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# Contents

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# STRONG CONVERGENCE RESULTS FOR VARIATIONAL INEQUALITY AND EQUILIBRIUM PROBLEM IN HADAMARD SPACES

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Abstract. The main purpose of this paper is to introduce and study a viscosity type algorithm in a Hadamard space which comprises of a demimetric mapping, a finite family of inverse strongly monotone mappings and an equilibrium problem for a bifunction. Strong convergence of the proposed algorithm to a common solution of variational inequality problem, fixed point problem and equilibrium problem is established in Hadamard spaces. Nontrivial Applications and numerical examples were given. Our results compliment some results in the literature.

## 1. Introduction

Let $X$ be a metric space and $C$ be a nonempty closed and convex subset of $X$. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if

$$Tx = x. \tag{1.1}$$

The set of fixed points of $T$ is denoted by F(T). With the recent rapid developments in fixed point theory, there has been a renewed interest in iterative schemes. The properties of iterations between the type of sequences and kind of operators have not been completely studied and are now under discussion. The theory of operators has occupied a central place in modern research using iterative schemes because of its promise of enormous utility in fixed point theory and its applications. In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate

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fixed point is more than enough, an approximate solution plays an important role in such situations. The theory of fixed points and consequently of approximate fixed points finds application in mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equations and several other areas of applicable analysis see for instance [9, 17, 23, 25, 26, 31, 33–35, 40, 45].

The mapping $T : C \to X$ is said to be:

(a) nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \text{for all } x, y \in C;$$

(b) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, q) \leq d(x, q), \quad \text{for all } x \in C \text{ and } q \in F(T);$$

(c) firmly nonexpansive if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle, \quad \text{for all } x, y \in C;$$

(d) $\alpha$-inverse strongly monotone if there exists $\alpha > 0$ such that

$$d^2(x, y) - \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \geq \alpha \Psi_T(x, y), \quad \text{for all } x, y \in C, \tag{1.2}$$

where $\Psi_T(x, y) = d^2(x, y) + d^2(Tx, Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$. It was established in [3] that the quantity $\Psi_T(x, y)$ is nonnegative.

Given a nonempty set $C$ and $f : C \times C \to \mathbb{R}$ a bifunction, the Equilibrium Problem (EP) is defined as follows:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.3}$$

The point $x^*$ in (1.3) is called an equilibrium point of $f$. We shall denote the solution set of problem (1.3) by $\mathrm{EP}(f, C)$. EPs have been widely studied in Hilbert, Banach and topological vector spaces [6, 12, 24] and Hadamard manifolds [11, 41]. One of the most popular and effective methods used for solving problem (1.3) and other related optimization problems is the Proximal Point Algorithm (PPA) which was introduced in a Hilbert space by Martinet [37] and was further studied by Rockafellar [47] in 1976. The PPA and its generalizations have also been studied extensively in Banach spaces and Hadamard manifolds (see [11, 36] and the references therein). Recently, many convergence results by the PPA for solving optimization problems were extended from the classical linear spaces to the setting of nonlinear space such as Riemannain manifolds and Hadamard spaces (see [4, 5, 10, 19, 46, 54] and reference therein). Numerous applications in computer vision, machine learning, electronic structure computation, system balancing, and robot manipulation can be reduced to find solution of optimization and equilibrium problems in nonlinear setting (see [1, 2, 27, 43, 50, 53]).

Very recently, Kumam and Chaipunya [36] studied EP (1.3) in Hadamard spaces. They established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity and coercivity assumptions ([36], Theorem 4.1). They also established some fundamental properties of the resolvent of a bifunction. Furthermore, they proved that the PPA $\Delta$-converges to an equilibrium point of a monotone bifunction in a Hadamard space. More precisely, they proved:

**Theorem 1.1.** ([36, Theorem 7.3]) *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f : C \times C \to \mathbb{R}$ be a monotone, $\Delta$-upper semicontinuous in the first variable such that $D(J_r^f) \supset C$ for all $r > 0$ where $D$ stands for the domain. Suppose that $EP(f, C) \neq \emptyset$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{r_n}^f(x_{n-1}), \quad n \in \mathbb{N},$$

*where $\{r_n\}$ is a sequence of positive real numbers bounded away from 0. Then $\{x_n\}$ $\Delta$-converges to an element of $EP(f, C)$.*

The Variational Inequality Problem (VIP) was first introduced by Stampacchia [49] for modeling problems arising in mechanics. To study the regularity problem for partial differential equations, Stampacchia [49] studied a generalization of the Lax-Milgram theorem and called all problems of this kind to be VIPs. The theory of VIP has numerous applications in diverse fields such as physics, engineering, economics, mathematical programming and others (see [8, 32, 39] and references therein). The VIP in a real Hilbert space $H$ is formulated as follows:

$$\text{find } x \in C \text{ such that } \langle Tx, y - x\rangle \geq 0, \quad \text{for all } y \in C, \tag{1.4}$$

where $C$ is a nonempty closed and convex subset of $H$ and $T$ is a nonlinear mapping defined on $C$. This formulation is recently extended to the framework of CAT(0) space $X$ by Alizadeh-Dehghan-Moradlou [3] as follows:

$$\text{find } x \in C \text{ such that } \langle \overrightarrow{Txx}, \overrightarrow{xy}\rangle \geq 0, \quad \text{for all } y \in C, \tag{1.5}$$

where $\overrightarrow{xy}$ stands for a vector in $X$ defined in (2.1).

They established the existence of VIP (1.5) when $T$ is an inverse strongly monotone mapping in a CAT(0) space. Furthermore, they introduced the following iterative algorithm for solving VIP (1.5): For arbitrary $x_1 \in C$, generate sequence $\{x_n\}$ as

$$\begin{cases} y_n = P_C(\beta_n x_n \oplus (1 - \beta_n)Tx_n), \\ x_{n+1} = P_C(\alpha_n x_n \oplus (1 - \alpha_n)Sy_n), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$, $S$ and $T$ are nonexpansive and inverse strongly monotone mappings, respectively. They also obtained $\Delta$-convergence of Algorithm (1.6) to a solution of the VIP (1.5), which is also a fixed point of the nonexpansive mapping $S$.

*Remark* 1.1. If $X = H$ is a real Hilbert space, then $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle b - a, d - c\rangle$ for all $a, b, c, d \in H$. Thus, the VIP (1.6) reduces to the VIP (1.5) when $X = H$.

Motivated by the work of Kumam and Chaipunya [36] and Alizadeh-Dehghan -Moradlou [3], we introduce and study a viscosity type algorithm which comprises of demimetric mapping, equilibrium problem for a monotone bifunction and a finite family of inverse strongly monotone mappings. Strong convergence of the proposed algorithm to common solution of a fixed point of a demimetric mapping, an equilibrium problem of a bifunction and variational inequality problem for a finite family of certain monotone mappings is established in a Hadamard space $X$. Furthermore, we applied our results to approximate solutions of minimization problems in $X$.

## 2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. Throughout this paper, we shall denote the strong and $\Delta$-convergence by $\longrightarrow$ and $\rightharpoonup$, respectively.

Let $(X, d)$ be a metric space and $x, y \in X$. A geodesic path joining $x$ to $y$ (or, a geodesic from $x$ to $y$) is a map $\gamma : [a, b] \subseteq \mathbb{R} \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$, and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [a, b]$. In particular, $\gamma$ is an isometry and $d(x, y) = b - a$. We say that a metric space X is uniquely geodesic if every two points of X are joined by only one geodesic segment (i.e., CAT(0) space). Examples of CAT(0) spaces are Euclidean spaces $\mathbb{R}^n$ and Hilbert spaces. For more details, please see [12, 20, 21, 28, 48]. Complete CAT(0) spaces are often called Hadamard spaces.

Let $(1-t)x \oplus ty$ denote the unique point $z$ in the geodesic segment joining $x$ to $y$ for each $x, y$ in a CAT(0) space such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, where $t \in [0, 1]$. Let $[x; y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$, then a subset $C$ of $X$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

In 2008, Breg and Nikolaev [6] introduced the concept of quailinearization mapping in CAT(0) spaces. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ which they called a vector and defined a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad a, b, c, d \in X, \tag{2.1}$$

called the quasilinearization mapping. It is easy to verify that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. It has been established that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [6]). Recall that the space $X$ is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$ for all $a, b, c, d \in X$.

Let $\{x_n\}$ be a bounded sequence in CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\})\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see [16, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to $\Delta$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

**Definition 2.1** ([4])**.** Let $X$ be a CAT(0) space and $C$ be a nonempty closed and convex subset of $X$. A mapping $T : C \to X$ is said to be $k$-demimetric if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$, such that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx), \quad \text{for all } x \in X \text{ and } y \in F(T). \tag{2.2}$$

**Definition 2.2.** Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$. The metric projection $P_C : X \to C$ assigns to each $x \in X$, the unique point $P_C x$ in $C$ such that

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}.$$

The map $P_C$ is nonexpansive [13].

**Definition 2.3.** Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$. A mapping $T : C \to C$ is said to be $\Delta$-demiclosed, if for any bounded sequence $\{x_n\}$ in $X$ such that $\Delta - \lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

**Lemma 2.1** ([4])**.** *Let $X$ be a CAT(0) space and $S : X \to X$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ with $F(S) \neq \emptyset$ and $\lambda \in (0, 1)$. Suppose that $S_\lambda = \lambda x \oplus (1-\lambda)Sx$. Then $S_\lambda$ is quasi-nonexpansive and $F(S_\lambda) = F(S)$.*

In [36], the authors introduce resolvent of a bifunction $f$ associated with the EP (1.3). They defined a perturbation bifunction $\bar{f}_x : C \times C \to \mathbb{R}$ of $f$ by

$$\bar{f}_x(x, y) := f(x, y) - \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \quad \text{for all } x, y \in C. \tag{2.3}$$

The perturbed bifunction $\bar{f}$ has a unique equilibrium point called resolvent operator $J^f : X \to 2^C$ of the bifunction $f$ (see [36]) and is defined by

$$\begin{aligned} J^f(x) &:= EP(C, \bar{f}_x) = \{z \in C : f(z, y) - \langle \overrightarrow{z\bar{x}}, \overrightarrow{zy} \rangle \geq 0,\, y \in C\} \\ &= \left\{ z \in C : f(z, y) + \frac{1}{2}(d^2(x, y) - d^2(x, z) - d^2(y, z)) \geq 0 \text{ for all } y \in C \right\}, \end{aligned} \tag{2.4}$$

$x \in X$. It was established in [36] that $J^f$ is well-defined.

**Lemma 2.2** ([36])**.** *Suppose that $f$ is monotone and $D(J^f) \neq \emptyset$. Then, the following properties hold.*

(i) *$J^f$ is singled-valued.*
(ii) *If $D(J^f) \supset C$, then $J^f$ is nonexpansive restricted to $C$.*
(iii) *If $D(J^f) \supset C$, $F(J^f) = EP(C, f)$.*

**Lemma 2.3** ([36])**.** *Suppose that $f$ has the following properties:*

(A1) $f(x,x) = 0$ *for all* $x \in C$;
(A2) $f$ *is monotone, i.e.,* $f(x,y) + f(y,x) \leq 0$ *for all* $x, y \in C$;
(A3) *for each* $x \in C$, $y \mapsto f(x,y)$ *is convex and lower semicontinuous;*
(A4) *for each* $x \in C$, $f(x,y) \geq \limsup_{t \downarrow 0} f((1-t)x \oplus tz, y)$ *for all* $x, z \in C$.

*Then* $D(J^f) = X$ *and* $J^f$ *is single-valued.*

*Remark* 2.1 ([23]). It follows from (2.4) that the resolvent $J_r^f$ of the bifunction $f$ $(r > 0)$ is given by

$$J_r^f(x) := EP(C, \bar{f}_x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle \overrightarrow{xz}, \overrightarrow{zy} \rangle \geq 0,\ y \in C \right\}, \quad x \in X, \tag{2.5}$$

where $\bar{f}$ in this case is defined as

$$\bar{f}_x(x,y) := f(x,y) + \frac{1}{r}\langle \overrightarrow{\bar{x}x}, \overrightarrow{xy} \rangle, \quad \text{for all } x, y \in C,\ \bar{x} \in X. \tag{2.6}$$

**Lemma 2.4** ([23]). *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f : C \times C \to \mathbb{R}$ be a monotone bifunction such that $C \subset D(J_r^f)$ for $r > 0$. Then, $J_r^f$ is firmly nonexpansive restricted to $C$. That is*

$$d^2(J_r^f x, J_r^f y) \leq \langle \overrightarrow{xy}, \overrightarrow{J_r^f x J_r^f y} \rangle. \tag{2.7}$$

**Lemma 2.5** ([15]). *Every bounded sequence in a Hadamard space always has a $\Delta$-convergent subsequence.*

**Lemma 2.6** ([29]). *Let $X$ be a Hadamard space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ $\Delta-$ converges to $x$ if and only if $\limsup\limits_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

**Lemma 2.7** ([56]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,$$

*where*
(*i*) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$;
(*ii*) $\limsup \sigma_n \leq 0$;
(*iii*) $\gamma_n \geq 0$, $n \geq 0$, $\sum \gamma_n < \infty$.

*Then $a_n \to 0$ as $n \to \infty$.*

**Lemma 2.8** ([3]). *Let $C$ be a nonempty closed and convex subset of Hadamard space $X$ and $T : C \to X$ be an $\alpha$-inverse strongly monotone mapping. Assume $\mu \in [0,1]$ and define $T_\mu : C \to X$ by $T_\mu x = (1-\mu)x \oplus \mu Tx$. If $0 < \mu < 2\alpha$, then $T_\mu$ is nonexpansive mapping and $F(T_\mu) = F(T)$.*

**Lemma 2.9** ([3]). *Let $C$ be a nonempty bounded closed and convex subset of a Hadamard space $X$ and $T : C \to X$ be an $\alpha$-inverse strongly monotone. Then $VI(C,T)$ is nonempty, closed and convex.*

**Lemma 2.10.** ([7, Lemma 3]) *Let $X$ be a uniformly convex hyperbolic space with modulus of uniform convexity $\eta$. For any $c > 0$, $\epsilon \in (0, 2]$, $\lambda \in [0, 1]$ and $v, x, y \in X$, $d(x, v) \leq c$, $d(y, v) \leq c$ and $d(x, y) \geq \epsilon c$ implies that*

$$d((1-\lambda)x \oplus \lambda y, v) \leq (1 - 2\lambda(1-\lambda)\eta(c, \epsilon))c.$$

If $X$ is a CAT(0) space, then $X$ is uniformly convex hyperbolic space ([30]).

**Lemma 2.11** ([3])**.** *Let $C$ be a nonempty convex subset of a Hadamard space $X$ and $T : C \to X$ be a mapping. Then,*

$$VI(C, T) = VI(C, T_\mu),$$

*where $\mu \in (0, 1]$ and $T_\mu : C \to X$ is a mapping defined by $T_\mu x = (1-\mu)x \oplus \mu T x$ for all $x \in C$.*

*Remark* 2.2 ([42])*.* It follows from Lemma 2.11 that

$$F(P_C T) = VI(C, T) = VI(C, T_\mu) = F(P_C T_\mu).$$

**Lemma 2.12.** *Let $X$ be a CAT(0) space, $x, y, z \in X$ and $t \in [0, 1]$. Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$ (see [15])*;*
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$ (see [15])*;*
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x, z) + (1-t)^2 d^2(y, z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [13]).

**Lemma 2.13** ([51])**.** *Let $X$ be a CAT(0) space, $\{x_i : i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$, be such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, z\right) \leq \sum_{i=1}^N \alpha_i d(x_i, z), \quad \textit{for all } x \in X.$$

**Lemma 2.14** ([14])**.** *Let $X$ be a CAT(0) space, $\{x_i : i = 1, 2, \ldots, N\} \subset X$, $\{y_i : i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$, be such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, \bigoplus_{i=1}^N \alpha_i y_i\right) \leq \sum_{i=1}^N \alpha_i d(x_i, y_i). \tag{2.8}$$

**Lemma 2.15** ([17])**.** *Let $X$ be a Hadamard space and $S : X \to X$ be a nonexpansive mapping. Then the conditions $\{x_n\}$ $\Delta$-converges to $x$ and $d(x_n, Sx_n) \to 0$, imply $x = Sx$.*

**Lemma 2.16** ([38])**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$. and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

*In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

## 3. Main Results

We begin with a technical results which will be used to prove our main results.

**Lemma 3.1** ([42]). *Let $C$ be a nonempty closed and convex subset of a $CAT(0)$ space $X$, $T_i : C \to X$, $i = 1, 2, \ldots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings and $\Psi_\mu : C \to C$ be defined by $\Psi_\mu x := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} x$ for all $x \in C$ and $\beta_i \in (0,1)$, where $T_{\mu_i} x := (1-\mu_i)x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$ with $\mu_i \in [0,1]$. If $\sum_{i=1}^N \beta_i = 1$, then the mapping $\Psi_\mu$ is nonexpansive. If in addition, $\cap_{i=1}^N F(P_C T_{\mu_i}) \neq \emptyset$, then $F(\Psi_\mu) = \bigcap_{i=1}^N F(P_C T_{\mu_i})$.*

**Proposition 3.1** ([23]). *Let $X$ be a Hadamard space and $f : C \times C \to \mathbb{R}$ be a monotone bifunction operator. Then*

$$d^2(u, J_r^f x) + d^2(J_r^f x, x) \leq d^2(u, x), \quad \text{for all } u \in F(J_r^f),\ x \in X \text{ and } r > 0.$$

**Theorem 3.1.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $f : C \times C \to \mathbb{R}$ be a monotone and upper semicontinuous bifunction such that conditions (A1)-(A4) of Lemma 2.3 are satisfied, $C \subset D(J_r^f)$ for $r > 0$ and $T_i : C \to X$, $i = 1, 2, \ldots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0,1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0,1)$. Suppose that $\Upsilon := F(S) \cap EP(f, C) \cap \left( \bigcap_{i=1}^N VI(C, T_i) \right)$ is nonempty and $\{x_n\}$ is a sequence generated by an arbitrary $x_1 \in X$ as follows:*

$$\begin{cases} u_n = J_{r_n}^f x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1-\alpha_n)[\beta_n y_n \oplus (1-\beta_n) S_\lambda y_n], \quad n \geq 1, \end{cases} \tag{3.1}$$

*where $S_\lambda x = \lambda x \oplus (1-\lambda) Sx$ is $\Delta$-demiclosed and $T_{\mu_i} x = (1-\mu_i)x \oplus \mu_i T_i x$, $0 < \mu_i < 2\beta_i$, for each $i = 1, 2, \ldots, N$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{\beta_i\} \subset (0,1)$ and $r_n \in (0, \infty)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Upsilon$, where $p = P_\Upsilon h(p)$.*

*Proof.* Let $p \in F(S) \cap EP(f, C) \cap \left( \bigcap_{i=1}^N VI(C, T_i) \right)$. By Lemma 3.1, we have that $\Psi_\mu$ is nonexpansive, that is,

$$d(y_n, p) = d(\Psi_\mu u_n, p) \leq d(u_n, p). \tag{3.2}$$

Since $J_{r_n}^f$ is firmly nonexpansive, we have

$$d(u_n, p) = d(J_{r_n}^f(x_n), p) \leq d(x_n, p). \tag{3.3}$$

Let $v_n = \beta_n y_n \oplus (1-\beta_n) S_\lambda y_n$, then we obtain

$$d(v_n, p) = d(\beta_n y_n \oplus (1-\beta_n) S_\lambda y_n, p)$$

$$
\begin{aligned}
&\leq \beta_n d(y_n, p) + (1-\beta_n) d(S_\lambda y_n, p) \\
&\leq \beta_n d(y_n, p) + (1-\beta_n) d(y_n, p) \\
&= d(y_n, p). \qquad (3.4)
\end{aligned}
$$

It follows from (3.1), (3.3) and Lemma 2.12 (i) that

$$
\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n h(x_n) \oplus (1-\alpha_n)) v_n, p) \\
&\leq \alpha_n d(h(x_n), p) + (1-\alpha_n) d(v_n, p) \\
&\leq \alpha_n d(h(x_n), p) + (1-\alpha_n) d(y_n, p) \\
&\leq \alpha_n d(h(x_n), h(p)) + \alpha_n d(h(p), p) + (1-\alpha_n) d(x_n, p) \\
&\leq \alpha_n \theta d(x_n, p) + \alpha_n d(h(p), p) + (1-\alpha_n) d(x_n, p) \\
&= [1 - \alpha_n(1-\theta)] d(x_n, p) + \alpha_n (1-\theta) \frac{d(h(p), p)}{1-\theta} \\
&\leq \max \left\{ d(x_n, p), \frac{d(h(p), p)}{1-\theta} \right\}.
\end{aligned}
$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{u_n\}$, $\{J^f_{r_n} x_n\}$ and $\{S_\lambda y_n\}$ are all bounded.

We now divide the rest of the proof into two cases.

**Case 1.** Suppose that $\{d(x_n, p)\}$ is monotonically non-increasing. Then there exists $\lim_{n\to\infty} \{d(x_n, p)\}$. This shows that

$$
\lim_{n\to\infty} [d(x_{n+1}, p) - d(x_n, p)] = 0. \qquad (3.5)
$$

Hence, we obtain from (3.1), Lemma 2.12 (ii), (3.2) and Proposition 3.1 that

$$
\begin{aligned}
d^2(x_{n+1}, p) &\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(v_n, p) - \alpha_n (1-\alpha_n) d^2(h(x_n), v_n) \\
&\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(y_n, p) \\
&\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(u_n, p) \\
&\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) \left[ d^2(y_n, p) - d^2(u_n, y_n) \right] \\
&= \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(x_n, p) - (1-\alpha_n) d^2(u_n, y_n). \qquad (3.6)
\end{aligned}
$$

From (3.6), we get

$$
(1-\alpha_n) d^2(y_n, u_n) \leq \alpha_n d^2(h(x_n), p) + d^2(x_n, p) - d^2(x_{n+1}, p).
$$

Hence, we obtain from (3.5) and condition (i) that

$$
\lim_{n\to\infty} d(y_n, u_n) = 0 = \lim_{n\to\infty} d(\Psi_\mu u_n, u_n). \qquad (3.7)
$$

Also, from (3.1), Lemma 2.12 (ii) and Proposition 3.1 we get

$$
\begin{aligned}
d^2(x_{n+1}, p) &\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(v_n, p) - \alpha_n (1-\alpha_n) d^2(h(x_n), v_n) \\
&\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(y_n, p) \\
&\leq \alpha_n d^2(h(x_n), p) + (1-\alpha_n) d^2(u_n, p)
\end{aligned}
$$

$$\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n)\left[d^2(x_n,p) - d^2(u_n,x_n)\right]$$

$$= \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(x_n,p) - (1-\alpha_n)d^2(u_n,x_n). \tag{3.8}$$

Thus,

$$(1-\alpha_n)d^2(u_n,x_n) \leq \alpha_n d^2(h(x_n),p) + d^2(x_n,p) - d^2(x_{n+1},p).$$

Hence, from condition (i) and (3.5), we have

$$\lim_{n\to\infty} d(u_n,x_n) = \lim_{n\to\infty} d(J^f_{r_n}x_n,x_n) = 0. \tag{3.9}$$

By (3.1) and Lemma 2.12 (ii), we get

$$\begin{aligned}
d^2(x_{n+1},p) \leq& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(v_n,p) - \alpha_n(1-\alpha_n)d^2(h(x_n),v_n)\\
\leq& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)[\beta_n d^2(y_n,p) + (1-\beta_n)d^2(S_\lambda y_n,p)\\
&- \beta_n(1-\beta_n)d^2(y_n,S_\lambda y_n)]\\
\leq& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(x_n,p) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(y_n,S_\lambda y_n).
\end{aligned}$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)d^2(S_\lambda y_n,y_n) \leq \alpha_n[d^2(h(x_n),p) - d^2(x_n,p)] + d^2(x_n,p) - d^2(x_{n+1},p).$$

By condition (i) and (3.5), we obtain

$$\lim_{n\to\infty} d(S_\lambda y_n,y_n) = 0. \tag{3.10}$$

Also, by (3.1), (3.7) and (3.9)

$$d(y_n,x_n) \leq d(\Psi_\mu u_n,u_n) + d(u_n,x_n) \to 0, \quad n\to\infty. \tag{3.11}$$

We also obtain from (3.1) and condition (i) that

$$d(x_{n+1},v_n) = d(\alpha_n h(x_n) \oplus (1-\alpha_n)v_n, v_n) \leq \alpha_n d(h(x_n),v_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.12}$$

Also, from the definition of $v_n$ and (3.10), we obtain

$$d(v_n,y_n) \leq \beta_n d(y_n,y_n) + (1-\beta_n)d(S_\lambda y_n,y_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.13}$$

Thus, from (3.11), (3.12) and (3.13), we get

$$d(x_{n+1},x_n) \leq d(x_{n+1},v_n) + d(v_n,y_n) + d(y_n,x_n) \to 0. \tag{3.14}$$

Next we show that

$$\limsup\langle\overrightarrow{h(z)z}, \overrightarrow{x_n z}\rangle \leq 0.$$

As $\{u_n\}$ is bounded, so by Lemma 2.5, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = z$. Also since $\Psi_\mu$ is nonexpansive, we obtain from (3.7), Lemma 2.15, Lemma 3.1 and Remark 2.2 that $z \in F(\Psi_\mu) = \cap_{i=1}^N F(P_C T_{\mu_i}) = \cap_{i=1}^N VI(C,T_{\mu_i})$. Let us show that $z \in EP(f,C)$. Since $\{J^f_{r_n}(x_n)\}$ is bounded, there exists a subsequence $\{w_k\}$ of $\{J^f_{r_n}(x_n)\}$ such that

$$\lim_{k\to\infty} d(w_k,p) = \liminf_{n\to\infty} d(J^f_{r_n}x_n,p)$$

and that $\{w_k\}$ $\Delta$-converges to some $z \in X$, where $w_k = J^f_{r_{n_k}} x_{n_k}$ for all $k \in \mathbb{N}$. By the definition of the resolvent $J^f_{r_n}$, we have

$$r_{n_k} f(w_k, y) + \frac{1}{2}(d^2(x_{n_k}, y) - d^2(x_{n_k}, w_k) - d^2(y, w_k)) \geq 0,$$

for all $y \in C$. In particular, letting $y = J^f z$, we have

$$d^2(x_{n_k}, J^f z) - d^2(x_{n_k}, w_k) - d^2(J^f z, w_k) \geq -2r_{n_k} f(w_k, J^f z).$$

Similarly, by the definition of $J^f$, we have

$$d^2(w_k, z) - d^2(J^f z, z) - d^2(J^f z, w_k) \geq -2f(w_k, J^f z).$$

Since $f$ is monotone, we have

$$\begin{aligned} &d^2(J^f z, x_{n_k}) - d^2(w_k, x_{n_k}) - d^2(w_k, J^f z) - r_{n_k} d^2(w_k, z) - r_{n_k} d^2(J^f z, z) \\ &- r_{n_k} d^2(J^f z, w_k) \geq 0, \end{aligned}$$

and hence

$$\begin{aligned} (1 + r_{n_k}) d^2(J^f z, w_k) &\leq d^2(J^f z, x_{n_k}) - d^2(w_k, x_{n_k}) + r_{n_k} d^2(w_k, z) - r_{n_k} d^2(J^f z, z) \\ &\leq d^2(J^f z, x_{n_k}) + d^2(w_k, z). \end{aligned}$$

It follows that

$$\begin{aligned} d^2(J^f z, w_k) &\leq \frac{1}{r_{n_k}}(d^2(J^f z, x_{n_k}) - d^2(J^f z, w_k) + d^2(z, w_k)) \\ &\leq \frac{1}{r_{n_k}} d(w_k, x_{n_k})(d(J^f z, x_{n_k}) - d(J^f z, w_k)) + d^2(z, w_k), \end{aligned}$$

for all $k \in \mathbb{N}$ and consequently, we obtain

$$\limsup_{k\to\infty} d^2(J^f z, w_k) \leq \limsup_{k\to\infty} d^2(z, w_k).$$

Since the asymptotic center of $\{w_k\}$ is unique point $z$, we have $z = J^f z$, that is, $z \in EP(C, f)$.

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = z$. It follows from (3.11) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta - \lim_{k\to\infty} y_{n_k} = z$. Since $S_\lambda$ is $\Delta$-demiclosed, it follows from (3.10) and Lemma 2.1 that $z \in F(S_\lambda) = F(S)$. Hence, $z \in \Upsilon := F(S) \cap EP(f, C) \cap \bigcap_{i=1}^N VI(C, T_{\mu_i})$.

Observe that

$$\limsup_{n\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_n z} \rangle \leq \limsup_{k\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_{n_k} z} \rangle. \tag{3.15}$$

Since $\{x_{n_k}\}$ $\Delta$-converges to $z$, therefore by Lemma 2.6, we have

$$\limsup_{k\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_n z} \rangle \leq 0.$$

This together with (3.15) gives

$$\limsup_{n\to\infty}\langle\overrightarrow{h(z)z},\overrightarrow{x_nz}\rangle \leq 0. \tag{3.16}$$

Also, we have

$$\begin{aligned}\langle\overrightarrow{h(z)z},\overrightarrow{zx_{n+1}}\rangle &= \langle\overrightarrow{h(z)z},\overrightarrow{zx_n}\rangle + \langle\overrightarrow{h(z)z},\overrightarrow{x_nx_{n+1}}\rangle\\ &= \langle\overrightarrow{h(z)z},\overrightarrow{zx_n}\rangle + d(z,h(z))d(x_n,x_{n+1}).\end{aligned} \tag{3.17}$$

Hence from (3.14), (3.16) and (3.17) we have that

$$\limsup_{n\to\infty}\langle\overrightarrow{h(z)z},\overrightarrow{zx_{n+1}}\rangle \leq 0. \tag{3.18}$$

Finally, we prove that $x_n \to z$ as $n\to\infty$. For any $n\in\mathbb{N}$, we set $\vartheta_n = \alpha_n z\oplus(1-\alpha_n)v_n$,

$$\begin{aligned}
d^2(x_{n+1},z) =& d^2(\alpha_n h(x_n)\oplus(1-\alpha_n)v_n, z)\\
\leq& d^2(\vartheta_n,z) + 2\langle\overrightarrow{x_{n+1}\vartheta_n},\overrightarrow{x_{n+1}z}\rangle\\
\leq& [\alpha_n d(z,z) + (1-\alpha_n)d(v_n,z)]^2\\
&+ 2\left[\alpha_n\langle\overrightarrow{h(x_n)\vartheta_n},\overrightarrow{x_{n+1}z}\rangle + (1-\alpha_n)\langle\overrightarrow{v_n\vartheta_n},\overrightarrow{x_{n+1}z}\rangle\right]\\
\leq& (1-\alpha_n)^2d^2(v_n,z) + 2\Big[\alpha_n^2\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\
&+ \alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle + (1-\alpha_n)^2\langle\overrightarrow{v_nv_n},\overrightarrow{x_{n+1}z}\rangle\Big]\\
\leq& (1-\alpha_n)^2d^2(y_n,z) + 2\Big[\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\
&+ \alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle + (1-\alpha_n)^2d(v_n,v_n)d(x_{n+1},z)\Big]\\
\leq& (1-\alpha_n)^2d^2(x_n,z) + 2\Big[\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\
&+ \alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle\Big]\\
\leq& (1-\alpha_n)^2d^2(x_n,z) + 2\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle\\
\leq& (1-\alpha_n)^2d^2(x_n,z) + 2\alpha_n\langle\overrightarrow{h(x_n)h(z)},\overrightarrow{x_{n+1}z}\rangle + 2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle\\
\leq& (1-\alpha_n)^2d^2(x_n,z) + 2\alpha_n\theta d(x_n,z)d(x_{n+1},z) + 2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle\\
\leq& (1-\alpha_n)^2d^2(x_n,z) + 2\alpha_n\theta\left(d^2(x_n,z) + d^2(x_{n+1},z)\right)\\
&+ 2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle.
\end{aligned}$$

As $\{\alpha_n\}$ and $\{x_n\}$ are bounded so there is $M>0$ such that $\frac{1}{1-\theta\alpha_n}d^2(x_n,z)\leq M$. It now follows that

$$d^2(x_{n+1},z) \leq \frac{(1-\alpha_n)^2+\theta\alpha_n}{1-\theta\alpha_n}d^2(x_n,z) + \frac{2\alpha_n}{1-\theta\alpha_n}\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle$$

$$
\begin{aligned}
&\leq \frac{(1-\alpha_n)^2+\theta\alpha_n}{1-\theta\alpha_n} d^2(x_n,z) + \frac{2\alpha_n}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n^2 M \\
&\leq \left[1-\frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}\right] d^2(x_n,z) + \frac{2\alpha_n}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n^2 M \\
&\leq \left[1-\frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}\right] d^2(x_n,z) \\
&\quad + \alpha_n\left[\frac{2}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n M\right]. \qquad (3.19)
\end{aligned}
$$

Set $\gamma_n = \frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}$, $\delta_n = \alpha_n\left[\frac{2}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n M\right]$. Now it follows from (3.18), (3.19) and Lemma 2.7 that $\{x_n\}$ converges strongly to $z$.

**Case 2.** Suppose that $\{d(x_n,p)\}$ is monotonically non-decreasing. There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $d(x_{n_j},z) < d(x_{n_j+1},z)$ for all $j \in \mathbb{N}$. Then by Lemma 2.16, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$.

$$
d^2(x_{m_k},z) \leq d^2(x_{m_k+1},z) \quad \text{and} \quad d^2(x_k,z) \leq d^2(x_{m_k+1},z), \qquad (3.20)
$$

for all $k \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
0 &\leq \liminf_{k\to\infty}\,[d(x_{m_k+1},z) - d(x_{m_k,z})] \\
&\leq \limsup_{k\to\infty}\,[d(x_{m_k+1},z) - d(x_{m_k},z)] \\
&\leq \limsup_{k\to\infty}\,[\alpha_{m_k} d(h(x_{m_k}),z) + (1-\alpha_{m_k})d(v_{m_k},z) - d(x_{m_k},z)] \\
&\leq \limsup_{k\to\infty}\,[\alpha_{m_k} d(h(x_{m_k}),z) + (1-\alpha_{m_k})d(x_{m_k},z) - d(x_{m_k},z)] \\
&= \limsup_{k\to\infty}\,[\alpha_{m_k}\left(d(h(x_{m_k}),z) - d(x_{m_k},z)\right)] = 0.
\end{aligned}
$$

This implies that

$$
\lim_{k\to\infty}\,[d(x_{m_k+1},z) - d(x_{m_k},z)] = 0. \qquad (3.21)
$$

By an argument as in **Case 1**, we get

$$
\limsup_{k\to\infty}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{m_k+1}z}\rangle \leq 0 \qquad (3.22)
$$

and

$$
d^2(x_{m_k+1},z) \leq (1-\gamma_{m_k})d^2(x_{m_k},z) + \gamma_{m_k}\delta_{m_k}.
$$

Since $d^2(x_{m_k},z) \leq d^2(x_{m_k+1},z)$ we get

$$
\gamma_{m_k} d^2(x_{m_k},z) \leq d^2(x_{m_k},z) - d^2(x_{m_k+1},z) + \gamma_{m_k}\delta_{m_k} \leq \gamma_{m_k}\delta_{m_k}. \qquad (3.23)
$$

Thus, from (3.20), we get

$$
\lim_{k\to\infty} d^2(x_{m_k},z) = 0. \qquad (3.24)
$$

It follows from (3.20), (3.22) and (3.24) the $\lim_{k\to\infty} d^2(x_k, z) = 0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges strongly to $z \in \Upsilon$. □

**Lemma 3.2.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f_j : C \times C \to \mathbb{R}$, $j = 1, 2, \dots, m$, be a finite family of monotone bifunctions such that* (A1)-(A4) *are satisfied. Then for $r > 0$, we have $F\left(\bigcap_{j=1}^m J_r^{f_m}\right) = \cap_{i=1}^m (J_r^{f_m})$, where*

$$\bigcap_{j=1}^{m} J_r^{f_j} = J_r^{fm} \circ J_r^{f_{m-1}} \circ \cdots \circ J_r^{f_2} \circ J_r^{f_1}.$$

The proof of Lemma 3.2, follows immediately from the proof of Theorem 3.1 in [55].

**Theorem 3.2.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $f_j : C \times C \to \mathbb{R}$, $j = 1, 2, \dots, m$, be monotone and upper semicontinuous bifunctions such that conditions* (A1)-(A4) *are satisfied, $C \subset D(J_r^f)$ for $r > 0$ and $T_i : C \to X$, $i = 1, 2, \dots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0, 1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Gamma := F(S) \cap EP(f_j, C) \cap \cap_{i=1}^N VI(C, T_i)$ is nonempty and $\{x_n\}$ is the sequence generated by an arbitrary $x_1 \in X$ as:*

$$\begin{cases} u_n = \Pi_{j=1}^m J_{r_n}^{f_j} x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n)[\beta_n y_n \oplus (1 - \beta_n) S_\lambda y_n], \quad n \geq 1, \end{cases} \tag{3.25}$$

*where $S_\lambda x = \lambda x \oplus (1 - \lambda) Sx$ is $\Delta$-demiclosed, $T_{\mu_i} x = (1 - \mu_i) x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$, for each $i = 1, 2, \dots, N$, and $\bigcap_{j=1}^m J_r^{f_j} = J_r^{fm} \circ J_r^{f_{m-1}} \circ \cdots \circ J_r^{f_2} \circ J_r^{f_1}$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, $\{\beta_i\} \subset (0, 1)$ and $r_n \in (0, \infty)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma h(p)$.*

*Proof.* Follows immediately from Theorem 3.1 and Lemma 3.2. □

## 4. Application to Minimization Problems

In this section, we give an application of our results to solve Minimization Problems. Let $X$ be a Hadamard space and $f : X \to (-\infty, \infty]$ be a proper and convex function. The problems in optimization require to find $x \in X$ such that

$$f(x) = \arg\min_{y \in X} g(y).$$

So $\arg\min_{y \in X} g(y)$ denotes the set of minimizers of $g$.

Let $v : X \to \mathbb{R}$ be a proper convex and lower semicontinuous function. Consider the bifunction $f_v : C \times C \to \mathbb{R}$ defined by

$$f_v(x, y) = v(y) - v(x), \quad \text{for all } x, y \in C.$$

Then, $f_v$ is monotone and upper semi continuous (see [3]). Moreover, $EP(f_v, C) = \arg\min_C v$, $J^{f_v} = prox^v$ and $D(prox^v) = X$ (see [3]), where $prox^v : X \to X$ is given by

$$prox^v(x) := \arg\min_{x \in X} \left[ v(y) + \frac{1}{2} d^2(y, x) \right], \quad \text{for all } x \in X.$$

Now we consider the following minimization and fixed point problems:

(4.1)

$$\text{find } x \in F(S) \cap F(\Psi_\mu) \text{ such that } v(x) \leq v(y), \quad \text{for all } y \in C,\ i = 1, 2, \ldots, m,$$

where $S$ is a demimetric mapping and $\Psi_\mu$ is as defined in Lemma 3.1.

Let us denote the solution set of problem (4.1) by $\Omega$.

**Theorem 4.1.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $v_j : X \to \mathbb{R}$, $j = 1, 2, \ldots, m$, be proper convex lower semicontinuous functions and $T_i : C \to X$, $i = 1, 2, \ldots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0, 1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Omega$ is nonempty and $\{x_n\}$ is the sequence generated by an arbitrary $x_1 \in X$ as*

$$\begin{cases} u_n = \Pi_{j=1}^m prox_{r_n}^{v_i} x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n)[\beta_n y_n \oplus (1 - \beta_n) S_\lambda y_n], \quad n \geq 1, \end{cases} \tag{4.2}$$

*where $S_\lambda x = \lambda x \oplus (1 - \lambda) S x$ is $\Delta$-demiclosed and $T_{\mu_i} x = (1 - \mu_i) x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$, for each $i = 1, 2, \ldots, N$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, $\{\beta_i\} \subset (0, 1)$ and $r_n \in (0, \infty)$ satisfy the following conditions:*

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega h(p)$.*

*Proof.* Set $J_{r_n}^{f_i} = prox_{r_n}^{v_i}$ in Algorithm 3.25 and apply Theorem 3.2 to approximate solutions of problem (4.1). □

*Remark* 4.1. (i) If we replace $h(x_n)$ by "$u$" (for arbitrary $u$) in our Algorithm 3.1 and Algorithm 3.25 (which are viscosity type), then we get the Halpern-type algorithm and the conclusion of our theorems still hold. However, we use a viscosity-type algorithm instead of Halpern-type algorithm due to the fact that viscosity-type algorithms have higher rate of convergence than Halpern-type.

(ii) A characterization of metric projection goes as follows:

$$p = P_\Gamma h(p) \Leftrightarrow \langle \overrightarrow{ph(p)}, \overrightarrow{yp} \rangle \geq 0, \quad \text{for all } y \in C. \tag{4.3}$$

Therefore, one advantage of adopting Algorithm 3.1 for our convergence analysis, is that it also converges to the variational inequality (4.3) (see for example [22]).

(iii) In Theorem 1.1, $\Delta$-convergence to an element of $EP(f, C)$ was obtained while we obtained strong convergence result which is also a solution of some variational inequality problems. Hence, Theorem (3.1) provides genuine extension of Theorem 1.1.

(iv) Theorem 4.1 generalizes Theorem 10 of [52] and Theorem 3.1 of [44] from Hilbert space to CAT(0) spaces.

## 5. Numerical Example

*Example* 5.1. We give numerical in $(\mathbb{R}^2, \|\cdot\|_2)$ (where $\mathbb{R}^2$ is the Euclidean plane) to support our main result.

Let $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ defined by

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}, \quad x, y \in \mathbb{R}^2.$$

Then $(\mathbb{R}^2, \rho)$ is an Hadamard space (see, for instance, [18, Example 5.2]) with geodesic joining $x$ to $y$ given by

(5.1)
$$(1-t)x \oplus ty = ((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2).$$

Now, define $\Phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Phi(x_1, x_2) = (100(x_2 - 2) - (x_1 - 2)^2)^2 + (x_1 - 3)^2.$$

Then, it follows from [18, Example 5.2] that $\Phi$ is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, \rho)$ but not convex in the classical sense.

Let $S : \mathbb{R}^2 \to \mathbb{R}$ be defined by $S\bar{x} = S(x_1, x_2) = (-2x_1, 3x_1^2 + x_2)$. Then $S$ is 3-generalized demimetric mapping in the sense $\rho$ with $F(S) = (0, 0)$, $\lambda = \frac{1}{4}$.

Let $X = \mathbb{R}^2$ and be an $R$-tree with radical metric $d_r$, where $d_r(x, y) = d(x, y)$ if $x$ and $y$ are situated on a Euclidean straight line passing through the origin and $d_r(x, y) = d(x, 0) + d(y, 0)$, otherwise. We put $p = (0, 1)$, $q = (1, 0)$ and $C = A \cup B \cup D$, where $A = \{(0, t) : t \in [2/3, 1]\}$, $B = \{(t, 0) : t \in [2/3, 1]\}$, $D = \{(t, s) : t + s = 1, t \in (0, 1)\}$ and defined $T : C \to C$ by

$$Tx := \begin{cases} q, & \text{if } x \in A, \\ p, & \text{if } x \in B, \\ x, & \text{if } x \in D. \end{cases} \tag{5.2}$$

Then, $T$ is $\frac{1}{4}-$ inverse strongly monotone in $(X, d_r)$ but not inverse strongly monotone in the classical sense.

In what follows, we choose $r_n = \frac{1}{5}$, $\beta_i = \frac{1}{N}$, $\mu_i = 0.035$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{3n}{5n+2}$ for $n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$. We study the behaviour of the sequence generated by Algorithm 3.1 for following initial values with $N = 10$.

Case I: $x_0 = (-2, -7)'$,

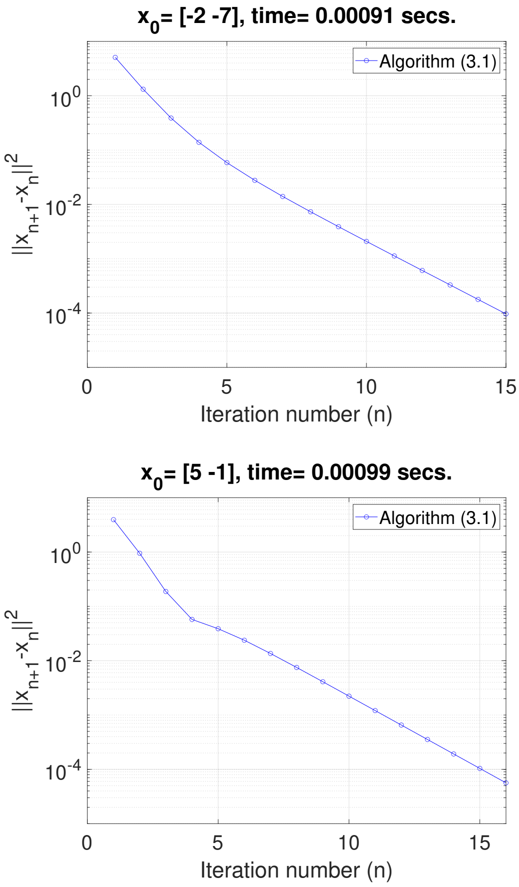


Figure 1. Example 5.1: Case I – Case II.

Case II: $x_0 = (5, -1)'$,
Case III: $x_0 = (3, 6)'$,
Case IV: $x_0 = (-4, 1)'$.

We also used $\|x_{n+1} - x_n\|^2 < 10^{-4}$ as stopping criterion and plot the graphs of error $||x_{n+1} - x_n||^2$ against number of iteration in each case. The computation results are shown in Figure 1–2. The numerical results show that the change in the initial values does not have significant effects on the number of iteration and CPU time taken for computation by Algorithm 3.1.

## 6. Conclusion

In this paper, we investigate a priori on the resolvent operator for a given bifunction, demimetric mapping and a finite family of inverse strongly monotone mappings. Main results here are that the resolvent operator here is single-valued and firmly nonexpansive. We then define proximal viscosity algorithm by iterating the resolvent of different bifurcating parameters. Strong convergence of the proposed algorithm to

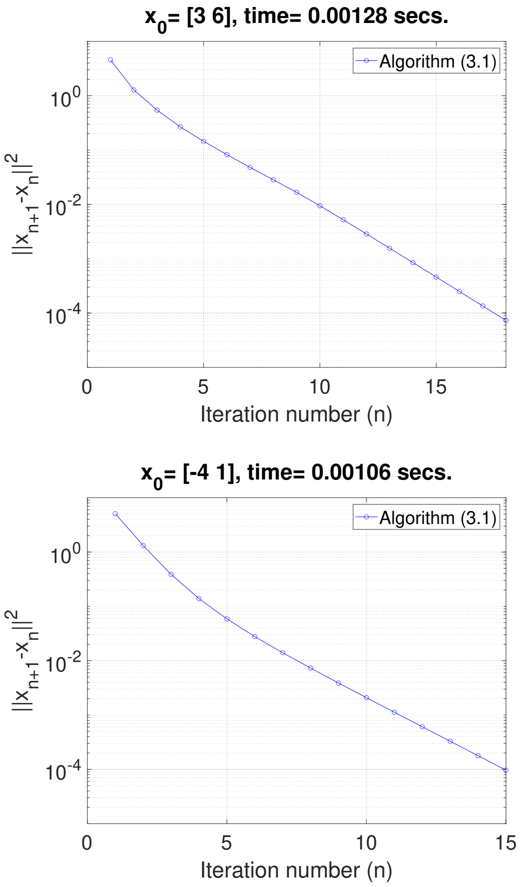


Figure 2. Example 5.1: Case III – Case IV

a common solution of variational inequality problem, fixed point problem and equilibrium problem is established in Hadamard spaces. Some applications and numerical example were also given.

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# ON A DETERMINANTAL FORMULA FOR DERANGEMENT NUMBERS

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Abstract. The aim of this note is to provide succinct proofs for a recent formula of the derangement numbers in terms of the determinant of a tridiagonal matrix.

## 1. Preliminaries

The $n$th derangement number $!n$, also known as subfactorial of $n$, is the number of permutations on $n$ elements, such that no element appears in its original position, i.e., is a permutation that has no fixed points.

Derangement numbers were first combinatorially studied by the French mathematician and Fellow of the Royal Society, Pierre Rémond de Montmort in his celebrated book *Essay d'analyse sur les jeux de hazard* published in 1708.

The two well-known recurrence relations

$$!n = (n-1)(!(n-1)+!(n-2)), \quad \text{for } n \geqslant 2, \tag{1.1}$$

and

$$!n = n\,(!(n-1)) + (-1)^n, \quad \text{for } n \geqslant 1, \tag{1.2}$$

with $!0 = 1$ and $!1 = 0$, were established and proved by Euler. They can be written in the explicit forms

$$!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i!,$$

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which coincide with the permanent of the all ones matrix minus the identity matrix, all of order $n$ [4].

The arithmetic properties of the sequence of derangements are very interesting, as we can find in [5]. There, they are studied in terms of the periodicity modulo a positive integer, $p$-adic valuations, and prime divisors. We can also find attractive relations to other number sequences. For example, in [11], for any prime number $p$ co-prime with a positive integer $m$, we have

$$\sum_{0<k<p} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1}\, !(m-1) \pmod{p},$$

where $B_k$ denotes the $k$th Bell number.

Among the most relevant generalizations we have the so-called $r$-derangement numbers [12], when some of the elements are restricted to be in distinct cycles in the cycle decomposition. For more details on this matter, recent formulas, and interpretations, the reader is referred to [1, 6, 10].

The first terms of this sequence are

$$1\,,0\,,1\,,2\,,9\,,44\,,265\,,1854\,,14833\,,133496\,,1334961\,,14684570$$

and it was coined by The On-Line Encyclopedia of Integer Sequences [9] as the sequence A000166.

Another interesting representation of the derangement numbers is in terms of the determinant of a certain family tridiagonal matrices. Kittappa [3] and Janjić [2] showed independently two similar formulas:

$$!(n+1) = \begin{vmatrix} 2 & -1 & & & \\ 3 & 3 & -1 & & \\ & 4 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & n & n \end{vmatrix}, \tag{1.3}$$

for $n \geqslant 2$, and

$$!(n+1) = \begin{vmatrix} 1 & -1 & & & & \\ 1 & 1 & -1 & & & \\ & 3 & 3 & -1 & & \\ & & 4 & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & n & n \end{vmatrix}, \tag{1.4}$$

for any positive integer $n$, respectively. Subtracting to the second row the first one, in (1.4), it is a straightforward exercise to check that both representations are exactly the same. Moreover they trivially satisfy (1.1)–(1.2).

In two recent replicated papers [7, 8], Qi, Wang, and Guo claim the discovery of a new representation for the derangement numbers in terms of the determinant of a new tridiagonal matrix. The aim of this short note is to show that this can be proven using elementary matrix theory and the above well-known representations.

## 2. Derangement Numbers and Tridiagonal Matrices

In [7, 8] it is simultaneously claimed the discovery of a new representation for $!n$ in terms of the determinant of the tridiagonal matrix of order $n+1$, namely,

$$!n = -\begin{vmatrix} -1 & -1 & & & & & \\ 0 & 0 & -1 & & & & \\ & 1 & 1 & -1 & & & \\ & & 2 & 2 & -1 & & \\ & & & 3 & \ddots & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & n-1 & n-1 \end{vmatrix}, \tag{2.1}$$

for any nonnegative integer. The proof is intricate and based on the higher derivatives of the generating function of $!n$.

However, using the elementary operations on rows $R_i$ and columns $C_i$

$$R_1 \leftarrow -R_1\,, \quad C_2 \leftarrow C_2 - C_1\,, \quad R_4 \leftarrow R_4 + 2R_2\,, \quad C_4 \leftarrow C_4 + C_2\,,$$

it follows that (2.1) equals

$$\left|\begin{array}{c|cc|cccc} 1 & & & & & & \\ \hline & 0 & -1 & & & & \\ & 1 & 1 & & & & \\ \hline & & & 2 & -1 & & \\ & & & 3 & \ddots & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & n-1 & n-1 \end{array}\right|$$

and this determinant is exactly (1.3).

Yet, there is also another way to check (2.1). For, expanding of the determinant along last row (or column) we immediately get (1.3). The conclusion now follows from the fact that for $n=0$ and $n=1$ the determinant (2.1) is, respectively, 1 and 0.

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# STABILITY OF AN $l$-VARIABLE CUBIC FUNCTIONAL EQUATION

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Abstract. Using the direct and fixed point methods, we obtain the solution and prove the Hyers-Ulam stability of the $l$-variable cubic functional equation

$$\begin{aligned}&f\left(\sum_{i=1}^{l} x_i\right)+\sum_{j=1}^{l} f\left(-lx_j+\sum_{i=1,i\neq j}^{l} x_i\right)\\ =&-2(l+1)\sum_{i=1,i\neq j\neq k}^{l} f(x_i+x_j+x_k)+(3l^2-2l-5)\sum_{i=1,i\neq j}^{l} f(x_i+x_j)\\ &-3(l^3-l^2-l+1)\sum_{i=1}^{l} f(x_i),\end{aligned}$$

$l\in\mathbb{N}$, $l\geq 3$, in random normed spaces.

## 1. Introduction

The theory of random normed space (briefly, RN-space) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. It is a practical tool for handling situations where classical theories fail to explain. Random theory has much application in several fields, for example, population dynamics, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence and so forth. The Cauchy additive equation

$$f(x+y)=f(x)+f(y)$$

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has been studied by many authors [7, 9, 13, 15, 19]. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping (see [5, 8, 11, 12, 20]). A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [23] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [4] noticed that the theorem of Skof [23] is still true if the relevant domain $X$ is replaced by an Abelian group. Jun and Kim [14] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.1}$$

and they established the solution and the Hyers-Ulam stability for the functional equation. The function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation (see [3, 6, 10, 16, 21, 22]). Czerwik [5] proved the Hyers-Ulam stability of the additive, quadratic and cubic functional equation.

Using the direct and fixed point methods, we obtain the solution and prove the Hyers-Ulam stability of the $l$-variable cubic functional equation

$$\begin{aligned} &f\left(\sum_{i=1}^{l} x_i\right)+\sum_{j=1}^{l} f\left(-lx_j+\sum_{i=1,i\neq j}^{l} x_i\right) \\ &=-2(l+1)\sum_{i=1,i\neq j\neq k}^{l} f(x_i+x_j+x_k)+(3l^2-2l-5)\sum_{i=1,i\neq j}^{l} f(x_i+x_j) \\ &\quad-3(l^3-l^2-l+1)\sum_{i=1}^{l} f(x_i), \end{aligned} \tag{1.2}$$

$l\in\mathbb{N}$, $l\geq 3$, in random normed spaces.

## 2. Preliminaries

In this section, we present some notations and basic definitions used in this article.

**Definition 2.1.** A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm if $T$ satisfies the following condition:

a) $T$ is commutative and associative;
b) $T$ is continuous;
c) $T(a,1)=a$ for all $a\in[0,1]$;
d) $T(a,b)\leq T(c,d)$ when $a\leq c$ and $b\leq d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous $t$-norms are $T_p(a,b)=ab$, $T_m(a,b)=\min\{a,b\}$ and $T_L(a,b)=\max\{a+b-1,0\}$ (The Lukasiewicz $t$-norm). Recall [7] that if $T$ is a $t$-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n}x_{n+i}$ is defined recurrently by $T_{i=1}^{1}x_i = x_i$ and $T_{i=1}^{n}x_i = T\left(T_{i=1}^{n-1}x_i, x_n\right)$ for $n \geq 2$, $T_{i=1}^{\infty}x_i$ is defined as $T_{i=1}^{\infty}x_{n+i}$. It is known that, for the Lukasiewicz $t$-norm, the following holds:

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

**Definition 2.2.** A random normed space (briefly, RN-space) is a triple, where $X$ is a vector space. $T$ is a continuous $t$-norm and $\mu$ is a mapping from $X$ into $D^+$ satisfying the following conditions:

(RN1) $\mu_x(t) = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{ax}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

**Definition 2.3.** Let $(X, \mu, T)$ be an RN-space.

1) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ for all $n > N$.
2) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x_m}(\epsilon) > 1-\lambda$ for all $n \geq m \geq N$.
3) The RN-space $(X, \mu, T)$ is said to be complete if every Cauchy sequence in $X$ is convergent to a point in $X$. For more details we can go through [1, 2, 4, 13, 18].

Throughout this paper, assume that $X$ is a vector space and $(Y, \mu, T)$ is a complete random normed space. All over this paper we use the following notation for a given mapping $f : X \to Y$

$$\begin{aligned} Df(x_1, \dots, x_l) =& f\left(\sum_{i=1}^{l} x_i\right) + \sum_{j=1}^{l} f\left(-lx_j + \sum_{i=1, i\neq j}^{l} x_i\right) \\ &+ 2(l+1)\sum_{i=1, i\neq j\neq k}^{l} f(x_i + x_j + x_k) - (3l^2 - 2l - 5)\sum_{i=1, i\neq j}^{l} f(x_i + x_j) \\ &+ 3(l^3 - l^2 - l + 1)\sum_{i=1}^{l} f(x_i), \end{aligned}$$

for all $x_1, x_2, x_3, \dots, x_l \in X$.

## 3. Solution of the $l$-Variable Cubic Functional Equation in (1.2)

In this section, we investigate the solution of the $l$-variable cubic functional equation (1.2).

**Lemma 3.1.** *If a mapping $f : X \to Y$ satisfies* (1.2)*, then the mapping $f : X \to Y$ is cubic.*

*Proof.* Letting $x_1 = x_2 = \cdots = x_l = 0$ in (1.2), we get

$$\begin{aligned}2f(0) = &-2(l+1)\left(1+3(l-3)+\frac{3(l-3)(l-4)}{2}+\frac{(l-3)(l-4)(l-5)}{6}\right)f(0)\\ &+(3l^2-2l-5)\left(3+3(l-3)+\frac{(l-3)(l-4)}{2}\right)f(0)\\ &-3l(l^3-l^2-l+2)f(0). \qquad (3.1)\end{aligned}$$

It follows from (3.1) that $f(0) = 0$. Setting $x_1 = x_3 = \cdots = x_l = 0$ and $x_2 = x$ in (1.2), we have

$$\begin{aligned}lf(x)+f(-lx) = &-2(l+1)\left(1+2(l-3)+\frac{(l-3)(l-4)}{2}\right)f(x)\\ &+(3l^2-2l-5)\left(1+2(l-3)+\frac{(l-3)(l-4)}{2}\right)f(x)\\ &-3(l-1)(l^3-l^2-l+1)f(x), \qquad (3.2)\end{aligned}$$

for all $x \in X$. It follows from (3.2) that

$$f(-x) = -f(x), \qquad (3.3)$$

for all $x \in X$. Letting $x_2 = x_3 = \cdots = x_l = 0$ and $x_1 = x$ in (1.2), we get

$$\begin{aligned}lf(x)+f(-lx) =&(3l^2-2l-5)\left(1+2(l-3)+\frac{l^2-7l+12}{2}\right)f(x)\\ &-3(l-1)(l^3-l^2-l+1)f(x)\\ &-2(l+1)\left(1+2(l-3)+\frac{l^2-7l+12}{2}\right)f(x), \qquad (3.4)\end{aligned}$$

for all $x \in X$. It follows from (3.4) and the oddness of $f$ that

$$f(lx) = l^3 f(x), \qquad (3.5)$$

for all $x \in X$. Letting $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_l = 0$, we get

$$\begin{aligned}&(l-1)f(x)+2f((-l+1)x)\\ =&-2(l+1)(l-2)f(2x)-2(l+1)\left(2(n-3)+\frac{2(l-3)(l-4)}{2}\right)f(x)\\ &+(3l^2-2l-5)f(2x)+2(3l^2-2l-5)(n-2)f(x)-6(l^3-l^2-l+1)f(x), \qquad (3.6)\end{aligned}$$

for all $x \in X$. It follows from (3.6), (3.5) and the oddness of $f$ that

$$f(2x) = 8f(x), \qquad (3.7)$$

for all $x \in X$. Setting $x_1 = x_2 = x_3 = x$ and $x_4 = x_5 = \cdots = x_l = 0$ in (1.2), we have

$$\begin{aligned}
&(l-2)f(3x) + 3f((-l+2)x) \\
=& -2(l+1)f(3x) - 6(l+1)f(2x) - 3(l+1)(l-3)(l-4)f(x) \\
&+ 3(3l^2-2l-5)f(2x) + 3(3l^2-2l-5)(n-3)f(x) - 9(l^3-l^2-l+1)f(x),
\end{aligned} \tag{3.8}$$

for all $x \in X$. It follows from (3.8) that

$$f(3x) = 27f(x), \tag{3.9}$$

for all $x, y \in X$. Setting $x_1 = x_3 = x_4 = x$ and $x_2 = x_5 = \cdots = x_l = 0$ in (1.2), we get

$$\begin{aligned}
&(l-2)f(2x+y) + 2f(-2x+y) + f(2x-3y) \\
=& -2(l+1)\Big(f(2x+y) + (l-3)f(2x) + 2(l-3)f(x+y)\Big) \\
&- 2(l+1)\Big((l-3)(l-4)\Big)f(x) + (l-3)(l-4)f(y) \\
&+ (3l^2-2l-5)\Big(f(2x) + 2(l-3)f(x) + (n-3)f(y) + 2f(x+y)\Big) \\
&- 3(l^3-l^2-l+1)(2f(x)+f(y)),
\end{aligned} \tag{3.10}$$

for all $x, y \in X$. It follows from (3.10) and the oddness of $f$ that

$$\begin{aligned}
&f(2x+y) - 2f(2x-y) + f(2x-3y) \\
=& -8f(2x+y) + 128f(x) + 32f(x+y) - 96f(x) - 48f(y),
\end{aligned} \tag{3.11}$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (3.11), we get

$$\begin{aligned}
&f(2x-y) - 2f(2x+y) + f(2x+3y) \\
=& -8f(2x-y) + 128f(x) + 32f(x-y) - 96f(x) + 48f(y),
\end{aligned} \tag{3.12}$$

for all $x, y \in X$. Adding (3.11) and (3.12), we have

$$\begin{aligned}
&f(2x+3y) + f(2x+3y) - f(2x-y) - f(2x+y) \\
=& -8f(2x+y) - 8f(2x+y) + 32f(x+y) + 32f(x-y) + 64f(x),
\end{aligned} \tag{3.13}$$

for all $x, y \in X$. It follows from (3.13) and (1.1) that

$$7f(2x+y) + 7f(2x-y) = 14(f(x+y) + f(x-y)) + 84f(x), \tag{3.14}$$

for all $x, y \in X$. It follows from (3.14) that

$$f(2x+y) + f(2x-y) = 2(f(x+y) + f(x-y)) + 12f(x),$$

for all $x, y \in X$. Therefore, the mapping $f : X \to Y$ is cubic. $\square$

## 4. Hyers-Ulam Stability of the $l$-Variablel Cubic Functional Equation (1.2): Direct Approach

In this setion, we prove the Hyers-Ulam stability of the $l$-variablel cubic functional equation (1.2) in RN-spaces by using the direct method.

**Theorem 4.1.** *Let $j = \pm 1$ and $f : X \to Y$ be a mapping for which there exists a function $\eta : X^l \to D^+$ with the condition*

$$\begin{aligned}(4.1)\qquad &\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{l^{(k+i)}x_1, l^{(k+i)}x_2, l^{(k+i)}x_3, \ldots, l^{(k+i)}x_l}\left(l^{(k+i+1)j}t\right)\right)\\ &= \lim_{k\to\infty} \eta_{l^{(kj)}x_1, l^{(kj)}x_2, l^{(kj)}x_3, \ldots, l^{(kj)}x_l}\left(l^{kj}t\right) = 1,\end{aligned}$$

*such that $f(0) = 0$ and*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \eta_{(x_1,x_2,\ldots,x_l)}(t), \tag{4.2}$$

*for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation* (1.2) *and*

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{m^{(i+1)j}x, \underbrace{0, \ldots, 0}_{(l-1)-times}}\left(l^{(i+1)j}t\right)\right), \tag{4.3}$$

*for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by*

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(l^{kj}x)}{l^{3kj}}}(t), \tag{4.4}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Assume $j = 1$. Setting $(x_1, x_2, \ldots, x_n) = (x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}})$ in (4.3), we have

$$\mu_{f(lx)-l^3 f(x)}(t) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}(t), \tag{4.5}$$

for all $x \in X$ and all $t > 0$. It follows from (4.4) and (RN2) that

$$\mu_{\frac{f(lx)}{l^3}-f(x)}(t) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(l^3 t\right),$$

for all $x \in X$ and all $t > 0$. Replacing $x$ by $l^k x$ in (4.5), we catch

$$\mu_{\frac{f(l^{k+1}x)}{l^{3(k+1)}}-\frac{f(l^k x)}{l^{3k}}}(t) \geq \eta_{l^k x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(l^{3k} l^3 t\right) \geq \eta_{x, \underbrace{0, \ldots, 0}_{(l-1)-\text{times}}}\left(\frac{l^{3k} l}{\alpha^k} t\right), \tag{4.6}$$

for all $x \in X$ and all $t > 0$. It follows from

$$\frac{f\left(l^n x\right)}{l^{3n}} - f(x) = \sum_{k=0}^{n-1} \frac{f\left(l^{k+1}x\right)}{l^{3(k+1)}} - \frac{f\left(l^k x\right)}{l^{3k}}$$

and (4.6) that

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}\right) \geq T_{k=0}^{n-1}\left(\eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t)\right) = \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t),$$

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right), \tag{4.7}$$

for all $x \in X$ and all $t > 0$. Replacing $x$ by $l^m x$ in (4.7), we get

$$\mu_{\frac{f(l^{n+m} x)}{l^{n+m}}-\frac{f(l^m x)}{l^{3m}}}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{l^{3k}l^3}}\right). \tag{4.8}$$

Since $\eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{l^{3k}l^3}}\right) \to 1$ as $m,n\to\infty$, $\left\{\frac{f(l^n x)}{l^{3n}}\right\}$ is a Cauchy sequence in $(Y,\mu,T)$. Since $(Y,\mu,T)$ is complete, this sequence converges to some point $C(x)\in Y$. Fix $x \in X$ and put $m = 0$ in (4.8). Then we have

$$\mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right)$$

and so, for every $\delta > 0$, we have

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\delta) &\geq T\left(\mu_{C(x)-\frac{f(l^n x)}{l^{3n}}}(\delta), \mu_{\frac{f(l^n x)}{l^{3n}}-f(x)}(t)\right)\\ &\geq T\left(\mu_{C(x)-\frac{f(l^n x)}{l^{3n}}}(\delta), \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{l^{3k}l^3}}\right)\right). \end{aligned}\tag{4.9}$$

Taking limit as $n \to \infty$ and using (4.9), we have

$$\mu_{C(x)-f(x)}(t+\delta) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left((l^3-\alpha)t\right). \tag{4.10}$$

Since $\delta$ is arbitrary, by taking $\delta \to 0$ in (4.10), we have

$$\mu_{C(x)-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left((l^3-\alpha)t\right). \tag{4.11}$$

Replacing $(x_1, x_2, \ldots, x_l)$ by $(2^n x_1, 2^n x_2, \ldots, 2^n x_l)$ in (4.2), we have

$$\mu_{Df(l^n x_1, l^n x_2, \ldots, l^n x_l)}(t) \geq \eta_{l^n x_1, l^n x_2, \ldots, l^n x_l}\left(l^{3n} t\right),$$

for all $x_1, x_2, \ldots, x_l \in X$ and all $t > 0$. Since

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{l^{(k+i)}x_1, l^{(k+i)}x_2, \ldots, l^{(k+i)}x_l}\left(l^{3(k+i+1)j}t\right)\right) = 1,$$

we conclude that $C$ fulfills (1.2).

To prove the uniqueness of the cubic mapping $C$, assume that there exists another cubic mapping $D$ from $X$ to $Y$, which satisfies (4.11). Fix $x \in X$. Clearly, $C(l^n x) = l^{3n}C(x)$ and $D(l^n x) = l^{3n}D(x)$ for all $x \in X$. It follows from (4.11) that

$$\begin{aligned}
\mu_{C(x)-D(x)}(t) &= \lim_{n\to\infty} \mu_{\frac{C(l^n x)}{l^{3n}} - \frac{D(l^n x)}{l^{3n}}}(t), \\
\mu_{C(x)-D(x)}(t) &\geq \min\left\{\mu_{\frac{C(l^n x)}{l^{3n}} - \frac{f(l^n x)}{l^{3n}}}\left(\frac{t}{2}\right), \mu_{\frac{D(l^n x)}{l^{3n}} - \frac{f(l^n x)}{l^{3n}}}\left(\frac{t}{2}\right)\right\} \\
&\geq \eta_{l^n x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(l^{3n}\left(l^3-\alpha\right)t\right) \\
&\geq \eta_{x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\frac{l^{3n}\left(l^3-\alpha\right)t}{\alpha^n}\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{l^{3n}(l^3-\alpha)t}{\alpha^n}\right) = \infty$, we get

$$\lim_{n\to\infty} \eta_{x, \underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(\left(\frac{l^{3n}(l^3-\alpha)t}{\alpha^n}\right)\right) = 1.$$

Therefore, it follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $C(x) = D(x)$.

For $j = -1$, we can prove the theorem by a similar way. This completes the proof. $\square$

The following corollary is an immediate consequence of Theorem 4.1, concerning the stability of (1.2).

**Corollary 4.1.** *Let $\xi$ and $\rho$ be nonnegative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \begin{cases} \eta_{\xi}(t), & \\ \eta_{\xi\sum_{i=1}^{n}\|x_i\|^{\rho}}(t), & s \neq 3, \\ \eta_{\xi\left(\prod_{i=1}^{n}\|x_i\|^{\rho}+\sum_{i=1}^{n}\|x_i\|^{n\rho}\right)}(t), & p \neq \frac{3}{n}, \end{cases}$$

*for all $x_1, x_2, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\xi}{|l^3-1|}}(t), \\ \eta_{\frac{\xi\|x\|^{\rho}}{|l^3-1^{\rho}|}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{|l^3-1^{n\rho}|}}(t), \end{cases}$$

*for all $x \in X$ and all $t > 0$.*

## 5. Hyers-Ulam Stability of the $l$-Variablel Cubic Functional Equation (1.2): Fixed Point Approach

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2) in random normed spaces by using the fixed point approach.

**Theorem 5.1.** *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^l \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \ldots, \delta_i^k x_l}(\delta_i^k t) = 1,$$

*for all $x_1, x_2, \ldots, x_l \in X$, $t > 0$ and $\delta_i = \begin{cases} l, \ i = 0, \\ \frac{1}{l}, \ i = 1, \end{cases}$ satisfying the functional inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \eta_{x_1,x_2,\ldots,x_l}(t),$$

*for all $x_1, x_2, \ldots, x_l \in X$ and $t > 0$. If there exists $L = L(i)$ such that the function*

$$x \mapsto \beta(x,t) = \eta_{\frac{x}{l}, \underbrace{0, \ldots, 0}_{(l-1)-times}}(t)$$

*has the property that*

$$\beta(x,t) \leq L \frac{1}{\delta_i^3} \beta(\delta_i x, t), \tag{5.1}$$

*for all $x \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.2) and*

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t),$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Let $\Omega := \{f : X \to Y : f \text{ is a function}\}$ and $d$ be a generalized metric on $\Omega$ such that

$$d(g,h) = \inf\left\{k \in (0,\infty) / \mu_{(g(x)-h(x))}(kt) \geq \beta(x,t) : x \in X, t > 0\right\}.$$

It is easy to see that $(\Omega, d)$ is complete (see [17]). Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x)$ for all $x \in X$. Now, for $g, h \in \Omega$ we have $d(g,h) \leq K$, which implies

$$\begin{aligned} \mu_{(g(x)-h(x))}(Kt) &\geq \beta(x,t), \\ \mu_{(Tg(x)-Th(x))}\left(\frac{Kt}{\delta_i}\right) &\geq \beta(x,t), \\ d(Tg(x), Th(x)) &\leq KL, \\ d(Tg, Th) &\leq L d(g,h), \end{aligned}$$

for all $g, h \in \Omega$. Therefore, $T$ is a strictly contractive mapping on $\Omega$ with Lipschitz constant $L$. It follows from (4.5) that

$$\mu_{f(lx)-l^3 f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(t),$$

for all $x \in X$. It follows from (4.5) that

$$\mu_{\frac{f(lx)}{l^3}-f(x)}(t) \geq \eta_{x,\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}(l^3 t),$$

for all $x \in X$. Using (5.1) for the case $i = 0$, we get

$$\mu_{\frac{f(lx)}{l^3}-f(x)}(t) \geq L\beta(x,t),$$

for all $x \in X$. Hence, we obtain

$$d(\mu_{Tf,f}) \leq L = L^{1-i} < \infty, \tag{5.2}$$

for all $x \in X$. Replacing $x$ by $\frac{x}{l}$ in (4), we get

$$\mu_{\frac{f(x)}{l}-f\left(\frac{x}{l}\right)}(t) \geq \eta_{\frac{x}{l},\underbrace{0,\ldots,0}_{(l-1)-\text{times}}}\left(l^3 t\right),$$

for all $x \in X$. By using (5.1) for the case $i = 1$, it reduce to

$$\mu_{l^3 f\left(\frac{x}{l}\right)-f(x)}(t) \geq \beta(x,t) \Rightarrow \mu_{Tf(x)-f(x)}(t) \geq \beta(x,t),$$

for all $x \in X$. Hence, we get

$$d\left(\mu_{Tf,f}\right) \leq L = L^{1-i} < \infty, \tag{5.3}$$

for all $x \in X$. From (5.2) and (5.3), we can conclude

$$d\left(\mu_{Tf,f}\right) \leq L = L^{1-i} < \infty,$$

for all $x \in X$.

The remaining proof is similar to the proof of Theorem 4.1. Since $C$ is a unique fixed point of $T$ in the set $\Delta = \{f \in \Omega \mid d(f,C) < \infty\}$, $C$ is a unique mapping such that

$$\mu_{f(x)-C(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t),$$

for all $x \in X$ and $t > 0$. This completes the proof. □

From Theorem 5.1, we obtain the following corollary concerning the stability for the functional equation (1.2).

**Corollary 5.1.** *Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_l)}(t) \geq \begin{cases} \eta_\xi(t), & \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho}(t), & \rho \neq 3, \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho + \sum_{i=1}^n \|x_i\|^{n\rho}\right)}(t), & p \neq \frac{3}{n}, \end{cases}$$

*for all $x_1, x_2, \dots, x_n \in X$ and $t > 0$, where $\rho, \xi$ are constants with $\xi > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\xi}{|l^3-1|}}(t), \\ \eta_{\frac{\xi\|x\|^\rho}{|l^3-1^\rho|}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{|l^3-1^{n\rho}|}}(t), \end{cases}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Set

$$\mu_{Df(x_1,x_2,\dots,x_l)}(t) \geq \begin{cases} \eta_\xi(t), \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho}(t), \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho + \sum_{i=1}^n \|x_i\|^{n\rho}\right)}(t), \end{cases}$$

for all $x_1, x_2, \dots, x_n \in X$ and $t > 0$. Then

$$\eta_{(\delta_i^k x_1, \delta_i^k x_2, \dots, \delta_i^k x_l)}(\delta_i^k t) = \begin{cases} \eta_\xi \delta_i^{3k}(t) \\ \eta_{\xi \sum_{i=1}^n \|x_i\|^\rho \delta_i^{(3-\rho)k}}(t) \\ \eta_{\xi\left(\prod_{i=1}^n \|x_i\|^\rho \delta_i^{(3-\rho)k} + \sum_{i=1}^n \|x_i\|^{n\rho} \delta_i^{(3-n\rho)k}\right)}(t) \end{cases}$$
$$\to \begin{cases} 1 \quad \text{as} \quad k \to \infty, \\ 1 \quad \text{as} \quad k \to \infty, \\ 1 \quad \text{as} \quad k \to \infty. \end{cases}$$

But we have that $\beta(x,t) = \eta_{\frac{x}{l}, \underbrace{0,\dots,0}_{(l-1)-\text{times}}}(t)$ has the property $L\frac{1}{\delta_i^3}\beta(\delta_i x, t)$ for all $x \in X$ and $t > 0$.

Now,

$$\beta(x,t) = \begin{cases} \eta_\xi(t), \\ \eta_{\frac{\xi\|x\|^\rho}{l^{3s}}}(t), \\ \eta_{\frac{\xi\|x\|^{n\rho}}{l^{3ns}}}(t), \end{cases}$$

$$L\frac{1}{\delta_i^3}\beta(\delta_i x, t) = \begin{cases} \eta_{\delta_i^{-3}\beta(x)}(t), \\ \eta_{\delta_i^{\rho-3}\beta(x)}(t), \\ \eta_{\delta_i^{n\rho-3}\beta(x)}(t). \end{cases}$$

Using Theorem 4.1, we prove the following six cases.

$L = l^{-3}$ if $i = 0$; $L = l$ if $i = 1$; $L = l^{\rho-3}$ for $\rho < 1$ if $i = 0$; $L = l^{3-\rho}$ for $s > 1$ if $i = 1$; $L = l^{n\rho-3}$ for $\rho < \frac{1}{n}$ if $i = 0$; $L = l^{3-n\rho}$ for $\rho > \frac{1}{n}$ if $i = 1$.

**Case 1.** $L = l^{-3}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi}{l^3-l}\right)}(t).$$

**Case 2.** $L = l^3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi}{1-l^3}\right)}(t).$$

**Case 3.** $L = l^{\rho-3}$ for $\rho < 1$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^\rho}{(l^3-l^{3\rho})}\right)}(t).$$

**Case 4.** $L = l^{3-\rho}$ for $\rho > 1$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^\rho}{(l^{3\rho}-l^3)}\right)}(t).$$

**Case 5.** $L = l^{n\rho-3}$ for $\rho < \frac{1}{n}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^{n\rho}}{(l^3-l^{3n\rho})}\right)}(t).$$

**Case 6.** $L = l^{3-n\rho}$ for $\rho > \frac{1}{n}$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\xi\|x\|^{n\rho}}{(l^{3n\rho}-l^3)}\right)}(t).$$

Hence, the proof is complete. □

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# INEQUALITIES FOR MAXIMUM MODULUS OF RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract. In this paper we prove some results concerning the rational functions with prescribed poles and restricted zeros. These results in fact generalize or strengthen some known inequalities for rational functions with prescribed poles and in turn produce new results besides the refinements of some known polynomial inequalities. Our method of proof may be useful for proving other inequalities for polynomials and rational functions.

## 1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most $n$ and $P'(z)$ be the derivative of $P(z)$. Let $D_k^- := \{z : |z| < k\}$, $D_k^+ := \{z : |z| > k\}$ and $T_k := \{z : \ |z| = k\}$. For a function $f$ defined on the circle $T_1$ in the complex plane $\mathcal{C}$, we write

$$\|f\| := \sup_{z\in T_1} |f(z)|, \quad w(z) := \prod_{j=1}^{n} (z - a_j)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) := \left\{ \frac{p(z)}{w(z)} : p \in \mathcal{P}_n \right\},$$

where $a_j \in D_1^+, \ j = 1, 2, \ldots n$.

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Thus, $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at $\infty$. We observe that the Blashke product $B(z) \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left( \frac{1 - \overline{a_j} z}{z - a_j} \right).$$

For every $P \in \mathcal{P}_n$, the following inequality is due to Bernstein [5]:

$$\|P'\| \leq n\|P\|,$$

where as by an application of maximum modulus principle

$$\|P(R, \cdot)\| \leq R^n \|P\|,$$

where $\|P(R, \cdot)\| = \sup_{z \in T_R} |P(z)|$. Both these inequalities are sharp and equality holds for polynomials having all zeros at the origin. In case $P(z) \neq 0$ for $z \in D_1^-$, then we have for $z \in T_1$

$$\|P'\| \leq \frac{n}{2} \|P\| \tag{1.1}$$

and

$$\|P(R, \cdot)\| \leq \frac{R^n + 1}{2} \|P\|, \tag{1.2}$$

whereas if $P(z) \neq 0$ for $z \in D_1^+$, then

$$\|P'\| \geq \frac{n}{2} \|P\|. \tag{1.3}$$

Inequality (1.1) was conjectured by Erdös and proved by Lax [9], whereas inequality (1.2) is due to Ankeny and Rivilin [1]. Inequality (1.3) is due to Turán [14]. In all the inequalities (1.1), (1.2), and (1.3) equality holds for polynomials having all zeros on the unit disk.

Li, Mohapatra and Rodriguez [10] extended inequalities (1.1) and (1.3) to rational functions $r \in \mathcal{R}_n$ and proved the following results.

**Theorem 1.1.** *Suppose $r \in \mathcal{R}_n$ and all the zeroes of $r$ lie in $T_1 \cup D_1^+$. Then for $z \in T_1$*

$$|r'(z)| \leq \frac{1}{2} |B'(z)| \|r\|.$$

**Theorem 1.2.** *Suppose $r \in \mathcal{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, \ldots, a_n$ and all the zeros of $r$ lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2} \{ |B'(z)| - (n - m) \} |r(z)|,$$

*where $m$ is the number of zeros of $r$.*

The inequality (1.2) was extended to rational functions by Govil and Mohapatra [7] (see also Aziz and Rather [3]) to read as follows.

**Theorem 1.3.** *Suppose $r \in \mathcal{R}_n$ and all the zeroes of $r$ lie in $T_1 \cup D_1^+$. Then for $z \in T_1$*

$$|r'(z)| \leq \frac{|B(Rz)| + 1}{2} |r(z)|.$$

## 2. Main Results

**Theorem 2.1.** *Suppose $r \in \mathcal{R}_n$, where $r$ has $n$ poles at $a_1, a_2, \ldots, a_n$ and all the zeros of $r$ lie in $T_1 \cup D_1^-$ with a zero of multiplicity $s$ at origin, then the following inequality holds for each point $z \in T_1$, such that $r(z) \neq 0$*

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{1}{2}\left\{|B'(z)| + (s+m-n) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|}\right\}, \tag{2.1}$$

*where $m$ is the number of zeros of $r$. Inequality* (2.1) *is sharp and equality holds for*

$$r(z) = \frac{z^s(z^{m-s} - 1)}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad z \in T_1,\ a \geq 1.$$

Since $\left|\frac{zr'(z)}{r(z)}\right| \geq \operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\}$, from Theorem 2.1, we immediately have the following.

**Corollary 2.1.** *Suppose $r \in \mathcal{R}_n$, where $r$ has $n$ poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_1 \cup D_1^-$, with $s$-fold zeros at origin, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + (s+m-n) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|}\right\}|r(z)|, \tag{2.2}$$

*where $m$ is the number of zeros of $r$. The result is sharp and equality holds for*

$$r(z) = \frac{z^s(z^{m-s} - 1)}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n,$$

*at $z = 1$ and $a \geq 1$.*

**Note.** Inequality (2.2) is trivally true in case $r(z) = 0$ for $z \in T_1$.

If we take $s = 0$ in Corollary 2.1, we get the following result, which is an improvement of Theorem 1.2, earlier proved by Li, Mohapatra and Rodriguez [10, Theorem 4].

**Corollary 2.2.** *Suppose $r \in \mathcal{R}_n$, where $r$ has $n$ poles and all the zeros of $r$ lie in $T_1 \cup D_1^-$. Then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - (n-m) + \frac{|c_m| - |c_0|}{|c_m| + |c_0|}\right\}|r(z)|,$$

*where $m$ is the number of zeros of $r$. The result is sharp and equality holds for*

$$r(z) = \frac{(z+1)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n$$

*at $z = 1$ and $a \geq 1$.*

Since $|c_m| \geq |c_0|$, therefore as mentioned above, Corollary 2.2 is an improvement of Theorem 1.2. In case number of poles of $r$ is same as its zeros, that is, when $m = n$ then Corollary 2.2 gives an improvement of [4, inequality (12)].

Taking $a_i = \alpha,\ i = 1, 2, \ldots, n$, in Corollary 1, we have the following.

**Corollary 2.3.** *If $P(z)$ is a polynomial of degree $m$ having all zeros in $T_1 \cup D_1^-$ with $s$-fold zero at origin, then*

$$|D_\alpha P(z)| \geq \frac{|\alpha|-1}{2}\left\{m+s+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}|P(z)|, \tag{2.3}$$

*where $D_\alpha P(z) := nP(z)+(\alpha-z)P'(z)$ is called the polar derivative of the polynomial $P(z)$ with respect to the point $\alpha$ and it generalizes the ordinary derivative of $P(z)$ of degree $n$ in the sense that*

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z).$$

By taking $s=0$ in Corollary 2.3, we get the following sharp result which is also an extension of a result of Dubinin [6] to the polar derivative of $P(z)$.

**Corollary 2.4.** *If $P(z) := \sum_{j=0}^n c_j z^j$ is a polynomial of degree $n$ having all zeros in $T_1 \cup D_1^-$, then*

$$|D_\alpha P(z)| \geq \frac{|\alpha|-1}{2}\left\{n+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}|P(z)|. \tag{2.4}$$

Equality in (2.4) holds for a polynomial $P(z)=(z-1)^n$ with $\alpha>1$. Since $|c_n| \geq |c_0|$, it follows that Corollary 2.4 is a refinement of a result of Shah [13].

Dividing both sides of (2.3) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following result.

**Corollary 2.5.** *If $P \in \mathcal{P}_n$ is such that $P(z)$ has all its zeros in $T_1 \cup D_1^-$ with $s$-fold zero at origin, then*

$$|P'(z)| \geq \frac{1}{2}\left(n+s+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right)|P(z)|. \tag{2.5}$$

For $s=0$ (2.5) reduces to the result of Dubinin [6] and is an improvement of a classical result of Turán [14].

Next we prove the following refinement of a result of Aziz and Shah [4, Theorem 1].

**Theorem 2.2.** *Suppose $r \in \mathcal{R}_n$, where $r$ has exactly $n$ poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r$ lie in $T_k \cup D_k^-$, $k \leq 1$, with a zero of order $s$ at origin. Then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)|-n+\frac{2(m+sk)}{1+k}\right\}|r(z)|, \tag{2.6}$$

*where $m$ is the number of zeros of $r$. The result is sharp and equality holds for*

$$r(z)=\frac{z^s(z+k)^{m-s}}{(z-a)^n} \quad \text{and} \quad B(z)=\left(\frac{1-az}{z-a}\right)^n$$

*at $z=1$ and $a \geq 1$.*

The result of Aziz and Shah [4, Theorem 1] is a special case of Theorem 2.2, if we take $s=0$.

As in previous case, if we take $a_i=\alpha$, $i=1,2,\ldots,n$, in Theorem 2.2, we get the following result on the polar derivatives of a polynomial.

**Corollary 2.6.** *If $P \in P_n$ is such that $P(z) \neq 0$ in $D_k^+$, $k \leq 1$ with s-fold zero at origin, then for every $\alpha$ with $|\alpha| \geq 1$*

$$|D_\alpha P(z)| \geq \frac{n(|\alpha|-1)(1+ks)}{1+k}|P(z)|. \tag{2.7}$$

*Remark* 2.1. In Corollary 2.6 if we take $s = 0$, we have the following generalization of a result of Shah [13].

**Corollary 2.7.** *If $P \in \mathcal{P}_n$ is such that $P(z)$ has all zeros in $T_k \cup D_k^-$, then for $|\alpha| \geq 1$ and $z \in T_1$*

$$|D_\alpha P(z)| \geq \frac{n(|\alpha|-1)}{1+k}|P(z)|.$$

*Remark* 2.2. Dividing both sides of (2.7) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following sharp result.

**Corollary 2.8.** *If $P \in \mathcal{P}_n$ is such that $P(z) \neq 0$ for $z \in D_k^+$ with s-fold zero at origin, then for $z \in T_1$*

$$|P'(z)| \geq \frac{n+ks}{1+k}|P(z)|.$$

Equality holds for $P(z) = z^s(z-k)^{n-s}$.

For $s = 0$, this gives result of Malik [11], whereas for $k = 1$, $s = 0$, it reduces to the classical theorem of Turán [14].

**Theorem 2.3.** *Suppose $r \in \mathcal{R}_n$ and all the zeros of $r$ lie in $T_k \cup D_k^+$. Then for $z \in T_1$*

$$|r(Rz)| \leq \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n}\left\{|B(Rz)| + 1\right\}\|r\|. \tag{2.8}$$

*Remark* 2.3. Theorem 1.3 is a special case of Theorem 2.3, when $k = 1$.

*Remark* 2.4. Let $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, so that

$$r(z) = \frac{P(z)}{(z-\alpha)^n} \text{ and } B(z) = \prod_1^n \frac{1-\overline{\alpha}z}{z-\alpha} = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n.$$

Using this in Theorem 2.3, it can be easily verified that for $|\alpha| \geq R > 1$ and $z \in T_1$

$$|P(Rz)| \leq \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n}\left\{\left|\frac{1-\overline{\alpha}Rz}{Rz-\alpha}\right|^n + 1\right\}\|P\|.$$

Letting $|\alpha| \to \infty$, we get the following result.

If $P(z)$ is a polynomial of degree $n$, which does not vanish in $|z| < k$, $k \geq 1$, then for $R > 1$ and $z \in T_1$

$$\|P(R, \cdot)\| \leq \frac{(R+k)^n(R^n+1)}{(R+k)^n + (1+Rk)^n}\|P\|.$$

This result was earlier proved by Aziz and Mohammad [2].

## 3. Lemmas and Proofs

For the proofs of these theorems we need the following lemmas.

**Lemma 3.1** ([8]). *Let $f : D \to D$ be holomorphic. Assume that $f(0) = 0$. Further assume that there is a $b \in \partial D$, the boundary of $D$, so that $f$ extends continuously to $b$, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

The next lemma is due to Aziz and Rather.

**Lemma 3.2** ([3]). *If $r \in \mathcal{R}_n$ and $z \in T_1$, then for every $R \geq 0$,*

$$|r(Rz)| + |r^*(Rz)| \leq \{|B(Rz)| + 1\}\|r\|,$$

*where $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$.*

*Proof of Theorem 2.1.* Suppose that $r(z) \neq 0$ for $z \in T_1$ and all the poles of $r(z)$ lie in $D_1^+$. Since $r(z)$ has a zero at origin of multiplicity $s$. Therefore,

$$r(z) = \frac{P(z)}{w(z)} = \frac{z^s h(z)}{w(z)},$$

where

$$h(z) := \sum_{j=0}^{m-s} c_{s+j} z^j = c_m \prod_{j=1}^{m-s} (z - z_j), \quad z_j \in D_1^-, \, j = 1, 2, \ldots, m-s,$$

and

$$w(z) = \prod_{j=1}^{n} (z - a_j).$$

This gives

$$\frac{r'(z)}{r(z)} = \frac{s}{z} + \frac{h'(z)}{h(z)} - \frac{w'(z)}{w(z)}.$$

Equivalently,

$$\text{Re}\left(\frac{zr'(z)}{r(z)}\right) = s + \text{Re}\left(\frac{zh'(z)}{h(z)}\right) - \text{Re}\left(\frac{zw'(z)}{w(z)}\right). \tag{3.1}$$

Since $h(z)$ has all zeros in $D_1^-$, therefore

$$h^*(z) = z^{m-s}\overline{h\left(\frac{1}{\bar{z}}\right)}$$

has all zeros in $D_1^+$, and hence

$$G(z) = \frac{zh(z)}{h^*(z)} = z\frac{c_m}{\overline{c_m}} \prod_{j=1}^{m-s} \left(\frac{z - z_j}{1 - z\overline{z_j}}\right) \tag{3.2}$$

is analytic in $T_1 \cup D_1^-$, with $G(0) = 0$ and $|G(z)| = 1$ for $z \in T_1$.

Applying Lemma 3.1 to $G(z)$, we get for $z \in T_1$

$$|G'(z)| \geq \frac{2}{1+|G'(0)|}. \tag{3.3}$$

Since

$$|G'(0)| = \left|\prod_{j=1}^{m-s} z_j\right| = \frac{|c_s|}{|c_m|},$$

it can be easily verified (see [15, proof of Lemma 1]) that for every $z \in T_1$

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \geq \frac{m-s-1}{2} + \frac{|c_m|}{|c_m|+|c_s|}. \tag{3.4}$$

Again we have

$$B(z) = \frac{w^*(z)}{w(z)},$$

where

$$w^*(z) = z^n \overline{w\left(\frac{1}{\overline{z}}\right)}.$$

This gives (see [15, Lemma 1])

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \frac{n-|B'(z)|}{2}. \tag{3.5}$$

Now using (3.4) and (3.5) in (3.1), we conclude that

$$\operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left\{s + |B'(z)| - (n-m) + \frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}.$$

The proof of Theorem 2.1 is completed. □

*Proof of Theorem 2.2.* Suppose that for each point $z \in T_1$, $r(z) \neq 0$ and all the poles of $r(z)$ lie in $D_1^+$. Since $r(z)$ has a zero of order $s$ at origin, therefore

$$r(z) = \frac{P(z)}{w(z)} = \frac{z^s Q(z)}{w(z)},$$

where

$$Q(z) = \sum_{j=0}^{m-s} c_{s+j} z^j$$

is a polynomial of degree $m-s$ having all zeros in $|z| \leq k$. This gives

$$\operatorname{Re}\frac{zr'(z)}{r(z)} = s + \operatorname{Re}\frac{zQ'(z)}{Q(z)} - \operatorname{Re}\frac{zw'(z)}{w(z)}. \tag{3.6}$$

We write $Q(z) = c_m \prod_{j=1}^{m-s}(z-z_j)$, $|z_j| \leq k \leq 1$, and it can be easily verified that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} \geq \frac{m-s}{1+k}.$$

Also, as in (3.5)

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \frac{n - |B'(z)|}{2}.$$

Using this in (3.6), we get

$$\operatorname{Re}\frac{zr'(z)}{r(z)} \geq s + \frac{m-s}{1+k} - \frac{n-|B'(z)|}{2}.$$

Now for $z \in T_1$ such that $r(z) \neq 0$, we have

$$\left|\frac{zr'(z)}{r(z)}\right| \geq \operatorname{Re}\frac{zr'(z)}{r(z)} \geq \frac{1}{2}\left\{|B'(z)| + \frac{2(m+sk) - n(1+k)}{1+k}\right\}.$$

This gives for $z \in T_1$ such that $r(z) \neq 0$,

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - n + \frac{2(m+sk)}{1+k}\right\}|r(z)|.$$

Since the result is trivially true if $r(z) = 0$ for $z \in T_1$, it follows that

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - n + \frac{2(m+sk)}{(1+k)}\right\}|r(z)|,$$

for all $z \in T_1$. The proof of Theorem 2.2 is completed. □

*Proof of Theorem 2.3.* Since zeros of $r(z)$ lie in $T_k \cup D_k^+$, therefore all zeros of $P(z)$ lie in $T_k \cup D_k^+$, $k \geq 1$ and $w(z) = \prod_{j=1}^{n}(z - \alpha_j)$, $|\alpha_j| > 1$ for all $j = 1, 2, \dots, n$.

Also

$$\begin{aligned} r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)} &= \prod_{j=1}^{n}\left(\frac{1-\overline{\alpha_j}z}{z-\alpha_j}\right)\overline{\left(\frac{P(\frac{1}{\bar{z}})}{w(\frac{1}{\bar{z}})}\right)} \\ &= \prod_{j=1}^{n}\left(\frac{1-\overline{\alpha_j}z}{z-\alpha_j}\right)\frac{z^n\overline{P(\frac{1}{\bar{z}})}}{\prod_{j=1}^{n}(1-\overline{\alpha_j}z)} = \frac{z^n\overline{P(\frac{1}{\bar{z}})}}{\prod_{j=1}^{n}(z-\alpha_j)} = \frac{P^*(z)}{w(z)}. \end{aligned}$$

Therefore,

$$\frac{r(z)}{r^*(z)} = \frac{P(z)}{P^*(z)}. \tag{3.7}$$

We write

$$P(z) = \prod_{j=1}^{n}(z - r_j e^{i\theta_j}), \quad \text{where } r_j \geq k \geq 1,\ j = 1, 2, \dots, n,$$

so that $P^*(z) = z^n\overline{P(\frac{1}{\bar{z}})} = \prod_{j=1}^{n}(1 - zr_je^{i\theta_j})$.

For $0 \leq \theta < 2\pi$ and $R > 1$ we have

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_je^{i\theta_j}}{1 - r_jRe^{i(\theta-\theta_j)}}\right|. \tag{3.8}$$

Now

$$\begin{aligned}\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{1-r_jRe^{i(\theta-\theta_j)}}\right|^2 &= \prod_{j=1}^{n}\left|\frac{Re^{i(\theta-\theta_j)}-r_j}{1-r_jRe^{i(\theta-\theta_j)}}\right|^2\\ &= \prod_{j=1}^{n}\left(\frac{Re^{i(\theta-\theta_j)}-r_j}{1-r_jRe^{i(\theta-\theta_j)}}\cdot\frac{Re^{-i(\theta-\theta_j)}-r_j}{1-r_jRe^{-i(\theta-\theta_j)}}\right)\\ &= \prod_{j=1}^{n}\frac{R^2-2Rr_j\cos(\theta-\theta_j)+r_j^2}{1-2Rr_j\cos(\theta-\theta_j)+r_j^2R^2}\le\prod_{j=1}^{n}\left(\frac{R+r_j}{1+Rr_j}\right)^2.\end{aligned}$$

Therefore, from (3.8), we have

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right|^2\le\prod_{j=1}^{n}\left(\frac{R+r_j}{1+Rr_j}\right)^2. \tag{3.9}$$

Since $|r_j|\ge k,\ k\ge 1$, it can be easily verified that

$$\frac{R+r_j}{1+Rr_j}\le\frac{R+k}{1+Rk}.$$

Using this in (3.9), we get

$$\left|\frac{P(Re^{i\theta})}{P^*(Re^{i\theta})}\right|\le\prod_{j=1}^{n}\left(\frac{R+k}{1+Rk}\right)=\left(\frac{R+k}{1+Rk}\right)^n. \tag{3.10}$$

Combining (3.7) and (3.10), we get

$$\left|\frac{r(Rz)}{r^*(Rz)}\right|\le\left(\frac{R+k}{1+Rk}\right)^n,\quad \text{for } z\in T_1,\ k\ge 1,\ R>1.$$

Equivalently,

$$\left(\frac{R+k}{1+Rk}\right)^{-n}|r(Rz)|\le|r^*(Rz)|. \tag{3.11}$$

Now using Lemma 3.2, we get from (3.11)

$$\left\{\left(\frac{R+k}{1+Rk}\right)^{-n}+1\right\}|r(Rz)|\le|r(Rz)|+|r^*(Rz)|.$$

That is,

$$|r(Rz)|\le\frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\{|B(Rz)|+1\}|r|.$$

The proof of Theorem 2.3 is completed. □

*Proof of Corollary 2.3.* Since $r(z)$ has a pole of order $n$ at $z=\alpha,\ |\alpha|>1$, we have

$$r(z)=\frac{P(z)}{(z-\alpha)^n}.$$

Therefore,

$$r'(z)=\frac{(z-\alpha)^nP'(z)-n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}}=\frac{-[nP(z)+(\alpha-z)P'(z)]}{(z-\alpha)^{n+1}}=\frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}.$$

Also,

$$B(z)=\left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$$

gives

$$B'(z)=\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using these facts, we immediately get for $z\in T_1$ from inequality (2.2)

$$\left|\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}\right|\geq\frac{1}{2}\left\{\frac{n|z-\alpha|^{n-1}(|\alpha|^2-1)}{|z-\alpha|^{n+1}}+\left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)\right\}\frac{|P(z)|}{|z-\alpha|^n}.$$

This gives

$$\begin{aligned}|D_\alpha P(z)|&\geq\frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|}+\left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)|z-\alpha|\right\}|P(z)|\\&\geq\frac{(|\alpha|-1)}{2}\left\{n+\left(s+m-n+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right)\right\}|P(z)|\\&=\frac{(|\alpha|-1)}{2}\left\{s+m+\frac{|c_m|-|c_s|}{|c_m|+|c_s|}\right\}|P(z)|,\quad z\in T_1,\ |\alpha|\geq1.\end{aligned}$$

Using the argument of continuity in case of poles the proof of Corollary 2.3 completes. □

*Proof of Corollary 2.6.* Since we have

$$r(z)=\frac{P(z)}{(z-\alpha)^n}\quad\text{and}\quad B(z)=\left(\frac{1-\alpha z}{z-\alpha}\right)^n,$$

therefore

$$r'(z)=\frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}$$

and

$$B'(z)=\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using in inequality (2.6), with $m=n$, we get

$$\left|\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}\right|\geq\frac{1}{2}\left\{\frac{n|z-\alpha|^{n-1}(|\alpha|^2-1)}{|z-\alpha|^{n+1}}+\frac{n(1-k)+2ks}{1+k}\right\}\frac{|P(z)|}{|z-\alpha|^n}.$$

This gives

$$|D_\alpha P(z)|\geq\frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|}+\frac{n(1-k)+2ks}{1+k}|z-\alpha|\right\}|P(z)|$$

$$\begin{aligned}&\geq \frac{1}{2}\left\{n(|\alpha|-1)+\frac{n(1-k)+2ks}{1+k}(|\alpha|-1)\right\}|P(z)|\\&=\frac{n(|\alpha|-1)(1+ks)}{1+k}|P(z)|, \quad z\in T_1,\ |\alpha|\geq 1.\end{aligned}$$

Using the argument of continuity in case of poles the proof of Corollary 2.6 completes. □

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# CONTROLLED INTEGRAL FRAMES FOR HILBERT $C^*$-MODULES

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Abstract. The notion of controlled frames for Hilbert spaces were introduced by Balazs, Antoine and Grybos to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Controlled frame theory has a great revolution in recent years. This theory have been extended from Hilbert spaces to Hilbert $C^*$-modules. In this paper we introduce and study the extension of this notion to integral frame for Hilbert $C^*$-modules. Also we give some characterizations between integral frame in Hilbert $C^*$-modules.

## 1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frames theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [12].

Hilbert $C^*$-module arose as generalization of the Hilbert space notion. The basic idea was to consider modules over $C^*$-algebras instead of linear spaces and to allow the inner product to take values in the $C^*$-algebras [17]. Continuous frames defined by Ali, Antoine and Gazeau [1]. Gabardo and Han in [11] called these kinds frames or frames associated with measurable spaces. For more details, the reader can refer to [4, 13–16, 20–32].

*Key words and phrases.* Integral frames, integral $*$-frame, controlled integral frames, controlled integral $*$-frame, $C^*$-algebra, Hilbert $\mathcal{A}$-modules.

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The goal of this article is the introduction and the study of the concept of controlled integral frames for Hilbert $C^*$-modules. Also we give some characterizations between integral frame in Hilbert $C^*$-modules.

In the following we briefly recall the definitions and basic properties of $C^*$-algebra and Hilbert $\mathcal{A}$-modules. Our references for $C^*$-algebras are [6,8]. For a $C^*$-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

**Definition 1.1** ([6]). Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$, and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$;

(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;

(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $|| \cdot ||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-modules over $\mathcal{A}$.

For every $a$ in $C^*$-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for all $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, a map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

The following lemmas will be used to prove our mains results.

**Lemma 1.1** ([19]). *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules. If $T \in End_{\mathcal{A}}^*(\mathcal{H})$, then*

$$\langle Tx, Tx \rangle_{\mathcal{A}} \leq \|T\|^2 \langle x, x \rangle_{\mathcal{A}}, \qquad x \in \mathcal{H}.$$

**Lemma 1.2** ([3]). *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) *$T$ is surjective;*

(ii) *$T^*$ is bounded below associted to the norm, i.e., there is $m > 0$ such that $m\|x\| \leq \|T^*x\|$ for all $x \in \mathcal{K}$;*

(iii) *$T^*$ is bounded below associted to the inner product, i.e., there is $m' > 0$ such that $m'\langle x, x \rangle_{\mathcal{A}} \leq \langle T^*x, T^*x \rangle_{\mathcal{A}}$ for all $x \in \mathcal{K}$.*

**Lemma 1.3** ([2]). *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$.*

(i) *If $T$ is injective and $T$ has closed range, then the adjointable map $T^*T$ is invertible and*

$$\|(T^*T)^{-1}\|^{-1} I_{\mathcal{H}} \leq T^*T \leq \|T\|^2 I_{\mathcal{H}}.$$

(ii) *If $T$ is surjective, then the adjointable map $TT^*$ is invertible and*

$$\|(TT^*)^{-1}\|^{-1} I_{\mathcal{K}} \leq TT^* \leq \|T\|^2 I_{\mathcal{K}}.$$

**Lemma 1.4** ([33]). *Let $(\Omega, \mu)$ be a measure spaces, $X$ and $Y$ are two Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to X$ is a measurable function, then*
$$\lambda\left(\int_\Omega f d\mu\right) = \int_\Omega (\lambda f) d\mu.$$

**Theorem 1.1** ([5]). *Let $X$ be a Banach spaces, $U : X \to X$ a bounded operator and $\|I - U\| < 1$. Then $U$ is invertible.*

## 2. Controlled Integral Frames for Hilbert $C^*$-Modules

Let $X$ be a Banach spaces, $(\Omega, \mu)$ a measure space, and $f : \Omega \to X$ be a measurable function. Integral of Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [10, 33]). Since every $C^*$-algebra and Hilbert $C^*$-module are Banach spaces, we can use this integral and its properties.

Let $(\Omega, \mu)$ be a measure space, $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $C^*$-modules over a unital $C^*$-algebra and $\{\mathcal{H}_w\}_{w\in\Omega}$ is a family of submodules of $\mathcal{H}$. $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ into $\mathcal{H}_w$.

We define the following:
$$l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega}) = \left\{x = \{x_w\}_{w\in\Omega} : x_w \in \mathcal{H}_w, \left\|\int_\Omega \langle x_w, x_w\rangle_{\mathcal{A}} d\mu(w)\right\| < \infty\right\}.$$
For any $x = \{x_w\}_{w\in\Omega}$ and $y = \{y_w\}_{w\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle x, y\rangle_{\mathcal{A}} = \int_\Omega \langle x_w, y_w\rangle_{\mathcal{A}} d\mu(w)$ and the norm is defined by $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$.

In this case, the $l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ is a Hilbert $C^*$-modules (see [17]).

In what follows, let $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse and let $F$ be a function from $\Omega$ to $\mathcal{H}$.

The following definitions was introduced by Mohamed Rossafi, Frej Chouchene and Samir Kabbaj in the paper entitled *Integral frame in Hilbert $C^*$-module* (see arXiv preprint- arXiv:2005.09995v2 [math.FA] 30 Nov 2020).

**Definition 2.1.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules and $(\Omega, \mu)$ be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called an integral frame associted to $(\Omega, \mu)$ if

- for all $x \in \mathcal{H}$, $w \to \langle x, F_\omega\rangle_{\mathcal{A}}$ is a measurable function on $\Omega$;
- there exists a pair of constants $0 < A, B$ such that
$$A\langle x, x\rangle_{\mathcal{A}} \le \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \le B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

**Definition 2.2.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega, \mu)$ be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called a $*$-integral frame associted to $(\Omega, \mu)$ if

- for all $x \in \mathcal{H}$, $w \to \langle x, F_\omega\rangle_{\mathcal{A}}$ is a measurable function on $\Omega$;
- there exist two non-zero elements $A, B$ in $\mathcal{A}$ such that
$$A\langle x, x\rangle_{\mathcal{A}} A^* \le \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \le B\langle x, x\rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}.$$

## 3. Main Results

**Definition 3.1.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-modules and $(\Omega, \mu)$ be a measure space. A $C$-controlled integral frame in $C^*$-module $\mathcal{H}$ is a map $F : \Omega \to \mathcal{H}$ such that there exist $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{3.1}$$

The elements $A$ and $B$ are called the $C$-controlled integral frame bounds. If $A = B$, we call this a $C$-controlled integral tight frame. If $A = B = 1$, it's called a $C$-controlled integral parseval frame. If only the right hand inequality of (3.1) is satisfied, we call $F$ a $C$-controlled integral Bessel mapping with bound $B$.

*Example* 3.1. Let $\mathcal{H} = \left\{ X = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ and $\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$ which is a $C^*$-algebra. We define the inner product:

$$\begin{array}{rcl} \mathcal{H} \times \mathcal{H} & \to & \mathcal{A}, \\ (A, B) & \mapsto & A(\overline{B})^t. \end{array}$$

This inner product makes $\mathcal{H}$ a $C^*$-module over $\mathcal{A}$. Let $C$ be an operator defined by

$$\begin{aligned} C : \mathcal{H} &\to \mathcal{H}, \\ X &\to \alpha X, \end{aligned}$$

where $\alpha$ is a reel number strictly greater than zero. It's clair that $C \in Gl^+(\mathcal{H})$. Let $\Omega = [0, 1]$ endowed with the Lebesgue's measure. It's clear that it is a measure space.

We consider

$$\begin{aligned} F : [0, 1] &\to \mathcal{H}, \\ w &\to F_w = \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & \frac{w}{2} \end{pmatrix}. \end{aligned}$$

In addition, for $X \in \mathcal{H}$, we have

$$\int_{\Omega} \langle X, F_w\rangle_{\mathcal{A}} \langle CF_w, X\rangle_{\mathcal{A}} d\mu(\omega) = \int_{\Omega} \alpha w^2 \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix} d\mu(\omega) = \frac{\alpha}{3} \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix}.$$

It's clear that

$$\frac{1}{4}\|X\|_{\mathcal{A}}^2 \leq \begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{|b|^2}{4} \end{pmatrix} \leq \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = \|X\|_{\mathcal{A}}^2.$$

Then we have

$$\frac{\alpha}{12}\|X\|_{\mathcal{A}}^2 \leq \int_{\Omega} \langle X, F_w\rangle_{\mathcal{A}} \langle CF_w, X\rangle_{\mathcal{A}} d\mu(\omega) \leq \frac{\alpha}{3}\|X\|_{\mathcal{A}}^2,$$

which show that $F$ is a $C$-controlled integral frame for the $C^*$-module $\mathcal{H}$.

**Definition 3.2.** Let $F$ be a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$. We define the frame operator $S_C : \mathcal{H} \to \mathcal{H}$ for $F$ by

$$S_C x = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} C F_\omega d\mu(\omega), \quad x \in \mathcal{H}.$$

**Proposition 3.1.** *The frame operator $S_C$ is positive, selfadjoint, bounded and invertible.*

*Proof.* For all $x \in \mathcal{H}$, by Lemma 1.4, we have

$$\langle S_C x, x\rangle_{\mathcal{A}} = \left\langle \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} C F_\omega d\mu(\omega), x \right\rangle_{\mathcal{A}} = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle C F_\omega, x\rangle_{\mathcal{A}} d\mu(\omega).$$

By left hand of inequality (3.1), we have

$$0 \leq A\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C x, x\rangle_{\mathcal{A}}.$$

Then $S_C$ is a positive operator, also, it's sefladjoint. From (3.1), we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C x, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S_C \leq B.I$$

Then $S_C$ is a bounded operator. Moreover,

$$0 \leq I - B^{-1} S_C \leq \frac{B-A}{B}.I,$$

Consequently,

$$\|I - B^{-1} S_C\| = \sup_{x \in \mathcal{H}, \|x\|=1} \|\langle (I - B^{-1} S_C)x, x\rangle_{\mathcal{A}}\| \leq \frac{B-A}{B} < 1.$$

The Theorem 1.1 shows that $S_C$ is invertible. ☐

**Corollary 3.1.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega, \mu)$ be a measure space. Let $F : \Omega \to \mathcal{H}$ be a mapping. Assume that $S$ is the frame operator for $F$. Then the following statements are equivalent:*

(1) *$F$ is an integral frame associted to $(\Omega, \mu)$ with integral frame bounds $A$ and $B$;*
(2) *we have $A.I \leq S \leq B.I$*

*Proof.* (1) $\Rightarrow$ (2) Let $F$ be an integral frame associted to $(\Omega, \mu)$ with integral frames bounds $A$ and $B$, then

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Since

$$Sx = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} F_\omega d\mu(\omega),$$

we have

$$\langle Sx, x\rangle_{\mathcal{A}} = \left\langle \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} F_\omega d\mu(\omega), x \right\rangle_{\mathcal{A}} = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle F_\omega, x\rangle_{\mathcal{A}} d\mu(\omega),$$

then

$$\langle Ax, x\rangle_{\mathcal{A}} \leq \langle Sx, x\rangle_{\mathcal{A}} \leq \langle Bx, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S \leq B.I.$$

(2) $\Rightarrow$ (1) Let $x \in \mathcal{H}$, then

$$\left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| = \|\langle Sx, x\rangle_{\mathcal{A}}\| \leq \|Sx\|\|x\| \leq B\|x\|^2. \tag{3.2}$$

Also,

$$\|\langle Sx, x\rangle_{\mathcal{A}}\| \geq \|\langle Ax, x\rangle_{\mathcal{A}}\| = A\|x\|^2. \tag{3.3}$$

By (3.2) and (3.3) we obtain

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2,$$

which ends the proof. □

**Theorem 3.1.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, $C \in GL^+(\mathcal{H})$ and $(\Omega, \mu)$ be a measure space and $F$ be a mapping for $\Omega$ to $\mathcal{H}$. Then $F$ is a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ if and only if there exist $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2 \quad x \in \mathcal{H}.$$

*Proof.* ($\Rightarrow$) obvious.

($\Leftarrow$) Supposes there exists $0 < A \leq B < \infty$, such that (3.1) holds. On one hand, for all $x \in \mathcal{H}$ we have

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(\omega) \right\| = \|\langle S_C x, x\rangle_{\mathcal{A}}\| = \|\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}}\| = \|S_C^{\frac{1}{2}} x\|^2.$$

By Lemma 1.2, there exists $0 < m$ such that

$$m\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}} = \langle S_C x, x\rangle_{\mathcal{A}}. \tag{3.4}$$

On other hand, for all $x \in \mathcal{H}$ we have

$$B\|x\|^2 \geq \left\| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle CF_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \right\|^2 = \|\langle S_C x, x\rangle_{\mathcal{A}}\| = \|\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}}\| = \|S_C^{\frac{1}{2}} x\|^2.$$

By Lemma 1.2, there exist $0 < m'$ such that

$$\langle S_C^{\frac{1}{2}} x, S_C^{\frac{1}{2}} x\rangle_{\mathcal{A}} = \langle S_C x, x\rangle_{\mathcal{A}} \leq m'\langle x, x\rangle_{\mathcal{A}}. \tag{3.5}$$

From (3.4) and (3.5), we conclude that $F$ is a $C$-controlled integral frame. □

*Remark* 3.1. If $F$ is a mapping from $\Omega$ to $\mathcal{H}$, then $F$ is an integral frame associted to $(\Omega, \mu)$ if and only if there exist $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \| \int_{\Omega} \langle x, F_{\omega}\rangle_{\mathcal{A}} \langle F_{\omega}, x\rangle_{\mathcal{A}} d\mu(w) \| \leq B\|x\|^2, \quad x \in \mathcal{H}.$$

**Corollary 3.2.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega, \mu)$ be a measure space. Let $F : \Omega \to \mathcal{H}$ be a mapping and $C \in GL^+(\mathcal{H})$. Then the following statements are equivalent.*

(1) *$F$ is a $C$-controlled integral frame associted to $(\Omega, \mu)$.*
(2) *We have $A.I \leq S_C \leq B.I$, where $S_C$ is the frame operator for $F$, for $A$ and $B$ given.*

*Proof.* (1) $\Rightarrow$ (2) Let $F$ be a $C$-controlled integral frame associted to $(\Omega, \mu)$ with $C$-controlled integral frames bounds $A$ and $B$, then

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

Since,

$$S_C x = \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega).$$

We have

$$\langle S_C x, x\rangle_{\mathcal{A}} = \left\langle \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega), x \right\rangle_{\mathcal{A}} = \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(\omega),$$

then

$$\langle Ax, x\rangle_{\mathcal{A}} \leq \langle Sx, x\rangle_{\mathcal{A}} \leq \langle Bx, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{H}.$$

So,

$$A.I \leq S \leq B.I.$$

(2) $\Rightarrow$ (1) Let $x \in \mathcal{H}$, then

$$\left\| \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \right\| = \|\langle S_C x, x\rangle_{\mathcal{A}}\| \leq \|S_C x\| \|x\| \leq B\|x\|^2. \tag{3.6}$$

Also,

$$\|\langle S_C x, x\rangle_{\mathcal{A}}\| \geq \|\langle Ax, x\rangle_{\mathcal{A}}\| = A\|x\|^2. \tag{3.7}$$

By (3.6) and (3.7) we obtain

$$A\|x\|^2 \leq \left\| \int_{\Omega} \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2,$$

which ends the proof. □

**Proposition 3.2.** *Let $C \in GL^+(\mathcal{H})$ and $F$ be a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A$ and $B$. Then $F$ is an integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$.*

*Proof.* Let $F$ be a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A$ and $B$.

On one hand we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \langle S_C x, x\rangle_{\mathcal{A}} = \langle CSx, x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}} Sx, C^{\frac{1}{2}} x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2 \langle Sx, x\rangle_{\mathcal{A}}.$$

So,

$$A\|C^{\frac{1}{2}}\|^{-2}\langle x,x\rangle_{\mathcal{A}} \leq \int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}}d\mu(w). \tag{3.8}$$

On other hand, for all $x\in\mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}}d\mu(w) &= \langle Sx,x\rangle_{\mathcal{A}}\\ &= \langle C^{-1}CSx,x\rangle_{\mathcal{A}}\\ &= \langle (C^{-1}CS)^{\frac{1}{2}}x,(C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x,(CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x,(S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &= \|C^{\frac{-1}{2}}\|^2\langle S_Cx,x\rangle_{\mathcal{A}}\\ &\leq \|C^{\frac{-1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}.\end{aligned}$$

Then

$$\int_\Omega \langle x,F_\omega\rangle_{\mathcal{A}}\langle F_\omega,x\rangle_{\mathcal{A}}d\mu(w) \leq \|C^{\frac{-1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}. \tag{3.9}$$

From (3.8) and (3.9) we conclude that $F$ is an integral frame $\mathcal{H}$ associted to $(\Omega,\mu)$ with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$. □

**Proposition 3.3.** *Let $C\in GL^+(\mathcal{H})$ and $F$ be an integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$ with bounds $A$ and $B$. Then $F$ is a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$ with bounds $A\|C^{\frac{-1}{2}}\|^2$ and $B\|C^{\frac{1}{2}}\|^2$.*

*Proof.* Let $F$ be an integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$ with bounds $A$ and $B$. Then for all $x\in\mathcal{H}$, we have

$$\begin{aligned}A\langle x,x\rangle_{\mathcal{A}} &\leq \langle Sx,x\rangle_{\mathcal{A}}\\ &= \langle C^{-1}CSx,x\rangle_{\mathcal{A}}\\ &= \langle (C^{-1}CS)^{\frac{1}{2}}x,(C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x,(CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x,(S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &= \|C^{\frac{-1}{2}}\|^2\langle S_Cx,x\rangle_{\mathcal{A}}.\end{aligned}$$

So,

$$A\|C^{\frac{-1}{2}}\|^{-2}\langle x,x\rangle_{\mathcal{A}} \leq \langle S_Cx,x\rangle_{\mathcal{A}}.$$

Hence, for all $x\in\mathcal{H}$, we have

$$\langle S_Cx,x\rangle_{\mathcal{A}} = \langle CSx,x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}}Sx,C^{\frac{1}{2}}x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2\langle Sx,x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2B\langle x,x\rangle_{\mathcal{A}}.$$

Therefore we conclude that $F$ is a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$ with bounds $A\|C^{\frac{-1}{2}}\|^{-2}$ and $B\|C^{\frac{1}{2}}\|^2$. □

**Theorem 3.2.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega,\mu)$ be a measure space. Let $F$ be a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$ with the frame operator $S_C$ and bounds $A$ and $B$. Let $K\in End^*_{\mathcal{A}}(\mathcal{H})$ be a surjective operator such that $KC=CK$. Then $KF$ is a $C$-controlled integral frame for $\mathcal{H}$ with the operator frame $KS_CK^*$.*

*Proof.* Let $F$ be a $C$-controlled integral frame for $\mathcal{H}$ associted to $(\Omega,\mu)$, then

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}}\leq\int_{\Omega}\langle K^*x,F_\omega\rangle_{\mathcal{A}}\langle CF_\omega,K^*x\rangle_{\mathcal{A}}d\mu(w)\leq B\langle K^*x,K^*x\rangle_{\mathcal{A}},\quad x\in\mathcal{H}.$$

By Lemma 1.1 and Lemma 1.3, we obtain

$$A\|(KK^*)^{-1}\|^{-1}\langle x,x\rangle_{\mathcal{A}}\leq\int_{\Omega}\langle x,KF_\omega\rangle_{\mathcal{A}}\langle CKF_\omega,x\rangle_{\mathcal{A}}d\mu(w)\leq B\|K^*\|^2\langle x,x\rangle_{\mathcal{A}},\quad x\in\mathcal{H},$$

which shows that $KF$ is a $C$-controlled integral frame.

Moreover, by Lemma 1.4, we have

$$KS_CK^*x=K\int_{\Omega}\langle K^*x,F_\omega\rangle_{\mathcal{A}}CF_\omega d\mu(\omega)=\int_{\Omega}\langle x,KF_\omega\rangle_{\mathcal{A}}CKF_\omega d\mu(\omega),$$

which ends the proof. □

## 4. Controlled $*$-Integral Frames

**Definition 4.1.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega,\mu)$ be a measure space. A $C$-controlled $*$-integral frame in $\mathcal{A}$-module $\mathcal{H}$ is a map $F:\Omega\to\mathcal{H}$ such that there exist two strictly nonzero elements A,B in $\mathcal{A}$ such that

$$A\langle x,x\rangle_{\mathcal{A}}A^*\leq\int_{\Omega}\langle x,F_\omega\rangle_{\mathcal{A}}\langle CF_\omega,x\rangle_{\mathcal{A}}d\mu(w)\leq B\langle x,x\rangle_{\mathcal{A}}B^*,\quad x\in\mathcal{H}. \tag{4.1}$$

The elements $A$ and $B$ are called the $C$-controlled $*$-integral frame bounds. If $A=B$, we call this a $C$-controlled $*$-integral tight frame. If $A=B=1$, it's called a $C$-controlled $*$-integral parseval frame. If only the right hand inequality of (4.1) is satisfied, we call $F$ a $C$-controlled $*$-integral Bessel mapping with bound $B$.

*Example* 4.1. Let $\mathcal{H}=\mathcal{A}=\{(a_n)_{n\in\mathbb{N}}\subset\mathbb{C}\mid\sum_{n\geq0}|a_n|<\infty\}$. Endowed with the product and the inner product defined as follow.

$$\begin{array}{lcl}\mathcal{A}\times\mathcal{A} & \to & \mathcal{A},\\ ((a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}) & \mapsto & (a_n)_{n\in\mathbb{N}}.(b_n)_{n\in\mathbb{N}}=(a_nb_n)_{n\in\mathbb{N}},\end{array}$$

and

$$\begin{array}{lcl}\mathcal{H}\times\mathcal{H} & \to & \mathcal{A},\\ ((a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}) & \mapsto & \langle(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}}=(a_n\overline{b_n})_{n\in\mathbb{N}}.\end{array}$$

Let $\Omega=[0,+\infty[$ endowed with the Lebesgue's measure which is a measure space

$$F:[0,+\infty[\,\to\mathcal{H},$$
$$w\mapsto F_w=(F_n^w)_{n\in\mathbb{N}},$$

where

$$F_n^w=\frac{1}{n+1},\quad\text{if}\quad n=[w],\quad\text{and}\quad F_n^w=0,\quad\text{elsewhere},$$

where $[w]$ is the whole part of $w$.

On the other hand, we consider the measure space $(\Omega, \mu)$, where $\mu$ is the Lebesgue measure restricted to $[0, +\infty[$, and the operator

$$C : \mathcal{H} \to \mathcal{H},$$
$$(a_n)_{n\in\mathbb{N}} \to (\alpha a_n)_{n\in\mathbb{N}},$$

where $\alpha$ is a strictly positive real number.

It's clear that $C$ is an invertible and both operators and $C$ and $C^{-1}$ are bounded. So,

$$\begin{aligned}
&\int_\Omega \langle (a_n)_{n\in\mathbb{N}}, F_w\rangle_{\mathcal{A}} \langle CF_w, (a_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}} d\mu(w)\\
=&\int_0^{+\infty} \left(0, 0, \dots, \frac{a_{[w]}}{[w]+1}, 0, \dots\right) \alpha \left(0, 0, \dots, \overline{\frac{a_{[w]}}{[w]+1}}, 0, \dots\right) d\mu(w)\\
=&\alpha \sum_{p=0}^{+\infty} \int_p^{p+1} \left(0, 0, \dots, \frac{|a_{[w]}|^2}{([w]+1)^2}, 0, \dots\right) d\mu(w)\\
=&\alpha \sum_{p=0}^{+\infty} \left(0, 0, \dots, \frac{|a_{[p]}|^2}{(p+1)^2}, 0, \dots\right)\\
=&\alpha \left(\frac{|a_n|^2}{(n+1)^2}\right)_{n\in\mathbb{N}}\\
=&\sqrt{\alpha}\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \langle (a_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}\rangle_{\mathcal{A}} \sqrt{\alpha}\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right),
\end{aligned}$$

which shows that $F$ is a $C$-controlled $*$-integral tight frame for $\mathcal{H}$ with bound $A = \sqrt{\alpha}\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \mathcal{A}$.

**Definition 4.2.** Let $F$ be a $C$-controlled $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$. We define the frame operator $S_C : \mathcal{H} \to \mathcal{H}$ for $F$ by

$$S_C x = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega), \quad x \in \mathcal{H}.$$

**Proposition 4.1.** *The frame operator $S_C$ is positive, selfadjoint, bounded and invertible.*

*Proof.* For all $x \in \mathcal{H}$, by Lemma 1.4, we have

$$\langle S_C x, x\rangle_{\mathcal{A}} = \left\langle \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega), x\right\rangle_{\mathcal{A}} = \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}} \langle CF_\omega, x\rangle_{\mathcal{A}} d\mu(\omega).$$

By left hand of inequality (4.1), we deduce that $S_C$ is a positive operator, also, it's sefladjoint. From (4.1), we have

$$A\langle x, x\rangle_{\mathcal{A}} A^* \le \langle S_C x, x\rangle_{\mathcal{A}} \le B\langle x, x\rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}.$$

The Theorem 2.5 in [18] shows that $S_C$ is invertible. $\square$

**Proposition 4.2.** *Let $C \in GL^+(\mathcal{H})$ and $F$ be a $C$-controlled $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A$ and $B$. Then $F$ is a $*$-integral frame $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $\|C^{\frac{1}{2}}\|^{-1}A$ and $\|C^{\frac{-1}{2}}\|B$.*

*Proof.* Let $F$ be a $C$-controlled $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$, with bounds $A$ and $B$.

On one hand we have

$$A\langle x, x\rangle_{\mathcal{A}}A^* \leq \langle S_C x, x\rangle_{\mathcal{A}} = \langle CSx, x\rangle_{\mathcal{A}} = \langle C^{\frac{1}{2}}Sx, C^{\frac{1}{2}}x\rangle_{\mathcal{A}} \leq \|C^{\frac{1}{2}}\|^2\langle Sx, x\rangle_{\mathcal{A}}.$$

So,

$$(\|C^{\frac{1}{2}}\|^{-1}A)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{1}{2}}\|^{-1}A)^* \leq \int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}}\langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w). \tag{4.2}$$

On other hand, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
\int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}}\langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) &= \langle Sx, x\rangle_{\mathcal{A}} \\
&= \langle C^{-1}CSx, x\rangle_{\mathcal{A}} \\
&= \langle (C^{-1}CS)^{\frac{1}{2}}x, (C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x, (CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x, (S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2\langle S_C x, x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2 B\langle x, x\rangle_{\mathcal{A}}B^*.
\end{aligned}$$

Then

$$\int_\Omega \langle x, F_\omega\rangle_{\mathcal{A}}\langle F_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq (\|C^{\frac{-1}{2}}\|B)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{-1}{2}}\|B)^*. \tag{4.3}$$

From (4.2) and (4.3) we conclude that $F$ is a $*$-integral frame $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$. $\square$

**Proposition 4.3.** *Let $C \in GL^+(\mathcal{H})$ and $F$ be an $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A$ and $B$. Then $F$ is a $C$-controlled $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $\|C^{\frac{-1}{2}}\|^{-1}A$ and $\|C^{\frac{1}{2}}\|B$.*

*Proof.* Let $F$ be an integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $A$ and $B$. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned}
A\langle x, x\rangle_{\mathcal{A}}A^* &\leq \langle Sx, x\rangle_{\mathcal{A}} \\
&= \langle C^{-1}CSx, x\rangle_{\mathcal{A}} \\
&= \langle (C^{-1}CS)^{\frac{1}{2}}x, (C^{-1}CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\
&\leq \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}x, (CS)^{\frac{1}{2}}x\rangle_{\mathcal{A}} \\
&= \|C^{\frac{-1}{2}}\|^2\langle (S_C)^{\frac{1}{2}}x, (S_C)^{\frac{1}{2}}x\rangle_{\mathcal{A}}
\end{aligned}$$

$$= \|C^{\frac{-1}{2}}\|^2 \langle S_C x, x\rangle_{\mathcal{A}}.$$

So,

$$(\|C^{\frac{-1}{2}}\|^{-1}A)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{-1}{2}}\|^{-1}A)^* \leq \langle S_C x, x\rangle_{\mathcal{A}}.$$

Hence, for all $x \in \mathcal{H}$,

$$\begin{aligned}\langle S_C x, x\rangle_{\mathcal{A}} &= \langle CSx, x\rangle_{\mathcal{A}}\\ &= \langle C^{\frac{1}{2}}Sx, C^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\ &\leq \|C^{\frac{1}{2}}\|^2\langle Sx, x\rangle_{\mathcal{A}}\\ &\leq \|C^{\frac{1}{2}}\|^2 B\langle x, x\rangle_{\mathcal{A}}B^*\\ &= (\|C^{\frac{1}{2}}\|B)\langle x, x\rangle_{\mathcal{A}}(\|C^{\frac{1}{2}}\|B)^*.\end{aligned}$$

Therefore, we conclude that $F$ is a $C$-controlled $*$-integral frame $\mathcal{H}$ associted to $(\Omega, \mu)$ with bounds $\|C^{\frac{-1}{2}}\|^{-1}A$ and $\|C^{\frac{1}{2}}\|B$. □

**Theorem 4.1.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\Omega, \mu)$ be a measure space. Let $F$ a $C$-controlled $*$-integral frame for $\mathcal{H}$ associted to $(\Omega, \mu)$ with the frame operator $S_C$ and bounds $A$ and $B$. Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ a surjective operator such that $KC = CK$. Then $KF$ is a $C$-controlled $*$-integral frame for $\mathcal{H}$ with the operator frame $KS_CK^*$.*

*Proof.* By (4.1), we have

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}}\langle CF_\omega, K^*x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle K^*x, K^*x\rangle_{\mathcal{A}}B^*, \quad x \in \mathcal{H}.$$

By Lemma 1.1 and Lemma 1.3, we obtain

$$A\|(KK^*)^{-1}\|^{-1}\langle x, x\rangle_{\mathcal{A}}A^* \leq \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}}\langle CKF_\omega, x\rangle_{\mathcal{A}} d\mu(w) \leq B\|K^*\|^2\langle x, x\rangle_{\mathcal{A}}B^*,$$

which shows that $KF$ is a $C$-controlled $*$-integral operator. Moreover, by Lemma 1.4, we have

$$KS_CK^*x = K\int_\Omega \langle K^*x, F_\omega\rangle_{\mathcal{A}} CF_\omega d\mu(\omega) = \int_\Omega \langle x, KF_\omega\rangle_{\mathcal{A}} CKF_\omega d\mu(\omega),$$

which ends the proof. □

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# STRUCTURE OF 3-PRIME NEAR RINGS WITH GENERALIZED $(\sigma, \tau)$-$n$-DERIVATIONS

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Abstract. In this paper, we define generalized $(\sigma, \tau)$-$n$-derivation for any mappings $\sigma$ and $\tau$ of a near ring $N$ and also investigate the structure of a 3-prime near ring satisfying certain identities with generalized $(\sigma, \tau)$-$n$-derivation. Moreover, we characterize the aforementioned mappings.

## 1. Introduction

A left near ring $N$ is a triplet $(N, +, .)$, where $+$ and $.$ are two binary operations such that $(i)\,(N, +)$ is a group (not necessarily abelian); $(ii)\,(N, .)$ is a semigroup, and $(iii)\,x.(y + z) = x.y + x.z$ for all $x, y, z \in N$. Analogously, if $N$ satisfies the right distributive law, i.e., $(x + y).z = x.z + y.z$ for all $x, y \in N$, then $N$ is said to be a right near ring. The most natural example of a left near ring is the set of all identity preserving mappings acting from right of an additive group $G$ (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication. If these mappings act from left on $G$, then we get a right near ring (Pilz [10, Example 1.4]). Throughout the paper, $N$ denotes a zero-symmetric left near ring with multiplicative centre $Z$ and for any pair of elements $x, y \in N$, $[x, y] = xy - yx$, $x \circ y = xy + yx$ and $(x, y) = x + y - x - y$ stand for the Lie product, Jordan Product and additive commutator respectively. Let $\sigma$ and $\tau$ be mappings on $N$. For any $x, y \in N$, set the symbol $[x, y]_{\sigma,\tau}$ will denote $x\sigma(y) - \tau(y)x$, while the symbol $(x \circ y)_{\sigma,\tau}$ will denote $x\sigma(y) + \tau(y)x$. The terminology multiplicative mappings on a near ring $N$ is used for the mappings $\sigma, \tau : N \to N$ satisfying $\sigma(xy) = \sigma(x)\sigma(y)$

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and $\tau(xy) = \tau(x)\tau(y)$ for all $x, y \in N$. A near ring $N$ is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields that $x0 = 0$). A near ring $N$ is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring $N$ is called 2-torsion free if $(N, +)$ has no element of order 2. A nonempty subset $U$ of $N$ is called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$) and if $U$ is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal.

Let $n \geq 2$ be a fixed positive integer and $N^n = \underbrace{N \times N \times \cdots \times N}_{n-\text{times}}$. A map $\Delta : N^n \to N$ is said to be permuting (symmetric) on a near ring $N$ if the relation $\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds for all $x_i \in N$, $i = 1, 2, \ldots, n$, and for every permutation $\pi \in S_n$, where $S_n$ is the permutation group on $\{1, 2, \ldots, n\}$. An additive mapping $F : N \to N$ is said to be a right (resp. left) generalized derivation with associated derivation $d$ if $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$), for all $x, y \in N$ and $F$ is said to be a generalized derivation with associated derivation $d$ on $N$ if it is both a right generalized derivation and a left generalized derivation on $N$ with associated derivation $d$.

Ozturk et al. [9] and Park et al. [6] studied bi-derivations and tri-derivations in near rings. Further, Ceven et al. [4] and Ozturk et al. [8] defined $(\sigma, \tau)$ bi-derivations and $(\sigma, \tau)$ tri-derivations in near rings. Let $\sigma$, $\tau$ be automorphisms on a near ring $N$. A symmetric bi-additive (additive in both arguments) mapping $d : N \times N \to N$ is said to be a $(\sigma, \tau)$ bi-derivation if $d(xx', y) = d(x, y)\sigma(x') + \tau(x)d(x', y)$ holds for all $x, x', y \in N$. A symmetric tri-additive (additive in each argument) mapping $d : N \times N \times N \to N$ is said to be a $(\sigma, \tau)$ tri-derivation if $d(xx', y, z) = d(x, y, z)\sigma(x') + \tau(x)d(x', y, z)$ holds for all $x, x', y, z \in N$.

Motivated by these concepts, we define $(\sigma, \tau)$-$n$-derivation and generalized $(\sigma, \tau)$-$n$-derivation for any arbitrary mappings $\sigma$ and $\tau$ of a near ring $N$ in place of automorphisms.

**Definition 1.1** (($\sigma, \tau$)-$n$-derivation)**.** Let $\sigma, \tau : N \to N$ be mappings on $N$. An $n$-additive (additive in each argument) mapping $d : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is called a $(\sigma, \tau)$-$n$-derivation of $N$ if the following equations

$$\begin{aligned}
d(x_1x_1', x_2, \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)d(x_1', x_2, \ldots, x_n),\\
d(x_1, x_2x_2', \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_2') + \tau(x_2)d(x_1, x_2', \ldots, x_n),\\
&\vdots\\
d(x_1, x_2, \ldots, x_nx_n') =& d(x_1, x_2, \ldots, x_n)\sigma(x_n') + \tau(x_n)d(x_1, x_2, \ldots, x_n')
\end{aligned}$$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

**Definition 1.2** (Right generalized $(\sigma, \tau)$-$n$-derivation)**.** An $n$-additive (additive in each argument) mapping $F : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is called a right generalized

$(\sigma,\tau)$-$n$-derivation associated with $(\sigma,\tau)$-$n$-derivation $d$ on $N$ if the relations

$$\begin{aligned}F(x_1x_1',x_2,\dots,x_n) =&F(x_1,x_2,...,x_n)\sigma(x_1')+\tau(x_1)d(x_1',x_2,\dots,x_n),\\ F(x_1,x_2x_2',\dots,x_n) =&F(x_1,x_2,\dots,x_n)\sigma(x_2')+\tau(x_2)d(x_1,x_2',\dots,x_n),\\ &\vdots\\ F(x_1,x_2,\dots,x_nx_n') =&F(x_1,x_2,\dots,x_n)\sigma(x_n')+\tau(x_n)d(x_1,x_2,\dots,x_n')\end{aligned}$$

hold for all $x_1,x_1',x_2,x_2',\dots,x_n,x_n'\in N$.

**Definition 1.3** (Left generalized $(\sigma,\tau)$-$n$-derivation)**.** An $n$-additive (additive in each argument) mapping $F:\underbrace{N\times N\times\cdots\times N}_{n-\text{times}}\to N$ is called a left generalized $(\sigma,\tau)$-$n$-derivation associated with $(\sigma,\tau)$-$n$-derivation $d$ on $N$ if the relations

$$\begin{aligned}F(x_1x_1',x_2,\dots,x_n) =&d(x_1,x_2,\dots,x_n)\sigma(x_1')+\tau(x_1)F(x_1',x_2,\dots,x_n),\\ F(x_1,x_2x_2',\dots,x_n) =&d(x_1,x_2,\dots,x_n)\sigma(x_2')+\tau(x_2)F(x_1,x_2',\dots,x_n),\\ &\vdots\\ F(x_1,x_2,\dots,x_nx_n') =&d(x_1,x_2,\dots,x_n)\sigma(x_n')+\tau(x_n)F(x_1,x_2,\dots,x_n')\end{aligned}$$

hold for all $x_1,x_1',x_2,x_2',\dots,x_n,x_n'\in N$.

A mapping $F:\underbrace{N\times N\times\cdots\times N}_{n-\text{times}}\to N$ is called a generalized $(\sigma,\tau)$-$n$-derivation associated with $(\sigma,\tau)$-$n$-derivation $d$ on $N$ if $F$ is both a right generalized $(\sigma,\tau)$-$n$-derivation and a left generalized $(\sigma,\tau)$-$n$-derivation associated with $(\sigma,\tau)$-$n$-derivation $d$ on $N$.

*Example* 1.1. *Let $S$ be a zero-symmetric left near ring and*

$$N=\left\{\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}\mid x,y,z,0\in S\right\}.$$

*Then $N$ is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings $F,d:\underbrace{N\times N\times\cdots\times N}_{n-times}\to N$ by*

$$F\left(\begin{pmatrix}0&x_1&y_1\\0&0&z_1\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&z_2\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&z_n\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&0\\0&0&z_1z_2\dots z_n\\0&0&0\end{pmatrix},$$

$$d\left(\begin{pmatrix}0&x_1&y_1\\0&0&z_1\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&z_2\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&z_n\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&x_1x_2\dots x_n&0\\0&0&0\\0&0&0\end{pmatrix}.$$

*Define $\sigma,\tau : N \to N$ by*

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad and \quad \tau\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

*It is easy to check that $F$ is a nonzero right (but not left) generalized $(\sigma,\tau)$-$n$-derivation associated with a nonzero $(\sigma,\tau)$-$n$-derivation $d$ of $N$, where $\sigma$ and $\tau$ are any arbitrary mappings on $N$.*

*Example* 1.2. *Let $N$ be a zero-symmetric left near ring as in Example* 1.1. *Define mappings $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ by*

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1x_2\dots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_1z_2\dots z_n \\ 0 & 0 & 0 \end{pmatrix}.$$

*Define $\sigma,\tau : N \to N$ by*

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^2 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \quad and \quad \tau\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

*It can be easily seen that $F$ is a nonzero left (but not right) generalized $(\sigma,\tau)$-$n$-derivation associated with a nonzero $(\sigma,\tau)$-$n$-derivation $d$ of $N$ for any arbitrary mappings $\sigma$ and $\tau$ on $N$.*

*Example* 1.3. *Let $S$ be a zero-symmetric left near ring and*

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

*It is easy to see that $N$ is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ by*

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & y_1y_2\dots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z_1z_2\dots z_n & 0 \end{pmatrix}.$$

*Define* $\sigma, \tau : N \to N$ *by*

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^2 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tau\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & yz & 0 \end{pmatrix}.$$

*It can be easily verified that $F$ is a nonzero right as well as left generalized $(\sigma,\tau)$-$n$-derivation associated with a nonzero $(\sigma,\tau)$-$n$-derivation $d$ of $N$, where $\sigma$ and $\tau$ are any arbitrary mappings on $N$.*

Obviously this notion covers the notion of a generalized $n$-derivation (in case $\sigma = \tau = I$), notion of an $n$-derivation (in case $F = d$, $\sigma = \tau = I$), notion of a left $n$-centralizer (in case $d = 0$, $\sigma = I$), notion of a $(\sigma,\tau)$-$n$-derivation (in case $F = d$) and the notion of a left $\sigma$-$n$-multiplier (in case $d = 0$). Thus, it is interesting to investigate the properties of this general notion. In [7], Bresar has proved that if $R$ is a 2-torsion free semiprime ring and $F : R \to R$ is an additive map on $R$ such that $F(x)x + xF(x) = 0$ for all $x \in R$, then $F = 0$. Further, Vukman [5] proved that if there exist a derivation $d : R \to R$ and an automorphism $\alpha : R \to R$, where $R$ is 2-torsion free semiprime ring such that $[d(x)x + xd(x), x] = 0$ for all $x \in R$, then $d$ and $\alpha - I$, $I$ denotes the identity mapping on $R$, map $R$ into its centre. Motivated by the mentioned results we prove that if a 3-prime near ring $N$ with a generalized $(\sigma,\tau)$-$n$-derivation $F$ satisfies certain identity, then $N$ is a commutative ring and $F$ is a left $\sigma$-$n$-multiplier on $N$.

## 2. Some Preliminaries

**Lemma 2.1.** ([1, Lemmas 1.2]). *Let $N$ be 3-prime near ring.*

(i) *If $z \in Z \setminus \{0\}$, then $z$ is not a zero divisor.*
(ii) *If $Z \setminus \{0\}$ and $x$ is an element of $N$ for which $xz \in Z$, then $x \in Z$.*

**Lemma 2.2.** ([1, Lemmas 1.3 and Lemma 1.4]). *Let $N$ be 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$.*

(i) *If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) *If $x \in N$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.*

**Lemma 2.3.** ([1, Lemma 1.5]). *If $N$ is a 3-prime near ring and $Z$ contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then $N$ is a commutative ring.*

**Lemma 2.4.** If $N$ is a 3-prime near ring admitting a generalized $(\sigma,\tau)$-$n$-derivation $F$ associated with a $(\sigma,\tau)$-$n$-derivation $d$ of $N$ such that $\sigma$ and $\tau$ are multiplicative mappings on $N$, then

$$\begin{aligned}
&\{d(x_1, x_2, \ldots, x_n)\sigma(y_1) + \tau(x_1)F(y_1, x_2, \ldots, x_n)\}\sigma(z_1) \\
=&d(x_1, x_2, \ldots, x_n)\sigma(y_1)\sigma(z_1) + \tau(x_1)F(y_1, x_2, \ldots, x_n)\sigma(z_1), \\
&\{d(x_1, x_2, \ldots, x_n)\sigma(y_2) + \tau(x_2)F(x_1, y_2, \ldots, x_n)\}\sigma(z_2) \\
=&d(x_1, x_2, \ldots, x_n)\sigma(y_2)\sigma(z_2) + \tau(x_2)F(x_1, y_2, \ldots, x_n)\sigma(z_2),
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\{d(x_1,x_2,\dots,x_n)\sigma(y_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\}\sigma(z_n)\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_n)\sigma(z_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\sigma(z_n),\end{aligned}$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,\dots,x_n,y_n,z_n \in N$.

*Proof.* For all $x_1,y_1,z_1,x_2,y_2,z_2,\dots,x_n,y_n,z_n \in N$

$$\begin{aligned}F(x_1y_1z_1,x_2,\dots,x_n)=&F(x_1y_1,x_2,\dots,x_n)\sigma(z_1)+\tau(x_1y_1)d(z_1,x_2,\dots,x_n)\\ =&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}\sigma(z_1)\\ &+\tau(x_1)\tau(y_1)d(z,u_2,\dots,u_n)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}F(x_1y_1z_1,x_2,\dots,x_n)=&d(x_1,x_2,\dots,x_n)\sigma(y_1z_1)+\tau(x_1)F(y_1z_1,x_2,\dots,x_n)\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_1)\sigma(z_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\sigma(z_1)\\ &+\tau(x_1)\tau(y_1)d(z_1,x_2,\dots,x_n).\end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\begin{aligned}&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}\sigma(z_1)\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_1)\sigma(z_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\sigma(z_1).\end{aligned}$$

Similarly, we can prove other relations for $i=2,3,\dots,n$. □

*Remark* 2.1. If $\sigma$ is an onto map on $N$, then Lemma 2.4 becomes

$$\begin{aligned}&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}a\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_1)a+\tau(x_1)F(y_1,x_2,\dots,x_n)a,\\ &\{d(x_1,x_2,\dots,x_n)\sigma(y_2)+\tau(x_2)F(x_1,y_2,\dots,x_n)\}a\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_2)a+\tau(x_2)F(x_1,y_2,\dots,x_n)a,\\ &\vdots\\ &\{d(x_1,x_2,\dots,x_n)\sigma(y_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\}a\\ =&d(x_1,x_2,\dots,x_n)\sigma(y_n)a+\tau(x_n)F(x_1,x_2,\dots,y_n)a,\end{aligned}$$

for all $x_1,y_1,x_2,y_2,\dots,x_n,y_n,a \in N$.

**Lemma 2.5.** Let $N$ be a 3-prime near ring and $U_1,U_2,\dots,U_n$ be nonzero semigroup ideals of $N$. Let $\sigma$ and $\tau$ be mappings on $N$ such that $U_i \subseteq \tau(U_i)$ for $i=1,2,\dots,n$. If $d$ is a nonzero $(\sigma,\tau)$-$n$-derivation on $N$, then $d(U_1,U_2,\dots,U_n)\neq\{0\}$.

*Proof.* Assume that

$$d(u_1,u_2,\dots,u_n)=0,\quad \text{for all } u_1\in U_1,u_2\in U_2,\dots,u_n\in U_n.\tag{2.3}$$

Replacing $u_1$ by $u_1r_1$, where $r_1\in N$ in (2.3) and using (2.3), we get

$$\tau(u_1)d(r_1,u_2,\dots,u_n)=0.$$

Since $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$, we have $U_1 d(r_1, u_2, \ldots, u_n) = \{0\}$. Applying Lemma 2.2 (ii), we obtain $d(r_1, u_2, \ldots, u_n) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$ and $r_1 \in N$. Replacing $u_2$ by $u_2 r_2$, where $r_2 \in N$ in the last expression and another application of Lemma 2.2(ii) yields that $d(r_1, r_2, \ldots, u_n) = 0$. Proceeding inductively, we conclude that $d(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_2, \ldots, r_n \in N$, a contradiction which completes the proof. □

**Lemma 2.6.** Let $N$ be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of $N$. Let $\sigma, \tau$ be multiplicative mappings on $U_i$ such that $U_1 \subseteq \sigma(U_1)$. If $d$ is a nonzero $(\sigma, \tau)$-$n$-derivation on $N$ such that $d(U_1, U_2, \ldots U_n)\sigma(a) = \{0\}$ or $\sigma(a)d(U_1, U_2, \ldots U_n) = \{0\}$ for all $a \in N$, then $\sigma(a) = 0$.

*Proof.* Suppose that $d(U_1, U_2, \ldots, U_n)\sigma(a) = \{0\}$. Then

$$d(u_1, u_2, \ldots, u_n)\sigma(a) = 0, \quad \text{for all } u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{2.4}$$

Replacing $u_1$ by $u_1 u_1'$ in (2.4) and using it again yields that

$$d(u_1, u_2, \ldots, u_n)\sigma(u_1')\sigma(a) = 0, \quad \text{for all } u_1, u_1' \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Equivalently,

$$d(u_1, u_2, \ldots, u_n)\sigma(U_1)\sigma(a) = \{0\}, \quad \text{for all } u_1, \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Since $U_1 \subseteq \sigma(U_1)$, we obtain

$$d(u_1, u_2, \ldots, u_n)U_1\sigma(a) = \{0\}, \quad \text{for all } u_1, \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Applying Lemma 2.2 (i) and Lemma 2.5, we obtain $\sigma(a) = 0$. Similarly, we can prove the result for later case. □

**Lemma 2.7.** Let $N$ be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of $N$. Let $\sigma$ be a onto map on $N$ such that $U_1 \subseteq \sigma(U_1)$ and $U_1 \cap Z \neq \emptyset$. If $d$ is a $(\sigma, \sigma)$-$n$-derivation on $N$, then $d(Z, U_2, U_3, \ldots, U_n) \subseteq Z$.

*Proof.* Suppose that $z \in U_1 \cap Z$. Then

$$d(zx_1, x_2, \ldots, x_n) = d(x_1 z, x_2, \ldots, x_n), \quad \text{for all } x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n,$$

and

$$\begin{aligned} &d(z, x_2, \ldots, x_n)\sigma(x_1) + \sigma(z)d(x_1, x_2, \ldots, x_n) \\ =&\sigma(x_1)d(z, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n)\sigma(z). \end{aligned}$$

Substituting $x_1' \in U_1$ and $z' \in U_1 \cap Z$ for $\sigma(x_1)$ and $\sigma(z)$ respectively, we get

$$d(z, x_2, \ldots, x_n)x_1' = x_1' d(z, x_2, \ldots, x_n), \quad \text{for all } x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Replacing $x_1'$ by $x_1' r$ for $r \in N$ in above expression and using it again, we find that $x_1'[d(z, x_2, \ldots, x_n), r] = 0$. Hence, $d(Z, U_2, U_3, \ldots, U_n) \subseteq Z$ by Lemma 2.2 (ii). □

**Lemma 2.8.** Let $N$ be a 3-prime near ring and $U_1, U_2, \dots, U_n$ be nonzero semigroup ideals of $N$. Let $\sigma, \tau$ be mappings on $N$ such that $U_i \subseteq \sigma(U_i)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \dots, n$. If $F$ is a nonzero right generalized $(\sigma, \tau)$-$n$-derivation associated with a $(\sigma, \tau)$-$n$-derivation $d$ on $N$, then $F(U_1, U_2, \dots, U_n) \neq \{0\}$.

*Proof.* Let

$$F(u_1, u_2, \dots, u_n) = 0, \quad \text{for all } u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{2.5}$$

Replacing $u_1$ by $u_1 r_1$, where $r_1 \in N$ in (2.5) and using (2.5), we get

$$\tau(u_1) d(r_1, u_2, \dots, u_n) = \{0\}.$$

Since $U_1 \subseteq \tau(U_1)$, we have

$$U_1 d(r_1, u_2, \dots, u_n) = \{0\}, \quad \text{for all } u_2 \in U_2, \dots, u_n \in U_n \text{ and } r_1 \in N.$$

Applying Lemma 2.2(ii), we find

$$d(r_1, u_2, \dots, u_n) = 0, \quad \text{for all } u_2 \in U_2, \dots, u_n \in U_n \text{ and } r_1 \in N. \tag{2.6}$$

Now replacing $u_2$ by $u_2 r_2$ in (2.6) for $r_2 \in N$ and another application of Lemma 2.2 (ii) yields that $d(r_1, r_2, u_3, \dots, u_n) = 0$ for all $u_3 \in U_3, \dots, u_n \in U_n$ and $r_1, r_2 \in N$. Proceeding inductively, we get $d(r_1, r_2, \dots, r_n) = 0$ for all $r_1, r_2, \dots, r_n \in N$, i.e., $d = 0$. Therefore, our hypothesis reduces to

$$F(r_1 u_1, u_2, \dots, u_n) = F(r_1, u_2, \dots, u_n)\sigma(u_1) = 0,$$

for all $u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n$ and $r_1 \in N$ which implies that

$$F(r_1, u_2, \dots, u_n) = 0, \quad \text{for all } u_2 \in U_2, \dots, u_n \in U_n \text{ and } r_1 \in N. \tag{2.7}$$

Replacing $u_2$ by $r_2 u_2$ in (2.7), we get $F(r_1, r_2, \dots, u_n)U_2 = \{0\}$ and Lemma 2.2 (ii) gives $F(r_1, r_2, u_3, \dots, u_n) = 0$ for all $u_3 \in U_3, \dots, u_n \in U_n$ and $r_1, r_2 \in N$. Proceeding inductively, we obtain $F = 0$ on $N$, a contradiction. □

## 3. Main Results

**Theorem 3.1.** *Let $N$ be a 3-prime near ring and $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of $N$. Suppose that $\sigma$, $\tau$ are multiplicative mappings on $U_i$ for $i = 1, 2, \dots, n$, such that $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \dots, n$, and $\sigma$ is onto on $N$. If $N$ admits a generalized $(\sigma, \tau)$-$n$-derivation $F$ associated with a $(\sigma, \tau)$-$n$-derivation $d$ such that $F(x_1 x_1', x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)F(x_1', x_2, \dots, x_n)$ for all $x_1, x_1' \in U_1, x_2 \in U_2, \dots, x_n \in U_n$, then $F$ is a left $\sigma$-$n$-multiplier on $N$.*

*Proof.* By hypothesis

$$\begin{aligned} F(x_1 x_1', x_2, \dots, x_n) &= d(x_1, x_2, \dots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \dots, x_n) \\ &= F(x_1, x_2, \dots, x_n)F(x_1', x_2, \dots, x_n), \end{aligned}$$

for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Replacing $x_1'$ by $x_1'z$ for $z \in U_1$ in the above relation, we get

$$\begin{aligned} &\{d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \ldots, x_n)\}F(z, x_2, \ldots, x_n) \\ =&d(x_1, x_2, \ldots, x_n)\sigma(x_1'z) + \tau(x_1)\{d(x_1', x_2, \ldots, x_n)\sigma(z) + \tau(x_1')F(z, x_2, \ldots, x_n)\}. \end{aligned}$$

Applying Lemma 2.4 and using the hypothesis, we obtain

$$\begin{aligned} &d(x_1, x_2, \ldots, x_n)\sigma(x_1')F(z, x_2, \ldots, x_n) + \tau(x_1)d(x_1', x_2, \ldots, x_n)\sigma(z) \\ &+ \tau(x_1)\tau(x_1')F(z, x_2, \ldots, x_n) \\ =&d(x_1, x_2, \ldots, x_n)\sigma(x_1'z) + \tau(x_1)d(x_1', x_2, \ldots, x_n)\sigma(z) + \tau(x_1)\tau(x_1')F(z, x_2, \ldots, x_n), \end{aligned}$$

which reduces to

$$d(x_1, x_2, \ldots, x_n)\sigma(x_1')(F(z, x_2, \ldots, x_n) - \sigma(z)) = 0,$$

for all $x_1, x_1', z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. This implies that

$$d(x_1, x_2, \ldots, x_n)U_1(F(z, x_2, \ldots, x_n) - \sigma(z)) = \{0\}.$$

By Lemma 2.2 (i), we obtain $d(x_1, x_2, \ldots, x_n) = 0$ or $F(z, x_2, \ldots, x_n) = \sigma(z)$ for all $z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$.

If $F(z, x_2, \ldots, x_n) = \sigma(z)$ for all $z \in U_1$, replacing $z$ by $zt$, we get

$$\tau(z)d(t, x_2, \ldots, x_n) = 0.$$

Putting $u \in U_1$ in place of $\tau(z)$ and using Lemma 2.2 (ii), we obtain $d(t, x_2, \ldots, x_n) = 0$ for all $t \in U_1$. Therefore, in both cases we arrive at $d(U_1, U_2, \ldots, U_n) = \{0\}$. Now arguing in the similar manner as we have done in Lemma 2.5, we can get $d = 0$ on $N$, which completes the proof. ☐

**Theorem 3.2.** *Let $N$ be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of $N$. Suppose that $\sigma$ is a multiplicative mapping on $U_i$ for $i = 1, 2, \ldots, n$, such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If $N$ admits a nonzero generalized $(\sigma, \sigma)$-$n$-derivation $F$ associated with a $(\sigma, \sigma)$-$n$-derivation $d$ such that $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$, then $N$ is a commutative ring.*

*Proof.* If $d \neq 0$, then for all $u_1, u_1' \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(u_1u_1', u_2, \ldots, u_n) = d(u_1, u_2, \ldots, u_n)\sigma(u_1') + \sigma(u_1)F(u_1', u_2, \ldots, u_n) \in Z(N). \tag{3.1}$$

Now commuting (3.1) with the element $\sigma(u_1)$ and using Lemma 2.4, we get

$$d(u_1, u_2, \ldots, u_n)\sigma(u_1')\sigma(u_1) = \sigma(u_1)d(u_1, u_2, \ldots, u_n)\sigma(u_1').$$

Since $\sigma$ is an onto map on $N$, replacing $\sigma(u_1')$ by $r_1 \in N$ in above expression, we find that

$$d(u_1, u_2, \ldots, u_n)r_1\sigma(u_1) = \sigma(u_1)d(u_1, u_2, \ldots, u_n)r_1. \tag{3.2}$$

Substituting $r_1r_2$ where $r_2 \in N$ in place of $r_1$ in (3.2) and using it again, we obtain

$$d(u_1, u_2, \ldots, u_n)N[\sigma(u_1), r_2] = \{0\}.$$

By 3-primeness of $N$, we get $d(u_1, u_2, \ldots, u_n) = 0$ or $[\sigma(u_1), r] = 0$ for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ and $r \in N$.

**Case 1.** Suppose there exists $x_0 \in U_1$ such that $d(x_0, u_2, \ldots, u_n) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. Then

$$F(u_1 x_0, u_2, \ldots, u_n) = F(u_1, u_2, \ldots, u_n)\sigma(x_0) \in Z(N),$$

for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Since $F(u_1, u_2, \ldots, u_n) \neq 0$, then $\sigma(x_0) \in Z(N)$ by Lemma 2.1 (ii).

**Case 2.** Suppose there exists $x_0 \in U_1$ such that $[\sigma(x_0), r] = 0$ for all $r \in N$, then $\sigma(x_0) \in Z(N)$.

In both cases, we obtain $\sigma(U_1) \subseteq Z(N)$ which implies that $U_1 \subseteq Z(N)$. Hence, by Lemma 2.3, we conclude that $N$ is a commutative ring.

Assume that $d = 0$, then another application of Lemma 2.1 (ii) and Lemma 2.8, our hypothesis gives $U_1 \subseteq Z(N)$ and $N$ is a commutative ring by Lemma 2.3. ☐

The following example shows that the 3-primeness hypothesis in Theorem 3.2 can not be omitted.

*Example* 3.1. *Let us consider Example* 1.3. *Consider*

$$U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

*Then clearly $U$ is a nonzero semigroup ideal of a non* 3*-prime zero-symmetric left near ring $N$. If we choose $U_1 = U_2 = \cdots = U_n = U$, then $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$. However, $N$ is not commutative.*

**Theorem 3.3.** Let $N$ be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Suppose that $\sigma$, $\tau$ are multiplicative mappings on $U_i$ for $i = 1, 2, \ldots, n$, such that $U_i \subseteq \sigma(U_i)$, $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$, and $\sigma$ is onto on $N$. If $N$ admits a generalized $(\sigma, \tau)$-$n$-derivation $F$ associated with a $(\sigma, \tau)$-$n$-derivation $d$ such that $F(x_1 x_1', x_2, \ldots, x_n) = F(x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n)$ for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then $N$ is commutative ring.

*Proof.* By hypothesis,

$$\begin{aligned} F(x_1 x_1', x_2, \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \ldots, x_n) \\ &= F(x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n), \end{aligned} \tag{3.3}$$

for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Substituting $x_1 x_1'$ for $x_1'$ in (3.3) and using Remark 2.1, we obtain

$$\begin{aligned} F(x_1(x_1 x_1'), x_2, \ldots, x_n) =& F(x_1 x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n) \\ =& d(x_1, x_2, \ldots, x_n)\sigma(x_1')F(x_1, x_2, \ldots, x_n) \\ &+ \tau(x_1)F(x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n). \end{aligned}$$

Also, using the definition of $F$, we get

$$\begin{aligned}F(x_1(x_1x_1'), x_2, \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_1x_1') + \tau(x_1)F(x_1x_1', x_2, \ldots, x_n)\\ =& d(x_1, x_2, \ldots, x_n)\sigma(x_1)\sigma(x_1')\\ &+ \tau(x_1)F(x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n).\end{aligned}$$

By comparing the last two equations, we can easily arrive at

$$d(x_1, x_2, \ldots, x_n)\sigma(x_1')F(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)\sigma(x_1)\sigma(x_1'). \tag{3.4}$$

Since $\sigma$ is onto on $N$, we get

$$d(x_1, x_2, \ldots, x_n)r_1F(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)\sigma(x_1)r_1.$$

Now substituting $r_1r_2$ for $r_1$ in above expression and using it again, we find that

$$d(x_1, x_2, \ldots, x_n)N[F(x_1, x_2, \ldots, x_n), r_2] = \{0\},$$

for all $x_1, \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$ and $r_2 \in N$. Since $N$ is 3-prime, we have $d(x_1, x_2, \ldots, x_n) = 0$ or $F(x_1, x_2, \ldots, x_n) \in Z(N)$ for all $x_1, \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Using the same argument as used in the proof of the Lemma 2.5 and Theorem 3.2, we conclude that $N$ is a commutative ring. □

**Theorem 3.4.** *Let $N$ be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an automorphism and $\tau$ be a homomorphism on $N$ such that $U_1 \subseteq \sigma(U_1)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$. If $N$ admits a left generalized $(\sigma, \tau)$-n-derivation $F$ associated with a $(\sigma, \tau)$-n-derivation $d$ such that $F([x, y], u_2, \ldots, u_n) = \pm\tau([x, y])$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then $N$ is a commutative ring.*

*Proof.* By hypothesis

$$F([x, y], u_2, \ldots, u_n) = \pm\tau([x, y]), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.5}$$

Replacing $y$ by $xy$ in (3.5) and using $[x, xy] = x[x, y]$, we get

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) + \tau(x)F([x, y], u_2, \ldots, u_n) = \pm(\tau(x)\tau(xy) - \tau(x)\tau(yx)),$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.6}$$

This implies that

$$d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y) = d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x).$$

Substituting $yz$ in place of $y$, where $z \in N$ in the last expression and using it again, we find that

$$d(x, u_2, \ldots, u_n)\sigma(y)[\sigma(x), \sigma(z)] = 0.$$

Since $U_1 \subseteq \sigma(U_1)$, then Lemma 2.2 (i) yields that $d(x, u_2, \ldots, u_n) = 0$ or $\sigma(x) \in Z(N)$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Since $\sigma$ is an automorphism on $N$, then $d(x, u_2, \ldots, u_n) = 0$ or $x \in Z(N)$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Using Lemma 2.7, we get $d(U_1, U_2, \ldots, U_n) \in Z(N)$ which forces that $N$ is a commutative ring by Theorem 3.2 which completes the proof. □

**Theorem 3.5.** *Let $N$ be a 2-torsion free 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an automorphism on $N$ and $\tau$ be a homomorphism on $N$ such that $U_1 \subseteq \sigma(U_1)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$. Then $N$ admits no left generalized $(\sigma, \tau)$-$n$-derivation $F$ associated with a nonzero $(\sigma, \tau)$-$n$-derivation $d$ satisfying one of the following conditions:*

(i) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x, y])$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$;
(ii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x \circ y)$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$;
(iii) $F(x \circ y, u_2, \ldots, u_n) = 0$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$.

*Proof.* (i) Assume that

$$F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x, y]), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.7}$$

Replacing $y$ by $xy$ in (3.7), we get

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \ldots, u_n) = \pm(\tau(x)\tau(xy) - \tau(x)\tau(yx)),$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x)\tau([x, y]).$$

Using the hypothesis, we find that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n,$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y). \tag{3.8}$$

Substituting $yz$ for $y$ in (3.8) where $z \in N$, we have

$$\begin{aligned} d(x, u_2, \ldots, u_n)\sigma(y)\sigma(z)\sigma(x) &= -d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y)\sigma(z) \\ &= d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y)(-\sigma(z)) \\ &= (-d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x))(-\sigma(z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(x))(-\sigma(z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)\sigma(-x)\sigma(-z), \end{aligned}$$

which implies that

$$\begin{aligned} 0 &= d(x, u_2, \ldots, u_n)\sigma(y)(\sigma(z)\sigma(x) - \sigma(-x)\sigma(-z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(z)\sigma(-x) + \sigma(-x)\sigma(z)). \end{aligned}$$

Since $U_1 \subseteq \sigma(U_1)$, Lemma 2.2 (i) yields that

$$d(x, u_2, \ldots, u_n) = 0 \text{ or } \sigma(-x) \in Z(N), \quad \text{for all } x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.9}$$

Suppose there exists $x_0 \in U_1$ such that $\sigma(-x_0) \in Z(N)$. Since $-U_1$ is a nonzero semigroup left ideal of $N$, replacing $x$ and $y$ by $-x_0$ in (3.8), we get

$$2d(-x_0, u_2, \ldots, u_n)\sigma(-x_0)\sigma(-x_0) = 0,$$

for all $u_2 \in U_2, \ldots, u_n \in U_n$. Using 2-torsion freeness of $N$, we conclude that $d(-x_0, u_2, \ldots, u_n)N\sigma(-x_0)N\sigma(-x_0) = \{0\}$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. By 3-primeness of $N$, we arrive at $d(-x_0, u_2, \ldots, u_n) = 0$ or $\sigma(-x_0) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. Since $\sigma$ is an automorphism of $N$, by (3.9) we get $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, so $d(U_1, U_2, \ldots, U_n) = \{0\}$, which contradicts Lemma 2.5.

(ii) Suppose that

$$F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x \circ y), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.10}$$

Replacing $y$ by $xy$ in (3.10), we get

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x)\tau(x \circ y),$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y). \tag{3.11}$$

Since (3.11) is same as (3.8), arguing in the similar manner as in (i), we find a contradiction with our hypothesis.

Using the same techniques, we can prove the result for (iii). ☐

**Theorem 3.6.** Let $N$ be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an homomorphism on $N$ such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If $N$ admits a left generalized $(\sigma, \sigma)$-$n$-derivation $F$ associated with a $(\sigma, \sigma)$-$n$-derivation $d$ such that $F([x, y], u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then $F$ is a right $\sigma$-$n$-multiplier on $N$ or $N$ is commutative.

*Proof.* By hypothesis

$$F([x, y], u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.12}$$

Replacing $y$ by $xy$ in (3.12), we get

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) + \sigma(x)F([x, y], u_2, \ldots, u_n) = \sigma(x)[\sigma(x), y]_{\sigma,\sigma},$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.13}$$

As (3.13) is same as (3.6), arguing in the similar manner as in Theorem 3.4, we obtain the result. ☐

**Theorem 3.7.** *Let $N$ be a 2-torsion free 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be a homomorphism on $N$ such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. Then $N$ admits no left generalized $(\sigma, \sigma)$-$n$-derivation $F$ associated with a nonzero $(\sigma, \sigma)$-$n$-derivation $d$ satisfying one of the following conditions:*

(i) *$F(x \circ y, u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$;*
(ii) *$F(x \circ y, u_2, \ldots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$.*

*Proof.* (i) Suppose that

$$F(x \circ y, u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.14}$$

Replacing $y$ by $xy$ in (3.14), we get

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = \sigma(x)[\sigma(x), y]_{\sigma,\sigma},$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.15}$$

Since (3.15) is same as (3.8), arguing as in the proof of Theorem 3.5, we find that $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ or $N$ is a commutative ring. If $N$ is a commutative ring, then our hypothesis becomes

$$2F(xy, u_2, \ldots, u_n) = 0,$$

for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. By 2-torsion freeness of $N$, we have $F(xy, u_2, \ldots, u_n) = 0$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. This implies that

$$d(x, u_2, \ldots, u_n)\sigma(y) + \sigma(x)F(y, u_2, \ldots, u_n) = 0.$$

Replacing $y$ by $yz$ in last expression, we obtain $d(x, u_2, \ldots, u_n)\sigma(y)\sigma(z) = 0$ for all $x, y, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ which implies that $d(x, u_2, \ldots, u_n)\sigma(U_1)\sigma(z) = \{0\}$ for all $x, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Since $U_1 \subseteq \sigma(U_1)$, we get

$$d(x, u_2, \ldots, u_n)U_1\sigma(z) = \{0\},$$

for all $x, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Using Lemma 2.2 (i), we have $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ or $\sigma(U_1) = U_1 = \{0\}$. Since $U_1 \neq \{0\}$, we conclude that $d(U_1, U_2, \ldots, U_n) = \{0\}$ which contradicts Lemma 2.5.

(ii) Assume that

$$F(x \circ y, u_2, \ldots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.16}$$

Substituting $xy$ for $y$ in (3.16), we have

$$\begin{aligned} F(x(x \circ y), u_2, \ldots, u_n) &= \sigma(x)\sigma(xy) + \sigma(xy)\sigma(x), \\ d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) &= \sigma(x)(\sigma(x) \circ y)_{\sigma,\sigma}, \end{aligned}$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Arguing in the similar manner as we have done above, we obtain $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, we again get a contradiction. □

**Theorem 3.8.** *Let $N$ be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an homomorphism on $N$ such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If $N$ admits a left generalized $(\sigma, \sigma)$-$n$-derivation $F$ associated with a nonzero $(\sigma, \sigma)$-$n$-derivation $d$ such that $F([x, y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then $N$ is a commutative ring.*

*Proof.* Suppose that for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F([x,y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]. \tag{3.17}$$

Replacing $y$ by $xy$ in (3.17), we get

$$d(x, u_2, \ldots, u_n)\sigma([x,y]) + \sigma(x)F([x,y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(xy)].$$

In view of our hypothesis, the above expression gives

$$\begin{aligned} &d(x, u_2, \ldots, u_n)\sigma(xy) - d(x, u_2, \ldots, u_n)\sigma(yx) + \sigma(x)d(x, u_2, \ldots, u_n)\sigma(y) \\ &\quad - \sigma(x)\sigma(y)d(x, u_2, \ldots, u_n) \\ &= d(x, u_2, \ldots, u_n)\sigma(xy) - \sigma(xy)d(x, u_2, \ldots, u_n), \end{aligned}$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = \sigma(x)d(x, u_2, \ldots, u_n)\sigma(y). \tag{3.18}$$

Replacing $y$ by $yu$ in the last equation and using it, we can easily arrive at

$$d(x, u_2, \ldots, u_n)\sigma(y)[\sigma(x), \sigma(u)] = 0.$$

Since $U_1 \subseteq \sigma(U_1)$, by Lemma 2.2 (i), we conclude that

$$d(x, u_2, \ldots, u_n) = 0 \quad \text{or} \quad \sigma(x) \in Z(U_1), \quad \text{for all } x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.19}$$

Suppose there exists $x_0 \in U$ such that $\sigma(x_0) \in Z(U_1)$. Then $\sigma(x_0)v = v\sigma(x_0)$ for all $v \in U_1$ and replacing $v$ by $vn$, where $n \in N$ and using it, we conclude that $U[\sigma(x_0), n] = \{0\}$ for all $n \in N$ by Lemma 2.2 (ii), we conclude that $\sigma(x_0) \in Z(N)$. In this case, (3.19) becomes

$$d(x, u_2, \ldots, u_n) = 0 \quad \text{or} \quad \sigma(x) \in Z(N) \quad \text{for all } x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.20}$$

In all cases, the equation (3.17) becomes

$$F([x,y], u_2, \ldots, u_n) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.21}$$

This equation is a special case of Theorem 3.4 with $\tau = 0$, which is already treated previously. ☐

**Theorem 3.9.** *Let $N$ be a 2-torsion free 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an automorphism on $N$ such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. Then $N$ admits no left generalized $(\sigma,\sigma)$-$n$-derivation $F$ associated with a nonzero $(\sigma,\sigma)$-$n$-derivation $d$ satisfying one of the following conditions:*

(i) $F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(y)$;
(ii) $F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$,

*for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$.*

*Proof.* $(i)$ By hypothesis, for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(y). \tag{3.22}$$

Substituting $xy$ for $y$ in (3.22) and using $(x \circ xy) = x(x \circ y)$, we obtain

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(xy).$$

Using the hypothesis, we find that

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -\sigma(x)d(x, u_2, \ldots, u_n)\sigma(y). \tag{3.23}$$

Replacing $y$ by $yz$ where $z \in N$ in the last expression and using the same steps that we introduced previously, we obtain $d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(z)\sigma(-x) + \sigma(-x)\sigma(z)) = 0$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n, z \in N$. Since $\sigma(U_1) = U_1$ and invoking Lemma 2.2 (i) and Lemma 2.3, we conclude that $d(x, u_2, \ldots, u_n) = 0$ or $\sigma(-x) \in Z(N)$.

Suppose there exists $x_0 \in U$ such that $\sigma(-x_0) \in Z(N)$. Since $-U_1$ is a nonzero semigroup left ideal of $N$, replacing $x$ and $y$ by $-x_0$ in (3.23), we get

$$2d(-x_0, u_2, \ldots, u_n)\sigma(-x_0)\sigma(-x_0) = 0, \quad \text{for all } u_2 \in U_2, \ldots, u_n \in U_n.$$

Using 2-torsion freeness of $N$, we conclude that

$$d(-x_0, u_2, \ldots, u_n)N\sigma(-x_0)N\sigma(-x_0) = \{0\},$$

for all $u_2 \in U_2, \ldots, u_n \in U_n$. By 3-primeness of $N$, we arrive at $d(-x_0, u_2, \ldots, u_n) = 0$ or $\sigma(-x_0) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. Since $\sigma$ is an automorphism of $N$, by (3.9) we get $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, so $d(U_1, U_2, \ldots, U_n) = \{0\}$, which contradicts Lemma 2.5.

(ii) By hypothesis, we have for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]. \tag{3.24}$$

Substituting $xy$ for $y$ in (3.24) and using $(x \circ xy) = x(x \circ y)$, we obtain

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(xy)],$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -\sigma(x)d(x, u_2, \ldots, u_n)\sigma(y). \tag{3.25}$$

(3.25) is same as (3.23), arguing in the similar manner as above, we conclude that $d(U_1, U_2, \ldots, U_n) = \{0\}$, which leads to a contradiction. □

**Theorem 3.10.** *Let $N$ be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of $N$. Let $\sigma$ be an homomorphism on $N$ such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If $F$ is a left generalized $(\sigma, \sigma)$-$n$-derivation associated with a nonzero $(\sigma, \sigma)$-$n$-derivation $d$ on $N$ such that $d([x, y], u_2, \ldots, u_n) = [F(x, u_2, \ldots, u_n), \sigma(y)]$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then $F$ is a right $\sigma$-$n$-multiplier on $N$ or $N$ is a commutative ring.*

*Proof.* Assume that

$$d([x,y],u_2,\dots,u_n) = [F(x,u_2,\dots,u_n),\sigma(y)], \tag{3.26}$$

for all $x,y \in U_1, u_2 \in U_2,\dots,u_n \in U_n$. Replacing $y$ by $xy$ in (3.26), we get

$$d(x[x,y],u_2,\dots,u_n) = [F(x,u_2,\dots,u_n),\sigma(xy)],$$

which implies that

$$d(x,u_2,\dots,u_n)\sigma([x,y]) + \sigma(x)d([x,y],u_2,\dots,u_n) = [F(x,u_2,\dots,u_n),\sigma(x)\sigma(y)].$$

Using (3.26), the last equation becomes

$$d(x,u_2,\dots,u_n)\sigma([x,y]) + \sigma(x)F(x,u_2,\dots,u_n)\sigma(y) = F(x,u_2,\dots,u_n)\sigma(x)\sigma(y).$$

For $x = y$, (3.26) gives $F(x,u_2,\dots,u_n)\sigma(x) = \sigma(x)F(x,u_2,\dots,u_n)$ which implies that $d(x,u_2,\dots,u_n)\sigma([x,y]) = 0$. As this equation is same as (3.6), arguing in the similar manner as in Theorem 3.4, we obtain the result. $\square$

**Theorem 3.11.** *Let $N$ be a 2-torsion free 3-prime near ring and $U_1,U_2,\dots,U_n$ are nonzero semigroup ideals of $N$ such that $U_1$ is closed under addition. Let $\sigma$ be a onto homomorphism on $N$ such that $U_1 \subseteq \sigma(U_1)$. Then $N$ admits no generalized $(\sigma,\sigma)$-$n$-derivation $F$ associated with a $(\sigma,\sigma)$-$n$-derivation $d$ such that $U_1 \cap Z \neq \emptyset$, $d(U_1 \cap Z, U_2, U_3,\dots,U_n) \neq \{0\}$ and $d(x \circ y,u_2,\dots,u_n) = F(x,u_2,\dots,u_n) \circ \sigma(y)$ for all $x,y \in U_1, u_2 \in U_2,\dots,u_n \in U_n$.*

*Proof.* Suppose that

$$d(x \circ y,u_2,\dots,u_n) = F(x,u_2,\dots,u_n) \circ \sigma(y), \tag{3.27}$$

for all $x,y \in U_1, u_2 \in U_2,\dots,u_n \in U_n$. Let $z \in U_1 \cap Z$ such that $d(z,u_2,u_3,\dots,u_n) \neq 0$ and replacing $y$ by $zy$ in (3.27), we get

$$d(z,u_2,\dots,u_n)\sigma(x \circ y) + \sigma(z)d(x \circ y,u_2,\dots,u_n) = F(x,u_2,\dots,u_n) \circ \sigma(z)\sigma(y).$$

Substituting arbitrary element $z' \in U_1 \cap Z$ for $\sigma(z)$ in above expression and using (3.27), we obtain $d(z,u_2,\dots,u_n)\sigma(x \circ y) = 0$. By Lemma 2.7, it is clear that $d(z,u_2,\dots,u_n) \in Z \setminus \{0\}$ which means that $d(z,u_2,\dots,u_n)N\sigma(x \circ y) = \{0\}$. By 3-primeness of $N$, we conclude that $\sigma(x \circ y) = 0$ for all $x,y \in U_1$ which implies that $\sigma(x) \circ \sigma(y) = 0$. Now replacing $\sigma(x)$ and $\sigma(y)$ by $x'$ and $y'$ for all $x',y' \in U_1$ respectively, we have $x' \circ y' = 0$. In particular $x'^2 = 0$ for all $x' \in U_1$. Since $U_1$ is closed under addition, we have $u(u+u')^2 = 0$ for all $u,u' \in U_1$ this gives $uu'u = 0$ for all $u,u' \in U_1$, i.e., $uU_1u = \{0\}$. Thus, $U_1 = \{0\}$, which contradicts our hypothesis. $\square$

The following example shows that the 3-primeness hypothesis in Theorems 3.4 to 3.11 can not be omitted.

*Example 3.2. Let $S$ be a zero-symmetric left near-ring which is not abelian. Consider*

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x,y,0 \in S \right\}$$

*and*

$$U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\}.$$

*Then clearly $U$ is a nonzero semigroup ideal of a non 3-prime zero-symmetric left near ring $N$. Define mappings $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ by*

$$F\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2\ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & y_1y_2\ldots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Define $\sigma, \tau : N \to N$ by*

$$\tau \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & -y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = id_N.$$

*If we choose $U_1 = U_2 = \cdots = U_n = U$, then it is easy to see that $F$ is a nonzero generalized $(\sigma,\sigma)$-$n$-derivation associated with a nonzero $(\sigma,\sigma)$-$n$-derivation $d$ and also a nonzero generalized $(\sigma,\tau)$-$n$-derivation associated with a nonzero $(\sigma,\tau)$-$n$-derivation $d$ of $N$ satisfying*

(i) $F(x \circ y, u_2, \ldots, u_n) = 0$;

(ii) $F([x,y], u_2, \ldots, u_n) = \pm\tau([x,y])$;

(iii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x,y])$;

(iv) $F(x \circ y, u_2, \ldots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}$;

(v) $F([x,y], u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$;

(vi) $F(x \circ y, u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$;

(vii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x \circ y)$;

(viii) $F([x,y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$;

(ix) $d([x,y], u_2, \ldots, u_n) = [F(x, u_2, \ldots, u_n), \sigma(y)]$;

(x) $F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$;

(xi) $F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(y)$;

(xii) $d(x \circ y, u_2, \ldots, u_n) = F(x, u_2, \ldots, u_n) \circ \sigma(y)$,

*for all $x, y, u_2, \ldots, u_n \in U$. However, $N$ is not a commutative ring.*

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# PICTURE FUZZY SUBGROUP

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Abstract. Picture fuzzy subgroup of a crisp group is established here and some properties connected to it are investigated. Also, normalized restricted picture fuzzy set, conjugate picture fuzzy subgroup, picture fuzzy coset, picture fuzzy normal subgroup and the order of picture fuzzy subgroup are defined. The order of picture fuzzy subgroup is defined using the cardinality of a special type of crisp subgroup. Some corresponding properties are established in this regard.

**Significant Statement.** Subgroup is an important algebraic structure in the field of Pure Mathematics. Study of different properties of subgroup in fuzzy sense is an interesting fact to the readers because fuzzy sense is the extension of classical sense. Readers can easily observe how the properties of subgroup hold in fuzzy sense like classical sense. Picture fuzzy sense is the generalization of fuzzy sense. In other words, picture fuzzy sense can be treated as advanced fuzzy sense. Readers will be interested to study how the properties of subgroup hold when the number of components increases in fuzzy environment. Our study is actually the study of an important type of advanced fuzzy algebraic structure.

## 1. Introduction

Generalizing the concept of classical set theory, Zadeh [12] initiated fuzzy set theory which leads a vital role for handling uncertainty in practical field. Considering the limitation of fuzzy set and generalizing fuzzy set, Atanassov [1] introduced intuitionistic fuzzy set. After the invention of fuzzy set, Rosenfeld [9] introduced fuzzy group. Intuitionistic fuzzy subgroup came in the light of study by Zhan and Tan [13]. Sharma

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[10] investigated $t$-intuitionistic fuzzy subgroup. As the time goes, different researchers have done a lot of research works in the context of fuzzy set and intuitionistic fuzzy set. Intuitionistic fuzzy set deals with the measure of membership and the measure of non-membership such that their sum does not exceed unity. It was observed that the measure of neutrality was not taken into account in intuitionistic fuzzy set. Cuong and Kreinovich [4] initiated the notion of picture fuzzy set including the measure of neutral membership with the intuitionistic fuzzy set. So, picture fuzzy set can be treated as an immediate generalization of intuitionistic fuzzy set by togethering three components namely positive, neutral and negative. With the advancement of time, different kinds of research works under picture fuzzy environment were performed by several researchers [2, 3, 5–8, 11].

Here an attempt has been made to define picture fuzzy subgroup, normalized restricted picture fuzzy set, conjugate picture fuzzy subgroup, picture fuzzy coset, picture fuzzy normal subgroup and the order of picture fuzzy subgroup. Different corresponding properties have also been studied.

## 2. Preliminaries

Here, some primary concepts of fuzzy set (FS), fuzzy subgroup (FSG), intuitionistic fuzzy set (IFS), intuitionistic fuzzy subgroup (IFSG), picture fuzzy set (PFS) and some basic operations on picture fuzzy sets (PFSs) are recapitulated.

**Definition 2.1** ([12])**.** Let $A$ be the set of universe. Then a FS $P$ over $A$ is defined as $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of membership of $a$ in $P$.

Realizing the absence of non-membership, Atanassov [1] included it in IFS.

**Definition 2.2** ([1])**.** Let $A$ be the set of universe. An IFS $P$ over $A$ is defined by $P = \{(a, \mu_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of membership of $a$ in $P$ and $v_P(a) \in [0, 1]$ is the measure of non-membership of $a$ in $P$ with the condition $0 \leqslant \mu_P(a) + v_P(a) \leqslant 1$ for all $a \in A$.

Here, $S_P(a)= 1 - (\mu_P(a) + v_P(a))$ is the measure of suspicion of $a$ in $P$, which excludes the measure of membership and non-membership.

Based on the notion of FS given by Zadeh, Rosenfeld [9] defined FSG.

**Definition 2.3** ([9])**.** Let $(G, *)$ be a group and $P = \{(a, \mu_P(a)) : a \in G\}$ be a FS in $G$. Then $P$ is said to be FSG of $G$ if $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$ and $\mu_P(a^{-1}) \geqslant \mu_P(a)$ for all $a, b \in G$. Here $a^{-1}$ is the inverse of $a$ in $G$.

**Definition 2.4** ([13])**.** Let $(G, *)$ be a crisp group and $P = \{(a, \mu_P(a), v_P(a)) : a \in G\}$ be an IFS in $G$. Then $P$ is said to be IFSG of $G$ if

(i) $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $v_P(a * b) \leqslant v_P(a) \vee v_P(b)$;

(ii) $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $v_P(a^{-1}) \leqslant v_P(a)$ for all $a, b \in G$. Here $a^{-1}$ is the inverse of $a$ in $G$.

Cuong and Kreinovich [4] included more possible types of uncertainty upon IFS and initiated a new set namely PFS.

**Definition 2.5** ([4])**.** Let $A$ be the set of universe. Then a PFS $P$ over the universe $A$ is defined as $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the measure of positive membership of $a$ in $P$, $\eta_P(a) \in [0,1]$ is the measure of neutral membership of $a$ in $P$ and $v_P(a) \in [0,1]$ is the measure of negative membership of $a$ in $P$ with the condition $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$ for all $a \in A$. For all $a \in A$ $1 - (\mu_P(a) + \eta_P(a) + v_P(a))$ is the measure of denial membership $a$ in $P$.

The basic operations on PFSs consisting equality, union and intersection are defined below.

**Definition 2.6** ([4])**.** Let $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), v_Q(a)) : a \in A\}$ be two PFSs over the universe $A$. Then

(i) $P \subseteq Q$ if and only if $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$, $v_P(a) \geqslant v_Q(a)$ for all $a \in A$;

(ii) $P = Q$ if and only if $\mu_P(a) = \mu_Q(a)$, $\eta_P(a) = \eta_P(a)$, $v_P(a) = v_Q(a)$ for all $a \in A$;

(iii) $P \cup Q = \{(a, \max(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \min(v_P(a), v_Q(a))) : a \in A\}$;

(iv) $P \cap Q = \{(a, \min(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \max(v_P(a), v_Q(a))) : a \in A\}$.

**Definition 2.7.** Let $P = \{(a, \mu_P, \eta_P, v_P) : a \in A\}$ be a PFS over the universe $A$. Then $(\theta, \phi, \psi)$-cut of $P$ is the crisp set in $A$ denoted by $C_{\theta,\phi,\psi}(P)$ and is defined by $C_{\theta,\phi,\psi}(P) = \{a \in A : \mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi, v_P(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0,1]$ with the condition $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Throughout the paper, we write PFS $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ as $P = (\mu_P, \eta_P, v_P)$.

## 3. Picture Fuzzy Subgroup

Now, we are going to define PFSG of a crisp group as the extension of FSG and IFSG.

**Definition 3.1.** Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in $G$. Then $P$ is said to be a PFSG of $G$ if

(i) $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b) \geqslant \eta_P(a) \wedge \eta_P(b)$, $v_P(a * b) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$;

(ii) $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $\eta_P(a^{-1}) \geqslant \eta_P(a)$, $v_P(a^{-1}) \leqslant v_P(a)$ for all $a \in G$, where $a^{-1}$ is the inverse of $a$ in $G$.

*Example* 3.1. A PFS $P = (\mu_P, \eta_P, v_P)$ in a group $G = (\mathbb{Z}, +)$ is considered here in the following way:

$$\mu_P(a) = \begin{cases} 0.35, & \text{when } a \in 2\mathbb{Z}, \\ 0.2, & \text{when } a \in 2\mathbb{Z} + 1, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.45, & \text{when } a \in 2\mathbb{Z}, \\ 0.2, & \text{when } a \in 2\mathbb{Z}+1, \end{cases}$$

$$v_P(a) = \begin{cases} 0.2, & \text{when } a \in 2\mathbb{Z}, \\ 0.4, & \text{when } a \in 2\mathbb{Z}+1. \end{cases}$$

It is not very tough to show that $P$ is a PFSG of $G$.

Now, we will develop a proposition in two parts. First part gives the relationship between the identity element and any other element of the universal group in case of a PFSG while the second part gives the relationship between the inverse of an element and the element itself of the universal group in case of a PFSG.

**Proposition 3.1.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then*

(i) *$\mu_P(e) \geqslant \mu_P(a)$, $\eta_P(e) \geqslant \eta_P(a)$, $v_P(e) \leqslant v_P(a)$ for all $a \in G$, where $e$ is the identity in $G$;*

(ii) *$\mu_P(a^{-1}) = \mu_P(a)$, $\eta_P(a^{-1}) = \eta_P(a)$, $v_P(a^{-1}) = v_P(a)$ for all $a \in G$. Here, $a^{-1}$ is the inverse of $a$ in $G$.*

*Proof.* (i) It is observed that

$$\begin{aligned} \mu_P(e) &= \mu_P(a * a^{-1}) \\ &\geqslant \mu_P(a) \wedge \mu_P(a^{-1}) \quad \text{[because } P \text{ is a PFSG of } G\text{]} \\ &= \mu_P(a) \quad \text{[because } \mu_P(a^{-1}) \geqslant \mu_P(a) \text{ as } P \text{ is a PFSG of } G\text{]}, \\ \eta_P(e) &= \eta_P(a * a^{-1}) \\ &\geqslant \eta_P(a) \wedge \eta_P(a^{-1}) \quad \text{[because } P \text{ is a PFSG of } G\text{]} \\ &= \eta_P(a) \quad \text{[because } \eta_P(a^{-1}) \geqslant \eta_P(a) \text{ as } P \text{ is a PFSG of } G\text{]}, \\ \text{and } v_P(e) &= v_P(a * a^{-1}) \\ &\leqslant v_P(a) \vee v_P(a^{-1}) \quad \text{[because } P \text{ is a PFSG of G]} \\ &= v_P(a) \quad \text{[because } v_P(a^{-1}) \leqslant v_P(a) \text{ as } P \text{ is a PFSG of G]}, \end{aligned}$$

for all $a \in G$.

(ii) Since $P$ is a PFSG of $G$, therefore $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $\eta_P(a^{-1}) \geqslant \eta_P(a)$ and $v_P(a^{-1}) \leqslant v_P(a)$ for all $a \in G$. Replacing $a$ by $a^{-1}$, it is obtained that $\mu_P(a) \geqslant \mu_P(a^{-1})$, $\eta_P(a) \geqslant \eta_P(a^{-1})$ and $v_P(a) \leqslant v_P(a^{-1})$ for all $a \in G$. Thus, $\mu_P(a^{-1}) = \mu_P(a)$, $\eta_P(a^{-1}) = \eta_P(a)$ and $v_P(a^{-1}) = v_P(a)$ for all $a \in G$. □

The following proposition suggests the necessary and sufficient condition under which a PFS will be a PFSG.

**Proposition 3.2.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in $G$. Then $P$ is a PFSG of $G$ if and only if $\mu_P(a * b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a * b^{-1}) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$.*

*Proof.* Since $P$ is a PFSG of G, therefore $\mu_P(a*b^{-1}) \geqslant \mu_P(a)\wedge\mu_P(b^{-1}) \geqslant \mu_P(a)\wedge\mu_P(b)$, $\eta_P(a*b^{-1}) \geqslant \eta_P(a)\wedge\eta_P(b^{-1}) \geqslant \eta_P(a)\wedge\eta_P(b)$ and $v_P(a*b^{-1}) \leqslant v_P(a)\vee v_P(b^{-1}) \leqslant v_P(a)\vee v_P(b)$ for all $a,b\in G$.

Conversely, let the condition be hold. Then

$$\begin{aligned}\mu_P(e) &= \mu_P(a*a^{-1}) \geqslant \mu_P(a)\wedge\mu_P(a) = \mu_P(a),\\ \eta_P(e) &= \eta_P(a*a^{-1}) \geqslant \eta_P(a)\wedge\eta_P(a) = \eta_P(a),\\ v_P(e) &= v_P(a*a^{-1}) \leqslant v_P(a)\vee v_P(a) = v_P(a),\end{aligned}$$

for all $a\in G$, $e$ is the identity in $G$. Thus, $\mu_P(e)\geqslant\mu_P(a)$, $\eta_P(e)\geqslant\eta_P(a)$ and $v_P(e)\leqslant v_P(a)$ for all $a\in G$.

Now,

$$\begin{aligned}\mu_P(b^{-1}) &= \mu_P(e*b^{-1}) \geqslant \mu_P(e)\wedge\mu_P(b) = \mu_P(b),\\ \eta_P(b^{-1}) &= \eta_P(e*b^{-1}) \geqslant \eta_P(e)\wedge\eta_P(b) = \eta_P(b),\\ v_P(b^{-1}) &= v_P(e*b^{-1}) \leqslant v_P(e)\vee v_P(b) = v_P(b), \quad \text{for all } b\in G.\end{aligned}$$

Thus, $\mu_P(b^{-1})\geqslant\mu_P(b)$, $\eta_P(b^{-1})\geqslant\eta_P(b)$, $v_P(b^{-1})\leqslant v_P(b)$ for all $b\in G$.

It is observed that

$$\begin{aligned}\mu_P(a*b) &= \mu_P(a*(b^{-1})^{-1}) \geqslant \mu_P(a)\wedge\mu_P(b^{-1}) \geqslant \mu_P(a)\wedge\mu_P(b),\\ \eta_P(a*b) &= \eta_P(a*(b^{-1})^{-1}) \geqslant \eta_P(a)\wedge\eta_P(b^{-1}) \geqslant \eta_P(a)\wedge\eta_P(b),\\ v_P(a*b) &= v_P(a*(b^{-1})^{-1}) \leqslant v_P(a)\vee v_P(b^{-1}) \leqslant v_P(a)\vee v_P(b), \quad \text{for all } a,b\in G.\end{aligned}$$

Consequently, $P$ is a PFSG of $G$. ☐

**Proposition 3.3.** *Let $(G,*)$ be a group and $P=(\mu_P,\eta_P,v_P)$, $Q=(\mu_Q,\eta_Q,v_Q)$ be two PFSGs in $G$. Then $P\cap Q$ is a PFSG of $G$.*

*Proof.* Let $P\cap Q = R = (\mu_R,\eta_R,v_R)$. Then $\mu_R(a)=\mu_P(a)\wedge\mu_Q(a)$, $\mu_R(a) = \eta_P(a)\wedge\eta_Q(a)$ and $v_R(a)=v_P(a)\vee v_Q(a)$ for all $a\in G$. Since $P,Q$ are PFSGs of $G$, therefore

$$\begin{aligned}\mu_R(a*b^{-1}) &= \mu_P(a*b^{-1})\wedge\mu_Q(a*b^{-1})\\ &\geqslant (\mu_P(a)\wedge\mu_P(b))\wedge(\mu_Q(a)\wedge\mu_Q(b))\\ &= (\mu_P(a)\wedge\mu_Q(a))\wedge(\mu_P(b)\wedge\mu_Q(b)) = \mu_R(a)\wedge\mu_R(b),\\ \eta_R(a*b^{-1}) &= \eta_P(a*b^{-1})\wedge\eta_Q(a*b^{-1})\\ &\geqslant (\eta_P(a)\wedge\eta_P(b))\wedge(\eta_Q(a)\wedge\eta_Q(b))\\ &= (\eta_P(a)\wedge\eta_Q(a))\wedge(\eta_P(b)\wedge\eta_Q(b)) = \eta_R(a)\wedge\eta_R(b),\\ v_R(a*b^{-1}) &= v_P(a*b^{-1})\vee v_Q(a*b^{-1})\\ &\leqslant (v_P(a)\vee v_P(b))\vee(v_Q(a)\vee v_Q(b))\\ &= (v_P(a)\vee v_Q(a))\vee(v_P(b)\vee v_Q(b)) = v_R(a)\vee v_R(b) \text{ for all } a,b\in G.\end{aligned}$$

Consequently, $R=P\cap Q$ is a PFSG of $G$. ☐

We have proved that the intersection of two PFSGs is also a PFSG. But, this is not true for union. If $P$ and $Q$ are two PFSGs then $P \cup Q$ may or may not be PFSG. This observation is proved by examples. Below we consider two examples. Example 3.2 shows that $P \cup Q$ is not a PFSG and Example 3.3 shows that $P \cup Q$ is a PFSG.

*Example* 3.2. Two PFSGs $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ in a group $G = (\mathbb{Z}, +)$ considered here in the following way:

$$\mu_P(a) = \begin{cases} 0.25, & \text{when } a \in 7\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{otherwise}, \end{cases}$$

$$v_P(a) = \begin{cases} 0, & \text{when } a \in 7\mathbb{Z}, \\ 0.5, & \text{otherwise}, \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.15, & \text{when } a \in 5\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a \in 5\mathbb{Z}, \\ 0.15, & \text{otherwise}, \end{cases}$$

$$v_Q(a) = \begin{cases} 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.3, & \text{otherwise}. \end{cases}$$

Then

$$\mu_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a \in 7\mathbb{Z}, \\ 0.15, & \text{when } a \in 5\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.15, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.15, & \text{otherwise}, \end{cases}$$

$$v_{P\cup Q}(a) = \begin{cases} 0, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.3, & \text{otherwise}. \end{cases}$$

Here, $\mu_{P\cup Q}(7 + (-5)) = \mu_{P\cup Q}(2) = 0 \ngeq \mu_{P\cup Q}(7) \wedge \mu_{P\cup Q}(5) = 0.25 \wedge 0.15 = 0.15$ and $v_{P\cup Q}(7 + (-5)) = v_{P\cup Q}(2) = 0.3 \nleq v_{P\cup Q}(7) \vee v_{P\cup Q}(5) = 0 \vee 0.2 = 0.2$. But, $\eta_{P\cup Q}(7 + (-5)) = \eta_{P\cup Q}(2) = 0.15 \geq \eta_{P\cup Q}(7) \wedge \eta_{P\cup Q}(5) = 0.15 \wedge 0.2 = 0.15$. Thus, $P \cup Q$ is not a PFSG.

*Example* 3.3. A PFS P=($\mu_P$, $\eta_P$, $v_P$) in a group $G$ is considered in the following way:

$$\mu_P(a) = \begin{cases} 0.45, & \text{when } a = 0, \\ 0.3, & \text{when } a \neq 0, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.4, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$v_P(a) = \begin{cases} 0.1, & \text{when } a = 0, \\ 0.15, & \text{when } a \neq 0, \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.35, & \text{when } a = 0, \\ 0.25, & \text{when } a \neq 0, \end{cases}$$

$$\mu_Q(a) = \begin{cases} 0.25, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$v_Q(a) = \begin{cases} 0.15, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0. \end{cases}$$

Therefore,

$$\mu_{P\cup Q}(a) = \begin{cases} 0.45, & \text{when } a = 0, \\ 0.3, & \text{when } a \neq 0, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.1, & \text{when } a = 0, \\ 0.15, & \text{when } a \neq 0. \end{cases}$$

Clearly, $P \cup Q$ is a PFSG of $G$.

**Proposition 3.4.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be PFSGs in $G$. Then $P \cup Q$ is a PFSG of $G$ if $P \subseteq Q$ or $Q \subseteq P$.*

*Proof.* Let $P \cup Q = R = (\mu_R, \eta_R, v_R)$. Then $\mu_R(a) = \mu_P(a) \vee \mu_Q(a)$, $\eta_R(a) = \eta_P(a) \wedge \eta_Q(a)$ and $v_R(a) = v_P(a) \wedge v_Q(a)$ for $a \in G$.

Case 1. Let $P \subseteq Q$. Then $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$ and $v_P(a) \geqslant v_Q(a)$ for all $a \in G$. Now,

$$\begin{aligned} \mu_R(a * b^{-1}) &= \mu_P(a * b^{-1}) \vee \mu_Q(a * b^{-1}) \\ &= \mu_Q(a * b^{-1}) \\ &\geqslant \mu_Q(a) \wedge \mu_Q(b) \quad [\text{because } Q \text{ is a PFSG of } G] \\ &= (\mu_P(a) \vee \mu_Q(a)) \wedge (\mu_P(b) \vee \mu_Q(b)) \\ &= \mu_R(a) \wedge \mu_R(b), \\ \eta_R(a * b^{-1}) &= \eta_P(a * b^{-1}) \wedge \eta_Q(a * b^{-1}) \\ &= \eta_P(a * b^{-1}) \\ &\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } Q \text{ is a PFSG of } G] \\ &= (\eta_P(a) \wedge \eta_Q(a)) \wedge (\eta_P(b) \wedge \eta_Q(b)) \\ &= \eta_R(a) \wedge \eta_R(b), \end{aligned}$$

$$\begin{aligned} v_R(a * b^{-1}) &= v_P(a * b^{-1}) \wedge v_Q(a * b^{-1}) \\ &= v_Q(a * b^{-1}) \\ &\leqslant v_Q(a) \vee v_Q(b) \quad \text{[because } Q \text{ is a PFSG of } G] \\ &= (v_P(a) \wedge v_Q(a)) \vee (v_P(b) \wedge v_Q(b)) \\ &= v_R(a) \vee v_R(b), \quad \text{for all } a, b \in G. \end{aligned}$$

Consequently, $R$ is a PFSG of $G$.

Case 2. When $Q \subseteq P$ then it can be proceeded in the similar way to get $\mu_R(a*b) \geqslant \mu_R(a) \wedge \mu_R(b)$, $\eta_R(a*b) \geqslant \eta_R(a) \wedge \eta_R(b)$ and $v_R(a*b) \leqslant v_R(a) \vee v_R(b)$ for all $a, b \in G$. ☐

**Definition 3.2.** Let $P = (\mu_P, \eta_P, v_P)$ and $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSs over the universe $A$. Then the Cartesian product of $P$ and $Q$ is the PFS $P \times Q = (\mu_{P\times Q}, \eta_{P\times Q}, v_{P\times Q})$, where $\mu_{P\times Q}((a,b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a,b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_{P\times Q}((a,b)) = v_P(a) \vee v_Q(b)$ for all $(a,b) \in A \times A$.

**Proposition 3.5.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs in $G$. Then $P \times Q$ is a PFSG of $G \times G$.*

*Proof.* Let $P \times Q = R = (\mu_R, \eta_R, v_R)$. Then $\mu_R((a,b)) = \mu_P(a) \wedge \mu_Q(b)$ , $\eta_R((a,b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_R((a,b)) = v_P(a) \vee v_Q(b)$ for all $(a,b) \in G \times G$.

Now,

$$\begin{aligned} \mu_R((a,b) * (c,d)^{-1}) &= \mu_R((a,b) * (c^{-1}, d^{-1})) = \mu_P(a * c^{-1}) \wedge \mu_Q(b * d^{-1}) \\ &\geqslant (\mu_P(a) \wedge \mu_P(c)) \wedge (\mu_Q(b) \wedge \mu_Q(d)) \quad \text{[as } P, Q \text{ are PFSGs of } G] \\ &= (\mu_P(a) \wedge \mu_Q(b)) \wedge (\mu_P(c) \wedge \mu_Q(d)) \\ &= \mu_R((a,b)) \wedge \mu_R((c,d)), \\ \eta_R((a,b) * (c,d)^{-1}) &= \eta_R((a,b) * (c^{-1}, d^{-1})) = \eta_P(a * c^{-1}) \wedge \eta_Q(b * d^{-1}) \\ &\geqslant (\eta_P(a) \wedge \eta_P(c)) \wedge (\eta_Q(b) \wedge \eta_Q(d)) \quad \text{[as } P, Q \text{ are PFSGs of } G] \\ &= (\eta_P(a) \wedge \eta_Q(b)) \wedge (\eta_P(c) \wedge \eta_Q(d)) \\ &= \eta_R((a,b)) \wedge \eta_R((c,d)), \\ v_R((a,b) * (c,d)^{-1}) &= v_R((a,b) * (c^{-1}, d^{-1})) = v_P(a * c^{-1}) \vee v_Q(b * d^{-1}) \\ &\leqslant (v_P(a) \vee v_P(c)) \vee (v_Q(b) \vee v_Q(d)) \quad \text{[as } P, Q \text{ are PFSGs of } G] \\ &= (v_P(a) \vee v_Q(b)) \vee (v_P(c) \vee v_Q(d)) \\ &= v_R((a,b)) \vee v_Q((c,d)), \quad \text{for all } (a,b), (c,d) \in G \times G. \end{aligned}$$

Consequently, $P \times Q$ is a PFSG of $G \times G$. ☐

The following proposition gives the relationship between the identity element and any other element in case of Cartesian product of two PFSGs.

**Proposition 3.6.** *Let $(G_1, *)$ and $(G_2, *)$ be two crisp groups and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs of $G_1$ and $G_2$, respectively. Then $\mu_{P\times Q}((e_1, e_2)) \geqslant$*

*$\mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$, where $(e_1,e_2)$ is the identity in $G_1 \times G_2$.*

*Proof.* Here, we have

$$\begin{aligned}
\mu_{P\times Q}((e_1,e_2)) &= \mu_P(e_1) \wedge \mu_Q(e_2)\\
&\geqslant \mu_P(a_1) \wedge \mu_Q(a_2) \quad \text{[by Proposition 3.1]}\\
&= \mu_{P\times Q}((a_1,a_2)),\\
\eta_{P\times Q}((e_1,e_2)) &= \eta_P(e_1) \wedge \eta_Q(e_2)\\
&\geqslant \eta_P(a_1) \wedge \eta_Q(a_2) \quad \text{[by Proposition 3.1]}\\
&= \eta_{P\times Q}((a_1,a_2)),\\
\text{and } v_{P\times Q}((e_1,e_2)) &= v_P(e_1) \vee v_Q(e_2)\\
&\leqslant v_P(a_1) \vee v_Q(a_2) \quad \text{[by Proposition 3.1]}\\
&= v_{P\times Q}((a_1,a_2)),
\end{aligned}$$

for all $a_1 \in G_1$ and for all $a_2 \in G_2$. Thus, it is obtained that $\mu_{P\times Q}((e_1,e_2)) \geqslant \mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$. □

**Proposition 3.7.** *Let $(G_1,*)$ and $(G_2,*)$ be two crisp groups and $P = (\mu_P,\eta_P,v_P)$, $Q = (\mu_Q,\eta_Q,v_Q)$ be two PFSs of $G_1$ and $G_2$ respectively such that $P \times Q$ is PFSG of $G_1 \times G_2$. Then one of the following conditions must hold:*

(i) *$\mu_Q(e_2) \geqslant \mu_P(a)$, $\eta_Q(e_2) \geqslant \eta_P(a)$, $v_Q(e_2) \leqslant v_P(a)$ for all $a \in G_1$, where $e_2$ is the identity in $G_2$;*

(ii) *$\mu_P(e_1) \geqslant \mu_Q(b)$, $\eta_P(e_1) \geqslant \eta_Q(b)$, $v_P(e_1) \leqslant v_Q(b)$ for all $b \in G_2$, where $e_1$ is the identity in $G_1$.*

*Proof.* Let none of the conditions be hold. Then there exists some $a \in G_1$ and some $b \in G_2$ such that $\mu_Q(e_2) < \mu_P(a)$, $\mu_P(e_1) < \mu_Q(b)$, $\eta_Q(e_2) < \eta_P(a)$, $\eta_P(e_1) < \eta_Q(b)$, $v_Q(e_2) > v_P(a)$, $v_P(e_1) > v_Q(b)$. Then we have

$$\begin{aligned}
\mu_{P\times Q}((a,b)) &= \mu_P(a) \wedge \mu_Q(b) > \mu_Q(e_2) \wedge \mu_P(e_1) = \mu_{P\times Q}((e_1,e_2)),\\
\eta_{P\times Q}((a,b)) &= \eta_P(a) \wedge \eta_Q(b) > \eta_Q(e_2) \wedge \eta_P(e_1) = \eta_{P\times Q}(e_1,e_2),\\
v_{P\times Q}((a,b)) &= v_P(a) \vee v_Q(b) < v_Q(e_2) \vee v_P(e_1) = v_{P\times Q}((e_1,e_2)).
\end{aligned}$$

Thus, it is obtained that $\mu_{P\times Q}((a,b)) > \mu_{P\times Q}((e_1,e_2))$, $\eta_{P\times Q}((a,b)) > \eta_{P\times Q}((e_1,e_2))$ and $v_{P\times Q}((a,b)) < v_{P\times Q}((e_1,e_2))$. This is a contradiction because $(e_1,e_2)$ is the identity in $G_1\times G_2$ and by Proposition 3.6, it is known that $\mu_{P\times Q}((e_1,e_2)) \geqslant \mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$. Hence, one of the conditions must hold.

The power of a PFS $P$ can be defined by taking the power of measure of three types of membership of each element. It is easy to verify that $k$-th power $P^k$ of $P$ is also a PFS. Now, it is the time to define power of a PFS below. □

**Definition 3.3.** Let $A$ be the set of universe and $P = (\mu_P, \eta_P, v_P)$ be a PFS in $A$. Then for a positive integer $k$, $k$-th power of the PFS $P$ is the PFS $P^k = (\mu_P^k, \eta_P^k, v_P^k)$, where $\mu_P^k(a) = (\mu_P(a))^k$, $\eta_P^k(a) = (\eta_P(a))^k$ and $v_P^k(a) = (v_P(a))^k$ for all $a \in A$.

Obviously, $(\mu_P(a))^k \leqslant \mu_P(a)$, $(\eta_P(a))^k \leqslant \eta_P(a)$ and $(v_P(a))^k \leqslant v_P(a)$ and $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$ for all $a \in A$. So, clearly, $0 \leqslant (\mu_P(a))^k + (\eta_P(a))^k + (v_P(a))^k \leqslant 1$ for all $a \in A$.

**Proposition 3.8.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $P^k = (\mu_P^k, \eta_P^k, v_P^k) = ((\mu_P(a))^k, (\eta_P(a))^k, (v_P(a))^k)$ is a PFSG of $G$ for a positive integer $k$.*

*Proof.* Since $P$ is a PFSG, therefore

$$\begin{aligned}
\mu_P^k(a * b^{-1}) &= (\mu_P(a * b^{-1}))^k \\
&\geqslant (\mu_P(a) \wedge \mu_P(b))^k \\
&= (\mu_P(a))^k \wedge (\mu_P(b))^k = \mu_P^k(a) \wedge \mu_P^k(b), \\
\eta_P^k(a * b^{-1}) &= (\eta_P(a * b^{-1}))^k \\
&\geqslant (\eta_P(a) \wedge \eta_P(b))^k \\
&= (\eta_P(a))^k \wedge (\eta_P(b))^k = \eta_P^k(a) \wedge \eta_P^k(b), \\
v_P^k(a * b^{-1}) &= (v_P(a * b^{-1}))^k \\
&\leqslant (v_P(a) \vee v_P(b))^k \\
&= (v_P(a))^k \vee (v_P(b))^k = v_P^k(a) \vee v_P^k(b), \quad \text{for all } a, b \in G.
\end{aligned}$$

Consequently, $P^k$ is a PFSG of $G$. □

**Definition 3.4.** For three chosen real numbers $\varepsilon_1 \in [0, 1]$, $\varepsilon_2 \in [0, 1]$ and $\varepsilon_3 \in [0, 1]$ with $\varepsilon_1 + \varepsilon_2 = 1$ and $\varepsilon_2 + \varepsilon_3 = 1$, we define restricted PFS $P$ over the set of universe $A$ as $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, \varepsilon_1]$, $\eta_P(a) \in [0, \varepsilon_2]$ and $v_P \in [0, \varepsilon_3]$ such that $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$. For any $a \in A$, $(\mu_P(a), \eta_P(a), v_P(a))$ is called picture fuzzy value (PFV). In case of here defined restricted PFS, $(\varepsilon_1, \varepsilon_2, 0)$ is the largest PFV.

Now, let us define a new type of restricted PFS called normalized restricted PFS as an extension of normalized IFS.

**Definition 3.5.** Let $P = (\mu_P, \eta_P, v_P)$ be a restricted PFS in $A$. Then $P$ is said to be normalized restricted PFS if there exists $a \in A$ such that $\mu_P(a) = \varepsilon_1$, $\eta_P(a) = \varepsilon_2$ and $v_P(a) = 0$.

Depending upon three real numbers $\varepsilon_1 \in [0, 1]$, $\varepsilon_2 \in [0, 1]$ and $\varepsilon_3 \in [0, 1]$ with the proposed conditions $\varepsilon_1 + \varepsilon_2 = 1$ and $\varepsilon_2 + \varepsilon_3 = 1$, many restricted PFSs are obtained and also many corresponding normalized restricted PFSs are obtained. Choose $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ and $\varepsilon_3 = 1$. Then $\mu_P(a) \in [0, 1]$, $\eta_P(a) = 0$ and $v_P(a) \in [0, 1]$. Thus, the neutral component is removed completely. So, restricted PFS reduces to IFS and it becomes normalized when there exists $a \in A$ such that $\mu_P(a) = \varepsilon_1 = 1$ and $v_P(a) = 0$, which is familiar to the concept of normalized IFS. So, normalized restricted PFS can be treated as an extension of normalized IFS.

**Proposition 3.9.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a normalized restricted PFS which forms a PFSG of $G$. Then $\mu_P(e) = \varepsilon_1$, $\eta_P(e) = \varepsilon_2$ and $v_P(e) = 0$, where $e$ is the identity in $G$.*

*Proof.* Since $P$ is a normalized restricted PFS therefore there exists some $a \in G$ such that $\mu_P(a) = \varepsilon_1$, $\eta_P(a) = \varepsilon_2$ and $v_P(a) = 0$. Now, by Proposition 3.1, it is known that $\mu_P(e) \geqslant \mu_P(a) = \varepsilon_1$, $\eta_P(e) \geqslant \eta_P(a) = \varepsilon_2$ and $v_P(e) \leqslant v_P(a) = 0$. It follows that $\mu_P(e) = \varepsilon_1$, $\eta_P(e) = \varepsilon_2$ and $v_P(e) = 0$. □

A new kind of group relation called conjugate is defined below for PFSGs.

**Definition 3.6.** Let $(G, *)$ be a crisp group of $G$ and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs $G$. Then $P$ is conjugate to $Q$ if there exists $a \in G$ such that $\mu_P(u) = \mu_Q(a * u * a^{-1})$, $\eta_P(u) = \eta_Q(a * u * a^{-1})$, $v_P(u) = v_Q(a * u * a^{-1})$ for all $u \in G$.

**Proposition 3.10.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$, $R = (\mu_R, \eta_R, v_R)$, $S = (\mu_S, \eta_S, v_S)$ be four PFSGs of $G$ such that $P$ is conjugate to $R$ and $Q$ is conjugate to $S$. Then $P \times Q$ is conjugate to $R \times S$.*

*Proof.* Since $P$ is conjugate to $R$, therefore $\mu_P(u_1) = \mu_R(a * u_1 * a^{-1})$, $\eta_P(u_1) = \eta_R(a * u_1 * a^{-1})$ and $v_P(u_1) = v_R(a * u_1 * a^{-1})$ for some $a \in G$ and for all $u_1 \in G$. Since $Q$ is conjugate to $S$, therefore $\mu_Q(u_2) = \mu_S(b * u_2 * b^{-1})$, $\eta_Q(u_2) = \eta_S(b * u_2 * b^{-1})$ and $v_Q(u_2) = v_S(b * u_2 * b^{-1})$ for some $b \in G$ and for all $u_2 \in G$.

Now,

$$\begin{aligned}
\mu_{P\times Q}((u_1, u_2)) &= \mu_P(u_1) \wedge \mu_Q(u_2) = \mu_R(a * u_1 * a^{-1}) \wedge \mu_S(b * u_2 * b^{-1}) \\
&= \mu_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}), \\
\eta_{P\times Q}((u_1, u_2)) &= \eta_P(u_1) \wedge \eta_Q(u_2) = \eta_R(a * u_1 * a^{-1}) \wedge \eta_S(b * u_2 * b^{-1}) \\
&= \eta_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}), \\
v_{P\times Q}((u_1, u_2)) &= v_P(u_1) \vee v_Q(u_2) = v_R(a * u_1 * a^{-1}) \vee v_S(b * u_2 * b^{-1}) \\
&= v_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}),
\end{aligned}$$

for some $(a, b) \in G \times G$ and for all $(u_1, u_2) \in G \times G$. Therefore, $P \times Q$ is conjugate to $R \times S$. □

The following proposition reflects on $(\theta, \phi, \psi)$-cut of a PFS. It actually tells about the condition imposed on $(\theta, \phi, \psi)$-cut of a PFS under which a PFS will be a PFSG.

**Proposition 3.11.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in $G$. Then $P$ is a PFSG of $G$ if all $(\theta, \phi, \psi)$-cuts of $P$ are crisp subgroups of $G$.*

*Proof.* Let $a, b \in G$, with $\theta = \mu_P(a) \wedge \mu_P(b)$, $\phi = \eta_P(a) \wedge \eta_P(b)$ and $\psi = v_P(a) \vee v_P(b)$. Then $\theta \in [0,1]$, $\phi \in [0,1]$ and $\psi \in [0,1]$ such that $\theta + \phi + \psi \in [0,1]$ is satisfied. It is observed that

$$\begin{aligned}
\mu_P(a) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(a) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
v_P(a) &\leqslant v_P(a) \vee v_P(b) = \psi.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_P(b) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta,\\
\eta_P(b) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi,\\
v_P(b) &\leqslant v_P(a) \vee v_P(b) = \psi.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_P(a) \geqslant \theta, \quad \eta_P(a) \geqslant \phi, \quad v_P(a) \leqslant \psi,\\
\mu_P(b) \geqslant \theta, \quad \eta_P(b) \geqslant \phi, \quad v_P(b) \leqslant \psi.
\end{aligned}$$

It follows that $a, b \in C_{\theta,\phi,\psi}(P)$. Since $C_{\theta,\phi,\psi}(P)$ is a crisp subgroup of $G$, therefore $a * b^{-1} \in C_{\theta,\phi,\psi}(P)$. This yields

$$\begin{aligned}
\mu_P(a * b^{-1}) &\geqslant \theta = \mu_P(a) \wedge \mu_P(b),\\
\eta_P(a * b^{-1}) &\geqslant \phi = \eta_P(a) \wedge \eta_P(b),\\
v_P(a * b^{-1}) &\leqslant \psi = v_P(a) \vee v_P(b).
\end{aligned}$$

Since $a, b$ are arbitrary elements of $G$, therefore $\mu_P(a * b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a * b^{-1}) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$. Consequently, $P$ is a PFSG of $G$. □

**Proposition 3.12.** *Let $G$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then the set $S = \{a \in G : \mu_P(a) = \mu_P(e), \eta_P(a) = \eta_P(e), v_P(a) = v_P(e)\}$ forms a crisp subgroup of $G$, where $e$ plays the role of identity in the group $G$.*

*Proof.* Let $S$ is non-empty because $e \in S$, where $e$ is the identity in $G$. Let $a, b \in S$. Then $\mu_P(a) = \mu_P(b) = \mu_P(e)$, $\eta_P(a) = \eta_P(b) = \eta_P(e)$ and $v_P(a) = v_P(b) = v_P(e)$.

Since $P$ be a PFSG of $G$, therefore

$$\begin{aligned}
\mu_P(a * b^{-1}) &\geqslant \mu_P(a) \wedge \mu_P(b) = \mu_P(e) \wedge \mu_P(e) = \mu_P(e),\\
\eta_P(a * b^{-1}) &\geqslant \eta_P(a) \wedge \eta_P(b) = \eta_P(e) \wedge \eta_P(e) = \eta_P(e),\\
v_P(a * b^{-1}) &\leqslant v_P(a) \vee v_P(b) = v_P(e) \vee v_P(e) = v_P(e).
\end{aligned}$$

From Proposition 3.1, $\mu_P(e) \geqslant \mu_P(a*b^{-1})$, $\eta_P(e) \geqslant \eta_P(a*b^{-1})$ and $v_P(e) \leqslant v_P(a*b^{-1})$. Consequently, $\mu_P(e) = \mu_P(a*b^{-1})$, $\eta_P(e) = \eta_P(a*b^{-1})$ and $v_P(e) = v_P(a*b^{-1})$. Thus,

$$a, b \in S \Rightarrow a * b^{-1} \in S.$$

Therefore, $S$ is a crisp subgroup of $G$. □

The following proposition reflects on $(\theta, \phi, \psi)$-cut of a PFSG. From the definition of $(\theta, \phi, \psi)$-cut of a PFS, we have noticed that $(\theta, \phi, \psi)$-cut of a PFS is a crisp set. From the following proposition, we will know $(\theta, \phi, \psi)$-cut of a PFSG is a crisp subgroup of the universal group.

**Proposition 3.13.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $C_{\theta,\phi,\psi}(P)$ is crisp subgroup of $G$.*

*Proof.* Let $a, b \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$, $v_P(a) \leqslant \psi$ and $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$, $v_P(b) \leqslant \psi$. Since $P$ is a PFSG, therefore

$$\begin{aligned}
\mu_P(a * b^{-1}) &\geqslant \mu_P(a) \wedge \mu_P(b) \geqslant \theta \wedge \theta = \theta, \\
\eta_P(a * b^{-1}) &\geqslant \eta_P(a) \wedge \eta_P(b) \geqslant \phi \wedge \phi = \phi, \\
v_P(a * b^{-1}) &\leqslant v_P(a) \vee v_P(b) \leqslant \psi \vee \psi = \psi.
\end{aligned}$$

Thus,

$$a, b \in C_{\theta,\phi,\psi}(P) \Rightarrow a * b^{-1} \in C_{\theta,\phi,\psi}(P).$$

Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp subgroup of $G$. □

The following proposition gives the relationship between the $r$-th power of an element and the element itself of the universal group in case of a PFSG. The relationship is given in terms of picture fuzzy membership values.

**Proposition 3.14.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$, $v_P(a^r) \leqslant v_P(a)$ for all $a \in G$ and for all integers $r$, where $a^r = a * a * \cdots * a$ ($r$ times).*

*Proof.* Case 1. Let $r$ be a positive integer. Then $r \geqslant 1$. Let us suppose $P(r)$: $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$ and $v_P(a^r) \leqslant v_P(a)$ for all $a \in G$. Here, $P(1)$ is trivially true. Now, since $P$ is a PFSG of $G$, therefore

$$\begin{aligned}
\mu_P(a^2) &= \mu_P(a * a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a), \\
\eta_P(a^2) &= \eta_P(a * a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a), \\
v_P(a^2) &= v_P(a * a) \leqslant v_P(a) \vee v_P(a) = v_P(a), \quad \text{for all } a \in G.
\end{aligned}$$

So, $P(2)$ is true. Let us assume that $P(r)$ is true for $r = m$, i.e., $\mu_P(a^m) \geqslant \mu_P(a)$, $\eta_P(a^m) \geqslant \eta_P(a)$ and $v_P(a^m) \leqslant v_P(a)$ for all $a \in G$.

Now,

$$\begin{aligned}
\mu_P(a^{m+1}) &= \mu_P(a^m * a) \geqslant \mu_P(a^m) \wedge \mu_P(a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a), \\
\eta_P(a^{m+1}) &= \eta_P(a^m * a) \geqslant \eta_P(a^m) \wedge \eta_P(a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a), \\
v_P(a^{m+1}) &= v_P(a^m * a) \leqslant v_P(a^m) \vee v_P(a) \leqslant v_P(a) \vee v_P(a) = v_P(a), \quad \text{for all } a \in G.
\end{aligned}$$

So, $P(r)$ is true for $r = m + 1$. Hence, $P(r)$ is true for all positive integers $r$.

Case 2. Let $r$ be a negative integer. Then $r \leqslant -1$. Say $t= -r$. Then $t \geqslant 1$. Now, $\mu_P(a^r) = \mu_P(a^{-t}) = \mu_P(a^t)$, $\eta_P(a^r) = \eta_P(a^{-t}) = \eta_P(a^t)$ and $v_P(a^r) = v_P(a^{-t}) =$

$v_P(a^t)$ for all $a \in G$ [by Proposition 3.1, because $a^{-t}$ is the inverse of $a^t$ in $G$]. As $t$ is a positive integer therefore the case is similar as Case 1. Thus finally, $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$ and $v_P(a^r) \leqslant v_P(a)$ for all $a \in G$.

Case 3. When $r = 0$, then it is trivially true as $\mu_P(e) \geqslant \mu_P(a)$, $\eta_P(e) \geqslant \eta_P(a)$ and $v_P(e) \leqslant v_P(a)$ for all $a \in G$, by Proposition 3.1. □

**Proposition 3.15.** *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then for $a \in G$ $\mu_P(a*b) = \mu_P(b)$, $\eta_P(a*b) = \eta_P(b)$ and $v_P(a*b) = v_P(b)$ for all $b \in G$ if and only if $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $v_P(a) = v_P(e)$, where $e$ plays the role of identity in $G$.*

*Proof.* Let for $a \in G$, $\mu_P(a*b) = \mu_P(b)$, $\eta_P(a*b) = \eta_P(b)$ and $v_P(a*b) = v_P(b)$ for all $b \in G$. When $b = e$, then $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $v_P(a) = v_P(e)$.

Conversely, let $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $v_P(a) = v_P(e)$. It is observed that

$$
\begin{aligned}
\mu_P(a*b) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= \mu_P(e) \wedge \mu_P(b) = \mu_P(b) \quad \text{[by Proposition 3.1]},\\
\eta_P(a*b) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= \eta_P(e) \wedge \eta_P(b) = \eta_P(b) \quad \text{[by Proposition 3.1]},\\
\text{and } v_P(a*b) &\leqslant v_P(a) \vee v_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= v_P(e) \vee v_P(b) = v_P(b), \quad \text{for all } b \in G \quad \text{[by Proposition 3.1]}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mu_P(b) &= \mu_P(a^{-1} * a * b) = \mu_P(a^{-1} * (a*b))\\
&\geqslant \mu_P(a^{-1}) \wedge \mu_P(a*b) \geqslant \mu_P(a) \wedge \mu_P(a*b)\\
&= \mu_P(e) \wedge \mu_P(a*b)\\
&= \mu_P(a*b) \quad \text{[by Proposition 3.1]},\\
\eta_P(b) &= \eta_P(a^{-1} * a * b) = \eta_P(a^{-1} * (a*b))\\
&\geqslant \eta_P(a^{-1}) \wedge \eta_P(a*b) \geqslant \eta_P(a) \wedge \eta_P(a*b)\\
&= \eta_P(e) \wedge \eta_P(a*b)\\
&= \eta_P(a*b) \quad \text{[by Proposition 3.1]},\\
v_P(b) &= v_P(a^{-1} * a * b) = v_P(a^{-1} * (a*b))\\
&\leqslant v_P(a^{-1}) \vee v_P(a*b) \leqslant v_P(a) \vee v_P(a*b)\\
&= v_P(e) \vee v_P(a*b)\\
&= v_P(a*b), \quad \text{for all } b \in G \quad \text{[by Proposition 3.1]}.
\end{aligned}
$$

Thus, $\mu_P(a*b) = \mu_P(b)$, $\eta_P(a*b) = \eta_P(b)$ and $v_P(a*b) = v_P(b)$ for all $b \in G$. □

**Proposition 3.16.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a restricted PFSG of $G$ for three chosen non-negative real numbers $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$. Let $\{a_k\}$*

*be a sequence of elements in $G$ such that* $\lim_{k\to\infty}\mu_P(a_k)=\varepsilon_1$, $\lim_{k\to\infty}\eta_P(a_k)=\varepsilon_2$ *and* $\lim_{k\to\infty}v_P(a_k)=0$. *Then* $\mu_P(e)=\varepsilon_1$, $\eta_P(e)=\varepsilon_2$ *and* $v_P(e)=0$, *where $e$ is the identity in $G$.*

*Proof.* From Proposition 3.1, $\mu_P(e)\geqslant\mu_P(a_k)$, $\eta_P(e)\geqslant\eta_P(a_k)$ and $v_P(e)\leqslant v_P(a_k)$ for all $k\in N$. Therefore, $\mu_P(e)\geqslant\lim_{k\to\infty}\mu_P(a_k)=\varepsilon_1$, $\eta_P(e)\geqslant\lim_{k\to\infty}\eta_P(a_k)=\varepsilon_2$ and $v_P(e)\leqslant\lim_{k\to\infty}v_P(a_k)=0$. Thus, $\mu_P(e)\geqslant\varepsilon_1$, $\eta_P(e)\geqslant\varepsilon_2$ and $v_P(e)\leqslant 0$. Consequently, $\mu_P(e)=\varepsilon_1$, $\eta_P(e)=\varepsilon_2$ and $v_P(e)=0$. ☐

**Proposition 3.17.** *Let $(G,*)$ be a cyclic group and $P=(\mu_P,\eta_P,v_P)$ be a PFSG of $G$. Let $a$ be any element in $G$ such that it generates the group $G$ with $a\in C_{\theta,\phi,\psi}(P)$. Then $C_{\theta,\phi,\psi}(P)=G$.*

*Proof.* Here $G=\langle a\rangle$. Let $a\in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a)\geqslant\theta$, $\eta_P(a)\geqslant\phi$ and $v_P(a)\leqslant\psi$. Let $t\in G$. Then $t=a^k$ for some integer $k$. Now,

$$\begin{aligned}
\mu_P(t)&=\mu_P(a^k)\\
&\geqslant\mu_P(a)\quad\text{[by Proposition 3.14]}\\
&\geqslant\theta,\\
\eta_P(t)&=\eta_P(a^k)\\
&\geqslant\eta_P(a)\quad\text{[by Proposition 3.14]}\\
&\geqslant\phi,\\
v_P(t)&=v_P(a^k)\\
&\leqslant v_P(a)\quad\text{[by Proposition 3.14]}\\
&\leqslant\psi.
\end{aligned}$$

Thus, $t\in G$ implies $t\in C_{\theta,\phi,\psi}(P)$. Therefore, $G\subseteq C_{\theta,\phi,\psi}(P)$. Already, it is known that $C_{\theta,\phi,\psi}(P)\subseteq G$. Consequently, $G=C_{\theta,\phi,\psi}(P)$. ☐

## 4. Homomorphism of Picture Fuzzy Subgroups

Here, we study some properties of PFSG under the classical group-homomorphism and anti-group homomorphism.

**Definition 4.1.** Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups. Then a mapping $h:G_1\to G_2$ is said to be a group homomorphism if $h(a*b)=h(a)\circ h(b)$ for all $a,b\in G_1$.

**Definition 4.2.** Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $h:G_1\to G_2$ be a surjective group-homomorphism. Then for a PFS $P=(\mu_P,\eta_P,v_P)$, the image of $P$ is the PFS $h(P)=(\mu_{h(P)},\eta_{h(P)},v_{h(P)})$ defined by

$$\mu_{h(P)}(b)=\bigvee_{a\in h^{-1}(b)}\mu_P(a),\quad \eta_{h(P)}(b)=\bigwedge_{a\in h^{-1}(b)}\eta_P(a),\quad v_{h(P)}(b)=\bigwedge_{a\in h^{-1}(b)}v_P(a),$$

for all $b\in G_2$.

**Proposition 4.1.** *$(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $h: G_1 \to G_2$ be a bijective group homomorphism. Then for a PFSG $P$ in $G_1$, $h(P)$ is a PFSG of $G_2$.*

*Proof.* It is observed that for $b_1 \in G_2$,

$$\mu_{h(P)}(b_1) = \bigvee_{a_1 \in h^{-1}(b_1)} \mu_P(a_1), \quad \eta_{h(P)}(b_1) = \bigwedge_{a_1 \in h^{-1}(b_1)} \eta_P(a_1), \quad v_{h(P)}(b_1) = \bigvee_{a_1 \in h^{-1}(b_1)} v_P(a_1).$$

Since $h$ is bijective, therefore $h^{-1}(b_1)$ is a singleton set. So, it can be written as $h^{-1}(b_1) = a_1$, i.e., $h(a_1) = b_1$ for unique $a_1 \in G_1$. Therefore, $\mu_{h(P)}(b_1) = \mu_{h(P)}(h(a_1)) = \mu_P(a_1)$, $\eta_{h(P)}(b_1) = \mu_{h(P)}(h(a_1)) = \eta_P(a_1)$ and $v_{h(P)}(b_1) = v_{h(P)}(h(a_1)) = v_P(a_1)$ for unique $a_1 \in G_1$.

Now,

$$\begin{aligned}
\mu_{h(P)}(b_1 \circ b_2^{-1}) &= \mu_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&\quad [\text{because } b_1 = h(a_1) \text{ and } b_2 = h(a_2) \text{ for unique } a_1 \text{ and } a_2 \in G_1] \\
&= \mu_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= \mu_P(a_1 * a_2^{-1}) \\
&\geqslant \mu_P(a_1) \wedge \mu_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= \mu_{h(P)}(h(a_1)) \wedge \mu_{h(P)}(h(a_2)) = \mu_{h(P)}(b_1) \wedge \mu_{h(P)}(b_2), \\
\eta_{h(P)}(b_1 \circ b_2^{-1}) &= \eta_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&= \eta_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= \eta_P(a_1 * a_2^{-1}) \\
&\geqslant \eta_P(a_1) \wedge \eta_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= \eta_{h(P)}(h(a_1)) \wedge \eta_{h(P)}(h(a_2)) = \eta_{h(P)}(b_1) \wedge \eta_{h(P)}(b_2), \\
v_{h(P)}(b_1 \circ b_2^{-1}) &= v_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&= v_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= v_P(a_1 * a_2^{-1}) \\
&\leqslant v_P(a_1) \vee v_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= v_{h(P)}(h(a_1)) \vee v_{h(P)}(h(a_2)) \\
&= v_{h(P)}(b_1) \vee v_{h(P)}(b_2), \quad \text{for all } b_1, b_2 \in G_2.
\end{aligned}$$

Consequently, $h(P)$ is a PFSG of $G_2$. □

**Definition 4.3.** Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups. Then a mapping $h: G_1 \to G_2$ is said to be an anti group homomorphism if $h(a * b) = h(b) \circ h(a)$ for all $a, b \in G_1$.

**Definition 4.4.** Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $Q = (\mu_Q, \eta_Q, v_Q)$ be a PFSG of $G_2$. Then for a mapping $h: G_1 \to G_2$, $h^{-1}(Q)$ is the PFS $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, v_{h^{-1}(Q)})$ defined by $\mu_{h^{-1}(Q)}(a) = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $v_{h^{-1}(Q)}(a) = v_Q(h(a))$ for all $a \in G_1$.

**Proposition 4.2.** *Let $(G_1, *)$ and $(G_2, \circ)$ be two crisp groups and $Q = (\mu_Q, \eta_Q, v_Q)$ be a PFSG of $G_2$. Then for an anti group-homomorphism $h$, $h^{-1}(Q)$ is a PFSG of $G_1$.*

*Proof.* Let $h^{-1}(Q)$=($\mu_{h^{-1}(Q)}$, $\eta_{h^{-1}(Q)}$, $v_{h^{-1}(Q)}$), where $\mu_{h^{-1}(Q)}(a) = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$, $v_{h^{-1}(Q)}(a) = v_Q(h(a))$ for all $a \in G_1$. Now, we have

$$\begin{aligned}
\mu_{h^{-1}(Q)}(a * b^{-1}) &= \mu_Q(h(a * b^{-1})) \\
&= \mu_Q(h(b^{-1}) \circ h(a)) \quad \text{[because $h$ is an anti group-homomorphism]} \\
&= \mu_Q((h(b))^{-1} \circ h(a)) \\
&\geqslant \mu_Q((h(b))^{-1}) \wedge \mu_Q(h(a)) \quad \text{[because $Q$ is a PFSG of $G_2$]} \\
&\geqslant \mu_Q(h(b)) \wedge \mu_Q(h(a)) \quad \text{[because $Q$ is PFSG of $G_2$]} \\
&= \mu_Q(h(a)) \wedge \mu_Q(h(b)) = \mu_{h^{-1}(Q)}(a) \wedge \mu_{h^{-1}(Q)}(b), \\
\eta_{h^{-1}(Q)}(a * b^{-1}) &= \eta_Q(h(a * b^{-1})) \\
&= \eta_Q(h(b^{-1}) \circ h(a)) \quad \text{[because $h$ is an anti group-homomorphism]} \\
&= \eta_Q((h(b))^{-1}) \circ h(a)) \\
&\geqslant \eta_Q((h(b))^{-1}) \wedge \eta_Q(h(a)) \quad \text{[because $Q$ is a PFSG of $G_2$]} \\
&\geqslant \eta_Q(h(b)) \wedge \eta_Q(h(a)) \quad \text{[because $Q$ is a PFSG of $G_2$]} \\
&= \eta_Q(h(a)) \wedge \eta_Q(h(b)) = \eta_{h^{-1}(Q)}(a) \wedge \eta_{h^{-1}(Q)}(b), \\
v_{h^{-1}(Q)}(a * b^{-1}) &= v_Q(h(a * b^{-1})) \\
&= v_Q(h(b^{-1}) \circ h(a)) \quad \text{[because $h$ is an anti group-homomorphism]} \\
&= v_Q((h(b))^{-1} \circ h(a)) \\
&\leqslant v_Q((h(b))^{-1}) \vee v_Q(h(a)) \quad \text{[because $Q$ is a PFSG of $G_2$]} \\
&\leqslant v_Q(h(b)) \vee v_Q(h(a)) \quad \text{[because $Q$ is a PFSG of $G_2$]} \\
&= v_Q(h(a)) \vee v_Q(h(b)) = v_{h^{-1}(Q)}(a) \vee v_{h^{-1}(Q)}(b), \quad \text{for all } a, b \in G_1.
\end{aligned}$$

Consequently, $h^{-1}(Q)$ is a PFSG of $G_1$. ☐

## 5. Picture Fuzzy Coset and Picture Fuzzy Normal Subgroup

Here, we define different kinds of picture fuzzy cosets (PFCSs) and picture fuzzy normal subgroup (PFNSG). Also, we investigate some related properties.

**Definition 5.1.** Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then for any $a \in G$ the picture fuzzy left coset of $P$ in $G$ is the PFS $aP = (\mu_{aP}, \eta_{aP}, v_{aP})$ defined by $\mu_{aP}(u) = \mu_P(a^{-1} * u)$, $\eta_{aP}(u) = \mu_P(a^{-1} * u)$ and $v_{aP}(u) = v_P(a^{-1} * u)$ for all $u \in G$.

**Definition 5.2.** Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then for any $a \in G$ the picture fuzzy right coset of $P$ in $G$ is the PFS $Pa =$

$(\mu_{Pa}, \eta_{Pa}, v_{Pa})$ defined by $\mu_{Pa}(u) = \mu_P(u * a^{-1})$, $\eta_{Pa}(u) = \mu_P(u * a^{-1})$ and $v_{Pa}(u) = v_P(u * a^{-1})$ for all $u \in G$.

**Definition 5.3.** Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then for any $a \in G$ the picture fuzzy middle coset of $P$ in $G$ is the PFS $aPa^{-1} = (\mu_{aPa^{-1}}, \eta_{aPa^{-1}}, v_{aPa^{-1}})$ defined by $\mu_{aPa^{-1}}(u) = \mu_P(a^{-1} * u * a)$, $\eta_{aPa^{-1}}(u) = \eta_P(a^{-1} * u * a)$ and $v_{aPa^{-1}}(u) = v_P(a^{-1} * u * a)$ for all $u \in G$.

In classical sense, any subgroup of a classical group is said to be normal if left coset and right coset of the subgroup for any element of the classical group are equal. In picture fuzzy sense, a PFSG is said to be PFNSG if picture fuzzy membership values of left coset and right coset of PFSG for any element of the universal group are equal.

**Definition 5.4.** Let $(g, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $P$ is called a PFNSG of $G$ if $\mu_{Pa}(u) = \mu_{aP}(u)$, $\eta_{Pa}(u) = \eta_{aP}(u)$, $v_{Pa}(u) = v_{aP}(u)$ for all $a, u \in G$.

**Proposition 5.1.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $P$ is a PFNSG of $G$ if and only if $\mu_P(a * b) = \mu_P(b * a)$, $\eta_P(a * b) = \eta_P(b * a)$ and $v_P(a * b) = v_P(b * a)$ for all $a, b \in G$.*

*Proof.* Let $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of $G$. Therefore, $\mu_{Pa}(u) = \mu_{aP}(u)$, $\eta_{Pa}(u) = \eta_{aP}(u)$ and $v_{Pa}(u) = v_{aP}(u)$ for all $a, u \in G$, i.e., $\mu_P(u * a^{-1}) = \mu_P(a^{-1} * u)$, $\eta_P(u * a^{-1}) = \eta_P(a^{-1} * u)$ and $v_P(u * a^{-1}) = v_P(a^{-1} * u)$ for all $a, u \in G$.

Now, $\mu_P(a * b) = \mu_P(a * (b^{-1})^{-1}) = \mu_P((b^{-1})^{-1} * a) = \mu_P(b * a)$, $\eta_P(a * b) = \eta_P(a * (b^{-1})^{-1}) = \eta_P((b^{-1})^{-1} * a) = \eta_P(b * a)$ and $v_P(a * b) = v_P(a * (b^{-1})^{-1}) = v_P((b^{-1})^{-1} * a) = v_P(b * a)$ for all $a, b \in G$.

Conversely, let $\mu_P(a*b) = \mu_P(b*a)$, $\eta_P(a*b) = \eta_P(b*a)$ and $v_P(a*b) = v_P(b*a)$ for all $a, b \in G$, i.e., $\mu_P(a * (b^{-1})^{-1}) = \mu_P((b^{-1})^{-1} * a)$, $\eta_P(a * (b^{-1})^{-1}) = \eta_P((b^{-1})^{-1} * a)$ and $v_P(a * (b^{-1})^{-1}) = v_P((b^{-1})^{-1} * a)$ for all $a, b \in G$. Letting $z = b^{-1}$ we get $\mu_P(a * z^{-1}) = \mu_P(z^{-1} * a)$, $\eta_P(a * z^{-1}) = \eta_P(z^{-1} * a)$ and $v_P(a * z^{-1}) = v_P(z^{-1} * a)$ for all $a, z \in G$. It follows that $\mu_{Pz}(a) = \mu_{zP}(a)$, $\eta_{Pz}(a) = \eta_{zP}(a)$ and $v_{Pz}(a) = v_{zP}(a)$ for all $a, z \in G$. Consequently, $P$ is a PFNSG of $G$. $\square$

**Proposition 5.2.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then $P$ is a PFNSG of $G$ if and only if $\mu_P(a*u*a^{-1}) = \mu_P(u)$, $\eta_P(a*u*a^{-1}) = \eta_P(u)$ and $v_P(a * u * a^{-1}) = v_P(u)$ for all $a, u \in G$.*

*Proof.* Let $P$ be a PFNSG of $G$. Then

$$\begin{aligned}
\mu_P(a * u * a^{-1}) &= \mu_P((a * u) * a^{-1}) \\
&= \mu_P(a^{-1} * (a * u)) \quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G] \\
&= \mu_P((a^{-1} * a) * u) = \mu_P(u), \\
\eta_P(a * u * a^{-1}) &= \eta_P((a * u) * a^{-1}) \\
&= \eta_P(a^{-1} * (a * u)) \quad \text{[using Proposition 5.1, as } P \text{ is a PFNSG of } G]
\end{aligned}$$

$$
\begin{aligned}
&= \eta_P((a^{-1} * a) * u) = \eta_P(u),\\
v_P(a * u * a^{-1}) &= v_P((a * u) * a^{-1})\\
&= v_P(a^{-1} * (a * u)) \quad \text{[using Proposition 5.1, as } P \text{ is a PFNSG of G]}\\
&= v_P((a^{-1} * a) * u) = v_P(u), \quad \text{for all } a, u \in G.
\end{aligned}
$$

Conversely, let the conditions be hold. Then $\mu_P(a*b) = \mu_P(b^{-1}*(b*a)*b) = \mu_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = \mu_P(b*a)$, $\eta_P(a*b) = \eta_P(b^{-1}*(b*a)*b) = \eta_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = \eta_P(b*a)$ and $v_P(a*b) = v_P(b^{-1}*(b*a)*b) = v_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = v_P(b*a)$ for all $a, b \in G$. Therefore, by Proposition 5.1, $P$ is a PFNSG of $G$. □

**Proposition 5.3.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of $G$. Then $S = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ is a crisp normal subgroup of $G$.*

*Proof.* By Proposition 3.12, $S$ is a crisp subgroup of $G$. Let $a \in G$ and $u \in S$. Then $\mu_P(u) = \mu_P(e)$, $\eta_P(u) = \eta_P(e)$, $v_P(u) = v_P(e)$. Since $P$ is a PFNSG of $G$, therefore, by Proposition 5.2, $\mu_P(a*u*a^{-1}) = \mu_P(u)$, $\eta_P(a*u*a^{-1}) = \eta_P(u)$ and $v_P(a*u*a^{-1}) = v_P(u)$. It follows that $\mu_P(a*u*a^{-1}) = \mu_P(e)$, $\eta_P(a*u*a^{-1}) = \eta_P(e)$ and $v_P(a*u*a^{-1}) = v_P(e)$. Thus, $a*u*a^{-1} \in S$. Hence, $S$ is a crisp normal subgroup of $G$. □

**Proposition 5.4.** *Let $(G, *)$ and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of $G$. Then for any $a \in G$, $aPa^{-1}$ is a PFNSG of $G$.*

*Proof.* Let $aPa^{-1}$=$(\mu_{aPa^{-1}}, \eta_{aPa^{-1}}, v_{aPa^{-1}})$, where $\mu_{aPa^{-1}}(u) = \mu_P(a^{-1}*u*a)$, $\eta_{aPa^{-1}}(u) = \eta_P(a^{-1}*u*a)$ and $v_{aPa^{-1}}(u) = v_P(a^{-1}*u*a)$ for all $u \in G$. Now,

$$
\begin{aligned}
\mu_{aPa^{-1}}(u_1 * u_2) &= \mu_P(a^{-1} * (u_1 * u_2) * a)\\
&= \mu_P(a^{-1} * (u_1 * u_2 * a))\\
&= \mu_P((u_1 * u_2 * a) * a^{-1})\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
&= \mu_P((u_1 * u_2) * (a * a^{-1}))\\
&= \mu_P(u_1 * u_2) = \mu_P(u_2 * u_1)\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],\\
\eta_{aPa^{-1}}(u_1 * u_2) &= \eta_P(a^{-1} * (u_1 * u_2) * a)\\
&= \eta_P(a^{-1} * (u_1 * u_2 * a))\\
&= \eta_P((u_1 * u_2 * a) * a^{-1})\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
&= \eta_P((u_1 * u_2) * (a * a^{-1}))\\
&= \eta_P(u_1 * u_2) = \eta_P(u_2 * u_1)\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],
\end{aligned}
$$

$$\begin{aligned}
v_{aPa^{-1}}(u_1 * u_2) =& v_P(a^{-1} * (u_1 * u_2) * a)\\
=& v_P(a^{-1} * (u_1 * u_2 * a))\\
=& v_P((u_1 * u_2 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is PFNSG of } G]\\
=& v_P((u_1 * u_2) * (a * a^{-1}))\\
=& v_P(u_1 * u_2) = v_P(u_2 * u_1)\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],
\end{aligned}$$

for all $u_1, u_2 \in G$. Also,

$$\begin{aligned}
\mu_{aPa^{-1}}(u_2 * u_1) =& \mu_P(a^{-1} * (u_2 * u_1) * a)\\
=& \mu_P(a^{-1} * (u_2 * u_1 * a))\\
=& \mu_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
=& \mu_P((u_2 * u_1) * (a * a^{-1})) = \mu_P(u_2 * u_1),\\
\eta_{aPa^{-1}}(u_2 * u_1) =& \eta_P(a^{-1} * (u_2 * u_1) * a)\\
=& \eta_P(a^{-1} * (u_2 * u_1 * a))\\
=& \eta_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
=& \eta_P((u_2 * u_1) * (a * a^{-1})) = \eta_P(u_2 * u_1),\\
v_{aPa^{-1}}(u_2 * u_1) =& v_P(a^{-1} * (u_2 * u_1) * a)\\
=& v_P(a^{-1} * (u_2 * u_1 * a))\\
=& v_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is PFNSG of } G]\\
=& v_P((u_2 * u_1) * (a * a^{-1})) = v_P(u_2 * u_1), \quad \text{for all } u_1, u_2 \in G.
\end{aligned}$$

Thus, it is obtained that $\mu_{aPa^{-1}}(u_1 * u_2) = \mu_{aPa^{-1}}(u_2 * u_1)$, $\eta_{aPa^{-1}}(u_1 * u_2) = \eta_{aPa^{-1}}(u_2 * u_1)$ and $v_{aPa^{-1}}(u_1 * u_2) = v_{aPa^{-1}}(u_2 * u_1)$ for all $u_1, u_2 \in G$. By Proposition 5.1, $aPa^{-1}$ is a PFNSG of $G$. □

## 6. Order of Picture Fuzzy Subgroup

Here, we define the order of a PFSG with the help of the cardinality of a special type of crisp subgroup. Also, we explore some results that correspond to the order of PFSG.

**Definition 6.1.** Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of $G$. Then the order of the PFSG $P$ is denoted by $O(P)$ and is defined as the cardinality of the crisp set $H_P = \{u \in G : \mu_P(u) = \mu_P(e),\ \eta_P(u) = \eta_P(e),\ v_P(u) = v_P(e)\}$, where $e$ plays the role of identity in $G$.

**Proposition 6.1.** *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of $G$. Then $O(P) = O(aPa^{-1})$ for any $a \in G$.*

*Proof.* From Definition 6.1, $O(P) = |H_P|$ and $O(aPa^{-1}) = |H_{aPa^{-1}}|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ and $H_{aPa^{-1}} = \{u \in G : \mu_{aPa^{-1}}(u) = \mu_{aPa^{-1}}(e), \eta_{aPa^{-1}}(u) = \eta_{aPa^{-1}}(e), v_{aPa^{-1}}(u) = v_{aPa^{-1}}(e)\}$. Now,

$$\begin{aligned}
\mu_{aPa^{-1}}(q) = \mu_{aPa^{-1}}(e) &\Leftrightarrow \mu_P(a^{-1} * q * a) = \mu_P(a^{-1} * e * a)\\
&\Leftrightarrow \mu_P((a^{-1} * q) * a) = \mu_P(e)\\
&\Leftrightarrow \mu_P(a * (a^{-1} * q)) = \mu_P(e)\\
&\quad\ [\text{by Proposition 5.1, because } P \text{ is a PFNSG of } G]\\
&\Leftrightarrow \mu_P((a * a^{-1}) * q) = \mu_P(e)\\
&\Leftrightarrow \mu_P(q) = \mu_P(e),\\
\eta_{aPa^{-1}}(q) = \eta_{aPa^{-1}}(e) &\Leftrightarrow \eta_P(a^{-1} * q * a) = \eta_P(a^{-1} * e * a)\\
&\Leftrightarrow \eta_P((a^{-1} * q) * a) = \eta_P(e)\\
&\Leftrightarrow \eta_P(a * (a^{-1} * q)) = \eta_P(e)\\
&\quad\ [\text{by Proposition 5.1, because } P \text{ is a PFNSG of } G]\\
&\Leftrightarrow \eta_P((a * a^{-1}) * q) = \eta_P(e)\\
&\Leftrightarrow \eta_P(q) = \eta_P(e),\\
v_{aPa^{-1}}(q) = v_{aPa^{-1}}(e) &\Leftrightarrow v_P(a^{-1} * q * a) = v_P(a^{-1} * e * a)\\
&\Leftrightarrow v_P((a^{-1} * q) * a) = v_P(e)\\
&\Leftrightarrow v_P(a * (a^{-1} * q)) = v_P(e)\\
&\quad\ [\text{by Proposition 5.1, because } P \text{ is PFNSG of } G]\\
&\Leftrightarrow v_P((a * a^{-1}) * q) = v_P(e)\\
&\Leftrightarrow v_P(q) = v_P(e), \quad \text{for all } q \in G.
\end{aligned}$$

Thus, if $r \in H_{aPa^{-1}}$ then $r \in H_P$ and if $s \in H_P$ then $s \in H_{aPa^{-1}}$. So, $H_{aPa^{-1}} \subseteq H_P$ and $H_P \subseteq H_{aPa^{-1}}$. Consequently, $H_P = H_{aPa^{-1}}$ which indicates that $H_P$ and $H_{aPa^{-1}}$ have the same cardinality, i.e., $O(P) = O(aPa^{-1})$. ☐

**Proposition 6.2.** *Let $(G, *)$ be a crisp abelian group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG which is conjugate to $Q = (\mu_Q, \eta_Q, v_Q)$. Then $P$ and $Q$ have the same order.*

*Proof.* From Definition 6.1, it is known that $O(P) = |H_P|$ and $O(Q) = |H_Q|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ and $H_Q = \{u \in G : \mu_Q(u) = \mu_Q(e), \eta_Q(u) = \eta_Q(e), v_Q(u) = v_Q(e)\}$, where $e$ is the identity in $G$. Since $P$ is conjugate to $Q$, therefore $\mu_P(u) = \mu_Q(a * u * a^{-1})$, $\eta_P(u) = \eta_Q(a * u * a^{-1})$, $v_P(u) = v_Q(a * u * a^{-1})$ for some $a \in G$ and for all $u \in G$.

Now, it is observed that

$$\begin{aligned} a*u*a^{-1} &= (a*u)*a^{-1} \\ &= a^{-1}*(a*u) \quad \text{[because } G \text{ is abelian]} \\ &= (a^{-1}*a)*u = u, \quad \text{for all } a,u \in G. \end{aligned}$$

It follows that

$$\begin{aligned} &\mu_P(u) = \mu_Q(a*u*a^{-1}) = \mu_Q(u), \\ &\eta_P(u) = \eta_Q(a*u*a^{-1}) = \eta_Q(u), \\ &v_P(u) = v_Q(a*u*a^{-1}) = v_Q(u), \quad \text{for all } u \in G. \end{aligned}$$

Thus, $H_P$=$\{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$=$\{u \in G : \mu_Q(u) = \mu_Q(e), \eta_Q(u) = \eta_Q(e), v_Q(u) = v_Q(e)\}$=$H_Q$. Therefore, $H_P$ and $H_Q$ have the same cardinality. Hence, $P$ and $Q$ have the same order. □

**Theorem 6.1** (Lagrange's theorem on PFSG)**.** *Let* $(G,*)$ *be a crisp group and* $P = (\mu_P, \eta_P, v_P)$ *be a PFSG of* $G$*. Then* $O(P)$ *is a divisor of* $O(G)$*.*

*Proof.* From Definition 6.1, it is known that $O(P) = |H_P|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(a) = \eta_P(e), v_P(a) = v_P(e)\}$, $e$ plays the role of identity in $G$. Now, by Proposition 3.12, it is known that $H_P$ is a crisp subgroup of $G$. By Lagrange's theorem on crisp group, $|H_P|$ is a divisor of $O(G)$, i.e., $O(P)$ is a divisor of $O(G)$. □

## 7. Conclusion

Investigation of the structure of algebraic system leads a significant in the field of Mathematics, Computer Science and other different areas. Here we have studied the theory of subgroup in the context of picture fuzzy set. In this paper, notion of PFSG has been established and different properties of PFSG have been investigated. Also, different notions related to PFSG such as PFCS, PFNSG, the order of PFSG have been brought into the light of our study. We expect that this paper will be fruitful to the researchers for further study of the theory of subgroup under some other types of set environment.

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# EXISTENCE RESULTS FOR A FRACTIONAL DIFFERENTIAL INCLUSION OF ARBITRARY ORDER WITH THREE-POINT BOUNDARY CONDITIONS

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ABSTRACT. This paper studies existence of solutions for a new class of fractional differential inclusions of arbitrary order with three-point fractional integral boundary conditions. Our results are based on Bohnenblust-Karlin's fixed point theorem.

## 1. INTRODUCTION

Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, viscoelasticity, electromagnetics, physics, biophysics, fitting of experimental data, blood flow phenomena, electrical circuits, biology, porous media etc. [11, 12, 18]. Due to these features, models of fractional order become more practical and realistic than the models of integer-order.

A generalization of differential inequalities and equations are known as differential inclusions. Some recent development on fractional differential equations and inclusions can be found in [2, 4–6, 8–10, 14–17, 20, 22, 23]. Interesting and important applications of differential inclusions are in problems arising from stochastic processes, optimal control theory, economics and so on. If the velocity of a dynamical system cannot be uniquely determined by the state of the system, then such a system can be modeled as a differential inclusion.

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In [14], Benchohra and Hamidi studied the boundary value problem for fractional differential inclusions given by

$$\begin{cases} {}^cD^\alpha w(\xi) \in Z(\xi, w(\xi)), \\ w(0) = w_0, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1, 2]$ and $Z : [0, \infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map with compact and convex values.

Ntouyas [20] investigated the existence of solutions for fractional order differential inclusions of the form

$$\begin{cases} {}^cD^q w(\xi) \in Z(\xi, w(\xi)), \quad 0 < \xi < 1, \\ w(0) = 0, w(1) = \alpha J^p w(\nu), \quad 0 < \nu < 1, \end{cases}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (1, 2]$, $J^p$ is the Riemann-Liouville fractional integral of order $p$, $Z : [0, 1) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map.

In this paper, we consider the multi-valued version of [21]. We study existence results for solutions of the following fractional differential inclusion

$$\begin{cases} {}^cD^{\beta_2} w(\xi) \in Z(\xi, w(\xi)), \quad \xi \in [0, 1], \\ w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, \quad I^{\beta_1} w(1) = 0, \end{cases} \tag{1.1}$$

where $\beta_1 > 0$, $n - 1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbb{N}$, and ${}^cD^{\beta_2}$ is the Caputo derivative of fractional order $\beta_2$, $I^{\beta_1}$ is the Riemann-Liouville integral of fractional order $\beta_1$, $Z : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ and $\nu^{n-1} \neq \frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}$.

## 2. Preliminaries

Let us recall some notations, definitions and lemmas from multi-valued analysis [13, 19].

Let $W = C([0, 1], \mathbb{R})$ denote the standard Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm

$$\|w\| = \max\{|w(\xi)| : \xi \in [0, 1]\}.$$

A fixed point of a multi-valued map $Z : W \to \mathcal{P}(W)$ is $w \in W$ such that $w \in Z(w)$. $Z$ is bounded on bounded sets if for any bounded subset $D$ of $W$, $Z(D) = \bigcup_{w \in D} Z(w)$ is bounded in $W$. $Z$ is said to be completely continuous if for every bounded subset $D$ of $W$, $\overline{Z(D)}$ is compact. $Z$ is closed (convex) valued if $Z(w)$ is closed (convex) for all $w \in W$. $Z$ is called u.s.c. (upper semi-continuous) on $W$ if the set $Z(w_0)$ is a nonempty closed subset of $W$ for each $w_0 \in W$ and if there exists an open neighborhood $E$ of $w_0$ such that $Z(E) \subseteq D$ for each open subset $D$ of $W$ containing $Z(w_0)$. $Z$ has a closed graph if

$$w_n \to w^\star, z_n \to z^\star, w_n \in W, z_n \in Z(w_n) \;\Rightarrow\; z^\star \in Z(w^\star).$$

If $Z$ has nonempty compact values and is completely continuous, then $Z$ has a closed graph if and only if $Z$ is u.s.c.

Throughout this paper, $BCC(W)$ is the set of all nonempty, convex, closed and bounded subsets of $W$. Let $L^1([0,1],\mathbb{R})$ be the standard Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|z\|_{L^1} = \int_0^1 |z(\xi)|\, d\xi.$$

The following definitions are well known [1, 11, 18].

**Definition 2.1.** The Caputo fractional derivative of order $\beta$ for at least $n$-times differentiable function $w : [0,\infty) \to \mathbb{R}$ is defined as

$${}^cD^{\beta} w(\xi) = \frac{1}{\Gamma(n-\beta)} \int_0^{\xi} (\xi - s)^{n-\beta-1} w^{(n)}(s)\, ds, \quad n-1 < \beta < n,\ n = \lceil \beta \rceil,$$

where $\lceil \beta \rceil$ denotes the least integer function of real number $\beta$.

**Definition 2.2.** The Riemann-Liouville integral of fractional order $\beta$ is defined as

$$I^{\beta} w(\xi) = \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} w(s)\, ds, \quad \beta > 0,$$

provided the integral exists.

**Lemma 2.1** ([21]). *Let* $\nu^{n-1} \neq \frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}$, $\beta_1 > 0$, $n-1 < \beta_2 \leq n$, $0 < \nu < 1$. *Then for* $z \in C([0,1],\mathbb{R})$, *the fractional differential system*

$$\begin{cases} {}^cD^{\beta_2} w(\xi) = z(\xi), \quad \xi \in [0,1], \\ w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, \quad I^{\beta_1} w(1) = 0, \end{cases} \tag{2.1}$$

*is equivalent to the integral equation*

$$\begin{aligned} w(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2-1} z(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2-1} z(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} z(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2-1} z(s)\, ds, \end{aligned} \tag{2.2}$$

*where*

$$Q = \frac{\Gamma(\beta_1+n)}{\Gamma(n) - \nu^{n-1}(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}. \tag{2.3}$$

**Lemma 2.2** ([20]). *A function* $w \in AC^n([0,1],\mathbb{R})$ *satisfying boundary conditions*

$$w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, \quad I^{\beta_1} w(1) = 0,$$

*is a solution of fractional differential inclusion* (1.1) *if* $z(\xi) \in Z(\xi, w(\xi))$ *on* $[0,1]$ *for some function* $z \in L^1([0,1],\mathbb{R})$ *and*

$$w(\xi) = \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2-1} z(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2-1} z(s)\, ds$$

$$+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} z(s)\, ds$$
$$- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z(s)\, ds.$$

For the forthcoming analysis, we need the following assumptions.

(A) $Z : [0,1] \times \mathbb{R} \to BCC(\mathbb{R})$ for each $w \in \mathbb{R}$, $(\xi, w) \mapsto z(\xi, w)$ is u.s.c. with respect to $w$ for a.e. $\xi \in [0,1]$ and is measurable with respect to $\xi$ and the set $S_{Z,w}$ is non-empty for each fixed $w \in \mathbb{R}$.

(B) There exists a function $m_\epsilon \in L^1([0,1], \mathbb{R}_+)$ for each $\epsilon > 0$ such that

$$\|Z(\xi, w)\| = \sup\{|v| : v(\xi) \in Z(\xi, w)\} \leq m_\epsilon(\xi),$$

for each $(\xi, w) \in [0,1] \times \mathbb{R}$ with $|w| \leq \epsilon$ and

$$\liminf_{\epsilon \to +\infty} \frac{\int_0^1 m_\epsilon(\xi)\, d\xi}{\epsilon} = \gamma < \infty.$$

**Lemma 2.3** ([3])**.** *Let $J$ be a compact real interval and $Z$ be a multi-valued map satisfying assumption* (A) *and let $\zeta$ be a continuous and linear function from $L^1(J, \mathbb{R})$ into $C(J)$. Then the operator*

$$\zeta \circ S_Z : C(J) \to BCC(J), \quad y \mapsto (\zeta \circ S_Z)(y) = \zeta(S_{Z,y}),$$

*is a closed graph operator in $C(J) \times C(J)$.*

**Lemma 2.4** ([7])**.** *Let $W$ be a Banach space and $D$ be a nonempty, convex, closed and bounded subset of $W$. Let $Z : D \to \mathcal{P}(W) \setminus \{\emptyset\}$ has convex, closed values and is u.s.c. with $Z(D) \subset D$ and $Z(\overline{D})$ is compact. Then $Z$ has a fixed point.*

Let us define a multi-valued map $\psi : W \to \mathcal{P}(W)$ as

$$\psi(w) = \bigg\{ y \in W : y(\xi) = \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2-1} z(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z(s)\, ds$$
$$+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} z(s)\, ds$$
$$- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z(s)\, ds \bigg\},$$

for $z \in S_{Z,w} = \{z(\xi) \in L^1([0,1], \mathbb{R}) : z(\xi) \in Z(\xi, y) \text{ for a.e. } \xi \in [0,1]\}$.

Observe that a fixed point of $\psi$ is a solution of (1.1). For convenience, we put

$$\Lambda = \frac{2}{\Gamma(\beta_2 + 1)} + \frac{|Q|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)} + \frac{|Q|}{\Gamma(\beta_1 + \beta_2 + 1)}.$$

## 3. Main Results

**Theorem 3.1.** *Assume that* (A) *and* (B) *hold with* $\Lambda\gamma < 1$. *Then the fractional differential inclusion* (1.1) *has at least one solution.*

*Proof.* The proof is divided into four steps.

**Step I**. $\psi(w)$ is convex for each $w \in C[0,1]$.

Let $\lambda \in [0,1]$ and $y_1, y_2 \in \psi(w)$. Then there exist $z_1, z_2 \in S_{Z,w}$ such that for each $\xi \in [0,1]$, we have

$$\begin{aligned} y_i(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z_i(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z_i(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1 + \beta_2 - 1} z_i(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z_i(s)\, ds. \end{aligned}$$

Now,

$$\begin{aligned} (\lambda y_1 + (1-\lambda) y_2)(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &- \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1 + \beta_2 - 1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds. \end{aligned}$$

Since $Z$ has convex values, $S_{Z,w}$ is also convex. Thus, for $z_1, z_2 \in S_{Z,w}$ and $\lambda \in [0,1]$, we have $\lambda z_1 + (1-\lambda) z_2 \in S_{Z,w}$. Hence, $\lambda y_1 + (1-\lambda) y_2 \in \psi(w)$, i.e., $\psi(w)$ is convex.

**Step II**. Let $\epsilon > 0$ and $B_\epsilon = \{w \in C[0,1] : \|w\| \leq \epsilon\}$. Then $B_\epsilon$ is a closed, convex and bounded set in $C[0,1]$. We shall prove that there exists $\epsilon > 0$ such that $\psi(B_\epsilon) \subseteq B_\epsilon$. Suppose it is not true. Then for each $\epsilon > 0$, there exist $w_\epsilon \in B_\epsilon$ and $y_\epsilon \in \psi(w_\epsilon)$ with $\|\psi(w_\epsilon)\| > \epsilon$ and

$$\begin{aligned} y_\epsilon(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z_\epsilon(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z_\epsilon(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1 + \beta_2 - 1} z_\epsilon(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z_\epsilon(s)\, ds, \end{aligned}$$

for some $z_\epsilon \in S_{Z,w_\epsilon}$.

Now,

$$\epsilon < \|\psi(w_\epsilon)\|$$

$$\begin{aligned}
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds + \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds \\
& + \frac{(\nu^{n-1} - \xi^{n-1}) Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} |z_\epsilon(s)| \, ds \\
& + \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds \\
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds + \frac{1}{\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds \\
& + \frac{|Q|}{\Gamma(\beta_1 + \beta_2)} \int_0^1 m_\epsilon(s) \, ds + \frac{|Q|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds.
\end{aligned}$$

Dividing both sides by $\epsilon$ and letting $\epsilon \to \infty$, we get

$$\left[ \frac{2}{\Gamma(\beta_2)} + \frac{|Q|}{\Gamma(\beta_1 + \beta_2)} + \frac{|Q|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \right] \gamma \geq 1,$$

implying $\Lambda\gamma \geq 1$, which contradicts the given assumption. Therefore, there exists $\epsilon > 0$ such that $\psi(B_\epsilon) \subseteq B_\epsilon$.

**Step III**. $\psi(B_\epsilon)$ is equicontinuous.

Let $\xi_1, \xi_2 \in [0, 1]$ with $\xi_1 < \xi_2$ and $w \in B_\epsilon$, $y \in \psi(w)$. Then there exists $z \in S_{Z,w}$ such that for each $\xi \in [0, 1]$, we have

$$\begin{aligned}
y(\xi) = & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2 - 1} z(s) \, ds - \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z(s) \, ds \\
& + \frac{(\nu^{n-1} - \xi^{n-1}) Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} z(s) \, ds \\
& - \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z(s) \, ds.
\end{aligned}$$

Now,

$$\begin{aligned}
|y(\xi_1) - y(\xi_2)| \leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi_1} |(\xi_2 - s)^{\beta_2 - 1} - (\xi_1 - s)^{\beta_2 - 1}| |z(s)| \, ds \\
& + \frac{1}{\Gamma(\beta_2)} \int_{\xi_1}^{\xi_2} |\xi_2 - s|^{\beta_2 - 1} |z(s)| \, ds \\
& + \frac{|Q| |\xi_1^{n-1} - \xi_2^{n-1}|}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} |z(s)| \, ds \\
& + \frac{|Q| |\xi_1^{n-1} - \xi_2^{n-1}|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z(s)| \, ds \\
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi_1} |(\xi_2 - s)^{\beta_2 - 1} - (\xi_1 - s)^{\beta_2 - 1}| m_\epsilon(s) \, ds \\
& + \frac{1}{\Gamma(\beta_2)} \int_{\xi_1}^{\xi_2} |\xi_2 - s|^{\beta_2 - 1} m_\epsilon(s) \, ds
\end{aligned}$$

$$+\frac{|Q||\xi_1^{n-1}-\xi_2^{n-1}|}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}m_\epsilon(s)\,ds$$
$$+\frac{|Q||\xi_1^{n-1}-\xi_2^{n-1}|}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}m_\epsilon(s)\,ds.$$

Now, the right-hand side approaches zero when $\xi_1$ approaches $\xi_2$, independently of $w\in B_\epsilon$. Hence, $\psi(B_\epsilon)$ is equicontinuous.

Combining Steps I to III and by a consequence of Arzelá-Ascoli theorem, we get that $\psi$ is a compact valued map.

**Step IV**. $\psi$ has a closed graph.

Let $w_n\to w^*,\ y_n\in\psi(w_n)$ and $y_n\to y^*$. We shall prove that $y^*\in\psi(w^*)$.

Now, $y_n\in\psi(w_n)$ implies that there exists $z_n\in S_{Z,w_n}$ such that for each $\xi\in[0,1]$, we have

$$\begin{aligned}y_n(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z_n(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z_n(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z_n(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z_n(s)\,ds.\end{aligned}$$

We shall show that there exists $z^*\in S_{Z,w^*}$ such that for each $\xi\in[0,1]$, we have

$$\begin{aligned}y^*(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z^*(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z^*(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z^*(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z^*(s)\,ds.\end{aligned}$$

Consider the continuous linear operator $\zeta:L^1([0,1],\mathbb{R})\to C[0,1]$ given by

$$\begin{aligned}\zeta(z)(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds.\end{aligned}$$

Now, it is clear that $\|y_n(\xi)-y^*(\xi)\|\to 0$ as $n\to\infty$.

As a consequence of Lemma 2.3, we deduce that $\zeta\circ S_Z$ is a closed graph operator with $y_n(\xi)\in\zeta(S_{Z,w_n})$.

Since $w_n \to w^*$, we have from Lemma 2.3

$$\begin{aligned} y^*(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2 - 1} z^*(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z^*(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} z^*(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z^*(s)\, ds, \end{aligned}$$

for some $z^* \in S_{Z,w^*}$.

Thus, the compact operator $\psi$ is u.s.c. with closed, convex values. From Lemma 2.4, we conclude that there exists a fixed point $w$ of $\psi$, which is a solution of (1.1). □

**Theorem 3.2.** *Assume that* (A) *and the following condition hold.*

(C) *There exist functions* $k_1(\xi), k_2(\xi) \in L^1([0,1], \mathbb{R}^+)$ *such that*

$$\|Z(\xi, w)\| \leq k_1(\xi)|w| + k_2(\xi),$$

*for each* $(\xi, w) \in [0,1] \times \mathbb{R}$, *with* $\Lambda \|k_1\|_{L^1} < 1$.

*Then the BVP* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* The proof follows by taking $k_1(\xi)\epsilon + k_2(\xi)$ in place of $m_\epsilon(\xi)$ in the proof of Theorem 3.1. □

**Theorem 3.3.** *Assume that* (A) *and the following condition hold.*

(D) *There exist functions* $k_1(\xi), k_2(\xi) \in L^1([0,1], \mathbb{R}^+)$, $\sigma \in [0,1]$ *such that*

$$\|Z(\xi, w)\| \leq k_1(\xi)|w|^{\sigma} + k_2(\xi),$$

*for each* $(\xi, w) \in [0,1] \times \mathbb{R}$.

*Then the BVP* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* The proof is obvious. Here we have $k_1(\xi)\epsilon^{\sigma} + k_2(\xi)$ in place of $m_\epsilon(\xi)$. □

## 4. Examples

In this section, we give some examples in order to illustrate our results.

*Example* 4.1. As the first example, let us consider the following fractional differential inclusion

$$\begin{cases} {}^cD^{\frac{9}{2}} w(\xi) \in Z(\xi, w(\xi)), \quad \xi \in [0,1], \\ w(\frac{1}{10}) = 0, \quad w'(0) = 0, \quad I^{\frac{7}{2}} w(1) = 0, \end{cases} \tag{4.1}$$

where $Z(\xi, w(\xi))$ is such that $\|Z(\xi, w)\| \leq \frac{1}{8(\xi+1)}|w| + e^{-\xi}$.

Here $\beta_2 = \frac{9}{2}$, implying $n = 5$, $\nu = \frac{1}{10}$, $\beta_1 = \frac{7}{2}$,

$$\nu^{n-1} = \nu^4 = \frac{1}{10000} \neq \frac{\Gamma(n)}{(\beta_1 + n - 1)(\beta_1 + n - 2) \cdots (\beta_1 + 1)}$$

$$=\frac{4}{(\beta_1+1)(\beta_1+2)(\beta_1+3)(\beta_1+4)}=\frac{64}{19305}=0.003315.$$

As $\|Z(\xi,w)\|\le\frac{1}{8(\xi+1)}|w|+e^{-\xi}$, therefore (C) is satisfied with $\|k_1\|_{L^1}=\frac{1}{8}\ln 2$. Further,

$$\begin{aligned}
&\Lambda\|k_1\|_{L^1}\\
=&\|k_1\|_{L^1}\Bigg[\frac{2}{\Gamma(\beta_2+1)}+\frac{\Gamma(\beta_1+5)}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)|\Gamma(5)-\nu^4(\beta_1+4)(\beta_1+3)(\beta_1+2)(\beta_1+1)|}\\
&+\frac{\Gamma(\beta_1+5)}{\Gamma(\beta_1+\beta_2+1)|\Gamma(5)-\nu^4(\beta_1+4)(\beta_1+3)(\beta_1+2)(\beta_1+1)|}\Bigg]\\
\approx&\frac{1}{8}\ln 2\left[\frac{64}{945\sqrt{\pi}}+\frac{286}{7\sqrt{\pi}\times 3.879344}+\frac{2027025\sqrt{\pi}}{2^8\times 7!\times 3.879344}\right]\\
\approx&\frac{1}{8}\ln 2[0.03821+5.942029+0.717803]\\
\approx&0.58034<1.
\end{aligned}$$

Thus, by Theorem 3.2, there exists at least one solution of the fractional differential inclusion (4.1).

*Example* 4.2. Now, consider the following fractional inclusion

$$\begin{cases} {}^cD^{\frac{5}{2}}w(\xi)\in Z(\xi,w(\xi)), & \xi\in[0,1],\\ w(\frac{1}{2})=0,\quad w'(0)=0, & I^{\frac{3}{2}}w(1)=0,\end{cases}\tag{4.2}$$

where $Z(\xi,w(\xi))$ is such that $\|Z(\xi,w)\|\le\frac{1}{4(\xi+1)^2}|w|^{\frac{1}{3}}+e^{-\xi}$.

Here $\beta_2=\frac{5}{2}$ implies $n=3$, $\nu=\frac{1}{2}$, $\beta_1=\frac{3}{2}$,

$$\nu^{n-1}=\nu^2=\frac{1}{4}\neq\frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}=\frac{2}{(\beta_1+2)(\beta_1+1)}=\frac{8}{35}.$$

Also, (D) is satisfied with $k_1(\xi)=\frac{1}{4(\xi+1)^2}$ and $k_2(\xi)=e^{-\xi}$ with $\sigma=\frac{1}{3}$. Therefore, it follows from Theorem 3.3 that there exists at least one solution of (4.2).

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# CONSTRUCTION OF SIMULTANEOUS COSPECTRAL GRAPHS FOR ADJACENCY, LAPLACIAN AND NORMALIZED LAPLACIAN MATRICES

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Abstract. In this paper we construct several classes of non-regular graphs which are co-spectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian, and hence we answer a question asked by Butler [2]. We make these constructions starting with two pairs $(G_1,\ H_1)$ and $(G_2,\ H_2)$ of $A$-cospectral regular graphs, then considering the subdivision graphs $S(G_i)$ and R-graphs $\mathcal{R}(H_i)$, $i=1,2$, and finally making some kind of partial joins between $S(G_1)$ and $\mathcal{R}(G_2)$ and $S(H_1)$ and $\mathcal{R}(H_2)$. Moreover, we determine the number of spanning trees and the Kirchhoff index of the newly constructed graphs.

## 1. Introduction

Cospectral graphs are non-isomorphic graphs which share the same eigenvalues of the same matrices associated with them. Several cospectral graphs are known for adjacency, combinatorial Laplacian and normalized Laplacian matrices separately. In 2010, Butler [2] asked that "Is there an example of two non-regular graphs which are cospectral with respect to the adjacency, combinatorial Laplacian and normalized Laplacian at the same time?" Normally regular graphs are always cospectral for all the matrices mentioned in the question. Here we construct some non-regular cospectral graphs for all the three matrices and hence give an answer to the above question of Butler. To present the results of the paper we need some definitions and terminology as follow. All graphs considered in the paper are simple and undirected. For any graph $G$, we take $V(G)$ and $E(G)$ as the vertex set and edge set of $G$

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respectively. The *adjacency matrix* of graph $G$, denoted by $A(G)$, is a square matrix whose rows and columns are indexed by vertices of graph $G$, and $(u, v)^{\text{th}}$ entry is 1 if and only if vertex $u$ is adjacent to vertex $v$ and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees in $G$, then the *Laplacian matrix* $L(G)$ is defined as $L(G) = D(G) - A(G)$ and the *normalized Laplacian matrix* $\mathcal{L}(G)$ of $G$ is defined as $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ with the convention that $D(G)^{-1}(u, u) = 0$ if degree of $u$ is zero. For a given square matrix $M$ of size $n$, we denote the characteristic polynomial $\det(xI_n - M)$ by $f_M(x)$. The eigenvalues of $A(G)$, $L(G)$ and $\mathcal{L}(G)$ are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, and $0 = \delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G) \leq 2$ respectively, where $n$ is the number of vertices of $G$. The multiset of eigenvalues of $A(G)$ (respectively $L(G)$, $\mathcal{L}(G)$) is called the *adjacency* (respectively *Laplacian*, *normalized Laplacian*) *spectrum* of $G$, and denoted by $A$-spectrum (respectively $L$-spectrum, $\mathcal{L}$-spectrum). Two graphs are said to be $A$-cospectral (respectively $L$-cospectral, $\mathcal{L}$-cospectral) if they have the same $A$-spectrum (respectively $L$-spectrum, $\mathcal{L}$-spectrum).

The adjacency, Laplacian and normalized Laplacian spectra of different kinds of graphs have been computed by several researchers [4, 7, 11, 12]. The *subdivision graph* $S(G)$ [6] of a graph $G$ is obtained by inserting a new vertex into every edge of $G$. The *R-graph* $\mathcal{R}(G)$ [5] of a graph $G$ is the graph obtained from $G$ by introducing a new vertex $u_e$ for each $e \in E(G)$ and making $u_e$ adjacent to both the end vertices of $e$. The set of such new vertices is denoted by $I(G)$, i.e., $I(G) = V(S(G))\backslash V(G) = V(\mathcal{R}(G))\backslash V(G)$. The partial joins of subdivision graph and $R$-graph which are considered in the paper are given in the definition below.

**Definition 1.1.** Let $G_1$ and $G_2$ be two vertex-disjoint graphs with number of vertices $n_1$ and $n_2$, and edges $m_1$ and $m_2$, respectively. Then the following hold.

(i) The *subdivision-vertex-R-vertex join* of $G_1$ and $G_2$, denoted by $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $2m_1 + n_1n_2 + 3m_2$ edges.

(ii) The *subdivision-edge-R-edge join* of $G_1$ and $G_2$, denoted by $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $I(G_2)$. The graph $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(2 + m_2) + 3m_2$ edges.

(iii) The *subdivision-edge-R-vertex join* of $G_1$ and $G_2$, denoted by $S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1(2 + n_2) + 3m_2$ edges.

(iv) The *subdivision-vertex-R-edge join* of $G_1$ and $G_2$, denoted by $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$, is the graph obtained from $S(G_1)$ and $\mathcal{R}(G_2)$ by joining each vertex of $V(G_1)$ with every vertex of $I(G_2)$. The graph $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $2m_1 + m_2(3 + n_1)$ edges.

*Example* 1.1. Let us consider two graphs $G_1 = P_4$ and $G_2 = P_3$. The set of dark vertices of $G_1$ and $G_2$ are $I(G_1)$ and $I(G_2)$, respectively.

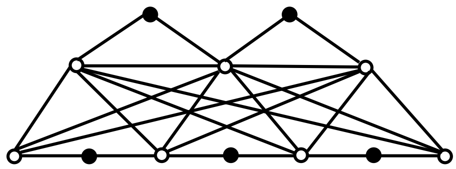

FIGURE 1. Subdivision-vertex-$R$-vertex join of $P_4$ and $P_3$

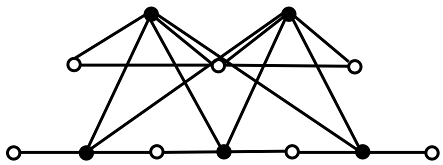

FIGURE 2. Subdivision-edge-$R$-edge join of $P_4$ and $P_3$

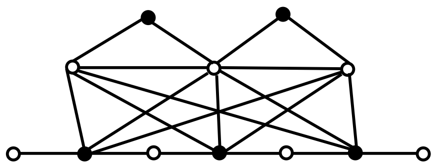

FIGURE 3. Subdivision-edge-$R$-vertex join of $P_4$ and $P_3$

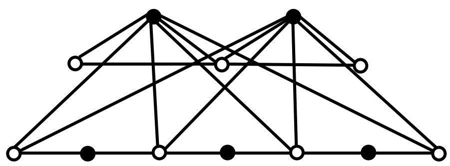

FIGURE 4. Subdivision-vertex-$R$-edge join of $P_4$ and $P_3$

In the following lemma we find the degrees of vertices in the above constructed graphs.

**Lemma 1.1.** $(i)$ *The degree of any vertex $v$ in $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\ddot{\vee}\mathcal{R}(G_2)}(v) = \begin{cases} n_2 + d_{G_1}(v), & \textit{if } v \in V(G_1), \\ 2, & \textit{if } v \in I(G_1) \bigcup I(G_2), \\ n_1 + 2d_{G_2}(v), & \textit{if } v \in V(G_2). \end{cases}$$

$(ii)$ *The degree of any vertex $v$ in $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)}(v) = \begin{cases} d_{G_1}(v), & \textit{if } v \in V(G_1), \\ 2 + m_2, & \textit{if } v \in I(G_1), \\ 2d_{G_2}(v), & \textit{if } v \in V(G_2), \\ 2 + m_1, & \textit{if } v \in I(G_2). \end{cases}$$

$(iii)$ *The degree of any vertex $v$ in $S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\overline{\dot{\vee}}\mathcal{R}(G_2)}(v) = \begin{cases} d_{G_1}(v), & \textit{if } v \in V(G_1), \\ 2 + n_2, & \textit{if } v \in I(G_1), \\ 2d_{G_2}(v) + m_1, & \textit{if } v \in V(G_2), \\ 2, & \textit{if } v \in I(G_2). \end{cases}$$

$(iv)$ *The degree of any vertex $v$ in $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ is given by*

$$d_{S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)}(v) = \begin{cases} d_{G_1}(v) + m_2, & \textit{if } v \in V(G_1), \\ 2, & \textit{if } v \in I(G_1), \\ 2d_{G_2}(v), & \textit{if } v \in V(G_2), \\ 2 + n_1, & \textit{if } v \in I(G_2). \end{cases}$$

For two matrices $A$ and $B$, of same size $m \times n$, the *Hadamard product* $A \bullet B$ of $A$ and $B$ is a matrix of the same size $m \times n$ with entries given by $(A \bullet B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (that is entrywise multiplication). Hadamard product is commutative, that is $A \bullet B = B \bullet A$.

**Notation.** Throughout the paper, for any positive integers $k$, $n_1$ and $n_2$, $I_k$ denotes the identity matrix of size $k$, $J_{n_1 \times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1, $\mathbf{1}_n$ stands for the column vector of size $n$ with all entries equal to 1, $K_{n \times n}$ denotes an $n \times n$ matrix whose all entries are the same. In other words, $K_{n\times n} = \alpha J_{n \times n}$, for a real number $\alpha$. For any positive integers $s$ and $t$, $O_{s \times t}$ denotes the zero matrix of size $s \times t$.

To prove our results we need some basics as given below.

**Lemma 1.2** (Schur Complement [6])**.** *Suppose that the order of all four matrices $M$, $N$, $P$ and $Q$ satisfy the rules of operations on matrices. Then we have*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = \begin{cases} |Q||M - NQ^{-1}P|, & \textit{if } Q \textit{ is a non-singular square matrix}, \\ |M||Q - PM^{-1}N|, & \textit{if } M \textit{ is a non-singular square matrix}. \end{cases}$$

**Lemma 1.3** ([6])**.** *For a square matrix $A$ of size $n$ and a scalar $\alpha$,*

$$\det(A + \alpha J_{n\times n}) = \det(A) + \alpha \mathbf{1}_n^T \operatorname{adj}(A) \mathbf{1}_n,$$

*where* $\operatorname{adj}(A)$ *is the adjugate matrix of $A$.*

**Lemma 1.4.** *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.$$

*Proof.*

$$\begin{aligned}(cI_n - dJ_{n\times n})^{-1} =& \frac{\operatorname{adj}(cI_n - dJ_{n\times n})}{\det(cI_n - dJ_{n\times n})} = \frac{c^{n-2}(c-nd)I_n + c^{n-2}dJ_{n\times n}}{c^{n-1}(c-nd)} \\ =& \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.\end{aligned}$$ □

For a graph $G$ on $n$ vertices and $m$ edges, the *vertex-edge incidence matrix* [8] $R(G)$ of $G$ is a matrix of size $n \times m$, with entry $r_{ij} = 1$ if the $i^{th}$ vertex is incident to the $j^{th}$ edge, and 0 otherwise. The *line graph* [8] of a graph $G$ is the graph $\mathbf{L}_G$, whose vertices are the edges of $G$ and two of these are adjacent in $\mathbf{L}_G$ if and only if they are incident on a common vertex in $G$.

The following is an well known result, may be found in [6].

**Lemma 1.5.** *Let $G$ be an $r$-regular graph. Then*
*(i) $R(G)^T R(G) = A(\mathbf{L}_G) + 2I_m$ and $R(G)R(G)^T = A(G) + rI_n$;*
*(ii) the eigenvalues of $A(\mathbf{L}_G)$ are the eigenvalues of $A(G)+(r-2)I_n$ and $-2$ repeated $m-n$ times.*

**Notation.** The $M$-coronal of an $n\times n$ matrix $M$, denoted by $\Gamma_M(x)$, is defined [3,13] as the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n$.

**Lemma 1.6** ([3]). *If $M$ is an $n \times n$ matrix with each row sum equal to a constant $t$, then $\Gamma_M(x) = \frac{n}{x-t}$.*

Butler [2] constructed non-regular bipartite graphs which are cospectral with respect to both the adjacency and normalized Laplacian matrices, and then asked for existence of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this paper we construct several classes of such graphs taking help of the operations subdivision-vertex-$R$-vertex join, subdivision-edge-$R$-edge join, subdivision-edge-$R$-vertex join and subdivision-vertex-$R$-edge join. We also find the number of spanning trees and Kirchhoff index for all the partial join of subdivision graph and $R$-graph constructed here.

## 2. Adjacency, Laplacian and Normalized Laplacian Spectra of the Graphs

In this section we consider regular graphs $G_i$ on $n_i$ vertices, $m_i$ edges, and with degree of regularity $r_i$, $i = 1, 2$. To obtain the required matrices we label the vertices of the graphs in the following way. Let $V(G_1) = \{v_1, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, \ldots, u_{n_2}\}$, $I(G_2) = \{f_1, \ldots, f_{m_2}\}$. Then $V(G_1) \cup I(G_1) \cup V(G_2) \cup$

$I(G_2)$ is a partition for all $V(S(G_1)\ddot{\nabla}\mathcal{R}(G_2))$, $V(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2))$, $V(S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2))$ and $V(S(G_1)\dot{\nabla}\mathcal{R}(G_2))$.

**Lemma 2.1.** *For $i = 1, 2$, let $G_i$ be a graph with $n_i$ vertices and $m_i$ edges. Then we have the following:*

$$(i)\ A(S(G_1)\ddot{\nabla}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & J_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(ii)\ A(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & J_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & J_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(iii)\ A(S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & J_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & J_{n_2\times m_1} & A(G_2) & R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix};$$

$$(iv)\ A(S(G_1)\dot{\nabla}\mathcal{R}(G_2)) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{n_1\times n_2} & J_{n_1\times m_2} \\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) & R(G_2) \\ J_{m_2\times n_1} & O_{m_2\times m_1} & R(G_2)^T & O_{m_2} \end{pmatrix}.$$

**Theorem 2.1.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ consists of:*

*(i) the eigenvalue $\pm\sqrt{r_1 + \lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $A(G_1)$;*

*(ii) roots of the equation $x^2 - \lambda_j(G_2)x - r_2 - \lambda_j(G_2) = 0$ for every eigenvalue $\lambda_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $A(G_2)$;*

*(iii) the eigenvalue $0$ with multiplicity $m_1 + m_2 - n_1 - n_2$;*

*(iv) four roots of the equation $x^4 - r_2x^3 - (2r_1 + n_1n_2 + 2r_2)x^2 + 2r_1r_2x + 4r_1r_2 = 0$.*

*Proof.* The adjacency characteristic polynomial of $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ is

$$f_{A(S(G_1)\ddot{\nabla}\mathcal{R}(G_2))}(x) = \det\begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} & O_{n_1\times m_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) & -R(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & xI_{m_2} \end{pmatrix} = x^{m_2}\det(S),$$

where

$$S = \begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) \end{pmatrix}$$

$$
\begin{aligned}
&-\begin{pmatrix} O_{n_1\times m_2} \\ O_{m_1\times m_2} \\ -R(G_2)\end{pmatrix}\frac{1}{x}\begin{pmatrix} O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T\end{pmatrix} \\
&=\begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\end{pmatrix}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\det(S) &= \det\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\det(W) \\
&=\prod_{j=1}^{n_2}\left(x-\lambda_j(G_2)-\frac{r_2}{x}-\frac{\lambda_j(G_2)}{x}\right)\det(W),
\end{aligned}
$$

where

$$
\begin{aligned}
W = &\begin{pmatrix} xI_{n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1}\end{pmatrix} \\
&-\begin{pmatrix} -J_{n_1\times n_2} \\ O_{m_1\times n_2}\end{pmatrix}\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)^{-1}\begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times m_1}\end{pmatrix} \\
= &\begin{pmatrix} xI_{n_1}-\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1\times n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1}\end{pmatrix}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\det(W) =& x^{m_1}\det\left(xI_{n_1}-\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)J_{n_1\times n_1}-\frac{1}{x}R(G_1)R(G_1)^T\right) \\
=& x^{m_1}\Bigg[\det\left(xI_{n_1}-\frac{1}{x}R(G_1)R(G_1)^T\right) \\
&-\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\mathbf{1}_{n_1}^T\operatorname{adj}\left(xI_{n_1}-\frac{1}{x}R(G_1)R(G_1)^T\right)\mathbf{1}_{n_1}\Bigg] \\
=& x^{m_1}\det\left(xI_{n_1}-\frac{1}{x}R(G_1)R(G_1)^T\right) \\
&\times\left[1-\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\mathbf{1}_{n_1}^T\left(xI_{n_1}-\frac{1}{x}R(G_1)R(G_1)^T\right)^{-1}\mathbf{1}_{n_1}\right] \\
=& x^{m_1}\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\left[1-\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)\Gamma_{\frac{1}{x}R(G_1)R(G_1)^T}(x)\right] \\
=& x^{m_1}\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\left[1-\frac{n_2}{x-r_2-\frac{2r_2}{x}}\frac{n_1}{x-\frac{2r_1}{x}}\right].
\end{aligned}
$$

Therefore,

$$
f_{A(S(G_1)\ddot{\vee}\mathcal{R}(G_2))}(x) = x^{m_1}x^{m_2}\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\prod_{j=1}^{n_2}\left(x-\lambda_j(G_2)-\frac{r_2}{x}-\frac{\lambda_j(G_2)}{x}\right)
$$

$$
\begin{aligned}
&\times\left[1-\frac{n_2}{x-r_2-\frac{2r_2}{x}}\frac{n_1}{x-\frac{2r_1}{x}}\right]\\
=&x^{m_1-n_1}x^{m_2-n_2}\prod_{i=2}^{n_1}\{x^2-r_1-\lambda_i(G_1)\}\\
&\times\prod_{j=2}^{n_2}\{x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)\}\\
&\times\{x^4-r_2x^3-(2r_1+n_1n_2+2r_2)x^2+2r_1r_2x+4r_1r_2\},
\end{aligned}
$$

and the result follows immediately. □

**Theorem 2.2.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ consists of:*

*(i) the eigenvalue $\pm\sqrt{r_1+\lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i=2,3,\ldots,n_1$, of $A(G_1)$;*

*(ii) roots of the equation $x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)=0$ for every eigenvalue $\lambda_j(G_2)$, $j=2,3,\ldots,n_2$, of $A(G_2)$;*

*(iii) the eigenvalue $0$ with multiplicity $m_1+m_2-n_1-n_2$;*

*(iv) four roots of the equation $x^4-r_2x^3-(2r_1+m_1m_2+2r_2)x^2+(2r_1r_2+m_1m_2r_2)x+4r_1r_2=0$.*

*Proof.* The adjacency characteristic polynomial of $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ is

$$
f_{A(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2))}(x)=\det\begin{pmatrix} xI_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2}-A(G_2) & -R(G_2)\\ O_{m_2\times n_1} & -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2}\end{pmatrix}=x^{n_1}\det(S),
$$

where

$$
\begin{aligned}
S=&\begin{pmatrix} xI_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2}\\ O_{n_2\times m_1} & xI_{n_2}-A(G_2) & -R(G_2)\\ -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2}\end{pmatrix}\\
&-\begin{pmatrix} -R(G_1)^T\\ O_{n_2\times n_1}\\ O_{m_2\times n_1}\end{pmatrix}\frac{1}{x}\begin{pmatrix} -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\end{pmatrix}\\
=&\begin{pmatrix} xI_{m_1}-\frac{1}{x}R(G_1)^TR(G_1) & O_{m_1\times n_2} & -J_{m_1\times m_2}\\ O_{n_2\times m_1} & xI_{n_2}-A(G_2) & -R(G_2)\\ -J_{m_2\times m_1} & -R(G_2)^T & xI_{m_2}\end{pmatrix}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\det(S)&=\det\left(xI_{m_1}-\frac{1}{x}R(G_1)^TR(G_1)\right)\det(W)\\
&=\det\left(xI_{m_1}-\frac{1}{x}(A(\mathbf{L}_{G_1})+2I_{m_1})\right)\det(W)
\end{aligned}
$$

$$= x^{m_1-n_1}\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\det(W),$$

where

$$\begin{aligned}W &= \begin{pmatrix} xI_{n_2}-A(G_2) & -R(G_2)\\ -R(G_2)^T & xI_{m_2}\end{pmatrix}\\ &\quad - \begin{pmatrix} O_{n_2\times m_1}\\ -J_{m_2\times m_1}\end{pmatrix}\left(xI_{m_1}-\frac{1}{x}R(G_1)^TR(G_1)\right)^{-1}\begin{pmatrix} O_{m_1\times n_2} & -J_{m_1\times m_2}\end{pmatrix}\\ &= \begin{pmatrix} xI_{n_2}-A(G_2) & -R(G_2)\\ -R(G_2)^T & xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2}\end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}\det(S) &= \det(xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2})\\ &\quad\times \det(xI_{n_2}-A(G_2)-R(G_2)(xI_{m_2}-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)J_{m_2\times m_2})^{-1}R(G_2)^T)\\ &= x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg[xI_{n_2}-A(G_2)\\ &\quad - R(G_2)\left\{\frac{1}{x}I_{m_2}+\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}J_{m_2\times m_2}\right\}R(G_2)^T\Bigg]\\ &= x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg(xI_{n_2}-A(G_2)\\ &\quad -\frac{1}{x}R(G_2)R(G_2)^T-\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}R(G_2)J_{m_2\times m_2}R(G_2)^T\Bigg)\\ &= x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\Bigg(xI_{n_2}-A(G_2)\\ &\quad -\frac{1}{x}R(G_2)R(G_2)^T-r_2^2\frac{\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}J_{n_2\times n_2}\Bigg)\\ &= x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\Bigg[\det\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\\ &\quad\times\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\mathbf{1}_{n_2}^T\operatorname{adj}\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\mathbf{1}_{n_2}\Bigg]\\ &= x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)\\ &\quad\times\Bigg[1-\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1))}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\mathbf{1}_{n_2}^T\\ &\quad\times\left(xI_{n_2}-A(G_2)-\frac{1}{x}R(G_2)R(G_2)^T\right)^{-1}\mathbf{1}_{n_2}\Bigg]\end{aligned}$$

$$
\begin{aligned}
=&x^{m_2}\left(1-\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\frac{m_2}{x}\right)\det\left(xI_{n_2}-A(G_2)-\frac{1}{x}(r_2I_{n_2}+A(G_2))\right)\\
&\times\left[1-\frac{r_2^2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x)\Gamma_{A(G_2)+\frac{1}{x}R(G_2)R(G_2)^T}(x)}{x(x-m_2\Gamma_{\frac{1}{x}R(G_1)^TR(G_1)}(x))}\right]\\
=&x^{m_2}\left(1-\frac{m_1m_2}{x(x-\frac{2r_1}{x})}\right)\prod_{j=1}^{n_2}\left\{x-\lambda_j(G_2)-\frac{1}{x}(r_2+\lambda_j(G_2))\right\}\\
&\times\left[1-\frac{r_2^2m_1n_2}{x(x-\frac{2r_1}{x})(x-\frac{m_1m_2}{x-\frac{2r_1}{x}})(x-r_2-\frac{2r_2}{x})}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{A(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2))}(x)=&x^{n_1}x^{m_1-n_1}x^{m_2}\left(1-\frac{m_1m_2}{x(x-\frac{2r_1}{x})}\right)\prod_{i=1}^{n_1}\left(x-\frac{r_1}{x}-\frac{\lambda_i(G_1)}{x}\right)\\
&\times\prod_{j=1}^{n_2}\left\{x-\lambda_j(G_2)-\frac{1}{x}(r_2+\lambda_j(G_2))\right\}\\
&\times\left[1-\frac{r_2^2m_1n_2}{x(x-\frac{2r_1}{x})(x-\frac{m_1m_2}{x-\frac{2r_1}{x}})(x-r_2-\frac{2r_2}{x})}\right]\\
=&x^{m_1-n_1}x^{m_2-n_2}\prod_{i=2}^{n_1}\{x^2-r_1-\lambda_i(G_1)\}\\
&\times\prod_{j=2}^{n_2}\{x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)\}\\
&\times\{x^4-r_2x^3-(2r_1+m_1m_2+2r_2)x^2\\
&+(2r_1r_2+m_1m_2r_2)x+4r_1r_2\},
\end{aligned}
$$

and hence the result follows. ☐

**Theorem 2.3.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $S(G_1)\overline{\nabla}\mathcal{R}(G_2)$ consists of:*

*(i) the eigenvalue $\pm\sqrt{r_1+\lambda_i(G_1)}$ for every eigenvalue $\lambda_i(G_1)$, $i=2,3,\dots,n_1$, of $A(G_1)$;*

*(ii) roots of the equation $x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)=0$ for every eigenvalue $\lambda_j(G_2)$, $j=2,3,\dots,n_2$, of $A(G_2)$;*

*(iii) the eigenvalue $0$ with multiplicity $m_1+m_2-n_1-n_2$;*

*(iv) four roots of the equation $x^4-r_2x^3-(2r_1+m_1n_2+2r_2)x^2+2r_1r_2x+4r_1r_2=0$.*

*Proof.* The proof is similar to that of proof of Theorem 2.2. ☐

**Theorem 2.4.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $S(G_1)\dot{\nabla}\mathcal{R}(G_2)$ consists of:*

*(i) the eigenvalue* $\pm\sqrt{r_1+\lambda_i(G_1)}$ *for every eigenvalue* $\lambda_i(G_1)$, $i=2,3,\dots,n_1$, *of* $A(G_1)$*;*

*(ii) roots of the equation* $x^2-\lambda_j(G_2)x-r_2-\lambda_j(G_2)=0$ *for every eigenvalue* $\lambda_j(G_2)$, $j=2,3,\dots,n_2$, *of* $A(G_2)$*;*

*(iii) the eigenvalue* $0$ *with multiplicity* $m_1+m_2-n_1-n_2$*;*

*(iv) four roots of the equation* $x^4-r_2x^3-(2r_1+m_1n_2+2r_2)x^2+(2r_1r_2+r_2n_1m_2)x+4r_1r_2=0$.

*Proof.* The proof is similar to that of proof of Theorem 2.1. $\square$

In the similar way as above we obtain Laplacian and normalized Laplacian spectra of the partial join graphs, which are given below.

**Lemma 2.2.** *We have the following Laplacian matrices:*

$$(i)\ L(S(G_1)\ddot{\bigtriangledown}\mathcal{R}(G_2))=\begin{pmatrix}(r_1+n_2)I_{n_1} & -R(G_1) & -J_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ -J_{n_2\times n_1} & O_{n_2\times m_1} & (r_2+n_1)I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & 2I_{m_2}\end{pmatrix};$$

$$(ii)\ L(S(G_1)\overline{\overline{\bigtriangledown}}R(G_2))=\begin{pmatrix}r_1I_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & (2+m_2)I_{m_1} & O_{m_1\times n_2} & -J_{m_1\times m_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & r_2I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & -J_{m_2\times m_1} & -R(G_2)^T & (2+m_1)I_{m_2}\end{pmatrix};$$

$$(iii)\ L(S(G_1)\overline{\dot{\bigtriangledown}}\mathcal{R}(G_2))=\begin{pmatrix}r_1I_{n_1} & -R(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2}\\ -R(G_1)^T & (2+n_2)I_{m_1} & -J_{m_1\times n_2} & O_{m_1\times m_2}\\ O_{n_2\times n_1} & -J_{n_2\times m_1} & (r_2+m_1)I_{n_2}+L(G_2) & -R(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & 2I_{m_2}\end{pmatrix};$$

$$(iv)\ L(S(G_1)\dot{\bigtriangledown}\mathcal{R}(G_2))=\begin{pmatrix}(r_1+m_2)I_{n_1} & -R(G_1) & O_{n_1\times n_2} & -J_{n_1\times m_2}\\ -R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & r_2I_{n_2}+L(G_2) & -R(G_2)\\ -J_{m_2\times n_1} & O_{m_2\times m_1} & -R(G_2)^T & (2+n_1)I_{m_2}\end{pmatrix}.$$

**Theorem 2.5.** *For* $i=1,2$, *let* $G_i$ *be an* $r_i$*-regular graph with* $n_i$ *vertices and* $m_i$ *edges. Then the Laplacian spectrum of* $S(G_1)\ddot{\bigtriangledown}\mathcal{R}(G_2)$ *consists of:*

*(i) roots of the equation* $x^2-(2+r_1+n_2)x+2n_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$ *,*$i=2,3,\dots,n_1$, *of* $L(G_1)$*;*

*(ii) roots of the equation* $x^2-(2+r_2+n_1+\mu_j(G_2))x+2n_1+3\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\dots,n_2$, *of* $L(G_2)$*;*

*(iii) the eigenvalue* $2$ *with multiplicity* $m_1+m_2-n_1-n_2$*;*

*(iv) four roots of the equation* $x^4-(4+r_1+r_2+n_1+n_2)x^3+(4+4n_1+4n_2+2r_1+2r_2+r_1r_2+r_1n_1+r_2n_2)x^2-2(2n_1+2n_2+r_1n_1+r_2n_2)x=0$.

**Theorem 2.6.** *For* $i=1,2$, *let* $G_i$ *be an* $r_i$*-regular graph with* $n_i$ *vertices and* $m_i$ *edges. Then the Laplacian spectrum of* $S(G_1)\overline{\overline{\bigtriangledown}}\mathcal{R}(G_2)$ *consists of:*

(*i*) *roots of the equation* $x^2-(2+r_1+m_2)x+r_1m_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$, *of* $L(G_1)$;

(*ii*) *roots of the equation* $x^2-(2+r_2+m_1+\mu_j(G_2))x+r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(*iii*) *the eigenvalue* $2+m_2$ *with multiplicity* $m_1-n_1$;

(*iv*) *the eigenvalue* $2+m_1$ *with multiplicity* $m_2-n_2$;

(*v*) *four roots of the equation* $x^4-(4+r_1+r_2+m_1+m_2)x^3+(4+2r_1+2r_2+r_1r_2+r_1m_1+r_2m_2+2m_1+2m_2+r_1m_2+r_2m_1)x^2-(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2)x=0$.

**Theorem 2.7.** *For* $i=1,2$, *let* $G_i$ *be an* $r_i$*-regular graph with* $n_i$ *vertices and* $m_i$ *edges. Then the Laplacian spectrum of* $S(G_1)\bar{\dot{\vee}}\mathcal{R}(G_2)$ *consists of:*

(*i*) *roots of the equation* $x^2-(2+r_1+n_2)x+r_1n_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$, *of* $L(G_1)$;

(*ii*) *roots of the equation* $x^2-(2+r_2+m_1+\mu_j(G_2))x+2m_1+3\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(*iii*) *the eigenvalue* $2+n_2$ *with multiplicity* $m_1-n_1$;

(*iv*) *the eigenvalue* $2$ *with multiplicity* $m_2-n_2$;

(*v*) *four roots of the equation* $x^4-(4+r_1+r_2+m_1+n_2)x^3+(4+2r_1+2r_2+4m_1+2n_2+r_1r_2+r_1m_1+r_1n_2+r_2n_2)x^2-(4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2)x=0$.

**Theorem 2.8.** *For* $i=1,2$, *let* $G_i$ *be an* $r_i$*-regular graph with* $n_i$ *vertices and* $m_i$ *edges. Then the Laplacian spectrum of* $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ *consists of:*

(*i*) *roots of the equation* $x^2-(2+r_1+m_2)x+2m_2+\mu_i(G_1)=0$ *for every eigenvalue* $\mu_i(G_1)$, $i=2,3,\ldots,n_1$ *of* $L(G_1)$;

(*ii*) *roots of the equation* $x^2-(2+r_2+n_1+\mu_j(G_2))x+r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2)=0$ *for every eigenvalue* $\mu_j(G_2)$, $j=2,3,\ldots,n_2$, *of* $L(G_2)$;

(*iii*) *the eigenvalue* $2$ *with multiplicity* $m_1-n_1$;

(*iv*) *the eigenvalue* $2+n_1$ *with multiplicity* $m_2-n_2$;

(*v*) *four roots of the equation* $x^4-(4+r_1+r_2+m_2+n_1)x^3+(4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1)x^2-(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)x=0$.

**Lemma 2.3.** *We have the following normalized Laplacian matrices:*

(*i*)

$$\mathcal{L}(S(G_1)\ddot{\vee}\mathcal{R}(G_2))=\begin{pmatrix} I_{n_1} & -cR(G_1) & -K_{n_1\times n_2} & O_{n_1\times m_2}\\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2}\\ -K_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2)\\ O_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2}\end{pmatrix},$$

*where* $K_{n_1\times n_2}$ *is the matrix of size* $n_1\times n_2$ *with all entries equal to* $\frac{1}{\sqrt{(r_1+n_2)(2r_2+n_1)}}$, $B(G_2)$ *is the* $n_2\times n_2$ *matrix whose all diagonal entries are* $1$ *and off-diagonal entries are* $\frac{r_2}{2r_2+n_1}$, $c$ *is the constant whose value is* $\frac{1}{\sqrt{2(r_1+n_2)}}$, $d$ *is the constant whose value is* $\frac{1}{\sqrt{2(2r_2+n_1)}}$;

(*ii*)

$$\mathcal{L}(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & -K_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ O_{m_2\times n_1} & -K_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

*where $K_{m_1\times m_2}$ is the matrix of size $m_1 \times m_2$ with all entries equal to $\frac{1}{\sqrt{(2+m_2)(2+m_1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are $1$ and off-diagonal entries are $\frac{r_2}{2r_2}$, $c$ is the constant whose value is $\frac{1}{\sqrt{r_1(2+m_2)}}$, $d$ is the constant whose value is $\frac{1}{\sqrt{2r_2(2+m_1)}}$;*

(*iii*)

$$\mathcal{L}(S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & O_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & -K_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & -K_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ O_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

*where $K_{m_1\times n_2}$ is the matrix of size $m_1 \times n_2$ with all entries equal to $\frac{1}{\sqrt{(2+n_2)(2r_2+m_1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are $1$ and off-diagonal entries are $\frac{r_2}{2r_2+m_1}$, $c$ is the constant whose value is $\frac{1}{\sqrt{r_1(2+n_2)}}$, $d$ is the constant whose value is $\frac{1}{\sqrt{2(2r_2+m_1)}}$;*

(*iv*)

$$\mathcal{L}(S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} & -K_{n_1\times m_2} \\ -cR(G_1)^T & I_{m_1} & O_{m_1\times n_2} & O_{m_1\times m_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) & -dR(G_2) \\ -K_{m_2\times n_1} & O_{m_2\times m_1} & -dR(G_2)^T & I_{m_2} \end{pmatrix},$$

*where $K_{m_1\times n_2}$ is the matrix of size $m_1 \times n_2$ with all entries equal to $\frac{1}{\sqrt{(2+n_1)(r_1+m_2)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are $1$ and off-diagonal entries are $\frac{r_2}{2r_2}$, $c$ is the constant whose value is $\frac{1}{\sqrt{2(r_1+m_2)}}$, $d$ is the constant whose value is $\frac{1}{\sqrt{2r_2(2+n_1)}}$.*

**Theorem 2.9.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the normalized Laplacian spectrum of $S(G_1)\ddot{\nabla}\mathcal{R}(G_2)$ consists of:*

(*i*) *roots of the equation $2(r_1 + n_2)x^2 - 4(r_1 + n_2)x + 2n_2 + r_1\delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

(*ii*) *roots of the equation $2(2r_2+n_1)x^2-2(3r_2+2n_1+r_2\delta_j(G_2))x+2n_1+3r_2\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

(*iii*) *the eigenvalue $1$ with multiplicity $m_1 + m_2 - n_1 - n_2$;*

(*iv*) *four roots of the equation $(2r_1r_2 + r_1n_1 + 2r_2n_2 + n_1n_2)x^4 - (5r_1r_2 + 3r_1n_1 + 5r_2n_2 + 3n_1n_2)x^3 + (3r_1r_2 + 3r_1n_1 + 5r_2n_2 + 3n_1n_2)x^2 - (r_1n_1 + 3r_2n_2 + n_1n_2)x = 0$.*

**Theorem 2.10.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the normalized Laplacian spectrum of $S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)$ consists of:*

($i$) *roots of the equation $(2 + m_2)x^2 - 2(2 + m_2)x + m_2 + \delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

($ii$) *roots of the equation $2(2 + m_1)x^2 - (6 + 3m_1 + 2\delta_j(G_2) + m_1\delta_j(G_2))x + m_1 + 3\delta_j(G_2) + m_1\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

($iii$) *the eigenvalue $1$ with multiplicity $m_1 + m_2 - n_1 - n_2$;*

($iv$) *four roots of the equation $2(4 + 2m_1 + 2m_2 + m_1m_2)x^4 - 7(4 + 2m_1 + 2m_2 + m_1m_2)x^3 + (24 + 14m_1 + 16m_2 + 7m_1m_2)x^2 - 2(2m_1 + 3m_2 + m_1m_2)x = 0$.*

**Theorem 2.11.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the normalized Laplacian spectrum of $S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)$ consists of:*

($i$) *roots of the equation $(2+n_2)x^2-2(2+n_2)x+n_2+\delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

($ii$) *roots of the equation $2(2r_2+m_1)x^2-2(3r_2+2m_1+r_2\delta_j(G_2))x+2m_1+3r_2\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

($iii$) *the eigenvalue $1$ with multiplicity $m_1 + m_2 - n_1 - n_2$;*

($iv$) *four roots of the equation $(4r_2+2r_2n_2+2m_1+m_1n_2)x^4-(10r_2+5r_2n_2+6m_1+3m_1n_2)x^3 + (6r_2 + 5r_2n_2 + 6m_1 + 3m_1n_2)x^2 - (3r_2n_2 + 2m_1 + m_1n_2)x = 0$.*

**Theorem 2.12.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the normalized Laplacian spectrum of $S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)$ consists of:*

($i$) *roots of the equation $2(r_1 + m_2)x^2 - 4(r_1 + m_2)x + 2m_2 + r_2\delta_i(G_1) = 0$ for every eigenvalue $\delta_i(G_1)$, $i = 2, 3, \ldots, n_1$, of $\mathcal{L}(G_1)$;*

($ii$) *roots of the equation $2(2 + n_1)x^2 - (6 + 3n_2 + 2\delta_j(G_2) + n_1\delta_j(G_2))x + n_1 + 3\delta_j(G_2) + n_1\delta_j(G_2) = 0$ for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \ldots, n_2$, of $\mathcal{L}(G_2)$;*

($iii$) *the eigenvalue $1$ with multiplicity $m_1 + m_2 - n_1 - n_2$;*

($iv$) *four roots of the equation $2(2r_1 + r_1n_1 + 2m_2 + m_2n_1)x^4 - 7(2r_1 + r_1n_1 + 2m_2 + m_2n_1)x^3 + (12r_1 + 7r_1n_1 + 16m_2 + 7m_2n_1)x^2 - 2(r_1n_1 + 3m_2 + m_2n_1)x = 0$.*

## 3. Simultaneous Cospectral Graphs

In this section we present the main result of the paper. We construct several classes of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. For the construction of these graphs we consider two pairs of $A$-cospectral regular graphs, which are readily available in the literature, for example see [14]. Then we take partial join of subdivision graph and $R$-graph belong to different pairs.

The following lemma is immediate from the definition of Laplacian and normalized Laplacian matrices.

**Lemma 3.1.** ($i$) *If $G$ is an $r$-regular graph, then $L(G) = rI_n - A(G)$ and $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$.*

($ii$) *If $G_1$ and $G_2$ are $A$-cospectral regular graphs, then they are also cospectral with respect to the Laplacian and normalized Laplacian matrices.*

**Observation.** From all the theorems given in the previous section we observe that the adjacency, Laplacian and normalized Lpalacian spectra of all the partial join graphs $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$, $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$, $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$, and $S(G_1)\overline{\vee}\mathcal{R}(G_2)$, depend only on the number of vertices, number of edges, degree of regularities, and the corresponding spectrum of $G_1$ and $G_2$. Furthermore, we note that, although $G_1$ and $G_2$ are regular graphs, $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$, $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$, $S(G_1)\overline{\vee}\mathcal{R}(G_2)$ and $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$ are non-regular graphs.

The following theorem is the main result of the paper.

**Theorem 3.1.** *Let $G_i$, $H_i$, $i = 1, 2$ be regular graphs, where $G_1$ need not be different from $H_1$. If $G_1$ and $H_1$ are $A$-cospectral, and $G_2$ and $H_2$ are $A$-cospectral then $S(G_1)\ddot{\vee}\mathcal{R}(G_2)$ (respectively, $S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)$, $S(G_1)\overline{\vee}\mathcal{R}(G_2)$, $S(G_1)\dot{\overline{\vee}}\mathcal{R}(G_2)$) and $S(H_1)\ddot{\vee}\mathcal{R}(H_2)$ (respectively, $S(H_1)\overline{\overline{\vee}}\mathcal{R}(H_2)$, $S(H_1)\overline{\vee}\mathcal{R}(H_2)$, $S(H_1)\dot{\overline{\vee}}\mathcal{R}(H_2)$) are simultaneously $A$-cospectral, $L$-cospectral and $\mathcal{L}$-cospectral.*

*Proof.* Follows from Lemma 3.1 and the above observation. $\square$

## 4. Spanning Trees and Kirchhoff Indices

Applying the results on Laplacian and normalized Laplacian spectra given in Section 2, we find the number of spanning trees and Kirchhoff index of all the partial join graphs constructed in the paper.

Let $t(G)$ denote the number of *spanning trees* of $G$. It is well known [5] that if $G$ is a connected graph on $n$ vertices with Laplacian spectrum $0 = \mu_1(G) \le \mu_2(G) \le \cdots \le \mu_n(G)$, then $t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}$.

The *Kirchhoff index* of a graph $G$, denoted by $Kf(G)$, is defined as the sum of resistances between all pairs of vertices [1, 10] in $G$. For a connected graph $G$ on $n$ vertices, the Kirchhoff index [9] can be expressed as $Kf(G) = n\sum_{i=2}^{n}\frac{1}{\mu_i(G)}$.

**Theorem 4.1.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then*

(i) $$t(S(G_1)\ddot{\vee}\mathcal{R}(G_2)) = \frac{2^{m_1+m_2-n_1-n_2}\cdot 2(2n_1+2n_2+r_1n_1+r_2n_2)\cdot\prod_{i=2}^{n_1}(2n_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(2n_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2};$$

(ii)
$$\begin{aligned} t(S(G_1)\overline{\overline{\vee}}\mathcal{R}(G_2)) =& (2+m_2)^{m_1-n_1}\cdot(2+m_1)^{m_2-n_2} \\ &\times\frac{(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2)\cdot\prod_{i=2}^{n_1}(r_1m_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2))}{n_1+n_2+m_1+m_2}; \end{aligned}$$

(iii)
$$\begin{aligned} t(S(G_1)\overline{\vee}\mathcal{R}(G_2)) =& (2+n_2)^{m_1-n_1}\cdot 2^{m_2-n_2} \\ &\times\frac{(4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2)\cdot\prod_{i=2}^{n_1}(r_1n_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(2m_1+3\mu_j(G_2))}{n_1+n_2+m_1+m_2}; \end{aligned}$$

(iv)

$$t(S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)) = 2^{m_1-n_1}\cdot(2+n_1)^{m_2-n_2} \times \frac{(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)\cdot\prod_{i=2}^{n_1}(2m_2+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2))}{n_1+n_2+m_1+m_2}.$$

**Theorem 4.2.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then*

(i)

$$Kf(S(G_1)\ddot{\nabla}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2)\Bigg(\frac{m_1+m_2-n_1-n_2}{2} + \frac{4+4n_1+4n_2+2r_1+2r_2+r_1r_2+r_1n_1+r_2n_2}{2(2n_1+2n_2+r_1n_1+r_2n_2)} + \sum_{i=2}^{n_1}\frac{2+r_1+n_2}{2n_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+n_1+\mu_j(G_2)}{2n_1+3\mu_j(G_2)}\Bigg);$$

(ii)

$$Kf(S(G_1)\overline{\overline{\nabla}}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2)\times\Bigg(\frac{m_1-n_1}{2+m_2}+\frac{m_2-n_2}{2+m_1} + \frac{4+2r_1+2r_2+r_1r_2+r_1m_1+r_2m_2+2m_1+2m_2+r_1m_2+r_2m_1}{2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2} + \sum_{i=2}^{n_1}\frac{2+r_1+m_2}{r_1m_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+m_1+\mu_j(G_2)}{r_2m_1+3\mu_j(G_2)+m_1\mu_j(G_2)}\Bigg);$$

(iii)

$$Kf(S(G_1)\overline{\dot{\nabla}}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2) \times\Bigg(\frac{m_1-n_1}{2+n_2}+\frac{m_2-n_2}{2} + \frac{4+2r_1+2r_2+4m_1+2n_2+r_1r_2+r_1m_1+r_1n_2+r_2n_2}{4m_1+2r_1m_1+2r_1n_2+r_1r_2n_2} + \sum_{i=2}^{n_1}\frac{2+r_1+n_2}{r_1n_2+\mu_i(G_1)} + \sum_{j=2}^{n_2}\frac{2+r_2+m_1+\mu_j(G_2)}{2m_1+3\mu_j(G_2)}\Bigg);$$

(iv)

$$Kf(S(G_1)\dot{\overline{\nabla}}\mathcal{R}(G_2)) = (n_1+n_2+m_1+m_2) \times\Bigg(\frac{m_1-n_1}{2}+\frac{m_2-n_2}{2+n_1}$$

$$+\frac{4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1}{4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1}$$
$$\left.+\sum_{i=2}^{n_1}\frac{2+r_1+m_2}{2m_2+\mu_i(G_1)}+\sum_{j=2}^{n_2}\frac{2+r_2+n_1+\mu_j(G_2)}{r_2n_1+3\mu_j(G_2)+n_1\mu_j(G_2)}\right).$$

## 5. Concluding remarks

The main result of the paper is based on regular $A$-cospectral graphs and certain operations on a pair of these graphs so that the operated (or resultant) graphs are non-regular and having adjacency, Laplacian and normalized Laplacian spectra which depend on only the order, size, degree of regularity and spectrum of the original graphs. Thus one may search for some other graph operations to construct simultaneous cospectral graphs like in the paper.

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# ON TWO DIFFERENT CLASSES OF WARPED PRODUCT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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ABSTRACT. Warped product skew CR-submanifold of the form $M = M_1 \times_f M_\perp$ of a Kenmotsu manifold $\bar{M}$ (throughout the paper), where $M_1 = M_T \times M_\theta$ and $M_T,\ M_\perp,\ M_\theta$ represents invariant, anti-invariant and proper slant submanifold of $\bar{M}$, studied in [28] and another class of warped product skew CR-submanifold of the form $M = M_2 \times_f M_T$ of $\bar{M}$, where $M_2 = M_\perp \times M_\theta$ is studied in [19]. Also the warped product submanifold of the form $M = M_3 \times_f M_\theta$ of $\bar{M}$, where $M_3 = M_T \times M_\perp$ and $M_T,\ M_\perp,\ M_\theta$ represents invariant, anti-invariant and proper point wise slant submanifold of $\bar{M}$, were studied in [18]. As a generalization of the above mentioned three classes, we consider a class of warped product submanifold of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$ in which $M_{\theta_1}$ and $M_{\theta_2}$ are proper slant submanifolds of $\bar{M}$ and $M_{\theta_3}$ represents a proper pointwise slant submanifold of $\bar{M}$. A characterization is given on the existence of such warped product submanifolds which generalizes the characterization of warped product submanifolds of the form $M = M_1 \times_f M_\perp$, studied in [28], the characterization of warped product submanifolds of the form $M = M_2 \times_f M_T$, studied in [19], the characterization of warped product submanifolds of the form $M = M_3 \times_f M_\theta$, studied in [18] and also the characterization of warped product pointwise bi-slant submanifolds of $\bar{M}$, studied in [17]. Since warped product bi-slant submanifolds of $\bar{M}$ does not exist (Theorem 4.2 of [17]), the Riemannian product $M_4 = M_{\theta_1} \times M_{\theta_2}$ cannot be a warped product. So, for studying the bi-warped product submanifolds of $\bar{M}$ of the form $M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$, we have taken $M_{\theta_1},\ M_{\theta_2},\ M_{\theta_3}$ as pointwise slant submanifolds of $\bar{M}$ of distinct slant functions $\theta_1,\ \theta_2,\ \theta_3$ respectively. The existence of such type of bi-warped product submanifolds of $\bar{M}$ is ensured by an example. Finally, a Chen-type inequality on the squared norm of the second fundamental form of such bi-warped product submanifolds of $\bar{M}$ is obtained which also generalizes the inequalities obtained in [33], [18] and [17], respectively.

*Key words and phrases.* Kenmotsu manifold, pointwise slant submanifolds, warped product, submanifolds, bi-warped product submanifolds.
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## 1. Introduction

The warped product [5] between two Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where $\pi_1$ and $\pi_2$ are canonical projections of $M_1 \times M_2$ onto $M_1$ and $M_2$, respectively and $\pi_i^*(g_i)$ is the pullback of $g_i$ via $\pi_i$ for $i = 1, 2$ and $f : N_1 \to \mathbb{R}^+$ is a smooth function.

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if $f$ is constant. For $M = N_1 \times_f N_2$, we have [5]

$$\nabla_U X = \nabla_X U = (X \ln f)U, \tag{1.1}$$

for any $X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

The study of warped product submanifold was initiated in [8–10]. Then many authors have studied warped product submanifolds of different ambient manifolds, see [15–17, 20]. In [31], Tanno classified almost contact metric manifolds in three different classes among which the third class was picked up by Kenmotsu in 1972 and he studied its differential geometric properties [21]. This class later named after him by Kenmotsu manifold which is very important class to study. Warped product submanifolds of Kenmotsu manifolds are also studied in ([1–3], [22], [23], [26], [27], [32]-[38]). Multiply warped products (see [11, 12, 38]) are generalizations of warped product and Riemannian product manifolds and bi-warped products are special classes of multiply warped products. Bi-warped product submanifolds of different ambient manifolds are studied in [33, 35]. For the study of slant immersion and slant submanifolds in contact metric manifolds we refer [6, 7, 24]. In [29] Park studied pointwise slant and pointwise semi slant submanifolds of almost contact Riemannian manifolds.

Recently, Roy et al. studied the characterization theorem on warped product submanifold of Sasakian manifolds in [30]. Motivated by the above studies, in this present paper we have studied warped product submanifolds of $\bar{M}$ of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ such that $\xi \in \Gamma(TM_4)$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, $M_{\theta_1}$, $M_{\theta_2}$ are proper slant submanifolds of $\bar{M}$ and here $M_{\theta_3}$ represents a proper pointwise slant submanifold of $\bar{M}$. Next we have studied bi-warped product submanifolds of $\bar{M}$ of the form $M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$, where $M_{\theta_1}$, $M_{\theta_2}$, $M_{\theta_3}$ are pointwise slant submanifolds of $\bar{M}$ of distinct slant functions $\theta_1$, $\theta_2$ and $\theta_3$, respectively.

The paper is organized as follows. Section 2 deals with some preliminary useful results for construction of the paper, Section 3 is concerned with the study of a class of submanifold $M$ of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle$, where $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ are slant distributions and $\mathcal{D}^{\theta_3}$ is pointwise slant distribution. In Section 4, we have studied warped product submanifolds of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ such that $\xi$ is orthogonal to $M_{\theta_3}$ with an supporting example. In Section 5, a characterization theorem of the mentioned class has been obtained, Section 6 deals with bi-warped product submanifolds $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ of $\bar{M}$, where $M_{\theta_1},\ , M_{\theta_2}, M_{\theta_3}$ are pointwise slant submanifolds of $\bar{M}$ and constructed an example. In Section 7, we have obtained a generalized inequality for such class of bi-warped product submanifolds of $\bar{M}$. The last section is the conclusion part of the paper where we have shown how the results of this paper generalizes several results of different works.

## 2. Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2m+1}$ is said to be an almost contact metric manifold [4] if it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$, an 1-form $\eta$ and a Riemannian metric $g$ which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -\, g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields $X, Y$ on $\bar{M}^{2m+1}$.

An almost contact metric manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following conditions hold [21]:

$$\bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\bar{\nabla}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where $\bar{\nabla}$ denotes the Riemannian connection of $g$.

Let $M$ be an $n$-dimensional submanifold of a Kenmotsu manifold $\bar{M}$. Throughout the paper we assume that the submanifold $M$ of $\bar{M}$ is tangent to the structure vector field $\xi$.

Let $\nabla$ and $\nabla^{\perp}$ be the induced connections on the tangent bundle $TM$ and the normal bundle $T^{\perp}M$ of $M$ respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $h$ and $A_V$ are second fundamental form and the shape operator (corresponding to the normal vector field $V$) respectively for the immersion of $M$ into $\bar{M}$. The second fundamental form $h$ and the shape operator $A_V$ are related by $g(h(X,Y), V) = g(A_V X, Y)$ for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where g is the Riemannian metric on $\bar{M}$ as well as on $M$.

The mean curvature $H$ of $M$ is given by $H = \frac{1}{n}\mathrm{trace}\, h$. A submanifold of a Kenmotsu manifold $\bar{M}$ is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ for any $X, Y \in \Gamma(TM)$. If $h(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$, then $M$ is totally geodesic and if $H = 0$, then $M$ is minimal in $\bar{M}$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent bundle $TM$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ an orthonormal basis of the normal bundle $T^\perp M$. We put

$$h^r_{ij} = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = g(h(e_i, e_j), h(e_i, e_j)),$$

for $r \in \{n+1, \ldots, 2m+1\}$, $i, j = 1, 2, \ldots, n$.

For a differentiable function $f$ on $M$, the gradient $\boldsymbol{\nabla} f$ is defined by

$$g(\boldsymbol{\nabla} f, X) = Xf,$$

for any $X \in \Gamma(TM)$. As a consequence, we get

$$\|\boldsymbol{\nabla} f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{2.8}$$

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we can write

(a) $\phi X = PX + QX$;

(b) $\phi V = bV + cV$,

where $PX$, $bV$ are the tangential components and $QX$, $cV$ are the normal components.

A submanifold $M$ of an almost contact metric manifold $\bar{M}$ is said to be slant if for each non-zero vector $X \in T_pM$, the angle $\theta$ between $\phi X$ and $T_pM$ is constant, i.e., it does not depend on the choice of $p \in M$.

A submanifold $M$ of an almost contact metric manifold $\bar{M}$ is said to be pointwise slant [13] if for any non-zero vector $X \in T_pM$ at $p \in M$, such that $X$ is not proportional to $\xi_p$, the angle $\theta(X)$ between $\phi X$ and $T^*_pM = T_pM - \{0\}$ is independent of the choice of non-zero $X \in T^*_pM$.

For pointwise slant submanifold, $\theta$ is a function on $M$, which is known as slant function of $M$. Invariant and anti-invariant submanifolds are particular cases of pointwise slant submanifolds with slant function $\theta = 0$ and $\frac{\pi}{2}$ respectively. Also a pointwise slant submanifold $M$ will be slant if $\theta$ is constant on $M$. Thus a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor constant. It may be noted that [25] $M$ is a pointwise slant submanifold of $\bar{M}$ if and only if exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.9}$$

Furthermore, $\lambda = \cos^2\theta$ for slant function $\theta$. If $M$ be a pointwise slant submanifold of $\bar{M}$, then we have [34]:

$$bQX = \sin^2\theta\{-X + \eta(X)\xi\}, \quad cQX = -QPX. \tag{2.10}$$

Let $M_1$, $M_2$, $M_3$ be Riemannian manifolds and let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be the product manifold of $M_1$, $M_2$, $M_3$ such that $f_1, f_2 : M_1 \to \mathbb{R}^+$ are real valued smooth functions. For each $i$, denote by $\pi_i : M \to M_i$ the canonical projection of $M$ onto $M_i$, $i = 1, 2, 3$. Then the metric on $M$, called a bi-warped metric is given by

$$g(X, Y) = g(\pi_{1_*}X, \pi_{2_*}Y) + (f_1 \circ \pi_1)^2 g(\pi_{2_*}X, \pi_{2_*}Y) + (f_2 \circ \pi_1)^2 g(\pi_{3_*}X, \pi_{3_*}Y),$$

for any $X, Y \in \Gamma(TM)$ and $*$ denotes the symbol for tangent maps. The manifold $M$ endowed with this product metric is called a bi-warped product manifold. Here $f_1$, $f_2$ are non-constant functions, called warping functions on $M$. Clearly, if both $f_1$, $f_2$ are constant on $M$, then $M$ is simply a Riemannian product manifold and if anyone of the functions is constant, then $M$ is a single warped product manifold. If neither $f_1$ nor $f_2$ is constant, then $M$ is a proper bi-warped product manifold.

Let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be a warped product submanifold of $\bar{M}$. Then we have [35]

$$\nabla_X Z = \sum_{i=1}^{2} (X(\ln f_i)) Z^i,$$

for any $X \in \mathcal{D}^1$, the tangent space of $M_1$ and $Z \in \Gamma(TN)$, where $N =_{f_1} M_2 \times_{f_2} M_3$ and $Z^i$ is $M_i$ components of $Z$ for each $i = 2, 3$ and $\nabla$ is the Levi-Civita connection on $M$.

## 3. Submanifolds of $\bar{M}$

In this section we consider submanifold $M$ of $\bar{M}$ such that

$$\begin{aligned} TM =& \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle \xi \rangle, \\ T^\perp M =& Q\mathcal{D}^{\theta_1} \oplus Q\mathcal{D}^{\theta_2} \oplus Q\mathcal{D}^{\theta_3} \oplus \nu, \end{aligned}$$

where $\nu$ is a $\phi$-invariant normal subbundle of $T^\perp M$.

If $M$ is such submanifold of $\bar{M}$, then for any $X \in \Gamma(TM)$ we have

$$X = T_1 X + T_2 X + T_3 X, \tag{3.1}$$

where $T_1$, $T_2$ and $T_3$ are the projections from $TM$ onto $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively.

If we put $P_1 = T_1 \circ P$, $P_2 = T_2 \circ P$ and $P_3 = T_3 \circ P$ then from (3.1), we get

$$\phi X = P_1 X + P_2 X + P_3 X + QX, \tag{3.2}$$

for $X \in \Gamma(TM)$.

From (2.9) and (3.2), we get

$$P_i^2 = \cos^2\theta_i(-I + \eta \otimes \xi), \quad \text{for } i = 1, 2, 3. \tag{3.3}$$

Now for the sake of further study we obtain the following useful results.

**Lemma 3.1.** *Let $M$ be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ and $\xi \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2})$ then the following relations hold:*

$$\begin{aligned} (\sin^2\theta_1 - \sin^2\theta_3) g(\nabla_{X_1} Y_1, X_3) =& g(A_{QP_1Y_1} X_3 - A_{QY_1} P_3 X_3, X_1) \\ &+ g(A_{QP_3X_3} Y_1 - A_{QX_3} P_1 Y_1, X_1), \end{aligned} \tag{3.4}$$

$$\begin{aligned} (\sin^2\theta_2 - \sin^2\theta_3) g(\nabla_{X_2} Y_2, X_3) =& g(A_{QP_2Y_2} X_3 - A_{QY_2} P_3 X_3, X_2) \\ &+ g(A_{QP_3X_3} Y_2 - A_{QX_3} P_2 Y_2, X_2), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\sin^2\theta_2 - \sin^2\theta_3) g(\nabla_{X_1} X_2, X_3) =& g(A_{QP_2X_2} X_3 - A_{QX_2} P_3 X_3, X_1) \\ &+ g(A_{QP_3X_3} X_2 - A_{QX_3} P_2 X_2, X_1), \end{aligned} \tag{3.6}$$

$$(3.7)\qquad (\sin^2\theta_1-\sin^2\theta_3)g(\nabla_{X_2}X_1,X_3)=g(A_{QP_1X_1}X_3-A_{QX_1}P_3X_3,X_2)+g(A_{QP_3X_3}X_1-A_{QX_3}P_1X_1,X_2),$$

*for any* $X_1,Y_1\in\Gamma(\mathcal{D}^{\theta_1}\oplus\langle\xi\rangle)$, $X_2,Y_2\in\Gamma(\mathcal{D}^{\theta_2}\oplus\langle\xi\rangle)$ *and* $X_3\in\Gamma(\mathcal{D}^{\theta_3})$.

*Proof.* For any $X_1,Y_1\in\Gamma(\mathcal{D}^{\theta_1}\oplus\langle\xi\rangle)$ and $X_3\in\Gamma(\mathcal{D}^{\theta_3})$, we have from (2.3), (2.5) and (3.2) that

$$\begin{aligned}
g(\nabla_{X_1}Y_1,X_3)=&g(\bar\nabla_{X_1}P_1Y_1,\phi X_3)+g(\bar\nabla_{X_1}QY_1,\phi X_3)\\
=&-g(\phi\bar\nabla_{X_1}P_1Y_1,X_3)+g(\bar\nabla_{X_1}QY_1,P_3X_3)+g(\bar\nabla_{X_1}QY_1,QX_3)\\
=&-g(\bar\nabla_{X_1}P_1^2Y_1,X_3)-g(\bar\nabla_{X_1}QP_1Y_1,X_3)+g((\bar\nabla_{X_1}\phi)P_1Y_1,X_3)\\
&+g(\bar\nabla_{X_1}QY_1,P_3X_3)-g(\bar\nabla_{X_1}QX_3,\phi Y_1)+g(\bar\nabla_{X_1}QX_3,P_1Y_1)\\
=&-g(\bar\nabla_{X_1}P_1^2Y_1,X_3)-g(\bar\nabla_{X_1}QP_1Y_1,X_3)+g(\bar\nabla_{X_1}QY_1,P_3X_3)\\
&+g(\bar\nabla_{X_1}bQX_3,Y_1)+g(\bar\nabla_{X_1}cQX_3,Y_1)+g(\bar\nabla_{X_1}QX_3,P_1Y_1).
\end{aligned}$$

Using (2.7), (2.10) and (3.3), the above equation reduces to

$$\begin{aligned}
g(\nabla_{X_1}Y_1,X_3)=&\cos^2\theta_1 g(\bar\nabla_{X_1}Y_1,X_3)+g(A_{QP_1Y_1}X_3,X_1)-g(A_{QY_1}P_3X_3,X_1)\\
&+\sin^2\theta_3 g(\bar\nabla_{X_1}Y_1,X_3)+g(A_{QP_3X_3}Y_1,X_1)-g(A_{QX_3}P_1Y_1,X_1),
\end{aligned}$$

from which the relation (3.4) follows.

The relations (3.5)–(3.7) follow similarly. □

**Lemma 3.2.** *Let $M$ be a submanifold of $\bar M$ where $TM=\mathcal{D}^{\theta_1}\oplus\mathcal{D}^{\theta_2}\oplus\mathcal{D}^{\theta_3}$ such that $\xi\in\Gamma(\mathcal{D}^{\theta_1}\oplus\mathcal{D}^{\theta_2})$. Then the following relations hold:*

$$\begin{aligned}
(3.8)\qquad (\sin^2\theta_3-\sin^2\theta_1)g(\nabla_{X_3}Y_3,X_1)=&g(A_{QP_3Y_3}X_1-A_{QY_3}P_1X_1,X_3)\\
&+g(A_{QP_1X_1}Y_3-A_{QX_1}P_3Y_3,X_3)\\
&+(\cos^2\theta_3-\cos^2\theta_1)\eta(X_1)g(X_3,Y_3),
\end{aligned}$$

$$\begin{aligned}
(3.9)\qquad (\sin^2\theta_3-\sin^2\theta_2)g(\nabla_{X_3}Y_3,X_2)=&g(A_{QP_3Y_3}X_2-A_{QY_3}P_2X_2,X_3)\\
&+g(A_{QP_2X_2}Y_3-A_{QX_2}P_3Y_3,X_3)\\
&+(\cos^2\theta_3-\cos^2\theta_2)\eta(X_2)g(X_3,Y_3),
\end{aligned}$$

*for any* $X_1\in\Gamma(\mathcal{D}^{\theta_1}\oplus\langle\xi\rangle)$, $X_2\in\Gamma(\mathcal{D}^{\theta_2}\oplus\langle\xi\rangle)$ *and* $X_3,Y_3\in\Gamma(\mathcal{D}^{\theta_3})$.

*Proof.* For any $X_1\in\Gamma(\mathcal{D}^{\theta_1}\oplus\langle\xi\rangle)$ and $X_3,Y_3\in\Gamma(\mathcal{D}^{\theta_3})$, we have from (2.3), (2.5) and (3.2) that

$$\begin{aligned}
g(\nabla_{X_3}Y_3,X_1)=&g(\bar\nabla_{X_3}P_3Y_3,\phi X_1)+g(\bar\nabla_{X_3}QY_3,\phi X_1)-\eta(X_1)g(X_3,Y_3)\\
=&-g(\phi\bar\nabla_{X_3}P_3Y_3,X_1)+g(\bar\nabla_{X_3}QY_3,P_1X_1)\\
&+g(\bar\nabla_{X_3}QY_3,QX_1)-\eta(X_1)g(X_3,Y_3)\\
=&-g(\bar\nabla_{X_3}P_3^2Y_3,X_1)-g(\bar\nabla_{X_3}QP_3Y_3,X_1)+g((\bar\nabla_{X_3}\phi)P_3Y_3,X_1)\\
&+g(\bar\nabla_{X_3}QY_3,P_1X_1)-g(\bar\nabla_{X_3}QX_1,\phi Y_3)
\end{aligned}$$

$$
\begin{aligned}
&+ g(\bar{\nabla}_{X_3} QX_1, P_3Y_3) - \eta(X_1)g(X_3, Y_3)\\
=&\cos^2\theta_3 g(\bar{\nabla}_{X_3} Y_3, X_1) - \sin 2\theta_3 X_3(\theta_3)g(Y_3, X_1)\\
&+ \cos^2\theta_3 \eta(X_1)g(X_3, Y_3) - g(\bar{\nabla}_{X_3} QP_3Y_3, X_1)\\
&+ g(\bar{\nabla}_{X_3} QY_3, P_1X_1) + g(\bar{\nabla}_{X_3} bQX_1, Y_3) + g(\bar{\nabla}_{X_3} cQX_1, Y_3)\\
&- g((\bar{\nabla}_{X_3}\phi)QX_1, Y_3) + g(\bar{\nabla}_{X_3} QX_1, P_3Y_3) - \eta(X_1)g(X_3, Y_3).
\end{aligned}
$$

Using (2.5), (2.7), (2.10), orthogonality of the distributions and symmetry of the shape operator, the above equation reduces to

$$
\begin{aligned}
g(\nabla_{X_3} Y_3, X_1) =& \cos^2\theta_3 g(\bar{\nabla}_{X_3} Y_3, X_1) + \cos^2\theta_3 \eta(X_1)g(X_3, Y_3)\\
&+ g(A_{QP_3Y_3} X_1, X_3) - g(A_{QY_3} P_1X_1, X_3)\\
&+ \sin^2\theta_1 g(\bar{\nabla}_{X_1} Y_3, X_1) + g(A_{QP_1X_1} Y_3, X_3)\\
&- g(A_{QX_1} P_3Y_3, X_3) - \cos^2\theta_1 \eta(X_1)g(X_3, Y_3).
\end{aligned}
$$

Following the same computational procedure for any $X_2 \in \Gamma(\mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ we can establish the relation (3.9). And hence, the lemma is proved. $\square$

## 4. Warped Product Submanifolds of Kenmotsu Manifolds

In this section we study warped product submanifolds of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ such that $\xi$ is orthogonal to $M_{\theta_3}$. Here $M_{\theta_1}$, $M_{\theta_2}$ represents proper slant submanifolds of $\bar{M}$ with slant angles $\theta_1$, $\theta_2$, respectively and $M_{\theta_3}$ represents pointwise-slant submanifolds of $\bar{M}$ with slant function $\theta_3$.

Now we construct an example of a non-trivial warped product submanifold $M$ of $\bar{M}$ of the form $M_4 \times_f M_{\theta_3}$.

*Example* 4.1. Consider the Kenmotsu manifold $M = \mathbb{R} \times_f \mathbb{C}^7$ with the structure $(\phi, \xi, \eta, g)$ is given by

$$
\phi\left(\sum_{i=1}^{7}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial t}\right) = \sum_{i=1}^{7}\left(X_i\frac{\partial}{\partial y_i} - Y_i\frac{\partial}{\partial x_i}\right),
$$

$\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and $g = \eta\otimes\eta + \sum_{i=1}^{7}(dx^i\otimes dx^i + dy^i\otimes dy^i)$. Let $M$ be a submanifold of $\bar{M}$ defined by the immersion $\chi$ as follows:

$$
\begin{aligned}
&\chi(u, v, \theta, \phi, r, s, t)\\
=&(u\cos\theta, u\sin\theta, 2u + 3v, 3u + 2v, v\cos\phi, v\sin\phi, 3\theta + 5\phi, 5\theta + 3\phi, v\cos\theta, v\sin\theta,\\
&u\cos\phi, u\sin\phi, 2r + 5s, 5r + 2s, t).
\end{aligned}
$$

Then the local orthonormal frame of $TM$ is spanned by the following:

$$
\begin{aligned}
Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1} + 2\frac{\partial}{\partial x_2} + 3\frac{\partial}{\partial y_2} + \cos\phi\frac{\partial}{\partial x_6} + \sin\phi\frac{\partial}{\partial y_6},\\
Z_2 &= 3\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y_2} + \cos\phi\frac{\partial}{\partial x_3} + \sin\phi\frac{\partial}{\partial y_3} + \cos\theta\frac{\partial}{\partial x_5} + \sin\theta\frac{\partial}{\partial y_5},
\end{aligned}
$$

$$
\begin{aligned}
Z_3 &= -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial y_1} + 3\frac{\partial}{\partial x_4} + 5\frac{\partial}{\partial y_4} - v\sin\theta\frac{\partial}{\partial x_5} + v\cos\theta\frac{\partial}{\partial y_5},\\
Z_4 &= -v\sin\phi\frac{\partial}{\partial x_3} + v\cos\phi\frac{\partial}{\partial y_3} + 5\frac{\partial}{\partial x_4} + 3\frac{\partial}{\partial y_4} - u\sin\phi\frac{\partial}{\partial x_6} + u\cos\phi\frac{\partial}{\partial y_6},\\
Z_5 &= 2\frac{\partial}{\partial x_7} + 5\frac{\partial}{\partial y_7}, \quad Z_6 = 5\frac{\partial}{\partial x_7} + 2\frac{\partial}{\partial y_7} \quad \text{and} \quad Z_7 = \frac{\partial}{\partial t}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\phi Z_1 &= \cos\theta\frac{\partial}{\partial y_1} - \sin\theta\frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial y_2} - 3\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial y_6} - \sin\phi\frac{\partial}{\partial x_6},\\
\phi Z_2 &= 3\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial y_3} - \sin\phi\frac{\partial}{\partial x_3} + \cos\theta\frac{\partial}{\partial y_5} - \sin\theta\frac{\partial}{\partial x_5},\\
\phi Z_3 &= -u\sin\theta\frac{\partial}{\partial y_1} - u\cos\theta\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial y_4} - 5\frac{\partial}{\partial x_4} - v\sin\theta\frac{\partial}{\partial y_5} - v\cos\theta\frac{\partial}{\partial x_5},\\
\phi Z_4 &= -v\sin\phi\frac{\partial}{\partial y_3} - v\cos\phi\frac{\partial}{\partial x_3} + 5\frac{\partial}{\partial y_4} - 3\frac{\partial}{\partial x_4} - u\sin\phi\frac{\partial}{\partial y_6} - u\cos\phi\frac{\partial}{\partial x_6},\\
\phi Z_5 &= 2\frac{\partial}{\partial y_7} - 5\frac{\partial}{\partial x_7} \quad \text{and} \quad \phi Z_6 = 5\frac{\partial}{\partial y_7} - 2\frac{\partial}{\partial x_7}.
\end{aligned}
$$

We take, $\mathcal{D}^{\theta_1} = \mathrm{Span}\{Z_1, Z_2\}$ , $D^{\theta_2} = \mathrm{Span}\{Z_5, Z_6\}$ and $\mathcal{D}^{\theta_3} = \mathrm{Span}\{Z_3, Z_4\}$. Then it is clear that $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are proper slant distributions with slant angles $\cos^{-1}\frac{1}{3}$ and $\cos^{-1}\frac{21}{29}$, respectively. Also, $\mathcal{D}^{\theta_3}$ is a proper pointwise slant distribution with slant function $\cos^{-1}(\frac{16}{u^2+v^2+34})$.

Clearly, $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are integrable distributions. Let us say that $M_4$ and $M_{\theta_3}$ are integral submanifolds of $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle$ and $\mathcal{D}^{\theta_3}$, respectively. Then the metric tensor $g_M$ of $M$ is given by

$$
\begin{aligned}
g_M &= 15(du^2 + dv^2) + 29(dr^2 + ds^2) + (u^2 + v^2 + 34)(d\theta^2 + d\phi^2)\\
&= g_{M_4} + (u^2 + v^2 + 34)g_{M_{\theta_3}}.
\end{aligned}
$$

Thus $M = M_4 \times_f M_{\theta_3}$ is a warped product submanifold of $\bar{M}$ with the warping function $f = \sqrt{u^2 + v^2 + 34}$.

Next we obtain the following useful lemmas.

**Lemma 4.1.** *Let $M = M_4 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$ such that $\xi \in M_4$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, $M_{\theta_1}, M_{\theta_2}$ are proper slant submanifolds and $M_{\theta_3}$ is a proper pointwise slant submanifold of $\bar{M}$, then*

$$\xi \ln f = 1, \tag{4.1}$$

$$g(h(X_1, Y_1), QX_3) = g(h(X_1, X_3), QY_1), \tag{4.2}$$

$$g(h(X_2, Y_2), QX_3) = g(h(X_2, X_3), QY_2), \tag{4.3}$$

$$g(h(X_1, X_3), QX_2) = g(h(X_1, X_2), QX_3) = g(h(X_2, X_3), QX_1), \tag{4.4}$$

*for $X_1, Y_1 \in M_{\theta_1}$, $X_2, Y_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.*

*Proof.* The proof of (4.1) is similar as in [28].

Now, for $X_1, Y_1 \in M_{\theta_1}$ and $X_3 \in M_{\theta_3}$, we have from (2.5) and (3.3) that

$$g(h(X_1,X_3),QY_1) = -\,g(\bar{\nabla}_{X1}P_3X_3,Y_1) - g(\bar{\nabla}_{X1}QX_3,Y_1) - g(\bar{\nabla}_{X_1}X_3,P_1Y_1). \tag{4.5}$$

Then using (1.1) in (4.5), we get (4.2).

Proceeding the same, for any $X_2, Y_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$, we get (4.2).

Again, for any $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$ we have from (2.5) and (3.3) that

$$g(h(X_1,X_3),QX_2) = -\,g(\bar{\nabla}_{X_3}P_1X_1,X_2) - g(\bar{\nabla}_{X_3}QX_1,X_2) - g(\bar{\nabla}_{X_3}X_1,P_2X_2). \tag{4.6}$$

Using (1.1) in (4.6), we find

$$g(h(X_1,X_3),QX_2) = g(h(X_2,X_3),QX_1). \tag{4.7}$$

Also,

$$g(h(X_1,X_2),QX_3) = -\,g(\bar{\nabla}_{X_1}P_2X_2,X_3) - g(\bar{\nabla}_{X_1}P_2X_2,X_3) - g(\bar{\nabla}_{X_1}X_2,P_3X_3). \tag{4.8}$$

Using (1.1) in (4.8), we get

$$g(h(X_1,X_2),QX_3) = g(h(X_1,X_3),QX_2). \tag{4.9}$$

Combining (4.7) and (4.9), we obtain (4.4). This completes the proof. ☐

**Lemma 4.2.** *Let $M = M_4 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$ such that $\xi \in M_4$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, $M_{\theta_1}$, $M_{\theta_2}$ are proper slant submanifolds and $M_{\theta_3}$ is a proper pointwise slant submanifold of $\bar{M}$, then*

$$\begin{aligned} &g(h(X_3,X_1),QY_3) - g(h(X_3,Y_3),QX_1) \\ &=\{(X_1 \ln f) - \eta(X_1)\}g(P_3X_3,Y_3) - (P_1X_1 \ln f)g(X_3,Y_3), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &g(h(X_3,X_2),QY_3) - g(h(X_3,Y_3),QX_2) \\ &=\{(X_2 \ln f) - \eta(X_2)\}g(P_3X_3,Y_3) - (P_2X_2 \ln f)g(X_3,Y_3), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &g(h(X_3,Y_3),QP_1X_1) - g(h(P_3Y_3,X_3),QX_1) \\ &\quad + g(h(X_1,X_3),QP_3Y_3) - g(h(P_1X_1,X_3),QY_3) \\ &=(\cos^2\theta_1 - \cos^2\theta_3)[\eta(X_1) - (X_1 \ln f)]g(X_3,Y_3), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &g(h(X_3,Y_3),QP_2X_2) - g(h(P_3Y_3,X_3),QX_2) \\ &\quad + g(h(X_2,X_3),QP_3Y_3) - g(h(P_2X_2,X_3),QY_3) \\ &=(\cos^2\theta_2 - \cos^2\theta_3)[\eta(X_2) - (X_2 \ln f)]g(X_3,Y_3), \end{aligned} \tag{4.13}$$

*for $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.*

*Proof.* From (2.5) and (3.3), we have for $X_1 \in M_{\theta_1}$ and $X_3, Y_3 \in M_{\theta_3}$ that

$$\begin{aligned} g(h(X_3,Y_3),QX_1) = &-\,g(\bar{\nabla}_{X_3}X_1,P_3Y_3) - g(\bar{\nabla}_{X_3}QY_3,X_1) \\ &+ \eta(X_1)g(\phi X_3,Y_3) + g(\bar{\nabla}_{X_3}P_1X_1,Y_3). \end{aligned} \tag{4.14}$$

Using (2.7) and (1.1) in (4.14), we get (4.10). Following the same procedure, for any $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$ we easily obtain (4.11).

Next, replacing $X_1$ by $P_1X_1$ and $Y_3$ by $P_3Y_3$ in (4.10), respectively and then adding the obtained equations, we get (4.12). Similarly, replacing $X_2$ by $P_2X_2$ and $Y_3$ by $P_3Y_3$ in (4.11), respectively and then adding the obtained equations, we get (4.13). □

## 5. Characterization

We prove the following theorem.

**Theorem 5.1.** *Let $M$ be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with $\xi$ orthogonal to $\mathcal{D}^{\theta_3}$, then $M$ is locally a warped product submanifold of the form $M = M_4 \times_f M_{\theta_3}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ if and only if*

$$\begin{aligned} &A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 \\ &=(\cos^2\theta_3 - \cos^2\theta_1)[X_1\mu - \eta(X_1)]Y_3, \end{aligned} \tag{5.1}$$

$$\begin{aligned} &A_{QP_2X_2}Y_3 - A_{QX_2}P_3Y_3 + A_{QP_3Y_3}X_2 - A_{QY_3}P_2X_2 \\ &=(\cos^2\theta_3 - \cos^2\theta_2)[X_2\mu - \eta(X_2)]Y_3, \end{aligned} \tag{5.2}$$

$$\xi\mu = 1, \tag{5.3}$$

*for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and for some smooth function $\mu$ on $M$ satisfying where $(Y_3\mu) = 0$ for any $Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.*

*Proof.* Let $M = M_4 \times_f M_{\theta_3}$ be a proper warped product submanifold of $\bar{M}$ such that $M_4 = M_{\theta_1} \times M_{\theta_2}$. Denote the tangent space of $M_{\theta_1}$, $M_{\theta_2}$ and $M_{\theta_3}$ by $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ respectively. Then from (4.2) we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_1) = 0. \tag{5.4}$$

Similarly, from (4.4) we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_2) = 0. \tag{5.5}$$

So, from (5.4) and (5.5) we conclude that

$$A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 \in \mathcal{D}^{\theta_3}. \tag{5.6}$$

Hence, from (4.12) and (5.6), relation (5.1) follows.

In similar way, in view of (4.3), (4.4) and (4.13) we get (5.2). The relation (5.3) is directly obtained from (4.1).

Conversely, let $M$ be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with $\xi$ orthogonal to $\mathcal{D}^{\theta_3}$ and the conditions (5.1)–(5.3) satisfied. Then from (3.4) and (3.7), in view of (5.1), respectively we get

$$g(\nabla_{X_1}Y_1, X_3) = 0 \quad \text{and} \quad g(\nabla_{X_2}X_1, X_3) = 0, \tag{5.7}$$

and also from (3.5), (3.6) in view of (5.2), respectively we get

$$g(\nabla_{X_2}Y_2, X_3) = 0 \quad \text{and} \quad g(\nabla_{X_1}X_2, X_3) = 0. \tag{5.8}$$

Thus, from (5.7), (5.8) and the fact that $\nabla_{X_3}\xi = 0$ we conclude that $g(\nabla_E F, X_3) = 0$ for every $E, F \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$. Hence the leaves of $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle$ are totally geodesic in $M$.

Now, by virtue of (3.8), (5.1) yields

$$g([X_3, Y_3], X_1) = 0, \tag{5.9}$$

and by virtue of (3.9), (5.2) yields

$$g([X_3, Y_3], X_2) = 0. \tag{5.10}$$

Hence, from (5.9), (5.10) and the fact that $h(A, \xi) = 0$, for all $A \in TM$, we conclude that

$$g([X_3, Y_3], E) = 0, \quad \text{for all } X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3}),$$

and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$, consequently $\mathcal{D}^{\theta_3}$ is integrable.

Let $h^{\theta_3}$ be the second fundamental form of $M_{\theta_3}$ in $\bar{M}$. Then for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, from (3.8), we find

$$g(h^{\theta_3}(X_3, Y_3), X_1) = -(X_1\mu)g(X_3, Y_3). \tag{5.11}$$

Similarly, for $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, from (3.9) we get

$$g(h^{\theta_3}(X_3, Y_3), X_2) = -(X_2\mu)g(X_3, Y_3). \tag{5.12}$$

Again, for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$, in view of (5.3) we have

$$g(h^{\theta_3}(X_3, Y_3), \xi) = -(\xi\mu)g(X_3, Y_3). \tag{5.13}$$

Hence, from (5.11)–(5.13) we conclude that

$$g(h^{\theta}(X_3, Y_3), E) = -g(\boldsymbol{\nabla}\mu, E)g(X_3, Y_3),$$

for every $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus, \langle\xi\rangle)$. Consequently, $M_{\theta_3}$ is totally umbilical in $\bar{M}$ with mean curvature vector $H^{\theta_3} = -\boldsymbol{\nabla}\mu$.

Finally, we will show that $H^{\theta_3}$ is parallel with respect to the normal connection $\nabla^{\perp}$ of $M_{\theta_3}$ in $M$. We take $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$, then we have

$$g(\nabla^{\perp}_{X_3}\boldsymbol{\nabla}\mu, E) = g(\nabla_{X_3}\boldsymbol{\nabla}^{\theta_1}\mu, X_1) + g(\nabla_{X_3}\boldsymbol{\nabla}^{\theta_2}\mu, X_2) + g(\nabla_{X_3}\boldsymbol{\nabla}^{\xi}\mu, \xi),$$

where $\boldsymbol{\nabla}^{\theta_1}$, $\boldsymbol{\nabla}^{\theta_2}$ and $\boldsymbol{\nabla}^{\xi}$ are the gradient components of $\mu$ on $M$ along $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ and $\langle\xi\rangle$ respectively. Then by the property of Riemannian metric, the above equation reduces to

$$\begin{aligned}
g(\nabla^{\perp}_{U}\boldsymbol{\nabla}\mu, E) =& X_3 g(\boldsymbol{\nabla}^{\theta_1}\mu, X_1) - g(\boldsymbol{\nabla}^{\theta_1}\mu, \nabla_{X_3}X_1) + X_3 g(\boldsymbol{\nabla}^{\theta_2}\mu, X_2) \\
& - g(\boldsymbol{\nabla}^{\theta_2}\mu, \nabla_{X_3}X_2) + X_3 g(\boldsymbol{\nabla}^{\xi}\mu, \xi) - g(\boldsymbol{\nabla}^{\xi}\mu, \nabla_{X_3}\xi) \\
=& X_3(X_1\mu) - g(\boldsymbol{\nabla}^{\theta_1}\mu, [X_3, X_1]) - g(\boldsymbol{\nabla}^{\theta_1}\mu, \nabla_{X_1}X_3) \\
& + X_3(X_2\mu) - g(\boldsymbol{\nabla}^{\theta_2}\mu, [X_3, X_2]) - g(\boldsymbol{\nabla}^{\theta_2}\mu, \nabla_{X_2}X_3) \\
& + X_3(\xi\mu) - g(\boldsymbol{\nabla}^{\xi}\mu, [X_3, \xi]) - g(\boldsymbol{\nabla}^{\xi}\mu, \nabla_{\xi}X_3) \\
=& X_1(X_3\mu) + g(\nabla_{X_1}\boldsymbol{\nabla}^{\theta_1}\mu, X_3) + X_2(X_3\mu)
\end{aligned}$$

$$\begin{aligned} &+ g(\nabla_{X_2}\boldsymbol{\nabla}^{\theta_2}\mu, X_3) + \xi(X_3\mu) - g(\nabla_\xi \boldsymbol{\nabla}^\xi \mu, X_3) \\ =&0, \end{aligned}$$

since $(X_3\mu) = 0$ for every $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $\nabla_{X_1}\boldsymbol{\nabla}^{\theta_1}\mu + \nabla_{X_2}\boldsymbol{\nabla}^{\theta_2}\mu + \nabla_\xi\boldsymbol{\nabla}^\xi\mu = \nabla_E\boldsymbol{\nabla}\mu$ is orthogonal to $\mathcal{D}^{\theta_3}$ for any $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ and $\boldsymbol{\nabla}\mu$ is the gradient along $M_4$ and $M_4$ is totally geodesic in $\bar{M}$. Hence, the mean curvature vector $H^{\theta_3}$ of $M_{\theta_3}$ is parallel. Thus, $M_{\theta_3}$ is an extrinsic sphere in $M$. Hence, by Hiepko's Theorem (see [14]), $M$ is locally a warped product submanifold. Thus, the proof is complete. $\square$

## 6. Bi-Warped Product Submanifolds

In this section we have studied bi-warped product submanifolds $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ of $\bar{M}$, where $M_{\theta_1}$, $M_{\theta_2}$, $M_{\theta_3}$ are pointwise slant submanifolds of $\bar{M}$ and an supporting example has been constructed. We denote $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$, $\mathcal{D}^{\theta_3}$ as the tangent spaces of $M_{\theta_1}$, $M_{\theta_2}$, $M_{\theta_3}$, respectively.

Then we write

$$TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle$$

and

$$T^\perp M = Q\mathcal{D}^{\theta_1} \oplus Q\mathcal{D}^{\theta_2} \oplus Q\mathcal{D}^{\theta_3}.$$

*Example* 6.1. Consider the Kenmotsu manifold $\bar{M} = \mathbb{R} \times_f \mathbb{C}^{10}$ with the structure $(\phi, \xi, \eta, g)$ is given by

$$\phi\bigg(\sum_{i=1}^{10}\bigg(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\bigg) + Z\frac{\partial}{\partial t}\bigg) = \sum_{i=1}^{10}\bigg(X_i\frac{\partial}{\partial y_i} - Y_i\frac{\partial}{\partial x_i}\bigg),$$

$\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and $g = \eta\otimes\eta + \sum_{i=1}^{10}(dx^i \otimes dx^i + dy^i \otimes dy^i)$. Let $M$ be a submanifold of $\bar{M}$ defined by the immersion $\chi$ as follows:

$$\begin{aligned} &\chi(u, v, \theta, \phi, r, s, t) \\ =&(u\cos\theta, u\sin\theta, v\cos\phi, v\sin\phi, 3\theta + 5\phi, 5\theta + 3\phi, v\cos\theta, v\sin\theta, u\cos\phi, u\sin\phi, u\cos r, \\ &v\cos s, u\sin r, v\sin s, 3r + 2s, 2r + 3s, u\cos s, v\cos r, u\sin s, v\sin r, t). \end{aligned}$$

Then the local orthonormal frame of $TM$ is spanned by the following:

$$\begin{aligned} Z_1 =& \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1} + \cos\phi\frac{\partial}{\partial x_5} + \sin\phi\frac{\partial}{\partial y_5} \\ &+ \cos r\frac{\partial}{\partial x_6} + \sin r\frac{\partial}{\partial x_7} + \cos s\frac{\partial}{\partial x_9} + \sin s\frac{\partial}{\partial x_{10}}, \\ Z_2 =& \cos\phi\frac{\partial}{\partial x_2} + \sin\phi\frac{\partial}{\partial y_2} + \cos\theta\frac{\partial}{\partial x_4} + \sin\theta\frac{\partial}{\partial y_4} \\ &+ \cos s\frac{\partial}{\partial y_6} + \sin s\frac{\partial}{\partial y_7} + \cos r\frac{\partial}{\partial y_9} + \sin r\frac{\partial}{\partial y_{10}}, \\ Z_3 =& -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial y_1} + 3\frac{\partial}{\partial x_3} + 5\frac{\partial}{\partial y_3} - v\sin\theta\frac{\partial}{\partial x_4} + v\cos\theta\frac{\partial}{\partial y_4}, \end{aligned}$$

$$\begin{aligned}
Z_4 =& -v\sin\phi\frac{\partial}{\partial x_2}+v\cos\phi\frac{\partial}{\partial y_2}+5\frac{\partial}{\partial x_3}+3\frac{\partial}{\partial y_3}-u\sin\phi\frac{\partial}{\partial x_5}+u\cos\phi\frac{\partial}{\partial y_5},\\
Z_5 =& -u\sin r\frac{\partial}{\partial x_6}+u\cos r\frac{\partial}{\partial x_7}+3\frac{\partial}{\partial x_8}+2\frac{\partial}{\partial y_8}-v\sin r\frac{\partial}{\partial y_9}+v\cos r\frac{\partial}{\partial y_{10}},\\
Z_6 =& V-Xv\sin s\frac{\partial}{\partial y_6}+v\cos s\frac{\partial}{\partial y_7}+2\frac{\partial}{\partial x_8}+3\frac{\partial}{\partial y_8}-u\sin s\frac{\partial}{\partial x_9}+u\cos s\frac{\partial}{\partial x_{10}}
\end{aligned}$$

and

$$Z_7=\frac{\partial}{\partial t}.$$

Then

$$\begin{aligned}
\phi Z_1 =& \cos\theta\frac{\partial}{\partial y_1}-\sin\theta\frac{\partial}{\partial x_1}+\cos\phi\frac{\partial}{\partial y_5}-\sin\phi\frac{\partial}{\partial x_5}\\
&+\cos r\frac{\partial}{\partial y_6}+\sin r\frac{\partial}{\partial y_7}+\cos s\frac{\partial}{\partial y_9}+\sin s\frac{\partial}{\partial y_{10}},\\
\phi Z_2 =& \cos\phi\frac{\partial}{\partial y_2}-\sin\phi\frac{\partial}{\partial x_2}+\cos\theta\frac{\partial}{\partial y_4}-\sin\theta\frac{\partial}{\partial x_4}\\
&-\cos s\frac{\partial}{\partial x_6}-\sin s\frac{\partial}{\partial x_7}-\cos r\frac{\partial}{\partial x_9}-\sin r\frac{\partial}{\partial x_{10}},\\
\phi Z_3 =& -u\sin\theta\frac{\partial}{\partial y_1}-u\cos\theta\frac{\partial}{\partial x_1}+3\frac{\partial}{\partial y_3}-5\frac{\partial}{\partial x_3}-v\sin\theta\frac{\partial}{\partial y_4}-v\cos\theta\frac{\partial}{\partial x_4},\\
\phi Z_4 =& -v\sin\phi\frac{\partial}{\partial y_2}-v\cos\phi\frac{\partial}{\partial x_2}+5\frac{\partial}{\partial y_3}-3\frac{\partial}{\partial x_3}-u\sin\phi\frac{\partial}{\partial y_5}-u\cos\phi\frac{\partial}{\partial x_5},\\
\phi Z_5 =& -u\sin r\frac{\partial}{\partial y_6}+u\cos r\frac{\partial}{\partial y_7}+3\frac{\partial}{\partial y_8}-2\frac{\partial}{\partial x_8}+v\sin r\frac{\partial}{\partial x_9}-v\cos r\frac{\partial}{\partial x_{10}},\\
\phi Z_6 =& v\sin s\frac{\partial}{\partial x_6}-v\cos s\frac{\partial}{\partial x_7}+2\frac{\partial}{\partial y_8}-3\frac{\partial}{\partial x_8}-u\sin s\frac{\partial}{\partial y_9}+u\cos s\frac{\partial}{\partial y_{10}}.
\end{aligned}$$

We take $\mathcal{D}^{\theta_1}=\mathrm{Span}\{Z_1,Z_2\}$, $\mathcal{D}^{\theta_2}=\mathrm{Span}\{Z_3,Z_4\}$ and $\mathcal{D}^{\theta_3}=\mathrm{Span}\{Z_5,Z_6\}$. Then it is clear that $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are proper pointwise slant distributions with slant functions $\cos^{-1}\{\frac{1}{2}\cos(r-s)\}$, $\cos^{-1}(\frac{16}{u^2+v^2+34})$ and $\cos^{-1}(\frac{5}{u^2+v^2+13})$, respectively. Clearly, $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are integrable distributions. Let us say that $M_{\theta_1}$, $M_{\theta_2}$ and $M_{\theta_3}$ are integral submanifolds of $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Then the metric tensor $g_M$ of $M$ is given by

$$\begin{aligned}
g_M &= 4(du^2+dv^2)+(u^2+v^2+34)(d\theta^2+d\phi^2)+(u^2+v^2+13)(dr^2+ds^2)\\
&= g_{M_{\theta_1}}+(u^2+v^2+34)g_{M_{\theta_2}}+(u^2+v^2+13)g_{M_{\theta_3}}.
\end{aligned}$$

Thus, $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ is a bi-warped product submanifold of $\bar{M}$ with the warping functions $f_1=\sqrt{u^2+v^2+34}$ and $f_2=\sqrt{u^2+v^2+13}$.

**Proposition 6.1** ([33]). *Let $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$. Then $M$ is a single warped product if $\xi$ is orthogonal to $\mathcal{D}^{\theta_1}$.*

**Proposition 6.2** ([33]). *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ such that $M$ is tangent to $M_{\theta_1}$. Then*

$$\xi(\ln f_i) = 1, \quad \textit{for all } i = 1, 2. \tag{6.1}$$

**Lemma 6.1.** *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$. Then*

$$g(h(X_1, Y_1), QX_3) = g(h(X_1, X_3), QY_1), \tag{6.2}$$

$$g(h(X_2, Y_2), QX_3) = g(h(X_1, X_3), QY_2), \tag{6.3}$$

$$g(h(X_1, X_2), QX_3) = g(h(X_1, X_3), QX_2), \tag{6.4}$$

*for every $X_1, Y_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$.*

*Proof.* Proof is similar to the proof of Lemma 4.1. □

**Lemma 6.2.** *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$. Then*

$$\begin{aligned} &g(h(X_2, Y_2), QX_1) - g(h(X_1, X_2), QY_2) \\ =&(P_1X_1 \ln f_1)g(X_2, Y_2) + [X_1(\ln f_1) - \eta(X_1)]g(X_2, P_2Y_2), \end{aligned} \tag{6.5}$$

$$\begin{aligned} &g(h(X_3, Y_3), QX_1) - g(h(X_1, X_3), QY_3) \\ =&(P_1X_1 \ln f_2)g(X_3, Y_3) + [X_1(\ln f_2) - \eta(X_1)]g(X_3, P_3Y_3), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &g(h(X_3, Y_3), QX_2) - g(h(X_2, X_3), QY_3) \\ =&(P_2X_2 \ln f_2)g(X_3, Y_3) + X_2(\ln f_2)g(X_3, P_3Y_3), \end{aligned} \tag{6.7}$$

*for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.*

*Proof.* Proof is similar to the proof of Lemma 4.2. □

**Lemma 6.3.** *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$. Then*

$$\begin{aligned} &g(h(X_1, Y_2), QP_2X_2) - g(h(X_1, P_2X_2), QY_2) \\ =&2\cos^2\theta_2\{(X_1 \ln f_1) - \eta(X_1)\}g(X_2, Y_2), \end{aligned} \tag{6.8}$$

$$\begin{aligned} &g(h(X_1, X_3), QP_3Y_3) - g(h(X_1, P_3X_3), QY_3) \\ =&2\cos^2\theta_3\{(X_1 \ln f_2) - \eta(X_1)\}g(X_3, Y_3), \end{aligned} \tag{6.9}$$

$$\begin{aligned} &g(h(X_2, X_3), QP_3Y_3) - g(h(X_2, P_3X_3), QY_3) \\ =&2\cos^2\theta_3(X_2 \ln f_2)g(X_3, Y_3), \end{aligned} \tag{6.10}$$

*for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.*

*Proof.* By polarization of (6.5), we get

$$\begin{aligned} g(h(X_2, Y_2), QX_1) - g(h(X_1, Y_2), QZ) =&(P_1X_1 \ln f_1)g(X_2, Y_2) \\ &+ [X_1(\ln f_1) - \eta(X_1)]g(X_2, Y_2). \end{aligned} \tag{6.11}$$

Subtracting (6.11) from (6.4), we find

$$g(h(X_1, Y_2), QX_2) - g(h(X_1, X_2), QY_2) = 2[X_1(\ln f_1) - \eta(X_1)]g(X_2, P_2Y_2). \tag{6.12}$$

Replacing $X_2$ by $P_2X_2$ in (6.12), we get (6.8). Similarly, (6.9) follows from (6.6) and (6.10) follows from (6.7). □

**Theorem 6.1.** *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$. Then $M$ can be $\mathcal{D}^{\theta_1} - \mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1} - \mathcal{D}^{\theta_3}$ mixed totally geodesic but cannot be $\mathcal{D}^{\theta_2} - \mathcal{D}^{\theta_3}$ mixed totally geodesic.*

*Proof.* The theorem follows from Lemma 6.3. □

## 7. Inequality

In this section, we establish a Chen-type inequality on a bi-warped product submanifold $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ of $\bar{M}$ of dimension $n$ such that $\xi$ is tangent to $M_{\theta_1}$. We take $\dim M_{\theta_1} = 2p+1$, $\dim M_{\theta_2} = 2q$, $\dim M_{\theta_3} = 2s$ and their corresponding tangent spaces are $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Assume that $\{e_1, e_2, \ldots, e_p, e_{p+1} = \sec\theta_1 P_1 e_1, \ldots, e_{2p} = \sec\theta_1 P_1 e_p, e_{2p+1} = \xi\}$, $\{e_{2p+2} = e_1^*, \ldots, e_{2p+q+1} = e_q^*, e_{2p+q+2} = e_{q+1}^* = \sec\theta_2 P_2 e_1^*, \ldots, e_{2p+2q+1} = e_{2q}^* = \sec\theta_2 P_2 e_q^*\}$ and $\{e_{2p+2q+2} = \hat{e}_1, \ldots, e_{2p+2q+s+1} = \hat{e}_s, e_{2p+2q+s+2} = \hat{e}_{s+1} = \sec\theta_3 P_3 \hat{e}_1, \ldots, e_{2p+2q+2s+1} = \hat{e}_{2s} = \sec\theta_3 P_3 \hat{e}_s\}$ are local orthonormal frames of $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Then the local orthonormal frames for $Q\mathcal{D}^{\theta_1}$, $Q\mathcal{D}^{\theta_2}$, $Q\mathcal{D}^{\theta_3}$ and $\nu$ are $\{\tilde{e}_1 = \csc\theta_1 Q e_1, \ldots, \tilde{e}_p = \csc\theta_1 Q e_p, \tilde{e}_{p+1} = \csc\theta_1 \sec\theta_1 Q P_1 e_1, \ldots, \tilde{e}_{2p} \csc\theta_1 \sec\theta_1 Q P_1 e_p\}$, $\{\tilde{e}_{2p+1} = \tilde{e}_1^* = \csc\theta_2 Q e_1^*, \ldots, \tilde{e}_{2p+q} = \tilde{e}_q^* = \csc\theta_2 Q e_q^*, \tilde{e}_{2p+q+1} = \tilde{e}_{q+1}^* = \csc\theta_2 \sec\theta_2 Q P_2 e_1^*, \ldots, \tilde{e}_{2p+2q} = \tilde{e}_{2q}^* = \csc\theta_2 \sec\theta_2 Q P_2 e_q^*\}$, $\{\tilde{e}_{2p+2q+1} = \tilde{\hat{e}}_1 = \csc\theta_3 Q \hat{e}_1, \ldots, \tilde{e}_{2p+2q+s} = \tilde{\hat{e}}_s = \csc\theta_3 Q \hat{e}_s, \tilde{e}_{2p+2q+s+1} = \tilde{\hat{e}}_{s+1} = \csc\theta_3 \sec\theta_3 Q P_3 \hat{e}_1, \ldots, \tilde{e}_{2p+2q+2s} = \tilde{\hat{e}}_{2s} = \csc\theta_3 \sec\theta_3 Q P_3 \hat{e}_s\}$ and $\{\tilde{e}_{2p+2q+2s+1}, \ldots, \tilde{e}_{2m+1}\}$ of dimensions $2p$, $2q$, $2s$ and $(2m+1-n-2p-2q-2s)$, respectively.

**Theorem 7.1.** *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be both $\mathcal{D}^{\theta_1} - \mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1} - \mathcal{D}^{\theta_3}$ mixed totally geodesic bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$. Then the squared norm of the second fundamental form satisfies*

$$\begin{aligned} \|h\|^2 \geq & 2q\csc^2\theta_1(\cos^2\theta_1 + \cos^2\theta_2)(\|\boldsymbol{\nabla}\ln f_1\|^2 - 1) \\ & + 2s\csc^2\theta_1(\cos^2\theta_1 + \cos^2\theta_3)(\|\boldsymbol{\nabla}\ln f_2\|^2 - 1), \end{aligned} \tag{7.1}$$

*where $2q = \dim M_{\theta_1}$, $2s = \dim M_{\theta_3}$, $\boldsymbol{\nabla}\ln f_1$ and $\boldsymbol{\nabla}\ln f_2$ are the gradients of warping function $\ln f_1$ and $\ln f_2$ along $M_{\theta_1}$ and $M_{\theta_2}$, respectively.*

*If the equality sign of (7.1) holds, then $M_{\theta_1}$ is totally geodesic and $M_{\theta_2}$, $M_{\theta_3}$ are totally umbilical submanifolds of $\bar{M}$.*

*Proof.* From the definition of $h$, we have

$$\|h\|^2 = \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), h(e_i, e_j)). \tag{7.2}$$

Now by decomposing (7.2) in our constructed frame fields, we get

$$
\begin{aligned}
\|h\|^2 =& \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\
&+ 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\
&+ 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2.
\end{aligned}\tag{7.3}
$$

Neglecting the $\nu$ component terms of (7.3), we obtain

$$
\begin{aligned}
|h\|^2 \geq& \sum_{r=1}^{2p}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e_r})^2 + \sum_{r=1}^{2q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e_r})^2\\
&+ \sum_{r=1}^{2s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + 2\sum_{i,r=1}^{2p}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\
&+ 2\sum_{r,j=1}^{2q}\sum_{i=1}^{2p} g(h(e_i,e_j^*),\tilde{e}_r)^2 + 2\sum_{r=1}^{2s}\sum_{i=1}^{2p}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\
&+ 2\sum_{i,r=1}^{2p}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + 2\sum_{r=1}^{2q}\sum_{i=1}^{2p}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2\\
&+ 2\sum_{r,j=1}^{2s}\sum_{i=1}^{2p} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{2p}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\
&+ \sum_{i,j,r=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2 + \sum_{r=1}^{2s}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\
&+ 2\sum_{r=1}^{2p}\sum_{i=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + 2\sum_{i,r=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2\\
&+ 2\sum_{j,r=1}^{2s}\sum_{i=1}^{2q} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{2p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2\\
&+ \sum_{r=1}^{2q}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{i,j,r=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2.
\end{aligned}\tag{7.4}
$$

In view of Lemma (6.1), the second, third and thirteenth terms are equal to zero. Using the $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic condition, seventh to thirteenth terms are also equal to zero. Also we can not find any relation for $g(h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1}),Q\mathcal{D}^{\theta_1})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_1}),Q\mathcal{D}^{\theta_2})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2}),Q\mathcal{D}^{\theta_3})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3}),Q\mathcal{D}^{\theta_3})$, $g(h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3}),Q\mathcal{D}^{\theta_2})$ and $g(h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3}),Q\mathcal{D}^{\theta_3})$, so we neglect first, eleventh, twelfth, fourteenth, fifteenth, seventeenth and eighteenth terms of (7.4) and obtain

$$\begin{aligned}\|h\|^2 \geq& \csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g(h(e_i^*, e_j^*), Qe_r)^2 + \csc^2\theta_1 \sec^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g(h(e_i^*, P_1e_j^*), QP_1e_r)^2\\ &+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i, \hat{e}_j), Qe_r)^2 + \csc^2\theta_1 \sec^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i, P_1\hat{e}_j), QP_1e_r)^2.\end{aligned}$$

By virtue of Lemma 6.2, the above relation yields

$$\begin{aligned}\|h\|^2 \geq& \csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} (P_1e_r \ln f_1)^2 g(e_i^*, e_j^*)^2\\ &+ csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} [(e_r \ln f_1) - \eta(e_r)]^2 g(e_i^*, P_2e_j^*)^2\\ &+ \csc^2\theta_1 \cos^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} (e_r \ln f_1)^2 g(e_i^*, e_j^*)^2\\ &+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2q} (P_1e_r \ln f_1)^2 g(e_i^*, P_2e_j^*)^2\\ &+ \csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} (P_1e_r \ln f_2)^2 g(\hat{e}_i, \hat{e}_j)^2\\ &+ csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} [(e_r \ln f_2) - \eta(e_r)]^2 g(\hat{e}_i, P_3\hat{e}_j)^2\\ &+ \csc^2\theta_1 \cos^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} (e_r \ln f_2)^2 g(\hat{e}_i, \hat{e}_j)^2\\ &+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2s} (P_1e_r \ln f_2)^2 g(\hat{e}_i, P_3\hat{e}_j)^2\\ =&2q\csc^2\theta_1(1 + \sec^2\theta_1\cos^2\theta_2)\sum_{r=1}^{p}(P_1e_r \ln f_1)^2\\ &+ 2qcsc^2\theta_1(\cos^2\theta_1 + \cos^2\theta_2)\sum_{r=1}^{p}[(e_r \ln f_1) - \eta(e_r)]^2\\ &+ 2q\csc^2\theta_1(1 + \sec^2\theta_1\cos^2\theta_3)\sum_{r=1}^{p}(P_1e_r \ln f_2)^2\\ &+ 2qcsc^2\theta_1(\cos^2\theta_1 + \cos^2\theta_3)\sum_{r=1}^{p}[(e_r \ln f_2) - \eta(e_r)]^2.\end{aligned}$$

Thus, we find

$$\|h\|^2 \geq 2q\csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_2)\left(\sum_{r=1}^{2p+1}(P_1e_r\ln f_1)^2-(\xi\ln f_1)^2\right)$$
$$+2s\csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_3)\left(\sum_{r=1}^{2p+1}(P_1e_r\ln f_2)^2-(\xi\ln f_2)^2\right). \tag{7.5}$$

Using (2.8) and Proposition 6.2, in (7.5), we get the inequality (7.1). If equality of (7.1) holds, for omitting $\nu$ components terms of (6.3), we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp\nu,\quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp\nu,\quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp\nu,\quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp\nu.$$

Also, for neglecting terms of (7.4), we obtain $h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_1}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp Q\mathcal{D}^{\theta_3}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_3}$, $h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_3}$. Next, since $M$ is both $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic, we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})=0,\quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_3})=0. \tag{7.6}$$

Also, from Lemma 6.1 with (6.6), we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_2},\quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_3},\quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_2}.$$

Thus, we can say that

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})=0, \tag{7.7}$$
$$h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\subset Q\mathcal{D}^{\theta_1}, \tag{7.8}$$
$$h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\subset Q\mathcal{D}^{\theta_1}, \tag{7.9}$$
$$h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\subset Q\mathcal{D}^{\theta_1}. \tag{7.10}$$

From (7.6) and (7.7), $M_{\theta_1}$ is totally geodesic in $M$ and hence in $\bar{M}$ [5, 7]. Again, since $M_{\theta_2}$ and $M_{\theta_3}$ are totally umbilical in $M$ [5, 7], with the fact (7.8)–(7.10), we conclude that $M_{\theta_2}$ and $M_{\theta_3}$ are totally umbilical in $\bar{M}$. Hence, the theorem is proved completely. □

## 8. Some Applications

As consequences of Theorem 5.1 we have the following.

1. If we take $\dim M_{\theta_2}=0$ and replace $\theta_3$ by $\theta_2$, then $M$ changes to a warped product pointwise bi-slant submanifold of the form $M_{\theta_1}\times_f M_{\theta_2}$, studied in [17]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [17]).

Let $M$ be a proper pointwise bi-slant submanifold of $\bar{M}$ such that $\xi\in\Gamma(\mathcal{D}^{\theta_1})$, then $M$ is locally a warped product submanifold of the form $M_{\theta_1}\times_f M_{\theta_2}$ if and only if

$$\begin{aligned}&A_{QP_1X_1}Y_2-A_{QX_1}P_2Y_2+A_{QP_2Y_2}X_1-A_{QY_2}P_1X_1\\=&(\cos^2\theta_2-\cos^2\theta_1)[(X_1\mu)-\eta(X_1)]Y_2,\end{aligned}$$

for any $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, for some smooth function $\mu$ on $M$ satisfying $(Y\mu) = 0$, for any $W \in \Gamma(\mathcal{D}^{\theta_2})$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [17].

2. If we take $\theta_1 = 0$, $\theta_2 = constant = \theta$, $\theta_3 = \frac{\pi}{2}$, then $M$ changes to a warped product skew CR-submanifold of the form $M_1 \times_f M_\perp$, where $M_1 = M_T \times M_\theta$, studied in [28]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.3 of [28]).

Let $M$ be a proper skew CR-submanifold of $\bar{M}$, then $M$ is locally a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product submanifold of the form $M_1 \times_f M_\perp$, where $M_1 = M_T \times M_\theta$ if and only if

(i) $A_{\phi Z}X \in \Gamma(\mathcal{D}^\perp)$ for any $X \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^\theta) \oplus \{\xi\}$ and $Z \in \Gamma(\mathcal{D}^\perp)$;

(ii) for any $X_1 \in \Gamma(\mathcal{D}^T)$, $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, $A_{\phi Z}X_1 = -(\phi X_1\mu)$, $A_\phi Z X_2 = 0$, $A_{QX_2Z} = (P_2X_2\mu)Z$, $(\xi\mu) = 1$,

for some smooth function $\mu$ on $M$ satisfying $(V\mu) = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$. Thus, Theorem 5.1 of this paper is a generalization of Theorem 5.3 of [28].

3. If we take $\theta_1 = \frac{\pi}{2}$, $\theta_2 = constant = \theta$, $\theta_3 = 0$, then $M$ changes to a warped product skew CR-submanifold of the form $M_2 \times_f M_T$, where $M_2 = M_\perp \times M_\theta$, studied in [19]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [19]).

Let $M$ be a proper skew CR-submanifold of $\bar{M}$, then $M$ is locally a warped product submanifold of the form $M_2 \times_f M_T$, where $M_2 = M_\perp \times M_\theta$ if and only if

(i) $A_\phi Z X = \{\eta(Z) - (Z\mu)\}\phi X$;

(ii) $A_{QUX} = \{\eta(U) - (U\mu)\}\phi X + (P_2U\mu)X$;

(iii) $(\xi\mu) = 1$,

for any $X \in \Gamma(\mathcal{D}^T)$, $U \in \Gamma(\mathcal{D}^\theta)$, $Z \in \Gamma(\mathcal{D}^\perp)$, for some smooth function $\mu$ on $M$ satisfying $(Y\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^T)$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [19].

4. If we take $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$ and $\theta_3 = \theta$ then $M$ changes to a warped product submanifold of the form $M_3 \times_f M_\theta$, where $M_3 = M_T \times M_\perp$, studied in [18]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [18]).

Let $M$ be a submanifold of a Kenmotsu manifold $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$ with $\xi$ is orthogonal to $M_\theta$. Then $M$ is locally a warped product submanifold of the form $M = M_3 \times_f M_\theta$, where $M_3 = M_T \times M_\perp$, if and only if the following relations hold:

(i) $A_{QV}\phi X - A_{QPV}X = \sin^2\theta[(X\mu) - \eta(X)]V$;

(ii) $A_{\phi Z}PV - A_{QPV}Z = -\cos^2\theta[(Z\mu) - \eta(Z)]V$;

(iii) $(\xi\mu) = 1$,

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$ and $(V\mu) = 0$ for some function $\mu$ on $M$ satisfying $(W\mu) = 0$, for any $W \in \Gamma(\mathcal{D}^\theta)$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [18].

As consequences of Theorem 7.1, we have the following.

1. If we consider $\theta_1 = constant$, $\theta_2 = 0$, $\theta_3 = \frac{\pi}{2}$, then the submanifold $M$ changes to bi-warped product submanifold of the form $M_\theta \times_{f_1} M_T \times_{f_2} M_\perp$, studied in [33]. In this case Theorem 7.1 of this paper takes the following form.

Let $M = M_\theta \times_{f_1} M_T \times_{f_2} M_\perp$ be a bi-warped product submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_\theta$, then the squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2q \csc^2\theta(1 + \cos^2\theta)(\|\boldsymbol{\nabla} \ln f_1\|^2 - 1) + 2s \cot^2\theta(\|\boldsymbol{\nabla} \ln f_2\|^2 - 1),$$

where $2q = \dim M_T$, $2s = \dim M_\perp$, $\boldsymbol{\nabla} \ln f_1$ and $\boldsymbol{\nabla} \ln f_2$ are the gradients of warping function $\ln f_1$ and $\ln f_2$ along $M_T$ and $M_\perp$, respectively.

If the equality sign holds, then $M_\theta$ is totally geodesic and $M_T$, $, M_\perp$ are totally umbilical submanifold of $\bar{M}$. Taking $\dim M_T = 2q = m_1$ and $\dim M_\perp = 2s = m_2$, we see that this statement coincides with the statement of Theorem 6 of [33]. Thus, Theorem 7.1 of this paper is a generalisation of Theorem 6 of [33].

2. If we consider $\dim M_{\theta_2} = 0$, then the submanifold $M$ changes into warped product pointwise bi-slant submanifold of the form $M_{\theta_1} \times_f M_{\theta_2}$ studied in [17]. In this case Theorem 7.1 of this paper takes the following form.

Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product pointwise bi-slant submanifold of $\bar{M}$ such that $\xi$ is tangent to $M_{\theta_1}$, then the squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2q \csc^2\theta_1(\cos^2\theta_1 + \cos^2\theta_2)(\|\boldsymbol{\nabla} \ln f\|^2 - 1),$$

where $2q = \dim M_{\theta_2}$, $\boldsymbol{\nabla} \ln f$ is the gradient of warping function $\ln f$ along $M_{\theta_1}$. If the equality sign holds, then $M_{\theta_1}$ is totally geodesic and $M_{\theta_2}$ is totally umbilical submanifold of $\bar{M}$. Thus, we see that this statement coincides with the statement of Theorem 6.1 of [19]. Hence Theorem 7.1 of this paper is a generalization of Theorem 6.1 of [17].

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