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A PRODUCT FORMULA AND CERTAIN q-LAPLACE TYPE TRANSFORMS FOR THE q-HUMBERT FUNCTIONS

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Abstract. The present work deals with the mathematical investigation of the product formulas and several q-Laplace type integral transforms of certain q-Humbert functions. In our investigation, the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform of certain q^2-Humbert functions are considered. Several useful special cases have been deduced as applications of main results.

1. Introduction and preliminaries

Integral transforms have been widely used in many areas of science and engineering and therefore so much work has been done on the theory and applications of integral transforms. The integral transform method is a persuasive way to solve numerous differential equations. Thus, in the literature there are lots of works on several integral transforms such as Laplace, Fourier, Mellin, Hankel. Two of the most frequently used formulas in the area of integral transforms are the classical Laplace and Sumudu transform and their corresponding q-analogues, see for example [1–6, 20, 21]. The Laplace transform is the most popular and extensively used in applied mathematics. Yürekli and Sadek [24] introduced a new type of Laplace transform, known as the $\mathcal{L}_2$-transform. These transforms were studied in more details by Yürekli [22, 23]. After that Uçar and Albayrak [19] have investigated the q-analogue of this $\mathcal{L}_2$-transforms, which are called the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform and are defined as follows [19]:

$$(1.1)\qquad {}_qL_2\{f(t); s\} = \frac{1}{[2]_q} \cdot \frac{(q^2;q^2)_\infty}{s^2} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} f\left(q^n s^{-1}\right)$$

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and

$$ {}_q\mathcal{L}_2\{f(t);s\} = \frac{1}{[2]_q} \cdot \frac{1}{(-s^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} q^{2n} \left(-s^2;q^2\right)_n f(q^n), \tag{1.2} $$

respectively.

In order to better understand the work, some notations and preliminaries of the quantum theory are recollected. For any real number b, the q-analogues of the shifted factorial $(b)_s$ is given by [8, 16]:

$$ (b;q)_0 = 1, \quad (b;q)_s = \prod_{i=0}^{s-1}(1-q^i b), \quad (b;q)_\infty = \prod_{i=0}^{\infty}(1-q^i b), \quad b,q\in\mathbb{R},\ n\in\mathbb{N}, \tag{1.3} $$

and satisfy the following relations [7]:

$$ (q;q)_{s+l} = (q;q)_s\ (q^{s+1};q)_l, \tag{1.4} $$

$$ (q^{s+1};q)_\infty = (q^{l+s+1};q)_\infty\ (q^{s+1};q)_l. \tag{1.5} $$

The q-analogues of a complex number b is given by [8]:

$$ [b]_q = \frac{1-q^b}{1-q}, \quad q\in\mathbb{C}\setminus\{1\},\ b\in\mathbb{C}. \tag{1.6} $$

The q-exponential functions are defined as [8]:

$$ e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}, \quad E_q(u) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{u^n}{[n]_q!}. \tag{1.7} $$

These q-exponential functions are related as [8]:

$$ e_q(u)E_q(-u) = 1 \quad \text{and} \quad e_q(-u)E_q(u) = 1. \tag{1.8} $$

The q-gamma functions $\Gamma_q(\alpha)$ and ${}_q\Gamma(\alpha)$ have the following series representations [16]:

$$ \Gamma_q(\alpha) = \frac{(q;q)_\infty}{(1-q)^{\alpha-1}} \sum_{k=0}^{\infty} \frac{q^{k\alpha}}{(q;q)_k} = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha}, \tag{1.9} $$

$$ {}_q\Gamma(\alpha) = \frac{K(A;\alpha)}{(1-q)^{\alpha-1}(-(1/A);q)_\infty} \sum_{k\in\mathbb{Z}} \left(\frac{q^k}{A}\right)^\alpha \left(-\frac{1}{A};q\right)_k, \tag{1.10} $$

where $K(A;\alpha)$ is the following remarkable function [16]:

$$ K(A;\alpha) = A^{\alpha-1}\frac{(-q/\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty} \cdot \frac{(-\alpha;q)_\infty}{(-\alpha q^{1-t};q)_\infty}, \quad \alpha\in\mathbb{R}. \tag{1.11} $$

Investigating the q-analogues of the special functions and exploring their properties is a prevailing topic for mathematicians and physicists. It is familiar that the parameter q symbolize for "quantum", which is extensively used in quantum calculus (or q-calculus). For more details of quantum calculus, one can see the book of Kac and Cheung [8]. The theory of q-special functions play an indispensable role in the formalism of mathematical physics. The development in q-calculus has also led to the extension of several remarkable functions to their q-analogues, see for example [9–12, 17]. Recently, the q-analogues of the Humbert functions are introduced by Srivastava and Shehata [17] by means of the generating functions and series definitions.

The q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ are specified by means the following generating equation [17]:

$$e_q\left(\frac{xu}{3}\right)e_q\left(\frac{xt}{3}\right)e_q\left(-\frac{x}{3ut}\right)=\sum_{m,n=0}^{\infty}\mathfrak{J}^{(1)}_{m,n}(x|q)u^m t^n \tag{1.12}$$

and have the following series representation [17]:

$$\mathfrak{J}^{(1)}_{m,n}(x|q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\ (q;q)_{m+k}\ (q;q)_{n+k}}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.13}$$

The q-Humbert functions of the second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ are defined by the following series expansion [17]:

$$\mathfrak{J}^{(2)}_{m,n}(x|q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\ (q;q)_{m+k}\ (q;q)_{n+k}}q^{\frac{k}{2}(3k+2(m+n)-1)}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.14}$$

The q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ and second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ are related as [17]:

$$\mathfrak{J}^{(1)}_{m,n}\left(q^{\frac{1}{3}}x\middle|\frac{1}{q}\right) = q^{\frac{1}{3}(m+n)+\binom{m}{2}+\binom{n}{2}}\mathfrak{J}^{(2)}_{m,n}(x|q). \tag{1.15}$$

The series form of the q-Humbert functions of the third kind $\mathfrak{J}^{(3)}_{m,n}(x|q)$ is given as [17]:

$$\mathfrak{J}^{(3)}_{m,n}(x|q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k\ (q;q)_{m+k}\ (q;q)_{n+k}}q^{\binom{k+1}{k}}\left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{1.16}$$

Inspired by the works on the q-special functions in diverse fields, in this article, the product formulas for the q-Humbert functions of first, second and third kind are obtained. Certain q-Laplace type integral transforms are investigated for the q^2-Humbert functions of first, second and third kind. Some examples are considered in order to show effectiveness of the proposed results by taking some special cases.

2. Product Formula

The Product formulas for q-Bessel functions are investigated by Rahman [13], which are proved to be very useful in many branches of mathematics. After that, Swarttouw [18] derived the product formulas for the Hahn-Exton q-Bessel function, which opened the way to a rich harmonic analysis. Motivated by these works, the product formula for the generalized q-Bessel functions are also established in [14]. We follow the same technique of calculation developed by Swarttouw to derive a product formula for q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$.

Theorem 2.1. *Let $x > 0$, $\gamma, \delta > 0$ and $m, n, p, r \in \mathbb{N}$, then the following product formula for the q-Humbert functions of the first kind $\mathfrak{J}^{(1)}_{m,n}(x|q)$ holds true:*

(2.1)

$$\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) = B_{m,n,p,r}(x|q) \sum_{i=0}^{\infty} M_i(q) \left(-\frac{\delta x}{3}\right)^{3i} \times {}_3\varphi_2 \left(\begin{array}{c} q^{-p-i}, \quad q^{-r-i}, \ q^{-i}; \\ \\ q^{n+1}, \ q^{m+1}; \end{array} q; \ -\frac{\gamma^3}{\delta^3} q^{\frac{1}{2}(2(3i+p+r)+3(1-k))} \right),$$

where ${}_l\varphi_s$ is the basic hypergeometric function defined by [15]*:*

$$ {}_l\varphi_s \left(\begin{array}{c} a_1, a_2, \ldots, a_l; \\ \\ b_1, b_2, \ldots, b_s; \end{array} q; \ z \right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_l; q)_k z^k}{(q, b_1, b_2, \ldots, b_s; q)_k} \tag{2.2}$$

and $B_{m,n,p,r}(x|q) = \frac{(1-q)^{m+n+p+r}\gamma^{m+n}\delta^{p+r}}{(q;q)_m (q;q)_n (q;q)_p (q;q)_r} \left(\frac{x}{3}\right)^{m+n+p+r}$, $M_i(q) = \frac{(1-q)^{3i}}{(q^{p+1};q)_i (q^{r+1};q)_i (q;q)_i}$.

Proof. In view of series (1.13), we can write

$$\begin{aligned} &\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k \ (q;q)_{m+k} \ (q;q)_{n+k}} \\ &\times \left(\frac{(1-q)\gamma x}{3}\right)^{m+n+3k} \sum_{l=0}^{\infty} \frac{(-1)^l}{(q;q)_l \ (q;q)_{p+l} \ (q;q)_{r+l}} \left(\frac{(1-q)\delta x}{3}\right)^{p+r+3l}, \end{aligned} \tag{2.3}$$

which on using identity (1.4) becomes

$$\begin{aligned} &\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) \\ &= B_{m,n,p,r}(x|q) \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}(1-q)^{3k+3l}\gamma^{3k}\delta^{3l} \left(\frac{x^3}{27}\right)^{k+l}}{(q;q)_k \ (q^{m+1};q)_k \ (q^{n+1};q)_k (q;q)_l \ (q^{p+1};q)_l \ (q^{r+1};q)_l}, \end{aligned} \tag{2.4}$$

where $B_{m,n,p,r}(x|q) = \frac{(1-q)^{m+n+p+r}\gamma^{m+n}\delta^{p+r}}{(q;q)_m (q;q)_n (q;q)_p (q;q)_r} \left(\frac{x}{3}\right)^{m+n+p+r}$.

Replacing l by $i-k$ in equation (2.4), we get

$$\begin{aligned} \mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) = &B_{m,n,p,r}(x|q) \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \frac{(-1)^i (1-q)^{3i} \gamma^{3k} \delta^{3(i-k)}}{(q;q)_k \ (q^{m+1};q)_k \ (q^{n+1};q)_k} \\ &\times \frac{1}{(q;q)_{i-k} \ (q^{p+1};q)_{i-k} \ (q^{r+1};q)_{i-k}} \left(\frac{x^3}{27}\right)^i, \end{aligned} \tag{2.5}$$

which on using the following identity

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}$$

gives

$$\begin{aligned} &\mathfrak{J}^{(1)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(1)}_{p,r}(\delta x|q) \\ =&B_{m,n,p,r}(x|q) \sum_{i=0}^{\infty} \frac{(-1)^i \left(\frac{(1-q)\delta x}{3}\right)^{3i}}{(q;q)_i \ (q^{p+1};q)_i \ (q^{r+1};q)_i} \\ &\times \sum_{k=0}^{i} \left(-\frac{\gamma}{\delta}\right)^{3k} \frac{(q^{-p-i};q)_k \ (q^{-r-i};q)_k \ (q^{-i};q)_k}{(q;q)_k \ (q^{m+1};q)_k \ (q^{n+1};q)_k} \ q^{\frac{k}{2}(6i+2p+2r+3-3k)}. \end{aligned} \tag{2.6}$$

Letting $M_i(q) = \frac{(1-q)^{3i}}{(q^{p+1};q)_i (q^{r+1};q)_i (q;q)_i}$ and using equation (2.2) in equation (2.6), assertion (2.1) is proved. □

Similarly, we get the following product formulas for the q-Humbert functions of the second and third kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ and $\mathfrak{J}^{(3)}_{m,n}(x|q)$, respectively.

Remark 2.1. Let $x > 0$, $\gamma, \delta > 0$ and $m, n, p, r \in \mathbb{N}$, then the following product formula for the q-Humbert functions of the second kind $\mathfrak{J}^{(2)}_{m,n}(x|q)$ holds true:

(2.7)

$$\begin{aligned} \mathfrak{J}^{(2)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(2)}_{p,r}(\delta x|q) =& B_{m,n,p,r}(x|q) \sum_{i=0}^{\infty} N_i(q) \left(-\frac{\delta x}{3}\right)^{3i} \\ &\times {}_3\varphi_2 \left(\begin{matrix} q^{-p-i}, & q^{-r-i}, & q^{-i}; \\ & q^{n+1}, & q^{m+1}; \end{matrix} \ q; \ -\frac{\gamma^3}{\delta^3} q^{\frac{1}{2}(2(3i+p+r)+3(1-k))} \right), \end{aligned}$$

where $N_i(q) = \frac{(1-q)^{3i}}{(q^{p+1};q)_i (q^{r+1};q)_i (q;q)_i} q^{\frac{i}{2}(3i+2p+2r-1)}$ and $B_{m,n,p,r}(x|q)$ is same as earlier.

Remark 2.2. Let $x > 0$, $\gamma, \delta > 0$ and $m, n, p, r \in \mathbb{N}$, then the following product formula for the q-Humbert functions of the third kind $\mathfrak{J}^{(3)}_{m,n}(x|q)$ holds true:

(2.8)

$$\begin{aligned} \mathfrak{J}^{(3)}_{m,n}(\gamma x|q) \times \mathfrak{J}^{(3)}_{p,r}(\delta x|q) =& B_{m,n,p,r}(x|q) \sum_{i=0}^{\infty} L_i(q) \left(-\frac{\delta x}{3}\right)^{3i} \\ &\times {}_3\varphi_2 \left(\begin{matrix} q^{-p-i}, & q^{-r-i}, & q^{-i}; \\ & q^{n+1}, & q^{m+1}; \end{matrix} \ q; \ -\frac{\gamma^3}{\delta^3} q^{\frac{1}{2}(2(3i+p+r)+3(1-k))} \right), \end{aligned}$$

where $L_i(q) = \frac{(1-q)^{3i}}{(q^{p+1};q)_i (q^{r+1};q)_i (q;q)_i} q^{i+2}$ and $B_{m,n,p,r}(x|q)$ is same as earlier.

In the following section, the ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform for the q^2-Humbert functions are investigated.

3. ${}_qL_2$-Transform and ${}_q\mathcal{L}_2$-Transform

In this section, we evaluate ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform of t^{2w-2} weighted product of m different q^2-Humbert functions. The q^2-Humbert functions are more relevant than the original q-Humbert functions because of the mathematical nature of ${}_qL_2$-transform and ${}_q\mathcal{L}_2$-transform which contain q^2-shift factorials.

Theorem 3.1. *Let* $\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ $j=1,2,\dots,m$, *be a set of* q^2*-Humbert functions of first kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w,a_j,μ_j,ν_j *and* $j=1,2,\dots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is,*

$$\text{(3.1)}\qquad {}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}$$
$$=\prod_{j=1}^{m}B_j(s)\sum_{kj=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ H_{k_j}(q^2)\Gamma_{q^2}(w+\mu_j+\nu_j+k_j),$$

where $\operatorname{Re}(s)>0$, $\operatorname{Re}(w)>0$ *and*

$$\text{(3.2)}\qquad B_j(s)=\frac{(a_j)^{\mu_j+\nu_j}}{[2]_q\ 3^{\mu_j+\nu_j}\ s^{2(w+\mu_j+\nu_j)}},\qquad H_{k_j}(q)=\frac{(1-q)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q;q)_{k_j}\ (q;q)_{3\mu_j+k_j}\ (q;q)_{3\nu_j+k_j}}.$$

Proof. In order to prove the theorem, let $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ in equation (1.1), we get

$$\text{(3.3)}\qquad {}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}$$
$$=\frac{1}{[2]_q}\cdot\frac{(q^2;q^2)_\infty}{s^2}\prod_{j=1}^{m}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n}\Big(q^ns^{-1}\Big)^{2w-2}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}((a_jq^{2n}s^{-2})^{\frac{1}{3}}|\ q^2),$$

which in view of series expansion (1.13) becomes

$${}_qL_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}=\frac{1}{[2]_q}\cdot\frac{(q^2;q^2)_\infty}{s^{2w}}$$
$$\times\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n=0}^{\infty}\frac{q^{2nw}}{(q^2;q^2)_{k_j}\ (q^2;q^2)_n}\Big[\frac{(1-q^2)a_jq^{2n}s^{-2}}{3}\Big]^{\mu_j+\nu_j+k_j}$$
$$=\prod_{j=1}^{m}\frac{(q^2;q^2)_\infty\Big((1-q^2)a_j/3\Big)^{\mu_j+\nu_j}}{[2]_q\ s^{2(w+\mu_j+\nu_j)}}\sum_{n=0}^{\infty}\frac{q^{2n(w+\mu_j+\nu_j)}}{(q^2;q^2)_n}$$
$$\times\sum_{k_j=0}^{\infty}\frac{\Big(-\frac{a_j(1-q^2)(q^ns^{-1})^2}{3}\Big)^{k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}.$$

Letting $B_j(s) = \frac{(a_j)^{\mu_j+\nu_j}}{[2]_q\ 3^{\mu_j+\nu_j}\ s^{2(w+\mu_j+\nu_j)}}$ in the above equation, it follows that

$$
\begin{aligned}
&{}_qL_2\left\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2); s\right\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\frac{\left(\frac{-a_j}{3s^2}\right)^{k_j}(1-q^2)^{\mu_j+\nu_j+k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_\infty(q^2)^{n(w+\mu_j+\nu_j+k_j)}}{(q^2;q^2)_n},
\end{aligned}
$$

which on using relation (1.9) and setting $H_{k_j}(q) = \frac{(1-q)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q;q)_{k_j}\ (q;q)_{3\mu_j+k_j}\ (q;q)_{3\nu_j+k_j}}$ gives

$$
\begin{aligned}
&{}_qL_2\left\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2); s\right\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j).
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

The following corollaries are an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let* $\mathcal{J}^{(2)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j = 1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of second kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(2)}_{3\mu_j,3\nu_k}(3(a_jt)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j *and* $j = 1,2,\ldots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is*

$$
\begin{aligned}
&{}_qL_2\{f(t);s\}\\
&=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ (q^2)^{\frac{k_j}{2}(3k_j+6(\mu_j+\nu_j)-1)}H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j),
\end{aligned}
$$

where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$ *and* $B_j(s)$, $H_{k_j}(q)$ *are same as in equation* (3.2).

Corollary 3.2. *Let* $\mathcal{J}^{(3)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j = 1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of third kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(3)}_{3\mu_j,3\nu_k}(3(a_jt)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j *and* $j = 1,2,\ldots,m$, *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is*

$$
{}_qL_2\{f(t);s\}=\prod_{j=1}^{m}B_j(s)\sum_{k_j=0}^{\infty}\left(\frac{-a_j}{3s^2}\right)^{k_j}\ (q^2)^{k_j+1}\ H_{k_j}(q^2)\ \Gamma_{q^2}(w+\mu_j+\nu_j+k_j),
$$

where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$ *and* $B_j(s)$, $H_{k_j}(q)$ *are same as in equation* (3.2).

Next, the ${}_q\mathcal{L}_2$-transform for the q^2-Humbert functions are investigated.

Theorem 3.2. *Let* $\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, $j = 1,2,\ldots,m$, *be a set of* q^2*-Humbert functions of first kind and* $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$, *where* w, a_j, μ_j, ν_j

and $j = 1, 2, \ldots, m$, are constants then ${}_q\mathcal{L}_2$-transform of $f(t)$ is

(3.4)
$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&A_{q^2}(s)\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\\
&\times\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j}\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)},
\end{aligned}
$$

where $A_{q^2}(s) = \frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}$ and $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$.

Proof. Using $f(t) = t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$ in equation (1.2), it follows that

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
&=\frac{1}{[2]_q\ (-s^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}q^{2n}(-s^2;q^2)_n\times\prod_{j=1}^{m}(q^n)^{2w-2}\mathcal{J}^{(1)}_{3\mu_j,3\nu_j}((a_jq^{2n})^{\frac{1}{3}}|\ q^2),
\end{aligned}\tag{3.5}
$$

which in view of series expansion (1.13) becomes

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&\frac{1}{[2]_q\ (-s^2;q^2)_\infty}\prod_{j=1}^{m}\sum_{n\in\mathbb{Z}}q^{2nw}(-s^2;q^2)_n\\
&\times\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}}{(q^2;q^2)_{k_j}}\frac{1}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\left[\frac{(1-q^2)a_jq^{2n}}{3}\right]^{\mu_j+\nu_j+k_j}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (-s^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}q^{2n(w+\mu_j+\nu_j)}\sum_{k_j=0}^{\infty}\frac{(-s^2;q^2)_n\left(\frac{a_jq^{2n}}{3}\right)^{k_j}(1-q^2)^{\mu_j+\nu_j+k_j}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (-s^2;q^2)_\infty}\sum_{k_j=0}^{\infty}\frac{\left(\frac{a_j}{3}\right)^{k_j}}{(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}}\sum_{n\in\mathbb{Z}}\frac{(-s^2;q^2)_n\ q^{2n(w+\mu_j+\nu_j+k_j)}}{(q^2;q^2)_{k_j}\ (-s^2;q^2)_\infty}.
\end{aligned}
$$

Using relations (1.9) and (1.10) in the above equation, it follows that

$$
\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&\prod_{j=1}^{m}\frac{\left(\frac{a_j}{3}\right)^{\mu_j+\nu_j}}{[2]_q\ (s^{2(w+\mu_j+\nu_j)})}\sum_{k_j=0}^{\infty}\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}
\end{aligned}
$$

$$\times \frac{\left(\frac{-a_j}{3s^2}\right)^{k_j}(1-q^2)^{w+2(\mu_j+\nu_j+k_j)-1}}{(q^2;q^2)_{k_j}(q^2;q^2)_{3\mu_j+k_j}(q^2;q^2)_{3\nu_j+k_j}},$$

which in view of identity (1.5) gives

$$\begin{aligned}
&{}_q\mathcal{L}_2\Big\{t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(1)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2);s\Big\}\\
=&\frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}\left(\frac{a_j}{3s^2}\right)^{\mu_j+\nu_j+k_j}}{\left(\frac{(q^2;q^2)_\infty}{(q^{2(3\mu_j+k_j+1)};q^2)_\infty}\right)}\\
&\times\frac{(1-q^2)^{2(\mu_j+\nu_j+k_j)}}{\left(\frac{(q^2;q^2)_\infty}{(q^{2(3\nu_j+k_j+1)};q^2)_\infty}\right)\left(\frac{(q^2;q^2)_\infty}{(q^{2(k_j+1)};q^2)_\infty}\right)}\cdot\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}.
\end{aligned}$$

Letting $A_{q^2}(s)=\frac{(1-q^2)^{w-1}}{[2]_q\ s^{2w}}$ in the above equation and in view of expression (1.9), assertion (3.4) follows. ☐

The following corollaries are an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let* $\mathcal{J}^{(2)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$*,* $j=1,2,\ldots,m$*, be a set of* q^2*-Humbert functions of second kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(2)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$*, where* w,a_j,μ_j,ν_j *and* $j=1,2,\ldots,m$*, are constants then* ${}_q\mathcal{L}_2$*-transform of* $f(t)$ *is*

$$\begin{aligned}
{}_q\mathcal{L}_2\{f(t);s\}=&A_{q^2}(s)\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}(q^2)^{\frac{k_j}{2}(3k_j+6(\mu_j+\nu_j)-1)}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)}\\
&\times\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j},
\end{aligned}$$

where $\mathrm{Re}(s)>0$*,* $\mathrm{Re}(w)>0$ *and* $A_{q^2}(s)$ *is same as in Theorem* 3.2.

Corollary 3.4. *Let* $\mathcal{J}^{(3)}_{3\mu_j,3\nu_j}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$*,* $j=1,2,\ldots,m$*, be a set of* q^2*-Humbert functions of third kind and* $f(t)=t^{2w-2}\prod_{j=1}^{m}\mathcal{J}^{(3)}_{3\mu_j,3\nu_k}(3(a_jt^2)^{\frac{1}{3}}|\ q^2)$*, where* w,a_j,μ_j,ν_j *and* $j=1,2,\ldots,m$*, are constants then* ${}_q\mathcal{L}_2$*-transform of* $f(t)$ *is,*

$$\begin{aligned}
{}_q\mathcal{L}_2\{f(t);s\}=&A_{q^2}(s)\prod_{j=1}^{m}\sum_{k_j=0}^{\infty}\frac{(-1)^{k_j}(q^2)^{k_j+1}}{\Gamma_{q^2}(k_j+1)\Gamma_{q^2}(3\mu_j+k_j+1)\Gamma_{q^2}(3\nu_j+k_j+1)}\\
&\times\frac{\Gamma_{q^2}(w+\mu_j+\nu_j+k_j)}{K(1/s^2,w+\mu_j+\nu_j+k_j)}\left(\frac{a_j}{3s^2(1-q^2)}\right)^{\mu_j+\nu_j+k_j},
\end{aligned}$$

where $\mathrm{Re}(s)>0$*,* $\mathrm{Re}(w)>0$ *and* $A_{q^2}(s)$ *is same as in Theorem* 3.2.

In the next section, we give certain examples to show the applications of the results established in previous sections.

4. Special Cases

In consideration of $m=1$, $a_1=a$, $k_1=k$, $\mu_1=\mu$ and $\nu_1=\nu$ in Theorem 3.1, Corollary 3.1 and Corollary 3.2, respectively, the ${}_qL_2$-transforms for the q-Humbert functions of the first, second and third kind are obtained:

$$ {}_qL_2\{t^{2w-2}\mathcal{J}^{(1)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} = \mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k \; H_k(q^2)\;\Gamma_{q^2}(w+\mu+\nu+k), $$

$$ {}_qL_2\{t^{2w-2}\mathcal{J}^{(2)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} $$
$$ =\mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k \; (q^2)^{\frac{k}{2}(3k+6(\mu+\nu)-1)}H_k(q^2)\;\Gamma_{q^2}(w+\mu+\nu+k) $$

and

$$ {}_qL_2\{t^{2w-2}\mathcal{J}^{(3)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} = \mathcal{C}(s)\sum_{k=0}^{\infty}\left(\frac{-a}{3s^2}\right)^k \; (q^2)^{k+1}\; H_k(q^2)\;\Gamma_{q^2}(w+\mu+\nu+k), $$

respectively, where $\mathrm{Re}(s)>0$, $\mathrm{Re}(w)>0$ and

$$ \mathcal{C}(s)=\frac{(a)^{\mu+\nu}}{[2]_q\;3^{\mu+\nu}\;s^{2(w+\mu+\nu)}},\quad H_k(q)=\frac{(1-q)^{w+2(\mu+\nu+k)-1}}{(q;q)_k\;(q;q)_{3\mu+k}\;(q;q)_{3\nu+k}}. $$

Taking $m=1$, $a_1=a$, $k_1=k$, $\mu_1=\mu$ and $\nu_1=\nu$ in Theorem 3.2, Corollary 3.3 and Corollary 3.4, respectively, the ${}_q\mathcal{L}_2$-transforms for the q-Humbert functions of the first, second and third kind are obtained:

$$ {}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(1)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} $$
$$ =A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)} $$
$$ \times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k}, $$

$$ {}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(2)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} $$
$$ =A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k(q^2)^{\frac{k}{2}(3k+6(\mu+\nu)-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)} $$
$$ \times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k} $$

and

$$ {}_q\mathcal{L}_2\{t^{2w-2}\mathcal{J}^{(3)}_{3\mu,3\nu}(3(at^2)^{\frac{1}{3}}|\, q^2);s\} $$
$$ =A_{q^2}(s)\sum_{k=0}^{\infty}\frac{(-1)^k(q^2)^{k+1}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(3\mu+k+1)} $$
$$ \times\frac{\Gamma_{q^2}(w+\mu+\nu+k)}{\Gamma_{q^2}(3\nu+k+1)K(1/s^2,w+\mu+\nu+k)}\left(\frac{a}{3s^2(1-q^2)}\right)^{\mu+\nu+k}, $$

respectively, where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(w) > 0$ and $A_{q^2}(s)$ is same as in Theorem 3.2.

In the present investigation, we have constructed the product formulas and certain q-Laplace type integral transforms for the q-Humbert functions of first, second and third kind. The results established in this article might be useful for solving q^2-difference equations by means of the ${}_qL_2$-transforms and ${}_q\mathcal{L}_2$-transforms. In the forthcoming paper, we plan to deal with constructing q^2-difference equations to use the results obtained here.

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