

ON THE GENERALIZATION OF FRACTIONAL KINETIC EQUATION COMPRISING INCOMPLETE H -FUNCTION

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ABSTRACT. In the present work, a novel and even more generalized fractional kinetic equation has been formulated in terms of polynomial weighted incomplete H -function, incomplete Fox-Wright function and incomplete generalized hypergeometric function, considering the importance of the fractional kinetic equations arising in the various science and engineering problems. All the derived findings are of natural type and can produce a variety of fractional kinetic equations and their solutions.

1. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

In order to explain the memory effects of complicated systems, the fact that fractional derivatives add a convolution integral with a power-law memory kernel or exponential-law memory kernel strengthens the value of fractional differential equations. It can also be shown that in the last few decades, very fascinating and revolutionary applications of fractional calculus operators have been developed in physics, chemistry, biology, engineering, finance and other fields of research. Some of the applications include: diffusion processes, mechanics of materials, combinatorics, inequalities, signal processing, image processing, advection and dispersion of solutes in porous or fractured media, modeling of viscoelastic materials under external forces, bioengineering, relaxation and reaction kinetics of polymers, random walks, mathematical finance, modeling of combustion, control theory, heat propagation, modeling

Key words and phrases. Fractional kinetic equation, fractional calculus, incomplete H -functions, incomplete Fox-Wright functions, incomplete generalized hypergeometric functions.

2010 *Mathematics Subject Classification.* Primary: 26A33, 33C20, 33C45, 33C60. Secondary: 33C05.

DOI 10.46793/KgJMat2305.701J

Received: June 28, 2020.

Accepted: November 16, 2020.

of viscoelastic materials, in biological systems and many more. The recent work [1–4], and references therein, can be referred to for further information.

Because of the importance in astrophysics, control systems, and mathematical physics, the research on the fractional kinetic equations and their solution has attracted interest from many researchers [5–12]. The fractional kinetic equation has indeed been commonly used to test various physical phenomena regulating the diffusion in porous media, reactions and relaxation mechanisms in complex structures. As a consequence, a significant number of research papers (see [13–20]) focused on the solution of these equations including generalized Mittag-Leffler function, Bessel's function, Struve function, G -function, H -function, and Aleph-function have recently been written. In this new fractional generalization of the kinetic equation, use of the incomplete special functions gives a different dimension to this study. The equation involves a family of polynomials, incomplete H -function, incomplete Fox-Wright function and incomplete generalized hypergeometric function. For these fractional kinetic equations, the Laplace transformation technique is used to derive the solution. Special cases are also illustrated in brief.

Haubold and Mathai [16] set the fractional differential equation within the rate of change of reaction, $\mathcal{N} = \mathcal{N}(t)$, the rate of destruction, $\delta(\mathcal{N}_t)$, and the rate of growth, $p(\mathcal{N}_t)$, as follows:

$$(1.1) \quad \frac{d\mathcal{N}}{dt} = -\delta(\mathcal{N}_t) + p(\mathcal{N}_t),$$

where \mathcal{N}_t is given by $\mathcal{N}_t(t^*) = \mathcal{N}(t - t^*)$, $t^* > 0$.

In addition, Haubold and Mathai [16] gave the limiting case of (1.1) when $\mathcal{N}(t)$ in the quantity of spatial fluctuations or homogeneities is ignored and given as

$$(1.2) \quad \frac{d\mathcal{N}_j}{dt} = -c_j \mathcal{N}_j(t),$$

where $\mathcal{N}_j(t = 0) = \mathcal{N}_0$ is the amount of density of species j at time $t = 0$, $c_j > 0$. If the index j is dropped and the typical kinetic equation (1.2) is integrated, we receive

$$(1.3) \quad \mathcal{N}(t) - \mathcal{N}_0 = -c_0 D_t^{-1} \mathcal{N}(t),$$

where ${}_0D_t^{-1}$ is the specialized case of the Riemann-Liouville fractional integral operator ${}_0D_t^{-\nu}$ lay it out as

$$(1.4) \quad {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du, \quad t > 0, \operatorname{Re}(\nu) > 0.$$

Haubold and Mathai [16] gave the fractional thought to the classical kinetic equation by considering fractional derivative rather than the total derivative in (1.2)

$$(1.5) \quad \mathcal{N}(t) - \mathcal{N}_0 = -c^\nu {}_0D_t^{-\nu} \mathcal{N}(t).$$

Then the solution for $\mathcal{N}(t)$ is a Mittag-Leffler function $E_\nu(\cdot)$

$$(1.6) \quad \mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c t)^{\nu r}}{\Gamma(\nu r + 1)} = \mathcal{N}_0 E_\nu(-c^\nu t^\nu).$$

In addition, Saxena and Kalla [17] thought about the subsequent fractional kinetic equation

$$(1.7) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 f(\mathbf{t}) = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

where $f(\mathbf{t}) \in \mathcal{L}(0, \infty)$.

The Laplace transformation of the Riemann-Liouville fractional integration of $f(t)$ given in the equation (1.4) is specified as

$$(1.8) \quad L \left[{}_0D_{\mathbf{t}}^{-\nu} f(\mathbf{t}); \omega \right] = \omega^{-\nu} F(\omega), \quad \mathbf{t} > 0, \operatorname{Re}(\nu) > 0, \operatorname{Re}(\omega) > 0,$$

where $F(\omega)$ is the Laplace transform of the function $f(\mathbf{t})$ and given by

$$(1.9) \quad F(\omega) = L[f(\mathbf{t}); \omega] = \int_0^\infty e^{-\omega t} f(\mathbf{t}) dt, \quad \mathbf{t} > 0, \operatorname{Re}(\omega) > 0.$$

On the other hand, the familiar lower and upper incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, respectively are defined as (see [21]):

$$(1.10) \quad \gamma(\xi, x) = \int_0^x u^{\xi-1} e^{-u} du, \quad \operatorname{Re}(\xi) > 0, x \geq 0,$$

and

$$(1.11) \quad \Gamma(\xi, x) = \int_x^\infty u^{\xi-1} e^{-u} du, \quad x \geq 0, \operatorname{Re}(\xi) > 0 \text{ if } x = 0.$$

These functions fulfill the following relation:

$$(1.12) \quad \gamma(\xi, x) + \Gamma(\xi, x) = \Gamma(\xi), \quad \operatorname{Re}(\xi) > 0.$$

By the use of above defined incomplete gamma functions, Srivastava et al. [21] defined the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ given as

$$(1.13) \quad {}_p\gamma_q \left[\begin{matrix} (a_1, x), & a_2, \dots, & a_p; \\ & b_1, \dots, & b_q; \end{matrix} \middle| z \right] = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{\ell=0}^\infty \frac{\gamma(a_1 + \ell, x) \prod_{j=2}^p \Gamma(a_j + \ell)}{\prod_{j=1}^q \Gamma(b_j + \ell)} \cdot \frac{z^\ell}{\ell!}$$

$$= \sum_{\ell=0}^\infty \frac{(a_1, x)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell \cdots (b_q)_\ell} \cdot \frac{z^\ell}{\ell!}$$

and

$$(1.14) \quad {}_p\Gamma_q \left[\begin{matrix} (a_1, x), & a_2, \dots, & a_p; \\ & b_1, \dots, & b_q; \end{matrix} \middle| z \right] = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{\ell=0}^\infty \frac{\Gamma(a_1 + \ell, x) \prod_{j=2}^p \Gamma(a_j + \ell)}{\prod_{j=1}^q \Gamma(b_j + \ell)} \cdot \frac{z^\ell}{\ell!}$$

$$= \sum_{\ell=0}^\infty \frac{[a_1, x]_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell \cdots (b_q)_\ell} \cdot \frac{z^\ell}{\ell!},$$

where $(a, x)_\ell$ and $[a, x]_\ell$ are incomplete Pochhammer symbols defined below and $(a)_\ell$ is Pochhammer symbol

$$(1.15) \quad (a, x)_\ell = \frac{\gamma(a + \ell, x)}{\Gamma(a)}, \quad a, \ell \in \mathbb{C}, x \geq 0,$$

and

$$(1.16) \quad [a, x]_\ell = \frac{\Gamma(a + \ell, x)}{\Gamma(a)}, \quad a, \ell \in \mathbb{C}, x \geq 0.$$

The existence and convergence conditions of the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ are set out in [21].

The incomplete Fox-Wright functions, ${}_p\Psi_q^{(\gamma)}$ and ${}_p\Psi_q^{(\Gamma)}$, are the generalization of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, and defined as follows (see [22]):

$$(1.17) \quad {}_p\Psi_q^{(\gamma)} \left[\begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}; \end{matrix} z \right] = \sum_{\ell=0}^{\infty} \frac{\gamma(a_1 + A_1\ell, x) \prod_{j=2}^p \Gamma(a_j + A_j\ell)}{\prod_{j=1}^q \Gamma(b_j + B_j\ell)} \cdot \frac{z^\ell}{\ell!}$$

and

$$(1.18) \quad {}_p\Psi_q^{(\Gamma)} \left[\begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}; \end{matrix} z \right] = \sum_{\ell=0}^{\infty} \frac{\Gamma(a_1 + A_1\ell, x) \prod_{j=2}^p \Gamma(a_j + A_j\ell)}{\prod_{j=1}^q \Gamma(b_j + B_j\ell)} \cdot \frac{z^\ell}{\ell!},$$

where $A_j, B_j \in \mathbb{R}^+$, $a_j, b_j \in \mathbb{C}$ and series converges absolutely for all $z \in \mathbb{C}$ when $\Delta = 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0$.

The incomplete Fox-Wright functions, ${}_p\Psi_q^{(\gamma)}$ and ${}_p\Psi_q^{(\Gamma)}$ satisfy the following decomposition formula

$$(1.19) \quad {}_p\Psi_q^{(\gamma)}(z) + {}_p\Psi_q^{(\Gamma)}(z) = {}_p\Psi_q(z),$$

where ${}_p\Psi_q(z)$ is Fox-Wright function.

Inspired by the applications of ${}_p\gamma_q$ and ${}_p\Gamma_q$ functions (defined above) and their representation as Mellin-Barnes contour integrals, Srivastava et al. [22] presented and researched the incomplete H -functions as follows:

$$(1.20) \quad \gamma_{u,v}^{r,s}(z) = \gamma_{u,v}^{r,s} \left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} g(\xi, x) z^{-\xi} d\xi$$

and

$$(1.21) \quad \Gamma_{u,v}^{r,s}(z) = \Gamma_{u,v}^{r,s} \left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} G(\xi, x) z^{-\xi} d\xi,$$

where

$$(1.22) \quad g(\xi, x) = \frac{\gamma(1 - a_1 - A_1\xi, x) \prod_{j=1}^r \Gamma(b_j + B_j\xi) \prod_{j=2}^s \Gamma(1 - a_j - A_j\xi)}{\prod_{j=r+1}^v \Gamma(1 - b_j - B_j\xi) \prod_{j=s+1}^u \Gamma(a_j + A_j\xi)}$$

and

$$(1.23) \quad G(\xi, x) = \frac{\Gamma(1 - a_1 - A_1\xi, x) \prod_{j=1}^r \Gamma(b_j + B_j\xi) \prod_{j=2}^s \Gamma(1 - a_j - A_j\xi)}{\prod_{j=r+1}^v \Gamma(1 - b_j - B_j\xi) \prod_{j=s+1}^u \Gamma(a_j + A_j\xi)},$$

with the set of conditions set out in [22].

These incomplete H -functions fulfill the following relation (known as decomposition formula):

$$(1.24) \quad \gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z).$$

The general class of polynomials of index n , $n = 0, 1, 2, \dots$, was defined by Srivastava [23] as:

$$(1.25) \quad S_n^m[x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} x^s,$$

where m is positive integer and $A_{n,s} \in \mathbb{R}$ (or \mathbb{C}) are arbitrary positive constants. The notations $(-n)_m$ and $[\cdot]$, respectively represent the Pochhammer symbol and the greatest integer function. Srivastava's polynomials give a number of known polynomials as its special cases on suitably specializing the coefficients $A_{n,s}$.

Throughout this paper we assume that the incomplete H -functions, incomplete Fox-Wright functions and incomplete expanded hypergeometric functions exist under the same sets of conditions set out in [21, 22].

2. SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

Theorem 2.1. Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of

$$(2.1) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu-1} S_n^m[\mathbf{a} \mathbf{t}^\zeta] \Gamma_{u,v}^{r,s}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(2.2) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} \mathbf{t}^\zeta)^k \\ & \times \Gamma_{u+1,v+1}^{r,s+1} \left[\mathbf{b} \mathbf{t}^\eta \left| \begin{array}{l} (a_1, A_1, x), (1 - \mu - \zeta k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1 - \mu - \zeta k - \nu i, \eta) \end{array} \right. \right]. \end{aligned}$$

Proof. To prove the result, Laplace transform method has been used. Taking the Laplace transform of (2.1) and using (1.21), (1.25) and (1.8), after little simplification, we obtain

$$(2.3) \quad [1 + c^\nu \omega^{-\nu}] \mathcal{N}(\omega) = \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k} \mathbf{a}^k}{k!} \cdot \frac{1}{2\pi i} \int_{\mathcal{L}} G(\xi, x) \mathbf{b}^\xi \frac{\Gamma(\mu + \zeta k - \eta \xi)}{\omega^{\mu + \zeta k - \eta \xi}} d\xi,$$

where $\mathcal{N}(\omega) = \mathcal{L}\{\mathcal{N}(t); \omega\}$ and $G(\xi, x)$ is defined in (1.23). Since $(1 + x)^{-1} = \sum_{r=0}^{\infty} (-1)^r x^r$, therefore (2.3) implies that

$$\begin{aligned} \mathcal{N}(\omega) = & \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k} \mathbf{a}^k}{k!} \cdot \frac{1}{2\pi i} \int_{\mathcal{L}} G(\xi, x) \mathbf{b}^\xi \Gamma(\mu + \zeta k - \eta \xi) d\xi \\ (2.4) \quad & \times \sum_{i=0}^{\infty} (-c^\nu)^i \omega^{-(\mu + \zeta k - \eta \xi + \nu i)}. \end{aligned}$$

Now, take the inverse Laplace transform of (2.4), we obtain

$$\begin{aligned} \mathcal{N}(t) = & \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k} \mathbf{a}^k}{k!} \cdot \frac{1}{2\pi i} \int_{\mathcal{L}} G(\xi, x) \mathbf{b}^\xi \Gamma(\mu + \zeta k - \eta \xi) d\xi \\ (2.5) \quad & \times \sum_{i=0}^{\infty} (-c^\nu)^i \frac{t^{(\mu + \zeta k - \eta \xi + \nu i - 1)}}{\Gamma(\mu + \zeta k - \eta \xi + \nu i)}. \end{aligned}$$

Finally, using (1.21) therein, we get the required result (2.2). □

Theorem 2.2. Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of

$$(2.6) \quad \mathcal{N}(t) - \mathcal{N}_0 t^{\mu-1} S_n^m[\mathbf{a} t^\zeta] \gamma_{u,v}^{r,s}[\mathbf{b} t^\eta] = -c^\nu {}_0D_t^{-\nu} \mathcal{N}(t),$$

is provided as

$$\begin{aligned} \mathcal{N}(t) = & \mathcal{N}_0 t^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu t^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} t^\zeta)^k \\ (2.7) \quad & \times \gamma_{u+1,v+1}^{r,s+1} \left[\mathbf{b} t^\eta \left| \begin{matrix} (a_1, A_1, x), (1 - \mu - \zeta k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1 - \mu - \zeta k - \nu i, \eta) \end{matrix} \right. \right]. \end{aligned}$$

Proof. The proof is the immediate consequence of the definitions (1.20), (1.25) and parallel to the Theorem 2.1. Hence, we skip the proof. □

The incomplete H -functions and the incomplete Fox-Wright functions are connected with the following relations (see, [22, (6.3) and (6.4)]):

$$(2.8) \quad \Gamma_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1, A_1, x), (1 - a_j, A_j)_{2,p} \\ (0, 1), (1 - b_j, B_j)_{1,q} \end{matrix} \right. \right] = {}_p\Psi_q^{(\Gamma)} \left[\begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}; \end{matrix} z \right]$$

and

$$(2.9) \quad \gamma_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1, A_1, x), (1 - a_j, A_j)_{2,p} \\ (0, 1), (1 - b_j, B_j)_{1,q} \end{matrix} \right. \right] = {}_p\Psi_q^{(\gamma)} \left[\begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}; \end{matrix} z \right].$$

If we make the substitution $\mathbf{b} = -\mathbf{b}$, $r = 1$, $s = p$, $v = q + 1$, $a_j \mapsto (1 - a_j)$, $j = 1, \dots, p$, $b_j \mapsto (1 - b_j)$ $j = 1, \dots, q$, and multiply by $\Gamma(\xi)$ (i.e., put $b = 0$ and $B = 1$) in (2.1), (2.2), (2.6) and (2.7), use of the equations (2.8) and (2.9) respectively leads to the following corollaries.

Corollary 2.1. Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of

$$(2.10) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu-1} S_n^m[\mathbf{a} \mathbf{t}^\zeta]_p \Psi_q^{(\Gamma)}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(2.11) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} \mathbf{t}^\zeta)^k \\ & \times {}_{p+1}\Psi_{q+1}^{(\Gamma)} \left[\begin{matrix} (a_1, A_1, x), (\mu + \zeta k, \eta), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}, (\mu + \zeta k + \nu i, \eta); \end{matrix} \mathbf{b} \mathbf{t}^\eta \right]. \end{aligned}$$

Corollary 2.2. Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of

$$(2.12) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu-1} S_n^m[\mathbf{a} \mathbf{t}^\zeta]_p \Psi_q^{(\gamma)}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(2.13) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} \mathbf{t}^\zeta)^k \\ & \times {}_{p+1}\Psi_{q+1}^{(\gamma)} \left[\begin{matrix} (a_1, A_1, x), (\mu + \zeta k, \eta), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}, (\mu + \zeta k + \nu i, \eta); \end{matrix} \mathbf{b} \mathbf{t}^\eta \right]. \end{aligned}$$

Remark 2.1. If we set $x = 0, p = 1, q = 2, a_1 = 1, A_1 = 1, b_1 = l + 1 + \frac{b}{2}, b_2 = \frac{3}{2}, B_1 = B_2 = 1, \eta = 2, \mathbf{b} = -\frac{c}{4}, \mu = l + 2, c^\nu = d^\nu$ and $S_n^m[\mathbf{a} \mathbf{t}^\zeta] = \frac{1}{2^{l+1}}$ (i.e., $m = 1, \zeta = 0, \mathbf{a} = \frac{1}{2}, A_{n,k} = \frac{k!}{(-n)_k}$ for $k = l + 1$ and $A_{n,k} = 0$, otherwise) into (2.10) and (2.11), then the resulting equations would correspond to the kinetic equation and its solution involving generalized Struve function given by Nisar et al. [24, page 168, (14) and (15)].

The incomplete Fox-Wright functions are related to the incomplete generalized hypergeometric functions, ${}_p\Gamma_q$ and ${}_p\gamma_q$ (see [21]). In consequence of (2.8) and (2.9), the incomplete H -functions are related to the incomplete generalized hypergeometric functions as below:

$$(2.14) \quad {}_p\Psi_q^{(\Gamma)} \left[\begin{matrix} (a_1, 1, x), (a_j, 1)_{2,p}; \\ (b_j, 1)_{1,q}; \end{matrix} z \right] = \mathcal{C}_q^p {}_p\Gamma_q \left[\begin{matrix} (a_1, x), (a_j)_{2,p}; \\ (b_j)_{1,q}; \end{matrix} z \right]$$

and

$$(2.15) \quad {}_p\Psi_q^{(\gamma)} \left[\begin{matrix} (a_1, 1, x), (a_j, 1)_{2,p}; \\ (b_j, 1)_{1,q}; \end{matrix} z \right] = \mathcal{C}_q^p {}_p\gamma_q \left[\begin{matrix} (a_1, x), (a_j)_{2,p}; \\ (b_j)_{1,q}; \end{matrix} z \right],$$

where \mathcal{C}_q^p is defined by

$$(2.16) \quad \mathcal{C}_q^p = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)}.$$

Thus,

$$(2.17) \quad \Gamma_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1, 1, x), (1 - a_j, 1)_{2,p} \\ (0, 1), (1 - b_j, 1)_{1,q} \end{matrix} \right. \right] = \mathcal{C}_q^p {}_p\Gamma_q \left[\begin{matrix} (a_1, x), (a_j)_{2,p}; \\ (b_j)_{1,q}; \end{matrix} z \right]$$

and

$$(2.18) \quad \gamma_{p,q+1}^{1,p} \left[-z \left| \begin{array}{c} (1-a_1, 1, x), (1-a_j, 1)_{2,p} \\ (0, 1), (1-b_j, 1)_{1,q} \end{array} \right. \right] = \mathcal{C}_q^p \, {}_p\gamma_q \left[\begin{array}{c} (a_1, x), (a_j)_{2,p}; \\ (b_j)_{1,q}; \quad z \end{array} \right].$$

If we substitute $\mathbf{b} = -\mathbf{b}$, $r = 1$, $s = p$, $v = q + 1$, $a_j \mapsto (1 - a_j)$, $j = 1, \dots, p$, $b_j \mapsto (1 - b_j)$, $j = 1, \dots, q$, $A_j = 1$, $j = 1, \dots, p$, $B_j = 1$, $j = 2, \dots, q$, and multiply by $\Gamma(\xi)$ (i.e., put $b = 0$ and $B = 1$) in (2.1), (2.2), (2.6) and (2.7). Then the use of (2.17) and (2.18), respectively, leads to the following corollaries.

Corollary 2.3. *Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of*

$$(2.19) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathcal{C}_q^p \mathbf{t}^{\mu-1} S_n^m [\mathbf{a} \mathbf{t}^\zeta] {}_p\Gamma_q [\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(2.20) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} \mathbf{t}^\zeta)^k \\ & \times \mathcal{C}_{q+1}^{p+1} \, {}_{p+1}\Gamma_{q+1} \left[\begin{array}{c} (a_1, x), \mu + \zeta k, a_2, a_3, \dots, a_p; \\ \mu + \zeta k + \nu i, b_1, b_2, \dots, b_q; \end{array} \mathbf{b} \mathbf{t}^\eta \right], \end{aligned}$$

where \mathcal{C}_q^p is defined in (2.16) and $\mathcal{C}_{q+1}^{p+1} = \mathcal{C}_q^p \frac{\Gamma(\mu+\zeta k)}{\Gamma(\mu+\zeta k+\nu i)}$.

Corollary 2.4. *Assume that $\zeta, \eta, \mathbf{a}, \mathbf{b} > 0$ and $\mu > 0$, then the solution of*

$$(2.21) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathcal{C}_q^p \mathbf{t}^{\mu-1} S_n^m [\mathbf{a} \mathbf{t}^\zeta] {}_p\gamma_q [\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(2.22) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (\mathbf{a} \mathbf{t}^\zeta)^k \\ & \times \mathcal{C}_{q+1}^{p+1} \, {}_{p+1}\gamma_{q+1} \left[\begin{array}{c} (a_1, x), \mu + \zeta k, a_2, a_3, \dots, a_p; \\ \mu + \zeta k + \nu i, b_1, b_2, \dots, b_q; \end{array} \mathbf{b} \mathbf{t}^\eta \right], \end{aligned}$$

where \mathcal{C}_q^p is defined in (2.16) and $\mathcal{C}_{q+1}^{p+1} = \mathcal{C}_q^p \frac{\Gamma(\mu+\zeta k)}{\Gamma(\mu+\zeta k+\nu i)}$.

3. APPLICATIONS

In this section, some consequences and applications of the above results are considered. Specific special cases of the derived findings can be developed by suitably specializing the coefficient $A_{n,s}$ to obtain a large number of spectrum of the known polynomials. To illustrate that we consider the following examples.

Example 3.1. Show that the solution of

$$(3.1) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu-1} \Gamma_{u,v}^{r,s} [\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.2) \quad \mathcal{N}(\mathbf{t}) = \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \Gamma_{u+1,v+1}^{r,s+1} \left[\mathbf{b} \mathbf{t}^\eta \left| \begin{array}{c} (a_1, A_1, x), (1-\mu, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1-\mu-\nu i, \eta) \end{array} \right. \right],$$

$$(3.3) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu-1} {}_p\Psi_q^{(\Gamma)}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.4) \quad \mathcal{N}(\mathbf{t}) = \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i {}_{p+1}\Psi_{q+1}^{(\Gamma)} \left[\begin{matrix} (a_1, A_1, x), (\mu, \eta), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}, (\mu + \nu i, \eta); \end{matrix} \mathbf{b} \mathbf{t}^\eta \right],$$

$$(3.5) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathcal{C}_q^p \mathbf{t}^{\mu-1} {}_p\Gamma_q[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.6) \quad \mathcal{N}(\mathbf{t}) = \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \mathcal{C}_{q+1}^{p+1} {}_{p+1}\Gamma_{q+1} \left[\begin{matrix} (a_1, x), \mu, a_2, a_3, \dots, a_p; \\ \mu + \nu i, b_1, b_2, \dots, b_q; \end{matrix} \mathbf{b} \mathbf{t}^\eta \right],$$

with $\mathcal{C}_{q+1}^{p+1} = \mathcal{C}_q^p \frac{\Gamma(\mu)}{\Gamma(\mu + \nu i)}$.

Solution. Here, setting $m = 1$, $\mathbf{a} = 1$, $\zeta = 0$ and $A_{n,s} = \frac{s!}{(-n)_{ms}}$ for $s = 0$ and $A_{n,s} = 0$ for $s \neq 0$ (i.e., $S_n^m[\mathbf{a} \mathbf{t}^\zeta] = 1$) in (2.1), (2.10) and (2.19). The assertions (3.1), (3.3) and (3.5) of the example follow from the Theorem 2.1, Corollary 2.1 and Corollary 2.3, respectively.

Remark 3.1. It is important to note that for $x = 0$, the kinetic equation and its solution given by (3.1) and (3.2) respectively, would give the corresponding results given earlier by Choi and Kumar [13].

Example 3.2. Show that the solution of

$$(3.7) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu + \frac{n}{2} - 1} H_n \left(\frac{1}{2\sqrt{\mathbf{t}}} \right) \Gamma_{u,v}^{r,s}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.8) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathbf{t}^k}{k! (n - 2k)!} \\ & \times \Gamma_{u+1,v+1}^{r,s+1} \left[\mathbf{b} \mathbf{t}^\eta \left| \begin{matrix} (a_1, A_1, x), (1 - \mu - k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1 - \mu - k - \nu i, \eta) \end{matrix} \right. \right], \end{aligned}$$

$$(3.9) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathbf{t}^{\mu + \frac{n}{2} - 1} H_n \left(\frac{1}{2\sqrt{\mathbf{t}}} \right) {}_p\Psi_q^{(\Gamma)}[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.10) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathbf{t}^k}{k! (n - 2k)!} \\ & \times {}_{p+1}\Psi_{q+1}^{(\Gamma)} \left[\begin{matrix} (a_1, A_1, x), (\mu + k, \eta), (a_j, A_j)_{2,p}; \\ (b_j, B_j)_{1,q}, (\mu + k + \nu i, \eta); \end{matrix} \mathbf{b} \mathbf{t}^\eta \right], \end{aligned}$$

$$(3.11) \quad \mathcal{N}(\mathbf{t}) - \mathcal{N}_0 \mathcal{C}_q^p \mathbf{t}^{\mu + \frac{n}{2} - 1} \mathbf{H}_n \left(\frac{1}{2\sqrt{\mathbf{t}}} \right) {}_p\Gamma_q[\mathbf{b} \mathbf{t}^\eta] = -c^\nu {}_0D_{\mathbf{t}}^{-\nu} \mathcal{N}(\mathbf{t}),$$

is provided as

$$(3.12) \quad \begin{aligned} \mathcal{N}(\mathbf{t}) = & \mathcal{N}_0 \mathbf{t}^{\mu-1} \sum_{i=0}^{\infty} (-c^\nu \mathbf{t}^\nu)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathbf{t}^k}{k! (n-2k)!} \\ & \times \mathcal{C}_{q+1}^{p+1} {}_p\Gamma_{q+1} \left[\begin{matrix} (a_1, x), \mu + k, a_2, a_3, \dots, a_p; \\ \mu + k + \nu i, b_1, b_2, \dots, b_q; \end{matrix} \mathbf{b} \mathbf{t}^\eta \right], \end{aligned}$$

with $\mathcal{C}_{q+1}^{p+1} = \mathcal{C}_q^p \frac{\Gamma(\mu+k)}{\Gamma(\mu+k+\nu i)}$.

Solution. Set $m = 2$, $\mathbf{a} = 1$, $\zeta = 1$ and $A_{n,s} = (-1)^s$ (i.e., $S_n^2[\mathbf{t}] = \mathbf{t}^{n/2} \mathbf{H}_n \left(\frac{1}{2\sqrt{\mathbf{t}}} \right)$, where $\mathbf{H}_n(\mathbf{t})$ is Hermite polynomial) in (2.1), (2.10) and (2.19). Thus, assertions (3.7), (3.9) and (3.11) of the example follow from the Theorem 2.1, Corollary 2.1 and Corollary 2.3, respectively.

Remark 3.2. As an application of the results (2.6), (2.12) and (2.21), a number of consequent results can be derived.

4. CONCLUDING REMARKS

Our attempt in this paper is to propose a new fractional generalization of the standard kinetic equation and to use the integral transformation approach to analyze its solution. A study of several interesting fractional kinetic equations and their solutions has been made, which include a family of polynomials and the incomplete H -function, incomplete Fox-Wright function and incomplete generalized hypergeometric function. The main results contained in the Theorem 2.1, Theorem 2.2 and their corollaries are of general nature. Analogously, various fractional kinetic equations and their solutions available in literature (see, [15–20]) can be obtained as special cases of the main results. Through addition, a number of recognized polynomials are produced by the polynomials family as their specific cases on a properly specialized connected sequence $A_{n,s}$. As a consequence, by providing appropriate basic values to the arbitrary sequences and the constraints, the main results can be used to generate a set of kinetic equations and their potential solutions. We intend to continue this study of the more generalized kinetic equations and their proposed solutions in the future work.

Acknowledgements. The authors are thankful to the referees for their valuable suggestions that helped in the considerable improvement in the quality of the paper.

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