analogues of the shifted factorial $(b)_s$ is given by [8, 16]:

$$ (b;q)_0 = 1, \quad (b;q)_s = \prod_{i=0}^{s-1}(1-q^i b), \quad (b;q)_\infty = \prod_{i=0}^{\infty}(1-q^i b), \quad b,q\in\mathbb{R},\ n\in\mathbb{N}, \tag{1.3}$$

and satisfy the following relations [7]:

$$ (q;q)_{s+l} =(q;q)_s\ (q^{s+1};q)_l, \tag{1.4}$$

$$ (q^{s+1};q)_\infty =(q^{l+s+1};q)_\infty\ (q^{s+1};q)_l. \tag{1.5}$$

The q-analogues of a complex number b is given by [8]:

$$ [b]_q = \frac{1-q^b}{1-q}, \quad q\in\mathbb{C}\setminus\{1\},\ b\in\mathbb{C}. \tag{1.6}$$

The q-exponential functions are defined as [8]:

$$ e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}, \quad E_q(u) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{u^n}{[n]_q!}. \tag{1.7}$$

These q-exponential functions are related as [8]:

$$ e_q(u)E_q(-u) = 1 \quad \text{and} \quad e_q(-u)E_q(u) = 1. \tag{1.8}$$

The q-gamma functions $\Gamma_q(\alpha)$ and ${}_q\Gamma(\alpha)$ have the following series representations [16]:

$$ \Gamma_q(\alpha) =\frac{(q;q)_\infty}{(1-q)^{\alpha-1}} \sum_{k=0}^{\infty} \frac{q^{k\alpha}}{(q;q)_k} = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha}, \tag{1.9}$$

$$ {}_q\Gamma(\alpha) =\frac{K(A;\alpha)}{(1-q)^{\alpha-1}(-(1/A);q)_\infty} \sum_{k\in\mathbb{Z}} \left(\frac{q^k}{A}\right)^\alpha \left(-\frac{1}{A};q\right)_k, \tag{1.10}$$

where $K(A;\alpha)$ is the following remarkable function [16]:

$$ K(A;\alpha) = A^{\alpha-1}\frac{(-q/\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty} \cdot \frac{(-\alpha;q)_\infty}{(-\alpha q^{1-t};q)_\infty}, \quad \alpha\in\mathbb{R}. \tag{1.11}$$

Investigating the q-analogues of the special functions and exploring their properties is a prevailing topic for mathematicians and physicists. It is familiar that the parameter q symbolize for "quantum", which is extensively used in quantum calculus (or q-calculus). For more details of quantum calculus, one can see the book of Kac and Cheung [8]. The theory of q-special functions play an indispensable role in the formalism of mathematical physics. The development in q-calculus has also led to the extension of several remarkable functions to their q-analogues, see for example [9–12, 17]. Recently, the q-analogues of the Humbert functions are introduced by Srivastava and Shehata [17] by means of the generating functions and series definitions.

The q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ are specified by means the following generating equation [17]:

$$e_q\left(\frac{xu}{3}\right)e_q\left(\frac{xt}{3}\right)e_q\left(-\frac{x}{3ut}\right)=\sum_{m,n=0}^{\infty}\mathfrak{J}^{(1)}_{m,n}(x|q)u^m t^n \tag{1.12}$$

and have the following series representation [17]:

$$\mathfrak{J}^{(1)}_{m,n}(x|q)\;=\;\sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\;(q;q)_{m+k}\;(q;q)_{n+k}}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.13}$$

The q-Humbert functions of the second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ are defined by the following series expansion [17]:

(1.14)

$$\mathfrak{J}^{(2)}_{m,n}(x|q)\;=\;\sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\;(q;q)_{m+k}\;(q;q)_{n+k}}q^{\frac{k}{2}(3k+2(m+n)-1)}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}.$$

The q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ and second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ are related as [17]:

$$\mathfrak{J}^{(1)}_{m,n}\left(q^{\frac{1}{3}}x\middle|\frac{1}{q}\right)=q^{\frac{1}{3}(m+n)+\binom{m}{2}+\binom{n}{2}}\mathfrak{J}^{(2)}_{m,n}(x|q). \tag{1.15}$$

The series form of the q-Humbert functions of the third kind $\mathfrak{J}^{(3)}_{m,n}(x|q)$ is given as [17]:

$$\mathfrak{J}^{(3)}_{m,n}(x|q)\;=\;\sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\;(q;q)_{m+k}\;(q;q)_{n+k}}q^{\binom{k+1}{k}}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.16}$$

Inspired by the works on the q-special functions in diverse fields, in this article, the product formulas for the q-Humbert functions of first, second and third kind are obtained. Certain q-Laplace type integral transforms are investigated for the q^2-Humbert functions of first, second and third kind. Some examples are considered in order to show effectiveness of the proposed results by taking some special cases.

2. Product Formula

The Product formulas for q-Bessel functions are investigated by Rahman [13], which are proved to be very useful in many branches of mathematics. After that, Swarttouw [18] derived the product formulas for the Hahn-Exton q-Bessel function, which opened the way to a rich harmonic analysis. Motivated by these works, the product formula for the generalized q-Bessel functions are also established in [14]. We follow the same technique of calculation developed by Swarttouw to derive a product formula for q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$.

Theorem 2.1. *Let $x > 0$, $\gamma, \delta > 0$ and $m, n, p, r \in \mathbb{N}$, then the following product formula for the q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ holds true:*

(2.1)

$$\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) = B_{m,n,p,r}(x|q) \sum_{i=0}^{\infty} M_i(q) \left(-\frac{\delta x}{3}\right)^{3i} \times {}_3\varphi_2 \left(\begin{matrix} q^{-p-i}, \quad q^{-r-i}, \; q^{-i}; \\ \\ q^{n+1}, \; q^{m+1}; \end{matrix} \quad q; \quad -\frac{\gamma^3}{\delta^3} q^{\frac{1}{2}(2(3i+p+r)+3(1-k))} \right),$$

where ${}_l\varphi_s$ is the basic hypergeometric function defined by [15]*:*

$$ {}_l\varphi_s \left(\begin{matrix} a_1, a_2, \ldots, a_l; \\ \\ b_1, b_2, \ldots, b_s; \end{matrix} \quad q; \quad z \right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_l; q)_k z^k}{(q, b_1, b_2, \ldots, b_s; q)_k} \tag{2.2}$$

and $B_{m,n,p,r}(x|q) = \frac{(1-q)^{m+n+p+r}\gamma^{m+n}\delta^{p+r}}{(q;q)_m (q;q)_n (q;q)_p (q;q)_r} \left(\frac{x}{3}\right)^{m+n+p+r}$, $M_i(q) = \frac{(1-q)^{3i}}{(q^{p+1};q)_i (q^{r+1};q)_i (q;q)_i}$.

Proof. In view of series (1.13), we can write

$$\begin{aligned} &\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k \; (q;q)_{m+k} \; (q;q)_{n+k}} \\ &\times \left(\frac{(1-q)\gamma x}{3}\right)^{m+n+3k} \sum_{l=0}^{\infty} \frac{(-1)^l}{(q;q)_l \; (q;q)_{p+l} \; (q;q)_{r+l}} \left(\frac{(1-q)\delta x}{3}\right)^{p+r+3l}, \end{aligned} \tag{2.3}$$

which on using identity (1.4) becomes

$$\begin{aligned} &\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) \\ &= B_{m,n,p,r}(x|q) \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}(1-q)^{3k+3l}\gamma^{3k}\delta^{3l}\left(\frac{x^3}{27}\right)^{k+l}}{(q;q)_k \; (q^{m+1};q)_k \; (q^{n+1};q)_k (q;q)_l \; (q^{p+1};q)_l \; (q^{r+1};q)_l}, \end{aligned} \tag{2.4}$$

where $B_{m,n,p,r}(x|q) = \frac{(1-q)^{m+n+p+r}\gamma^{m+n}\delta^{p+r}}{(q;q)_m (q;q)_n (q;q)_p (q;q)_r} \left(\frac{x}{3}\right)^{m+n+p+r}$.

Replacing l by $i-k$ in equation (2.4), we get

$$\begin{aligned} \mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) = &B_{m,n,p,r}(x|q) \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \frac{(-1)^i(1-q)^{3i}\gamma^{3k}\delta^{3(i-k)}}{(q;q)_k \; (q^{m+1};q)_k \; (q^{n+1};q)_k} \\ &\times \frac{1}{(q;q)_{i-k} \; (q^{p+1};q)_{i-k} \; (q^{r+1};q)_{i-k}} \left(\frac{x^3}{27}\right)^i, \end{aligned} \tag{2.5}$$

which on using the following identity

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k} \left(-\frac{q}{a}\right)^k \; q^{\binom{k}{2}-nk}$$

gives

$$
\begin{aligned}
(2.6)\qquad & \mathfrak{J}^{(1)}_{m,n}(\gamma x|q)\times \mathfrak{J}^{(1)}_{p,r}(\delta x|q)\\
=&B_{m,n,p,r}(x|q)\sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{(1-q)\delta x}{3}\right)^{3i}}{(q;q)_i\;(q^{p+1};q)_i\;(q^{r+1};q)_i}\\
&\times\sum_{k=0}^{i}\left(-\frac{\gamma}{\delta}\right)^{3k}\frac{(q^{-p-i};q)_k\;(q^{-r-i};q)_k\;(q^{-i};q)_k}{(q;q)_k\;(q^{m+1};q)_k\;(q^{n+1};q)_k}\;q^{\frac{k}{2}(6i+2p+2r+3-3k)}.
\end{aligned}
$$

Letting $M_i(q)=\frac{(1-q)^{3i}}{(q^{p+1};q)_i(q^{r+1};q)_i(q;q)_i}$ and using equation (2.2) in equation (2.6), assertion (2.1) is proved. □

Similarly, we get the following product formulas for the q-Humbert functions of the second and third kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ and $\mathfrak{J}^{(3)}_{m,n}(x|q)$, respectively.

Remark 2.1. Let $x>0$, $\gamma,\delta>0$ and $m,n,p,r\in\mathbb{N}$, then the following product formula for the q-Humbert functions of the second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ holds true:

(2.7)

$$
\begin{aligned}
\mathfrak{J}^{(2)}_{m,n}(\gamma x|q)\times \mathfrak{J}^{(2)}_{p,r}(\delta x|q)=&B_{m,n,p,r}(x|q)\sum_{i=0}^{\infty}N_i(q)\left(-\frac{\delta x}{3}\right)^{3i}\\
&\times{}_3\varphi_2\left(\begin{matrix}q^{-p-i},\quad q^{-r-i},\; q^{-i};\\ q^{n+1},\; q^{m+1};\end{matrix}\quad q;\quad -\frac{\gamma^3}{\delta^3}q^{\frac{1}{2}(2(3i+p+r)+3(1-k))}\right),
\end{aligned}
$$

where $N_i(q)=\frac{(1-q)^{3i}}{(q^{p+1};q)_i(q^{r+1};q)_i(q;q)_i}q^{\frac{i}{2}(3i+2p+2r-1)}$ and $B_{m,n,p,r}(x|q)$ is same as earlier.

Remark 2.2. Let $x>0$, $\gamma,\delta>0$ and $m,n,p,r\in\mathbb{N}$, then the following product formula for the q-Humbert functions of the third kind $\mathfrak{J}^{(3)}_{m,n}(x|q)$ holds true:

(2.8)

$$
\begin{aligned}
\mathfrak{J}^{(3)}_{m,n}(\gamma x|q)\times \mathfrak{J}^{(3)}_{p,r}(\delta x|q)=&B_{m,n,p,r}(x|q)\sum_{i=0}^{\infty}L_i(q)\left(-\frac{\delta x}{3}\right)^{3i}\\
&\times{}_3\varphi_2\left(\begin{matrix}q^{-p-i},\quad q^{-r-i},\; q^{-i};\\ q^{n+1},\; q^{m+1};\end{matrix}\quad q;\quad -\frac{\gamma^3}{\delta^3}q^{\frac{1}{2}(2(3i+p+r)+3(1-k))}\right),
\end{aligned}
$$

where $L_i(q)=\frac{(1-q)^{3i}}{(q^{p+1};q)_i(q^{r+1};q)_i(q;q)_i}q^{i+2}$ and $B_{m,n,p,r}(x|q)$ is same as earlier.

In the following section, the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform for the q^2-Humbert functions are investigated.

3. ${}_qL_2$-Transform and ${}_q\mathcal{L}_2$-Transform

In this section, we evaluate ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform of t^{2w-2} weighted product of m different q^2-Humbert functions. The q^2-Humbert functions are more relevant than the original q-Humbert functions because of the mathematical nature of ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform which contain q^2-shift factorials.

Theorem 3.1. *Let* $\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ $j=1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of first kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w,a_j,μ_j,ν_j *and* $j=1,2,\ldots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is,*

$$
\begin{aligned}
(3.1)\qquad & {}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{kj=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ H_{k_j}(q^2)\Gamma_{q^2}(w+\mu_j+\nu_j+k_j),
\end{aligned}
$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ *and*

$$
(3.2)\qquad B_j(s)=\frac{(a_j)^{\mu_j+\nu_j}}{[2]_q\ 3^{\mu_j+\nu_j}\ s^{2(w+\mu_j+\nu_j)}},\qquad H_{k_j}(q)=\frac{(1-q)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q;q)_{k_j}\ (q;q)_{3\mu_j+k_j}\ (q;q)_{3\nu_j+k_j}}.
$$

Proof. In order to prove the theorem, let $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ in equation (1.1), we get

$$
\begin{aligned}
(3.3)\qquad & {}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
&=\frac{1}{[2]_q}\cdot\frac{(q^2;q^2)_\infty}{s^2}\prod_{j=1}^{m}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n}\left(q^ns^{-1}\right)^{2w-2}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}((a_jq^{2n}s^{-2})^{\frac{1}{3}}|\ q^2),
\end{aligned}
$$

which in view of series expansion (1.13) becomes

$$
\begin{aligned}
&{}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}=\frac{1}{[2]_q}\cdot\frac{(q^2;q^2)_\infty}{s^{2w}}\\
&\times\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n=0}^{\infty}\frac{q^{2nw}}{(q^2;q^2)_{k_j}\ (q^2;q^2)_n}\left[\frac{(1-q^2)a_jq^{2n}s^{-2}}{3}\right]^{\mu_j+\nu_j+k_j}\\
&=\prod_{j=1}^{m}\frac{(q^2;q^2)_\infty\Big((1-q^2)a_j/3\Big)^{\mu_j+\nu_j}}{[2]_q\ s^{2(w+\mu_j+\nu_j)}}\sum_{n=0}^{\infty}\frac{q^{2n(w+\mu_j+\nu_j)}}{(q^2;q^2)_n}\\
&\times\sum_{k_j=0}^{\infty}\frac{\left(-\frac{a_j(1-q^2)(q^ns^{-1})^2}{3}\right)^{k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}.
\end{aligned}
$$

Letting $B_j(s) = \frac{(a_j)^{\mu_j+\nu_j}}{[2]_q\ 3^{\mu_j+\nu_j}\ s^{2(w+\mu_j+\nu_j)}}$ in the above equation, it follows that

$$
\begin{aligned}
&{}_qL_2\left\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\right\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\frac{\left(\frac{-a_j}{3s^2}\right)^{k_j}(1-q^2)^{\mu_j+\nu_j+k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_\infty(q^2)^{n(w+\mu_j+\nu_j+k_j)}}{(q^2;q^2)_n},
\end{aligned}
$$

which on using relation (1.9) and setting $H_{k_j}(q) = \frac{(1-q)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q;q)_{k_j}\ (q;q)_{3\mu_j+k_j}\ (q;q)_{3\nu_j+k_j}}$ gives

$$
\begin{aligned}
&{}_qL_2\left\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\right\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j).
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

The following corollaries are an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let* $\mathcal{J}^{(2)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j=1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of second kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(2)}_{3\mu_j,3\nu_k}(3(a_jt)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j *and* $j=1,2,\ldots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is*

$$
\begin{aligned}
&{}_qL_2\{f(t);s\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ (q^2)^{\frac{k_j}{2}(3k_j+6(\mu_j+\nu_j)-1)}H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j),
\end{aligned}
$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ *and* $B_j(s)$, $H_{k_j}(q)$ *are same as in equation* (3.2).

Corollary 3.2. *Let* $\mathcal{J}^{(3)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j=1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of third kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(3)}_{3\mu_j,3\nu_k}(3(a_jt)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j *and* $j=1,2,\ldots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is*

$$
{}_qL_2\{f(t);s\}=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ (q^2)^{k_j+1}\ H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j),
$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ *and* $B_j(s)$, $H_{k_j}(q)$ *are same as in equation* (3.2).

Next, the ${}_q\mathcal{L}_2$-transform for the q^2-Humbert functions are investigated.

Theorem 3.2. *Let* $\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j=1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of first kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j

and $j=1,2,\ldots,m$, are constants then ${}_q\mathcal{L}_2$-transform of $f(t)$ is

(3.4)
$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&A_{q^2}(s)\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\\
&\times\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j}\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)},
\end{aligned}
$$

where $A_{q^2}(s)=\frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}$ and $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$.

Proof. Using $f(t)=t^{2w-2}\prod\limits_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ in equation (1.2), it follows that

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
&=\frac{1}{[2]_q\ (-s^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}q^{2n}(-s^2;q^2)_n\times\prod_{j=1}^{m}(q^n)^{2w-2}\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}((a_jq^{2n})^{\frac{1}{3}}|\ q^2),
\end{aligned}\tag{3.5}
$$

which in view of series expansion (1.13) becomes

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&\frac{1}{[2]_q\ (-s^2;q^2)_\infty}\prod_{j=1}^{m}\sum_{n\in\mathbb{Z}}q^{2nw}(-s^2;q^2)_n\\
&\times\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{(q^2;q^2)_{k_j}}\frac{1}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\left[\frac{(1-q^2)a_jq^{2n}}{3}\right]^{\mu_j+\nu_j+k_j}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (-s^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}q^{2n(w+\mu_j+\nu_j)}\sum_{k_j=0}^{\infty}\frac{(-s^2;q^2)_n\left(\frac{a_jq^{2n}}{3}\right)^{k_j}(1-q^2)^{\mu_j+\nu_j+k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (-s^2;q^2)_\infty}\sum_{k_j=0}^{\infty}\frac{\left(\frac{a_j}{3}\right)^{k_j}}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n\in\mathbb{Z}}\frac{(-s^2;q^2)_n\ q^{2n(w+\mu_j+\nu_j+k_j)}}{(q^2;q^2)_{k_j}\ (-s^2;q^2)_\infty}.
\end{aligned}
$$

Using relations (1.9) and (1.10) in the above equation, it follows that

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (s^{2(w+\mu_j+\nu_j)})}\sum_{k_j=0}^{\infty}\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}
\end{aligned}
$$

$$\times \frac{\left(\frac{-a_j}{3s^2}\right)^{k_j} (1-q^2)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}},$$

which in view of identity (1.5) gives

$$\begin{aligned}
&{}_q\mathcal{L}_2\left\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\right\}\\
=&\frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}\left(\frac{a_j}{3s^2}\right)^{\mu_j+\nu_j+k_j}}{\left(\frac{(q^2;q^2)_\infty}{(q^{2(3\mu_j+k_j+1)};q^2)_\infty}\right)}\\
&\times\frac{(1-q^2)^{2(\mu_j+\nu_j+k_j)}}{\left(\frac{(q^2;q^2)_\infty}{(q^{2(3\nu_j+k_j+1)};q^2)_\infty}\right)\left(\frac{(q^2;q^2)_\infty}{(q^{2(k_j+1)};q^2)_\infty}\right)}\cdot\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}.
\end{aligned}$$

Letting $A_{q^2}(s)=\frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}$ in the above equation and in view of expression (1.9), assertion (3.4) follows. □

The following corollaries are an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let* $\mathcal{J}^{(2)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j=1,2,\dots,m$, *be a set of* q^2*-Humbert functions of second kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(2)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w,a_j,μ_j,ν_j *and* $j=1,2,\dots,m$, *are constants then* ${}_q\mathcal{L}_2$*-transform of* $f(t)$ *is*

$$\begin{aligned}
{}_q\mathcal{L}_2\{f(t);s\}=&A_{q^2}(s)\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}(q^2)^{\frac{k_j}{2}(3k_j+6(\mu_j+\nu_j)-1)}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)}\\
&\times\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j},
\end{aligned}$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ *and* $A_{q^2}(s)$ *is same as in Theorem* 3.2.

Corollary 3.4. *Let* $\mathcal{J}^{(3)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j=1,2,\dots,m$, *be a set of* q^2*-Humbert functions of third kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(3)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w,a_j,μ_j,ν_j *and* $j=1,2,\dots,m$, *are constants then* ${}_q\mathcal{L}_2$*-transform of* $f(t)$ *is,*

$$\begin{aligned}
{}_q\mathcal{L}_2\{f(t);s\}=&A_{q^2}(s)\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}(q^2)^{k_j+1}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)}\\
&\times\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j},
\end{aligned}$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ *and* $A_{q^2}(s)$ *is same as in Theorem* 3.2.

In the next section, we give certain examples to show the applications of the results established in previous sections.

4. Special Cases

In consideration of $m = 1$, $a_1 = a$, $k_1 = k$, $\mu_1 = \mu$ and $\nu_1 = \nu$ in Theorem 3.1, Corollary 3.1 and Corollary 3.2, respectively, the ${}_qL_2$-transforms for the q-Humbert functions of the first, second and third kind are obtained:

$$ {}_qL_2\{t^{2w-2}\mathcal{J}^{(1)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\} = \mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k\ H_k(q^2)\ \Gamma_{q^2}(w+\mu+\nu+k), $$

$$ \begin{aligned} &{}_qL_2\{t^{2w-2}\mathcal{J}^{(2)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\}\\ =&\mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k\ (q^2)^{\frac{k}{2}(3k+6(\mu+\nu)-1)}H_k(q^2)\ \Gamma_{q^2}(w+\mu+\nu+k) \end{aligned} $$

and

$$ {}_qL_2\{t^{2w-2}\mathcal{J}^{(3)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\} = \mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k\ (q^2)^{k+1}\ H_k(q^2)\ \Gamma_{q^2}(w+\mu+\nu+k), $$

respectively, where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$ and

$$ \mathcal{C}(s) = \frac{(a)^{\mu+\nu}}{[2]_q\ 3^{\mu+\nu}\ s^{2(w+\mu+\nu)}},\quad H_k(q) = \frac{(1-q)^{w+2(\mu+\nu+k)-1}}{(q;q)_k\ (q;q)_{3\mu+k}\ (q;q)_{3\nu+k}}. $$

Taking $m = 1$, $a_1 = a$, $k_1 = k$, $\mu_1 = \mu$ and $\nu_1 = \nu$ in Theorem 3.2, Corollary 3.3 and Corollary 3.4, respectively, the ${}_q\mathcal{L}_2$-transforms for the q-Humbert functions of the first, second and third kind are obtained:

$$ \begin{aligned} &{}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(1)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\}\\ =&A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)}\\ &\times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k}, \end{aligned} $$

$$ \begin{aligned} &{}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(2)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\}\\ =&A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k(q^2)^{\frac{k}{2}(3k+6(\mu+\nu)-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)}\\ &\times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k} \end{aligned} $$

and

$$ \begin{aligned} &{}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(3)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2); s\}\\ =&A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k(q^2)^{k+1}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)}\\ &\times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k}, \end{aligned} $$

respectively, where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$ and $A_{q^2}(s)$ is same as in Theorem 3.2.

In the present investigation, we have constructed the product formulas and certain q-Laplace type integral transforms for the q-Humbert functions of first, second and third kind. The results established in this article might be useful for solving q^2-difference equations by means of the ${}_qL_2$-transforms and ${}_q\mathcal{L}_2$-transforms. In the forthcoming paper, we plan to deal with constructing q^2-difference equations to use the results obtained here.

References

[1] W. H. Abdi, *On q-Laplace transforms*, The Proceedings of the National Academy of Sciences, India **29** (1960), 389–408.
[2] W. H. Abdi, *Applications of q-Laplace transform to the solution of certain q-integral equations*, Rend. Circ. Mat. Palermo **11**(3) (1962), 245–257. https://doi.org/10.1007/BF02843870
[3] D. Albayrak, S. D. Purohit and F. Ucar, *On q-Sumudu transforms of certain q-polynomials*, Filomat **27**(2) (2013), 411–427. https://doi.org/10.2298/FIL1302411A
[4] D. Albayrak, S. D. Purohit and F. Ucar, *On q-analogues of Sumudu transform*, Analele Universitatii "Ovidius" Constanta - Seria Matematica **21**(1) (2013), 239–259. https://doi.org/10.2478/auom-2013-0016
[5] F. B. M. Belgacem, *Sumudu transform applications to Bessel functions and equations*, Appl. Math. Sci. **4**(74) (2010), 3665–3686.
[6] M. A. Chaudhry, *Laplace transform of certain functions with applications*, Internat. J. Math. Math. Sci. **23**(2) (2000), 99–102. https://doi.org/10.1155/S0161171200001150
[7] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd Edition, Cambridge University Press, Cambridge, 2004.
[8] V. G. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.
[9] S. Khan and T. Nahid, *Numerical computation of zeros of certain hybrid q-special sequences*, Procedia Computer Science **152** (2019), 166–171. https://doi.org/10.1016/j.procs.2019.05.039
[10] S. Khan and T. Nahid, *Determinants forms, difference equations and zeros of the q-Hermite Appell polynomials*, Mathematics **6**(11) (2018), 1–16. https://doi.org/10.3390/math6110258
[11] M. Riyasat, S. Khan and T. Nahid, *Quantum algebra $\mathcal{E}_q(2)$ and 2D q-Bessel functions*, Rep. Math. Phys. **83**(2) (2019), 191–206. https://doi.org/10.1016/S0034-4877(19)30039-4
[12] M. Riyasat, T. Nahid and S. Khan, *q-Tricomi functions and quantum algebra representations*, Georgian Math. J. (2020) (to appear). https://doi.org/10.1515/gmj-2020-2079
[13] M. Rahman, *An addition theorem and some product formulas for q-Bessel functions*, Canad. J. Math. **40**(5) (1988), 1203–1221. https://doi.org/10.4153/CJM-1988-051-7
[14] M. B. Said and J. El Kamel, *Product formula for the generalized q-Bessel function*, J. Difference Equ. Appl. **22**(11) (2016), 1663–1672. https://doi.org/10.1080/10236198.2016.1234615
[15] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
[16] A. De Sole and V. G. Kac, *On integral representations of q-gamma and q-beta functions*, Rendiconti Lincei - Matematica e Applicazioni **9** (2005), 11–29.
[17] H. M. Srivastava and A. Shehata, *A family of new q-extensions of the Humbert functions*, Eur. J. Math. Sci. **4**(1) (2018), 13–26.
[18] R. F. Swarttouw, *An addition theorem and some product formulas for the Hahn-Exton q-Bessel functions*, Canad. J. Math. **44**(4) (1992), 867–879. https://doi.org/10.4153/CJM-1992-052-6
[19] F. Uçar and D. Albayrak, *On q-Laplace type integral operators and their applications*, J. Difference Equ. Appl. **18**(6) (2012), 1001–1014. https://doi.org/10.1080/10236198.2010.540572

[20] G. K. Watugala, *Sumudu transform: A new integral transform to solve differential equations and control engineering problems*, Mathematical Engineering in Industry **24**(1) (1993), 35–43. https://doi.org/10.1080/0020739930240105

[21] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, New Jersey, 1946.

[22] O. Yürekli, *Theorems on L_2-transforms and its applications*, Complex Var. Elliptic Equ. **38**(2) (1999), 95–107. https://doi.org/10.1080/17476939908815157

[23] O. Yürekli, *New identities involving the Laplace and the L_2-transforms and their applications*, Appl. Math. Comput. **99**(2-3) (1999), 141–151. https://doi.org/10.1016/S0096-3003(98)00002-2

[24] O. Yürekli and I. Sadek, *A Parseval-Goldstein type theorem on the Widder potential transform and its applications*, Int. J. Math. Math. Sci. **14**(3) (1991), 517–524. https://doi.org/10.1155/S0161171291000704

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EXISTENCE AND STABILITY OF SOLUTIONS FOR NABLA FRACTIONAL DIFFERENCE SYSTEMS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, we propose sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions for coupled systems of fractional nabla difference equations with anti-periodic boundary conditions, by using fixed point theorems. We also support these results through a couple of examples.

1. Introduction

The study of anti-periodic boundary value problems garnered significant interest due to their occurrence in the mathematical modelling of a variety of real-world problems in engineering and science. For example, we refer [19, 31, 32, 40] and the references therein.

The boundary value problems (BVPs) connected with nabla fractional difference equations can be tackled with almost similar methods as their continuous counterparts. Peterson et al. [15, 24] have initiated the study of BVPs for linear and nonlinear nabla fractional difference equations with conjugate boundary conditions. Gholami et al. [20] studied the existence of solutions for a coupled system of two-point nabla fractional difference BVPs. Recently, the author [26, 27] obtained sufficient conditions on existence and uniqueness of solutions for nonlinear nabla fractional difference equations associated with different classes of boundary conditions. In spite of the

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existence of a substantial mathematical theory of the continuous fractional anti-periodic BVPs [5–7, 13, 16, 36, 42], there has been no progress in developing the theory of discrete fractional anti-periodic BVPs in nabla perspective.

On the other hand, Hyers responses to Ulam's questions have initiated the study of stability of functional equations [23, 38]. Rassias [35] generalized the Hyers result for linear mappings. Later, several mathematicians have extended Ulam's problem in different directions [28]. There were significant contributions towards the study of Ulam-Hyers stability of ordinary as well as fractional differential equations [33,41]. The study of Ulam-Hyers stability enriched the qualitative theory of fractional difference equations [17, 18, 25].

Motivated by these facts, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions:

$$\begin{cases} \Big(\nabla_0^{\alpha_1-1}\big(\nabla u_1\big)\Big)(t) + f_1(t, u_1(t), u_2(t)) = 0, & t \in \mathbb{N}_2^T, \\ \Big(\nabla_0^{\alpha_2-1}\big(\nabla u_2\big)\Big)(t) + f_2(t, u_1(t), u_2(t)) = 0, & t \in \mathbb{N}_2^T, \\ u_1(0) + u_1(T) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(T) = 0, \\ u_2(0) + u_2(T) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(T) = 0. \end{cases} \tag{1.1}$$

Here $T \in \mathbb{N}_2$, $1 < \alpha_1, \alpha_2 < 2$, $f_1, f_2 : \mathbb{N}_0^T \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, ∇_0^ν denotes the ν^{th}-th order Riemann-Liouville type backward (nabla) difference operator where $\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}$ and ∇ denotes the first order nabla difference operator.

The present paper is organized as follows. Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness and Ulam-Hyers stability of solutions of the BVP (1.1). We present a few examples in Section 4.

2. Preliminaries

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in [21].

Denote by $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by $\rho(t) = \max\{a, t-1\}$ for all $t \in \mathbb{N}_a$.

Definition 2.1 ([21])**.** Define the μ^{th}-order nabla fractional Taylor monomial by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}, \quad \mu \in \mathbb{R} \setminus \{\dots, -2, -1\}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Observe that

$$H_\mu(a, a) = 0$$

and

$$H_\mu(t, a) = 0, \quad \text{for all } \mu \in \{\dots, -2, -1\} \text{ and } t \in \mathbb{N}_a.$$

The first order backward (nabla) difference of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by $\Big(\nabla u\Big)(t) = u(t) - u(t-1)$ for $t \in \mathbb{N}_{a+1}$.

Definition 2.2 ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\Big(\nabla_a^{-\nu} u\Big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\Big(\nabla_a^{-\nu} u\Big)(a) = 0$.

Definition 2.3 ([21]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order nabla difference of u is given by

$$\Big(\nabla_a^{\nu} u\Big)(t) = \Big(\nabla\Big(\nabla_a^{-(1-\nu)} u\Big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.1 ([21]). *We observe the following properties of nabla fractional Taylor monomials:*

(a) $\nabla H_\mu(t, a) = H_{\mu-1}(t, a)$, $t \in \mathbb{N}_a$;
(b) $\sum_{s=a+1}^{t} H_\mu(s, a) = H_{\mu+1}(t, a)$, $t \in \mathbb{N}_a$;
(c) $\sum_{s=a+1}^{t} H_\mu(t, \rho(s)) = H_{\mu+1}(t, a)$, $t \in \mathbb{N}_a$.

Proposition 2.1 ([24]). *Let $s \in \mathbb{N}_a$ and $-1 < \mu$. The following properties hold.*

(a) $H_\mu(t, \rho(s)) \geq 0$ *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $H_\mu(t, \rho(s)) > 0$ *for* $t \in \mathbb{N}_s$.
(b) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* s *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in (0, \infty)$.
(c) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (-1, 0)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* s.
(d) $H_\mu(t, \rho(s))$ *is a non-decreasing function with respect to* t *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in [0, \infty)$.
(e) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (0, \infty)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* t.
(f) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{s+1}$ *and* $\mu \in (-1, 0)$.

Proposition 2.2 ([24]). *Let u and v be two nonnegative real-valued functions defined on a set S. Further, assume u and v achieve their maximum values in S. Then,*

$$|u(t) - v(t)| \leq \max\{u(t), v(t)\} \leq \max\Big\{\max_{t \in S} u(t), \max_{t \in S} v(t)\Big\},$$

for every fixed t in S.

3. Green's Function and Its Property

Assume $T \in \mathbb{N}_2$, $1 < \alpha < 2$ and $h : \mathbb{N}_2^T \to \mathbb{R}$. Consider the boundary value problem

$$\begin{cases} \Big(\nabla_0^{\alpha-1}\Big(\nabla u\Big)\Big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T, \\ u(0) + u(T) = 0, \quad \Big(\nabla u\Big)(1) + \Big(\nabla u\Big)(T) = 0. \end{cases} \tag{3.1}$$

First, we construct the Green's function, $G(t,s)$ corresponding to (3.1), and obtain an expression for its unique solution. Denote by

$$D_1 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \geq s\}, \quad D_2 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \leq \rho(s)\}$$

and

$$\xi_\alpha = 2\left[1 + H_{\alpha-2}(T,0)\right]. \tag{3.2}$$

Theorem 3.1. *The unique solution of the nabla fractional boundary value problem* (3.1) *is given by*

$$u(t) = \sum_{s=2}^{T} G_\alpha(t,s)h(s), \quad t \in \mathbb{N}_0^T, \tag{3.3}$$

where

$$G_\alpha(t,s) = \begin{cases} K_\alpha(t,s) - H_{\alpha-1}(t,\rho(s)), & (t,s) \in D_1, \\ K_\alpha(t,s), & (t,s) \in D_2. \end{cases} \tag{3.4}$$

Here

$$\begin{aligned} K_\alpha(t,s) =& \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \\ &+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) - H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big]. \end{aligned}$$

Proof. Denote by

$$\Big(\nabla u\Big)(t) = v(t), \quad t \in \mathbb{N}_1^T.$$

Subsequently, the difference equation in (3.1) takes the form

$$\Big(\nabla_0^{\alpha-1} v\Big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T. \tag{3.5}$$

Let $v(1) = c_2$. Then, by Lemma 5.1 of [4], the unique solution of (3.5) is given by

$$v(t) = H_{\alpha-2}(t,0)c_2 - \Big(\nabla_1^{-(\alpha-1)} h\Big)(t), \quad t \in \mathbb{N}_1^T.$$

That is,

$$\Big(\nabla u\Big)(t) = H_{\alpha-2}(t,0)c_2 - \Big(\nabla_1^{-(\alpha-1)} h\Big)(t), \quad t \in \mathbb{N}_1^T. \tag{3.6}$$

Applying the first order nabla sum operator, ∇_0^{-1} on both sides of (3.6), we obtain

$$u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - \Big(\nabla_1^{-\alpha} h\Big)(t), \quad t \in \mathbb{N}_0^T, \tag{3.7}$$

where $c_1 = u(0)$. We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants c_1 and c_2 in (3.7). It follows from the first boundary condition $u(0) + u(T) = 0$ that

$$2c_1 + H_{\alpha-1}(T,0)c_2 = \Big(\nabla_1^{-\alpha} h\Big)(T). \tag{3.8}$$

The second boundary condition $\Big(\nabla u\Big)(1) + \Big(\nabla u\Big)(T) = 0$ yields

$$\left[1 + H_{\alpha-2}(T,0)\right]c_2 = \Big(\nabla_1^{-(\alpha-1)} h\Big)(T). \tag{3.9}$$

Solving (3.8) and (3.9) for c_1 and c_2, we obtain

$$c_1 = \frac{1}{2}\left[\sum_{s=2}^{T} H_{\alpha-1}(T,\rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s)\right], \tag{3.10}$$

$$c_2 = \frac{2}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s). \tag{3.11}$$

Substituting these expressions in (3.7), we achieve (3.4). □

Lemma 3.1. *Observe that*

$$|K_\alpha(t,s)| \le \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big], \tag{3.12}$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$.

Proof. Denote by

$$\begin{aligned} K'_\alpha(t,s) =& \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) \\ & + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big] \end{aligned} \tag{3.13}$$

and

$$K''_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big], \tag{3.14}$$

so that

$$K_\alpha(t,s) = K'_\alpha(t,s) - K''_\alpha(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

Clearly, from Proposition 2.1,

$$K'_\alpha(t,s) \ge 0, \quad K''_\alpha(t,s) > 0, \quad \text{for all } (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

From Proposition 2.2, it is obvious that

$$|K_\alpha(t,s)| \le \left\{\max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s)\right\}. \tag{3.15}$$

First, we evaluate the first backward difference of $K'_\alpha(t,s)$ with respect to t for a fixed s. Consider

$$\nabla K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[2H_{\alpha-2}(t,0)H_{\alpha-2}(T,\rho(s))\Big] > 0,$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$, implying that $K'_\alpha(t,s)$ is an increasing function of t for a fixed s. Thus, we have

$$K'_\alpha(t,s) \le K'_\alpha(T,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \tag{3.16}$$

It follows from (3.13)–(3.16) that

$$
\begin{aligned}
&|K_\alpha(t,s)| \\
\le &\left\{ \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
\le &\left\{ \max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s), \max_{s\in\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
= &\max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s) \\
= &\frac{1}{\xi_\alpha} \max_{s\in\mathbb{N}_2^T} \Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \\
&+ H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) \Big] \\
\le &\frac{1}{\xi_\alpha} \Big[\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0) \max_{s\in\mathbb{N}_2^T} H_{\alpha-2}(T,\rho(s)) \\
&+ H_{\alpha-2}(T,0) \max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s)) \Big] \\
= &\frac{1}{\xi_\alpha} \Big[H_{\alpha-1}(T,\rho(2)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(T)) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,\rho(2)) \Big] \\
= &\frac{1}{\xi_\alpha} \Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1) \Big].
\end{aligned}
$$

The proof is complete. □

4. Existence and Uniqueness of Solutions of (1.1)

Let $X = \mathbb{R}^{T+1}$ be the Banach space of all real $(T+1)$-tuples equipped with the maximum norm

$$\|u\|_X = \max_{t\in\mathbb{N}_0^T} |u(t)|.$$

Obviously, the product space $\Big(X \times X, \|\cdot\|_{X\times X}\Big)$ is also a Banach space with the norm

$$\|(u_1,u_2)\|_{X\times X} = \|u_1\|_X + \|u_2\|_X.$$

A closed ball with radius R centred on the zero function in $X \times X$ is defined by

$$\mathcal{B}_R = \{(u_1,u_2) \in X \times X : \|(u_1,u_2)\|_{X\times X} \le R\}.$$

Define the operator $T : X \times X \to X \times X$ by

$$T(u_1,u_2)(t) = \begin{pmatrix} T_1(u_1,u_2)(t) \\ T_2(u_1,u_2)(t) \end{pmatrix}, \quad t \in \mathbb{N}_0^T, \tag{4.1}$$

where

$$
\begin{aligned}
T_1(u_1,u_2)(t) &= \sum_{s=2}^{T} G_{\alpha_1}(t,s) f_1(s,u_1(s),u_2(s)) \\
&= \sum_{s=2}^{T} K_{\alpha_1}(t,s) f_1(s,u_1(s),u_2(s)) - \sum_{s=2}^{t} H_{\alpha_1-1}(t,s) f_1(s,u_1(s),u_2(s)) \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
T_2(u_1,u_2)(t) &= \sum_{s=2}^{T} G_{\alpha_2}(t,s) f_2(s,u_1(s),u_2(s)) \\
&= \sum_{s=2}^{T} K_{\alpha_2}(t,s) f_2(s,u_1(s),u_2(s)) - \sum_{s=2}^{t} H_{\alpha_2-1}(t,s) f_2(s,u_1(s),u_2(s)). \qquad (4.3)
\end{aligned}
$$

Clearly, (u_1,u_2) is a fixed point of T if and only if (u_1,u_2) is a solution of (1.1). For our convenience, denote by

$$\Lambda_i = \frac{1}{\xi_{\alpha_i}}\Big[H_{\alpha_i-1}(T,1) + 2H_{\alpha_i-1}(T,0) + H_{\alpha_i-2}(T,0)H_{\alpha_i-1}(T,1)\Big], \qquad (4.4)$$

$$a_i = l_i\left[\Lambda_i(T-1) + H_{\alpha_i}(T,1)\right], \qquad (4.5)$$

$$b_i = m_i\left[\Lambda_i(T-1) + H_{\alpha_i}(T,1)\right], \qquad (4.6)$$

$$c_i = n_i\left[\Lambda_i(T-1) + H_{\alpha_i}(T,1)\right], \qquad (4.7)$$

$$d_i = M_i\left[\Lambda_i(T-1) + H_{\alpha_i}(T,1)\right], \qquad (4.8)$$

for $i = 1,2$. Assume

(H1)' for each $i \in \{1,2\}$, there exist nonnegative numbers l_i and m_i such that

$$|f_i(t,u_1,u_2) - f_i(t,v_1,v_2)| \le l_i\|u_1 - v_1\|_X + m_i\|u_2 - v_2\|_X,$$

for all (t,u_1,u_2), $(t,v_1,v_2) \in \mathbb{N}_0^T \times X \times X$;

(H1) for each $i \in \{1,2\}$, there exist nonnegative numbers l_i and m_i such that

$$|f_i(t,u_1,u_2) - f_i(t,v_1,v_2)| \le l_i\|u_1 - v_1\|_X + m_i\|u_2 - v_2\|_X,$$

for all (t,u_1,u_2), $(t,v_1,v_2) \in \mathbb{N}_0^T \times \mathcal{B}_R$;

(H2)' for each $i \in \{1,2\}$, there exist nonnegative numbers L_i such that

$$|f_i(t,u_1,u_2)| \le L_i,$$

for all $(t,u_1,u_2) \in \mathbb{N}_0^T \times X \times X$;

(H2) for each $i \in \{1,2\}$, there exist nonnegative numbers l_i, m_i, and n_i such that

$$|f_i(t,u_1,u_2)| \le l_i\|u_1\|_X + m_i\|u_2\|_X + n_i,$$

for all $(t,u_1,u_2) \in \mathbb{N}_0^T \times \mathcal{B}_R$;

(H3) for each $i \in \{1,2\}$,

$$\max_{t\in\mathbb{N}_0^T} |f_i(t,0,0)| = M_i;$$

(H4) $\lambda = (a_1 + a_2) + (b_1 + b_2) \in (0,1)$.

We apply Banach's fixed point theorem to establish existence and uniqueness of solutions of (1.1).

Theorem 4.1 ([37])**.** *Let S be a closed subset of a Banach space X. Then, any contraction mapping T of X into itself has a unique fixed point.*

Theorem 4.2. *Assume* (H1), (H3) *and* (H4) *hold. If we choose*

$$R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]},$$

then the system (1.1) *has a unique solution* $(u_1, u_2) \in \mathcal{B}_R$.

Proof. Clearly, $T : \mathcal{B}_R \to X \times X$. First, we show that T is a contraction mapping. To see this, let (u_1, u_2), $(v_1, v_2) \in \mathcal{B}_R$, and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$\begin{aligned}
&|T_i(u_1, u_2)(t) - T_i(v_1, v_2)(t)| \\
\leq &\sum_{s=2}^{T} |K_{\alpha_i}(t, s)|\, |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))| \\
&+ \sum_{s=2}^{t} H_{\alpha_i - 1}(t, s)\, |f_i(s, u_1(s), u_2(s)) - f_i(s, v_1(s), v_2(s))| \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t, s)| + \sum_{s=2}^{t} H_{\alpha_i - 1}(t, s)\right] \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] \left[\Lambda_i(T - 1) + H_{\alpha_i}(t, 1)\right] \\
\leq &[l_i \|u_1 - v_1\|_X + m_i \|u_2 - v_2\|_X] \left[\Lambda_i(T - 1) + H_{\alpha_i}(T, 1)\right] \\
\leq &a_i \|u_1 - v_1\|_X + b_i \|u_2 - v_2\|_X,
\end{aligned}$$

implying that, for each $i \in \{1, 2\}$,

$$\Big\|T_i(u_1, u_2) - T_i(v_1, v_2)\Big\|_X \leq \Big[a_i \|u_1 - v_1\|_X + b_i \|u_2 - v_2\|_X\Big]. \tag{4.9}$$

Thus, we have

$$\begin{aligned}
&\|T(u_1, u_2) - T(v_1, v_2)\|_{X \times X} \\
= &\Big\|T_1(u_1, u_2) - T_1(v_1, v_2)\Big\|_X + \Big\|T_2(u_1, u_2) - T_2(v_1, v_2)\Big\|_X \\
\leq &\Big[(a_1 + a_2)\|u_1 - v_1\|_X + (b_1 + b_2)\|u_2 - v_2\|_X\Big] \\
\leq &\lambda\Big[(\|u_1 - v_1\|_X + \|u_2 - v_2\|_X\Big] \\
= &\lambda\|(u_1, u_2) - (v_1, v_2)\|_{X \times X}.
\end{aligned}$$

Since $\lambda < 1$, T is a contraction mapping with contraction constant λ. Next, we show that

$$T : \mathcal{B}_R \to \mathcal{B}_R. \tag{4.10}$$

To see this, let $(u_1, u_2) \in \mathcal{B}_R$, and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$\begin{aligned}
&\Big|T_i(u_1, u_2)(t)\Big| \\
\leq &\sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\, |f_i(s, u_1(s), u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\, |f_i(s, u_1(s), u_2(s))| \\
\leq &\sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\, |f_i(s, u_1(s), u_2(s)) - f_i(s,0,0)| + \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\, |f_i(s,0,0)| \\
&+ \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\, |f_i(s, u_1(s), u_2(s)) - f_i(s,0,0)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\, |f_i(s,0,0)| \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X\Big] \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + M_i \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| \\
&+ \Big[l_i\|u_1\|_X + m_i\|u_2\|_X\Big] \sum_{s=2}^{t} H_{\alpha_i-1}(t,s) + M_i \sum_{s=2}^{t} H_{\alpha_i-1}(t,s) \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + M_i\Big]\, [\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
\leq &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + M_i\Big]\, [\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
\leq &a_i\|u_1\|_X + b_i\|u_2\|_X + d_i,
\end{aligned}$$

implying that, for each $i \in \{1, 2\}$,

$$\Big\|T_i(u_1, u_2)\Big\|_X \leq a_i\|u_1\|_X + b_i\|u_2\|_X + d_i. \tag{4.11}$$

Thus, we have

$$\begin{aligned}
\|T(u_1, u_2)\|_{X\times X} &= \Big\|T_1(u_1, u_2)\Big\|_X + \Big\|T_2(u_1, u_2)\Big\|_X \\
&\leq (a_1 + a_2)R + (b_1 + b_2)R + (d_1 + d_2) \leq R,
\end{aligned}$$

implying that (4.10) holds. Therefore, by Theorem 4.1, T has a unique fixed point $(u_1, u_2) \in \mathcal{B}_R$. The proof is complete. □

Corollary 4.1. *Assume* (H1)' *and* (H4) *hold. Then, the system* (1.1) *has a unique solution* $(u_1, u_2) \in X \times X$.

We apply Brouwer's fixed point theorem to establish existence of solutions of (1.1).

Theorem 4.3 ([37]). *Let C be a non-empty bounded closed convex subset of $\mathbb{R}^n$ and $T : C \to C$ be a continuous mapping. Then, T has a fixed point in C.*

Theorem 4.4. *Assume* (H2) *and* (H4) *hold. If we choose*

$$R \geq \frac{(c_1 + c_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]},$$

then the system (1.1) *has at least one solution* $(u_1, u_2) \in \mathcal{B}_R$.

Proof. We claim that $T : B_R \to B_R$. To see this, let $(u_1, u_2) \in \mathcal{B}_R$ and $t \in \mathbb{N}_0^T$. For each $i \in \{1, 2\}$, consider

$$
\begin{aligned}
&\Big|T_i(u_1, u_2)(t)\Big| \\
\le &\sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(s, u_1(s), u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(s, u_1(s), u_2(s))| \\
\le &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \sum_{s=2}^{T} |K_{\alpha_i}(t,s)| \\
&\quad + \Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big] \sum_{s=2}^{t} H_{\alpha_i-1}(t,s) \\
\le &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
\le &\Big[l_i\|u_1\|_X + m_i\|u_2\|_X + n_i\Big]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
\le &a_i\|u_1\|_X + b_i\|u_2\|_X + c_i,
\end{aligned}
$$

implying that, for each $i \in \{1, 2\}$,

$$
\Big\|T_i(u_1, u_2)\Big\|_X \le a_i\|u_1\|_X + b_i\|u_2\|_X + c_i. \tag{4.12}
$$

Thus, we have

$$
\begin{aligned}
\|T(u_1, u_2)\|_{X\times X} &= \Big\|T_1(u_1, u_2)\Big\|_X + \Big\|T_2(u_1, u_2)\Big\|_X \\
&\le (a_1 + a_2)R + (b_1 + b_2)R + (c_1 + c_2) \le R,
\end{aligned}
$$

implying that $T : B_R \to B_R$. Therefore, by Brouwer's fixed point theorem, T has a fixed point $(u_1, u_2) \in \mathcal{B}_R$. The proof is complete. ☐

Corollary 4.2. *Assume* (H2)' *hold. Then, the system* (1.1) *has at least one solution* $(u_1, u_2) \in X \times X$.

Urs [39] presented some Ulam-Hyers stability results for the coupled fixed point of a pair of contractive type operators on complete metric spaces. We use Urs's [39] approach to establish Ulam-Hyers stability of solutions of (1.1).

Definition 4.1 ([39])**.** Let X be a Banach space and T_1, $T_2 : X \times X \to X$ be two operators. Then, the operational equations system

$$
\begin{cases} u_1 = T_1(u_1, u_2), \\ u_2 = T_2(u_1, u_2), \end{cases} \tag{4.13}
$$

is said to be Ulam-Hyers stable if there exist C_1, C_2, C_3, $C_4 > 0$ such that for each ε_1, $\varepsilon_2 > 0$ and each solution-pair $(u_1^*, u_2^*) \in X \times X$ of the in-equations:

$$
\begin{cases} \|u_1 - T_1(u_1, u_2)\|_X \le \varepsilon_1, \\ \|u_2 - T_2(u_1, u_2)\|_X \le \varepsilon_2, \end{cases} \tag{4.14}
$$

there exists a solution $(v_1^*, v_2^*) \in X \times X$ of (4.13) such that

$$\begin{cases} \|u_1^* - v_1^*\|_X \le C_1\varepsilon_1 + C_2\varepsilon_2, \\ \|u_2^* - v_2^*\|_X \le C_3\varepsilon_1 + C_4\varepsilon_2. \end{cases} \tag{4.15}$$

Theorem 4.5 ([39]). *Let X be a Banach space, T_1, $T_2 : X \times X \to X$ be two operators such that*

$$\begin{cases} \|T_1(u_1, u_2) - T_1(v_1, v_2)\|_X \le k_1\|u_1 - v_1\|_X + k_2\|u_2 - v_2\|_X, \\ \|T_2(u_1, u_2) - T_2(v_1, v_2)\|_X \le k_3\|u_1 - v_1\|_X + k_4\|u_2 - v_2\|_X, \end{cases} \tag{4.16}$$

for all (u_1, u_2), $(v_1, v_2) \in X \times X$. Suppose

$$H = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

converges to zero. Then, the operational equations system (4.13) *is Ulam-Hyers stable.*

Set

$$H = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{4.17}$$

Theorem 4.6. *Assume the hypothesis of Theorem* 4.2 *holds. Further, assume the spectral radius of H is less than one. Then, the unique solution of the system* (1.1) *is Ulam-Hyers stable.*

Proof. In view of Theorem 4.2, we have

$$\begin{cases} \Big\|T_1(u_1, u_2) - T_1(v_1, v_2)\Big\|_X \le a_1\|u_1 - v_1\|_X + b_1\|u_2 - v_2\|_X, \\ \Big\|T_2(u_1, u_2) - T_2(v_1, v_2)\Big\|_X \le a_2\|u_1 - v_1\|_X + b_2\|u_2 - v_2\|_X, \end{cases} \tag{4.18}$$

which implies that

$$\|T(u_1, u_2) - T(v_1, v_2)\|_{X\times X} \le H \begin{pmatrix} \|u_1 - v_1\|_X \\ \|u_2 - v_2\|_X \end{pmatrix}. \tag{4.19}$$

Since the spectral radius of H is less than one, the unique solution of (1.1) is Ulam-Hyers stable. The proof is complete. $\square$

5. Examples

In this section, we provide two examples to illustrate the applicability of Theorem 4.2, Theorem 4.4 and Theorem 4.6.

Example 5.1. Consider the following coupled system of two-point nabla fractional difference boundary value problems

(5.1)
$$\begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.001)e^{-t}\left[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.002)\left[e^{-t} + \sin u_1(t) + \sin u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ u_1(0) + u_1(9) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(9) = 0, \\ u_2(0) + u_2(9) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(9) = 0. \end{cases}$$

Comparing (1.1) and (5.1), we have $T = 9$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(t, u_1, u_2) = (0.001)e^{-t}\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right]$$

and

$$f_2(t, u_1, u_2) = (0.002)\left[e^{-t} + \sin u_1 + \sin u_2\right],$$

for all $(t, u_1, u_2) \in \mathbb{N}_0^9 \times \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{N}_0^9 \times \mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (H1) with $l_1 = 0.001$, $m_1 = 0.001$, $l_2 = 0.002$ and $m_2 = 0.002$. We have

$$\begin{aligned} M_1 &= \max_{t \in \mathbb{N}_0^9} |f_1(t, 0, 0)| = 0.001, \\ M_2 &= \max_{t \in \mathbb{N}_0^9} |f_2(t, 0, 0)| = 0.002, \\ a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.1053, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.1053, \\ d_1 &= M_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.0527, \\ d_2 &= M_2\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.1053. \end{aligned}$$

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.316 \in (0, 1)$, implying that assumptions (H3) and (H4) hold. Choose

$$R \geq \frac{(d_1 + d_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]} = 0.231.$$

Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1, u_2) \in \mathcal{B}_R$. Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.0527 & 0.0527 \\ 0.1053 & 0.1053 \end{pmatrix}.$$

The spectral radius of M is 0.158, which is less than one, implying that M converges to zero. Thus, by Theorem 4.6, the unique solution of (5.1) is Ulam-Hyers stable.

Example 5.2. Consider the following coupled system of two-point nabla fractional difference boundary value problems

$$\begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.01)\left[e^{-t} + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t)\right] = 0, & t \in \mathbb{N}_2^4, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.02)\left[e^{-t} + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}}\right] = 0, & t \in \mathbb{N}_2^4, \\ u_1(0) + u_1(4) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(4) = 0, \\ u_2(0) + u_2(4) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(4) = 0. \end{cases} \tag{5.2}$$

Comparing (1.1) and (5.2), we have $T = 4$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(t, u_1, u_2) = (0.01)\left[e^{-t} + \frac{1}{\sqrt{1+u_1^2}} + u_2\right]$$

and

$$f_2(t, u_1, u_2) = (0.02)\left[e^{-t} + u_1 + \frac{1}{\sqrt{1+u_2^2}}\right],$$

for all $(t, u_1, u_2) \in \mathbb{N}_0^4 \times \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{N}_0^4 \times \mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (H2) with $l_1 = 0.01$, $m_1 = 0.01$, $l_2 = 0.02$, $m_2 = 0.02$, $n_1 = 0.01$ and $n_2 = 0.02$. We have

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ c_1 &= n_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ c_2 &= n_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438. \end{aligned}$$

Also, $\lambda = (a_1 + a_2) + (b_1 + b_2) = 0.7314 \in (0, 1)$, implying that assumption (H4) hold. Choose

$$R \geq \frac{(c_1 + c_2)}{1 - [(a_1 + a_2) + (b_1 + b_2)]} = 1.3615.$$

Hence, by Theorem 4.2, the system (5.1) has at least one solution $(u_1, u_2) \in \mathcal{B}_R$.

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References

[1] T. Abdeljawad, *On Riemann and Caputo fractional differences*, Comput. Math. Appl. **62**(3) (2011), 1602–1611. https://doi.org/10.1016/j.camwa.2011.03.036

[2] T. Abdeljawad and J. Alzabut, *On Riemann-Liouville fractional q-difference equations and their application to retarded logistic type model*, Math. Methods Appl. Sci. **41**(18) (2018), 8953–8962. https://doi.org/10.1002/mma.4743

[3] T. Abdeljawad, J. Alzabut and H. Zhou, *A Krasnoselskii existence result for nonlinear delay Caputo q-fractional difference equations with applications to Lotka-Volterra competition model*, Appl. Math. E-Notes **17** (2017), 307–318.

[4] T. Abdeljawad and F. M. Atici, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. **2012** (2012), Article ID 406757, 13 pages. https://doi.org/10.1155/2012/406757

[5] R. P. Agarwal, B. Ahmad and J. J. Nieto, *Fractional differential equations with nonlocal (parametric type) anti-periodic boundary conditions*, Filomat **31**(5) (2017), 1207–1214. https://doi.org/10.2298/FIL1705207A

[6] B. Ahmad and J. J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topol. Methods Nonlinear Anal. **35**(2) (2010), 295–304.

[7] B. Ahmad and J. J. Nieto, *Anti-periodic fractional boundary value problems*, Comput. Math. Appl. **62**(3) (2011), 1150–1156. https://doi.org/10.1016/j.camwa.2011.02.034

[8] K. Ahrendt, L. Castle, M. Holm and K. Yochman, *Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula*, Commun. Appl. Anal. **16**(3) (2012), 317–347.

[9] J. Alzabut, T. Abdeljawad and D. Baleanu, *Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model*, J. Comput. Anal. Appl. **25**(5) (2018), 889–898.

[10] J. Alzabut, T. Abdeljawad and H. Alrabaiah, *Oscillation criteria for forced and damped nabla fractional difference equations*, J. Comput. Anal. Appl. **24**(8) (2018), 1387–1394.

[11] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ. Special Edition I (3) (2009), 1–12. https://doi.org/10.14232/EJQTDE.2009.4.3

[12] H. Baghani, J. Alzabut and J. J. Nieto, *A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions*, Math. Methods Appl. Sci. (2020), 1–11. https://doi.org/10.1002/mma.6639

[13] M. Benchohra, N. Hamidi and J. Henderson, *Fractional differential equations with anti-periodic boundary conditions*, Numer. Funct. Anal. Optim. **34**(4) (2013), 404–414. https://doi.org/10.1080/01630563.2012.763140

[14] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, MA, 2001.

[15] A. Brackins, *Boundary value problems of nabla fractional difference equations*, Ph. D. Thesis, The University of Nebraska-Lincoln, Nebraska-Lincoln, 2014.

[16] G. Chai, *Anti-periodic boundary value problems of fractional differential equations with the Riemann-Liouville fractional derivative*, Adv. Difference Equ. **306** (2013), 1–24. https://doi.org/10.1186/1687-1847-2013-306

[17] C. Chen, M. Bohner and B. Jia, *Ulam-Hyers stability of Caputo fractional difference equations*, Math. Methods Appl. Sci. (2019), 1–10. https://doi.org/10.1002/mma.5869

[18] F. Chen and Y. Zhou, *Existence and Ulam stability of solutions for discrete fractional boundary value problem*, Discrete Dyn. Nat. Soc. (2013), Article ID 459161, 7 pages. https://doi.org/10.1155/2013/459161

[19] Y. Chen, J. J. Nieto and D. O'Regan, *Anti-periodic solutions for fully nonlinear first-order differential equations*, Math. Comput. Model. **46**(9–10) (2007), 1183–1190. `https://doi.org/10.1016/j.mcm.2006.12.006`

[20] Y. Gholami and K. Ghanbari, *Coupled systems of fractional* ∇*-difference boundary value problems*, Differ. Equ. Appl. **8**(4) (2016), 459–470. `https://dx.doi.org/10.7153/dea-08-26`

[21] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2015.

[22] J. Hein, S. McCarthy, N. Gaswick, B. McKain and K. Speer, *Laplace transforms for the nabla difference operator*, PanAmer. Math. J. **21** (2011), 79–96.

[23] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224. `https://doi.org/10.1073/pnas.27.4.222`

[24] A. Ikram, *Lyapunov inequalities for nabla Caputo boundary value problems*, J. Difference Equ. Appl. **25**(6) (2019), 757–775. `https://doi.org/10.1080/10236198.2018.1560433`

[25] J. J. Mohan, *Hyers-Ulam stability of fractional nabla difference equations*, Int. J. Anal. (2016), Article ID 7265307, 1–5. `https://doi.org/10.1155/2016/7265307`

[26] J. M. Jonnalagadda, *On two-point Riemann-Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13**(2) (2018), 141–166.

[27] J. M. Jonnalagadda, *Existence results for solutions of nabla fractional boundary value problems with general boundary conditions*, Advances in the Theory of Nonlinear Analysis and its Application **4**(1) (2020), 29–42. `https://doi.org/10.31197/atnaa.634557`

[28] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[29] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[30] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B. V., Amsterdam, 2006.

[31] J. Liu and Z. Liu, *On the existence of anti-periodic solutions for implicit differential equations*, Acta Math. Hungar. **132**(3) (2011), 294–305. `https://doi.org/10.1007/s10474-010-0054-2`

[32] J. W. Lyons and J. T. Neugebauer, *A difference equation with anti-periodic boundary conditions*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **22**(1) (2015), 47–60.

[33] M. M. Matar, J. Alzabut and J. M. Jonnalagadda, *A coupled system of nonlinear Caputo-Hadamard Langevin equations associated with nonperiodic boundary conditions*, Math. Methods Appl. Sci. (2020), 1–21. `https://doi.org/10.1002/mma.6711`

[34] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering **198**, Academic Press, San Diego, CA, 1999.

[35] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(2) (1978), 297–300. `https://doi.org/10.2307/2042795`

[36] J. Sun, Y. Liu and G. Liu, *Existence of solutions for fractional differential systems with anti-periodic boundary conditions*, Comput. Math. Appl. **64**(6) (2012), 1557–1566. `https://doi.org/10.1016/j.camwa.2011.12.083`

[37] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics **66**, Cambridge University Press, London-New York, 1974.

[38] S. A. Ulam, *Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics **8**, New York, London, Interscience Publishers, 1960.

[39] C. Urs, *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Math. Notes **14**(1) (2013), 323–333. `https://doi.org/10.18514/MMN.2013.598`

[40] Y. Wang and Y. Shi, *Eigenvalues of second-order difference equations with periodic and anti-periodic boundary conditions*, J. Math. Anal. Appl. **309**(1) (2005), 56–69. `https://doi.org/10.1016/j.jmaa.2004.12.010`

[41] A. Zada, H. Waheed, J. Alzabut and X. Wang, *Existence and stability of impulsive coupled system of fractional integrodifferential equations*, Demonstr. Math. **52**(1) (2019), 296–335. https://doi.org/10.1515/dema-2019-0035

[42] H. Zhang and W. Gao, *Existence and uniqueness results for a coupled system of nonlinear fractional differential equations with anti-periodic boundary conditions*, Abstr. Appl. Anal. (2014), Article ID 463517, 7 pages. https://doi.org/10.1155/2014/463517

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DIFFERENCE ANALOGUES OF SECOND MAIN THEOREM AND PICARD TYPE THEOREM FOR SLOWLY MOVING PERIODIC TARGETS

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Abstract. In this paper, we show some Second main theorems for linearly non-degenerate meromorphic mappings over the field $\mathcal{P}_c^1$ of c-periodic meromorphic functions having their hyper-orders strictly less than one in $\mathbb{C}^m$ intersecting slowly moving targets in $\mathbb{P}^n(\mathbb{C})$. As an application, we give some Picard type theorems for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ under the growth condition hyper-order less than one.

1. Introduction

In 2006, R. Halburd and R. Korhonen [5] considered the Second main theorem for complex difference operator with finite order in complex plane. Later, difference analogues of the Second main theorem for holomorphic curves or for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ were obtained independently by the authors such as P. M. Wong, H. F. Law, P. P. W. Wong, R. Halburd, R. Korhonen, K. Tohge, T. B. Cao (see [2, 6–8]). Recently, T. B. Cao and R. Korhonen [3] obtained a new natural difference analogue of the Cartan's theorem [1], in which the counting function $N(r, \nu^0_{W(f)})$ of Wronskian determinant of f is replaced by the counting function $N(r, \nu^0_{C(f)})$ of Casorati determinant of f (it was called the finite difference Wronskian determinant in [6]).

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In particular, under the growth condition hyper-order < 1, R. Korhonen, N. Li-K. Tohge [9] obtained the Second main theorem of holomorphic curves of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ for slowly moving periodic targets which is similar to the Cartan's theorem.

To state some results in this direction, we recall some notations in [3, 9].

Let $c \in \mathbb{C}^m$, we denote by $\mathcal{M}_m$ the set of all meromorphic functions on $\mathbb{C}^m$, by $\mathcal{P}_c$ the set of all meromorphic functions of $\mathcal{M}_m$ periodic with period c, and by $\mathcal{P}_c^\lambda$ the set of all meromorphic functions of $\mathcal{M}_m$ periodic with period c and having their hyper-orders strictly less than λ. Obviously, $\mathcal{P}_c^\lambda \subset \mathcal{P}_c \subset \mathcal{M}_m$.

Definition 1.1. Let f be a meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$. Then the map f is said to be linearly nondegenerate over a field $\mathcal{K}$ if the entire functions $f_0, \ldots, f_n$ are linearly independent over the field $\mathcal{K}$.

For $c = (c_1, \ldots, c_m)$ and $z = (z_1, \ldots, z_m)$, we write $c + z = (c_1 + z_1, \ldots, c_m + z_m)$. Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$. Denote

$$f(z) \equiv f := \bar{f}^{[0]}, f(z+c) \equiv \bar{f} := \bar{f}^{[1]}, f(z+2c) \equiv \bar{\bar{f}} := \bar{f}^{[2]}, \ldots, f(z+kc) \equiv \bar{f}^{[k]}.$$

Let

$$D^{(j)} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1(j)} \cdots \left(\frac{\partial}{\partial z_m}\right)^{\alpha_m(j)}$$

be a partial differentiation operator of order at most $j = \sum_{k=1}^m \alpha_k(j)$. Similarly as the Wronskian determinant

$$W(f) = W(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ D^{(1)} f_0 & D^{(1)} f_1 & \cdots & D^{(1)} f_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{(n)} f_0 & D^{(n)} f_1 & \cdots & D^{(n)} f_n \end{vmatrix},$$

the Casorati determinant is defined by

$$C(f) = C(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \bar{f}_0 & \bar{f}_1 & \cdots & \bar{f}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{f}_0^{[n]} & \bar{f}_1^{[n]} & \cdots & \bar{f}_n^{[n]} \end{vmatrix}.$$

Let $H_1, H_2, \ldots, H_q$ be (fixed) hyperplanes in $\mathbb{P}^n(\mathbb{C})$ given by

$$H_j = \{[\omega_0 : \cdots : \omega_n] \in \mathbb{P}^n(\mathbb{C}) : a_{j0}\omega_0 + \cdots + a_{jn}\omega_n = 0\} \quad (1 \le j \le q),$$

where the constants $a_{j0}, \ldots, a_{jn} \in \mathbb{C}$ are not simultaneously zero.

Definition 1.2. Let $N \geqslant n$ and $q \geqslant N + 1$. The family $\{H_j\}_{j=1}^q$ is said to be in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ if any $N + 1$ of the vectors $(a_{j0}, \ldots, a_{jn})$ $(1 \le j \le q)$ are linearly independent over $\mathbb{C}$.

If they are in n-subgeneral position, we simply say that they are in general position.

Let $a_1, \dots, a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$ $(1 \leq j \leq q)$. The moving hyperplane H_j associated with a_j is defined by

$$H_j(z) = \{[\omega_0 : \cdots : \omega_n] \in \mathbb{P}^n(\mathbb{C}) : L_{H_j}(z, a_j(z)) := a_{j0}(z)\omega_0 + \cdots + a_{jn}(z)\omega_n = 0\},$$

with $z \in \mathbb{C}^m \setminus I(a_j)$, where $I(a_j)$ is the locus of indeterminacy of a_j.

Similarly to the above definition, we have the following.

Definition 1.3. Let $k \geq n$ and $q \geq k+1$ and let $\mathcal{K}$ be a field such that $\mathbb{C} \subset \mathcal{K}$. We say that the moving targets $a_1, \dots, a_q$ (also say that the moving hyperplanes $H_1(z), \dots, H_q(z)$) are in k-subgeneral position over $\mathcal{K}$ if any $k+1$ of vectors $(a_{j0}(z), \dots, a_{jn}(z))$ $(1 \leq j \leq q)$ are linearly independent over $\mathcal{K}$.

If they are in n-subgeneral position over $\mathcal{K}$, we also simply say that they are in general position over $\mathcal{K}$.

Let f, a be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations $f = (f_0, \dots, f_n),\ a = (a_0, \dots, a_n)$, respectively. We define $(f, a) := \sum_{i=0}^n a_i f_i$.

Definition 1.4. We say that a is a small moving target or a slowly moving target with respect to f if $T(r, a) = o(T(r, f))$ as $r \to \infty$, where the notations $T(r, a)$ and $T(r, f)$ are characteristic functions of a and f, respectively.

When the entire functions a_j $(0 \leq j \leq n)$ are periodic with period c then we say that a_j are moving periodic targets with period c.

Definition 1.5. Let $n \in \mathbb{N}, c \in \mathbb{C}^m \setminus \{0\}$ and $a \in \mathbb{C}$. An a-point z_0 of a meromorphic function $f(z)$ is said to be n-successive with separation c, if the n mappings $f(z+kc)$ $(k = 1, \dots, n)$ take the value a at $z = z_0$ with multiplicity not less than that of $f(z)$ there. All the other a-points of $f(z)$ are called n-aperiodic of pace c.

By $\tilde{N}^{[n,c]}(r, (f, a))$, we denote the counting function of all n-aperiodic zeros of the function (f, a) of pace c.

Note that $\tilde{N}_{[n,c]}(r, (f, a)) \equiv 0$ when all zeros of (f, a) with taking their multiplicities into account are located periodically with period c. This is also the case when the moving target a is forward invariant by f with respect to the translation $\tau_c = z + c$, i.e., $\tau_c(f^{-1}(a)) \subset f^{-1}(a)$ and $f^{-1}(a)$ are considered to be multi-sets in which each point is repeated according to its multiplicity. In fact, it follows by the definition that any zero with a forward invariant preimage of the function (f, a) must be n-successive with separation c, since

$$f^{-1}(a) \subset \tau_{-c}(f^{-1}(a)) \subset \cdots \subset \tau_{-(n-1)c}(f^{-1}(a)).$$

With these definitions, the Second main theorem of holomorphic curves for slowly moving periodic hyperplanes is stated as follows.

Theorem A. ([9]) (Difference analogue of the Cartan's Second main theorem) *Let $n \geq 1$ and let $g = (g_0 : \cdots : g_n)$ be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with* *hyper-order* $\zeta = \zeta_2(g) < 1$*, where* $g_0, \ldots, g_n$ *are linearly independent over* $\mathcal{P}_c^1$*. Let*

$$a_j(z) = (a_{j0} : \cdots : a_{jn}) \quad (j \in \{0, \ldots, q\}),$$

where $a_{jk}(z)$ *are c-periodic entire functions satisfying* $T(r, a_{jk}) = o(T_g(r))$ *for all* $j, k \in \{0, \ldots, q\}$*. If the moving hyperplanes*

$$H_j(z) = \{(\omega_0, \ldots, \omega_n) : L_{H_j}(z, a_j(z)) = 0\} \quad (j \in \{0, \ldots, q\})$$

are located in general position over $\mathcal{P}_c^1$*, then*

$$\Big|\Big| \quad (q-n)T_g(r) \leq \sum_{j=0}^{q} \tilde{N}_g^{[n,c]}(r, L_{H_j}) + o\left(T_g(r)\right).$$

Here, by the notation "$||$ P“ we mean the assertion P holds for all $r \in [0, \infty)$ outside of an exceptional set with finite logarithmic measure.

Firstly, by using the idea proposed by D. D. Thai, S. D. Quang [11], we will extend Theorem A to meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Namely, we have the following.

Theorem 1.1. *Let* $c \in \mathbb{C}^m$ *and* $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping over* $\mathcal{P}_c^1$ *with hyper-order* $\zeta_2(f) < 1$*. Let* a_j $(1 \leq j \leq q)$ *be q slowly moving periodic targets with respect to f with period c, located in general position over* $\mathcal{P}_c^1$*. Then we have*

$$\Big|\Big| \quad \frac{q}{n+2}T(r,f) \leq \sum_{j=1}^{q} \tilde{N}_{(f,a_j)}^{[n,c]}(r) + o\left(T(r,f)\right),$$

where $\tilde{N}_{(f,a_j)}^{[n,c]}(r)$ *is the counting function of all n-aperiodic zeros of the function* (f, a_j)*.*

We now consider $\mathcal{A} = \{a_1, \ldots, a_q\}$ be the set of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$.

Definition 1.6. We say that $\mathcal{A}$ is nondegenerate over a field $\mathcal{K}$ if $\dim(\mathcal{A})_{\mathcal{K}} = n+1$ and for each nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$

$$(\mathcal{A}_1)_{\mathcal{K}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{K}} \cap \mathcal{A} \neq \emptyset,$$

where $(\mathcal{A})_{\mathcal{K}}$ is the linear span of $\mathcal{A}$ over the field $\mathcal{K}$.

With the above definitions, we have the following theorem.

Theorem 1.2. *Let* $c \in \mathbb{C}^m$ *and let* $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping over* $\mathcal{P}_c^1$ *with hyper-order* $\zeta_2(f) < 1$*. Let* a_j $(1 \leq j \leq q)$ *be q slowly moving periodic targets with respect to f with period c such that* $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$*. Assume that* $\mathcal{A} = \{a_1, \ldots, a_q\}$ *is nondegenerate over* $\mathcal{M}_m$*. Then we have*

$$\Big|\Big| \quad T(r,f) \leq \sum_{j=1}^{q} \tilde{N}_{(f,a_j)}^{[n,c]}(r) + o\left(T(r,f)\right),$$

where $\tilde{N}_{(f,a_j)}^{[n,c]}(r)$ *is the counting function of all n-aperiodic zeros of the function* (f, a_j)*.*

We would like note that if the mapping f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z+c$, i.e., $\tau((f, a_j)^{-1}) \subset (f, a_j)^{-1}$ (counting multiplicity) holds for all j, then $\tilde{N}^{[n,c]}_{(f,a_j)}(r) = 0$ for all j. It follows from Theorem 1.2 that there exists no linearly nondegenerate meromorphic mapping over $\mathcal{P}_c$ which is periodic with period c.

By considering the uniqueness problem for $f(z)$ and $f(z+c)$ intersecting hyperplanes, the authors of [3, 7] obtained an unicity theorem for linearly degenerate meromorphic mappings over $\mathcal{P}_c$. That result is an extension of Picard's theorem under the growth condition hyper-order less than 1.

Theorem B ([3]). *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$, and let $\tau(z) = z + c$, where $c \in \mathbb{C}^m$. Assume that $\tau((f, H_j)^{-1}) \subset (f, H_j)^{-1}$ (counting multiplicity) holds for q distinct hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$. If $q > 2N$, then $f(z) = f(z + c)$.*

Finally, we would like to extend the above result to the case of slowly moving periodic targets.

Theorem 1.3. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$. Let $\mathcal{A} = \{a_1, \dots, a_q\}$ be the set of slowly moving periodic targets with respect to f with period c which is nondegenerate over $\mathcal{M}_m$ and satisfying $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$. Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z + c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}^1_c$ of dimension $\le \left[\frac{n}{q-n}\right]$. Particularly, if $q > 2n$, then f is periodic with period c, i.e., $f(z) = f(z + c)$.*

Theorem 1.4. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with hyper-order $\zeta_2(f) < 1$. Let $c \in \mathbb{C}^m$ and $k \in \mathbb{N}$, $k \ge n$. Let $\mathcal{A} = \{a_1, \dots, a_q\}$ be the set of slowly moving periodic targets with respect to f with period c such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$ and satisfies condition: $\dim(\mathcal{A})_{\mathcal{M}_m} = n + 1$ and for each a proper subset $\mathcal{A}_1$ of $\mathcal{A}$ with $|\mathcal{A}_1| \ge k + 1$ then $(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1) \ne \emptyset$. Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z+c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}^1_c$ of dimension $\le \left[\frac{k}{q-k}\right]$. Particularly, if $q > 2k$ then f is periodic with period c, i.e., $f(z) = f(z + c)$.*

Denote by $\mathcal{R} = \mathcal{R}(\{a_i\}_{i=1}^q) \subset \mathcal{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\frac{a_{ik}}{a_{il}}$ with $a_{il} \not\equiv 0$. Obviously, $\mathcal{R} \subset \mathcal{P}^1_c \subset \mathcal{M}_m$. Since the proof of Theorem 1.4, we can see that this theorem also holds when the field M_m is replaced by the field $\mathcal{R}$ or the field $\mathcal{P}^1_c$. Therefore, when the moving targets are in k-subgeneral position over $\mathcal{P}^1_c$, it is easy to see that they satisfy the hypothesis of Theorem 1.4. Immediately, we have the following corollary which is an extension of Theorem B.

Corollary 1.1. *Let $c \in \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with hyper-order $\zeta_2(f) < 1$. Let a_j $(1 \le j \le q)$ be q slowly moving periodic targets with period c, located in k-subgeneral position over $\mathcal{P}^1_c$ such that $(f, a_j) \not\equiv 0$ $(1 \le j \le q)$.*

Assume that f is forward invariant over a_j for all $j \in \{1, \dots, q\}$ with respect to the translation $\tau_c(z) = z + c$. Then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq \left[\frac{n}{q-k}\right]$. Particularly, if $q \geq 2k+1$, then f is periodic with period c, i.e., $f(z) = f(z+c)$.

2. Preliminaries

2.1. **Divisor.** We set $||z|| = \Big(|z_1|^2 + \cdots + |z_n|^2\Big)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and define

$$B_m(r) := \{z \in \mathbb{C}^m : ||z|| < r\}, \quad S_m(r) := \{z \in \mathbb{C}^m : ||z|| = r\} \quad (0 < r < \infty).$$

Define

$$\sigma_m(z) := \Big(dd^c||z||^2\Big)^{m-1} \qquad \text{and}$$

$$\eta_m(z) := d^c\mathrm{log}||z||^2 \wedge \Big(dd^c\mathrm{log}||z||^2\Big)^{m-1} \quad \text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1\cdots\partial^{\alpha_m}z_m}$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\ \{n : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < n\} \quad (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbb{C}^m$ a map $\nu : \Omega \to \mathbb{Z}$ such that for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighbourhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$. For a divisor ν on Ω, we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighbourhood $U \subset \Omega$ such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m-2$, and we define the divisors $\nu^0_\varphi,\ \nu^\infty_\varphi$ by $\nu^0_\varphi := \nu_F,\ \nu^\infty_\varphi := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.2. **Counting function.** For a divisor ν on $\mathbb{C}^m$, we define the counting function of ν by

$$n(t) = \begin{cases} \int\limits_{|\nu|\,\cap B(t)} \nu(z)\sigma_{m-1}, & \text{if } m \geq 2, \\ \sum\limits_{|z| \leq t} \nu(z), & \text{if } m = 1. \end{cases}$$

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2m-1}}dt \quad (1 < r < \infty).$$

Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi), \quad N_\varphi^{(M)}(r) = N^{(M)}(r, \nu_\varphi).$$

2.3. **Characteristic function.** Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension greater or equal to 2. Set $\|f\| = \Big(|f_0|^2 + \cdots + |f_n|^2\Big)^{1/2}$.

The characteristic function of f is defined by

$$T(r, f) = \int\limits_{S_m(r)} \log\|f\|\eta_m - \int\limits_{S_m(1)} \log\|f\|\eta_m.$$

Note that $T(r, f)$ is independent of the choice of the representation of f. The order and hyper-order of f are respectively defined by

$$\zeta(f) := \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \zeta_2(f) := \limsup_{r\to\infty} \frac{\log^+\log^+ T(r, f)}{\log r},$$

where $\log^+ x := \max\{\log x, 0\}$ for any $x > 0$.

2.4. **Some lemmas.** It is known that the holomorphic functions $f_0, \dots, f_n$ on $\mathbb{C}^m$ are linearly dependent over $\mathbb{C}$ if and only if their Wronskian determinant $W(f_0, \dots, f_n)$ vanishes identically [10]. The similar result was proved by T. B. Cao, R. Korhonen [3] as follows.

Lemma 2.1 ([3])**.** (i) *Let $c \in \mathbb{C}^m$. A meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$ satisfies $C(f) \not\equiv 0$ if and only if f is linearly nondegenerate over the field $\mathcal{P}_c$.*

(ii) *Let $c \in \mathbb{C}^m$. If a meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$ satisfies $\zeta_2(f) < \lambda < +\infty$, then $C(f) \not\equiv 0$ if and only if f is linearly nondegenerate over the field $\mathcal{P}_c^\lambda \subset \mathcal{P}_c$.*

The lemma on the Logarithmic derivative [1, 4] plays an important role in the Nevanlinna theory. Here, it is replaced by the following lemma due to T. B. Cao, R. Korhonen [3].

Lemma 2.2 ([3])**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}^m$ such that $f(0) \neq 0, \infty$, and let $\epsilon > 0$. If $\zeta_2(f) := \zeta < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = \int_{S_m(r)} \log^+ \left|\frac{f(z+c)}{f(z)}\right| \eta_m(z) = o\left(\frac{T(r, f)}{r^{1-\zeta-\epsilon}}\right),$$

where ϵ is some positive constant.

Lemma 2.3. ([7, Lemma 8.3]). *Let $T : [0; +\infty) \to [0; +\infty)$ be a non-decreasing continuous function and let $s \in (0; +\infty)$. If the hyper-order $\zeta = \limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} < 1$ and $\delta \in (0; 1-\zeta)$, then*

$$|| \quad T(r+s) = T(r) + o\left(\frac{T(r,f)}{r^{\delta}}\right).$$

3. Proof of Theorem 1.1

We recall the Second main theorem of meromorphic mappings with hyper-order $\zeta_2(f) < 1$ intersecting hyperplanes in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$.

Lemma 3.1 ([3])**.** *Let $c \in \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping over $\mathcal{P}_c$ with hyper-order $\zeta = \zeta_2(f) < 1$, and let H_j $(1 \le j \le q)$ be q $(q > 2N - n + 1)$ hyperplanes in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$. Then we have*

$$\Big|\Big|(q - 2N + n - 1)T(r,f) \le \sum_{j=1}^{q} N(r, \nu^0_{(f,H_j)}) - \frac{N}{n} N(r, \nu^0_{(C(f))}) + o\left(\frac{T(r,f)}{r^{1-\zeta-\epsilon}}\right),$$

where $\nu^0_{(f,H_j)}$ is the zero divisor of function $H_j(f)$ and ϵ is some positive constant.

Proof of Theorem 1.1. Consider $n+2$ meromorphic mappings $a_{j_0}, \dots, a_{j_{n+1}}$ with reduced representations $a_{j_k} = (a_{j_k0} : \cdots : a_{j_kn})$ $(1 \le j_0 < \cdots < j_{n+1} \le q)$. We may assume that $a_{j_k0} \ne 0$ for all $0 \le k \le n+1$. We put $\tilde{a}_{j_ki} = \frac{a_{j_ki}}{a_{j_k0}}$, $\tilde{a}_{jk} = (\tilde{a}_{j_k0} : \cdots : \tilde{a}_{j_kn})$. Take a reduced representation $f = (f_0 : \cdots : f_n)$ of f and define $(f, \tilde{a}_{j_k}) = \sum_{i=0}^{n} f_i \tilde{a}_{j_ki}$. Since $\{a_j\}_{j=1}^q$ is in general position over $\mathcal{P}_c^1$, we have $\tilde{a}_{n+1} = \sum_{k=0}^{n} c_k \tilde{a}_{j_k}$, where

$$c_k \in \mathcal{R}(\{a_j\}_{j=1}^q) \setminus \{0\} \quad \text{and} \quad T(r, c_k) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r,f)).$$

Moreover, we can see that $c_k \in \mathcal{P}_c^1$ for all $0 \le k \le n$.

Define $\tilde{f} = (c_0(f, \tilde{a}_{j_0}) : \cdots : c_n(f, \tilde{a}_{j_n}))$. Then $\tilde{f}$ is a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ over $\mathcal{P}_c^1$. Indeed, assume that $\sum_{k=0}^{n} \lambda_k c_k (f, \tilde{a}_{j_k}) \equiv 0$ with $\lambda_k \in \mathcal{P}_c^1$ $(0 \le k \le n)$. This implies that

$$\left(\sum_{k=0}^{n} \lambda_k c_k \tilde{a}_{j_k0}\right) f_0 + \cdots + \left(\sum_{k=0}^{n} \lambda_k c_k \tilde{a}_{j_kn}\right) f_n \equiv 0.$$

Since f is linearly nondegenerate over $\mathcal{P}_c^1$, we have

$$\sum_{k=0}^{n} \lambda_k c_k \tilde{a}_{j_ki} = 0 \quad (i = 0, \dots, n).$$

By $\det(\tilde{a}_{j_ki})_{0\le k\le n, 0\le i\le n} \ne 0$, the above linearly equation system has solutions $\lambda_k c_k \equiv 0$ $(0 \le k \le n)$. Hence, $\lambda_k \equiv 0$ $(0 \le k \le n)$. This implies that $\tilde{f}$ is linearly nondegenerate over $\mathcal{P}_c^1$.

Let z_0 is a common zero of $c_k(f, \tilde{a}_{j_k})$ $(0 \le k \le n)$. There are two possibilities.

Case 1. If $(f, \tilde{a}_{j_k})(z_0) = 0$ for all $0 \le k \le n$, then z_0 is either in $I(f)$ which is an analytic subset of codim > 2 or z_0 is a zero of $\det(\tilde{a}_{j_ki})_{0\le k\le n, 0\le i\le n}$, where $I(f)$ is the locus of indeterminacy of f. Moreover, by the First main theorem and by the assumption of the theorem, we get

$$\begin{aligned}N_{\det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}}(r) &\le T(r, \det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}) + O(1)\\ &= O\left(\max_{1\le j\le q} T(r, a_j)\right)\\ &= o(T(r, f)).\end{aligned}$$

Case 2. If there exists $0 \le k \le n$ such that $c_k(z_0) = 0$, we also have

$$N_{c_k}(r) \le T(r, c_k) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f)).$$

We now take $\tilde{f} = (hc_0(f, \tilde{a}_{j_0}) : \cdots : hc_n(f, \tilde{a}_{j_n}))$ as a reduced representation of $\tilde{f}$, where h is a meromorphic function on $\mathbb{C}^m$. It is easy to see that

$$N_h(r) \le \sum_{k=0}^{n} \left(N_{1/c_k}(r) + N_{a_{j_k 0}}(r)\right) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f))$$

and

$$N_{1/h}(r) \le \sum_{k=0}^{n} N_{c_k}(r) + N_{\det(\tilde{a}_{j_k i})_{0\le k\le n, 0\le i\le n}}(r) = O\left(\max_{1\le j\le q} T(r, a_j)\right) = o(T(r, f)).$$

Define $F_k = (f, \tilde{a}_{j_k})$ $(0 \le k \le n)$. Then since the linearly equation system

$$\sum_{t=0}^{n} \tilde{a}_{j_k t} f_t = F_k \quad (0 \le k \le n),$$

we can see that $f_t = \sum_{i=0}^{n} b_{ti} F_i$ $(0 \le t \le n)$, where $b_{ti} \in \mathcal{R}(\{a_j\}_{j=1}^{q}) \cap \mathcal{P}_c^1$. Put

$$A = \left(\sum_{0\le k,t\le n} |\tilde{a}_{j_k t}|^2\right)^{1/2} \quad \text{and} \quad B = \left(\sum_{0\le k,t\le n} |b_{ti}|^2\right)^{1/2}.$$

Then

$$||f|| \le B\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \quad \text{and} \quad \left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \le A||f||.$$

Therefore, we have

$$\begin{aligned}T(r, f) &= \int_{S(r)} \log\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \eta_n + O\left(\max_{1\le j\le q} T(r, a_j)\right)\\ &= \int_{S(r)} \log\left(\sum_{t=0}^{n} |F_t|\right)^{1/2} \eta_n + o(T(r, f)).\end{aligned}$$

On the other hand, we have

$$\sum_{k=0}^{n} |(f, \tilde{a}_{j_k})|^2 \le \left(\sum_{k=0}^{n} |hc_k(f, \tilde{a}_{j_k})|^2\right)\left(\sum_{k=0}^{n} \left|\frac{1}{c_k}\right|^2\right)\left|\frac{1}{h}\right|^2$$

and

$$\sum_{k=0}^{n} |hc_k(f, \tilde{a}_{j_k})|^2 \le |h|^2 \left(\sum_{k=0}^{n} |c_k|^2\right) \left(\sum_{k=0}^{n} |(f, \tilde{a}_{j_k})|^2\right).$$

Hence, we have

$$\begin{aligned} T(r, f) &= T(r, \tilde{f}) + O\left(\max_{1\le j\le q} T(r, a_j)\right) - N_h(r) + N_{1/h}(r) \\ &= T(r, \tilde{f}) + o(T(r, f)). \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned} \zeta_2(\tilde{f}) &= \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, \tilde{f})}{\log r} = \limsup_{r\to\infty} \frac{\log^+ \log^+ (T(r, f) + o(T(r, f)))}{\log r} \\ &\le \limsup_{r\to\infty} \frac{\log^+ \log^+ (2T(r, f))}{\log r} \le \limsup_{r\to\infty} \frac{\log^+ (2 + \log^+ (T(r, f)))}{\log r} \\ &\le \limsup_{r\to\infty} \frac{\log^+ 2 + \log^+ \log^+ (T(r, f)))}{\log r} = \zeta_2(f) < 1. \end{aligned}$$

By applying Lemma 3.1 to the hyperplanes

$$H_0 = \{\omega_0 = 0\}, \dots, H_n = \{\omega_n = 0\}, H_{n+1} = \{\omega_0 + \cdots + \omega_n = 0\},$$

for $\tilde{f}$, we have

$$T(r, \tilde{f}) \le \sum_{k=0}^{n} N_{hc_k(f, \tilde{a}_{j_k})}(r) + N_{h(f, \tilde{a}_{j_{n+1}})}(r) - N(r, \nu^0_{(C(f))}) + o\left(\frac{T(r, \tilde{f})}{r^{1-\zeta-\epsilon}}\right).$$

This, by going through all points $z_0 \in \mathbb{C}^m$ and by the definitions of $\tilde{N}^{[n,c]}(r, H(f))$, we obtain

$$\begin{aligned} T(r, \tilde{f}) &\le \sum_{k=0}^{n} \tilde{N}^{[n,c]}_{hc_k(f, \tilde{a}_{j_k})}(r) + \tilde{N}^{[n,c]}_{h(f, \tilde{a}_{j_{n+1}})}(r) + o\left(\frac{T(r, \tilde{f})}{r^{1-\zeta-\epsilon}}\right) \\ &\le \sum_{k=0}^{n+1} \tilde{N}^{[n,c]}_{(f, a_{j_k})}(r) + O\left(\max_{1\le j\le q} T(r, a_j)\right) + o\left(\frac{T(r, \tilde{f})}{r^{1-\zeta-\epsilon}}\right) \\ &\le \sum_{k=0}^{n+1} \tilde{N}^{[n,c]}_{(f, a_{j_k})}(r) + o(T(r, f)). \end{aligned} \tag{3.2}$$

Combining inequality (3.1) with inequality (3.2), we get

$$T(r, f) \le \sum_{k=0}^{n+1} \tilde{N}^{[n,c]}_{(f, a_{j_k})}(r) + o(T(r, f)). \tag{3.3}$$

We now take the sum of both of sides of (3.3) over all combinations $(j_0, \dots, j_{n+1})$ with $1 \le j_0 < \cdots < j_{n+1} \le q$, we get

$$\Big\| \quad \frac{q}{n+2} T(r, f) \le \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f, a_j)}(r) + o(T(r, f)).$$

The proof of Theorem 1.1 is completed. $\square$

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following.

Lemma 4.1 ([11]). *Assume that $\mathcal{A} = \{a_1, \ldots, a_q\}$ is nondegenerate over $\mathcal{M}_m$. There exist subsets $I_1, \ldots, I_k$ of $(f, \tilde{a}_j)_{j=1}^q$ such that the following are satisfied:*

(i) *I_1 is minimal over $\mathcal{R}$, i.e., I_1 is linearly dependent over $\mathcal{R}$ and each proper subset of I_1 is linearly independent over $\mathcal{R}$;*

(ii) *I_i is linearly independent over $\mathcal{R}$ for all $2 \leq i \leq k$;*

(iii) $\left(\bigcup_{j=1}^k I_j\right)_{\mathcal{R}} = (\{(f, \tilde{a}_i)\}_{i=1}^q)_{\mathcal{R}}$*;*

(iv) *for each $2 \leq i \leq k$, there exist meromorphic functions $c_\alpha \in \mathcal{R} \setminus \{0\}$ such that*

$$\sum_{(f,\tilde{a}_\alpha)\in I_i} c_\alpha (f, \tilde{a}_\alpha) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}},$$

where $\mathcal{R} = \mathcal{R}(\{a_j\}_{j=1}^q)$.

Proof of Theorem 1.2. Take subsets $I_1 = \{(f, \tilde{a}_1), (f, \tilde{a}_2), \ldots, (f, \tilde{a}_{t_1})\}$ and $I_i = \{(f, \tilde{a}_{t_{i-1}+1}), \ldots, (f, \tilde{a}_{t_i})\}$ $(2 \leq i \leq k)$ as in Lemma 4.1. Since I_1 is minimal, there exist $c_{1j} \in \mathcal{R} \setminus \{0\}$ such that

$$\sum_{j=1}^{t_1} c_{1j}(f, \tilde{a}_j) = 0.$$

Put $c_{1j} = 0$ for all $j > t_1$. We have $\sum_{j=1}^{t_k} c_{1j}(f, \tilde{a}_j) = 0$. Lemma 4.1 yields that $\{c_{1j}(f, \tilde{a}_j)\}_{j=2}^{t_1}$ is linearly independent over $\mathcal{R}$. It is easy to see that $\{\tilde{a}_j\}_{j=2}^{t_1}$ is linearly independent over $\mathbb{C}$. Since the assumption that f is linearly nondegenerate over $\mathcal{P}_c^1$, and using the arguments as in Theorem 1.1, we can see that $\{c_{1j}(f, \tilde{a}_j)\}_{j=2}^{t_1}$ are linearly independent over $\mathcal{P}_c^1$. Therefore, by Lemma 2.1, the Casorati determinant

$$\begin{aligned} C_1 &= C(c_{12}(f, \tilde{a}_1), \ldots, c_{1t_1}(f, \tilde{a}_{t_1})) \\ &= \prod_{j=0}^{t_1-2} \bar{f}_0^{[j]} C\left(\frac{c_{12}(f, \tilde{a}_2)}{f_0}, \ldots, \frac{c_{1t_1}(f, \tilde{a}_{t_1})}{f_0}\right) \\ &= \prod_{j=0}^{t_1-t_0-1} \bar{f}_0^{[j]} \tilde{C}_1 \not\equiv 0, \end{aligned}$$

where $t_0 = 1$.

We now consider $i \geq 2$. By the property of subset I_i, there exist meromorphic functions $c_{ij} \not\equiv 0$, $t_{i-1} + 1 \leq j \leq t_i$ in $\mathcal{R}$ such that $\sum_{j=t_{i-1}+1}^{t_i} c_{ij}(f, \tilde{a}_j) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}}$. Therefore, there exist meromorphic functions $c_{ij} \in \mathcal{R}$ $(1 \leq j \leq t_i)$ such that $c_{ij} \not\equiv 0$, $t_{i-1} + 1 \leq j \leq t_i$ and $\sum_{j=1}^{t_i} c_{ij}(f, \tilde{a}_j) = 0$. Put $c_{ij} = 0$ for all $j > t_i$, then $\sum_{j=1}^{t_k} c_{ij}(f, \tilde{a}_j) = 0$. Since $c_{ij}(f, \tilde{a}_j)_{j=t_{i-1}+1}^{t_i}$ are linearly independent over $\mathcal{R}$, they are linearly independent over $\mathcal{P}^1_c$, we have

$$\begin{aligned} C_i &= C(c_{it_{i-1}+1}(f,\tilde{a}_{t_{i-1}+1}),\dots,c_{it_i}(f,\tilde{a}_{t_i}))\\ &= \prod_{j=0}^{t_i-t_{i-1}-1}\bar{f}_0^{[j]}C\left(\frac{c_{it_{i-1}+1}(f,\tilde{a}_{t_{i-1}+1})}{f_0},\dots,\frac{c_{it_i}(f,\tilde{a}_{t_i})}{f_0}\right)\\ &= \prod_{j=0}^{t_i-t_{i-1}-1}\bar{f}_0^{[j]}\tilde{C}_i\not\equiv 0. \end{aligned}$$

Consider the $t_k\times t_k+1$ minor matrices $\mathcal{T}$ and $\tilde{\mathcal{T}}$ given by

$$\mathcal{T}=\begin{pmatrix} c_{11}(f,\tilde{a}_1) & \cdots & c_{1t_k}(f,\tilde{a}_{t_k})\\ c_{11}(\bar{f},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f},\tilde{a}_{t_k})\\ \vdots & \vdots & \vdots\\ c_{11}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_{t_k})\\ c_{21}(f,\tilde{a}_1) & \cdots & c_{2t_k}(f,\tilde{a}_{t_k})\\ c_{21}(\bar{f},\tilde{a}_1) & \cdots & c_{2t_k}(\bar{f},\tilde{a}_{t_k})\\ \vdots & \vdots & \vdots\\ c_{21}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_1) & \cdots & c_{1t_k}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_{t_k})\\ c_{31}(f,\tilde{a}_1) & \cdots & c_{3t_k}(f,\tilde{a}_{t_k})\\ c_{31}(\bar{f},\tilde{a}_1) & \cdots & c_{3t_k}(\bar{f},\tilde{a}_{t_k})\\ \vdots & \vdots & \vdots\\ c_{k1}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_1) & \cdots & c_{kt_k}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_{t_k}) \end{pmatrix}$$

and

$$\tilde{\mathcal{T}}=\begin{pmatrix} \frac{c_{11}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{1t_k}(f,\tilde{a}_{t_k})}{f_0}\\ \frac{c_{11}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{1t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0}\\ \vdots & \vdots & \vdots\\ \frac{c_{11}(\bar{f}^{[t_1-t_0-1]},\tilde{a}_1)}{\bar{f}_0^{[t_1]}} & \cdots & \frac{c_{1t_k}(\bar{f}^{[t_1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_1-t_0-1]}}\\ \frac{c_{21}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{2t_k}(f,\tilde{a}_{t_k})}{f_0}\\ \frac{c_{21}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{2t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0}\\ \vdots & \vdots & \vdots\\ \frac{c_{21}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_1)}{\bar{f}_0^{[t_2-t_1-1]}} & \cdots & \frac{c_{1t_k}(\bar{f}^{[t_2-t_1-1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_2-t_1-1]}}\\ \frac{c_{31}(f,\tilde{a}_1)}{f_0} & \cdots & \frac{c_{3t_k}(f,\tilde{a}_{t_k})}{f_0}\\ \frac{c_{31}(\bar{f},\tilde{a}_1)}{\bar{f}_0} & \cdots & \frac{c_{3t_k}(\bar{f},\tilde{a}_{t_k})}{\bar{f}_0}\\ \vdots & \vdots & \vdots\\ \frac{c_{k1}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_1)}{\bar{f}_0^{[t_k-t_{k-1}-1]}} & \cdots & \frac{c_{kt_k}(\bar{f}^{[t_k-t_{k-1}-1]},\tilde{a}_{t_k})}{\bar{f}_0^{[t_k-t_{k-1}-1]}} \end{pmatrix}.$$

Denote by $\mathcal{D}_i$ (resp. $\tilde{\mathcal{D}}_i$) the determinant of the matrix obtained by deleting the $(i+1)$-th column of the minor matrix $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$). Since the sum of each row of $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$) is zero, we get

$$\mathcal{D}_i = (-1)^i \mathcal{D}_0 = (-1)^i \prod_{v=1}^{k} C_v = (-1)^i \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{C}_v = (-1)^i \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{\mathcal{D}}_0$$
$$= \prod_{v=1}^{k} \prod_{l=0}^{t_v - t_{v-1}-1} \bar{f}_0^{[l]} \tilde{\mathcal{D}}_i.$$

Since $\dim(\mathcal{A})_{\mathcal{R}} = n+1$, we can assume that $\tilde{a}_{i_1}, \dots, \tilde{a}_{i_{n+1}}$ is a basis of $(\mathcal{A})$ over $\mathcal{R}$. Then $\det(\tilde{a}_{i_j s})_{1 \le j \le n+1,\, 0 \le s \le n} \not\equiv 0$. By solving the linearly equation system

$$(f, \tilde{a}_{i_j}) = \tilde{a}_{i_j 0} f_0 + \cdots + \tilde{a}_{i_j n} f_n \quad (1 \le j \le n+1),$$

we get $f_v = \sum_{j=1}^{n+1} A_{vt}(f, \tilde{a}_{i_j})$ $(0 \le v \le n)$, with $A_{vt} \in \mathcal{R}$.

Take a basic $\{(f, \tilde{a}_{j_1})\}, \dots, (f, \tilde{a}_{j_d})\}$ of the space $\left(\bigcup_{i=1}^k I_i\right)_{\mathcal{R}}$. By

$$((f, \tilde{a}_{j_1})\}, \dots, (f, \tilde{a}_{j_d}))_{\mathcal{R}} = \left(\bigcup_{i=1}^{k} I_i\right)_{\mathcal{R}} = (\{(f, \tilde{a}_j)\}_{j=1}^q)_{\mathcal{R}},$$

we have $f_v = \sum_{t=1}^d B_{vt}(f, \tilde{a}_{j_t})$ $(0 \le v \le n)$, with $B_{tv} \in \mathcal{R}$. Hence,

$$|f_v(z)| \le \sum_{t=1}^{d} |B_{vt}(z)| \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z)|\} \quad (z \in \mathbb{C}^m).$$

Define $A(z) = \sum_{t=1}^d \sum_{v=0}^n |B_{vt}(z)|$, then we have

$$||f(z)|| \le A(z) \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z)|\}$$

and

$$\begin{aligned}
\int_{S(r)} \log^+ A(z) \eta_n &\le \sum_{t=1}^{d} \sum_{v=0}^{n} \log^+ |B_{tv}(z)| \eta_n + O(1) \\
&\le \sum_{t=1}^{d} \sum_{v=0}^{n} T(r, B_{vt}) + O(1) \\
&= O(\max_{1 \le j \le q} T(r, a_j)) + O(1) \\
&= o(T(r, f)).
\end{aligned} \tag{4.1}$$

We now fix $z_0 \in \mathbb{C}^m$ and take i $(1 \le i \le t_k)$ such that

$$|(f, \tilde{a}_i)(z_0)| = \max_{1 \le i \le t_k} \{|(f, \tilde{a}_i)(z_0)|\}.$$

Then

$$\frac{|\mathcal{D}_0(z_0)|\cdot||f(z_0)||}{\prod_{j=1}^{t_k}|(f,\tilde{a_j})(z_0)|}=\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a_j})(z_0)|}\cdot\frac{||f(z_0)||}{|(f,\tilde{a_i})(z_0)|}$$
$$\leq A(z_0)\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a_j})(z_0)|}.$$

Therefore, we have

$$\log\frac{|\mathcal{D}_0(z_0)|\cdot||f(z_0)||}{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z_0)|}\leq\log^+\left(A(z_0)\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)$$
$$\leq\log^+\left(\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)+\log^+A(z_0).$$

It implies that for each $z\in\mathbb{C}^m$, we have

$$\log\frac{|\mathcal{D}_0(z)|\cdot||f(z)||}{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z)|}\leq\sum_{i=1}^{t_k}\log^+\left(\frac{|\mathcal{D}_i(z)|}{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}\right)+\log^+A(z)$$
$$=\sum_{i=1}^{t_k}\log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\frac{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}|\bar{f}_0^{[l]}|}}\right)+\log^+A(z).$$

Hence, we get
(4.2)

$$\log||f(z)||\leq\log\frac{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)(z)|}{|\mathcal{D}_0(z)|}+\sum_{i=1}^{t_k}\log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\frac{\prod_{j=1,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}|\bar{f}_0^{[l]}(z)|}}\right)+\log^+A(z).$$

Note that each element of the matrix of the determinant

$$\frac{\tilde{\mathcal{D}}_i(z)}{\frac{\prod_{j=1,j\neq i}^{t_k}(f,\tilde{a}_j)(z)}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}\bar{f}_0^{[l]}(z)}}$$

has a form

$$c_{ij}\frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}}\cdot\frac{\bar{f}_0^{[l]}}{f_0}\quad(1\leq i\leq k,1\leq j\leq t_k).$$

On the other hand, by the definition of the counting functions and the Jensen's formula, we have

$$\begin{aligned}
T\left(r, \frac{(f,\tilde{a}_j)}{f_0}\right) &= \int_{S(r)} \log|1+\sum_{i=1}^{n}\tilde{f}_i\tilde{a}_{ji}|\eta_m + O(1)\\
&\le \int_{S(r)} \log((1+\sum_{i=1}^{n}|\tilde{f}_i|)^{1/2}(1+\sum_{i=1}^{n}|\tilde{a}_{ji}|)^{1/2})\eta_m\\
&\quad + N_{f_0}(r) + \sum_{i=1}^{n} N_{a_{i0}}(r) + O(1)\\
&= \int_{S(r)} \log(1+\sum_{i=1}^{n}|\tilde{f}_i|)^{1/2}\eta_m + N_{f_0}(r) + \int_{S(r)} \log(1+\sum_{i=1}^{n}|\tilde{a}_{ji}|)^{1/2}\eta_m\\
&\quad + \sum_{i=1}^{n} N_{a_{i0}}(r) + O(1)\\
&\le T(r,f) + T(r,a_j) + O(1)\\
&= T(r,f) + o(T(r,f)).
\end{aligned}$$

Therefore, the hyper-order $\zeta_2\left(\frac{(f,\tilde{a}_j)}{f_0}\right) < 1$ $(1 \le j \le q)$. By Lemma 2.2, we have

$$\begin{aligned}
\Big\| m\left(r, c_{ij}\frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}}\cdot\frac{\bar{f}_0^{[l]}}{f_0}\right) &\le m(r,c_{ij}) + m\left(r, \frac{\frac{(\bar{f}^{[l]},\tilde{a}_j)}{\bar{f}_0^{[l]}}}{\frac{(f,\tilde{a}_j)}{f_0}}\right) + m\left(r, \frac{\bar{f}_0^{[l]}}{f_0}\right)\\
&\le O(\max_{1\le j\le q}\{T(r,a_j)\}) + o(T(r,f))\\
&= o(T(r,f)).
\end{aligned}$$

Hence,

$$\Big\| \quad m\left(r, \frac{\tilde{\mathcal{D}}_i(z)}{\frac{\prod_{j=1,j\ne i}^{t_k}(f,\tilde{a}_j)(z)}{\prod_{v=1}^{k}\prod_{l=0}^{t_v-t_{v-1}-1}\bar{f}_0^{[l]}(z)}}\right) \le o(T(r,f)). \tag{4.3}$$

By taking integrating both sides of inequality (4.2) and together this with (4.1) and (4.3), we get

(4.4)

$$\Big\| \quad T(r,f) \le \int_{S(r)} \log\frac{\prod_{j=1}^{t_k}|(f,\tilde{a}_j)|}{|\mathcal{D}_0|}\eta_m + o(T(r,f)) \le N\left(r, \nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}\right) + o(T(r,f)).$$

Take z_0 as a zero of $\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}$. Then z_0 is a zero or a pole of some $(f,\tilde{a}_j)$ or a pole of some c_{sj}.

Case 1. Assume that z_0 is an n-successive with separation c of $(f,\tilde{a}_j)$ with multiplicity $v_j > 0$ for all j $(1 \le j \le t_k)$. Then $\nu^0_{(\bar{f}^{[v]},\tilde{a}_j)}(z_0) \ge \nu^0_{(f,\tilde{a}_j)}(z_0)$ for all $1 \le v \le n$

and $1 \le j \le t_k$. Without loss of generality, we may assume that $v_1 \le v_2 \le \cdots \le v_k$. For each $1 \le j \le t_k$, we have

$$C_i = (f, \tilde{a}_{t_{i-1}+1}) \cdots (f, \tilde{a}_{t_i}) c_{it_{i-1}+1} \cdots c_{it_i} \begin{vmatrix} 1 & \cdots & 1 \\ \frac{(\bar{f}, \tilde{a}_{t_{i-1}+1})}{(f, \tilde{a}_{t_{i-1}+1})} & \cdots & \frac{(\bar{f}, \tilde{a}_{t_i})}{(f, \tilde{a}_{t_i})} \\ \vdots & \ddots & \vdots \\ \frac{(\bar{f}^{[t_i - t_{i-1}-1]}, \tilde{a}_{t_{i-1}+1})}{(f, \tilde{a}_{t_{i-1}+1})} & \cdots & \frac{(\bar{f}^{[t_i - t_{i-1}-1]}, \tilde{a}_{t_i})}{(f, \tilde{a}_{t_i})} \end{vmatrix}$$
$$=: (f, \tilde{a}_{t_{i-1}+1}) \cdots (f, \tilde{a}_{t_i}) c_{it_{i-1}+1} \cdots c_{it_i} \hat{C}_i.$$

Put $J = \{(i, j) : c_{ij} \not\equiv 0, 1 \le i \le k, 1 \le j \le t_k\}$, then

$$\mathcal{D}_0 = \prod_{i=1}^{k} C_i = \prod_{j=2}^{t_k} (f, \tilde{a}_j) \prod_{(i,j)\in J} c_{ij} \prod_{i=1}^{k} \hat{C}_i.$$

Hence,

$$\nu_{\mathcal{D}_0}(z_0) = \sum_{j=2}^{t_k} \nu_{(f,\tilde{a}_j)}(z_0) + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) + \sum_{i=1}^{k} \nu_{\hat{C}_i}(z_0) \ge \sum_{j=2}^{t_k} v_j + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0). \tag{4.5}$$

Take $\{a_{d_0}, \ldots, a_{d_n}\}$ as a basis of $(\mathcal{A})_{\mathcal{M}_m}$. Since $(\{(f, \tilde{a}_i)\}_{i=1}^{t_k})_{\mathcal{R}} = (\{(f, \tilde{a}_i)\}_{a\in\mathcal{A}})_{\mathcal{R}}$,

$$(f, \tilde{a}_{d_j}) = \sum_{i=1}^{t_k} \alpha_{ij} (f, \tilde{a}_{ij}) \quad (0 \le j \le n).$$

Note that $\alpha_{ij} \in \mathcal{R}$. Put $I = \{\alpha_{ij}, 1 \le i \le t_k, 0 \le j \le n \text{ such that } \alpha_{ij} \ne 0\}$ and $m = \max_{\alpha_{ij}\in I} \nu^{\infty}_{\alpha_{ij}}(z_0)$.

• If $m \ge v_1$, then $v_1 \le \sum_{\alpha_{ij}\in I} \nu^{\infty}_{\alpha_{ij}}(z_0)$.

• Otherwise, we have z_0 is a zero of $(f, \tilde{a}_{d_j})$ with multiplicity at least $v_1 - m$ for $0 \le j \le n$. Then z_0 is a zero of (f, a_{d_j}) with multiplicity at least $v_1 - m$ for $0 \le j \le n$. If $z_0 \notin I(f)$, then z_0 is a zero of $\det(a_{d_j s})$ with multiplicity at least $v_1 - m$. It implies that $v_1 \le \sum_{\alpha_{ij}\in I} \nu^{\infty}_{\alpha_{ij}}(z_0) + \nu^0_{\det(a_{d_j s})}(z_0)$ if $z_0 \notin I(f)$. Therefore, together these with (4.5), we have

$$\nu^0_{\frac{\prod_{j=1}^{t_k}(f\tilde{a}_j)}{\mathcal{D}_0}}(z_0) \le \sum_{j=1}^{t_k} v_j - \left(\sum_{j=2}^{t_k} v_j + \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) \right)$$
$$\le \sum_{\alpha_{ij}\in I} \nu^{\infty}_{\alpha_{ij}}(z_0) + \nu^0_{\det(a_{d_j s})}(z_0) - \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0).$$

Case 2. Assume that there exists an index j_0 such that z_0 is not an n-successive with separation c of $(f, \tilde{a}_{j_0})$. Without loss of generality, we may assume that $j_0 = 1$. We can assume that z_0 is an n-successive with separation c of $(f, \tilde{a}_j)$ with all $2 \le j \le l$, and z_0 is an n-aperiodic with separation c of $(f, \tilde{a}_j)$ with all $l + 1 \le j \le t_{k_0}$, and z_0 is a pole of $(f, \tilde{a}_j)$ with all $t_{k_0} < j \leq t_k$, where $k_0 \leq k$. Take i_0 satisfying $t_{i_0-1} \leq l < t_{i_0}$, we have

$$C_i = (f, \tilde{a}_{t_{i-1}+1}) \cdots (f, \tilde{a}_{t_i}) c_{it_{i-1}+1} \cdots c_{it_i} \hat{C}_i \quad (1 \leq i \leq t_0 - 1)$$

and

$$\begin{aligned}
C_{i_0} =&(f, \tilde{a}_{t_{i_0-1}+1}) \cdots (f, \tilde{a}_l) c_{i_0 t_{i_0-1}+1} \cdots c_{i_0 t_{i_0}} \\
&\times \begin{pmatrix}
1 & \cdots & 1 & \cdots & (f, \tilde{a}_{l+1}) & \cdots & (f, \tilde{a}_{t_0}) \\
\frac{(\bar{f}, \tilde{a}_{t_{i_0-1}+1})}{(f, \tilde{a}_{t_{i_0-1}+1})} & \cdots & \frac{(\bar{f}, \tilde{a}_l)}{(f, \tilde{a}_l)} & \cdots & (\bar{f}, \tilde{a}_{l+1}) & \cdots & (\bar{f}, \tilde{a}_{t_0}) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{(\bar{f}^{[v]}, \tilde{a}_{t_{i_0-1}+1})}{(f, \tilde{a}_{t_{i_0-1}+1})} & \cdots & \frac{(\bar{f}^{[v]}, \tilde{a}_l)}{(f, \tilde{a}_l)} & \cdots & (\bar{f}^{[v]}, \tilde{a}_{l+1}) & \cdots & (\bar{f}^{[v]}, \tilde{a}_{t_0})
\end{pmatrix} \\
=&(f, \tilde{a}_{t_{i_0-1}+1}) \cdots (f, \tilde{a}_l) c_{i_0 t_{i_0-1}+1} \cdots c_{i_0 t_{i_0}} \hat{C}_{i_0},
\end{aligned}$$

where $v = t_0 - t_{i_0-1} - 1$ and

$$C_i = c_{it_{i-1}+1} \cdots c_{it_i} C((f, \tilde{a}_{t_{i-1}+1}), \ldots, (f, \tilde{a}_{t_i})) = c_{it_{i-1}+1} \cdots c_{it_i} \hat{C}_i,$$

for all $i_0 + 1 \leq i \leq k$. Then we have

$$\frac{\prod_{j=1}^{t_k} (f, \tilde{a}_j)}{\mathcal{D}_0} = \frac{(f, \tilde{a}_1) \prod_{i=l+1}^{t_k} (f, \tilde{a}_i)}{\prod_{(i,j)\in J} c_{ij} \cdot \prod_{i=1}^{k} \hat{C}_i}$$

and therefore

$$\begin{aligned}
\nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}(z_0) =&\nu^0_{(f,\tilde{a}_1)(z_0)} + \sum_{i=l+1}^{t_k} \nu^0_{(f,\tilde{a}_i)}(z_0) - \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) + \sum_{i_0+1\leq i\leq k, 0\leq j\leq n} \nu^\infty_{a_{t_i j}}(z_0) \\
&- \sum_{i=1}^{k} \nu^0_{\hat{C}_i}(z_0) \\
\leq&\nu^0_{(f,\tilde{a}_1)(z_0)} + \sum_{i=l+1}^{t_k} \nu^0_{(f,\tilde{a}_i)}(z_0) - \sum_{(i,j)\in J} \nu_{c_{ij}}(z_0) + \sum_{i_0+1\leq i\leq k, 0\leq j\leq n} \nu^\infty_{a_{t_i j}}(z_0).
\end{aligned}$$

By going through all points $z_0 \in \mathbb{C}^m$ and by the definition of $\tilde{N}^{[n,c]}_{(f,\tilde{a}_i)}(r)$, two cases above imply that

$$\begin{aligned}
N\left(r, \nu^0_{\frac{\prod_{j=1}^{t_k}(f,\tilde{a}_j)}{\mathcal{D}_0}}\right) \leq& \sum_{j=1}^{t_k} \tilde{N}^{[n,c]}(r, \nu^0_{(f,\tilde{a}_j)}) - \sum_{(i,j)\in J} N(r, \nu_{c_{ij}}) + \sum_{1\leq i\leq t_k, 0\leq j\leq n} N(r, \nu^\infty_{a_{ij}}) \\
&+ \sum_{\alpha_{ij}\in I} N(r, \nu^\infty_{\alpha_{ij}}) + N(r, \nu^0_{\det(a_{d_j s})}) \\
\leq& \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f,\tilde{a}_j)}(r) + o(T(r, f)).
\end{aligned}$$

Together this with (4.4), we have

$$\Big\| \quad T(r,f) \le \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f,\tilde{a}_j)}(r) + o(T(r,f)) \quad \le \sum_{j=1}^{q} \tilde{N}^{[n,c]}_{(f,a_j)}(r) + o(T(r,f)).$$

The proof of Theorem 1.2 is completed. □

5. Proof of Theorem 1.3

We recall the lemma due to T. B. Cao and R. Korhonen [3] as follows.

Lemma 5.1 ([3])**.** *Let $c \in \mathbb{C}^m$ and $f = (f_0 : \cdots : f_n)$ be a meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ such that hyper-order $\zeta_2(f) < \lambda \le 1$ and all zeros of $f_0, \ldots, f_n$ forward invariant with respect to the translation $\tau(z) = z + c$. Let $S_1 \cup \cdots \cup S_l$ be the partition of $\{0, 1, \ldots, n\}$ formed in such a way that i and j are in the same class S_k if and only if $\frac{f_i}{f_j} \in \mathcal{P}^\lambda_c$. If $f_0 + \cdots + f_n = 0$, then*

$$\sum_{j \in S_k} f_j = 0,$$

for all $k \in \{1, \ldots l\}$.

Lemma 5.2. *If $\mathcal{A} = \{a_1, \ldots, a_q\}$ is linearly nondegenerate over $\mathcal{M}_m$, then $\mathcal{A}$ is linearly nondegenerate over $\mathcal{R}$, i.e., $\dim(\mathcal{A})_{\mathcal{R}} = n+1$ and for each nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$, we have*

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \neq \emptyset.$$

Proof. By the assumption, $\dim(\mathcal{A})_{\mathcal{M}_m} = n+1$, so $\dim(\mathcal{A})_{\mathcal{R}} = n+1$. Take any nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$, we have

$$(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}_m} \cap \mathcal{A} \neq \emptyset.$$

We consider two possibilities.

Case 1. There exists $a_t \in (\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A} \setminus \mathcal{A}_1)$. Then there exist $b_1, \ldots, b_k \in \mathcal{A}_1$ which are linearly independent over $\mathcal{M}_m$ and there exist $c_1, \ldots, c_k \in \mathcal{M}_m \setminus \{0\}$ such that $a_t = \sum_{i=1}^k c_i b_i$. Take reduced representations $a_t = (a_{t0} : \cdots : a_{tn})$ and $b_i = (b_{i0} : \cdots : b_{in})$ $(1 \le i \le k)$. We have a linear equation system $\sum_{i=1}^k c_i b_{ij} = a_{tj}$ $(0 \le j \le n)$. Since $\{\tilde{b}_1, \ldots, \tilde{b}_k\}$ is linearly independent over $\mathcal{M}_m$, $\operatorname{rank}(b_{ij})_{1 \le i \le k, 0 \le j \le n} = k$. By solving the above linear equation system, we have $c_i \in \mathcal{R}$ $(1 \le i \le k)$. It follows that $a_t \in (\mathcal{A}_1)_{\mathcal{R}}$ and hence $a_t \in (\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)$, i.e.,

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \neq \emptyset.$$

Case 2. There exists $a_t \in \mathcal{A}_1 \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}_m}$. By the same arguments as in Case 1, we have $a_t \in (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}}$ and therefore we also have

$$(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{R}} \cap \mathcal{A} \neq \emptyset.$$

Lemma 5.2 is proved. □

Proof of Theorem 1.3. By the assumption of the theorem, the holomorphic functions $g_j = (f, a_j) = \sum_{i=0}^n f_i a_i$ satisfy $\{\tau(g_j^{-1}(0))\} \subset \{g_j^{-1}(0)\}, j \in \{1, \dots, q\}$, where $\{\cdot\}$ denotes a multiset with counting multiplicities of its elements. We say that $i \sim j$ if $g_i = \alpha g_j$ for some $\alpha \in \mathcal{P}_c^1 \setminus \{0\}$. Therefore, the set of indexes $\{1, \dots, q\}$ may be split into disjoint equivalent classes S_j, $\{1, \dots, q\} = \cup_{j=1}^l S_j$ for some $1 \leq l \leq q$.

We assume that the complement of S_j has at least $n+1$ elements for some $j \in 1, \dots, l$. Let $\mathcal{A}_1 = \{1, \dots, q\} \setminus S_j$. Then $\mathcal{A}_1$ contains at least $n+1$ elements. By Lemma 5.2, there exist $\{s_0\}$ and $\{s_1, \dots, s_u\}$ belonging to disjoint equivalent classes and $\alpha_1, \dots, \alpha_u \in \mathcal{R} \setminus \{0\}$ such that $a_{s_0} + \sum_{j=1}^u \alpha_j a_{s_j} \equiv 0$. So, we have

$$(f, a_{s_0}) + \sum_{j=1}^u \alpha_j (f, a_{s_j}) = g_{s_0} + \sum_{j=1}^u \alpha_j g_{s_j} \equiv 0.$$

By the assumption of the theorem again, we can see that all zeros of $\alpha_j g_{s_j}$ are forward invariant with respect to the translation $\tau(z) = z + c$. This implies that

$$g := (g_{s_0} : \alpha_1 g_{s_1} : \cdots : \alpha_u g_{s_u})$$

is a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^u(\mathbb{C})$. We have

$$\begin{aligned} T(r, g) &= \int_{S_m(r)} \log ||g|| \eta_m + O(1) \\ &\leq \int_{S_m(r)} \log ||f|| \eta_m + \sum_{j=0}^u \int_{S_m(r)} \log ||a_{s_j}|| \eta_m + \sum_{j=1}^u \int_{S_m(r)} \log ||\alpha_{s_j}|| \eta_m + O(1) \\ &= T(r, f) + \sum_{j=0}^u T(r, a_{s_j}) + \sum_{j=1}^u T(r, \alpha_{s_j}) + O(1) \\ &= T(r, f) + o(T(r, f)). \end{aligned}$$

Therefore, $\zeta_2(g) \leq \zeta_2(f) < 1$. By Lemma 5.1, we get $g_{s_0} \equiv 0$ and $\sum_{j=1}^u \alpha_j g_{s_j} \equiv 0$. This is a contradiction. It implies that the complement of S_j has at most n elements for all $j \in 1, \dots, l$, and hence S_j has at least $q - n$ elements for all $j \in 1, \dots, l$. From this, we have

$$l \leq \frac{q}{q-n}.$$

Since $\dim(\mathcal{A})_{\mathcal{M}_m} = n+1$, we have $\dim(\mathcal{A})_{\mathcal{P}_c^1} = n+1$. Therefore, we can take a subset $V \subset \{1, \dots, q\}$ with $|V| = n+1$ such that $\{a_j\}_{j \in V}$ is linearly independent. Put $V_j = V \cap S_j$ for each $1 \leq j \leq l$. Then we have $V = \cup_{j=1}^l V_j$. Since each V_j gives raise to $|V_j| - 1$ equations over the field $\mathcal{P}_c^1$, it is easy to see that there are at least

$$\sum_{j=1}^l (|V_j| - 1) = n + 1 - l \geq n + 1 - \frac{q}{q-n} = n - \frac{n}{q-n}$$

linear independent relations over the field $\mathcal{P}_c^1$. Hence, the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq \left[\frac{n}{q-n}\right]$. If $q > 2n$, obviously $\left[\frac{n}{q-n}\right] = 0$. It follows that $f(z) = f(z+c)$. Theorem 1.3 is proved. $\square$

6. Proof of Theorem 1.4

Repeat the arguments as in Theorem 1.3. Assume that the complement $\mathcal{A}_1 = \{1\dots,q\}\setminus S_j$ of S_j has at least $k+1$ elements for some $j \in 1,\dots,l$. By the assumption, we have $(\mathcal{A}_1)_{\mathcal{M}_m} \cap (\mathcal{A}\setminus\mathcal{A}_1) \neq \emptyset$. From Case 1 in the proof of Lemma 5.2, it is easy to see that $(\mathcal{A}_1)_{\mathcal{R}} \cap (\mathcal{A}\setminus\mathcal{A}_1) \neq \emptyset$. Therefore, there exist $s_0 \in S_j = \mathcal{A}\setminus\mathcal{A}_1, v_1,\dots,v_t \in \mathcal{A}_1$ and $\beta_1,\dots,\beta_t \in \mathcal{R}\setminus\{0\}$ such that $a_{s_0} + \sum_{j=1}^t \beta_j a_{v_j} = 0$. Similarly to the proof of Theorem 1.3, we can deduce that $(f, a_{s_0}) \equiv 0$. This is a contradiction. Therefore, $|\mathcal{A}_1| \le k$, so $|S_j| \ge q-k$. Again using the discussion as in Theorem 1.3, we can show that the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\le \left[\frac{k}{q-k}\right]$ and if $q > 2k$, then $\left[\frac{k}{q-k}\right] = 0$. Therefore, $f(z) = f(z+c)$. Theorem 1.4 is proved.

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References

[1] H. Cartan, *Sur lés zeros des combinaisons linéaires de pfonctions holomorphes données*, Mathematica Cluj **7** (1933), 5–31.

[2] T. B. Cao, *Difference analogues of the second main theorem for meromorphic functions in several complex variables*, Math. Nachr. **287**(5-6) (2014), 530–545. https://doi.org/10.1002/mana.201200234

[3] T. B. Cao and R. Korhonen, *A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables*, J. Math. Anal. Appl. **444**(2) (2016), 1114–1132. https://doi.org/10.1016/j.jmaa.2016.06.050

[4] H. Fujimoto, *The uniqueness problem of meromorphic maps into the complex projective space*, Nagoya Math. J. **58** (1975) 1–23.

[5] R. G. Halburd and R. J. Korhonen, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31**(2) (2006), 463–478.

[6] P. M. Wong, H. F. Law and P. P. W. Wong, *A second main theorem on $\mathbb{P}^n$ for difference operator*, Sci. China Ser. A-Math. **52**(12) (2009), 2751–2758. https://doi.org/10.1007/s11425-009-0213-5

[7] R. Halburd, R. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc. **366**(8) (2014), 4267–4298. https://doi.org/10.1090/S0002-9947-2014-05949-7

[8] R. Korhonen, *A difference Picard theorem for meromorphic functions of several variables*, Comput. Methods Funct. Theory **12**(1) (2012), 343–361. https://doi.org/10.1007/BF03321831

[9] R. Korhonen, N. Li and K. Tohge, *Difference an anlogue of Cartan's second main theorem for slowly moving periodic target*, Ann. Acad. Sci. Fenn. Math. **41** (2016), 523–549. https://doi.org/10.5186/aasfm.2016.4131

[10] E. I. Nochka, *On the theory of meromorphic functions*, Sov. Math. Dokl. **27** (1983), 377–381.

[11] D. D. Thai and S. D.Quang, *Second main theorem with truncated counting function in several complex variables for moving targets*, Forum Math. **20** (2008), 163–179. https://doi.org/10.1515/FORUM.2008.007

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APPROXIMATING SOLUTIONS OF MONOTONE VARIATIONAL INCLUSION, EQUILIBRIUM AND FIXED POINT PROBLEMS OF CERTAIN NONLINEAR MAPPINGS IN BANACH SPACES

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Abstract. In this paper, motivated by the works of Timnak et al. [Filomat **31**(15) (2017), 4673–4693], Ogbuisi and Izuchukwu [Numer. Funct. Anal. **40**(13) (2019)] and some other related results in literature, we introduce an iterative algorithm and employ a Bregman distance approach for approximating a zero of the sum of two monotone operators, which is also a common solution of equilibrium problem involving pseudomonotone bifunction and a fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in the framework of a reflexive Banach space. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of the aforementioned problems. Furthermore, we give some applications of the consequences of our main result to convex minimization problem and variational inequality problem. Lastly, we display a numerical example to show the applicability of our main result. The result presented in this paper extends and complements many related results in the literature.

1. Introduction

Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \quad x^* \in E^*.$$

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For more information on Legendre functions, see [37]. Let the domain of f be denoted as $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$, hence for any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, we define the right-hand derivative of f at x in the direction of y by

$$f^0(x,y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

Let $f : E \to (-\infty, +\infty]$ be a function, then f is said to be:

(i) essentially smooth, if the subdifferential of f denoted as ∂f is both locally bounded and single-valued on its domain;

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [10, 51] for more details on Legendre functions.

The function f is said to be:

(i) Gâteaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x,y)$ coincides with $\bigtriangledown f(x)$ (the value of the gradient $\bigtriangledown f$ of f at x);

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$;

(iii) Frechet differentiable at x, if its limit is attained uniformly in $||y|| = 1$.

f is said to be uniformly Frechet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $||y|| = 1$. The function f is said to be Legendre if it satisfies the following conditions.

(i) The $\operatorname{int}(\operatorname{dom} f)$ is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom} \bigtriangledown f = \operatorname{int}(\operatorname{dom} f)$.

(ii) The $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom} \bigtriangledown f^* = \operatorname{int}(\operatorname{dom} f)$.

Let E be a Banach space and $B_s := \{z \in E : ||z|| \leq s\}$ for $s > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [55, page 203 and 221] if $\rho_s(t) > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, +\infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, where ρ_s denote the gauge of uniform convexity of f. The function f is also said to be uniformly smooth on bounded subsets of E [55, page 221], if $\lim_{t \downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, +\infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B, y \in S_E, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha)ty) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function f is said to be uniformly convex if the function $\delta f : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta f(t) := \sup \left\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2} \right) : ||y - x|| = t \right\},$$

satisfies $\lim_{t \downarrow 0} \frac{\delta f(t)}{t} = 0$.

Recall that the function f is said to be totally convex at a point $x \in Domf$, if the function $v_f : \text{int}(\text{dom}\, f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{int}(\text{dom}\, f), ||y - x|| = t\},$$

is positive whenever $t > 0$. For details on uniformly convex and totally convex functions, see [12, 15, 18].

Definition 1.1 ([12])**.** Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : E \times E \to [0, +\infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \triangledown f(y), x - y\rangle$$

is called the Bregman distance with respect of f.

It is well-known that Bregman distance D_f does not satisfy the properties of a metric because D_f fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in \text{dom}\, f$ and $y, z \in \text{int}(\text{dom}\, f)$

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \triangledown f(z) - \triangledown f(y), x - y\rangle. \tag{1.1}$$

For more information on Bregman functions and Bregman distances, see [38, 45]. Let $T : Q \to Q$ be a mapping, a point $x \in Q$ is called a fixed point of T, if $Tx = x$. We denote by $F(T)$ the set of all fixed points of T. Moreso, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to+\infty} ||Tx_n - x_n|| = 0$. The notion of asymptotic fixed point was introduced by [40]. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

Let Q be a nonempty closed and convex subset of E. An operator $T : Q \to Q$ is said to be:

(i) Bregman relatively nonexpansive, if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \text{for all } p \in F(T),\ x \in Q \text{ and } F(\hat{T}) = F(T);$$

(ii) Bregman quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \text{for all } x \in Q \text{ and } p \in F(T);$$

(iii) Bregman Strongly Nonexpansive (BSNE) with $\hat{F}(T) \neq \emptyset$, if

$$D_f(p, Tx) \leq D_f(p, x), \quad \text{for all } x \in C,\ p \in F(\hat{T})$$

and for any bounded sequence $\{x_n\}_{n\geq1} \subset Q$,

$$\lim_{n\to+\infty}(D_f(p, x_n) - D_f(p, Tx_n)) = 0$$

implies that $\lim_{n\to+\infty} D_f(Tx_n, x_n) = 0$. For more information on these classes of mappings, see [29, 30].

Let $B : E \to 2^{E^*}$ be a set-valued mapping, the domain and range of B are denoted by $\text{dom}\, B = \{x \in E : Bx \neq \emptyset\}$ and $\text{ran}\, B = \cup_{x\in B} Bx$, respectively. The graph of B is denoted as $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. Recall that B is called a monotone mapping, if for any $x, y \in \operatorname{dom} B$, we have $\xi \in Bx$ and $\zeta \in By$ implies $\langle \xi - \zeta, x - y \rangle \geq 0$. B is said to be maximal monotone if it is monotone and its graph is not contained in the graph of any other monotone mapping. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function and B be a maximal monotone mapping from E to E^*. For any $\lambda > 0$, the mapping $Res^f_{\lambda B} : E \to \operatorname{dom} B$ defined by

$$Res^f_{\lambda B} = (\triangledown f + \lambda B)^{-1} \circ \triangledown f,$$

is called the f-resolvent of B. It is well known that $B^{-1}(0) = F(Res^f_{\lambda B})$ for each $\lambda > 0$.

Let Q be a nonempty closed and convex subset of a reflexive Banach space E, the mapping $A : E \to 2^{E^*}$ is called Bregman Inverse Strongly Monotone (BISM) on the set Q if

$$Q \cap (\operatorname{dom} f) \cap (\operatorname{int}(\operatorname{dom} f)) \neq \emptyset,$$

and for any $x, y \in Q \cap (\operatorname{int}(\operatorname{dom} f))$, $\xi \in Ax$ and $\zeta \in Ay$, we have that

$$\langle \xi - \zeta, \triangledown f^*(\triangledown f(x) - \xi) - \triangledown f^*(\triangledown f(y) - \zeta) \rangle \geq 0.$$

Let $A : E \to E^*$ be a single-valued monotone mapping and $B : E \to 2^{E^*}$ be a multivalued monotone mapping. Then, the Monotone Variational Inclusion Problem (MVIP) (also known as the problem of finding a zero of sum of two monotone mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{1.2}$$

We denote by Ω, the solution set of problem (1.2).

It is well known that many interesting problems arising from mechanics, economics, applied sciences, optimization such as equilibrium and variational inequality problems can be solved using MVIP.

Suppose $A = 0$ in (1.2), we obtain the following Monotone Inclusion Problem (MIP), which is to find $x \in E$ such that

$$0^* \in B(x). \tag{1.3}$$

Many algorithms have been introduced by several authors for solving the MVIP and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, see [1,7–9,17,28,31,34,35,44,53]. For instance, Reich and Sabach [27,42] introduced some iterative algorithms and proved two strong convergence results for approximating a common solution of a finite family of MIP (1.3) in a reflexive Banach space. Recently, Timnak et al. [51] introduced a new Halpern-type iterative scheme for finding a common zero of finitely many maximal monotone mappings in a reflexive Banach space and prove the following strong convergence theorem.

Theorem 1.1. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subset of E. Let $A_i : E \to 2^{E^*}$, $i = 1, 2, \ldots,$ be*

N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $(0,1)$ satisfying the following control conditions:

(i) $\lim_{n\to+\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{+\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E, \ x_1 \in E \ \text{chosen arbitrarily}, \\ y_n = \nabla f^*[\beta_n \nabla f(x_n) + (1-\beta_n) \nabla f(Res^f_{r_N A_N}) \cdots (Res^f_{r_1 A_1}(x_n))], \\ x_{n+1} = \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n)], \end{cases} \tag{1.4}$$

for $n \in \mathbb{N}$, where ∇f is the gradient of f. If $r_i > 0$, for each $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (1.4) converges strongly to $proj^f_Z u$ as $n \to +\infty$.

Very recently, Ogbuisi and Izuchukwu [33] introduced the following iterative algorithm to obtain a strong convergence result for approximating a zero of sum of two maximal monotone operators which is also a fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach space. Let $u, x_0 \in Q$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$\begin{cases} Q_0 = Q, \\ y_n = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(x_n) + \gamma_n \nabla f(T(x_n))), \\ u_n = (Res^f_{\lambda B} \circ A^f_\lambda) y_n, \\ Q_{n+1} = \{z \in Q_n : D_f(z, u_n) \leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n)\}, \\ x_{n+1} = P^f_{Q_{n+1}}(x_0), \quad n \geq 0, \end{cases} \tag{1.5}$$

with conditions $\lim_{n\to\infty} \alpha_n = 0, \ \alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \ \gamma_n < b < 1$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)\cap\Gamma} x_0$, where $\Gamma := (A+B)^{-1}(0)$.

Equilibrium Problem (EP) involving monotone bifunctions and related optimization problems have been studied extensively by many authors, (see [2, 3, 11, 19, 22, 23, 36, 39, 46, 47, 49, 50] and other references contained in). Very recently, Eskandani et al. [18] introduced an EP involving a pseudomonotone bifunction in the framework of a reflexive Banach space.

Let C be a nonempty closed and convex subset of a reflexive Banach space E, the EP for a bifunction $g : C \times C \to \mathbb{R}$ satisfying condition $g(x,x) = 0$ for every $x \in C$ is defined as follows: find $x^* \in C$ such that

$$g(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.6}$$

We denote by Δ, the set of solutions of (1.6).

Recall that a bifunction g is called monotone on C, if for all $x, y \in C, \ g(x,y) + g(y,x) \leq 0$ and the mapping $A : C \to E^*$ is pseudomonotone if and only if the bifunction $g(x,y) = \langle A(x), y - x\rangle$ is pseudomonotone on C (see [18]). To solve an EP involving a pseudomonotone bifunction, we need the following assumptions:

L1. g is pseudomonotone, i.e., for all $x, y \in C$:

$$g(x,y) \geq 0 \quad \text{implies} \quad g(y,x) \leq 0;$$

L2. g is Bregman-Lipschitz type continuous, i.e., there exist two positive constants c_1, c_2 such that

$$g(x,y) + g(y,z) \geq g(x,z) - c_1 D_f(y,x) - c_2 D_f(z,y), \quad \text{for all } x, y, z \in C;$$

L3. g is weakly continuous on $C \times C$, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y respectively, then $g(x_n, y_n) \to g(x,y)$;

L4. $g(x, \cdot)$ is convex, lower semicontinuous and subdifferential on C for every fixed $x \in C$;

L5. for each $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tx + (1-t)y, z) \leq g(y,z)$.

Using assumptions L1-L5, Eskandani et al. [18] introduced an hybrid iterative algorithm to approximate a common element of the set of solutions of finite family of EPs involving pseudomonotone bifunctions and the set of common fixed points for a finite family of Bregman relatively nonexpansive mappings in the framework of reflexive Banach spaces. They proved the following strong convergence theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ be a super coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let for $i = 1, 2, \ldots, N, g_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* L1-L5. *Assume that for each $1 \leq r \leq M$, $T_r : C \to CB(C)$ be a multivalued Bregman relatively nonexpansive mapping, such that $\Gamma = (\cap_{r=1}^{M} F(T_r)) \cap (\cap_{i=1}^{N} EP(g_i)) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in C$ and*

$$\begin{cases} w_n^i = \operatorname{argmin}\{\lambda_n g_i(x_n, w) + D_f(w, x_n) : w \in C\}, \quad i = 1, \ldots, N, \\ z_n^i = \operatorname{argmin}\{\lambda_n g_i(w_n^i, z) + D_f(z, x_n) : z \in C\}, \quad i = 1, \ldots, N, \\ i_n \in \operatorname{Argmax}\{D_f(z_n^i, x_n), i = 1, 2, \ldots, N\}, \quad \overline{z_n} := z_n^{i_n}, \\ y_n = \triangledown f^*(\beta_{n,0} \triangledown f(\overline{z_n}) + \sum_{r=1}^{M} \beta_{n,r} \triangledown f(z_{n,r})), \quad z_{n,r} \in T_r \overline{z_n}, \\ x_{n+1} = \overleftarrow{P_C^f}(\triangledown f^*(\alpha_n \triangledown f(u_n) + (1-\alpha_n) \triangledown f(z_{n,r})), \end{cases}$$

where $CB(C)$ denotes the family of a nonempty, closed and convex subsets of C, $\{\alpha_n\}, \{\beta_{n,r}\}, \{\lambda_n\}$ and $\{u_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n \to +\infty} \alpha_n = 0$, $\sum_{n=1}^{+\infty} \alpha_n = \infty$;

(ii) $\{\beta_{n,r}\} \subset (0,1)$, $\sum_{r=0}^{M} \beta_{n,r} = 1$, $\liminf_{n \to +\infty} \beta_{n,0}\beta_{n,r} > 0$ for all $1 \leq r \leq M$ and $n \in \mathbb{N}$;

(iii) $\{\lambda_n\} \subset [a,b] \subset (0,p)$, where $p = \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$, $c_1 = \max_{1 \leq i \leq N} c_{i,1}$;

$c_2 = \max_{1 \leq i \leq N} c_{i,2}$ and $c_{i,1}, c_{i,2}$ are the Bregman-Lipschitz coefficients of g_i for all $1 \leq i \leq N$;

(iv) $\{u_n\} \subset E, \lim_{n \to +\infty} u_n = u$ for some $u \in E$.

Then, the sequence $\{x_n\}$ converges strongly to $\overleftarrow{P_\Gamma^f} u$.

Remark 1.1. We will like to emphasize that approximating a common solution of MVIP and EP have some possible applications to mathematical models whose constraints can be expressed as MVIP and EP. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, to mention a few, (see [20]).

Inspired by the works of Eskandani et al. [18], Timnak et al. [51], Ogbuisi and Izuchukwu [33] and other related results in literature, we introduce a Halpern type iteration process to approximate a zero of sum of two monotone operators, which is also a common solution of equilibrium problem involving pseudomonotone bifunction and fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in the framework of a reflexive Banach space. We state and prove a strong convergence result for finding a common solution of the aforementioned problems and give applications to the consequences of our main results. Finally, we display a numerical example to show the applicability of our main result. The result presented in this paper improve and generalize some known results in the literature.

2. Preliminaries

We give some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively.

Definition 2.1. A function $f : E \to \mathbb{R}$ is said to be super coercive if

$$\lim_{x\to+\infty} \frac{f(x)}{||x||} = +\infty$$

and strongly coercive if

$$\lim_{||x_n||\to+\infty} \frac{f(x_n)}{||x_n||} = +\infty.$$

Definition 2.2. Let Q be a nonempty subset of a real Banach space E and $\{T_n\}_{n=1}^{+\infty}$ a sequence of mappings from Q into E such that $\cap_{n=1}^{+\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{+\infty}$ is said to satisfy the AKTT-condition if for each bounded subset K of Q,

$$\sum_{n=1}^{+\infty} \sup\{||T_{n+1}z - T_n z|| : z \in K\} < +\infty.$$

Lemma 2.1 ([6])**.** *Let C be a nonempty subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into E which satisfies the AKTT condition. Then, for each $x \in C$, $\{T_n x\}_{n=1}^{+\infty}$ is convergent. Furthermore, if we define a mapping $T : C \to E$ by*

$$Tx := \liminf_{n\to+\infty} T_n x, \quad \textit{for all } x \in C,$$

then, for each bounded subset K of C,

$$\limsup_{n\to+\infty} \{||T_n z - Tz|| : z \in K\} = 0.$$

In this sequel, we write that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{+\infty}$ satisfies the AKTT-condition and $F(T) = \cap_{n=1}^{+\infty} F(T_n)$.

Lemma 2.2 ([51]). *Let E be a Banach space, $s > 0$ a constant, ρ_s the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ a convex function which is uniformly convex on bounded subset of E. Then*

(i) for any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(||x-y||),$$

(ii) for any $x, y \in B_s$

$$\rho_s(||x-y||) \leq D_g(x,y).$$

Here, $B_s := \{z \in E : ||z|| \leq s\}$.

Lemma 2.3 ([14]). *Let E be a reflexive Banach space, $f : E \to \mathbb{R}$ a strongly coercive Bregman function and V a function defined by*

$$V(x,x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E,\, x^* \in E^*.$$

The following assertions also hold:

$$\begin{aligned} D_f(x, \bigtriangledown f^*(x^*)) =& V(x,x^*), \quad \text{for all } x \in E \text{ and } x^* \in E^*, \\ V(x,x^*) + \langle \bigtriangledown g^*(x^*) - x, y^* \rangle \leq& V(x, x^* + y^*), \quad \text{for all } x \in E \text{ and } x^*, y^* \in E^*. \end{aligned}$$

Lemma 2.4 ([14]). *Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} ||y_n - x_n|| = 0.$$

Lemma 2.5 ([18]). *Let C be a nonempty closed convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a Legendre and super coercive function. Suppose that $g : C \times C \to \mathbb{R}$ is a bifunction satisfying* L1-L4. *For arbitrary sequence $\{x_n\} \subset C$ and $\{\lambda_n\} \subset (0, +\infty)$, let $\{w_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} w_n = \operatorname{argmin}_{y\in C}\{\lambda_n g(x_n, y) + D_f(y, x_n)\}, \\ z_n = \operatorname{argmin}_{y\in C}\{\lambda_n g(w_n, y) + D_f(y, x_n)\}. \end{cases}$$

Then, for all $x^* \in \Delta$, we have that

$$D_f(x^*, z_n) \leq D_f(x^*, x_n) - (1 - \lambda_n c_1) D_f(w_n, x_n) - (1 - \lambda_n c_2) D_f(z_n, w_n),$$

where c_1 and c_2 are the Bregman-Lipschitz coefficients of g.

Lemma 2.6 ([42]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.7 ([55]). *Let $f : E \to \mathbb{R}$ be a continuous convex function which is bounded on bounded subsets of E. Then, the following are equivalent:*

(i) f is super coercive and uniformly convex on bounded subset of E;

(ii) $\operatorname{dom} f^* = E^*$, *$f^*$ is bounded and uniformly smooth on bounded subsets of E^*;*

(iii) $\operatorname{dom} f^* = E^*$, f^* *is Fréchet differentiable and* $\bigtriangledown f^*$ *is uniformly norm-to-norm continuous on bounded subset of* E^*.

Lemma 2.8 ([55]). *Let* $f : E \to \mathbb{R}$ *be a continuous convex function which is super coercive. Then, the following are equivalent:*

(i) *f is bounded and uniformly smooth on bounded subsets of E;*

(ii) *f is Fréchet differentiable and* $\bigtriangledown f$ *is uniformly norm-to-norm continuous on bounded subset of E;*

(iii) $\operatorname{dom} f^* = E^*$, *$f^*$ is super coercive and uniformly convex on bounded subsets of E^*.*

Lemma 2.9 ([33]). *Let* $B : E \to 2^{E^*}$ *be a maximal monotone operator and* $A : E \to E^*$ *be a BISM mapping such that* $(A + B)^{-1}(0^*) \neq \emptyset$. *Let* $f : E \to \mathbb{R}$ *be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then*

$$D_f(u, Res^f_{\lambda B} \circ A^f(x)) + D_f(Res^f_{\lambda B}(x), x) \leq D_f(u, x),$$

for any $u \in (A + B)^{-1}(0^*),\ x \in E$ *and* $\lambda > 0$.

Lemma 2.10 ([33]). *Let* $B : E \to 2^{E^*}$ *be a maximal monotone operator and* $A : E \to E^*$ *be a BISM mapping such that* $(A + B)^{-1}(0^*) \neq \emptyset$. *Let* $f : E \to \mathbb{R}$ *be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then*

(i) $(A + B)^{-1}(0^*) = F(Res^f_{\lambda B} \circ A^f_\lambda)$;

(ii) $Res^f_{\lambda B} \circ A^f_\lambda$ *is a BSNE operator with* $F(Res^f_{\lambda B} \circ A^f_\lambda) = \hat{F}(Res^f_{\lambda B} \circ A^f_\lambda)$.

Definition 2.3. Let E be a reflexive Banach space and C a nonempty closed and convex subset of E. A Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $P^f_C(x) \in C$ satisfying

$$D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Lemma 2.11 ([41]). *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. Then*

(i) $z = P^f_C(x)$ *if and only if* $\langle \bigtriangledown f(x) - \bigtriangledown f(z), y - z\rangle \leq 0$ *for all* $y \in C$;

(ii) $D_f(y, P^f_C(x)) + D_f(P^f_C(x), x) \leq D_f(y, x)$ *for all* $y \in C$.

Lemma 2.12 ([52]). *Let C be a nonempty convex subset of a reflexive Banach space E and* $f : C \to \mathbb{R}$ *be a convex and subdifferential function on C. Then f attains its minimum at $x \in C$ if and only if* $0 \in \partial f(x) + N_C(x)$, *where $N_C(x)$ is the normal cone of C at x, that is*

$$N_C(x) := \{x^* \in E^* : \langle x - z, x^*\rangle \geq 0 \text{ for all } z \in C\}.$$

Lemma 2.13 ([16])**.** *If f and g are two convex functions on E such that there is a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ where f is continuous, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad \text{for all } x \in E. \tag{2.1}$$

Lemma 2.14 ([48])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n, \quad n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that
(*i*) $\sum_{n=1}^{+\infty} \sigma_n = +\infty$;
(*ii*) $\limsup_{n\to+\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{+\infty} |\sigma_n\delta_n| < +\infty$.
Then $\lim_{n\to+\infty} a_n = 0$.

3. Main Results

In what follows, Ω and Δ denote the solution set of MVIP (1.2) and EP (1.6) respectively.

Algorithm 3.1. Choose $u, x_1 \in E$. Assume that the control parameters $\{\mu_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:
(*i*) $\alpha_n \in (0,1)$, $\lim_{n\to+\infty} \alpha_n = 0$ and $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(*ii*) $\beta_n \in (0,1)$ and $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$;
(*iii*) $0 < \underline{\mu} \leq \mu_n \leq \overline{\mu} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, where c_1, c_2 are positive constants.
Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n) \nabla f(T_n x_n)), \\ y_n = (Res^f_{\lambda B} \circ A^f_\lambda)u_n, \\ z_n = \operatorname{argmin}_{a\in Q}\{\mu_n g(y_n, a) + D_f(a, y_n)\}, \\ w_n = \operatorname{argmin}_{a\in Q}\{\mu_n g(z_n, a) + D_f(a, y_n)\}, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n)), \quad n \geq 1. \end{cases} \tag{3.1}$$

Theorem 3.2. *Let E be a reflexive Banach space with E^* its dual and $Q \subseteq E$ a nonempty closed convex set. For $n \in \mathbb{N}$, let $T_n : E \to E$ be an infinite family of Bregman quasi-nonexpansive mapping such that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfy the AKTT-condition and $F(T) = \hat{F}(T)$. Let $A : E \to E^*$ be a BISM mapping, $B : E \to 2^{E^*}$ a maximal monotone operator and $g : Q \times Q \to \mathbb{R}$ a bifunction satisfying* L1-L5. *Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $Q \subset \operatorname{int}(\operatorname{dom} f)$ with $\Gamma := \cap_{n=1}^{+\infty} F(T_n) \cap \Omega \cap \Delta \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to $v = P^f_\Gamma u$, where P^f_Γ is the Bregman projection from E to Γ.*

Proof. Let $p \in \Gamma$, then we have from (3.1), Lemma 2.5 and Lemma 2.10 (*ii*) that

$$D_f(p, w_n) \leq D_f(p, y_n) - (1-\mu_n c_1)D_f(z_n, y_n) - (1-\mu_n c_2)D_f(w_n, z_n)$$

$$
\begin{aligned}
&=D_f(p, Res^f_{\lambda B} \circ A^f_\lambda(u_n)) - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n)\\
&\leq D_f(p, u_n) - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n)\\
&= D_f(p, \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)))\\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n)\\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_n x_n)\\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n)\\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, x_n)\\
&\quad - (1-\mu_n c_1) D_f(z_n, y_n) - (1-\mu_n c_2) D_f(w_n, z_n)\\
&\leq D_f(p, x_n). \qquad (3.2)
\end{aligned}
$$

Now, we conclude from (3.1) and (3.2) that

$$
\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n)))\\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, w_n)\\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, u_n)\\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n)\\
&\leq \max\Big\{D_f(p, u), D_f(p, x_n)\Big\}\\
&\ \ \vdots\\
&\leq \max\Big\{D_f(p, u), D_f(p, x_1)\Big\}. \qquad (3.3)
\end{aligned}
$$

From Lemma 2.8, we have that f^* is bounded on bounded subset of E^*. Hence, $\bigtriangledown f^*$ is also bounded on bounded subset of E^*. From Lemma 2.6, the following sequences $\{x_n\}_{n=1}^{+\infty}$, $\{(T_n x_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* u_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* w_n)\}_{n=1}^{+\infty}$, $\{(\bigtriangledown f^* z_n)\}_{n=1}^{+\infty}$ and $\{(\bigtriangledown f^* y_n)\}_{n=1}^{+\infty}$ are all bounded. In view of Lemma 2.7 and Lemma 2.8, dom $f^* = E^*$ and f^* is super coercive and uniformly convex on bounded subset of E^*. Let $s \geq \sup\{||x_n||, || \bigtriangledown (T_n x_n)|| : n \in \mathbb{N}\}$ be large enough and let $\rho_s^* : [0, +\infty) \to [0, +\infty)$ be the gauge of uniform convexity of f^*. Now, we have from Lemma 2.2 and (3.1) that

$$
\begin{aligned}
D_f(p, u_n) &= D_f(p, \bigtriangledown f^*(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)))\\
&= f(p) + f(\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n))\\
&\quad - \langle p, \beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)\rangle\\
&\leq \beta_n f(p) + (1-\beta_n) f(p) + \beta_n f^*(\bigtriangledown f(x_n)) + (1-\beta_n) f^*(\bigtriangledown f(T_n x_n)\\
&\quad - \beta_n(1-\beta_n)\rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||)\\
&\quad - \langle p, \beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(T_n x_n)\rangle\\
&\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_n x_n)\\
&\quad - \beta_n(1-\beta_n)\rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||)\\
&\leq D_f(p, x_n) - \beta_n(1-\beta_n)\rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(T_n x_n)||). \qquad (3.4)
\end{aligned}
$$

From (3.1), (3.2) and (3.4) and Lemma 2.3, we obtain that

$$\begin{aligned}
D_f(p,x_{n+1}) =& D_f(p, \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)))\\
=& V_p(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n))\\
\leq& V_p(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) - \alpha_n(\triangledown f(u) - \triangledown f(p))\\
&- \langle \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) - p, -\alpha_n(\triangledown f(u) - \triangledown f(p))\rangle\\
=& V(p, \alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)) + \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle\\
=& D_f(p, \triangledown f^*(\alpha_n \triangledown f(p) + (1-\alpha_n) \triangledown f(w_n)))\\
&+ \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle\\
\leq& \alpha_n D_f(p,p) + (1-\alpha_n) D_f(p, w_n) + \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle\\
\leq& (1-\alpha_n) D_f(p,x_n) - (1-\alpha_n)(1-\mu_n c_1) D_f(z_n, y_n)\\
&- (1-\alpha_n)(1-\mu_n c_2) D_f(w_n, z_n)\\
&- \beta_n (1-\beta_n) \rho_s^*(|| \triangledown f(x_n) - \triangledown f(T_n x_n))||\\
&+ \alpha_n \langle x_{n+1} - p, \triangledown f(u) - \triangledown f(p)\rangle.
\end{aligned} \tag{3.5}$$

We now consider two cases to prove a strong convergence result.

CASE 1. Assume that the sequence $\{D_f(p,x_n)\}$ is a monotone decreasing sequence, then $\{D_f(p,x_n)\}$ is convergent. Clearly, we have that $D_f(p,x_n) - D_f(p,x_{n+1}) \to 0$, as $n \to +\infty$.

Now, we have from (3.5), Lemma 2.4, conditions (i)-(iii) of (3.1) that

$$\lim_{n\to+\infty} ||z_n - y_n|| = 0. \tag{3.6}$$

Also,

$$\lim_{n\to+\infty} ||w_n - z_n|| = 0 \tag{3.7}$$

and

$$\lim_{n\to+\infty} \rho_s^*(|| \triangledown f(x_n) - \triangledown f(T_n x_n)||) = 0. \tag{3.8}$$

Applying the property of ρ_s^* on (3.8), we obtain that

$$\lim_{n\to+\infty} || \triangledown f(x_n) - \triangledown f(T_n x_n)|| = 0.$$

Since $\triangledown f^*$ is uniformly norm-to-norm continuous on bounded subset of E^*, we obtain that

$$\lim_{n\to+\infty} ||x_n - T_n x_n|| = 0. \tag{3.9}$$

Since $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfies the AKTT condition, we then conclude that

$$\begin{aligned}
||x_n - Tx_n|| &\leq ||x_n - T_n x_n|| + ||T_n x_n - Tx_n||\\
&\leq ||x_n - T_n x_n|| + \sup\{||T_n x - Tx|| : x \in K\},
\end{aligned} \tag{3.10}$$

where $K = rB = \{x \in E : ||x|| \leq r\}$. By applying Lemma 2.1, (3.9) and (3.10), we have that

$$\lim_{n\to+\infty} ||x_n - Tx_n|| = 0. \tag{3.11}$$

From (3.9), the boundedness of $\bigtriangledown f$ and the uniform continuity of f on bounded subsets of E, we have that

$$D_f(T_nx_n, x_n) = f(T_nx_n) - f(x_n) - \langle T_nx_n - x_n, \bigtriangledown f x_n\rangle \to 0, \quad n \to +\infty. \tag{3.12}$$

From Lemma 2.9, (3.1), (3.2) and (3.3), we have that

$$\begin{aligned} D_f(y_n, u_n) &\leq D_f(p, u_n) - D_f(p, y_n) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_nx_n) - D_f(p, u_n) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, x_n) - D_f(p, w_n) \\ &= D_f(p, x_n) + \alpha_n D_f(p, u) - D_f(p, x_{n+1}). \end{aligned}$$

Using condition (i) and Lemma 2.4, we have that

$$\lim_{n\to+\infty} ||y_n - u_n|| = 0. \tag{3.13}$$

From (3.1) and (3.12), we have that

$$\begin{aligned} D_f(x_n, u_n) &= D_f(x_n, \bigtriangledown f^*(\beta_n f(x_n) + (1-\beta_n) \bigtriangledown f(T_nx_n))) \\ &\leq \beta_n D_f(x_n, x_n) + (1-\beta_n) D_f(x_n, T_nx_n) \to 0, \quad n \to +\infty. \end{aligned}$$

Hence, we have from Lemma 2.4 that

$$\lim_{n\to+\infty} ||u_n - x_n|| = 0. \tag{3.14}$$

From (3.13) and (3.14), we have that

$$\lim_{n\to+\infty} ||y_n - x_n|| = 0. \tag{3.15}$$

From (3.6) and (3.15), we obtain that

$$\lim_{n\to+\infty} ||z_n - x_n|| = 0. \tag{3.16}$$

From (3.7) and (3.16), we have that

$$\lim_{n\to+\infty} ||w_n - x_n|| = 0. \tag{3.17}$$

Using (3.1), we have that

$$\begin{aligned} D_f(w_n, x_{n+1}) &= D_f(w_n, \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n))) \\ &\leq \alpha_n D_f(w_n, u) + (1-\alpha_n) D_f(w_n, w_n) \to 0, \quad n \to +\infty. \end{aligned}$$

We have from Lemma 2.4 that

$$\lim_{n\to+\infty} ||x_{n+1} - w_n|| = 0. \tag{3.18}$$

We conclude from (3.17) and (3.18) that

$$\lim_{n\to+\infty} ||x_{n+1} - x_n|| = 0. \tag{3.19}$$

Since $\{x_n\}$ is bounded in E, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. From (3.14), (3.15), (3.16) and (3.17), we have that $\{u_{n_k}\}$, $\{z_{n_k}\}$, $\{y_{n_k}\}$ and $\{w_{n_k}\}$ converges weakly to x^*. Also, from (3.11), we obtain that $x^* \in F(\hat{T}) = F(T)$. Next, we show that $x^* \in \Omega$. Since

$$z_n = \text{argmin}_{a\in Q}\{\mu_n g(y_n, a) + D_f(a, y_n)\},$$

then by Lemma 2.12 and 2.13 and condition L4, we obtain that

$$0 \in \partial\mu_n g(y_n, z_n) + \bigtriangledown D_f(z_n, y_n) + N_C(z_n).$$

Therefore, there exist $\bar{\theta_n} \in \partial g(y_n, z_n)$ and $\theta_n \in N_C(z_n)$ such that

$$\mu_n\bar{\theta_n} + \bigtriangledown f(z_n) - \bigtriangledown f(y_n) + \theta_n = 0. \tag{3.20}$$

Observe that $\theta_n \in N_C(z_n)$ and $\langle q - z_n, \theta_n\rangle \leq 0$ for all $q \in Q$. Since $\bar{\theta_n} \in \partial g(y_n, z_n)$, we have

$$g(y_n, q) - g(y_n, z_n) \geq \langle q - z_n, \theta_n\rangle, \tag{3.21}$$

for all $q \in Q$. Using (3.20) and (3.21), we obtain that

$$\mu_n[g(y_n, q) - g(y_n, z_n)] \geq \langle z_n - q, \nabla f(z_n) - \bigtriangledown f(y_n)\rangle, \quad \text{for all } q \in C,$$

this implies that

$$[g(y_n, q) - g(y_n, z_n)] \geq \frac{1}{\mu_n}\langle z_n - q, \nabla f(z_n) - \bigtriangledown f(y_n)\rangle, \quad \text{for all } q \in C. \tag{3.22}$$

Using (3.6), (3.15), (3.16), condition L3 and letting $n \to \infty$ in (3.22), we conclude that $g(x^*, q) \geq 0$, for all $q \in Q$. Hence $x^* \in \Delta$. We will also show that $0^* \in A(x^*) + B(x^*)$. From (3.13) and Lemma 2.10, we have that $x^* \in \hat{F}(Res^f_{\lambda B} \circ A^f_\lambda) = F(Res^f_{\lambda B} \circ A^f_\lambda) = (A + B)^{-1}(0^*)$. That is $0^* \in A(x^*) + B(x^*)$. Hence, $x^* \in \Omega$. We conclude that $x^* \in \Gamma$. We prove that $\{x_n\}$ converges strongly to $v = P^f_\Gamma u$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and

$$\limsup_{n\to+\infty}\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle = \lim_{k\to\infty}\langle x_{n_k+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle. \tag{3.23}$$

Since $x^* \in \Gamma$, we obtain from Lemma 2.11 and (3.23) that

$$\limsup_{n\to+\infty}\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle = \langle x^* - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle \leq 0. \tag{3.24}$$

From (3.5), we have that

$$D_f(v, x_{n+1}) \leq (1 - \alpha_n)D_f(v, x_n) + \alpha_n\langle x_{n+1} - v, \bigtriangledown f(u) - \bigtriangledown f(v)\rangle. \tag{3.25}$$

On applying Lemma 2.14 in (3.25), we conclude that $D_f(x_n, v) \to 0,\ n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $v = P^f_\Gamma u$.

CASE 2. Suppose that $\{D_f(p,x_n)\}$ is not a monotone decreasing sequence. Let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping defined for $n\geq n_0$ for some sufficiently large n_0 by

$$\tau(n)=\max\{j\in\mathbb{N}: j\leq n,\ D_f(p,x_{n_k})\leq D_f(p,x_{n_j+1})\}.$$

Then τ is a non-decreasing sequence such that $\tau(n)\to+\infty$ and $D_f(p,x_{\tau(n)})\leq D_f(p,x_{\tau(n)+1})$ for $n\geq n_0$.

We have from (3.5), conditions (i), (ii), (iii) that

$$\lim_{\tau(n)\to+\infty}||z_{\tau(n)}-y_{\tau(n)}||=0=\lim_{\tau(n)\to+\infty}||w_{\tau(n)}-z_{\tau(n)}|| \tag{3.26}$$

and

$$\lim_{\tau(n)\to+\infty}||\bigtriangledown f(x_{\tau(n)})-\bigtriangledown(T_{\tau(n)}x_{\tau(n)})||=0. \tag{3.27}$$

Following the same argument as in CASE 1, we have

$$\begin{aligned}\lim_{\tau(n)\to+\infty}||u_{\tau(n)}-x_{\tau(n)}||&=0,\\ \lim_{\tau(n)\to+\infty}||y_{\tau(n)}-x_{\tau(n)}||&=0,\\ \lim_{\tau(n)\to+\infty}||x_{\tau(n+1)}-x_{\tau(n)}||&=0\end{aligned}$$

and

$$\limsup_{\tau(n)\to+\infty}\langle x_{\tau(n)+1}-v,\bigtriangledown f(u)-\bigtriangledown f(v)\rangle=\langle x^*-v,\bigtriangledown f(u)-\bigtriangledown f(v)\rangle\leq 0. \tag{3.28}$$

It follows from (3.5) that

$$D_f(v,x_{\tau(n)+1})\leq(1-\alpha_{\tau(n)})D_f(v,x_{\tau(n)})+\alpha_{\tau(n)}\langle x_{\tau(n)+1}-v,\bigtriangledown f(u)-\bigtriangledown f(v)\rangle. \tag{3.29}$$

Since $\alpha_{\tau(n)}>0$, we obtain that

$$D_f(v,x_{\tau(n)})\leq\langle x_{\tau(n)+1}-v,\bigtriangledown f(u)-\bigtriangledown f(v)\rangle.$$

Hence, we deduce from (3.28) that $D_f(v,x_{\tau(n)})=0$. This implies that $\{x_{\tau(n)}\}$ converges strongly to v. Thus, $\{x_n\}$ converges strongly to $v\in\Gamma$. $\square$

We give the following consequences of our main result. In the next result, we consider a fixed point problem of relatively nonexpansive mapping and an EP involving a pseudomonotone bifunction.

Corollary 3.1. *Let E be a reflexive Banach space with E^* its dual and $Q\subseteq E$ be a nonempty closed convex set. Let $T:E\to E$ be a Bregman relatively nonexpansive mapping with $F(T)=\hat{F}(T)$ and $g:Q\times Q\to\mathbb{R}$ be a bifunction satisfying* L1-L5. *Assume that $f:E\to\mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that*

$Q \subset \operatorname{int}(\operatorname{dom} f)$ *with* $\Gamma := F(T) \cap \Delta \neq \emptyset$. *Let the sequence* $\{x_n\}$ *be generated iteratively by*

$$
\begin{cases}
u_n = \triangledown f^*(\beta_n \triangledown f(x_n) + (1-\beta_n) \triangledown f(Tx_n)),\\
z_n = \operatorname{argmin}_{a \in Q}\{\mu_n g(u_n, a) + D_f(a, u_n)\},\\
w_n = \operatorname{argmin}_{a \in Q}\{\mu_n g(z_n, a) + D_f(a, u_n)\},\\
x_{n+1} = \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(w_n)), \quad n \geq 1.
\end{cases}
\tag{3.30}
$$

If conditions (i)*-*(iii) *in* (3.1) *still hold, then the sequence* $\{x_n\}$ *converges strongly to* $x^* = P_\Gamma^f u$.

Here, we consider a common solution of fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings which is a zero of sum of monotone operators.

Corollary 3.2. *Let* E *be a reflexive Banach space and* E^* *be its dual space. For* $n \in \mathbb{N}$, *let* $T_n : E \to E$ *be an infinite family of Bregman quasi-nonexpansive mapping such that* $(\{T_n\}_{n=1}^{+\infty}, T)$ *satisfy the AKTT-condition and* $F(T) = \hat{F}(T)$. *Let* $A : E \to E^*$ *be a BISM mapping and* $B : E \to 2^{E^*}$ *be a maximal monotone operator. Assume that* $f : E \to \mathbb{R}$ *is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* E *such that* $\Gamma := \cap_{n=1}^{+\infty} F(T_n) \cap \Omega \neq \emptyset$. *Let the sequence* $\{x_n\}$ *be generated iteratively by*

$$
\begin{cases}
u_n = \triangledown f^*(\beta_n \triangledown f(x_n) + (1-\beta_n) \triangledown f(T_n x_n)),\\
y_n = (Res_{\lambda B}^f \circ A_\lambda^f) u_n,\\
x_{n+1} = \triangledown f^*(\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(y_n)), \quad n \geq 1.
\end{cases}
\tag{3.31}
$$

If conditions (i)*-*(ii) *in* (3.1) *still hold, then the sequence* $\{x_n\}$ *converges strongly to* $v = P_\Gamma u$.

Remark 3.1. In our result, we employed a Halpern type iterative algorithm due to its flexibility in defining the algorithm parameters, which is important from the numerical implementation perspective. The iteration process employed in this result has an advantage over the ones used in [18, 33] and some known results in the literature in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in practical computation of the iterative sequence. In fact, the results presented in Corollary 3.1 and 3.2 coincide with the results of [18] and [33], and in one way or the other extend their result based on their choice of iterative algorithm.

4. Applications and Numerical Example

4.1. **Convex Minimization Problem (CMP).** Let Q be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function which attains its minimum over E. Let $T_n : Q \to E$ be an infinite family of Bregman quasi-nonexpansive mapping such that $(\{T_n\}_{n=1}^{+\infty}, T)$ satisfy the AKTT-condition with $F(T) = \hat{F(T)}$ and $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. Then, the CMP is to find $x \in F(T)$ such that

$$g(x) = \min_{y \in E} g(y). \tag{4.1}$$

It is generally known that (4.1) can be formulated as follows: find $x \in F(T)$ such that

$$0^* \in \partial g(x), \tag{4.2}$$

where $\partial g = \{\xi \in E^* : \langle \xi, y - x\rangle \leq g(y) - g(x) \text{ for all } x \in E\}$. It is known that ∂g is a maximal monotone operator whenever g is a proper, convex and lower semi-continuous function. Hence, by taking $\partial g = B$ and $A = 0$ in Theorem 3.1, we obtain a strong convergence result for approximation solutions of EP involving pseudomonotone bifunction and CMP (4.1).

4.2. **Variational Inequality Problem.** Let C be a nonempty closed and convex subset of a reflexive Banach space E with E^* its dual. Let $A : C \to E^*$ be a mapping and the function g defined as $g(x, y) = \langle y - x, Ax\rangle$. Then, the classical Variational Inequality Problem (VIP) is to find $z \in C$ such that

$$\langle y - z, Az\rangle \geq 0, \quad \text{for all } y \in C. \tag{4.3}$$

VIP is one of the most important problems in optimization as it is used in studying differential equations, minimax problems, and has certain applications to mechanics and economic theory, see [4, 5, 21, 24, 25]. We denote by $VI(C, A)$, the set of solutions of VIP (4.3).

Lemma 4.1 ([18])**.** *Let C be a nonempty closed convex subset of a reflexive Banach space E, $A : C \to E^*$ be a mapping and $f : E \to \mathbb{R}$ be a Legendre function. Then*

$$\overleftarrow{P}^f_C(\triangledown f^*[\triangledown f(x) - \mu A(y)]) = \operatorname{argmin}_{w \in C}\{\mu\langle w - y, A(y)\rangle\} + D_f(w, x)\},$$

for all $x \in E$, $y \in C$ and $\mu \in (0, +\infty)$.

Theorem 4.1. *Let E be a reflexive Banach space with E^* its dual and $Q \subseteq E$ be a nonempty closed convex set. Let $T : E \to E$ be a Bregman relatively nonexpansive mapping with $F(T) = \hat{F(T)}$. Let A is a pseudomonotone and L-Lipschitz continuous mapping from Q to E^*. Assume that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $Q \subset \operatorname{int}(\operatorname{dom} f)$ with $\Gamma := \{F(T) \cap VI(Q, A)\} \neq \emptyset$. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} u_n = \triangledown f^*(\beta_n \triangledown f(x_n) + (1 - \beta_n) \triangledown f(Tx_n)), \\ z_n = \overleftarrow{P}^f_Q(\triangledown f^*(\triangledown f(x_n) - \mu_n A(u_n))), \\ w_n = \overleftarrow{P}^f_Q(\triangledown f^*(\triangledown f(x_n) - \mu_n A(z_n))), \\ x_{n+1} = \triangledown f^*(\alpha_n \triangledown f(u) + (1 - \alpha_n) \triangledown f(w_n)), \quad n \geq 1. \end{cases} \tag{4.4}$$

Suppose that conditions (i)-(ii) *in* (3.1) *hold and* $\{\mu_n\} \subset [a,b] \subset (0,p)$, *where* $p = \min \frac{2\tau}{L}$ *and* τ *is given by* (1.2) *holds, then the sequence* $\{x_n\}$ *converges strongly to* $v = P_\Gamma^f u$.

4.3. **Numerical Example.** We now display a numerical example of our algorithm to show its applicability.

Let $X = \mathbb{R}$ and $C = [0,1]$. Now, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{2x^4}{27}$, then $\bigtriangledown f(x) = \frac{8x^3}{27}$. Thus, by the definition of Fenchel conjugate of f, we obtain that $f^*(x^*) = \frac{9}{8}x^{*\frac{4}{3}}$ and $\bigtriangledown f^*(x^*) = \frac{36}{24}x^{*\frac{1}{3}}$. Note that f satisfies the assumptions in Theorem 3.1 (see [13]), and that $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$. Let $B : \mathbb{R} \to \mathbb{R}$ be defined by $B(x) = 7x - 2$ and $A : \mathbb{R} \to \mathbb{R}$ be defined by$A(x) = 5x$, then A and B are BISM and maximal monotone mappings respectively. Therefore, we compute their resolvents as follows:

$$\begin{aligned} Res_\lambda^f(A_\lambda^f x) &= (\bigtriangledown f + \lambda B)^{-1} \bigtriangledown f(\bigtriangledown f^*(\bigtriangledown f - \lambda A)(x)) \\ &= (\bigtriangledown f + \lambda B)^{-1}((\bigtriangledown f - \lambda A)(x)) \\ &= (\bigtriangledown f + \lambda B)^{-1}(x^3 - 5\lambda x). \end{aligned}$$

Now, define $g : C \times C \to \mathbb{R}$ by $g(x,y) = M(x)(y-x)$, where

$$M(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{100}, \\ \sin(x - \frac{1}{100}), & \frac{1}{100} \le x \le 1, \end{cases}$$

then g satisfies assumptions L1-L5 with $c_1 = 1 = c_2$ (see [18]). Also, define $T_n : \mathbb{R} \to \mathbb{R}$ by $T_n = \frac{1}{n}x$ for all $x \in \mathbb{R}$. Take $\alpha_n = \frac{1}{100n+1}$ and $\beta_n = \frac{n+1}{2n+7}$. Then, all assumptions of Theorem 3.1 are satisfied. Hence, Algorithm 3.1 becomes

$$\begin{cases} u_n = \bigtriangledown f^*(\beta_n f(x_n) + (1-\beta_n) \bigtriangledown f(\frac{x_n}{n})), \\ y_n = (\bigtriangledown f + \lambda B)^{-1}(u_n^3 - 5\lambda u_n), \\ z_n = y_n - \mu_n M(y_n), \\ w_n = y_n - \mu_n M(z_n), \\ x_{n+1} = \bigtriangledown f^*(\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(w_n)). \end{cases}$$

Case 1: $x_1 = 2,\ u = 0.5,\ \lambda = 10$ and $\frac{n+1}{4n+5}$.
Case 2: $x_1 = 0.5,\ u = 2,\ \lambda = 0.1$ and $\frac{n+1}{4n+5}$.
Case 3: $x_1 = 0.5,\ u = 2,\ \lambda = 0.1$ and $\frac{2n+1}{6n+7}$.
Case 4: $x_1 = 3,\ u = -7,\ \lambda = 2$ and $\frac{2n+1}{6n+7}$.

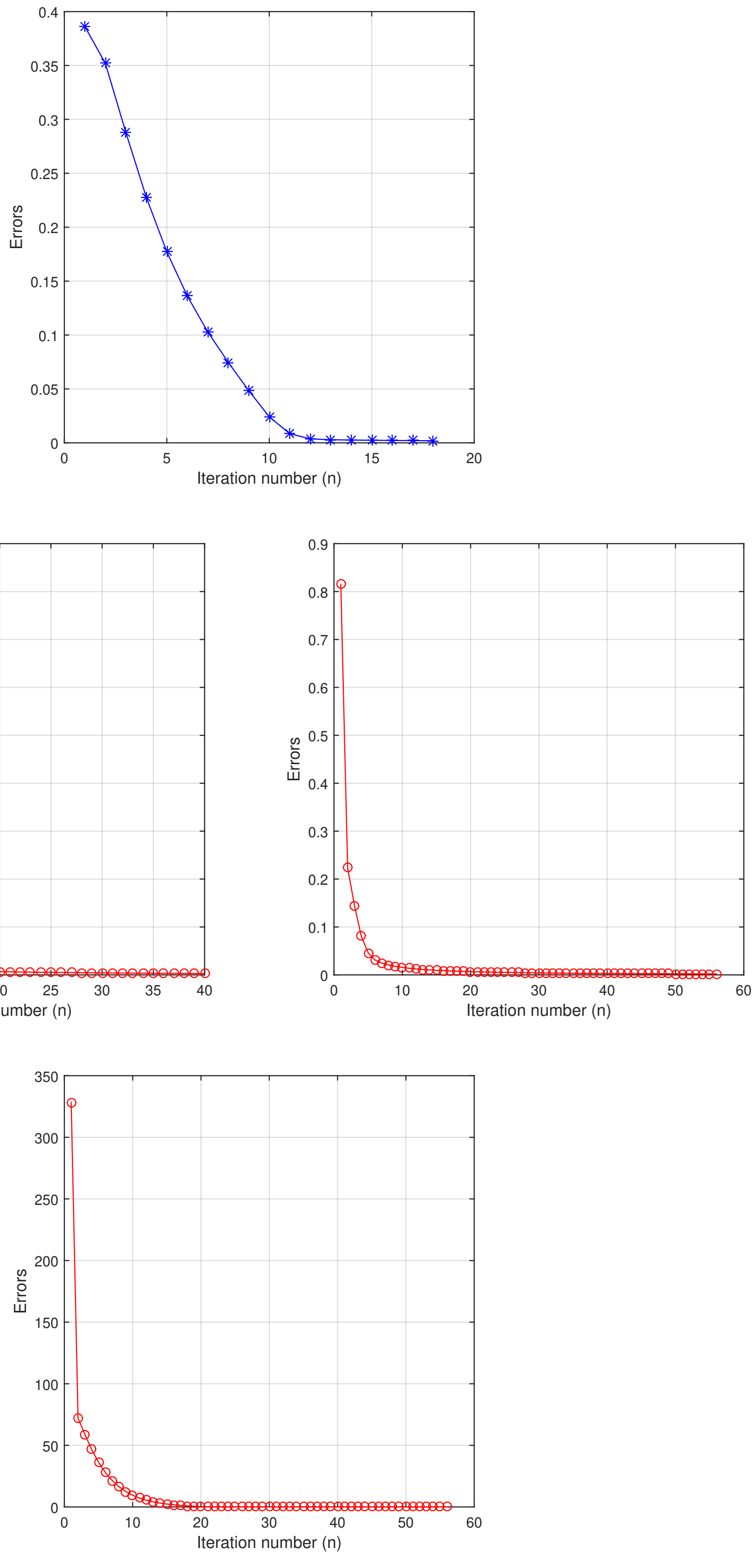

FIGURE 1. Errors vs Iteration numbers(n): Case 1 (top), Case 2 (middle left), Case 3 (middle right), Case 4 (bottom).

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References

[1] M. Abbas, B. Ali, T. Nazir, N. M. Dedović, B. Bin-Mhsin and S. N. Radenović, *Solutions and Ulam-Hyers stability of differential inclusions involving Suzuki type multivalued mappings on b-metric spaces*, Vojnotehnički Glasnik/Military Technical Courier **68**(3) (2020), 438–487.

[2] H. A. Abass, F. U. Ogbuisi and O. T. Mewomo, *Common solution of split equilibrium problem with no prior knowledge of operator norm*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **80**(1) (2018), 175–190.

[3] H. A. Abass, C. C. Okeke and O. T. Mewomo, *On split equality mixed equilibrium and fixed point problems of generalized k_i-strictly pseudo-contractive multivalued mappings*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **25**(6) (2018), 369–395.

[4] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, *Modified inertia subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems*, Optimization (2020). https://doi.org/10.1080/02331934.2020.1723586

[5] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, *A general iterative method for finding common fixed point of finite family of demicontractive mappings with accretive variational inequality problems in Banach spaces*, Nonlinear Stud. **27**(1) (2020), 1–24.

[6] K. Aoyama, Y. Kamimura, W. Takahashi and M. Toyoda, *Approximation of common fixed point of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360. https://doi.org/10.1016/j.na.2006.08.032

[7] K. O. Aremu, H. A. Abass, C. Izuchukwu and O. T. Mewomo, *A viscosity-type algorithm for an infinitely countable family of (f,g)-generalized k-strictly pseudononspreading mappings in $CAT(0)$ spaces*, Analysis (Berlin) **40**(1) (2020), 19–37. https://doi.org/10.1515/anly-2018-0078

[8] K. O. Aremu, C. Izuchukwu, G. N. Ogwo and O. T. Mewomo, *Multi-step iterative algorithm for minimization and fixed point problems in p-uniformly convex metric spaces*, J. Ind. Manag. Optim. (2020). https://doi.org/10.3934/jimo.2020063

[9] K. O. Aremu, L. O. Jolaoso, C. Izuchukwu and O. T. Mewomo, *Approximation of common solution of finite family of monotone inclusion and fixed point problems for demicontractive multivalued mappings in $CAT(0)$ spaces*, Ric. Mat. **69**(1) (2020), 13–34. https://doi.org/10.1007/s11587-019-00446-y

[10] H. H. Bauschke, J. M. Borwein and P. L. Combettes, *Essentially smoothness, essentially strict convexity and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3** (2001), 615–647.

[11] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. **63** (1994), 123–145.

[12] L. M. Bregman, *The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming*, USSR Computational Mathematics and Mathematical Physics **7** (1967), 200–217.

[13] M. Borwein, S. Reich and S. Sabach, *A characterization of Bregman firmly nonexpansive opertors using a new monotonicity concept*, J. Nonlinear Convex Anal. **12** (2011), 161–184.

[14] D. Butnairu and E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. **2006** (2006), Article ID 84919, 39 pages. https://doi.org/10.1155/AAA/2006/84919

[15] D. Butnairu, S. Reich and A. J. Zaslavski, *There are many totally convex functions*, J. Convex Anal. **13** (2006), 623–632.

[16] I. Cioranescu, *Geometry of Banach spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[17] H. Dehghan, C. Izuchukwu, O. T. Mewomo, D. A. Taba and G. C. Ugwunnadi, *Iterative algorithm for a family of monotone inclusion problems in $CAT(0)$ spaces*, Quaest. Math. **43**(7) (2020), 975–998. https://doi.org/10.2989/16073606.2019.1593255

[18] G. Z. Eskandani, M. Raeisi and T. M. Rassias, *A hybrid extragradient method for solving pseudomonotone equilibrium problem using Bregman distance*, J. Fixed Point Theory Appl. **20** (2018), Article ID 132, 27 pages. https://doi.org/10.1007/s11784-018-0611-9

[19] A. Gibali, L. O. Jolaoso, O. T. Mewomo and A. Taiwo, *Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces*, Results Math. **75** (2020), Article ID 179, 36 pages. https://doi.org/10.1007/s00025-020-01306-0

[20] H. Iiduka, *Acceleration method for convex optimization over the fixed point set of a non-expansive mappings*, Math. Prog. Series A **149** (2015), 131–165. https://doi.org/10.1007/s10107-013-0741-1

[21] C. Izuchukwu, A. A. Mebawondu and O. T. Mewomo, *A new method for solving split variational inequality problems without co-coerciveness*, J. Fixed Point Theory Appl. **22**(4) (2020), Article ID 98, 23 pages. https://doi.org/10.1007/s11784-020-00834-0

[22] C. Izuchukwu, G. N. Ogwo and O. T. Mewomo, *An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions*, Optimization (2020). https://doi.org/10.1080/02331934.2020.1808648

[23] L. O. Jolaoso, T. O. Alakoya, A. Taiwo and O. T. Mewomo, *A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems*, Rend. Circ. Mat. Palermo (2) **69**(3) (2020), 711–735. https://doi.org/10.1007/s12215-019-00431-2

[24] L. O. Jolaoso, A. Taiwo, T. O. Alakoya and O. T. Mewomo, *A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem*, Comput. Appl. Math. **39**(1) (2020), Article ID 38, 28 pages. https://doi.org/10.1007/s40314-019-1014-2

[25] L. O. Jolaoso, A. Taiwo, T. O. Alakoya and O. T. Mewomo, *Strong convergence theorem for solving pseudo-monotone variational inequality problem using projection method in a reflexive Banach space*, J. Optim. Theory Appl. **185**(3) (2020), 744–766. https://doi.org/10.1007/s10957-020-01672-3

[26] Z. Jouymandi and F. Moradlou, *Retraction algorithms for solving variational inequalities, pseudomonotone equilibrium problems and fixed point problems in Banach spaces*, Numer. Algorithms **78** (2018), 1153–1182. https://doi.org/10.1007/s11075-017-0417-7

[27] G. Kassay, S. Reich and S. Sabach, *Iterative methods for solving systems of variational inequalities in Banach spaces*, SIAM J. Optim. **21** (2011), 1319–1344.

[28] V. Manojlović, *On conformally invariant extremal problems*, Appl. Anal. Discrete Math. **3**(1) (2009), 97–119. https://doi.org/10.2298/AADM0901097M

[29] V. Martin Marquez, S. Reich and S. Sabach, *Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Math. Anal. Appl. **400** (2013), 597–614. https://doi.org/10.1016/j.jmaa.2012.11.059

[30] V. Martin Marquez, S. Reich and S. Sabach, *Iterative methods for approximating fixed point points of Bregman nonexpansive operators,* Discrete Contin. Dyn. Syst. Ser. S **6**(4) (2013), 1043–1063. https://doi:10.3934/dcdss.2013.6.1043

[31] Z. D. Mitrović, S. Radenović, S. Reich and A. Zaslavski, *Iterating nonlinear contractive mappings in Banach spaces,* Carpathian J. Math. **36** (2) (2020), 287–294.

[32] F. U. Ogbuisi and O. T. Mewomo, *Iterative solution of split variational inclusion problem in a real Banach spaces*, Afr. Mat. **28** (2017), 295–309. https://doi.org/10.1007/s13370-016-0450-z

[33] F. U. Ogbuisi and C. Izuchukwu, *Approxiamting a zero of sum of two monotone operators which solves a fixed point problem in reflexive Banach spaces*, Numer. Funct. Anal. **41**(3) (2020), 322–343. https://doi.org/10.1080/01630563.2019.1628050

[34] G. N. Ogwo, C. Izuchukwu, K. O. Aremu and O. T. Mewomo, *A viscosity iterative algorithm for a family of monotone inclusion problems in an Hadamard space*, Bull. Belg. Math. Soc. Simon Stevin **27** (2020), 127–152.

[35] O. K. Oyewole, H. A. Abass and O. T. Mewomo, *Strong convergence algorithm for a fixed point constraint split null point problem*, Rend. Circ. Mat. Palermo (2) (2020). https://doi.org/10.1007/s12215-020-00505-6

[36] O. K. Oyewole, L. O. Jolaoso, C. Izuchukwu and O. T. Mewomo, *On approximation of common solution of finite family of mixed equilibrium problems involving $\mu-\alpha$ relaxed monotone mapping in a Banach space*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81**(1) (2019), 19–34.

[37] D. Reem and S. Reich, *Solutions to inexact resolvent inclusion problems with applications to nonlinear analysis and optimization*, Rend. Circ. Mat. Palermo (2) **67** (2018), 337–371. https://doi.org/10.1007/s12215-017-0318-6

[38] D. Reem, S. Reich and A. De Pierro, *Re-examination of Bregman functions and new properties of their divergences*, Optimization **68** (2019), 279–348. https://doi.org/10.1080/02331934.2018.1543295

[39] S. Reich and S. Sabach, *Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces*, Contemp. Math. **568** (2012), 225–240.

[40] S. Reich, *A Weak Convergence Theorem for the Alternating Method with Bregman Distances*, Theory and Applications of Nonlinear Operators, Marcel Dekker, New York, 1996, 313–318.

[41] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 471–485.

[42] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **31** (2010), 24–44. https://doi.org/10.1080/01630560903499852

[43] F. Schopfer, T. Schuster and A. K. Louis, *An iterative regularization method for the solution of the split feasibilty problem in Banach spaces*, Inverse Probl. **24**(5) (2008), Article ID 055008.

[44] A. Taiwo, T. O. Alakoya and O. T. Mewomo, *Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces*, Numer. Algorithms (2020). https://doi.org/10.1007/s11075-020-00937-2

[45] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *A modified Halpern algorithm for approximating a common solution of split equality convex minimization problem and fixed point problem in uniformly convex Banach spaces*, Comput. Appl. Math. **38**(2) (2019), Article ID 77. https://doi.org/10.1007/s40314-019-0841-5

[46] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *Parallel hybrid algorithm for solving pseudomonotone equilibrium and split common fixed point problems*, Bull. Malays. Math. Sci. Soc. **43** (2020), 1893–1918. https://doi.org/10.1007/s40840-019-00781-1

[47] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *General alternative regularization method for solving split equality common fixed point problem for quasi-pseudocontractive mappings in Hilbert spaces*, Ric. Mat. **69**(1) (2020), 235–259. https://doi.org/10.1007/s11587-019-00460-0

[48] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *Viscosity approximation method for solving the multiple-set split equality common fixed-point problems for quasi-pseudocontractive mappings in Hilbert Spaces*, J. Ind. Manag. Optim. (2020). https://doi.org/10.3934/jimo.2020092

[49] A. Taiwo, L. O. Jolaoso, O. T. Mewomo and A. Gibali, *On generalized mixed equilibrium problem with α-β-μ bifunction and μ-τ monotone mapping*, J. Nonlinear Convex Anal. **21**(6) (2020), 1381–1401.
[50] A. Taiwo, A. O.-E. Owolabi, L. O. Jolaoso, O. T. Mewomo and A. Gibali, *A new approximation scheme for solving various split inverse problems*, Afr. Mat. (2020). https://doi.org/10.1007/s13370-020-00832-y
[51] S. Timnak, E. Naraghirad and N. Hussain, *Strong convergence of Halpern iteration for products of finitely many resolvents of maximal monotone operators in Banach spaces*, Filomat **31**(15) (2017), 4673–4693.
[52] J. V. Tie, *Convex Analysis: An Introductory Text*, Wiley, New York, 1984.
[53] V. Todorčević, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*, Springer Nature Switzerland AG, 2019.
[54] F. Q. Xia and N. J. Huang, *Variational inclusions with a general H-monotone operators in Banch spaces*, Comput. Math. Appl. **54**(1) (2010), 24-30. https://doi.org/10.1016/j.camwa.2006.10.028
[55] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co. Inc., River Edge NJ, 2002.

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ON A GENERALIZED DRYGAS FUNCTIONAL EQUATION AND ITS APPROXIMATE SOLUTIONS IN 2-BANACH SPACES

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ABSTRACT. In this paper, we introduce and solve the following generalized Drygas functional equation

$$f(x+ky)+f(x-ky)=2f(x)+k^2f(y)+k^2f(-y),$$

where $k \in \mathbb{N}$. Also, we discuss some stability and hyperstability results for the considered equation in 2-Banach spaces by using the fixed point approach.

1. INTRODUCTION AND PRELIMINARIES

We begin this paper by some notations and symbols. We will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+ = [0,\infty)$ and the set of all natural numbers greater than or equal to m will be denoted by $\mathbb{N}_m$, $m \in \mathbb{N}$. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B.

S. Gähler [23, 24] introduced the basic concept of linear 2-normed spaces. He gave some important facts concerning 2-normed spaces and some preliminary results as follows.

Definition 1.1. Let X be a real linear space with $\dim X > 1$ and $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ be a function satisfying the following properties:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\| = \|y, x\|$;
(c) $\|\lambda x, y\| = |\lambda|\|x, y\|$;

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(d) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *the* 2*-norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called *the linear* 2*-normed space*. Sometimes the condition (d) is called the *triangle inequality*.

Example 1.1. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_{\mathbb{R}^2}$ is defined by

$$\|x, y\|_{\mathbb{R}^2} = |x_1 y_2 - x_2 y_1| .$$

Lemma 1.1. *Let* $(X, \|\cdot,\cdot\|)$ *be a* 2*-normed space. If* $x \in X$ *and* $\|x, y\| = 0$ *for all* $y \in X$*, then* $x = 0$*.*

Definition 1.2. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k \to \infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \to x$ with $k \to \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.3. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l \to \infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a 2*-Banach space*.

The following lemma is one of the tools whose we need in our main results.

Lemma 1.2 ([31])**.** *Let* X *be a* 2*-normed space. Then*

(a) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all* $x, y, z \in X$*;*
(b) *if* $\|x, z\| = 0$ *for all* $z \in X$*, then* $x = 0$*;*
(c) *for a convergent sequence* x_n *in* X

$$\lim_{n \longrightarrow \infty} \|x_n, z\| = \left\| \lim_{n \longrightarrow \infty} x_n, z \right\|,$$

for all $z \in X$*.*

The problem of the stability of functional equations is caused by the question of S. M. Ulam [38] about the stability in group homomorphisms. The first affirmative partial answer to the Ulam's problem for Banach spaces was provided by D. H. Hyers [28]. The result of Hyers was generalizable. Namely, it was generalized by T. Aoki [3] for additive mappings and by Th. M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. In 1994, P. Găvruţa [25] introduced the generalization of the Th. M. Rassias theorem was obtained by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

Within that, a special kind of stability was introduced. This kind is the hyperstability which was given by the following definition.

Definition 1.4 ([13])**.** Let S be a nonempty set, (Y, d) be a metric space, $\mathcal{E} \subset \mathcal{C} \subset \mathbb{R}_+^{S^n}$ be nonempty, $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+^S$ and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset Y^S$ into Y^{S^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \dots, x_n) = \mathcal{F}_2\varphi(x_1, \dots, x_n), \quad x_1, \dots, x_n \in S, \tag{1.1}$$

is $(\mathcal{E}, \mathcal{T})$-hyperstable provided for any $\varepsilon \in \mathcal{E}$ and $\varphi_0 \in \mathcal{D}$ with

$$d\Big(\mathcal{F}_1\varphi_0(x_1, \dots, x_n), \mathcal{F}_2\varphi_0(x_1, \dots, x_n)\Big) \leq \varepsilon(x_1, \dots, x_n), \quad x_1, \dots, x_n \in S,$$

there is a solution $\varphi \in \mathcal{D}$ of equation (1.1) such that

$$d\Big(\varphi(x), \varphi_0(x)\Big) \leq \mathcal{T}\varepsilon(x), \quad x \in S.$$

In [5] the first result of hyperstability has been published, however, the term *hyperstability* was first used in [30].

There are many papers concerning the hyperstability of functional equations, see for example [4, 7–9, 13, 16–20, 26, 27, 30, 33]. In 2013, Brzdęk [6] gave an important result that will be a basic tool to study the stability and hyperstability of functional equations.

Theorem 1.1 ([6])**.** *Let X be a nonempty set, (Y, d) a complete metric space $f_1, \dots, f_s \colon X \to X$ and $L_1, \dots, L_s \colon X \to \mathbb{R}_+$ be given mappings. Let $\Lambda \colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)),$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$. If $\mathcal{T} \colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\Big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\Big) \leq \sum_{i=1}^{s} L_i(x) d\Big(\xi(f_i(x)), \mu(f_i(x))\Big), \quad \xi, \mu \in Y^X, x \in X,$$

and a function $\varepsilon \colon X \to \mathbb{R}_+$ and a mapping $\varphi \colon X \to Y$ satisfies

$$d\left(\mathcal{T}\varphi(x), \varphi(x)\right) \leq \varepsilon(x), \quad x \in X,$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k \varepsilon(x) < \infty, \quad x \in X,$$

then for every $x \in X$ the limit

$$\psi(x) := \lim_{n \to \infty} \mathcal{T}^n \varphi(x)$$

exists and the function $\psi \in Y^X$ is a unique fixed point of $\mathcal{T}$ with

$$d\left(\varphi(x), \psi(x)\right) \leq \varepsilon^*(x), \quad x \in X.$$

In 2019, M. Almahalebi et al. [2] introduced and proved an analogue of Theorem 1.1 in 2-Banach spaces.

Theorem 1.2 ([2])**.** *Let X be a nonempty set, $\Big(Y, \|\cdot,\cdot\|\Big)$ be a 2-Banach space, $g : X \to Y$ be a surjective mapping and let $f_1, \dots, f_r : X \to X$ and $L_1, \dots, L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ are two operators satisfying the conditions*

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z)\Big\| \le \sum_{i=1}^{r} L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big), g(z)\Big\|,$$

for all $\xi, \mu \in Y^X$, $x, z \in X$ and

$$\Lambda\delta(x,z) := \sum_{i=1}^{r} L_i(x)\delta\Big(f_i(x), z\Big), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \tag{1.2}$$

If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), g(z)\Big\| \le \varepsilon(x,z)$$

and

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty} \Big(\Lambda^n\varepsilon\Big)(x,z) < \infty,$$

for all $x, z \in X$, then the limit

$$\lim_{n\to\infty} \Big((\mathcal{T}^n\varphi)\Big)(x)$$

exists for each $x \in X$. Moreover, the function $\psi : X \to Y$ defined by

$$\psi(x) := \lim_{n\to\infty} \Big((\mathcal{T}^n\varphi)\Big)(x)$$

is a fixed point of $\mathcal{T}$ with

$$\Big\|\varphi(x) - \psi(x), g(z)\Big\| \le \varepsilon^*(x,z),$$

for all $x, z \in X$.

Another version of Theorem 1.2 in 2-Banach space can be found in [14]. Also, J. Brzdęk and K. Ciepliński extended their fixed point result to the n-normed spaces in [15].

In this paper, we consider and solve the following equation

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y), \tag{1.3}$$

where $k \in \mathbb{N}$. This equation can be reduced to the Drygas equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y). \tag{1.4}$$

In addition, we use Theorem 1.2 to investigate some stability and hyperstability results of equation (1.3) in 2-Banach spaces.

2. Solution of (1.3)

Throughout this section, X and Y will be real vector spaces. The functional equation (1.3) is connected with the functional equation (1.4) as it is shown below.

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if f satisfies the Drygas functional equation* (1.4) *for all $x, y \in X$.*

Proof. Suppose that $f : X \to Y$ satisfies (1.3) for all $x, y \in X$. Letting $x = y = 0$ in (1.3), we get $f(0) = 0$. Also, by setting $x = 0$ in (1.3), we obtain that

$$f(ky) + f(-ky) = k^2 f(y) + k^2 f(-y), \quad y \in X.$$

To prove that f satisfies the Drygas functional equation (1.4) for all $x, y \in X$, we assume that $x' = x$ and $y' = ky$ be two elements in X. Then we get

$$\begin{aligned} f(x'+y') + f(x'-y') &= f(x+ky) + f(x-ky) \\ &= 2f(x) + k^2 f(y) + k^2 f(-y) \\ &= 2f(x) + f(ky) + f(-ky) \\ &= 2f(x') + f(y') + f(-y'), \quad x, y \in X, \end{aligned}$$

which means that f satisfies the Drygas functional equation (1.4) for all $x, y \in X$. On the other hand, let f be a function satisfying the Drygas functional equation (1.4) for all $x, y \in X$ with $f(0) = 0$ and $f(x) = B(x, x) + A(x)$. Then

$$\begin{aligned} f(x+ky) + f(x-ky) &= 2f(x) + f(ky) + f(-ky) \\ &= 2f(x) + B(ky, ky) + A(ky) + B(-ky, -ky) + A(-ky) \\ &= 2f(x) + k^2 B(y, y) + k^2 B(-y, -y) + \underbrace{A(ky) + A(-ky)}_{=0} \\ &= 2f(x) + k^2 B(y, y) + k^2 B(-y, -y) + k^2 \underbrace{\Big(A(y) + A(-y)\Big)}_{=0} \\ &= 2f(x) + k^2 \Big(B(y, y) + A(y)\Big) + +k^2 \Big(B(-y, -y) + A(-y)\Big) \\ &= 2f(x) + k^2 f(y) + k^2 f(-y), \quad x, y \in X, \end{aligned}$$

which means that f satisfies (1.3) for all $x, y \in X$. □

3. Stability Results

In this section, we give some investigations on the stability and hyperstability results of the equation (1.3) by using Theorem 1.2 in 2-Banach spaces.

Theorem 3.1. *Let X be a normed space, $\Big(Y, \|\cdot, \cdot\|\Big)$ be a 2-Banach space and $h_1, h_2 : X_0^2 \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \Big\{n \in \mathbb{N} : \alpha_n < 1\Big\} \neq \emptyset,$$

where

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn)+\frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn)+\frac{k^2}{2}\lambda_1(n)\lambda_2(n)+\frac{k^2}{2}\lambda_1(-n)\lambda_2(-n)$$

and

$$\lambda_i(n) := \inf\Big\{t \in \mathbb{R}_+ : h_i(nx, z) \le th_i(x, z),\ x, z \in X_0\Big\},$$

for all $n \in \mathbb{N}$ with $i \in \{1, 2\}$. Assume that $f : X \to Y$ satisfies the inequality

$$\Big\|f(x + ky) + f(x - ky) - 2f(x) - k^2 f(y) - k^2 f(-y), g(z)\Big\| \le h_1(x, z)h_2(y, z), \tag{3.1}$$

for all $x, y, z \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, where $g : X \to Y$ is a surjective mapping. Then there exists a unique function $D : X \to Y$ that satisfies the equation (1.3) *such that*

$$\Big\|f(x) - D(x), g(z)\Big\| \le \lambda_0 h_1(x, z)h_2(x, z),$$

for all $x, z \in X_0$, where

$$\lambda_0 = \frac{\lambda_2(n)}{2(1 - \alpha_m)}.$$

Proof. Let us fix $m \in \mathbb{N}$. Replacing x by mx, where $x \in X_0$, in the inequality (3.1), we obtain

$$\begin{aligned}&\left\|\frac{1}{2}f\Big((1 + km)x\Big) + \frac{1}{2}f\Big((1 - km)x\Big) - \frac{k^2}{2}f(mx) - \frac{k^2}{2}f(-mx) - f(x)\ ,\ g(z)\right\|\\ &\le\frac{1}{2}h_1(x, z)h_2(mx, z),\end{aligned} \tag{3.2}$$

for all $x, z \in X_0$. Define the operator $\mathcal{T}_m\colon Y^{X_0} \to Y^{X_0}$ by

$$\mathcal{T}_m\xi(x) := \frac{1}{2}\xi\Big((1 + km)x\Big) + \frac{1}{2}\xi\Big((1 - km)x\Big) - \frac{k^2}{2}\xi(mx) - \frac{k^2}{2}\xi(-mx),$$

for all $x \in X_0$ and $\xi \in Y^{X_0}$. Further put

$$\varepsilon_m(x, z) := \frac{1}{2}h_1(x, z)h_2(mx, z), \quad x, z \in X_0, \tag{3.3}$$

and observe that

$$\varepsilon_m(x, z) = \frac{1}{2}h_1(x, z)h_2(mx, z) \le \frac{1}{2}\lambda_2(m)h_1(x, z)h_2(x, z), \quad x, z \in X_0,\ m \in \mathbb{N}. \tag{3.4}$$

Thus, the inequality (3.2) becomes

$$\|\mathcal{T}_m f(x) - f(x), g(z)\| \le \varepsilon_m(x, z), \quad x, z \in X_0.$$

Furthermore, for every $x, z \in X_0$ and $\xi, \mu \in Y^{X_0}$, we have

$$\begin{aligned}&\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), g(z)\Big\|\\ =&\left\|\frac{1}{2}\xi\Big((1 + km)x\Big) + \frac{1}{2}\xi\Big((1 - km)x\Big) - \frac{k^2}{2}\xi(mx)\right.\end{aligned}$$

$$-\frac{k^2}{2}\xi(-mx)-\frac{1}{2}\mu\Big((1+km)x\Big)-\frac{1}{2}\mu\Big((1-km)x\Big)+\frac{k^2}{2}\mu(mx)+\frac{k^2}{2}\mu(-mx),g(z)\Big\|$$
$$\le\frac{1}{2}\left\|(\xi-\mu)\Big((1+km)x\Big),g(z)\right\|+\frac{1}{2}\left\|(\xi-\mu)\Big((1-km)x\Big),g(z)\right\|$$
$$+\frac{k^2}{2}\left\|(\xi-\mu)(mx)\ ,\ g(z)\right\|+\frac{k^2}{2}\left\|(\xi-\mu)(-mx),g(z)\right\|,$$

for all $x,z\in X_0$ and $\xi,\mu\in Y^{X_0}$. It means that the condition (1.2) is satisfied and this brings us to define the operator $\Lambda_m\colon \mathbb{R}_+^{X_0\times X_0}\to\mathbb{R}_+^{X_0\times X_0}$ by

$$\Lambda_m\delta(x,z):=\frac{1}{2}\delta\Big((1+km)x,z\Big)+\frac{1}{2}\delta\Big((1-km)x,z\Big)+\frac{k^2}{2}\delta(mx,z)+\frac{k^2}{2}\delta(-mx,z),$$

for all $x,z\in X_0$ and $\delta\in\mathbb{R}_+^{X_0\times X_0}$. This operator has the form given by (1.2) with $f_1(x)=(1+km)x$, $f_2(x)=(1-km)x$, $f_3(x)=mx$, $f_4(x)=-mx$, $L_1(x)=L_2(x)=\frac{1}{2}$ and $L_3(x)=L_4(x)=\frac{k^2}{2}$ for all $x\in X_0$.

By induction on $n\in\mathbb{N}$, it is easy to show that

$$\Big(\Lambda_m^n\varepsilon_m\Big)(x,z)\le\frac{1}{2}\lambda_2(m)\alpha_m^n h_1(x,z)h_2(x,z), \tag{3.5}$$

for all $x,z\in X_0$ and all $m\in\mathcal{U}$, where

$$\alpha_m=\frac{1}{2}\lambda_1(1+km)\lambda_2(1+km)+\frac{1}{2}\lambda_1(1-km)\lambda_2(1-km)+\frac{k^2}{2}\lambda_1(m)\lambda_2(m)$$
$$+\frac{k^2}{2}\lambda_1(-m)\lambda_2(-m).$$

Indeed, (3.3) and (3.4) imply that the inequality (3.5) holds for $n=0$. Next, we assume that (3.5) holds for $n=r$, where $r\in\mathbb{N}_1$. Then we obtain

$$\begin{aligned}\Big(\Lambda_m^{r+1}\varepsilon_m\Big)(x,z)=&\Lambda_m\left(\Big(\Lambda_m^r\varepsilon_m\Big)(x,z)\right)\\
=&\frac{1}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big((1+km)x,z\Big)+\frac{1}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big((1-km)x,z\Big)\\
&+\frac{k^2}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big(mx,z\Big)+\frac{k^2}{2}\Big(\Lambda_m^r\varepsilon_m\Big)\Big(-mx,z\Big)\\
\le&\frac{1}{4}\lambda_2(m)\alpha_m^r h_1\Big((1+km)x,z\Big)h_2\Big((1+km)x,z\Big)\\
&+\frac{1}{4}\lambda_2(m)\alpha_m^r h_1\Big((1-km)x,z\Big)h_2\Big((1-km)x,z\Big)\\
&+\frac{k^2}{4}\lambda_2(m)\alpha_m^r h_1\Big(mx,z\Big)h_2\Big(mx,z\Big)\\
&+\frac{k^2}{4}\lambda_2(m)\alpha_m^r h_1\Big(-mx,z\Big)h_2\Big(-mx,z\Big)\\
\le&\frac{1}{2}\lambda_2(m)\Big(\frac{1}{2}\lambda_1(1+km)\lambda_2(1+km)+\frac{1}{2}\lambda_1(1-km)\lambda_2(1-km)\end{aligned}$$

$$
\begin{aligned}
&+\frac{k^2}{2}\lambda_1(m)\lambda_2(m)+\frac{k^2}{2}\lambda_1(-m)\lambda_2(-m)\Big)\alpha_m^r h_1(x,z)h_2(x,z)\\
=&\frac{1}{2}\lambda_2(m)\alpha_m^{r+1}h_1(x,z)h_2(x,z),
\end{aligned}
$$

for all $x, z \in X_0$ and all $m \in \mathcal{U}$. It means that (3.5) holds for $n = r + 1$ which implies that (3.5) holds for all $n \in \mathbb{N}$. Hence, in view of (3.5), we obtain

$$
\begin{aligned}
\varepsilon^*(x,z) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x,z)\\
&\leq \sum_{n=0}^{\infty} \frac{1}{2}\lambda_2(m)\alpha_m^n h_1(x,z)h_2(x,z)\\
&= \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)} < \infty,
\end{aligned}
$$

for all $x, z \in X_0$ and all $m \in \mathcal{U}$. Therefore, according to Theorem 1.2, with $\varphi = f$ and using the surjectivity of g, we get that the limit

$$\lim_{n\to\infty} \Big(\mathcal{T}_m^n f\Big)(x)$$

exists and defined a function $D_m : X \to Y$ such that

$$\Big\|f(x) - D_m(x), g(z)\Big\| \leq \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)}, \quad x, z \in X_0,\ m \in \mathcal{U}. \tag{3.6}$$

To prove that F_m satisfies the functional equation (1.3), just prove the following inequality by the induction on $n \in \mathbb{N}_0$

(3.7)

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^n f)\Big(x+ky)+(\mathcal{T}_m^n f)\Big(x-ky\Big)-2(\mathcal{T}_m^n f)\Big(x\Big)-k^2(\mathcal{T}_m^n f)\Big(y\Big)-k^2(\mathcal{T}_m^n f)(-y),g(z)\Big\|\\
\leq&\alpha_m^n h_1(x,z)h_2(y,z),
\end{aligned}
$$

for every $x, y, z \in X_0$ such that $x + ky \neq 0$, $x - ky \neq 0$ and every $m \in \mathcal{U}$.

First, for $n = 0$, we just find (3.1). Next, take $r \in \mathbb{N}$ and assume that (3.7) holds for $n = r$ and every $x, y, z \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, $m \in \mathcal{U}$. Then, for each $x, y, z \in X_0$ and $m \in \mathcal{U}$, we have

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^{r+1} f)\Big(x+ky)+(\mathcal{T}_m^{r+1} f)\Big(x-ky\Big)-2(\mathcal{T}_m^{r+1} f)\Big(x\Big)\\
&-k^2(\mathcal{T}_m^{r+1} f)\Big(y\Big)-k^2(\mathcal{T}_m^{r+1} f)(-y)\ ,\ g(z)\Big\|\\
=&\Big\|\frac{1}{2}(\mathcal{T}_m^r f)\Big((1+km)(x+ky)\Big)+\frac{1}{2}(\mathcal{T}_m^r f)\Big((1-km)(x+ky)\Big)\\
&-\frac{k^2}{2}(\mathcal{T}_m^r f)\Big(m(x+ky)\Big)-\frac{k^2}{2}(\mathcal{T}_m^r f)\Big(-m(x+ky)\Big)\\
&+\frac{1}{2}(\mathcal{T}_m^r f)\Big((1+km)(x-ky)\Big)+\frac{1}{2}(\mathcal{T}_m^r f)\Big((1-km)(x-ky)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(m(x-ky)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big(-m(x-ky)\Big) \\
&- (\mathcal{T}_m^r f)\Big((1+km)(x)\Big) - (\mathcal{T}_m^r f)\Big((1-km)(x)\Big) \\
&+ k^2(\mathcal{T}_m^r f)\Big(mx\Big) + k^2(\mathcal{T}_m^r f)\Big(-mx\Big) \\
&- \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1+km)(y)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1-km)(y)\Big) + \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(my\Big) \\
&+ \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(-my\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1+km)(-y)\Big) - \frac{k^2}{2}(\mathcal{T}_m^r f)\Big((1-km)(-y)\Big) \\
&+ \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(-my\Big) + \frac{k^4}{2}(\mathcal{T}_m^r f)\Big(my\Big), g(z)\Big\| \\
\le& \frac{1}{2}\Big\|(\mathcal{T}_m^r f)\Big((1+km)(x+ky)\Big) + (\mathcal{T}_m^r f)\Big((1+km)(x-ky)\Big) \\
&- 2(\mathcal{T}_m^r f)\Big((1+km)(x)\Big) - k^2(\mathcal{T}_m^r f)\Big((1+km)(y)\Big) \\
&- k^2(\mathcal{T}_m^r f)\Big((1+km)(-y)\Big), g(z)\Big\| \\
&+ \frac{1}{2}\Big\|(\mathcal{T}_m^r f)\Big((1-km)(x+ky)\Big) + (\mathcal{T}_m^r f)\Big((1-km)(x-ky)\Big) \\
&- 2(\mathcal{T}_m^r f)\Big((1-km)(x)\Big) - k^2(\mathcal{T}_m^r f)\Big((1-km)(y)\Big) \\
&- k^2(\mathcal{T}_m^r f)\Big((1-km)(-y)\Big), g(z)\Big\| \\
&+ \frac{k^2}{2}\Big\|(\mathcal{T}_m^r f)\Big(m(x+ky)\Big) + (\mathcal{T}_m^r f)\Big(m(x-ky)\Big) - 2(\mathcal{T}_m^r f)\Big(mx\Big) \\
&- k^2(\mathcal{T}_m^r f)\Big(my\Big) - k^2(\mathcal{T}_m^r f)\Big(-my\Big), g(z)\Big\| \\
&+ \frac{k^2}{2}\Big\|(\mathcal{T}_m^r f)\Big(-m(x+ky)\Big) + (\mathcal{T}_m^r f)\Big(-m(x-ky)\Big) - 2(\mathcal{T}_m^r f)\Big(-mx\Big) \\
&- k^2(\mathcal{T}_m^r f)\Big(-my\Big) - k^2(\mathcal{T}_m^r f)\Big(my\Big), g(z)\Big\| \\
\le& \frac{1}{2}\alpha_m^r h_1\Big((1+km)x, z\Big)h_2\Big((1+km)y, z\Big) + \frac{1}{2}\alpha_m^r h_1\Big((1-km)x, z\Big)h_2\Big((1-km)y, z\Big) \\
&+ \frac{k^2}{2}\alpha_m^r h_1\Big(mx, z\Big)h_2\Big(my, z\Big) + \frac{k^2}{2}\alpha_m^r h_1\Big(-mx, z\Big)h_2\Big(-my, z\Big) \\
=& \alpha_m^{r+1} h_1(x,z)h_2(y,z).
\end{aligned}
$$

Thus, by induction, we have shown that (3.7) holds for every $x, y, z \in X_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$ such that $x + ky \neq 0$ and $x - ky \neq 0$. Letting $n \to \infty$ in (3.7), we obtain the equality

$$D_m(x+ky) + D_m(x-ky) = 2D_m(x) + k^2 D_m(y) + k^2 D_m(-y),$$

for all $x, y \in X_0$ and $m \in \mathcal{U}$ such that $x + ky \neq 0$ and $x - ky \neq 0$. This implies that $D_m : X \to Y$, defined in this way, is a solution of the equation

$$D(x) = \frac{1}{2}D\Big((1+km)x\Big) + \frac{1}{2}D\Big((1-km)x\Big) - \frac{k^2}{2}D(mx) - \frac{k^2}{2}D\Big(-mx\Big), \tag{3.8}$$

for all $x \in X_0$ and all $m \in \mathcal{U}$. Next, we will prove that each cubic functional equation $D : X \to Y$ satisfying the inequality

$$\Big\|f(x) - D(x), g(z)\Big\| \leq L\, h_1(x,z)h_2(x,z), \quad x, z \in X_0, \tag{3.9}$$

with some $L > 0$, is equal to D_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $D : X \to Y$ satisfying (3.9). From (3.6), for each $x \in X_0$, we get

$$\begin{aligned} \Big\|D(x) - D_{m_0}(x), g(z)\Big\| &\leq \Big\|D(x) - f(x), g(z)\Big\| + \Big\|f(x) - D_{m_0}(x), g(z)\Big\| \\ &\leq L\, h_1(x,z)h_2(x,z) + \varepsilon^*_{m_0}(x,z) \\ &\leq L_0\, h_1(x,z)h_2(x,z) \sum_{n=0}^{\infty} \alpha^n_{m_0}, \end{aligned} \tag{3.10}$$

where $L_0 := 2(1-\alpha_{m_0})L + \lambda_2(m_0) > 0$ and we exclude the case that $h_1(x,z) \equiv 0$ or $h_2(x,z) \equiv 0$ which is trivial. Observe that D and D_{m_0} are solutions to equation (3.8) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\Big\|D(x) - D_{m_0}(x), g(z)\Big\| \leq L_0\, h_1(x,z)h_2(x,z) \sum_{n=j}^{\infty} \alpha^n_{m_0}, \quad x, z \in X_0. \tag{3.11}$$

The case $j = 0$ is exactly (3.10). We fix $r \in \mathbb{N}$ and assume that (3.11) holds for $j = r$. Then, in view of (3.10), for each $x, z \in X_0$, we get

$$\begin{aligned} \Big\|D(x) - D_{m_0}(x), g(z)\Big\| &= \Big\|\frac{1}{2}D\Big((1+km_0)x\Big) + \frac{1}{2}D\Big((1-km_0)x\Big) - \frac{k^2}{2}D(m_0x) \\ &\quad - \frac{k^2}{2}D\Big(-m_0x\Big) - \frac{1}{2}D_{m_0}\Big((1+km_0)x\Big) \\ &\quad - \frac{1}{2}D_{m_0}\Big((1-km_0)x\Big) + \frac{k^2}{2}D_{m_0}(m_0x) \\ &\quad + \frac{k^2}{2}D_{m_0}\Big(-m_0x\Big), g(z)\Big\| \\ &\leq \frac{1}{2}\Big\|D\Big((1+km_0)x\Big) - D_{m_0}\Big((1+km_0)x\Big), g(z)\Big\| \\ &\quad + \frac{1}{2}\Big\|D\Big((1-km_0)x\Big) - D_{m_0}\Big((1-km_0)x\Big), g(z)\Big\| \\ &\quad + \frac{k^2}{2}\Big\|D\Big(m_0x\Big) - D_{m_0}\Big(m_0x\Big), g(z)\Big\| \\ &\quad + \frac{k^2}{2}\Big\|D\Big(-m_0x\Big) - D_{m_0}\Big(-m_0x\Big), g(z)\Big\| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}L_0\ h_1\Big((1+km_0)x,z\Big)h_2\Big((1+km_0)x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n \\
&\quad + \frac{1}{2}L_0\ h_1\Big((1-km_0)x,z\Big)h_2\Big((1-km_0)x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n \\
&\quad + \frac{k^2}{2}L_0\ h_1\Big(m_0x,z\Big)h_2\Big(m_0x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n \\
&\quad + \frac{k^2}{2}L_0\ h_1\Big(-m_0x,z\Big)h_2\Big(-m_0x,z\Big)\sum_{n=r}^{\infty}\alpha_{m_0}^n \\
&\leq L_0\ \alpha_{m_0}\ h_1(x,z)h_2(x,z)\sum_{n=r}^{\infty}\alpha_{m_0}^n \\
&= L_0\ h_1(x,z)h_2(x,z)\sum_{n=r+1}^{\infty}\alpha_{m_0}^n.
\end{aligned}
$$

This shows that (3.11) holds for $j = k+1$. Now we can conclude that the inequality (3.11) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (3.11), we get

$$D = D_{m_0}. \tag{3.12}$$

Thus, we have also proved that $D_m = D_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (3.6)) yields

$$\Big\|f(x) - D_{m_0}(x), g(z)\Big\| \leq \frac{\lambda_2(m)h_1(x,z)h_2(x,z)}{2(1-\alpha_m)}, \quad x,z \in X_0,\ m \in \mathcal{U}.$$

This implies (1.3) with $D = D_{m_0}$ and (3.12) confirms the uniqueness of D. □

4. Hyperstaility Results

The following theorems and corollaries concern the η-hyperstability of (1.3) in 2-Banach spaces. Namely, we consider functions $f : X \to Y$ fulfilling (1.3) approximately, i.e., satisfying the inequality

$$\Big\|f(x+ky) + f(x-ky) - 2f(x) - k^2f(y) - k^2f(-y), g(z)\Big\| \leq \eta(x,y,z), \tag{4.1}$$

for all $x, y, z \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$ with $\eta : X_0 \times X_0 \times X_0 \to \mathbb{R}_+$ is a given mapping. Then we find a unique cubic function $F : X \to Y$ which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1.3) is η-hyperstable in the class of functions $f : X \to Y$, i.e., each $f : X \to Y$ satisfying inequality (4.1), with such η, must fulfil equation (1.3).

Theorem 4.1. *Let X be a normed space, $(Y, \|\cdot,\cdot\|)$ be a real 2-Banach space, h_1, h_2 and $\mathcal{U}$ be as in Theorem 3.1. Assume that*

$$\begin{cases} \lim_{n\to\infty}\lambda_2(n) = \lim_{n\to\infty}\lambda_1(1+kn)\lambda_2(1+kn) = \lim_{n\to\infty}\lambda_1(-n)\lambda_2(-n) = 0, \\ \lim_{n\to\infty}\lambda_1(1-kn)\lambda_2(1-kn) = \lim_{n\to\infty}\lambda_1(n)\lambda_2(n) = 0. \end{cases} \tag{4.2}$$

Then every $f : X \to Y$ satisfying (3.1) is a solution of (1.3) on X_0.

Proof. Suppose that $f : X \to Y$ satisfies (3.1). Then, by Theorem 3.1, there exists a mapping $D : X \to Y$ satisfying (1.3) and

$$\|f(x) - D(x), g(z)\| \leq \lambda_0 h_1(x,z) h_2(x,z),$$

for all $x, z \in X_0$, where $g : X \to Y$ is a surjective mapping and

$$\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)},$$

with

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn) + \frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn) + \frac{k^2}{2}\lambda_1(n)\lambda_2(n) + \frac{k^2}{2}\lambda_1(-n)\lambda_2(-n).$$

Since, in view of (4.2), $\lambda_0 = 0$, this means that $f(x) = D(x)$ for all $x \in X_0$, whence

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y),$$

for all $x, y \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, which implies that f satisfies the functional equation (1.3) on X_0. $\square$

Corollary 4.1. *Let $\big(X, \|\cdot\|\big)$ be a normed space, $\big(Y, \|\cdot,\cdot\|\big)$ be a real 2-Banach space and $\theta \geq 0$, $s \geq 0$, $p, q \in \mathbb{R}$ such that $p + q < 0$. Suppose that $f : X \to Y$ such that $f(0) = 0$ satisfies the inequality*

(4.3)
$$\Big\|f(x+ky) + f(x-ky) - 2f(x) - k^2 f(y) - k^2 f(-y), g(z)\Big\| \leq \theta \|x\|^p \, \|y\|^q \, \|z\|^s,$$

for all $x, y, z \in X_0$ such that $x + ky \neq 0$ and $x - ky \neq 0$, where $g : X \to Y$ is a surjective mapping. Then f satisfies (1.3) on X_0.

Proof. The proof follows from Theorem 3.1 by defining $h_1, h_2 : X_0 \times X_o \to \mathbb{R}_+$ by $h_1(x,z) = \theta_1 \|x\|^p \|z\|^{s_1}$, $h_2(y,z) = \theta_2 \|y\|^q \|z\|^{s_2}$ and $h_1(0,z) = h_2(0,z) = 0$ with $\theta_1, \theta_2 \in \mathbb{R}_+$, $s_1, s_2 \in \mathbb{R}_+$ and $p, q \in \mathbb{R}$ such that $\theta_1 \theta_2 = \theta$, $s_1 + s_2 = s$ and $p + q < 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\lambda_1(n) &= \inf\{t \in \mathbb{R}_+ \colon h_1(nx,z) \leq t\, h_1(x,z),\ x,z \in X_0\}\\ &= \inf\{t \in \mathbb{R}_+ \colon \theta_1 \|nx\|^p \|z\|^{s_1} \leq t\, \theta_1 \|x\|^p \|z\|^{s_1},\ x,z \in X_0\}\\ &= n^p.\end{aligned}$$

Also, we have $\lambda_2(n) = n^q$ for all $n \in \mathbb{N}$. Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn) + \frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn) + \frac{k^2}{2}\lambda_1(n)\lambda_2(n) + \frac{k^2}{2}\lambda_1(-n)\lambda_2(-n)\\ =&\frac{1}{2}(1+kn)^{p+q} + \frac{1}{2}(1-kn)^{p+q} + k^2 n^{p+q} < 1,\end{aligned}$$

for all $n \geq n_0$. According to Theorem 3.1, there exists a unique Drygas function $D : X \to Y$ such that

$$\Big\|f(x) - D(x), g(z)\Big\| \leq \theta \lambda_0 h_1(x,z) h_2(x,z),$$

for all $x, z \in X_0$, where

$$\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)},$$

with

$$\alpha_n = \frac{1}{2}\lambda_1(1+kn)\lambda_2(1+kn) + \frac{1}{2}\lambda_1(1-kn)\lambda_2(1-kn) + \frac{k^2}{2}\lambda_1(n)\lambda_2(n) + \frac{k^2}{2}\lambda_1(-n)\lambda_2(-n).$$

Since $p+q<0$, one of p and q must be negative. Assume that $q<0$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\lambda_2(n) = \lim_{n\to\infty} n^q = 0,\\
&\lim_{n\to\infty}\lambda_1(1+kn)\lambda_2(1+kn) = \lim_{n\to\infty}(1+kn)^{p+q} = 0,\\
&\lim_{n\to\infty}\lambda_1(1-kn)\lambda_2(1-kn) = \lim_{n\to\infty}(1+kn)^{p+q} = 0,\\
&\lim_{n\to\infty}\lambda_1(n)\lambda_2(n) = \lim_{n\to\infty} n^{p+q} = 0.
\end{aligned}$$

Thus by Theorem 4.1, we get the desired results. □

The next corollary prove the hyperstability results for the inhomogeneous Drygas functional equation

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y) + G(x,y).$$

Corollary 4.2. *Let $\big(X, \|\cdot\|\big)$ be a normed space, $\big(Y, \|\cdot,\cdot\|\big)$ be a real 2-Banach space and $\theta \geq 0$, $s \geq 0$, $p, q \in \mathbb{R}$ such that $p+q<0$. Assume that $G : X^2 \to Y$ and $f : X \to Y$ such that $f(0) = 0$ and satisfies the inequality*

$$\Big\|f(x+ky)+f(x-ky)-2f(x)-k^2f(y)-k^2f(-y)-G(x,y), g(z)\Big\| \leq \theta\|x\|^p\ \|y\|^q\ \|z\|^s, \tag{4.4}$$

for all $x, y, z \in X_0$ such that $x+ky \neq 0$ and $x-ky \neq 0$, where $g : X \to Y$ is a surjective mapping. If the functional equation

$$f(x+ky) + f(x-ky) = 2f(x) + k^2 f(y) + k^2 f(-y) + G(x,y), \tag{4.5}$$

for all $x, y \in X_0$ such that $x+ky \neq 0$ and $x-ky \neq 0$ has a solution $f_0 : X \to Y$ on X_0, then f is a solution to (4.5) on X_0.

Proof. From (4.4) we get that the function $K : X \to Y$ defined by $K := f - f_0$ satisfies (4.3). Consequently, Corollary 4.1 implies that K is a solution to Drygas functional equation (1.3) on X_0. Therefore,

$$\begin{aligned}
&f(x+ky) + f(x-ky) - 2f(x) - k^2f(y) - k^2f(-y) - G(x,y)\\
=&K(x+ky) + f_0(x+ky) + K(x-ky) + f_0(x-ky) - 2K(x) - 2f_0(x)\\
&- k^2K(y) - k^2f_0(y) - k^2K(-y) - k^2f_0(-y) - G(x,y)\\
=&0,
\end{aligned}$$

for all $x, y \in X_0$ such that $x+ky \neq 0$ and $x-ky \neq 0$ which means f is a solution to (4.5) on X_0. □

References

[1] M. Almahalebi, *On the hyperstability of σ-Drygas functional equation on semigroups*, Aequationes Math. **90**(4) (2016), 849–857. https://doi.org/10.1007/s00010-016-0422-2

[2] M. Almahalebi and A. Chahbi, *Approximate solution of p-radical functional equation in 2-Banach spaces*, Acta Math. Scientia. **39**(2) (2019), 551–566. https://doi.org/10.1007/s10473-019-0218-2

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66. https://doi.org/10.2969/jmsj/00210064

[4] A. Bahyrycz and M. Piszczek, *Hyperstability of the Jensen functional equation*, Acta Math. Hungar. **142** (2014), 353–365. https://doi.org/10.1007/s10474-013-0347-3

[5] D. G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397. https://doi.org/10.1215/S0012-7094-49-01639-7

[6] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory Appl. **2013** (2013), Article ID 265. https://doi.org/10.1186/1687-1812-2013-285

[7] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141** (2013), 58–67. https://doi.org/10.1007/s10474-013-0302-3

[8] J. Brzdęk, *Remarks on hyperstability of the Cauchy functional equation*, Aequationes Math. **86** (2013), 255–267. https://doi.org/10.1007/s00010-012-0168-4

[9] J. Brzdęk, *A hyperstability result for the Cauchy equation*, Bull. Aust. Math. Soc. **89** (2014), 33–40. https://doi.org/10.1017/S0004972713000683

[10] J. Brzdęk, L. Cădariu and K. Ciepliński, *Fixed point theory and the Ulam stability*, J. Funct. Spaces **2014** (2014), Article ID 829419. https://doi.org/16.10.1007/s11784-016-0288-x

[11] J. Brzdęk, W. Fechner, M. S. Moslehian and J. Sikorska, *Recent developments of the conditional stability of the homomorphism equation*, Banach J. Math. Anal. **9** (2015), 278–327. https://doi.org/10.15352/bjma/09-3-20

[12] J. Brzdęk, J. Chudziak and Zs. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal. **74** (2011), 6728–6732. https://doi.org/10.1016/j.na.2011.06.052

[13] J. Brzdęk and K. Ciepliński, *Hyperstability and superstability*, Abs. Appl. Anal. **2013** (2013), Article ID 401756, 13 pages. https://doi.org/10.1016/j.na.2011.06.052.10.1155/2013/401756

[14] J. Brzdęk and K. Ciepliński, *On a fixed point theorem in 2-Banach spaces and some of its applications*, Acta Math. Sci. **38**(2) (2018), 377–744.https://doi.org/10.1016/S0252-9602(18)30755-0

[15] J. Brzdęk and K. Ciepliński, *A fixed point theorem in n-Banach spaces and Ulam stability*, J. Math. Anal. Appl. **470** (2019), 632–646. https://doi.org/10.1016/j.jmaa.2018.10.028

[16] J. Brzdęk and El-S. El-Hady, *On approximately additive mappings in 2-Banach spaces*, Bull. Aust. Math. Soc. (2019). https://doi.org/10.1017/S0004972719000868

[17] Y. J. Cho, C. Park and M. Eshaghi Gordji, *Approximate additive and quadratic mappings in 2-Banach spaces and related topics*, Int. J. Nonlinear Anal Appl. **3**(2) (2012), 75–81. https://dx.doi.org/10.22075/ijnaa.2012.55

[18] S. C. Chung and W.-G. Park, *Hyers-Ulam stability of functional equations in 2-Banach spaces*, Int. J. Math. Anal. (Ruse) **6**(17/20) (2012), 951–961. https://dx.doi.org/10.22075/ijnaa.2012.55

[19] K. Ciepliński, *Approximate multi-additive mappings in 2-Banach spaces*, Bull. Iranian Math. Soc. **41**(3) (2015), 785–792.

[20] K. Ciepliński and T. Z. Xu, *Approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces*, Carpathian J. Math. **29**(2) (2013), 159–166.

[21] H. Drygas, *Quasi-Inner Products and their Applications*, Springer, Netherlands, 1987, 13–30.

[22] B. R. Ebanks, P. L. Kannappan and P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner-product spaces*, Canad. Math. Bull. **35**(3) (1992), 321–327. https://doi.org/10.4153/CMB-1992-044-6

[23] S. Gähler, 2-*metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

[24] S. Gähler, *Linear* 2-*normiete Räumen*, Math. Nachr. **28** (1964), 1–43.

[25] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436. https://doi.org/10.1006/jmaa.1994.1211

[26] J. Gao, *On the stability of the linear mapping in* 2-*normed spaces*, Nonlinear Funct. Anal. Appl. **14**(5) (2009), 801–807.

[27] E. Gselmann, *Hyperstability of a functional equation*, Acta Math. Hungar. **124** (2009), 179–188. https://doi.org/10.1007/s10474-009-8174-2

[28] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224. https://dx.doi.org/10.1073%2Fpnas.27.4.222

[29] S.-M. Jung and P. K. Sahoo, *Stability of a functional equation of Drygas*, Aequationes Math. **64**(3) (2002), 263–273. https://doi.org/10.1007/PL00012407

[30] Gy. Maksa and Zs. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17** (2001), 107–112.

[31] W.-G. Park, *Approximate additive mappings in* 2-*Banach spaces and related topics,* J. Math. Anal. Appl. **376** (2011), 193–202. https://doi.org/10.1016/j.jmaa.2010.10.004

[32] M. Piszczek and J. Szczawińska, *Hyperstability of the Drygas Functional Equation*, J. Funct. Spaces **2013** (2013). http://dx.doi.org/10.1155/2013/912718

[33] M. M. Piszczek, *Remark on hyperstability of the general linear equation*, Aequationes Math. **88** (2014), 163–168. https://doi.org/10.1007/s00010-013-0214-x

[34] Th. M. Rassias, *On the stability of linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300. https://doi.org/10.1090/S0002-9939-1978-0507327-1

[35] P. K. Sahoo and Pl. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, Florida, 2011.

[36] M. Sirouni and S. Kabbaj, *A fixed point approach to the hyperstability of Drygas functional equation in metric spaces*, J. Math. Comput. Sci. **4**(4) (2014), 705–715.

[37] W. Smajdor, *On set-valued solutions of a functional equation of Drygas*, Aequationes Math. **77** (2009), 89–97. https://doi.org/10.1007/s00010-008-2935-9

[38] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

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