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A NEW APPROACH FOR SOLVING A NEW CLASS OF NONLINEAR OPTIMAL CONTROL PROBLEMS GENERATED BY ATANGANA-BALEANU-CAPUTO VARIABLE ORDER FRACTIONAL DERIVATIVE AND FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

F. GHOMANJANI[1]

Abstract. In the sequel, the numerical solution of a new class of nonlinear optimal control problems (OCPs) generated by Atangana-Baleanu-Caputo (ABC) variable order (V-O) fractional derivative (FD) and fractional Volterra-Fredholm integro-differential equations (FVFIDEs) is found by Bezier curve method (BCM). The main idea behind this work is the use of the BCM. In this technique, the solution is found in the form of a rapid convergent series. Using this method, it is possible to obtain BCM solution of the general form of multipoint boundary value problems. To shown the efficiency of the developed method, numerical results are stated as the main results in this study.

1. Introduction

OCPs is one of the main topics refered to the V-O fractional. These problems are related to the V-O fractional operators in their cost functional and dynamical system. Recently, many numerical methods have been stated such as in [1] and [2], the Bernstein functions for nonlinear V-O fractional OCPs is stated. In [3], generalized polynomials is studied for a kind of V-O fractional 2D OCPs. B-splines (where Bezier form is a special case of B-splines), due to numerical stability and arbitrary order of accuracy, have become popular tools for solving differential equations. The use of Bezier curves for solving V-O fractional OCPs (2.1) and FVFIDEs is a novel idea.

Key words and phrases. variable order, Bezier curve, nonlinear optimal control problems (NOCPs), Volterra-Fredholm integro-differential equations.

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Additionally some papers spent the Bezier curves. In [4] and [5], the authors utilized the Bezier curves for solving delay differential equation (DDE) and optimal control of switched systems numerically. In [6], the authors proposed the utilization of Bezier curves on some linear optimal control systems with pantograph delays. Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, the Bezier control points strategy is utilized (see [7]). Some other uses of the Bezier functions are found in (see [8]). The organization of this study is classified as follows. Problem statement is introduced in Section 2. Also solving ABC V-O FD based on the Bezier curves is stated in Section 3. Convergence analysis is stated in Section 4. A numerical example is solved in Section 5, then a remark is stated about FVFIDEs. Solving FVFIDEs based on Bezier curves is presented in Section 6. Section 7 will give a problem statement for FVFIDEs. Numerical applications for FVFIDEs are presented in Section 8. Finally, Section 9 will give a conclusion briefly.

2. Problem Statement

In this paper, the following definition is considered.

Definition 2.1. Let $\alpha : [0, \tau_{max}] \to (0, 1)$ be a continuous function and $x \in C^1[0, \tau_{max}]$. The V-O FD of order $\alpha(\tau)$ in the ABC sense of $x(\tau)$ is defined as follows (see [9]):

$$ {}_0^{ABC}D_t^{\alpha(\tau)} x(\tau) = \frac{C(\alpha(\tau))}{1-\alpha(\tau)} \int_0^\tau x'(s) E_{\alpha(\tau)} \left(\frac{-\alpha(\tau)(\tau - s)^{\alpha(\tau)}}{1-\alpha(\tau)} \right) ds, \tag{2.1}$$

where

$$\begin{aligned} E_\lambda(\tau) =& \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma(j\lambda+1)}, \quad \lambda \in R^+, \, \tau \in R, \\ C(\alpha(\tau)) =& 1 - \alpha(\tau) + \frac{\alpha(\tau)}{\Gamma(\alpha(\tau))}, \\ {}_0^{ABC}D_t^{\alpha(\tau)} c =& 0, \quad \text{for any constant } c. \end{aligned}$$

So, we focus on the following problem

$$\min J = \int_0^{\tau_{\max}} L(\tau, x_1(\tau), x_2(\tau), \ldots, x_m(\tau), u(\tau)) d\tau,$$

such that

$$\begin{aligned} & {}_0^{ABC}D_t^{\alpha_i(\tau)}\, x(\tau) = G_i(\tau, x_1(\tau), x_2(\tau), \ldots, x_m(\tau), u(\tau)), \\ & \alpha_i(\tau) \in (0, 1), \quad i = 1, 2, \ldots, m, \, m \in \mathbb{N}, \\ & x_i(0) = x_{i,0}, \quad i = 0, 1, \ldots, m, \quad \tau_{max} \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where L, G_i for $i = 1, 2, \ldots, m$, are continuous operators, α_i for $i = 1, 2, \ldots, m$, is a continuous function on $[0, \tau_{max}]$ and $x_{i,0}$ is a given real constant.

3. Solving ABC V-O FD Based on the Bezeir Curves

Our aim is utilizing Bezier curves to approximate the solutions $x(\tau)$ and $u(\tau)$ where $x(\tau)$ and $u(\tau)$ are given below. Define the Bezier polynomials of degree n over the interval $[\tau_0, \tau_f]$ as follows:

$$x(\tau) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{\tau-\tau_0}{h}\right), \quad \tau_f = 1, \quad \tau_0 = 0, \quad u(\tau) = \sum_{r=0}^{n} b_r B_{r,n}\left(\frac{\tau-\tau_0}{h}\right), \tag{3.1}$$

where $h = \tau_f - \tau_0$ and

$$B_{r,n}\left(\frac{\tau-\tau_0}{h}\right) := \binom{n}{r}\frac{1}{h^n}(\tau_f-\tau)^{n-r}(\tau-\tau_0)^r$$

is the Bernstein polynomial of degree n over the interval $[\tau_0, \tau_f]$ and $a_r, b_r,\ r = 0, 1, \ldots, n$, and they are unknown control points. Also, we have

$$\begin{aligned}
\frac{dB_{r,n}(\tau)}{d\tau} =& n\Big(B_{r-1,n-1}(\tau) - B_{r,n-1}(\tau)\Big),\\
\frac{dx(\tau)}{d\tau} =& \sum_{r=0}^{n-1} na_r B_{r-1,n-1}(\tau) - \sum_{r=0}^{n-1} na_r B_{r,n-1}(\tau)\\
=& \sum_{r=0}^{n-1} na_{r+1} B_{r,n-1}(\tau) - \sum_{r=0}^{n-1} na_r B_{r,n-1}(t)\\
=& \sum_{r=0}^{n-1} B_{r,n-1}(\tau) n(a_{r+1} - a_r),
\end{aligned}$$

then

$$\begin{aligned}
{}_0^{ABC}D_t^{\alpha(\tau)} x(\tau) =& \frac{C(\alpha(\tau))}{1-\alpha(\tau)}\int_0^\tau \left(\sum_{r=0}^{n} a_r B_{r,n}(\tau)\right)' E_{\alpha(\tau)}\left(\frac{-\alpha(\tau)(\tau-s)^{\alpha(\tau)}}{1-\alpha(\tau)}\right) ds\\
=& \frac{C(\alpha(\tau))}{1-\alpha(\tau)}\int_0^\tau \sum_{i=0}^{n-1} B_{i,n-1}(\tau) n\left(a_{i+1} - a_i\right) E_{\alpha(\tau)}\left(\frac{-\alpha(\tau)(\tau-s)^{\alpha(\tau)}}{1-\alpha(\tau)}\right) ds.
\end{aligned} \tag{3.2}$$

By substituting $x(\tau)$ and $u(\tau)$ and (3.2) in (2.2), we obtain a simplified problem then we can solve this problem by Maple 16. Our goal is to solve the following optimization problem over the interval $[\tau_0, \tau_f]$ to find the entries of the vectors a_r, b_r, for $r = 0, 1, \ldots, n$.

4. Convergence Analysis

In this section, we can suppose the following problem

$$\min J = \int_0^{\tau_{\max}} x^T(\tau)P(\tau)(\tau) + u^T(\tau)Q(\tau)u(\tau)\ d\tau, \tag{4.1}$$

such that

$$
\begin{aligned}
&{}_0^{ABC}D_t^{\alpha(\tau)}\, x(\tau) - A(\tau)x(\tau) - B(\tau)u(\tau) = F(\tau),\\
&\alpha_i(\tau) \in (0,1), \quad i = 1, 2, \ldots, m,\ m \in \mathbb{N},\\
&x(0) = x_0 = a, \quad u(0) = u_0 = b, \quad a, b \in \mathbb{R}, \quad \tau_{max} = 1,
\end{aligned}
$$

where $P(\tau)$ and $Q(\tau)$ are given non-negative functions for $\tau \in [0, 1]$.

Lemma 4.1. For a polynomial in Bezier form

$$x(\tau) = \sum_{i=0}^{n_2} a_{i,n_2} B_{i,n_2}(\tau),$$

where a_{i,n_2+m_1} is the Bezier coefficient of $x(\tau)$ after being degree-elevated to degree $n_2 + m_1$. Now, we have

$$\frac{\sum_{i=0}^{n_2} a_{i,n_2}^2}{n_2+1} \geq \frac{\sum_{i=0}^{n_2+1} a_{i,n_2+1}^2}{n_2+2} \geq \cdots \geq \frac{\sum_{i=0}^{n_2+m_1} a_{i,n_2+m_1}^2}{n_2+m_1+1}.$$

Proof. See [10]. □

Theorem 4.1. If the problem (4.1) has a unique C^1 continuous solution $\bar{x}$, C^0 continuous control solution $\bar{u}$, then the approximate solution obtained by the control-point-based method converges to the exact solution $(\bar{x}, \bar{u})$ as the degree of the approximate solution tends to infinity.

Proof. Given an arbitrary small positive number $\epsilon > 0$, by the Weierstrass Theorem, one can find polynomials $Q_{1,N_1}(\tau)$ and $Q_{2,N_2}(\tau)$ of degree N_1 and N_2 such that (see [11])

$$
\begin{aligned}
\|Q_{1,N_1}(\tau) - \bar{x}(\tau)\|_\infty &\leq \frac{\epsilon}{16\|A(\tau)\|_\infty},\\
\|Q_{2,N_2}(\tau) - \bar{u}(\tau)\|_\infty &\leq \frac{\epsilon}{16\|B(\tau)\|_\infty},\\
\|\,{}_0^{ABC}D_t^{\alpha(\tau)}\, Q_{1,N_1}(\tau) - {}_0^{ABC}D_t^{\alpha(\tau)}\bar{x}(\tau)\|_\infty &\leq \frac{\epsilon}{16},
\end{aligned}
$$

where $\|\cdot\|_\infty$ stands for the L_∞-norm over $[0, 1]$. Now, we have

$$
\begin{aligned}
\|a - Q_{1,N_1}(0)\|_\infty &\leq \frac{\epsilon}{16},\\
\|b - Q_{2,N_2}(0)\|_\infty &\leq \frac{\epsilon}{16}.
\end{aligned}
\tag{4.2}
$$

In general, $Q_{1,N_1}(\tau)$ and $Q_{2,N_2}(\tau)$ do not satisfy the boundary conditions. After a small perturbation with linear and constant polynomials β and γ for $Q_{1,N_1}(\tau)$, $Q_{2,N_2}(\tau)$ we can obtain polynomials $P_{1,N_1}(\tau) = Q_{1,N_1}(\tau) + \beta$, $P_{2,N_1}(\tau) = Q_{2,N_2}(\tau) + \gamma$ such that $P_{1,N_1}(\tau)$ satisfy the boundary conditions $P_{1,N_1}(0) = a$, $P_{2,N_2}(0) = b$. Thus

$Q_{1,N_1}(0)+\beta=a$, $Q_{2,N_2}(0)+\gamma=b$ by utilizing (4.2), one have

$$\begin{aligned}\|a-Q_{1,N_1}(0)\|_\infty =&\|\beta\|_\infty \leq \frac{\epsilon}{16},\\ \|b-Q_{2,N_2}(0)\|_\infty =&\|\gamma\|_\infty \leq \frac{\epsilon}{16}.\end{aligned}$$

Now, we have

$$\begin{aligned}\|P_{1,N_1}(\tau)-\bar{x}(\tau)\|_\infty =&\|Q_{1,N_1}(\tau)+\beta-\bar{x}(\tau)\|_\infty\\ \leq&\|Q_{1,N_1}(\tau)-\bar{x}(\tau)\|_\infty+\|\beta\|_\infty \leq \frac{2\epsilon}{16},\\ \|\, {}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau)-{}_0^{ABC}D_t^\alpha \bar{x}(\tau)\,\|_\infty =&\|\, {}_0^{ABC}D_t^\alpha Q_{1,N_1}(\tau)-{}_0^{ABC}D_t^\alpha \bar{x}(\tau)\,\|_\infty < \frac{\epsilon}{16}.\end{aligned}$$

Now, let define

$$\begin{aligned}LP_N(x)=L\left(P_{1,N_1}(\tau),P_{2,N_2}(\tau),{}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau)\right)=&{}_0^{ABC}D_t^\alpha\, P_{1,N_1}(x)\\ &-A(\tau)x(\tau)-B(\tau)u(\tau)=F(\tau),\end{aligned}$$

for every $\tau\in[0,1]$. Thus, for $N\geq N_1$, one may find an upper bound for the following residual:

$$\begin{aligned}\|LP_N(x)-F(\tau)\|_\infty =&\|L\left(P_{1,N_1}(\tau),P_{2,N_2}(\tau),{}_0^{ABC}D_t^\alpha P_{1,N_1}(\tau)\right)-F(\tau)\|_\infty\\ \leq&\|\,{}_0^{ABC}D_t^\alpha\, P_{1,N_1}(\tau)-{}_0^{ABC}D_t^\alpha\bar{x}(\tau)\|_\infty\\ &+\|A(\tau)\|_\infty\|P_{1,N_1}(\tau)-\bar{x}(\tau)\|_\infty+\|B(\tau)\|_\infty\|P_{2,N_2}(\tau)-\bar{u}(\tau)\|_\infty\\ \leq&\frac{\epsilon}{16}+\|A(\tau)\|_\infty\frac{\epsilon}{16\|A(\tau)\|_\infty}+\|B(\tau)\|_\infty\frac{\epsilon}{16\|B(\tau)\|_\infty}\leq\epsilon.\end{aligned}$$

Since the residual $R(P_N):=LP_N(x)-F(x)$ is a polynomial, we can represent it by a Bezier form. Thus we have

$$R(P_N):=\sum_{i=0}^{m}d_{i,m}B_{i,m}(x). \tag{4.3}$$

Then from Lemma 1 in [10], there exists an integer M, $M\geq N$, such that when $m>M$, we have

$$\left|\frac{1}{m+1}\sum_{i=0}^{m}d_{i,m}^2-\int_0^1(R(P_N))^2dx\right|<\epsilon,$$

which gives

$$\frac{1}{m+1}\sum_{i=0}^{m}d_{i,m}^2<\epsilon+\int_0^1(R(P_N))^2\;dt\leq\epsilon. \tag{4.4}$$

Suppose $x(\tau)$ and $u(\tau)$ are approximated solutions of (4.1) obtained by the control-point-based method of degree k, $k \geq m \geq M$. Let

$$\begin{aligned}R(x(\tau), u(\tau), {}_0^{ABC}D_t^{\alpha} x(\tau)) =& L(x(\tau), u(\tau), D^{\alpha} x(\tau)) - F(\tau)\\ =& \sum_{i=0}^{k} c_{i,k} B_{i,k}(x), \quad k \geq m \geq M,\ x \in [0,1].\end{aligned}$$

Define the following norm for difference approximated solution $(x(\tau), u(\tau))$ and exact solution $(\bar{x}(\tau), \bar{u}(\tau))$:

$$\begin{aligned}\|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| :=& \int_0^1 |{}_0^{ABC}D_t^{\alpha}\, x(\tau) - {}_0^{ABC}D_t^{\alpha} \bar{x}(\tau)| d\tau\\ &+ \int_0^1 |x(\tau) - \bar{x}(\tau)| d\tau + \int_0^1 |u(0) - \bar{u}(0)| d\tau. \end{aligned} \tag{4.5}$$

It is easy to show that

$$\begin{aligned}&\|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| \qquad (4.6)\\ =& C(|R(x(\tau), u(\tau), {}_0^{ABC}D_t^{\alpha} x(\tau) - \bar{x}(\tau), \bar{u}(\tau), {}_0^{ABC}D_t^{\alpha} \bar{x}(\tau))|\\ &+ |x(0) - \bar{x}(0)| + |u(0) - \bar{u}(0)|)\\ =& \int_0^1 \sum_{i=0}^{k} (c_{i,k} B_{i,k}(t))^2 dx \leq \frac{C}{k+1} \sum_{i=0}^{k} c_{i,k}^2. \qquad (4.7)\end{aligned}$$

Last inequality in (4.6) is obtained from Lemma 1 in [10] which C is a constant positive number. Now from Lemma 4.1 and (4.3), one can easily show that:

$$\begin{aligned}\|(x(\tau), u(\tau)) - (\bar{x}(\tau), \bar{u}(\tau))\| \leq& \frac{C}{k+1} \sum_{i=0}^{k} c_{i,k}^2\\ \leq& \frac{C}{k+1} \sum_{i=0}^{k} d_{i,k}^2 \leq \cdots \leq \frac{C}{m+1} \sum_{i=0}^{m} d_{i,m}^2\\ \leq& C(\epsilon) = \epsilon_1, \quad m \geq M,\end{aligned} \tag{4.8}$$

where last inequality in (4.8) is coming from (4.4). This completes the proof. □

5. Numerical Application

Now, a numerical example of ABC V-O FD is stated to illustrate the BCM. All results are obtained by utilizing Maple 16.

Example 5.1. The following ABC V-O FD is considered (see [9])

$$\begin{aligned}&\min J[u] = \frac{1}{2} \int_0^1 x^2(\tau) + u^2(\tau) d\tau,\\ &{}_0^{ABC}D_t^{\alpha(\tau)}\, x(\tau) = u(\tau) - x(\tau),\\ &x(0) = 1,\end{aligned}$$

where the exact solution with $\alpha(\tau) = 1$ is followed as:

$$\begin{aligned}
x_{exact}(\tau) =& \cosh(\sqrt{2}\tau) + \nu \sinh(\sqrt{2}\tau),\\
u_{exact}(\tau) =&(1+\sqrt{2}\nu)\cosh(\sqrt{2}\tau) + (\sqrt{2}+\nu)\sinh(\sqrt{2}\tau),\\
\nu =& -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} = -0.98,\\
J_{exact} =&0.192909.
\end{aligned}$$

By BCM with $\alpha = 0.95 + 0.04\sin(\tau)$, we obtain the following solution:

$$\begin{aligned}
x_{approx}(\tau) =& -1.383424070\tau + 0.9779939940\tau^2 - 0.3971018840\tau^3\\
&+ 0.08450149506\tau^4 + 0.9999999998,\\
u_{approx}(\tau) =& -2.365740323 - 0.06762181093\tau + 2.572114069\tau^2 - 8.284294539\tau^3\\
&+ 3.832909652\tau^4,\\
J_{approx} =&0.1929636321.
\end{aligned}$$

The graphs of approximated and exact solution x and u are plotted in Figs. 1, 2 (with $n = 4$). Table 1 shows comparison of the values of $x(\tau)$ and $u(\tau)$ for proposed method, Chebyshev cardinal function method [9], and exact solution (with $\alpha = 0.95 + 0.04\sin(\tau)$).

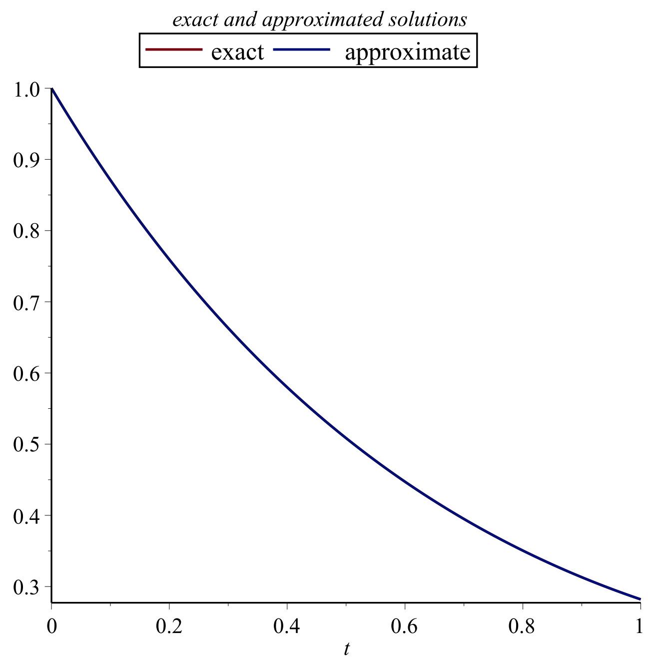

FIGURE 1. The graphs of approximated and exact solution x ($n = 4$) for Example 5.1

TABLE 1. Comparison of the value of $x(\tau)$ and $u(\tau)$ for proposed method, Chebyshev cardinal function method, and exact solution for Example 5.1

τ	$x(\tau)$ in proposed method	$x(\tau)$ in Chebyshev	$x(\tau)$ in exact	$u(\tau)$ in proposed method	$u(\tau)$ in Chebyshev	$u(\tau)$ in exact
0.2	0.7593933324	0.711981	0.759393	−0.2768738382	−0.305356	−0.276873
0.4	0.5798581284	0.549930	0.579944	−0.1854511779	−0.210872	−0.190227
0.6	0.4472007827	0.430800	0.447200	−0.1141705380	−0.129592	−0.118900
0.8	0.3504725480	0.343133	0.350472	−0.05714883910	−0.061000	−0.057148
1.0	0.2819695347	0.280370	0.281969	0.0	0.012271	0.0

Remark 5.1. In science, some problems such as earthquake engineering, biomedical engineering, can be modeled by fractional integro-differential equations (FIDEs). For analyzing these systems, it is required to obtain the solution of FIDEs. Finding the solution of them is not easy. For solving FIDEs, various techniques are suggested such as, Adomian decomposition method (ADM) [12, 13], Laplace decomposition method (LDM) [14], Taylor expansion method (TEM) [15], Spectral collocation method (SCM) [16]. In this paper, the following FVFIDEs are considered:

$$\begin{aligned}\mu_2(x)y'' + \mu_1(x)y' + \mu_\alpha(x)D_x^\alpha y + \mu_0(x)y =& g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt, \\ &0 < \alpha \leq 1,\ \lambda_1, \lambda_2 \in \mathbb{R}, \end{aligned} \tag{5.1}$$

where D_x^α is the Caputo sense fractional derivative. Here, the given functions g, μ_i, for $i = 1, 2, 3$, μ_α, K_1 and K_2 are supposed to be sufficiently smooth. Such equations

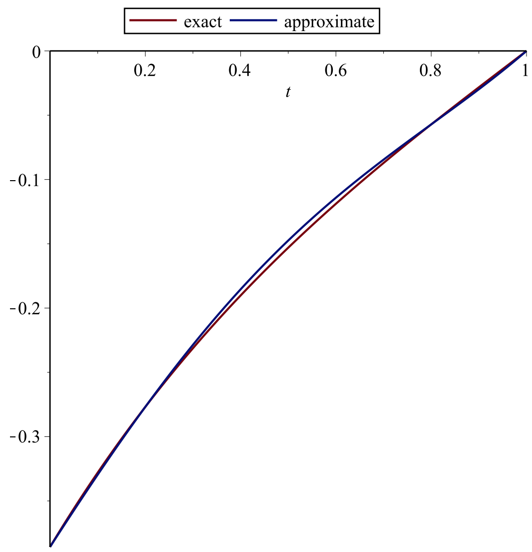

FIGURE 2. The graphs of approximated and exact solution u ($n = 4$) for Example 5.1

arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory (see [17]). In this work, one may utilize Bezier curves technique for solving FVFIDEs.

6. Solving FVFIDEs Based on Bezier Curves

Several definitions of a fractional derivative of order $\alpha > 0$ existed.

Definition 6.1. The Caputo's fractional derivative of order α is stated in [18]

$$(D^{\alpha}y)(x) = \frac{1}{\Gamma(n_1-\alpha)}\int_0^x (x-s)^{-\alpha-1+n_1}y^{(n_1)}(s)ds, \quad n_1 - 1 \leq \alpha \leq n_1,\, n_1 \in \mathbb{N},$$

where $\alpha > 0$ and n_1 is the smallest integer greater than α.

Definition 6.2. The Riemann-Liouville fractional integer operator of order α is presented in [18]

$$I^{\alpha}y(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}y(s)ds, & \alpha > 0, \\ y(x), & \alpha = 0. \end{cases}$$

6.1. **Function approximation.** Utilizing Bezier curves, this technique is to approximate the solutions $y(x)$ where $y(x)$ is given in (6.2). We define the Bezier polynomials of degree n that approximate over the interval $x \in [x_0, x_f]$ as follows:

$$y \approx P^n y = \sum_{i=0}^{n} c_i B_{i,n}\left(\frac{x-x_0}{h}\right) = C^T B(x),$$

where $h = x_f - x_0$,

$$\begin{aligned} C^T &= [c_0, c_1, \ldots, c_n]^T, \\ B^T(x) &= [B_{0,n}(x), B_{1,n}(x), \ldots, B_{n,n}(x)]^T, \\ B_{i,n}\left(\frac{x-x_0}{h}\right) &= \binom{n}{i}\frac{1}{h^n}(x_f - x)^{n-i}(x-x_0)^i, \end{aligned} \tag{6.1}$$

is the Bernstein polynomial with degree n for $x \in [x_0, x_f]$, and c_r is the control point [6]. Our technique is utilizing Bezier curves to approximate the solution $y(x)$ in Eq. (5.1). Define the Bezier polynomials of degree n over the interval $[x_0, x_f] = [0, 1]$ as follows:

$$y_n(x) \simeq \sum_{i=0}^{n} c_i B_{i,n}(x), \quad 0 \leq x \leq 1, \tag{6.2}$$

where

$$B_{i,n}(x) = \binom{n}{i}(1-x)^{n-i}x^i, \quad i = 0, 1, \ldots, n.$$

7. Problem Statement for FVFIDEs

From (5.1), one may have

$$\begin{aligned}\mu_\alpha(x)D_x^\alpha y =& g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y)\,, \quad 0 < \alpha \le 1.\end{aligned}$$

By utilizing Bezier curve method, one may have

$$y_n(x) \simeq \sum_{i=0}^{n} c_i B_{i,n}(x), \quad 0 \le x \le 1.$$

Therefore,

$$\begin{aligned}\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)' =& n\sum_{i=0}^{n-1}(x)(c_{i+1} - c_i), \\ \left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)'' =& n(n-1)\sum_{i=0}^{n-2} B_{i,n-2}(x)(c_{i+2} - 2c_{i+1} + c_i), \\ \frac{\partial}{\partial c_i}\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)'' =& \frac{\partial}{\partial c_i}\left(\sum_{i=2}^{n} B_{i+2,n}(x)(n+2)(n+1)c_i\right. \\ & \left. -2\sum_{i=1}^{n-1} B_{i+1,n-1}(x)n(n+1)c_i + \sum_{i=0}^{n-2} B_{i,n-2}(x)n(n+1)c_i\right) \\ =& \left(\sum_{i=2}^{n} B_{i+2,n}(x)(n+2)(n+1) - 2\sum_{i=1}^{n-1} B_{i+1,n-1}(x)n(n+1)\right. \\ & \left. + \sum_{i=0}^{n-2} B_{i,n-2}(x)n(n+1)\right).\end{aligned}$$

Now

$$\begin{aligned}D^\alpha\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right) =& g(x) + \lambda_1 \int_a^x K_1(x,t)\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)\left(\sum_{i=0}^{n} c_i B_{i,n}(t)\right)dt \\ &- \left(\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)\left(\sum_{i=0}^{n} c_i B_{i,n}(x)\right)\right).\end{aligned}$$

One may define

$$\begin{aligned}R(x, c_0, c_1, \ldots, n) =& \sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \left(g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt\right. \\ & \left. + \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y)\right).\end{aligned}$$

Also define the following equation

$$S(x, c_0, c_1, \dots, c_n) = \int_0^1 R(x, c_0, c_1, \dots, n)^2 w_1(x)dx, \quad w_1(x) = 1,$$

now, we have

$$\begin{aligned} S(x, c_0, c_1, \dots, c_n) = \int_0^1 \Bigg(\sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \Big(g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ + \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y) \Big) \Bigg)^2 dx. \end{aligned} \tag{7.1}$$

Then

$$\frac{\partial S}{\partial c_i} = 0, \quad 0 \le i \le n. \tag{7.2}$$

Using (7.1) and (7.2), we have

$$\begin{aligned} &\int_0^1 \Bigg(\sum_{i=0}^{n} c_i D^\alpha B_{i,n}(x) - \Big(g(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt \\ &+ \lambda_2 \int_a^b K_2(x,t)y(t)dt - (\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y) \Big) \Bigg)^2 dx \times \Big(D^\alpha B_{i,n}(x) \\ &- \lambda_1 \int_a^x K_1(x,t)B_{i,n}(t)dt - \lambda_2 \int_a^b K_2(x,t)B_{i,n}(t)dt - \mu_2(x)\left(B_{i,n}(x)\right)'' \Big) = 0. \end{aligned} \tag{7.3}$$

By (7.3), one can obtain a system of $n+1$ linear equations with $n+1$ unknown coefficients c_i. Also by utilizing many subroutine algorithm for solving this linear equations, one can find the unknown coefficients c_i, $i = 0, 1, \dots, n$.

8. Numerical Applications for FVFIDEs

In this section, we present some numerical examples to illustrate the proposed method.

Example 8.1. Consider the following problem (see [19])

$$\begin{aligned} &y''(x) + D_x^\alpha y(x) - g(x) + 2\int_0^x K_1(x,t)y(t)dt - \int_0^1 K_2(x,t)y(t)dt = 0, \\ &y(0) = 0, \quad y(1) = 0, \\ &g(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15}, \\ &K_1(x,t) = x - t, \quad K_2(x,t) = x^2 - t, \\ &y_{exact}(x) = x^3(x-1), \end{aligned}$$

using the described technique, one may have

$$y_{approx}(x) = -5.551115125 \times 10^{-15} x(1-x)^4 - 1.110223025 \times 10^{-14} x^2(1-x)^3 - 1.000000000 x^3(1-x)^2 - 1.000000000 x^4(1-x),$$

where the absolute error of the proposed method is zero (see Table 2). One may note that Alkan and Hatipoglu [19] obtained the absolute error around 10^{-3}, with $N = 32$. The graphs of approximated solution and exact solution $y(x)$ are plotted in Figure 3.

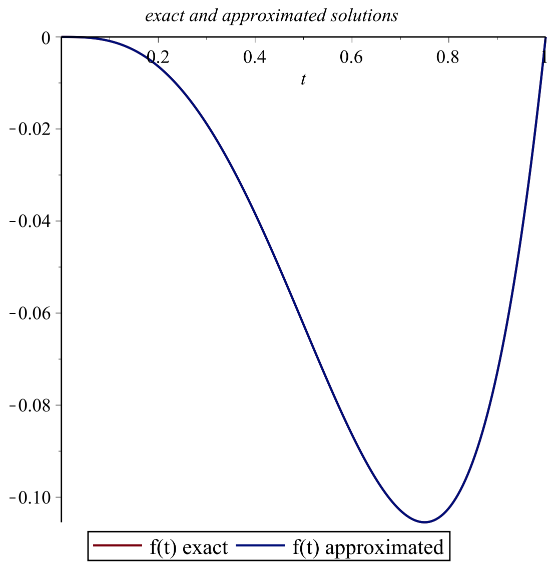

FIGURE 3. The graphs of approximated and exact solution $y(x)$ for Example 8.1

TABLE 2. Exact, estimated values and absolute error of $y(x)$ for Example 8.1

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	0.0009000000000	0.0009000000000	0.0
0.2	0.006400000000	0.006400000000	0.0
0.3	0.01890000000	0.01890000000	0.0
0.4	0.03840000000	0.03840000000	0.0
0.5	0.06250000000	0.06250000000	0.0
0.6	0.08640000000	0.08640000000	0.0
0.7	0.1029000000	0.1029000000	0.0
0.8	0.1024000000	0.1024000000	0.0
0.9	0.07290000000	0.07290000000	0.0
1.0	0.000000000000	0.000000000000	0.0

Example 8.2. Consider the following LFIDE (see [19])

$$y''(x) + \frac{1}{x}D_x^{0.5}y(x) + \frac{1}{x^2} - g(x) - \int_0^x K_1(x,t)y(t)dt - \int_0^1 K_2(x,t)y(t)dt = 0,$$
$$y(0) = 0, \quad y(1) = 0,$$
$$g(x) = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.0674\cos(x) + 5.95385\sin(x),$$
$$K_1(x,t) = \sin(x-t), \quad K_2(x,t) = \cos(x-t),$$
$$y_{exact}(x) = x^2(1-x),$$

using the described technique, one may have

$$y_{approx}(x) = x^2(1-x)^3 + 2x^3(1-x)^2 + x^4(1-x),$$

where the absolute error is zero (see Table 3). One may note that Alkan and Hatipoglu [19] obtained the absolute error around 10^{-7}, with $N = 64$. The graphs of approximated and exact solution $y(x)$ are plotted in Fig. 4.

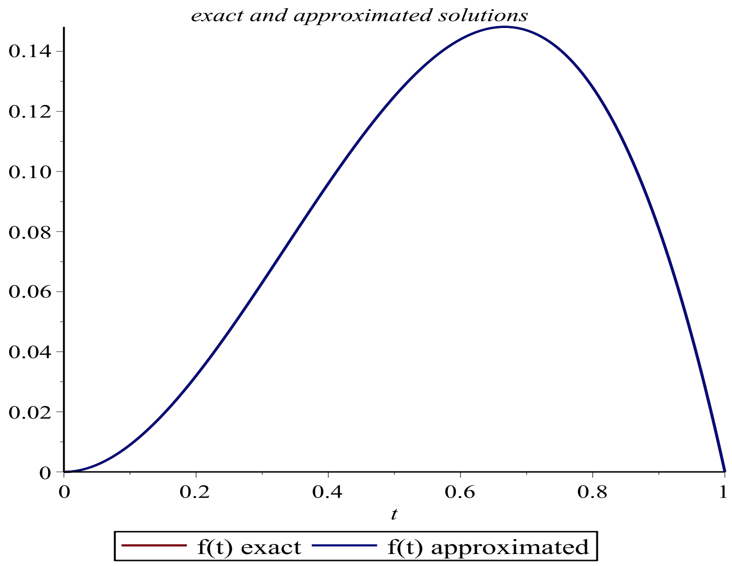

FIGURE 4. The graphs of approximated and exact solution $y(x)$ for Example 8.2

Example 8.3. Consider the following fractional integro-differential equation (see [17])

$$D^\alpha y(x) - x(1+e^x) - 3e^x - y(x) + \int_0^x y(t)dt = 0, \quad \alpha = 4,$$
$$y(0) = 1, \quad y''(0) = 2, \quad y(1) = 1 + e, \quad y''(1) = 3e,$$
$$y_{exact}(x) = 1 + xe^x,$$

using the described technique, one may have

$$y_{approx}(x) = -(1-x)^5 + 5.999923400x(1-x)^4 + 14.99969361x^2(1-x)^3 + 19.50425782x^3(1-x)^2 + 13.15738369x^4(1-x) + x^5(1+e),$$

where the absolute error is less 10^{-5} (see Table 4). The graphs of approximated and exact solution $y(x)$ are plotted in Figure 5.

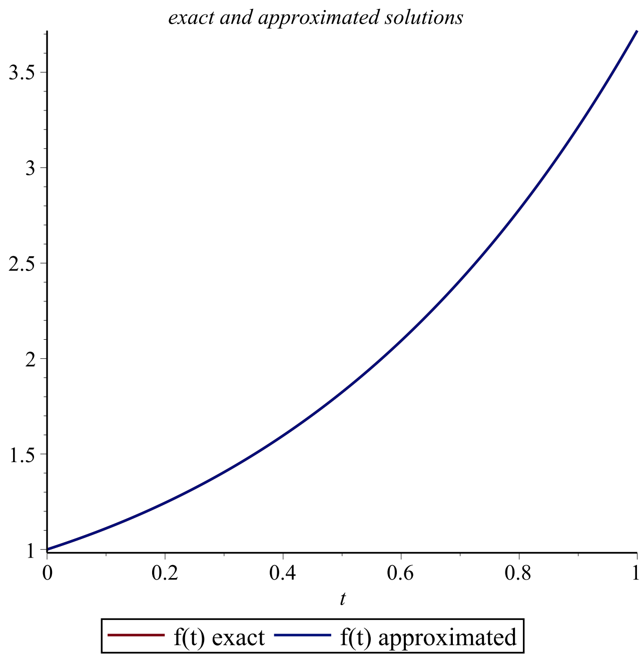

FIGURE 5. The graphs of approximated and exact solution $y(x)$ for Example 8.3

TABLE 3. Exact, estimated values and absolute error of $y(x)$ for Example 8.2

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	0.0009000000000	0.0009000000000	0.0
0.2	0.03200000000	0.03200000000	0.0
0.3	0.06300000000	0.06300000000	0.0
0.4	0.09600000000	0.09600000000	0.0
0.5	0.1250000000	0.1250000000	0.0
0.6	0.1440000000	0.1440000000	0.0
0.7	0.1470000000	0.1470000000	0.0
0.8	0.1280000000	0.1280000000	0.0
0.9	0.08100000000	0.08100000000	0.0
1.0	0.000000000000	0.000000000000	0.0

TABLE 4. Exact, estimated values and absolute error of $y(x)$ for Example 8.3

x	Exact $y(x)$	Present $y(x)$	Absolute error
0.1	1.110517092	1.110512538	$0.4554981720\times10^{-5}$
0.2	1.244280552	1.244280551	$3.150000000\times10^{-10}$
0.3	1.404957642	1.404964146	$0.6503142000\times10^{-5}$
0.4	1.596729879	1.596736160	$0.6280720000\times10^{-5}$
0.5	1.824360636	1.824360635	$3.200000000\times10^{-10}$
0.6	2.093271280	2.093271279	$1.000000000\times10^{-10}$
0.7	2.409626895	2.409649925	$0.2303010000\times10^{-4}$
0.8	2.780432742	2.780504994	$0.7225290000\times10^{-4}$
0.9	3.213642800	3.213749965	$0.1071650000\times10^{-3}$
1.0	3.718281828	3.718281828	0.0

Example 8.4. We consider the following fourth-order, nonlinear fractional integro-differential equation

$$\begin{aligned}
&D^{\alpha}y(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4,\\
&y(0) = 1, \quad y(1) = e, \quad y''(0) = 1, \quad y''(1) = e,\\
&y_{exact}(x) = e^x,
\end{aligned}$$

using this method with $n = 6$, we have

$$\begin{aligned}
y_{approx}(x) =& 1 + 0.002209653x^6 + 0.00723447x^5 + 0.50002963x^2 + 0.1664541100x^3\\
&+ 0.04235543000x^4 + 0.9999985750x.
\end{aligned}$$

From Table 5, we see that, the results obtained with the present method are in good agreement with the results of Momani and Noor [17] and Legendre method [17].

TABLE 5. Comparison of the value of $y(x)$ for Momani and Noor [17], Legendre method [17], stated method, exact value and absolute error of our method for Example 8.4 with $\alpha = 4$

x	Momani and Noor	Legendre	stated method	exact value	our absolute error
0.1	1.10516012	1.10517092	1.105170918	1.105170918	$2.818280000\times10^{-10}$
0.2	1.22138187	1.22140276	1.221402758	1.221402758	$3.700000000\times10^{-11}$
0.3	1.34982923	1.34985881	1.349858768	1.349858808	3.954700000×10^{-8}
0.4	1.49178854	1.49182470	1.491824662	1.491824698	3.563000000×10^{-8}
0.5	1.64868133	1.64872127	1.648721271	1.648721271	$4.400000000\times10^{-10}$
0.6	1.82207855	1.82211880	1.822118800	1.822118800	0.0
0.7	2.01371621	2.01375271	2.013752666	2.013752707	4.120000000×10^{-8}
0.8	2.22551265	2.22554093	2.225540928	2.225540928	$5.000000000\times10^{-10}$
0.9	2.45958740	2.45960311	2.459603313	2.459603111	2.020000000×10^{-7}
1.0	2.71828183	2.71828183	2.718281828	2.718281828	0.0

9. CONCLUSIONS

This paper deals with the approximate solution of ABC V-O fractional derivative and FVFIDEs via BCM. The solution obtained using the suggested method is in very good agreement with the already existing ones and state that this approach can solve the problem effectively. The stated technique reduces the CPU time and the computer memory comparing with existing methods (see some examples). Although the stated technique is very easy to utilize and the obtained results are satisfactory.

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[1]Department of Mathematics, Kashmar Higher Education Institute, Kashmar, Iran
Email address: f.ghomanjani@kashmar.ac.ir, fatemeghomanjani@gmail.com.