Kragujevac Journal of Mathematics
Volume 47(6) (2023), Pages 825–845.

# STRONG CONVERGENCE RESULTS FOR VARIATIONAL INEQUALITY AND EQUILIBRIUM PROBLEM IN HADAMARD SPACES

G. C. UGWUNNADI[1,3], C. C. OKEKE[1,3], A. R. KHAN[2], AND L. O. JOLAOSO[3]

Abstract. The main purpose of this paper is to introduce and study a viscosity type algorithm in a Hadamard space which comprises of a demimetric mapping, a finite family of inverse strongly monotone mappings and an equilibrium problem for a bifunction. Strong convergence of the proposed algorithm to a common solution of variational inequality problem, fixed point problem and equilibrium problem is established in Hadamard spaces. Nontrivial Applications and numerical examples were given. Our results compliment some results in the literature.

## 1. Introduction

Let $X$ be a metric space and $C$ be a nonempty closed and convex subset of $X$. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if

$$Tx = x. \tag{1.1}$$

The set of fixed points of $T$ is denoted by F(T). With the recent rapid developments in fixed point theory, there has been a renewed interest in iterative schemes. The properties of iterations between the type of sequences and kind of operators have not been completely studied and are now under discussion. The theory of operators has occupied a central place in modern research using iterative schemes because of its promise of enormous utility in fixed point theory and its applications. In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate

*Key words and phrases.* Variational inequality problem, inverse strongly monotone operator, viscosity iteration, equilibruim problem, demimetric mapping, Hadamard space.
2010 *Mathematics Subject Classification.* Primary: 47H09, 47H10. Secondary: 47J25.
DOI 10.46793/KgJMat2306.825U
*Received*: June 28, 2020.
*Accepted*: December 06, 2020.

fixed point is more than enough, an approximate solution plays an important role in such situations. The theory of fixed points and consequently of approximate fixed points finds application in mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equations and several other areas of applicable analysis see for instance [9, 17, 23, 25, 26, 31, 33–35, 40, 45].

The mapping $T : C \to X$ is said to be:

(a) nonexpansive if

$$d(Tx, Ty) \le d(x, y), \quad \text{for all } x, y \in C;$$

(b) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, q) \le d(x, q), \quad \text{for all } x \in C \text{ and } q \in F(T);$$

(c) firmly nonexpansive if

$$d^2(Tx, Ty) \le \langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle, \quad \text{for all } x, y \in C;$$

(d) $\alpha$-inverse strongly monotone if there exists $\alpha > 0$ such that

$$d^2(x, y) - \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \ge \alpha \Psi_T(x, y), \quad \text{for all } x, y \in C, \tag{1.2}$$

where $\Psi_T(x, y) = d^2(x, y) + d^2(Tx, Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$. It was established in [3] that the quantity $\Psi_T(x, y)$ is nonnegative.

Given a nonempty set $C$ and $f : C \times C \to \mathbb{R}$ a bifunction, the Equilibrium Problem (EP) is defined as follows:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \ge 0, \quad \text{for all } y \in C. \tag{1.3}$$

The point $x^*$ in (1.3) is called an equilibrium point of $f$. We shall denote the solution set of problem (1.3) by $\mathrm{EP}(f, C)$. EPs have been widely studied in Hilbert, Banach and topological vector spaces [6, 12, 24] and Hadamard manifolds [11, 41]. One of the most popular and effective methods used for solving problem (1.3) and other related optimization problems is the Proximal Point Algorithm (PPA) which was introduced in a Hilbert space by Martinet [37] and was further studied by Rockafellar [47] in 1976. The PPA and its generalizations have also been studied extensively in Banach spaces and Hadamard manifolds (see [11, 36] and the references therein). Recently, many convergence results by the PPA for solving optimization problems were extended from the classical linear spaces to the setting of nonlinear space such as Riemannain manifolds and Hadamard spaces (see [4, 5, 10, 19, 46, 54] and reference therein). Numerous applications in computer vision, machine learning, electronic structure computation, system balancing, and robot manipulation can be reduced to find solution of optimization and equilibrium problems in nonlinear setting (see [1, 2, 27, 43, 50, 53]).

Very recently, Kumam and Chaipunya [36] studied EP (1.3) in Hadamard spaces. They established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity and coercivity assumptions ([36], Theorem 4.1). They also established some fundamental properties of the resolvent of a bifunction. Furthermore, they proved that the PPA $\Delta$-converges to an equilibrium point of a monotone bifunction in a Hadamard space. More precisely, they proved:

**Theorem 1.1.** ([36, Theorem 7.3]) *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f : C \times C \to \mathbb{R}$ be a monotone, $\Delta$-upper semicontinuous in the first variable such that $D(J_r^f) \supset C$ for all $r > 0$ where $D$ stands for the domain. Suppose that $EP(f, C) \neq \emptyset$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{r_n}^f(x_{n-1}), \quad n \in \mathbb{N},$$

*where $\{r_n\}$ is a sequence of positive real numbers bounded away from 0. Then $\{x_n\}$ $\Delta$-converges to an element of $EP(f, C)$.*

The Variational Inequality Problem (VIP) was first introduced by Stampacchia [49] for modeling problems arising in mechanics. To study the regularity problem for partial differential equations, Stampacchia [49] studied a generalization of the Lax-Milgram theorem and called all problems of this kind to be VIPs. The theory of VIP has numerous applications in diverse fields such as physics, engineering, economics, mathematical programming and others (see [8, 32, 39] and references therein). The VIP in a real Hilbert space $H$ is formulated as follows:

$$\text{find } x \in C \text{ such that } \langle Tx, y - x\rangle \geq 0, \quad \text{for all } y \in C, \tag{1.4}$$

where $C$ is a nonempty closed and convex subset of $H$ and $T$ is a nonlinear mapping defined on $C$. This formulation is recently extended to the framework of CAT(0) space $X$ by Alizadeh-Dehghan-Moradlou [3] as follows:

$$\text{find } x \in C \text{ such that } \langle \overrightarrow{Txx}, \overrightarrow{xy}\rangle \geq 0, \quad \text{for all } y \in C, \tag{1.5}$$

where $\overrightarrow{xy}$ stands for a vector in $X$ defined in (2.1).

They established the existence of VIP (1.5) when $T$ is an inverse strongly monotone mapping in a CAT(0) space. Furthermore, they introduced the following iterative algorithm for solving VIP (1.5): For arbitrary $x_1 \in C$, generate sequence $\{x_n\}$ as

$$\begin{cases} y_n = P_C(\beta_n x_n \oplus (1-\beta_n)Tx_n), \\ x_{n+1} = P_C(\alpha_n x_n \oplus (1-\alpha_n)Sy_n), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$, $S$ and $T$ are nonexpansive and inverse strongly monotone mappings, respectively. They also obtained $\Delta$-convergence of Algorithm (1.6) to a solution of the VIP (1.5), which is also a fixed point of the nonexpansive mapping $S$.

*Remark* 1.1. If $X = H$ is a real Hilbert space, then $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle b - a, d - c\rangle$ for all $a, b, c, d \in H$. Thus, the VIP (1.6) reduces to the VIP (1.5) when $X = H$.

Motivated by the work of Kumam and Chaipunya [36] and Alizadeh-Dehghan -Moradlou [3], we introduce and study a viscosity type algorithm which comprises of demimetric mapping, equilibrium problem for a monotone bifunction and a finite family of inverse strongly monotone mappings. Strong convergence of the proposed algorithm to common solution of a fixed point of a demimetric mapping, an equilibrium problem of a bifunction and variational inequality problem for a finite family of certain monotone mappings is established in a Hadamard space $X$. Furthermore, we applied our results to approximate solutions of minimization problems in $X$.

## 2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. Throughout this paper, we shall denote the strong and $\Delta$-convergence by $\longrightarrow$ and $\rightharpoonup$, respectively.

Let $(X,d)$ be a metric space and $x,y \in X$. A geodesic path joining $x$ to $y$ (or, a geodesic from $x$ to $y$) is a map $\gamma : [a,b] \subseteq \mathbb{R} \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$, and $d(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [a,b]$. In particular, $\gamma$ is an isometry and $d(x,y) = b-a$. We say that a metric space X is uniquely geodesic if every two points of X are joined by only one geodesic segment (i.e., CAT(0) space). Examples of CAT(0) spaces are Euclidean spaces $\mathbb{R}^n$ and Hilbert spaces. For more details, please see [12, 20, 21, 28, 48]. Complete CAT(0) spaces are often called Hadamard spaces.

Let $(1-t)x \oplus ty$ denote the unique point $z$ in the geodesic segment joining $x$ to $y$ for each $x,y$ in a CAT(0) space such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$, where $t \in [0,1]$. Let $[x;y] := \{(1-t)x \oplus ty : t \in [0,1]\}$, then a subset $C$ of $X$ is convex if $[x,y] \subseteq C$ for all $x,y \in C$.

In 2008, Breg and Nikolaev [6] introduced the concept of quailinearization mapping in CAT(0) spaces. They denoted a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ which they called a vector and defined a mapping $\langle \cdot,\cdot \rangle : (X\times X)\times(X\times X) \to \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right), \quad a,b,c,d \in X, \tag{2.1}$$

called the quasilinearization mapping. It is easy to verify that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,d,e \in X$. It has been established that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [6]). Recall that the space $X$ is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d)$ for all $a,b,c,d \in X$.

Let $\{x_n\}$ be a bounded sequence in CAT(0) space $X$. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\})\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see [16, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to $\Delta$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

**Definition 2.1** ([4])**.** Let $X$ be a CAT(0) space and $C$ be a nonempty closed and convex subset of $X$. A mapping $T : C \to X$ is said to be $k$-demimetric if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$, such that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx), \quad \text{for all } x \in X \text{ and } y \in F(T). \tag{2.2}$$

**Definition 2.2.** Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$. The metric projection $P_C : X \to C$ assigns to each $x \in X$, the unique point $P_C x$ in $C$ such that

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}.$$

The map $P_C$ is nonexpansive [13].

**Definition 2.3.** Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$. A mapping $T : C \to C$ is said to be $\Delta$-demiclosed, if for any bounded sequence $\{x_n\}$ in $X$ such that $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

**Lemma 2.1** ([4])**.** *Let $X$ be a CAT$(0)$ space and $S : X \to X$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ with $F(S) \neq \emptyset$ and $\lambda \in (0, 1)$. Suppose that $S_\lambda = \lambda x \oplus (1 - \lambda)Sx$. Then $S_\lambda$ is quasi-nonexpansive and $F(S_\lambda) = F(S)$.*

In [36], the authors introduce resolvent of a bifunction $f$ associated with the EP (1.3). They defined a perturbation bifunction $\bar{f}_x : C \times C \to \mathbb{R}$ of $f$ by

$$\bar{f}_x(x, y) := f(x, y) - \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \quad \text{for all } x, y \in C. \tag{2.3}$$

The perturbed bifunction $\bar{f}$ has a unique equilibrium point called resolvent operator $J^f : X \to 2^C$ of the bifunction $f$ (see [36]) and is defined by

$$\begin{aligned} J^f(x) &:= EP(C, \bar{f}_x) = \{z \in C : f(z, y) - \langle \overrightarrow{z\bar{x}}, \overrightarrow{zy} \rangle \geq 0, \, y \in C\} \\ &= \left\{ z \in C : f(z, y) + \frac{1}{2}(d^2(x, y) - d^2(x, z) - d^2(y, z)) \geq 0 \text{ for all } y \in C \right\}, \end{aligned} \tag{2.4}$$

$x \in X$. It was established in [36] that $J^f$ is well-defined.

**Lemma 2.2** ([36])**.** *Suppose that $f$ is monotone and $D(J^f) \neq \emptyset$. Then, the following properties hold.*

(i) *$J^f$ is singled-valued.*
(ii) *If $D(J^f) \supset C$, then $J^f$ is nonexpansive restricted to $C$.*
(iii) *If $D(J^f) \supset C$, $F(J^f) = EP(C, f)$.*

**Lemma 2.3** ([36])**.** *Suppose that $f$ has the following properties:*

(A1) *$f(x,x)=0$ for all $x\in C$;*
(A2) *$f$ is monotone, i.e., $f(x,y)+f(y,x)\le 0$ for all $x,y\in C$;*
(A3) *for each $x\in C$, $y\mapsto f(x,y)$ is convex and lower semicontinuous;*
(A4) *for each $x\in C$, $f(x,y)\ge \limsup_{t\downarrow 0} f((1-t)x\oplus tz,y)$ for all $x,z\in C$.*

*Then $D(J^f)=X$ and $J^f$ is single-valued.*

*Remark* 2.1 ([23]). It follows from (2.4) that the resolvent $J_r^f$ of the bifunction $f$ $(r>0)$ is given by

$$J_r^f(x) := EP(C,\bar{f}_x) = \left\{ z\in C : f(z,y)+\frac{1}{r}\langle \overrightarrow{xz},\overrightarrow{zy}\rangle \ge 0,\ y\in C\right\},\quad x\in X, \tag{2.5}$$

where $\bar{f}$ in this case is defined as

$$\bar{f}_x(x,y) := f(x,y)+\frac{1}{r}\langle \overrightarrow{\bar{x}x},\overrightarrow{xy}\rangle,\quad \text{for all } x,y\in C,\ \bar{x}\in X. \tag{2.6}$$

**Lemma 2.4** ([23])**.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f: C\times C\to\mathbb{R}$ be a monotone bifunction such that $C\subset D(J_r^f)$ for $r>0$. Then, $J_r^f$ is firmly nonexpansive restricted to $C$. That is*

$$d^2(J_r^f x, J_r^f y)\le \langle \overrightarrow{xy},\overrightarrow{J_r^f x J_r^f y}\rangle. \tag{2.7}$$

**Lemma 2.5** ([15])**.** *Every bounded sequence in a Hadamard space always has a $\Delta$-convergent subsequence.*

**Lemma 2.6** ([29])**.** *Let $X$ be a Hadamard space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ $\Delta-$ converges to $x$ if and only if $\limsup\limits_{n\to\infty}\langle \overrightarrow{x_n x},\overrightarrow{xy}\rangle\le 0$ for all $y\in X$.*

**Lemma 2.7** ([56])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\le (1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n,\quad n\ge 0,$$

*where*
(*i*) $\{\alpha_n\}\subset[0,1]$, $\sum\alpha_n=\infty$;
(*ii*) $\limsup \sigma_n\le 0$;
(*iii*) $\gamma_n\ge 0$, $n\ge 0$, $\sum\gamma_n<\infty$.

*Then $a_n\to 0$ as $n\to\infty$.*

**Lemma 2.8** ([3])**.** *Let $C$ be a nonempty closed and convex subset of Hadamard space $X$ and $T: C\to X$ be an $\alpha$-inverse strongly monotone mapping. Assume $\mu\in[0,1]$ and define $T_\mu: C\to X$ by $T_\mu x=(1-\mu)x\oplus\mu Tx$. If $0<\mu<2\alpha$, then $T_\mu$ is nonexpansive mapping and $F(T_\mu)=F(T)$.*

**Lemma 2.9** ([3])**.** *Let $C$ be a nonempty bounded closed and convex subset of a Hadamard space $X$ and $T: C\to X$ be an $\alpha$-inverse strongly monotone. Then $VI(C,T)$ is nonempty, closed and convex.*

**Lemma 2.10.** ([7, Lemma 3]) *Let $X$ be a uniformly convex hyperbolic space with modulus of uniform convexity $\eta$. For any $c > 0$, $\epsilon \in (0, 2]$, $\lambda \in [0, 1]$ and $v, x, y \in X$, $d(x, v) \leq c$, $d(y, v) \leq c$ and $d(x, y) \geq \epsilon c$ implies that*

$$d((1 - \lambda)x \oplus \lambda y, v) \leq (1 - 2\lambda(1 - \lambda)\eta(c, \epsilon))c.$$

If $X$ is a CAT(0) space, then $X$ is uniformly convex hyperbolic space ([30]).

**Lemma 2.11** ([3]). *Let $C$ be a nonempty convex subset of a Hadamard space $X$ and $T : C \to X$ be a mapping. Then,*

$$VI(C, T) = VI(C, T_\mu),$$

*where $\mu \in (0, 1]$ and $T_\mu : C \to X$ is a mapping defined by $T_\mu x = (1 - \mu)x \oplus \mu Tx$ for all $x \in C$.*

*Remark* 2.2 ([42]). It follows from Lemma 2.11 that

$$F(P_C T) = VI(C, T) = VI(C, T_\mu) = F(P_C T_\mu).$$

**Lemma 2.12.** *Let $X$ be a CAT(0) space, $x, y, z \in X$ and $t \in [0, 1]$. Then*

(i) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$ (see [15]);

(ii) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$ (see [15]);

(iii) $d^2(tx \oplus (1 - t)y, z) \leq t^2 d^2(x, z) + (1 - t)^2 d^2(y, z) + 2t(1 - t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [13]).

**Lemma 2.13** ([51]). *Let $X$ be a CAT(0) space, $\{x_i : i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$, be such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, z\right) \leq \sum_{i=1}^N \alpha_i d(x_i, z), \quad \text{for all } x \in X.$$

**Lemma 2.14** ([14]). *Let $X$ be a CAT(0) space, $\{x_i : i = 1, 2, \ldots, N\} \subset X$, $\{y_i : i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$, be such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$d\left(\bigoplus_{i=1}^N \alpha_i x_i, \bigoplus_{i=1}^N \alpha_i y_i\right) \leq \sum_{i=1}^N \alpha_i d(x_i, y_i). \tag{2.8}$$

**Lemma 2.15** ([17]). *Let $X$ be a Hadamard space and $S : X \to X$ be a nonexpansive mapping. Then the conditions $\{x_n\}$ $\Delta$-converges to $x$ and $d(x_n, Sx_n) \to 0$, imply $x = Sx$.*

**Lemma 2.16** ([38]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$. and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

*In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

## 3. Main Results

We begin with a technical results which will be used to prove our main results.

**Lemma 3.1** ([42]). *Let $C$ be a nonempty closed and convex subset of a CAT(0) space $X$, $T_i : C \to X$, $i = 1, 2, \dots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings and $\Psi_\mu : C \to C$ be defined by $\Psi_\mu x := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} x$ for all $x \in C$ and $\beta_i \in (0,1)$, where $T_{\mu_i} x := (1-\mu_i)x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$ with $\mu_i \in [0,1]$. If $\sum_{i=1}^N \beta_i = 1$, then the mapping $\Psi_\mu$ is nonexpansive. If in addition, $\cap_{i=1}^N F(P_C T_{\mu_i}) \neq \emptyset$, then $F(\Psi_\mu) = \bigcap_{i=1}^N F(P_C T_{\mu_i})$.*

**Proposition 3.1** ([23]). *Let $X$ be a Hadamard space and $f : C \times C \to \mathbb{R}$ be a monotone bifunction operator. Then*

$$d^2(u, J_r^f x) + d^2(J_r^f x, x) \le d^2(u, x), \quad \text{for all } u \in F(J_r^f),\ x \in X \text{ and } r > 0.$$

**Theorem 3.1.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $f : C \times C \to \mathbb{R}$ be a monotone and upper semicontinuous bifunction such that conditions (A1)-(A4) of Lemma 2.3 are satisfied, $C \subset D(J_r^f)$ for $r > 0$ and $T_i : C \to X$, $i = 1, 2, \dots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0,1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0,1)$. Suppose that $\Upsilon := F(S) \cap EP(f, C) \cap \left(\bigcap_{i=1}^N VI(C, T_i)\right)$ is nonempty and $\{x_n\}$ is a sequence generated by an arbitrary $x_1 \in X$ as follows:*

$$\begin{cases} u_n = J_{r_n}^f x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1-\alpha_n)[\beta_n y_n \oplus (1-\beta_n) S_\lambda y_n], \quad n \ge 1, \end{cases} \tag{3.1}$$

*where $S_\lambda x = \lambda x \oplus (1-\lambda) S x$ is $\Delta$-demiclosed and $T_{\mu_i} x = (1-\mu_i)x \oplus \mu_i T_i x$, $0 < \mu_i < 2\beta_i$, for each $i = 1, 2, \dots, N$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{\beta_i\} \subset (0,1)$ and $r_n \in (0, \infty)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Upsilon$, where $p = P_\Upsilon h(p)$.*

*Proof.* Let $p \in F(S) \cap EP(f, C) \cap \left(\bigcap_{i=1}^N VI(C, T_i)\right)$. By Lemma 3.1, we have that $\Psi_\mu$ is nonexpansive, that is,

$$d(y_n, p) = d(\Psi_\mu u_n, p) \le d(u_n, p). \tag{3.2}$$

Since $J_{r_n}^f$ is firmly nonexpansive, we have

$$d(u_n, p) = d(J_{r_n}^f (x_n), p) \le d(x_n, p). \tag{3.3}$$

Let $v_n = \beta_n y_n \oplus (1-\beta_n) S_\lambda y_n$, then we obtain

$$d(v_n, p) = d(\beta_n y_n \oplus (1-\beta_n) S_\lambda y_n, p)$$

$$
\begin{aligned}
&\leq \beta_n d(y_n,p) + (1-\beta_n) d(S_\lambda y_n, p) \\
&\leq \beta_n d(y_n,p) + (1-\beta_n) d(y_n, p) \\
&= d(y_n,p). \qquad (3.4)
\end{aligned}
$$

It follows from (3.1), (3.3) and Lemma 2.12 (i) that

$$
\begin{aligned}
d(x_{n+1},p) &= d(\alpha_n h(x_n) \oplus (1-\alpha_n))v_n, p) \\
&\leq \alpha_n d(h(x_n),p) + (1-\alpha_n) d(v_n,p) \\
&\leq \alpha_n d(h(x_n),p) + (1-\alpha_n) d(y_n,p) \\
&\leq \alpha_n d(h(x_n),h(p)) + \alpha_n d(h(p),p) + (1-\alpha_n) d(x_n,p) \\
&\leq \alpha_n \theta d(x_n,p) + \alpha_n d(h(p),p) + (1-\alpha_n) d(x_n,p) \\
&= [1-\alpha_n(1-\theta)] d(x_n,p) + \alpha_n(1-\theta)\frac{d(h(p),p)}{1-\theta} \\
&\leq \max\left\{ d(x_n,p), \frac{d(h(p),p)}{1-\theta} \right\}.
\end{aligned}
$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{u_n\}$, $\{J^f_{r_n} x_n\}$ and $\{S_\lambda y_n\}$ are all bounded.

We now divide the rest of the proof into two cases.

**Case 1.** Suppose that $\{d(x_n,p)\}$ is monotonically non-increasing. Then there exists $\lim_{n\to\infty}\{d(x_n,p)\}$. This shows that

$$
\lim_{n\to\infty}[d(x_{n+1},p) - d(x_n,p)] = 0. \qquad (3.5)
$$

Hence, we obtain from (3.1), Lemma 2.12 (ii), (3.2) and Proposition 3.1 that

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(v_n,p) - \alpha_n(1-\alpha_n) d^2(h(x_n),v_n) \\
&\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(y_n,p) \\
&\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(u_n,p) \\
&\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n)\left[d^2(y_n,p) - d^2(u_n,y_n)\right] \\
&= \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(x_n,p) - (1-\alpha_n) d^2(u_n,y_n). \qquad (3.6)
\end{aligned}
$$

From (3.6), we get

$$
(1-\alpha_n) d^2(y_n,u_n) \leq \alpha_n d^2(h(x_n),p) + d^2(x_n,p) - d^2(x_{n+1},p).
$$

Hence, we obtain from (3.5) and condition (i) that

$$
\lim_{n\to\infty} d(y_n,u_n) = 0 = \lim_{n\to\infty} d(\Psi_\mu u_n, u_n). \qquad (3.7)
$$

Also, from (3.1), Lemma 2.12 (ii) and Proposition 3.1 we get

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(v_n,p) - \alpha_n(1-\alpha_n) d^2(h(x_n),v_n) \\
&\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(y_n,p) \\
&\leq \alpha_n d^2(h(x_n),p) + (1-\alpha_n) d^2(u_n,p)
\end{aligned}
$$

$$\le \alpha_n d^2(h(x_n),p) + (1-\alpha_n)\left[d^2(x_n,p) - d^2(u_n,x_n)\right]$$

$$= \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(x_n,p) - (1-\alpha_n)d^2(u_n,x_n). \tag{3.8}$$

Thus,

$$(1-\alpha_n)d^2(u_n,x_n) \le \alpha_n d^2(h(x_n),p) + d^2(x_n,p) - d^2(x_{n+1},p).$$

Hence, from condition (i) and (3.5), we have

$$\lim_{n\to\infty} d(u_n,x_n) = \lim_{n\to\infty} d(J^f_{r_n}x_n,x_n) = 0. \tag{3.9}$$

By (3.1) and Lemma 2.12 (ii), we get

$$\begin{aligned} d^2(x_{n+1},p) \le& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(v_n,p) - \alpha_n(1-\alpha_n)d^2(h(x_n),v_n) \\ \le& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)[\beta_n d^2(y_n,p) + (1-\beta_n)d^2(S_\lambda y_n,p) \\ &- \beta_n(1-\beta_n)d^2(y_n,S_\lambda y_n)] \\ \le& \alpha_n d^2(h(x_n),p) + (1-\alpha_n)d^2(x_n,p) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(y_n,S_\lambda y_n). \end{aligned}$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)d^2(S_\lambda y_n,y_n) \le \alpha_n[d^2(h(x_n),p) - d^2(x_n,p)] + d^2(x_n,p) - d^2(x_{n+1},p).$$

By condition (i) and (3.5), we obtain

$$\lim_{n\to\infty} d(S_\lambda y_n,y_n) = 0. \tag{3.10}$$

Also, by (3.1), (3.7) and (3.9)

$$d(y_n,x_n) \le d(\Psi_\mu u_n,u_n) + d(u_n,x_n) \to 0, \quad n\to\infty. \tag{3.11}$$

We also obtain from (3.1) and condition (i) that

$$d(x_{n+1},v_n) = d(\alpha_n h(x_n) \oplus (1-\alpha_n)v_n, v_n) \le \alpha_n d(h(x_n),v_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.12}$$

Also, from the definition of $v_n$ and (3.10), we obtain

$$d(v_n,y_n) \le \beta_n d(y_n,y_n) + (1-\beta_n)d(S_\lambda y_n,y_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.13}$$

Thus, from (3.11), (3.12) and (3.13), we get

$$d(x_{n+1},x_n) \le d(x_{n+1},v_n) + d(v_n,y_n) + d(y_n,x_n) \to 0. \tag{3.14}$$

Next we show that

$$\limsup \langle \overrightarrow{h(z)z}, \overrightarrow{x_n z}\rangle \le 0.$$

As $\{u_n\}$ is bounded, so by Lemma 2.5, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = z$. Also since $\Psi_\mu$ is nonexpansive, we obtain from (3.7), Lemma 2.15, Lemma 3.1 and Remark 2.2 that $z \in F(\Psi_\mu) = \cap_{i=1}^N F(P_C T_{\mu_i}) = \cap_{i=1}^N VI(C, T_{\mu_i})$. Let us show that $z \in EP(f,C)$. Since $\{J^f_{r_n}(x_n)\}$ is bounded, there exists a subsequence $\{w_k\}$ of $\{J^f_{r_n}(x_n)\}$ such that

$$\lim_{k\to\infty} d(w_k,p) = \liminf_{n\to\infty} d(J^f_{r_n}x_n,p)$$

and that $\{w_k\}$ $\Delta$-converges to some $z \in X$, where $w_k = J^f_{r_{n_k}} x_{n_k}$ for all $k \in \mathbb{N}$. By the definition of the resolvent $J^f_{r_n}$, we have

$$r_{n_k} f(w_k, y) + \frac{1}{2}(d^2(x_{n_k}, y) - d^2(x_{n_k}, w_k) - d^2(y, w_k)) \geq 0,$$

for all $y \in C$. In particular, letting $y = J^f z$, we have

$$d^2(x_{n_k}, J^f z) - d^2(x_{n_k}, w_k) - d^2(J^f z, w_k) \geq -2r_{n_k} f(w_k, J^f z).$$

Similarly, by the definition of $J^f$, we have

$$d^2(w_k, z) - d^2(J^f z, z) - d^2(J^f z, w_k) \geq -2f(w_k, J^f z).$$

Since $f$ is monotone, we have

$$\begin{aligned}
&d^2(J^f z, x_{n_k}) - d^2(w_k, x_{n_k}) - d^2(w_k, J^f z) - r_{n_k} d^2(w_k, z) - r_{n_k} d^2(J^f z, z) \\
&- r_{n_k} d^2(J^f z, w_k) \geq 0,
\end{aligned}$$

and hence

$$\begin{aligned}
(1 + r_{n_k}) d^2(J^f z, w_k) &\leq d^2(J^f z, x_{n_k}) - d^2(w_k, x_{n_k}) + r_{n_k} d^2(w_k, z) - r_{n_k} d^2(J^f z, z) \\
&\leq d^2(J^f z, x_{n_k}) + d^2(w_k, z).
\end{aligned}$$

It follows that

$$\begin{aligned}
d^2(J^f z, w_k) &\leq \frac{1}{r_{n_k}}(d^2(J^f z, x_{n_k}) - d^2(J^f z, w_k) + d^2(z, w_k)) \\
&\leq \frac{1}{r_{n_k}} d(w_k, x_{n_k})(d(J^f z, x_{n_k}) - d(J^f z, w_k)) + d^2(z, w_k),
\end{aligned}$$

for all $k \in \mathbb{N}$ and consequently, we obtain

$$\limsup_{k\to\infty} d^2(J^f z, w_k) \leq \limsup_{k\to\infty} d^2(z, w_k).$$

Since the asymptotic center of $\{w_k\}$ is unique point $z$, we have $z = J^f z$, that is, $z \in EP(C, f)$.

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = z$. It follows from (3.11) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta - \lim_{k\to\infty} y_{n_k} = z$. Since $S_\lambda$ is $\Delta$-demiclosed, it follows from (3.10) and Lemma 2.1 that $z \in F(S_\lambda) = F(S)$. Hence, $z \in \Upsilon := F(S) \cap EP(f, C) \cap \bigcap_{i=1}^N VI(C, T_{\mu_i})$.

Observe that

$$\limsup_{n\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_n z} \rangle \leq \limsup_{k\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_{n_k} z} \rangle. \tag{3.15}$$

Since $\{x_{n_k}\}$ $\Delta$-converges to $z$, therefore by Lemma 2.6, we have

$$\limsup_{k\to\infty} \langle \overrightarrow{h(z)z}, \overrightarrow{x_n z} \rangle \leq 0.$$

This together with (3.15) gives

$$\limsup_{n\to\infty}\langle\overrightarrow{h(z)z},\overrightarrow{x_nz}\rangle\le 0. \tag{3.16}$$

Also, we have

$$\begin{aligned}\langle\overrightarrow{h(z)z},\overrightarrow{zx_{n+1}}\rangle &= \langle\overrightarrow{h(z)z},\overrightarrow{zx_n}\rangle+\langle\overrightarrow{h(z)z},\overrightarrow{x_nx_{n+1}}\rangle\\ &=\langle\overrightarrow{h(z)z},\overrightarrow{zx_n}\rangle+d(z,h(z))d(x_n,x_{n+1}).\end{aligned} \tag{3.17}$$

Hence from (3.14), (3.16) and (3.17) we have that

$$\limsup_{n\to\infty}\langle\overrightarrow{h(z)z},\overrightarrow{zx_{n+1}}\rangle\le 0. \tag{3.18}$$

Finally, we prove that $x_n\to z$ as $n\to\infty$. For any $n\in\mathbb{N}$, we set $\vartheta_n=\alpha_n z\oplus(1-\alpha_n)v_n$,

$$\begin{aligned}d^2(x_{n+1},z)=&d^2(\alpha_nh(x_n)\oplus(1-\alpha_n)v_n,z)\\ \le& d^2(\vartheta_n,z)+2\langle\overrightarrow{x_{n+1}\vartheta_n},\overrightarrow{x_{n+1}z}\rangle\\ \le&[\alpha_nd(z,z)+(1-\alpha_n)d(v_n,z)]^2\\ &+2\left[\alpha_n\langle\overrightarrow{h(x_n)\vartheta_n},\overrightarrow{x_{n+1}z}\rangle+(1-\alpha_n)\langle\overrightarrow{v_n\vartheta_n},\overrightarrow{x_{n+1}z}\rangle\right]\\ \le&(1-\alpha_n)^2d^2(v_n,z)+2\Big[\alpha_n^2\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle+\alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\ &+\alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle+(1-\alpha_n)^2\langle\overrightarrow{v_nv_n},\overrightarrow{x_{n+1}z}\rangle\Big]\\ \le&(1-\alpha_n)^2d^2(y_n,z)+2\Big[\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle+\alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\ &+\alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle+(1-\alpha_n)^2d(v_n,v_n)d(x_{n+1},z)\Big]\\ \le&(1-\alpha_n)^2d^2(x_n,z)+2\Big[\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle+\alpha_n(1-\alpha_n)\langle\overrightarrow{h(x_n)v_n},\overrightarrow{x_{n+1}z}\rangle\\ &+\alpha_n(1-\alpha_n)\langle\overrightarrow{v_nz},\overrightarrow{x_{n+1}z}\rangle\Big]\\ \le&(1-\alpha_n)^2d^2(x_n,z)+2\alpha_n\langle\overrightarrow{h(x_n)z},\overrightarrow{x_{n+1}z}\rangle\\ \le&(1-\alpha_n)^2d^2(x_n,z)+2\alpha_n\langle\overrightarrow{h(x_n)h(z)},\overrightarrow{x_{n+1}z}\rangle+2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle\\ \le&(1-\alpha_n)^2d^2(x_n,z)+2\alpha_n\theta d(x_n,z)d(x_{n+1},z)+2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle\\ \le&(1-\alpha_n)^2d^2(x_n,z)+2\alpha_n\theta\left(d^2(x_n,z)+d^2(x_{n+1},z)\right)\\ &+2\alpha_n\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle.\end{aligned}$$

As $\{\alpha_n\}$ and $\{x_n\}$ are bounded so there is $M>0$ such that $\frac{1}{1-\theta\alpha_n}d^2(x_n,z)\le M$. It now follows that

$$d^2(x_{n+1},z)\le\frac{(1-\alpha_n)^2+\theta\alpha_n}{1-\theta\alpha_n}d^2(x_n,z)+\frac{2\alpha_n}{1-\theta\alpha_n}\langle\overrightarrow{h(z)z},\overrightarrow{x_{n+1}z}\rangle$$

$$
\begin{aligned}
&\leq \frac{(1-\alpha_n)^2+\theta\alpha_n}{1-\theta\alpha_n} d^2(x_n,z) + \frac{2\alpha_n}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n^2 M \\
&\leq \left[1 - \frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}\right] d^2(x_n,z) + \frac{2\alpha_n}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n^2 M \\
&\leq \left[1 - \frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}\right] d^2(x_n,z) \\
&\quad + \alpha_n\left[\frac{2}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n M\right]. \qquad (3.19)
\end{aligned}
$$

Set $\gamma_n = \frac{1-2\theta\alpha_n-(1-2\alpha_n)}{1-\theta\alpha_n}$, $\delta_n = \alpha_n\left[\frac{2}{1-\theta\alpha_n}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{n+1}z}\rangle + \alpha_n M\right]$. Now it follows from (3.18), (3.19) and Lemma 2.7 that $\{x_n\}$ converges strongly to $z$.

**Case 2.** Suppose that $\{d(x_n,p)\}$ is monotonically non-decreasing. There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $d(x_{n_j},z) < d(x_{n_j+1},z)$ for all $j\in\mathbb{N}$. Then by Lemma 2.16, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$.

$$
d^2(x_{m_k},z) \leq d^2(x_{m_k+1},z) \quad \text{and} \quad d^2(x_k,z)\leq d^2(x_{m_k+1},z), \qquad (3.20)
$$

for all $k\in\mathbb{N}$. Therefore,

$$
\begin{aligned}
0 &\leq \liminf_{k\to\infty}[d(x_{m_k+1},z) - d(x_{m_k,z})] \\
&\leq \limsup_{k\to\infty}[d(x_{m_k+1},z) - d(x_{m_k},z)] \\
&\leq \limsup_{k\to\infty}[\alpha_{m_k} d(h(x_{m_k}),z) + (1-\alpha_{m_k})d(v_{m_k},z) - d(x_{m_k},z)] \\
&\leq \limsup_{k\to\infty}[\alpha_{m_k} d(h(x_{m_k}),z) + (1-\alpha_{m_k})d(x_{m_k},z) - d(x_{m_k},z)] \\
&= \limsup_{k\to\infty}[\alpha_{m_k}(d(h(x_{m_k}),z) - d(x_{m_k},z))] = 0.
\end{aligned}
$$

This implies that

$$
\lim_{k\to\infty}[d(x_{m_k+1},z) - d(x_{m_k},z)] = 0. \qquad (3.21)
$$

By an argument as in **Case 1**, we get

$$
\limsup_{k\to\infty}\langle \overrightarrow{h(z)z}, \overrightarrow{x_{m_k+1}z}\rangle \leq 0 \qquad (3.22)
$$

and

$$
d^2(x_{m_k+1},z) \leq (1-\gamma_{m_k})d^2(x_{m_k},z) + \gamma_{m_k}\delta_{m_k}.
$$

Since $d^2(x_{m_k},z)\leq d^2(x_{m_k+1},z)$ we get

$$
\gamma_{m_k} d^2(x_{m_k},z) \leq d^2(x_{m_k},z) - d^2(x_{m_k+1},z) + \gamma_{m_k}\delta_{m_k} \leq \gamma_{m_k}\delta_{m_k}. \qquad (3.23)
$$

Thus, from (3.20), we get

$$
\lim_{k\to\infty} d^2(x_{m_k},z) = 0. \qquad (3.24)
$$

It follows from (3.20), (3.22) and (3.24) the $\lim_{k\to\infty} d^2(x_k, z) = 0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges strongly to $z \in \Upsilon$. $\square$

**Lemma 3.2.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$ and $f_j : C \times C \to \mathbb{R}$, $j = 1, 2, \dots, m$, be a finite family of monotone bifunctions such that (A1)-(A4) are satisfied. Then for $r > 0$, we have $F\left(\bigcap_{j=1}^m J_r^{f_m}\right) = \cap_{i=1}^m (J_r^{f_m})$, where*

$$\bigcap_{j=1}^{m} J_r^{f_j} = J_r^{f_m} \circ J_r^{f_{m-1}} \circ \dots \circ J_r^{f_2} \circ J_r^{f_1}.$$

The proof of Lemma 3.2, follows immediately from the proof of Theorem 3.1 in [55].

**Theorem 3.2.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $f_j : C \times C \to \mathbb{R}$, $j = 1, 2, \dots, m$, be monotone and upper semicontinuous bifunctions such that conditions (A1)-(A4) are satisfied, $C \subset D(J_r^f)$ for $r > 0$ and $T_i : C \to X$, $i = 1, 2, \dots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0, 1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Gamma := F(S) \cap EP(f_j, C) \cap \cap_{i=1}^N VI(C, T_i)$ is nonempty and $\{x_n\}$ is the sequence generated by an arbitrary $x_1 \in X$ as:*

$$\begin{cases} u_n = \Pi_{j=1}^m J_{r_n}^{f_j} x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n)[\beta_n y_n \oplus (1 - \beta_n) S_\lambda y_n], \quad n \geq 1, \end{cases} \tag{3.25}$$

*where $S_\lambda x = \lambda x \oplus (1 - \lambda) S x$ is $\Delta$-demiclosed, $T_{\mu_i} x = (1 - \mu_i) x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$, for each $i = 1, 2, \dots, N$, and $\bigcap_{j=1}^m J_r^{f_j} = J_r^{f_m} \circ J_r^{f_{m-1}} \circ \dots \circ J_r^{f_2} \circ J_r^{f_1}$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, $\{\beta_i\} \subset (0, 1)$ and $r_n \in (0, \infty)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma h(p)$.*

*Proof.* Follows immediately from Theorem 3.1 and Lemma 3.2. $\square$

## 4. Application to Minimization Problems

In this section, we give an application of our results to solve Minimization Problems. Let $X$ be a Hadamard space and $f : X \to (-\infty, \infty]$ be a proper and convex function. The problems in optimization require to find $x \in X$ such that

$$f(x) = \arg\min_{y \in X} g(y).$$

So $\arg\min_{y\in X} g(y)$ denotes the set of minimizers of $g$.

Let $v : X \to \mathbb{R}$ be a proper convex and lower semicontinuous function. Consider the bifunction $f_v : C \times C \to \mathbb{R}$ defined by

$$f_v(x, y) = v(y) - v(x), \quad \text{for all } x, y \in C.$$

Then, $f_v$ is monotone and upper semi continuous (see [3]). Moreover, $EP(f_v, C) = \arg\min_C v$, $J^{f_v} = prox^v$ and $D(prox^v) = X$ (see [3]), where $prox^v : X \to X$ is given by

$$prox^v(x) := \arg\min_{x \in X} \left[ v(y) + \frac{1}{2} d^2(y, x) \right], \quad \text{for all } x \in X.$$

Now we consider the following minimization and fixed point problems:

(4.1)

$$\text{find } x \in F(S) \cap F(\Psi_\mu) \text{ such that } v(x) \le v(y), \quad \text{for all } y \in C,\ i = 1, 2, \ldots, m,$$

where $S$ is a demimetric mapping and $\Psi_\mu$ is as defined in Lemma 3.1.

Let us denote the solution set of problem (4.1) by $\Omega$.

**Theorem 4.1.** *Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$, $v_j : X \to \mathbb{R}$, $j = 1, 2, \ldots, m$, be proper convex lower semicontinuous functions and $T_i : C \to X$, $i = 1, 2, \ldots, N$, be a finite family of $\alpha_i$-inverse strongly monotone mappings. Let $h$ be a contraction of $C$ into itself with coefficient $\theta \in (0, 1)$ and $S : C \to C$ be a $k$-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Omega$ is nonempty and $\{x_n\}$ is the sequence generated by an arbitrary $x_1 \in X$ as*

$$\begin{cases} u_n = \Pi_{j=1}^m prox_{r_n}^{v_i} x_n, \\ y_n = \Psi_\mu u_n := \bigoplus_{i=1}^N \beta_i P_C T_{\mu_i} u_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n)[\beta_n y_n \oplus (1 - \beta_n) S_\lambda y_n], \quad n \ge 1, \end{cases} \tag{4.2}$$

*where $S_\lambda x = \lambda x \oplus (1 - \lambda) S x$ is $\Delta$-demiclosed and $T_{\mu_i} x = (1 - \mu_i) x \oplus \mu_i T_i x$, $0 < \mu_i < 2\alpha_i$, for each $i = 1, 2, \ldots, N$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, $\{\beta_i\} \subset (0, 1)$ and $r_n \in (0, \infty)$ satisfy the following conditions:*

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{i=1}^N \beta_i = 1$.

*Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega h(p)$.*

*Proof.* Set $J_{r_n}^{f_i} = prox_{r_n}^{v_i}$ in Algorithm 3.25 and apply Theorem 3.2 to approximate solutions of problem (4.1). □

*Remark* 4.1. (i) If we replace $h(x_n)$ by "$u$" (for arbitrary $u$) in our Algorithm 3.1 and Algorithm 3.25 (which are viscosity type), then we get the Halpern-type algorithm and the conclusion of our theorems still hold. However, we use a viscosity-type algorithm instead of Halpern-type algorithm due to the fact that viscosity-type algorithms have higher rate of convergence than Halpern-type.

(ii) A characterization of metric projection goes as follows:

$$p = P_\Gamma h(p) \Leftrightarrow \langle \overrightarrow{ph(p)}, \overrightarrow{yp} \rangle \ge 0, \quad \text{for all } y \in C. \tag{4.3}$$

Therefore, one advantage of adopting Algorithm 3.1 for our convergence analysis, is that it also converges to the variational inequality (4.3) (see for example [22]).

(iii) In Theorem 1.1, $\Delta$-convergence to an element of $EP(f,C)$ was obtained while we obtained strong convergence result which is also a solution of some variational inequality problems. Hence, Theorem (3.1) provides genuine extension of Theorem 1.1.

(iv) Theorem 4.1 generalizes Theorem 10 of [52] and Theorem 3.1 of [44] from Hilbert space to CAT(0) spaces.

## 5. Numerical Example

*Example* 5.1. We give numerical in $(\mathbb{R}^2, \|\cdot\|_2)$ (where $\mathbb{R}^2$ is the Euclidean plane) to support our main result.

Let $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ defined by

$$\rho(x,y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}, \quad x, y \in \mathbb{R}^2.$$

Then $(\mathbb{R}^2, \rho)$ is an Hadamard space (see, for instance, [18, Example 5.2]) with geodesic joining $x$ to $y$ given by

$$(1-t)x \oplus ty = ((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2). \tag{5.1}$$

Now, define $\Phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Phi(x_1, x_2) = (100(x_2 - 2) - (x_1 - 2)^2)^2 + (x_1 - 3)^2.$$

Then, it follows from [18, Example 5.2] that $\Phi$ is a proper convex and lower semicontinuous function in $(\mathbb{R}^2,\ \rho)$ but not convex in the classical sense.

Let $S : \mathbb{R}^2 \to \mathbb{R}$ be defined by $S\bar{x} = S(x_1, x_2) = (-2x_1, 3x_1^2 + x_2)$. Then $S$ is 3-generalized demimetric mapping in the sense $\rho$ with $F(S) = (0,0)$, $\lambda = \frac{1}{4}$.

Let $X = \mathbb{R}^2$ and be an $R$-tree with radical metric $d_r$, where $d_r(x,y) = d(x,y)$ if $x$ and $y$ are situated on a Euclidean straight line passing through the origin and $d_r(x,y) = d(x,0) + d(y,0)$, otherwise. We put $p = (0,1)$, $q = (1,0)$ and $C = A \cup B \cup D$, where $A = \{(0,t) : t \in [2/3, 1]\}$, $B = \{(t,0) : t \in [2/3, 1]\}$, $D = \{(t,s) : t + s = 1, t \in (0,1)\}$ and defined $T : C \to C$ by

$$Tx := \begin{cases} q, & \text{if } x \in A, \\ p, & \text{if } x \in B, \\ x, & \text{if } x \in D. \end{cases} \tag{5.2}$$

Then, $T$ is $\frac{1}{4}-$ inverse strongly monotone in $(X, d_r)$ but not inverse strongly monotone in the classical sense.

In what follows, we choose $r_n = \frac{1}{5}$, $\beta_i = \frac{1}{N}$, $\mu_i = 0.035$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{3n}{5n+2}$ for $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$. We study the behaviour of the sequence generated by Algorithm 3.1 for following initial values with $N = 10$.

Case I: $x_0 = (-2, -7)'$,

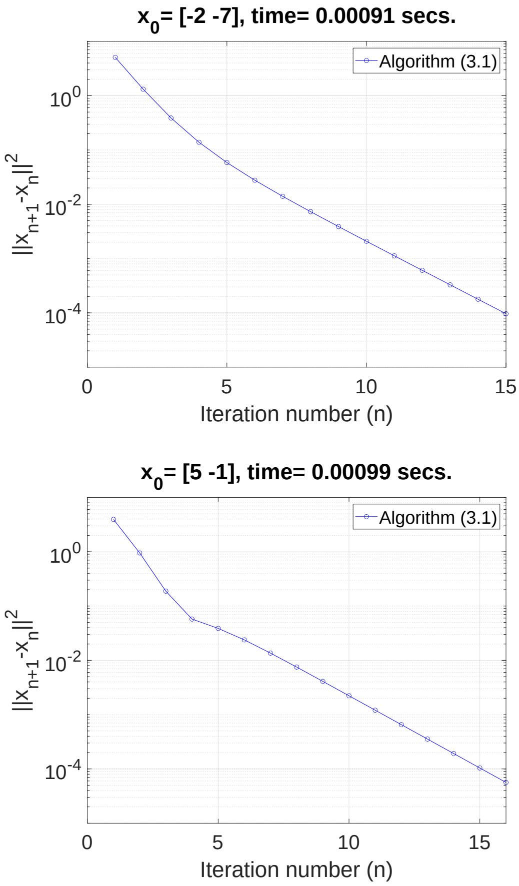


FIGURE 1. Example 5.1: Case I – Case II.

Case II: $x_0 = (5, -1)'$,
Case III: $x_0 = (3, 6)'$,
Case IV: $x_0 = (-4, 1)'$.

We also used $\|x_{n+1} - x_n\|^2 < 10^{-4}$ as stopping criterion and plot the graphs of error $||x_{n+1} - x_n||^2$ against number of iteration in each case. The computation results are shown in Figure 1–2. The numerical results show that the change in the initial values does not have significant effects on the number of iteration and CPU time taken for computation by Algorithm 3.1.

## 6. Conclusion

In this paper, we investigate a priori on the resolvent operator for a given bifunction, demimetric mapping and a finite family of inverse strongly monotone mappings. Main results here are that the resolvent operator here is single-valued and firmly nonexpansive. We then define proximal viscosity algorithm by iterating the resolvent of different bifurcating parameters. Strong convergence of the proposed algorithm to

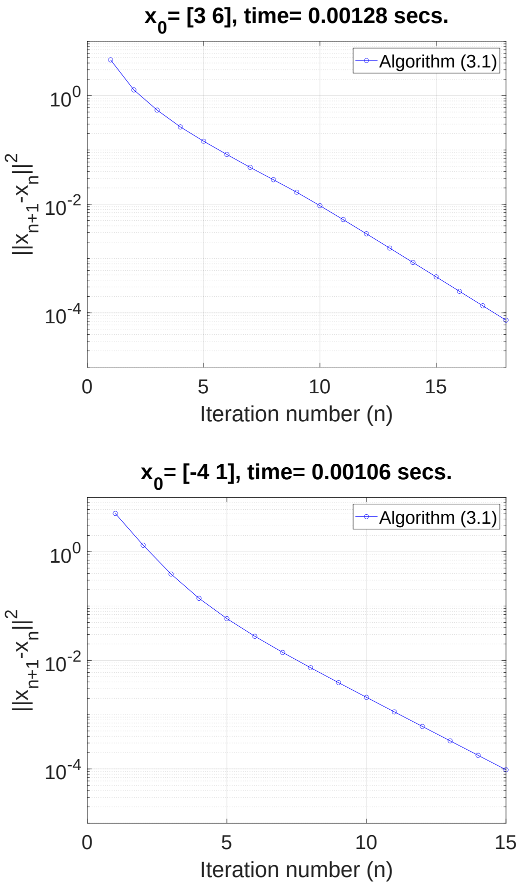


Figure 2. Example 5.1: Case III – Case IV

a common solution of variational inequality problem, fixed point problem and equilibrium problem is established in Hadamard spaces. Some applications and numerical example were also given.

## References


[1] R. L. Adler, J. P. Dedieu, J. Y. Margulies, M. Martens and M. Shub, *Newtons method on Reimannan manifolds and a geometric model for human spine*, IMA J. Numer. Anal. **22** (2002), 359–390. https://doi.org/10.1093/imanum/22.3.359

[2] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, *Modified inertia subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems*, Optimization (2020). https://doi.org/10.1080/02331934.2020.1723586

[3] S. Alizadeh, H. Dehghan and F. Moradlou, *$\Delta$-convergence theorem for inverse strongly monotone mappings in $CAT(0)$ spaces*, Fixed Point Theory **19**(1) (2018), 45–56.

[4] K. O. Aremu, C. Izuchukwu, G. C. Ugwunnadi and O. T. Mewomo, *On proximal point algorithm and demimetric mappings in $CAT(0)$ spaces*, Demostr. Math. **51** (2018), 277–294. https://doi.org/10.1515/dema-2018-0022

[5] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701. https://doi.org/10.1007/s11856-012-0091-3

[6] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl. **90** (1996), 31–43. https://doi.org/10.1007/BF02192244

[7] F. Bruhat and J. Tits, *Groupes réductifs sur un corp local, I. Donneés Radicielles Valuées*, Institut des Hautes Études Scientifiques **41** (1972), 5–251.

[8] C. E. Chidume and C. O. Chidume, *Iterative approximation of fixed points of nonexpansive mappings*, J. Math. Anal. Appl. **318** (2006), 288–295. https://doi.org/10.1016/j.jmaa.2005.05.023

[9] C. E. Chidume, A. U. Bello and P. Ndambomve, *Strong and $\Delta$-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces*, Abstr. Appl. Anal. **2014** (2014). https://doi.org/10.1155/2014/805168

[10] P. Cholamjiak, *The modified proximal point algorithm in CAT(0) spaces*, Optim. Letters **9** (2015), 1401–1410. https://doi.org/10.1007/s11590-014-0841-8

[11] V. Colao, G. López, G. Marino and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds*, J. Math. Anal. Appl. **388** (2012), 61–77. https://doi.org/10.1016/j.jmaa.2011.11.001

[12] P. L. Combetes and S. A. Histoage, *Equilibrium programmer in Hilbert space*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[13] H. Dehghan and J. Rooin, *A characterization of metric projection in CAT(0) spaces*, International Conference on Functional Equation. Geometric Functions and Applications (ICFGA 2012), 10–12th May 2012, Payame Noor University, Tabriz, 2012, 41–43.

[14] H. Dehghan, C. Izuchukwu, O. T. Mewomo, D. A. Taba and G. C. Ugwunnadi, *Iterative algorithm for a family of monotone inclusion problems in CAT(0) spaces*, Quest. Mat. **24** (2019), 1–24. https://doi.org/10.2989/16073606.2019.1593255

[15] S. Dhompongesa, and B. Panyanak, *On $\Delta$-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (2008), 2572–2579. https://doi.org/10.1016/j.camwa.2008.05.036

[16] S. Dhompongsa, W. A. Kirk and B. Sims, *Fixed points of uniformly Lipschizian mappings*, Nonlinear Anal. **65** (2006), 762–772. https://doi.org/10.1016/j.na.2005.09.044

[17] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 35–45.

[18] G. Z. Eskandani and M. Raeisi, *On the zero pointproblem of monotone operators in Hadamard spaces*, Numer. Algorithms **80**(4) (2018), 1155–1179. https://doi.org/10.1007/s11075-018-0521-3

[19] O. P. Ferreira and P. R. Oliveira, *Proximal point algorithm on Riemannian manifolds*, Optimization **51** (2002), 257–270. https://doi.org/10.1080/02331930290019413

[20] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[21] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhauser, Boston, 1999.

[22] C. Izuchukwu, A. A. Mebawondu, K. O. Aremu, H. A. Abass and O. T. Mewomo, *Viscosity iterative techniques for approximating a common zero of monotone operators in Hadamard space*, Rend. Circ. Mat. Palermo (2) **2** (2019), 1–21. https://doi.org/10.1007/s12215-019-00415-2

[23] C. Izuchukwu, K. O. Aremu, A. A. Mebawondu and O. T. Mewomo, *A viscosity iterative technique for equilibrium and fixed point problems in a Hadamard space*, Appl. Gen Topol. **20**(1) (2019), 193–210. https://doi.org/10.4995/agt.2019.10635

[24] A. N. Iusem, G. Kassay and W. Sosa, *On certain conditions for the existence of solutions of equilibrium problems*, Math. Program. **116** (2009), 259–273. https://doi.org/10.1007/s10107-007-0125-5

[25] S. Jain and L. B. Jain, *On Banach contraction principle in a cone metric space*, J. Nonlinear Sci. Appl. **5** (2012), 252–258.

[26] L. O. Jolaoso, T. O. Alakoya, A. Taiwo and O. T. Mewomo, *Inertial extragradient method via viscosity approximation approach for solving equilibrium problem in Hilbert space*, Optimization (2020). https://doi.org/10.1080/02331934.2020.1716752

[27] L. O. Jolaoso, A. Taiwo, T. O. Alakoya and O. T. Mewomo, *Strong convergence theorem for solving pseudo-monotone variational inequality using projection method in a reflexive Banach spaces*, J. Optim. Theory Appl. **185**(3) (2020), 744–766. https://doi.org/10.1007/s10957-020-01672-3

[28] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Lectures Math., ETH ZNurich, Birkhauser, Basel, 1997.

[29] M. A. Kakavandi, *Duality and subdifferential for convex functions in complete CAT(0) metric spaces*, Nonlinear Anal. **73** (2010), 3450–3455. https://doi.org/10.1016/j.na.2010.07.033

[30] M. A. Khamsi and A. R. Khan, *Inequalities in metric spaces with applications*, Nonlinear Anal. **74**(12) (2011), 4036–4045. https://doi.org/10.1016/j.na.2011.03.034

[31] K. S. Kim, *Some convergence theorems for contractive type mappings in CAT(0) spaces*, Abstr. Appl. Anal. (2013), Article ID 381715, 9 pages. https://doi.org/10.1155/2013/381715

[32] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math. **5** (2001), 387–404.

[33] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696. https://doi.org/10.1016/j.na.2007.04.011

[34] W. A. Kirk, *Geometry and fixed point theory II*, International Conference on Fixed Point Theory and Applications, Yokohama Publishers, Yokohama, 2004, 113–142.

[35] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. **60** (1976) 179–182.

[36] P. Kumam and P. Chaipuya, *Equilibrium problems and proximal algorithms in Hadamard spaces*, J. Nonlinear Anal. Optim. **8**(2) (2017), 155–172.

[37] B. Martinet, *Régularisation d'Inéquations variationnelles par approximations successives*, Modélisation Mathématique et Analyse Numérique **4** (1970), 154–158.

[38] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Analysis **16** (2008), 899–912. https://doi.org/10.1007/s11228-008-0102-z

[39] G. Marino and H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **318** (2006), 43–52. https://doi.org/10.1016/j.jmaa.2005.05.028

[40] J. Merryfield, *Generalization of the Banach contraction principle*, J. Math. Anal. Appl. **273** (2002), 112–120.

[41] M. A. Noor and K. I. Noor, *Some algorithms for equilibrium problems on Hadamard manifolds*, J. Inequal. Appl. **230** (2012), 8 pages. https://doi.org/10.1186/1029-242X-2012-230

[42] U. A. Osisiogu, F. L. Adum and T. E. Efor, *Strong convergence results for variational inequality problem in CAT(0) spaces*, Advances in Nonlinear Variational Inequalities **23**(1) (2020), 84–101.

[43] C. C. Okeke and C. Izuchukwu, *A strong convergence theorem for monotone inclusion and minimization problems in complete CAT(0) spaces*, Optim. Methods Softw. **34** (2018), 1168–1183. https://doi.org/10.1080/10556788.2018.1472259

[44] C. C. Okeke, C. Izuchukwu and O. T. Mewomo, *Strong convergence results from convex minimization and monotone variational inclusion problem in Hilbert space*, Rend. Circ. Mat. Palermo (2) **69** (2020), 675–693. https://doi.org/10.1007/s12215-019-00427-y

[45] R. S. Palais, *A simple proof of the Banach contraction principle*, J. Fixed Point Theory Appl. **2** (2007), 221–223.

[46] E. A. Papa Quiroz and P. R. Oliveira, *Proximal point methods for quasiconvex and convex functions with Bregman distance on Hadamard manifolds*, J. Convex Anal. **16** (2009), 49–69.

[47] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[48] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), 537–558. https://doi.org/10.1016/0362-546X(90)90058-O

[49] G. Stampacchia, *Variational Inequalities*, in: *Theory and Application of Monotone Operators*, Proceedings of the NATO Advanced Study Institute, Venice, Edizioni Odersi, Gubbio, Italy, 1968, 102–192.

[50] S. T. Smith, *Optimization techniques on Riemannian manifolds*, in: *Field Institute Communications*, RI: Amer. Math. Soc. **3**, Providence, 1994, 113–146.

[51] J. Tang, *Viscosity approximation methods for a family of nonexpansive mappings in $CAT(0)$ spaces*, Abstr. Appl. Anal. (2014), Article ID 389804, 9 pages. https://doi.org/10.1155/2014/389804

[52] M. Tian and M. Li, *A general iterative method for constrained convex minimization problems in Hilbert space*, WSEAS Transactions on Mathematics **13** (2014), 271–281.

[53] C. Udriste, *Convex Functions and Optimization Methods Riemannian Manifolds*, Mathematics and its Applications **297**, Springer, Dordrecht, 1994.

[54] G. C. Ugwunnadi, A. R. Khan and M. Abbas, *A hybrid proximal point algorithm for finding minimizers and fixed points in $CAT(0)$ spaces*, J. Fixed Point Appl. **20**(82) (2018). https://doi.org/10.1007/s11784-018-0555-0

[55] G. C. Ugwunnadi, C. Izuchukwu and O. T. Mewomo, *Strong convergence theorem for monotone inclusion problem in $CAT(0)$ spaces*, Afr. Mat. **30** (2019), 151–161. https://doi.org/10.1007/s13370-018-0633-x

[56] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332

[1]Department of Mathematics,
University of Eswatini,
Private Bag 4,
Kwaluseni, Eswatini
*Email address*: ugwunnadi4u@yahoo.com
*Email address*: chibueze.okeke87@yahoo.com

[2]Department of mathematics and Statistics,
King Fahd University of Petroleum and Minerals Dhahran,
31261, Saudi Arabia
*Email address*: arahim@kfupm.edu.sa

[3]Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Sciences University,
P.O. Box 94 Medunsa 0204, Pretoria, South Africa
*Email address*: jollatanu@yahoo.co.uk