ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 47, Number 7, 2023

Special Issue: Analysis, Topology and Applications 2022, Vrnjačka Banja, Serbia, June 29–July 02, 2022

Guest Editors: Dragan Đurčić, Ljubiša Kočinac

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić . - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Niš : Grafika Galeb). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2307

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
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Email: `krag_j_math@kg.ac.rs`
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Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: Grafika Galeb, Niš, Serbia
From 2021 the journal appears in one volume and six issues per annum.

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Volume 47(7) (2023), Pages 995–1010.

SOME PROPERTIES OF RANGE OPERATORS ON LCA GROUPS

RUCHIKA VERMA[1] AND KUMARI TEENA[2]

Abstract. In this paper, we study the structure of shift preserving operators acting on shift-invariant spaces in $L^2(G)$, where G is a locally compact Abelian group. We generalize some results related to shift-preserving operator and its associated range operator from $L^2(\mathbb{R}^d)$ to $L^2(G)$. We investigate the matrix structure of range operator $R(\xi)$ on range function J associated to shift-invariant space V, in the case of a locally compact Abelian group G. We also focus on some properties like as normal and unitary operator for range operator on $L^2(G)$. We show that shift preserving operator U is invertible if and only if fiber of corresponding range operator R is invertible and investigate the measurability of inverse $R^{-1}(\xi)$ of range operator on $L^2(G)$.

1. Introduction

Many authors such as Aldroubi, Benedetto, Bownik, de Boor, De Vore, Li, Ron, Rzeszotnik, Shen, Weiss and Wilson have studied shift-invariant subspaces of $L^2(\mathbb{R}^n)$ cf. [2, 3, 5, 8–10, 24–27]. The theory of shift-invariant spaces plays an important role in many areas such as theory of wavelets, Gabor systems, multi-resolution analysis, frames, approximation theory etc. Shift-invariant spaces of $L^2(\mathbb{R}^n)$ are the spaces which are invariant under integer translations. After that, the structural properties of shift-invariant spaces are studied by R. A. Kamyabi Gol and R. Raisi Tousi [18–22], by C. Cabrelli and V. Paternostro [12], and by M. Bownik and K. A. Ross [4], in locally compact Abelian groups. The locally compact Abelian group framework has several advantages because it has a valid theory for the classical groups such as $\mathbb{Z}^d$, $\mathbb{T}^d$ and $\mathbb{Z}_n$ (see [13–17]).

Key words and phrases. Shift-invariant space, range function, range operator, locally compact Abelian group, shift preserving operator, frame, Parseval frame, normal operator, unitary operator.
2020 *Mathematics Subject Classification.* Primary: 47A15, 22B99. Secondary: 42C15, 43A25.
DOI 10.46793/KgJMat2307.0995V
Received: September 29, 2022.
Accepted: November 08, 2022.

Let G be locally compact Abelian (which will be abbreviates as "LCA") group with a Haar measure m_G. We shall use the constructions and notations from [18–21], associated to LCA groups. The dual group of G is denoted by $\widehat{G}$. Let L be a uniform lattice in G. A subspace V of $L^2(G)$ is shift-invariant if it is invariant under translation operations, that is $T_k V \subseteq V$, where $T_k f(x) = f(k^{-1}x)$ for all $x \in G$, $f \in V$ and $k \in L$. For any function $f \in L^1(G)$, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_G f(x)\bar{\xi}(x) dm_G(x),$$

where $\xi \in \widehat{G}$ is a character on G. If L is a uniform lattice L in G, then a fundamental domain is defined by a measurable set S_L in G such that every element $x \in G$ can be uniquely represented as $x = kz$, where $k \in L$ and $z \in S_L$. There always exists a fundamental domain for a uniform lattice in a LCA group, see [23]. Let $\Phi \subseteq L^2(G)$ be a countable set of functions, then

$$S(\Phi) = \overline{\text{Span}}\{T_k\phi : \phi \in \Phi, k \in L\}$$

is a shift-invariant space generated by Φ. If the set of generators Φ is finite, then the space $S(\Phi)$ is called finitely generated shift-invariant space. A range function J [3,8,9,19] is associated to each shift-invariant space V, which represents that the space V as a measurable field of closed subspaces of $\ell^2(L^\perp)$, where $L^\perp$ is annihilator of L. These subspaces are called the fiber spaces. There is an isometric isomorphism T (see [19]) between shift-invariant space V and its associated range function J. A bounded linear operator $U : L^2(G) \to L^2(G)$ is called a shift-preserving operator with respect to uniform lattice L, if $UT_k = T_k U$ for all $k \in L$. Every shift-preserving operator U has a corresponding range operator $R(\xi)$.

Our paper is organised as follows. Section 2 includes some background results on LCA groups. In Section 3, we prove our main results. The eigenvalues of a shift-preserving operator are named as s-eigenvalues and eigenspaces as s-eigenspaces. We show that when $\dim J(\xi) < +\infty$, then, the operator $R(\xi)$ can be represented by a matrix with measurable entries for a.e. (almost every) $\xi \in S_{L^\perp}$. We see that the invertibility of shift preserving operator U can be deduced from invertibility of its fibers $R(\xi)$ and $\xi \mapsto R^{-1}(\xi)$ is measurable for a.e. $\xi \in S_{L^\perp}$, where $R^{-1}(\xi)$ is the inverse of range operator associated to shift-preserving operator U. We have taken the ideas of our main results from the paper [1]. We proved similar results for locally compact abelian groups.

2. Background on LCA Groups

First we provide the background and notations on the LCA group, which we will use later in our main results.

Let G be an LCA group and $\widehat{G}$ is dual of G, elements of $\widehat{G}$ are usually denoted by ξ (called characters on G). Throughout this paper we assume that G is a second countable LCA group. A subgroup L of LCA group G is called uniform lattice if it is discrete and co-compact (i.e., G/L is compact). Let $L^{\perp}$ is annihilator of L in $\widehat{G}$, where

$$L^{\perp}=\{\xi\in\widehat{G}:\xi(L)=\{1\}\}.$$

Then, "the identities $L^{\perp}=\widehat{G/L}$ and $\widehat{G}/L^{\perp}$, together with the fact that a locally compact Abelian group is compact if and only if its dual group is discrete, imply that the subgroup $L^{\perp}$ is a uniform lattice in $\widehat{G}$“, see [18].

Definition 2.1 ([18])**.** Let G be a LCA group and L be a uniform lattice in G. A closed subspace $V\subseteq L^2(G)$ is called a shift-invariant space if $f\in V$ implies that $T_kf\in V$ for any $k\in L$, where T_k is translation operator defined by $T_kf(x)=f(k^{-1}x)$ for all $x\in G$.

Let $\phi\in L^2(G)$, then $V_\phi=\overline{\operatorname{Span}}\{T_k\phi:k\in L\}$ is called the principal shift-invariant space generated by ϕ.

The following proposition (cf. [18, Proposition 2.2]) characterizes the elements in a principal shift-invariant subspace V_ϕ of $L^2(G)$ in terms of their Fourier transforms.

Proposition 2.1 ([18])**.** *Let $\phi\in L^2(G)$, then $f\in V_\phi$ if and only if $\widehat{f}(\xi)=r(\xi)\widehat{\phi}(\xi)$ for some $r\in L^2\left(\widehat{L},\omega_\phi\right)$, which is given by $r(\xi)=\sum_{i=1}^{n}a_i\overline{\xi}(k_i)$, $a_i\in\mathbb{C}$, where $L^2\left(\widehat{L},\omega_\phi\right)$ is the space of all functions $r:\widehat{L}\to\mathbb{C}$ satisfying*

$$\int_{\widehat{L}}|r(\xi)|^2\omega_\phi(\xi)d\xi<+\infty$$

and $\omega_\phi(\xi)=\sum_{\eta\in L^{\perp}}\left|\hat{\phi}(\xi\eta)\right|^2$.

The following proposition gives necessary and sufficient condition for the shifts of function ϕ to be an orthonormal system in space $L^2(G)$.

Proposition 2.2 ([18])**.** *Suppose $\phi\in L^2(G)$. Then $\{T_k\phi:k\in L\}$ is an orthonormal system in $L^2(G)$ if and only if $\omega_\phi=1$ a.e. on $\widehat{G}$.*

Remark 2.1 ([18]). If V_ϕ is a principal shift-invariant space, and ω_ϕ is as in Proposition 2.1, then the spectrum of $V_\phi=\operatorname{supp}(\omega_\phi)$, where $\operatorname{supp}(\omega_\phi)$ denote the support of ω_ϕ. That is, $\Omega_\phi=\{\xi\in\widehat{G}:\omega_\phi(\xi)\neq 0\}$. Also, when the set $\{T_k\phi:k\in L\}$ is an orthonormal system for $L^2(G)$, then $\Omega_\phi=\widehat{G}$.

Definition 2.2 ([18])**.** A subset X of a Hilbert space H is called a frame for H if there exist two numbers $0<A\leq B<+\infty$ which satisfy the following inequality

$$A\|h\|^2\leq\sum_{\eta_0\in X}|\langle h,\eta_0\rangle|^2\leq B\|h\|^2,\quad h\in H.$$

If $A=B=1$, then X is called Parseval frame.

The next theorem (cf. [18, Theorem 3.6]) shows that for every principal shift-invariant space V_ϕ, shifts of its generator ϕ form a Parseval frame.

Theorem 2.1 ([18])**.** *Let $\phi \in L^2(G)$. Then the set $\{T_k\phi : k \in L\}$ forms a Parseval frame for space V_ϕ if and only if*

$$\omega_\phi = \chi_{\Omega_\phi} \quad a.e.\ on\ \widehat{G},$$

and in this case ϕ is said to be a Parseval frame generator for the space V_ϕ.

The following proposition shows that the spaces $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ are isometrically isomorphic to each other.

Proposition 2.3 ([19])**.** *The mapping $T : L^2(G) \to L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ defined by $Tf(\xi) = \left(\widehat{f}(\xi\eta)\right)_{\eta\in L^\perp}$ is an isometric isomorphism between $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$, where $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ is the space of square integrable functions $f : S_{L^\perp} \to \ell^2(L^\perp)$ with inner product defined by*

$$\langle f, g\rangle = \int_{S_{L^\perp}} \langle f(\xi), g(\xi)\rangle_{\ell^2 \in L^\perp} d(\xi),$$

and $S_{L^\perp}$ is fundamental domain of $L^\perp$ in space $\widehat{G}$ with measure $d\xi$ on it.

Definition 2.3 ([19])**.** Let G be a LCA group and L be an uniform lattice in G. A range function associated to a shift-invariant space V is a mapping

$$J : S_{L^\perp} \to \{\text{closed subspaces of } \ell^2(L^\perp)\}.$$

"The range function J is called measurable if associated orthogonal projections $P(\xi) : \ell^2(L^\perp) \to J(\xi)$ are measurable that is $\xi \mapsto \langle P(\xi)a, b\rangle$ is measurable for each $a, b \in \ell^2(L^\perp)$" [19].

The shift-invariant space V can be defined in terms of a measurable range function J as follows

$$V = \{f \in L^2(G) : Tf(\xi) \in J(\xi) \text{ for a.e. } \xi \in S_{L^\perp}\}.$$

There is a one to one correspondence between V and J under the convention that the range functions are identified if they are equal a.e.

The following theorem (cf. [19, Theorem 4.1]) allows to reduce the problem of checking whether the shifts of a generator ϕ form a frame in a subspace of the space $L^2(G)$, to analyzing the elements in the subspaces of $\ell^2(L^\perp)$, which are parameterized by the base space $S_{L^\perp}$.

Theorem 2.2 ([19])**.** *Let G be a second countable LCA group, L be a uniform lattice in group G, $S_{L^\perp}$ be a fundamental domain for annihilator $L^\perp$ in the dual group $\widehat{G}$ of G, $\Phi \subseteq L^2(G)$ is a countable set and T is the mapping defined in Proposition 2.3. Then the set $\{T_k\phi : \phi \in \Phi, k \in L\}$ forms a frame for $S(\phi)$ with frame bounds A and B if and only if the set $\{T\phi(\xi) : \phi \in \Phi\}$ forms a frame for $J(\xi)$ with same frame bounds A and B for a.e. $\xi \in S_{L^\perp}$ (same result holds for fundamental frame and Riesz family).*

Definition 2.4 ([19])**.** A bounded linear operator $U : L^2(G) \to L^2(G)$ is called shift preserving if $UT_k = T_kU$ for all $k \in L$, where $T_kf(x) = f(k^{-1}x)$ for all $x \in G$.

Definition 2.5 ([21])**.** A range operator R on a range functon J is a mapping defined as

$$R : S_{L^\perp} \to \{\text{bounded linear operator on closed subspaces of } \ell^2(L^\perp)\},$$

that is, the domain of $R(\xi)$ is $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Range operator R is said to be measurable if the mapping $\xi \mapsto \langle R(\xi)P(\xi)a, b\rangle$ is measurable for all $a, b \in l^2(L^\perp)$.

The following theorem (cf. [21, Theorem 6.1]) gives a characterization of shift preserving operators in terms of its range operators.

Theorem 2.3 ([21])**.** *Suppose V is a shift-invariant space in $L^2(G)$ and J is its associated range function. Then for every shift preserving operator $U : V \to L^2(G)$, there exists a measurable range operator R on range function J such that*

$$(T \circ U)\, f(\xi) = R(\xi)\,(Tf(\xi))\,, \tag{2.1}$$

for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$, where T is an isometric isomorphism between the spaces $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$. Conversely, for a measurable range operator R on J satisfying the condition $\operatorname{ess\,sup}_{\xi\in S_{L^\perp}} \|R(\xi)\| < +\infty$, there exists a bounded shift preserving operator $U : V \to L^2(G)$, such that equation (2.1) holds. There is a one-to-one correspondence between U and R, under the convention that the range operators are identified if they are equal a.e.

The following proposition (cf. [20, Proposition 2.2]) characterized all Parseval frame generators of shift-invariant space $S(\phi)$ as follows.

Proposition 2.4 ([20])**.** *Let $\phi \in L^2(G)$. Then ϕ is a Parseval frame generator of the space $S(\phi)$ if and only if $\|T\phi(\xi)\|^2_{l^2(L^\perp)} = \sum_{\eta\in L^\perp} |\widehat{\phi}(\xi\eta)|^2 = \chi_{\Omega_\phi}(\xi)$ for a.e. $\xi \in S_{L^\perp}$, where $\Omega_\phi = \{\xi \in S_{L^\perp} : T\phi(\xi) \neq 0\}$.*

Remark 2.2 ([20]). $TT_k\phi(\xi) = \widehat{T_k\phi}(\xi\eta) = \overline{\xi}(k)\widehat{\phi}(\xi\eta) = M_k(\xi)T\phi(\xi)$, where $\eta \in L^\perp$ and $M_k \in L^2(S_{L^\perp})$ is defined by $M_k(\xi) = \overline{\xi}(k)$, $\xi \in S_{L^\perp}$.

The next theorem (cf. [12, Theorem 4.11]) shows that a shift-invariant space V of $L^2(G)$ can be orthogonally decomposed in the sum of principal shift-invariant spaces having some additional properties.

Theorem 2.4 ([12])**.** *Let V be a shift-invariant space in space $L^2(G)$. Then V can be decomposed as an orthonormal sum*

$$V = \bigoplus_{i\in\mathbb{N}} S(\phi_i),$$

where each function ϕ_i is a Parseval frame generator for space $S(\phi_i)$ and $\sigma\left(S(\phi_{i+1})\right) \subseteq \sigma\left(S(\phi_i)\right)$ for all $i \in \mathbb{N}$. Moreover, $\dim J_{S(\phi_i)}(\xi) = \|T\phi_i(\xi)\|$ for all $i \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}} \|T\phi_i(\xi)\| = \dim J_V(\xi)$ for a.e. $\xi \in S_{L^\perp}$.

The following proposition (cf. [20, Proposition 2.4]) shows that the orthogonality of $S(\phi_1)$ and $S(\phi_2)$ depends upon the relation of their generators ϕ_1 and ϕ_2 in $L^2(G)$.

Proposition 2.5 ([20])**.** *The shift-invariant spaces $S(\phi_1)$ and $S(\phi_2)$ are orthogonal if and only if the following condition holds*

$$\sum_{\eta \in L^{\perp}} \widehat{\phi}_1(\xi\eta)\overline{\widehat{\phi}_2}(\xi\eta) = 0 \quad a.e.\ \xi \in \widehat{G}.$$

3. Main Results

Some of the properties of a shift preserving operator U which are related to the properties of its fibers are already proved in [20] like the following theorem.

Theorem 3.1 ([20])**.** *Suppose V is a shift-invariant space in $L^2(G)$. Let J be range function associated to V and $U : V \to V$ be a shift preserving operator with its corresponding range operator R. Then the following statements hold.*

(a) *If U is compact, then $R(\xi)$ is compact for a.e. $\xi \in S_{L^{\perp}}$.*
(b) *The operator U is an isometry if and only if the operator $R(\xi)$ is an isometry for a.e. $\xi \in S_{L^{\perp}}$.*
(c) *The adjoint operator $U^* : V \to V$ of U is also a shift preserving operator and its corresponding range operator R^* is given by $R^*(\xi) = R(\xi)^*$ for a.e. $\xi \in S_{L^{\perp}}$.*
(d) *The operator U is self adjoint if and only if the operator $R(\xi)$ is self adjoint.*

In this section, we prove the similar results for normal and unitary operators. We begin with the following.

3.1. **s-eigenvalue and s-eigenspace** ([1])**.** Let $a = \{a_k\}_{k\in L} \in l^2(L)$, define $\Lambda_a : L^2(G) \to L^2(G)$ as

$$\Lambda_a = \sum_{k \in L} a_k T_k.$$

Then operator Λ_a is well defined and bounded if and only if the spectrum of the sequence a is bounded.

Remark 3.1. If $a \in l^2(L)$ is of bounded spectrum, that is $\widehat{a} \in L^\infty(S_{L^{\perp}})$, then

$$T(\Lambda_a f)(\xi) = T\left(\sum_{k\in L} a_k T_k f(\xi)\right) = \sum_{k\in L} a_k T T_k f(\xi) = \sum_{k \in L} a_k M_k(\xi) T f(\xi),$$

that is

$$T(\Lambda_a f)(\xi) = \sum_{k\in L} a_k \overline{\xi}(k) T f(\xi).$$

Thus, $T(\Lambda_a f)(\xi) = \widehat{a}(\xi) T f(\xi)$ for each $f \in L^2(G)$ and for a.e. $\xi \in S_{L^{\perp}}$.

We see that the operator $\Lambda_a : V \to V$ is a shift preserving operator if V is shift-invariant space, and its corresponding range operator is given by $R_a(\xi) = \widehat{a}(\xi) I$ for a.e. $\xi \in S_{L^{\perp}}$, where I is the identity operator on space $J(\xi)$.

If the set $E(\Phi) = \{T_k\phi : \phi \in \Phi, k \in L\}$ forms a frame for space V, where $\Phi \subseteq L^2(G)$ is a countable set, then every function $f \in V$ can be expressed as

$$f = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)T_k\phi_\alpha, \quad b_\alpha \in l^2(L^\perp),$$

where I is index set and $\Phi = (\phi_\alpha)_{\alpha\in I} \subseteq L^2(G)$. Then

$$\widehat{\Lambda_a f}(\xi) = \left(\sum_{k\in L}\widehat{a_k T_k f}(\xi)\right) = \sum_{k\in L} a_k\widehat{T_k f}(\xi) = \sum_{k\in L} a_k\overline{\xi}(k)\widehat{f}(\xi) = \widehat{a}(\xi)\widehat{f}(\xi),$$

implies

$$\widehat{\Lambda_a f}(\xi) = \widehat{a}(\xi)\sum_{\alpha\in I}\widehat{b_\alpha}(\xi)\widehat{\phi_\alpha}(\xi),$$

$$\because f(x) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)T_k\phi_\alpha(x) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)\phi_\alpha(k^{-1}x).$$

Then $\widehat{f}(\xi) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)\overline{\xi}(k_\alpha)\widehat{\phi}_\alpha(\xi) = \sum_{\alpha\in I}\widehat{b}_\alpha(\xi)\widehat{\phi}_\alpha(\xi)$. Therefore, we have

$$\Lambda_a f = \sum_{\alpha\in I}\sum_{k\in L}(a * b_\alpha)(k)T_k\phi_\alpha.$$

Definition 3.1 ([1])**.** Let V be a shift-invariant space and $U : V \to V$ be a bounded shift-preserving operator. Let $a \in l^2(L)$ be a sequence with bounded spectrum. Then, Λ_a is called an s-eigenvalue of operator U if the following condition holds

$$V_a = \ker(U - \Lambda_a) \neq \{0\},$$

and V_a is called the s-eigenspace corresponding to s-eigenvalue Λ_a. s-eigenspace V_a is always a shift-invariant subspace of V with respect to operator U, that is $UV_a \subseteq V_a$ and for each $f \in V_a$, we have $Uf = \Lambda_a f$.

The next result establishes a relation between the s-eigenvalues of shift preserving operator U and eigenvalues of corresponding range operator of U.

Proposition 3.1. *Let V be a shift-invariant space and J be its associated range function J with* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a bounded shift-preserving operator with corresponding range operator R and $a \in \ell^2(L^\perp)$ is a sequence with bounded spectrum. If Λ_a is an s-eigenvalue of operator U, then $\Lambda_a(\xi) = \widehat{a}(\xi)$ is an eigenvalue of corresponding range operator $R(\xi)$ for a.e. $\xi \in \Omega_{\phi_a} = \sigma(V_a)$, the spectrum of V_a.*

Proof. Let G be a second countable LCA group and let V be a shift-invariant subspace of $L^2(G)$. By using Theorem 2.4, there exists a family of functions $\{\phi_n\} \subseteq L^2(G)$ such that

$$V = \bigoplus_{n=1}^{+\infty} S(\phi_n),$$

where each ϕ_n is a Parseval frame generator of $S(\phi_n)$ for every $n \in \mathbb{N}$.

Since every principal shift-invariant space $S(\phi)$ has a Parseval frame generator ϕ, by using Proposition 2.4, a function ϕ is a Parseval frame generator of space $S(\phi)$ if and only if $\|T\phi(\xi)\|^2_{l^2(L^\perp)} = \sum_{\eta\in L^\perp} |\widehat{\phi}(\xi\eta)|^2 = \chi_{\Omega_\phi}(\xi)$ for a.e. $\xi \in S_{L^\perp}$, where $\Omega_\phi = \sigma(S(\phi)) = \{\xi \in S_{L^\perp} : T\phi(\xi) \neq 0\}$ is spectrum of $S(\phi)$. This implies that

$$\operatorname{supp}(\|T\phi(\xi)\|^2_{l^2(L^\perp)}) = \Omega_\phi.$$

Now, since s-eigenspace V_a is a shift-invariant subspace of V. So, using the above results for V_a, there exists Parseval frame generator $\phi_a \in V_a$ such that the following condition holds

$$\operatorname{supp}(\|T\phi_a(\xi)\|^2_{l^2(L^\perp)}) = \Omega_{\phi_a} = \sigma(V_a).$$

So,

$$T\phi_a(\xi) \neq 0, \quad \text{for a.e. } \xi \in \Omega_{\phi_a}. \tag{3.1}$$

Also, since $T(\Lambda_a\phi_a)(\xi) = \widehat{a}(\xi)T\phi_a(\xi)$, this implies that $T(U\phi_a)(\xi) = \lambda_a(\xi)T\phi_a(\xi)$, that is, $R(\xi)T\phi_a(\xi) = \lambda_a(\xi)T\phi_a(\xi)$, that is, $(R(\xi) - \lambda_a(\xi)I)T\phi_a(\xi) = 0$. So, $T\phi_a(\xi) \in \ker(R(\xi) - \lambda_a(\xi)I)$. Thus, using equation (3.1), for a.e. $\xi \in \Omega_{\phi_a}$

$$\ker(R(\xi) - \lambda_a(\xi)I) \neq \{0\}.$$

Hence, $\lambda_a(\xi) = \widehat{a}(\xi)$ is an eigenvalue of $R(\xi)$ for a.e. $\xi \in \Omega_{\phi_a} = \sigma(V_a)$. ☐

Remark 3.2. ([20, Remark 3.2]). Let V be a shift-invariant space in $L^2(G)$. Suppose $V = \bigoplus_{n=1}^{\infty} S(\phi_n)$ is orthonormal decomposition of V, where each $(\phi_n)_{n\in\mathbb{N}}$ is a Parseval frame generator of space $S(\phi_n)$. Then

(a) the set $\{T_k\phi_n : k \in L, n \in \mathbb{N}\}$ forms a Parseval frame for space V;
(b) $\{T\phi_n(\xi) : \ n \in \mathbb{N}\} - \{0\}$ is an orthonormal basis for $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$;
(c) for $\phi_n \neq 0$, $n \in \mathbb{N}$ and $k \in L$, we have

$$\|T_k\phi_n\|_2^2 = \|\phi_n\|_2^2 = \|T\phi_n\|^2_{L\left(S_{L^\perp}, l^2(L^\perp)\right)} = \int_{S_{L^\perp}} \|T\phi_n(\xi)\|^2_{l^2(L^\perp)} d\xi = 1;$$

(d) $\{T_k\phi_n : k \in L, n \in \mathbb{N}\}$ is an orthonormal basis for space V.

The following lemma will be a key working for our main results. It will be used at many places in this article.

Lemma 3.1. *Let V be a shift-invariant space in $L^2(G)$ with associated range function J such that* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$. Then, there are disjoint measurable sets $\{A_n\}_{n\in\mathbb{N}_0}$ and functions $\{\phi_i\}_{i\in\mathbb{N}}$ in $L^2(G)$ such that $S_{L^\perp} = \cup_{n\in\mathbb{N}_0} A_n$ and the following statements hold:*

(*i*) *set $\{T_k\phi_i : i \in \mathbb{N}, k \in L\}$ forms a Parseval frame for space V;*
(*ii*) $T\phi_i(\xi) = 0$ *for a.e. $\xi \in A_n$ and $i > n$;*
(*iii*) $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ *is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$;*
(*iv*) $\dim J(\xi) = n$ *for a.e. $\xi \in A_n$.*

Proof. Since V is a shift-invariant subspace of $L^2(G)$, therefore using Theorem 2.4 there exist functions $\{\phi_i\}_{i\in\mathbb{N}}$ in space $L^2(G)$ satisfying

$$V = \bigoplus_{i\in\mathbb{N}} S(\phi_i), \tag{3.2}$$

where for each i, function ϕ_i is a Parseval frame generator of the space $S(\phi_i)$, $i \in \mathbb{N}$. This shows that the set $\{T_k\phi_i : i \in \mathbb{N},\ k \in L\}$ forms a Parseval frame for shift-invariant space V. So, by using Theorem 2.2, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ also forms a Parseval frame for space $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$.

Now, define $A_0 = S_{L^\perp} \backslash \sigma(V)$ and $A_n = \sigma(S(\phi_n)) \backslash \sigma(S(\phi_{n+1}))$, for $n > 0$. These sets A_n are pairwise disjoint as $\sigma(S(\phi_{i+1}))$ is contained in $\sigma(S(\phi_i))$ for all $n \in \mathbb{N}$. Also, it is given that $\dim J(\xi) < +\infty$. This implies that $\sum_{i\in\mathbb{N}} \|T\phi_i(\xi)\| < +\infty$ for a.e. $\xi \in S_{L^\perp}$. Since $\sigma(S(\phi_i)) = \{\xi \in S_{L^\perp} : T\phi_i(\xi) \neq 0\}$, so, if $\xi \in \cap_{i\in\mathbb{N}}\sigma(S(\phi_i))$, then this implies that $T\phi_i(\xi) \neq 0$ for all $i \in \mathbb{N}$ and therefore, $\sum_{i\in\mathbb{N}} \|T\phi_i(\xi)\|$ is not finite, which is not possible. Thus, $\cap_{i\in\mathbb{N}}\sigma(S(\phi_i)) = \emptyset$, this implies $\cup_{n\in\mathbb{N}_0} A_n = S_{L^\perp}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now, by definition of A_n, $\xi \in A_n$ implies $\xi \in \sigma(S(\phi_n))$ and $\xi \notin \sigma(S(\phi_{n+1})) \supseteq \sigma(S(\phi_{n+2})) \supseteq \cdots$. Thus, $T\phi_i(\xi) = 0$ for $i > n$ and a.e. $\xi \in A_n$.

From equation (3.2), spaces $S(\phi_i)$ and $S(\phi_j)$ are orthogonal to each other for all $i \neq j$. So, using Proposition 2.5, we have

$$\sum_{\eta\in L^\perp} \widehat{\phi}_i(\xi\eta)\overline{\widehat{\phi}_j}(\xi\eta) = 0, \quad \text{for } i \neq j \text{ and for a.e. } \xi \in S_{L^\perp} \subset \widehat{G},$$

this implies

$$\langle T\phi_i(\xi), T\phi_j(\xi)\rangle = 0, \quad \text{for } i \neq j \text{ and for a.e. } \xi \in S_{L^\perp}.$$

So, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ is an orthogonal set. Also, by Proposition 2.4 for $\xi \in A_n$, we have $\|T\phi_i(\xi)\|^2_{l^2(L^\perp)} = 1$. Thus, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ is an orthonormal set in $J(\xi)$. Hence, $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$ and $\dim J(\xi) = n$ for a.e. $\xi \in A_n$. This completes the proof. □

The following remark is taken from paper [20] which is useful in our main results while using the properties related to Parseval frame of the shift-invariant space V.

Remark 3.3. If A is a measurable set such that $A \subseteq S_{L^\perp}$ and $\dim J(\xi) = n$ a.e. $\xi \in S_{L^\perp}$, then $A \subseteq A_n$, $n \in \mathbb{N}$ or $A_n = \{\xi \in S_{L^\perp} : \dim J(\xi) = n\}$.

In the next proposition, we see that $R(\xi)$ can be written in the form of a matrix when $\dim J(\xi) < +\infty$ for a.e. $\xi \in S_{L^\perp}$.

Proposition 3.2. *Let V be a shift-invariant space in $L^2(G)$ and J be the associated range function J with* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a shift preserving operator and R is its associated range operator. Then, $R(\xi)$ has a matrix representation for a.e. $\xi \in S_{L^\perp}$. If* $\dim J(\xi) = n$ *for a.e. $\xi \in A$, where $A \subseteq S_{L^\perp}$ is a* *measurable set, then the $n \times n$ matrix representation of $R(\xi)$ is given by*

$$R(\xi) = \begin{bmatrix} r_{1,1}(\xi) & r_{1,2}(\xi) & \cdots & r_{1,n}(\xi) \\ r_{2,1}(\xi) & r_{2,2}(\xi) & \cdots & r_{2,n}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1}(\xi) & r_{n,2}(\xi) & \cdots & r_{n,n}(\xi) \end{bmatrix}, \tag{3.3}$$

where $\{r_{i,j}\}_{i,j=1}^{n}$ are measurable bounded functions defined on A.

Proof. Since $\dim J(\xi) < +\infty$, then by using Lemma 3.1, there exist functions $\{\phi_i\}_{\in \mathbb{N}} \subseteq L^2(G)$ and a family of disjoint measurable sets $\{A_n\}_{n\in\mathbb{N}_0}$ satisfying $S_{L^\perp} = \cup_{n\in\mathbb{N}_0} A_n$ and $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of $J(\xi)$, for a fixed $n \in \mathbb{N}$. Since the domain of $R(\xi)$ is $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. So, $R(\xi)$ has a matrix representation given in (3.3) with respect to orthonormal basis of space $J(\xi)$.

Now, we show that entries of matrix given in (3.3) are measurable and bounded. Since the set $\{T_k\phi_i : i \in \mathbb{N}, k \in L\}$ is a Parseval frame for space V, then for every $j \in \mathbb{N}$, we get

$$U\phi_j = \sum_{i\in\mathbb{N}} \sum_{k\in L} d_i^j(k) T_k \phi_i, \tag{3.4}$$

where $d_i^j(k) \in l^2(\mathbb{N} \times L)$, $i, j, \in \mathbb{N}$ and $k \in L$. This implies that

$$\begin{aligned} TU\phi_j(\xi) &= T\left(\sum_{i\in\mathbb{N}} \sum_{k\in L} d_i^j(k) T_k \phi_i(\xi)\right) = \sum_{i\in\mathbb{N}} \sum_{k\in L} d_i^j(k) T T_k \phi_i(\xi) \\ &= \sum_{i\in\mathbb{N}} \sum_{k\in L} d_i^j(k) M_k(\xi) T\phi_i(\xi). \end{aligned}$$

Now, $T\phi_i(\xi) = 0$ for $i > n$ as a.e. $\xi \in A_n$ (Lemma 3.1). So,

$$TU\phi_j(\xi) = \sum_{i=1}^{n} \sum_{k\in L} d_i^j(k) M_k(\xi) T\phi_i(\xi).$$

Let $r_{i,j}(\xi) = \sum_{k\in L} d_i^j(k) M_k(\xi)$, $1 \le i, j \le n$. Then

$$TU\phi_j(\xi) = \sum_{i=1}^{n} r_{i,j}(\xi) T\phi_i(\xi). \tag{3.5}$$

Since, $M_k \in L^2(S_{L^\perp})$, this implies that M_k is square integrable with respect to the Haar measure defined on G. So, the functions $r_{i,j}(\xi)$ defined above are measurable for a.e. $\xi \in A_n$.

Let $[R](\xi)$ denote the matrix form of operator $R(\xi)$ relative to basis $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ for a.e. $\xi \in A_n$. Then

$$([R](\xi))_{i,j} = (R(\xi)T\phi_j(\xi))_i = (TU\phi_j(\xi))_i = r_{i,j}(\xi).$$

So, the matrix $[R](\xi)$ can be described in terms of measurable entries $\{r_{i,j}(\xi)\}_{i,j=1}^{n}$, $\xi \in A_n$ for fixed $n \in \mathbb{N}$. Now using the fact that the function T is an isometry, we have

$$|r_{i,j}(\xi)| = |\,(TU\phi_j(\xi))_i\,| \leq \|\,(TU\phi_j(\xi))_i\,\| = \|\,(U\phi_j(\xi))_i\,\| \leq \|U\|.$$

This implies that $\{r_{i,j}(\xi)\}_{i,j=1}^n$ are bounded functions for a.e. $\xi \in A_n$, because U is a bounded operator. Since, $A \subseteq S_{L^\perp}$ is measurable, where $\dim J(\xi) = n$ for a.e. $\xi \in A$, then $A \subseteq A_n$. Hence, the proposition also holds for A. □

Remark 3.4. The entries $r_{i,j}(\xi)$ in matrix $[R](\xi)$, may not be L-periodic in case of LCA groups. It is explained in the following example.

Example 3.1. Let $G = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the circle group. Then, its dual group is $\mathbb{Z}$, i.e., $\widehat{\mathbb{T}} = \mathbb{Z}$ and in this case $\xi(x) = x^\xi$ where $\xi \in \mathbb{Z}$ and $x \in \mathbb{T}$. So,

$$r_{i,j}(\xi) = \sum_{k \in L} d_i^j(k)\overline{\xi}(k) = \sum_{k \in L} d_i^j(k) k^{\overline{\xi}},$$

where $L = \{z \in \mathbb{T} : z = x + iy, x, y \in \mathbb{Q}\}$ is a discrete subgroup of $\mathbb{T}$. In this case, the entries $\{r_{i,j}\}_{i,j=1}^n$ of matrix $[R](\xi)$ are not L-periodic. That is, $r_{i,j}(\xi k_1) \neq r_{i,j}(\xi)$ for any $k_1 \in L$, as $k^{\overline{\xi k_1}} \neq k^{\overline{\xi}}$ in general for any $k_1 \in L$.

In the next result, we prove that under certain conditions the inverse of a range operator is measurable.

Proposition 3.3. *Let $R(\xi) : J(\xi) \to J(\xi)$ be a measurable range operator, where J is corresponding range function J satisfying* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$. Then $\xi \mapsto (R(\xi))^{-1}$, $\xi \in S_{L^\perp}$, is a measurable range operator.*

Proof. Since $\dim J(\xi) < +\infty$, therefore, using Lemma 3.1, there exist a family of functions $\{\phi_i\}_{i \in \mathbb{N}} \subseteq L^2(G)$ and measurable sets $\{A_n\}_{n \in \mathbb{N}_0}$. So, to prove $\xi \mapsto (R(\xi))^{-1}$, $\xi \in S_{L^\perp}$, is measurable range operator, it is sufficient to prove that function $\xi \mapsto (R(\xi))^{-1}$, $\xi \in A_n$, is measurable for each $n \in \mathbb{N}_0$. The result trivially holds for A_0.

Now let $n \geq 1$, then to prove $\xi \mapsto (R(\xi))^{-1}$ is measurable, we need to show that $\langle (R(\xi))^{-1} P_{J(\xi)} u, v \rangle$ is measurable for a.e. $\xi \in S_{L^\perp}$ and for all $u, v \in l^2(L^\perp)$. Since, the set $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$. So, it is sufficient to prove that $\langle (R(\xi))^{-1} T\phi_i(\xi), T\phi_j(\xi) \rangle$ is measurable for every $i, j = 1, 2, \ldots, n$.

Let $\{e_1, e_2, \ldots, e_n\}$ denote the canonical basis of $\mathbb{C}^n$. Define $u(\xi) : J(\xi) \to \mathbb{C}^n$ as

$$u(\xi)(T\phi_i(\xi)) = e_i \quad \text{a.e. } \xi \in A_n.$$

Here, the operator $u(\xi)$ is change of basis operator from $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ to $\{e_1, e_2, \ldots, e_n\}$. Then

$$([R](\xi))^{-1} = u(\xi)(R(\xi))^{-1}(u(\xi))^{-1},$$

where $[R](\xi)$ denotes the matrix of range operator $R(\xi)$ relative to basis $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$. Now

$$\langle (R(\xi))^{-1} T\phi_i(\xi), T\phi_j(\xi) \rangle = \langle (u(\xi))^{-1}([R](\xi))^{-1} u(\xi) T\phi_i(\xi), T\phi_j(\xi) \rangle = \langle (R(\xi))^{-1} e_i, e_j \rangle.$$

Let entries of matrix $([R](\xi))^{-1}$ are $s_{i,j}(\xi)$, which can be obtained as

$$s_{i,j}(\xi) = (-1)^{i+j}\frac{\det\left(([R](\xi))_{ij}\right)}{\det([R](\xi))}, \tag{3.6}$$

where $([R](\xi))_{ij}$ is the minor matrix which is obtained after removing i-th row and j-th column from matrix $[R](\xi)$. Since the entries of the matrix $[R](\xi)$ are measurable, therefore measurablity is preserved under these operations. This implies that $([R](\xi))^{-1}$ has measurable entries. So, $\xi \mapsto \langle (R(\xi))^{-1}e_i, e_j\rangle$ is measurable for a.e. $\xi \in A_n$ and therefore $\xi \mapsto (R(\xi))^{-1}$ is measurable for a.e. $\xi \in A_n$ for all $n \in \mathbb{N}_0$. Hence, $\xi \mapsto (R(\xi))^{-1}$ is measurable for a.e. $\xi \in S_{L^\perp}$. $\square$

In the following theorem, we show the relation between invertiblity of a shift preserving operator and invertiblity of its fibers with $\dim J(\xi) < +\infty$ for a.e. $\xi \in S_{L^\perp}$.

Theorem 3.2. *Let V be a shift-invariant space and J be its range function with $\dim J(\xi) < +\infty$ a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a shift preserving operator with corresponding range operator R. Then, the following statements are true.*

(a) *The inverse U^{-1} of U is also a shift preserving operator, if operator U is invertible.*

(b) *The shift preserving operator U is invertible if and only if operator $R(\xi)$ is invertible for a.e. $\xi \in S_{L^\perp}$ and there exists a constant $K > 0$ such that $R(\xi)$ is uniformly bounded from below by K. In that case, range operator of U^{-1} is denoted by R^{-1}, and $(R(\xi))^{-1} = R^{-1}(\xi)$ for a.e. $\xi \in S_{L^\perp}$.*

Proof. (a) Let U be invertible shift preserving operator, then U^{-1} is a bounded operator. Now, for each function $f \in V$ and $k \in L$ we get

$$U^{-1}T_kf = U^{-1}T_kUU^{-1}f = U^{-1}UT_kU^{-1}f = T_kU^{-1}f.$$

This implies that $U^{-1}T_k = T_kU^{-1}$. So, U^{-1} is also shift preserving operator.

(b) First suppose that U is invertible. Let U^{-1} is inverse of U and R^{-1} be corresponding range operator of U^{-1}. We prove that $\xi \mapsto R^{-1}(\xi)R(\xi)$ and $\xi \mapsto R(\xi)R^{-1}(\xi)$ are measurable and uniformly bounded range operators on J for a.e. $\xi \in S_{L^\perp}$. Then it is enough to show that the mapping $\xi \mapsto \langle R^{-1}(\xi)R(\xi)P_{(J(\xi))^{-1}J(\xi)}u, v\rangle$ is measurable for all $u, v \in l^2(L^\perp)$. Let $u(\xi)$ be change of basis operator defined in Proposition 3.3. Then

$$([R](\xi))^{-1} = u(\xi)(R(\xi))^{-1}(u(\xi))^{-1}.$$

This implies that $(R(\xi))^{-1} = (u(\xi))^{-1}([R](\xi))^{-1}u(\xi)$ and $R(\xi) = (u(\xi))^{-1}[R](\xi)u(\xi)$. Since, $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis for $J(\xi)$ a.e. $\xi \in A_n$. Then, it is sufficient to prove that $\xi \mapsto \langle R^{-1}(\xi)R(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle$ is measurable for a.e. $\xi \in A_n$. Let $([R](\xi))^{-1}[R](\xi)$ denotes matrix form of $(R^{-1}(\xi))R(\xi)$, then

$$([R](\xi))^{-1}[R](\xi) = u(\xi)R^{-1}(\xi)R(\xi)(u(\xi))^{-1}.$$

So,

$$\langle R^{-1}(\xi)R(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle = \langle (u(\xi))^{-1}([R](\xi))^{-1}[R](\xi)u(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle$$

$$= \langle (([R](\xi))^{-1}[R](\xi)e_i, e_j \rangle.$$

Since $[R](\xi)$ is matrix corresponding to operator $R(\xi)$ with measurable entries $\{r_{i,j}\}_{i,j=1}^n$ and the matrix $([R](\xi))^{-1}$ has measurable entries $\{s_{i,j}\}_{i,j=1}^n$ defined in (3.6). Then, the matrix $([R](\xi))^{-1}[R](\xi)$ has measurable entries $\{t_{i,j}\}_{i,j=1}^n$, where

$$t_{i,j} = \sum_{k=1}^{n} s_{i,j} r_{k,j}, \quad \text{for } i,j = 1,2,\ldots,n.$$

This shows that $\xi \mapsto \langle (([R](\xi))^{-1}[R](\xi)e_i, e_j \rangle$ is a measurable function for a.e. $\xi \in A_n$. Thus, $\xi \mapsto R^{-1}(\xi)R(\xi)$ is measurable for a.e. $\xi \in A_n$. Similarly, $\xi \mapsto R(\xi)R^{-1}(\xi)$ is measurable for a.e. $\xi \in A_n$. Also, the operators $R^{-1}(\xi)R(\xi)$ and $R(\xi)R^{-1}(\xi)$ are uniformly bounded range operators on J and their corresponding shift preserving operators are $U^{-1}U$ and UU^{-1}, respectively. So,

$$R^{-1}(\xi)R(\xi) = R(\xi)R^{-1}(\xi) = I,$$

where I denotes the identity range operator on space $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Thus, $R^{-1}(\xi)$ is inverse of $R(\xi)$. Next we prove that the operator $R(\xi)$ is uniformly bounded below.

Since R^{-1} is range operator corresponding to shift preserving operator U^{-1}. Therefore,

$$\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R^{-1}(\xi)\| \leq \|U^{-1}\|.$$

Thus, $R^{-1}(\xi)$ is bounded uniformly from above by $\|U^{-1}\|$ as U^{-1} is bounded. This implies that $R(\xi)$ is bounded uniformly from below.

Converse, assume that the operator $R(\xi)$ is invertible for a.e. $\xi \in S_{L^\perp}$ and uniformly bounded by a constant $K > 0$. Then $\xi \mapsto (R(\xi))^{-1}$ is uniformly bounded below by constant K and it is a measurable range operator on J. So, there is a corresponding shift preserving operator $\tilde{U}$ such that for every $f \in V$ and a.e. $\xi \in S_{L^\perp}$, we have

$$(R(\xi))^{-1}Tf(\xi) = T\tilde{U}f(\xi).$$

Now $Tf(\xi) = R^{-1}(\xi)R(\xi)Tf(\xi) = T\tilde{U}Uf(\xi)$ and $Tf(\xi) = R(\xi)R^{-1}(\xi) = TU\tilde{U}f(\xi)$. Thus, $U\tilde{U} = \tilde{U}U = I$. Hence, U is invertible and $U^{-1} = \tilde{U}$. This completes the proof. □

Theorem 3.3. *Let V be a shift-invariant space in space $L^2(G)$, where G is a LCA group and let J be the range function associated to V. Let $U : V \to V$ be a bounded shift preserving operator with corresponding range operator R on $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Then the following conditions hold.*

(1) *The operator U is normal if and only if $R(\xi)$ is a normal operator for a.e. $\xi \in S_{L^\perp}$.*

(2) *The operator U is unitary if and only if $R(\xi)$ is unitary for a.e. $\xi \in S_{L^\perp}$. (where $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\widehat{G}$).*

Proof. Since $R(\xi)$ is a range operator on $J(\xi)$ corresponding to shift preserving operator U. So,

$$(T \circ U)(f(\xi)) = R(\xi)(T(f(\xi)) \quad \text{a.e. } \xi \in S_{L^\perp}, \tag{3.7}$$

where $T : L^2(G) \to L^2(S_{L^\perp}, l^2(L^\perp))$ is fiberization mapping which is defined by $Tf(\xi) = (\widehat{f}(\xi\eta)_{\eta \in L^\perp}$. Also, we know that if U is shift preserving operator then its adjoint operator $U^* : V \to V$ is also a shift-preserving operator, and the associated range operator is R^* which is given by $R^*(\xi) = (R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$ (by Proposition 3.5 in [20]).

(1) First we show that the range operators corresponding to U^*U and UU^* are R^*R and RR^*, respectively given by $R^*(\xi)R(\xi) = (R(\xi))^*(R(\xi))$ and $R(\xi)R(\xi)^* = (R(\xi))(R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$. Note that the operators R^*R and RR^* given by $R^*(\xi)R(\xi) = (R(\xi))^*(R(\xi))$ and $R(\xi)R(\xi)^* = (R(\xi))(R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$ are measurable. Also, $\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R^*(\xi)R(\xi)\| < +\infty$ and $\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R(\xi)R^*(\xi)\| < +\infty$. Then, by Theorem 2.3, there exist shift preserving operators W_1 and W_2 on V which satisfies

$$(T \circ W_1)(f(\xi)) = R^*(\xi)R(\xi)(T(f(\xi))$$

and

$$(T \circ W_2)(f(\xi)) = R(\xi)R^*(\xi)(T(f(\xi)),$$

for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$. Now for all functions $f, g \in V$, consider

$$\begin{aligned}
\langle U^*Uf, g\rangle &= \langle Uf, Ug\rangle, \\
&= \langle T \circ Uf, T \circ Ug\rangle, \\
&= \int_{S_{L^\perp}} \langle T \circ Uf(\xi), T \circ Ug(\xi)\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle R(\xi)(Tf(\xi)), R(\xi)(Tg(\xi))\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle R^*(\xi)R(\xi)(Tf(\xi)), Tg(\xi)\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle (ToW_1)f(\xi), Tg(\xi)\rangle d\xi,
\end{aligned}$$

$$\langle U^*Uf, g\rangle = \langle T \circ W_1 f, Tg\rangle. \tag{3.8}$$

This implies that $\langle U^*Uf, g\rangle = \langle W_1 f, Tg\rangle$ for all $f, g \in V$. So, $U^*U = W_1$. That is,

$$(T \circ (U^*U))f(\xi) = R(\xi)(Tf(\xi)), \quad \text{for a.e. } \xi \in S_{L^\perp} \text{ and for all } f \in V. \tag{3.9}$$

Similarly,

$$(T \circ (UU^*))f(\xi) = R(\xi)(Tf(\xi)), \quad \text{for a.e. } \xi \in S_{L^\perp} \text{ and for all } f \in V. \tag{3.10}$$

Now suppose that U is a normal operator, so $U^*U = UU^*$, then from (3.9) and (3.10)

$$R^*(\xi)R(\xi) = R(\xi)R^*(\xi) \quad \text{a.e. } \xi \in S_{L^\perp}.$$

This implies that $R(\xi)$ is a normal operator for a.e. $\xi \in S_{L^\perp}$.

(2) Using Theorem 27, if U is invertible then U^{-1} is also a shift preserving operator and the corresponding range operator R^{-1} is given by $R^{-1}(\xi) = (R(\xi))^{-1}$ for a.e. $\xi \in S_{L^\perp}$. Also the adjoint operator U^* is a shift-preserving and the corresponding range operator is R^* given by $R^*(\xi) = (R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$. So, we have $(T \circ U^{-1})f(\xi) = R^{-1}(\xi)(Tf(\xi))$ and $(T \circ U^*)f(\xi) = R^*(\xi)(Tf(\xi))$ for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$. Since, U is unitary this implies that $U^* = U^{-1}$. Then, we get

$$R^{-1}(\xi) = R^*(\xi) \quad \text{a.e. } \xi \in S_{L^\perp}.$$

Thus, $R(\xi)$ is unitary for a.e. $\xi \in S_{L^\perp}$. □

4. Conclusion

In this paper, we used the concept of s-eigenvalue and s-eigenspace to see the relation between the eigenvalues of a shift preserving operator and the corresponding range operator on LCA group. We also characterized the matrix structure of range operator, in the finite dimensional case. We got the conditions which ensure that invertibility of shift preserving operator implies the invertibility of the fiber of the corresponding range operator and vice versa. In the end, we got some conditions which show that a shift preserving operator and the fiber of corresponding range operator both share the same properties like as unitary, normal, isometry, self adjoint etc.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1011–1019.

ORBITAL CONTINUITY AND COMMON FIXED POINTS IN MENGER PM-SPACES

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Abstract. In this paper, we prove that if a pair of semi R-commuting self-mappings defined on Menger PM-spaces with a nonlinear contractive condition posses a unique common fixed point, then these mappings are orbitally continuous. Also, we investigate whether this assertion and it converse holds if we replace semi R-commutativity with some other concept of commutativity in the weaker sense.

1. Introduction

The notion of orbital continuity was defined by Ćirić [4]. Shastri et al. [25] introduced the notion of orbital continuity for a pair of self-mappings.

Definition 1.1 ([25]). Let f and g be two self-mappings of a metric space (X, d) and let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence in X such that $gx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$ Then the set $O(x_0, g, f) = \{gx_n \mid n = 0, 1, 2, \ldots\}$ is called the (g, f)-orbit at x_0 and f (or g) is called (g, f)-orbitally continuous if $gx_n \to u$ implies $x_n \to fu$, as $n \to +\infty$ (or $gx_n \to u$ implies $ggx_n \to gu$, as $n \to +\infty$).

Following the results obtained by Machuca [13] and Goebel [6], Jungck [9] generalized Banach contraction principle [1] by proving common fixed theorem for a pair of commutative self-mappings. Since then many common fixed point theorems have been obtained using various generalizations of commutativity (see e.g. [7, 10–12, 17–19, 21, 24]). Overview of weaker forms of commuting mappings and their systematic comparison can be found in [26].

Key words and phrases. Orbital continuity, probabilistic metric spaces, common fixed point, nonlinear contractive condition.

2020 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54H25, 54E70.

DOI 10.46793/KgJMat2307.1011N

Received: September 29, 2022.

Accepted: December 14, 2022.

Pant [19] introduced the notion of semi R-commutativity. Ješić et al. [8] extended this notion to Menger PM-spaces and obtained common fixed point theorem for two semi R-commuting self-mappings (Theorem 3.3. in [8]).

Patak et al. [20] defined the notion of R-weak commutativity of type A_f and of type A_g. Using the probabilistic version of this notion, Nikolić et al. [16] proved that orbital continuity for two self-mappings is a necessary and sufficient condition for the existence of a unique common fixed point for these mappings if they are R-weakly commuting of type A_f (or of type A_g) with nonlinear contractive condition in the sense of Boyd and Wong [2], for Menger PM-spaces.

In this paper we prove that converse of a slight modification of Theorem 3.3. in [8] (see Theorem 3.1. given below) holds under additional condition. Also, we investigate whether this theorem and it converse remain true if we replace a pair of semi R-commuting mappings with mappings that satisfy some other weaker form of commutativity. Nikolić et al. [16] gave a positive answer in this sense for a pair of R-weakly commutative mappings of type A_f (or of type A_g).

2. Preliminaries

In 1906, Fréchet introduced the concept of distance on an arbitrary set, described the properties of the distance function and thus founded the axiomatics of metric spaces. This abstractly introduced mathematical object found great applications in the study of not only mathematical objects in which the concept of distance appears. However, in many cases where metric spaces are used, assigning a unique real non-negative number to each pair of elements of a set is not sufficient to describe the observed phenomenon or problem. Namely, in many situations the concept of distance is more suitable to be viewed probabilistically, than as a quantity determined by a real number. In this way, in 1942, Menger [14] gave the definition of the statistical metric space using the notion of distribution function (in 1964, in the name of this spaces the adjective “statistical” was changed to “probabilistic”). Continuing the study of probabilistic metric spaces Schweizer and Sklar [22, 23] introduced some topological notions for this space and gave some properties devoted to the axiomatics of probabilistic metric spaces (in particular for triangle inequality).

Some function $F : \mathbb{R} \to [0, 1]$ is a distribution function if F is a left-continuous and non-decreasing mapping, which satisfies $F(0) = 0$ and $\sup_{x \in \mathbb{R}} F(x) = 1$. With ε_0 we will denote the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([23]). A mapping $T : [0, 1]^2 \mapsto [0, 1]$ is continuous t-norm if T satisfies the following conditions:

a) T is commutative and associative;
b) T is continuous;
c) $T(a, 1) = a$, for any $a \in [0, 1]$;

d) $T(a_1, b_1) \leq T(a_2, b_2)$ whenever $a_1 \leq a_2$ and $b_1 \leq b_2$, and $a_1, b_1, a_2, b_2 \in [0, 1]$.

Definition 2.2. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into the set of all distribution functions ($\mathcal{F}(x, y) = F_{x,y}$ for any $(x, y) \in X \times X$) if and only if the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq T\Big(F_{x,y}(t), F_{y,z}(s)\Big)$, for all $x, y, z \in X$ and all $s, t \geq 0$.

In 1960, Schweizer and Sklar [22] defined (ε, λ)-topology in a Menger PM-space $(X, \mathcal{F}, T)$ and proved that this topology is a Hausdorff topology. Since 1960 many other topics related to PM-spaces have been studied by various authors, such as convergence of sequences, continuity of mappings, completion, etc. We will only state the following definition.

Definition 2.3. Let $(X, \mathcal{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if, for any $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.
(3) A Menger PM-space is said to be complete if any Cauchy sequence in X is convergent to a point in X.

Also, the following two lemmas are stated and proved by Schweizer and Sklar [22].

Lemma 2.1 ([22]). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to+\infty} x_n = x$. Then $F_{x,x_n}(t) \to F_{x,x}(t) = \varepsilon_0(t)$, for any $t > 0$, as $n \to +\infty$ and conversely.*

Lemma 2.2 ([22]). *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and T is continuous. Then the function $\mathcal{F}$ is lower semi-continuous for any fixed $t > 0$, i.e., for any fixed $t > 0$ and any two convergent sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, it follows that*

$$\liminf_{n\to+\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Lemma 2.3 ([23]). *Let y be a fixed point and suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence such that $\lim_{n\to+\infty} x_n = x$. Then*

$$\liminf_{n\to+\infty} F_{x_n,y}(t) = F_{x,y}(t).$$

Remark 2.1. Lemma 2.3 is a corollary of Lemma 2.2.

3. Main Results

Fang et al. [5] defined the notion of algebraic sum for two distribution functions.

Definition 3.1 ([5])**.** The algebraic sum of distribution functions F and G, in denotation $F \oplus G$, is defined by:

$$(F \oplus G)(t) = \sup_{s_1+s_2=t} \min\{F(s_1), G(s_2)\}, \tag{3.1}$$

for any $t \in \mathbb{R}$.

From the previous definition, it is obvious that the following inequality

$$(F \oplus G)(t) \geq \min\{F(s_1), G(s_2)\} \tag{3.2}$$

holds for any $t > 0$, and arbitrary and fixed $s_1, s_2 > 0$, such that $s_1 + s_2 = t$.

Ješić et al. [8] extended definition of semi R-commutativity to Menger PM-spaces.

Definition 3.2 ([8])**.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f and g be two self-mappings of X. The mappings f and g will be called semi R-commuting if there exists $R > 0$ such that:

i) $F_{ffx,gfx}(Rt) \geq F_{fx,gx}(t)$ or
ii) $F_{fgx,gfx}(Rt) \geq F_{fx,gx}(t)$ or
iii) $F_{fgx,ggx}(Rt) \geq F_{fx,gx}(t)$ or
iv) $F_{ffx,ggx}(Rt) \geq F_{fx,gx}(t)$

is true for any $t > 0$, and for any $x \in X$ such that $fx, gx \in f(X) \cap g(X)$.

Using this notion Ješić et al. [8] proved the next theorem.

Theorem 3.1 ([8])**.** *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f and g be semi R-commuting mappings, $g(X)$ is a probabilistic bounded set and $g(X) \subseteq f(X)$ satisfying the condition*

$$F_{gx,gy}\Big(\varphi(t)\Big) \geq \min\Big\{F_{fx,fy}(2t), F_{fx,gx}(t), F_{fy,gy}(t), \Big(F_{fx,gy} \oplus F_{gx,fy}\Big)(\alpha t)\Big\}, \tag{3.3}$$

for all $x, y \in X$, any $t > 0$ and any $\alpha > 3$, and for some continuous function $\varphi : (0, +\infty) \to (0, +\infty)$ which satisfies condition $\varphi(t) < t$, for any $t > 0$. If (g, f)-orbitally continuous self-mappings on X, then f and g have a unique common fixed point.

Remark 3.1. In Theorem 3.3. from [8] the assumption for function φ is more general than assumption for function φ from assertion of Theorem 3.1 (see condition (1.1), page 2 in [8]).

For the proof of the main result, we need the following lemmas.

Lemma 3.1 ([16])**.** *Suppose that the function $\varphi : (0, +\infty) \to (0, +\infty)$ is continuous and satisfies condition $\varphi(t) < t$, for any $t > 0$ and let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the following assertion holds: if for $x, y \in X$ we have $F_{x,y}\Big(\varphi(t)\Big) \geq F_{x,y}(t)$ for any $t > 0$, then $x = y$.*

Lemma 3.2 ([16]). *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. If for two convergent sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ holds that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = p$, then $F_{x_n,y_n}(t) \to 1$, as $n \to +\infty$, for any $t > 0$.*

In the following theorem, we will prove that the converse of Theorem 3.1 holds.

Theorem 3.2. *Let the functions f and g satisfy all the assumptions of the Theorem 3.1 and let ggx_n converges for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X whenever gx_n converges. If f and g have a unique common fixed point, then mappings f and g are (g, f)-orbitally continuous.*

Proof. Let us suppose that z is a common fixed point for mappings f and g. Since, (g, f)-orbit of any point x_0 defined by $gx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$ converges to z, it follows that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$. According to the definition of semi R-commutativity, we can observe next four cases.

Case 1. Firstly, we will suppose that mappings f and g satisfy condition *iii*) from Definition 3.2, i.e., we will suppose that there exists $R > 0$ such that inequality $F_{fgx,ggx}(Rt) \geq F_{fx,gx}(t)$ holds for any $t > 0$, and for any $x \in X$ such that $fx, gx \in f(X) \cap g(X)$. Then, it follows that there exists $R > 0$ such that $F_{fgx_n,ggx_n}(Rt) \geq F_{fx_n,gx_n}(t)$ holds, for any $t > 0$. If we apply Lemma 3.2 for such R and any $t > 0$, it follows

$$F_{fgx_n,ggx_n}(Rt) \to 1, \quad \text{as } n \to +\infty. \tag{3.4}$$

Now, if we put $x = gx_n$, $y = z$, and $gz = fz$ in contractive condition (3.3) and if we apply condition (PM3) from Definition 2.2, then we get that

$$\begin{aligned} F_{ggx_n,gz}\Big(\varphi(t)\Big) &\geq \min\Big\{F_{fgx_n,gz}(2t), F_{fgx_n,ggx_n}(t), F_{gz,gz}(t), \Big(F_{fgx_n,gz} \oplus F_{ggx_n,fz}\Big)(\alpha t)\Big\} \\ &\geq \min\Big\{F_{fgx_n,gz}(2t), F_{fgx_n,ggx_n}(t), F_{gz,gz}(t), F_{fgx_n,gz}(2t), F_{gz,ggx_n}(t)\Big\} \\ &= \min\Big\{F_{fgx_n,gz}(2t), F_{fgx_n,ggx_n}(t), F_{gz,ggx_n}(t)\Big\} \\ &\geq \min\Big\{T\Big(F_{fgx_n,ggx_n}(t), F_{ggx_n,gz}(t)\Big), F_{fgx_n,ggx_n}(t), F_{gz,ggx_n}(t)\Big\} \end{aligned} \tag{3.5}$$

holds, for all $x, y \in X$, any $t > 0$ and any $\alpha > 3$. Using assumption that ggx_n converges, having in mind condition (3.4) and conditions b), c) and d) from Definition 2.1, if take $\liminf$ as $n \to +\infty$ in inequality (3.5) and apply Lemma 2.3, we get

$$F_{\lim\limits_{n\to+\infty} ggx_n,gz}\Big(\varphi(t)\Big) \geq F_{\lim\limits_{n\to+\infty} ggx_n,gz}(t).$$

Finally, if we apply Lemma 3.1, then we get that $\lim_{n\to+\infty} ggx_n = gz$. Hence, g is (g, f)-orbitally continuous. Now, we will show that f is (g, f)-orbitally continuous. Indeed, using condition (PM3) from Definition 2.2 it follows that

$$F_{fgx_n,gz}(t) \geq T\left(F_{fgx_n,ggx_n}\left(\frac{t}{2}\right), F_{ggx_n,gz}\left(\frac{t}{2}\right)\right)$$

holds for any $t > 0$. Letting $n \to +\infty$ in previous inequality, from condition (3.4) and Lemma 2.1 we get that $F_{fgx_n,gz}(t) \to 1$, for any $t > 0$. Finally, applying Lemma 2.1 we get $\lim_{n\to+\infty} fgx_n = gz = fz$. Hence, f and g are orbitally continuous.

Case 2. Now, we will suppose that mappings f and g satisfy condition $i)$ from Definition 3.2, i.e., we will suppose that there exists $R > 0$ such that inequality $F_{ffx,gfx}(Rt) \geq F_{fx,gx}(t)$ holds for any $t > 0$, and for any $x \in X$ such that $fx, gx \in f(X) \cap g(X)$. In this case, for such $R > 0$, it follows that

$$F_{ffx_n,gfx_n}(Rt) \to 1, \quad \text{as } n \to +\infty \tag{3.6}$$

holds for any $t > 0$. Similarly, as in Case 1, if we put $x = fx_n$, $y = z$, and $gz = fz$ in contractive condition (3.3) and if we apply condition (PM3) from Definition 2.2, then we obtain

$$F_{gfx_n,gz}\Big(\varphi(t)\Big) \geq \min\Big\{T\Big(F_{ffx_n,gfx_n}(t), F_{gfx_n,gz}(t)\Big), F_{ffx_n,gfx_n}(t), F_{gz,gfx_n}(t)\Big\}, \tag{3.7}$$

for all $x, y \in X$, and any $t > 0$. Having in mind condition (3.6) and conditions b), c) and d) from Definition 2.1, if taking $\liminf$ as $n \to +\infty$ in inequality (3.7) we get

$$\liminf_{n\to+\infty} F_{gfx_n,gz}\Big(\varphi(t)\Big) \geq \liminf_{n\to+\infty} F_{gfx_n,gz}(t). \tag{3.8}$$

From assumption that ggx_n converges, then from $gfx_n = ggx_{n-1}$ we get that gfx_n converges. Now, having in mind Lemma 2.3, and applying Lemma 3.1 for condition (3.8) we get that $\lim_{n\to+\infty} ggx_n = gz$. The remaining part of the proof is analogous as in the previous case.

Case 3. We will suppose that mappings f and g satisfy condition $ii)$ from Definition 3.2, i.e., we will suppose that there exists $R > 0$ such that inequality $F_{fgx,gfx}(Rt) \geq F_{fx,gx}(t)$ holds for any $t > 0$, and for any $x \in X$ such that $fx, gx \in f(X) \cap g(X)$. In this case, for such $R > 0$, it follows that

$$F_{fgx_n,gfx_n}(Rt) \to 1, \quad \text{as } n \to +\infty,$$

i.e.,

$$F_{fgx_n,ggx_{n-1}}(Rt) \to 1, \quad \text{as } n \to +\infty, \tag{3.9}$$

for any $t > 0$. Applying condition (PM3) from Definition 2.2 it follows that

$$F_{fgx_n,ggx_n}(t) \geq T\left(F_{fgx_n,ggx_{n-1}}\left(\frac{t}{2}\right), F_{ggx_{n-1},ggx_n}\left(\frac{t}{2}\right)\right)$$

holds, for any $t > 0$. Letting $n \to +\infty$ in previous inequality, using assumption that ggx_n converges, condition (3.9) and Lemma 3.2, and having in mind conditions b) and c) from Definition 2.1 we get that $F_{fgx_n,ggx_n}(t) \to 1$, for any $t > 0$, i.e., we obtain condition (3.4). Therefore, the proof of this case reduces to the proof of Case 1.

Case 4. Finally, in this case we will suppose that mappings f and g satisfy condition $iv)$ from definition of semi R-commutativity for Menger PM-spaces, i.e., we will suppose that there exists $R > 0$ such that inequality $F_{ffx,ggx}(Rt) \geq F_{fx,gx}(t)$ holds for any $t > 0$, and for any $x \in X$ such that $fx, gx \in f(X) \cap g(X)$. In this case, for such $R > 0$, we get that

$$F_{ffx_n,ggx_n}(Rt) \to 1, \quad \text{as } n \to +\infty,$$

i.e.,

$$F_{fgx_{n-1},ggx_n}(Rt) \to 1, \quad \text{as } n \to +\infty, \tag{3.10}$$

for any $t > 0$. Now, similarly as in previous case, condition

$$F_{fgx_{n-1},ggx_{n-1}}(t) \geq T\left(F_{fgx_{n-1},ggx_n}\left(\frac{t}{2}\right), F_{ggx_n,ggx_{n-1}}\left(\frac{t}{2}\right)\right)$$

is satisfied, for every $t > 0$. Letting $n \to +\infty$ in previous inequality, using assumption that ggx_n converges, condition (3.10) and Lemma 3.2, and having in mind conditions b) and c) from Definition 2.1 we get that $F_{fgx_{n-1},ggx_{n-1}}(t) \to 1$, i.e., we get $F_{fgx_n,ggx_n}(t) \to 1$, for any $t > 0$. The rest of the proof is the same as in the previous cases.

Now, the proof is completed. □

Now, we list some definitions of weaker forms of commuting mappings introduced for Menger PM-spaces by various authors.

Definition 3.3 ([7])**.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f and g be self-mappings of X. The mappings f and g will be called R-weakly commuting if there exists some positive real number R such that

$$F_{fgx,gfx}(Rt) \geq F_{fx,gx}(t),$$

for any $t > 0$ and any $x \in X$.

Definition 3.4 ([15])**.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f and g be self-mappings of X. The mappings f and g will be called compatible if

$$\lim_{n \to +\infty} F_{fgx_n,gfx_n}(t) = 1,$$

for any $t > 0$, whenever $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = u$, for some u in X.

Definition 3.5 ([3])**.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f and g be self-mappings of X. The mappings f and g will be called compatible of type (A) if

$$\lim_{n \to +\infty} F_{fgx_n,ggx_n}(t) = 1 \quad \text{and} \quad \lim_{n \to +\infty} F_{gfx_n,ffx_n}(t) = 1,$$

for any $t > 0$, whenever $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = u$, for some u in X.

Theorem 3.1 and Theorem 3.2 remain true if we replace assumption that a pair of mappings is semi R-commutative with assumption that these self-mappings are R-weakly commuting or compatible or compatible of type (A). These theorems can also be proved under the assumption that a pair of self-mappings satisfies some other types of compatibility (for instance type (E) or type (P) (in their probabilistic versions)).

Also, positive answer for Theorem 3.1 and Theorem 3.2 in this sense was obtained by Nikolić et al. [16] for a pair of R-weakly commutative mappings of type A_f (or type A_g).

Acknowledgements. The first author acknowledge the support of the Ministry of Education, Science and Technological Development of the Republic of Serbia, institutionally funded through the Military Academy, University of Defence.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1021–1028.

A FIXED POINT THEOREM FOR MAPPINGS SATISFYING CYCLICAL CONTRACTIVE CONDITIONS IN $(3,2)$-W-SYMMETRIZABLE SPACES

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Abstract. In this paper we are concerned with $(3,2)$-symmetrics and $(3,2)$-W-symmetrizable spaces. First we give the basic definitions, the notation, some examples and elementary results about these spaces, then we prove the existence of a fixed point for self mappings satisfying cyclical contractive conditions in $(3,2)$-W-symmetrizable spaces.

1. Introduction

The geometric properties of the metric spaces, their axiomatic classification and generalizations have been considered in a lot of papers: [1, 5, 11, 13–16, 18–20].

The notion of an (n, m, ρ)-metric, $n > m$, as a generalization of the usual notion of a pseudometric (the case $n = 2$, $m = 1$), and the notion of an $(n+1)$-metric (as in [14] and [11]) was introduced in [6]. Some connections between the topologies induced by a $(3, 1, \rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [19]) are given in [7]. Other characterizations of $(3, j, \rho)$-metrizable topological spaces, $j \in \{1, 2\}$ are given in [3, 4, 8, 9].

Fixed points theory plays a basic role in applications of many branches of mathematics. The Banach fixed point theorem [2] is a very simple and powerful theorem with a wide range of applications. Several extensions and generalizations of this result have appeared in the literature. Through the years this theorem has been generalized and extended by many authors in various ways and directions. In [12] Kirk, Srinivasan

Key words and phrases. $(3,2)$-W-symmetrizable space, cyclical contractive condition, self mapping, fixed point.

2020 *Mathematics Subject Classification.* Primary: 45H10. Secondary: 54H25.

DOI 10.46793/KgJMat2307.1021D

Received: October 03, 2022.

Accepted: January 18, 2023.

and Veeramani introduced the notion of cyclical representation and characterized the Banach's contraction mapping principle in context of a mapping which satisfy a cyclical contractive condition.

Here we consider only $(3,2)$-W-metrizable spaces. The purpose of this paper is to prove, using a new type of implicit relation, a fixed point theorem for mappings which satisfy a cyclical contractive condition.

2. Preliminaries

We give the basic definitions for $(3,2,\rho)$-metric spaces and $(3,2)$-metric spaces, as in [3].

Let $M \neq \emptyset$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x,y,z)\alpha(u,v,w) \Leftrightarrow \pi(u,v,w) = (x,y,z),$$

where π is a permutation. We will use the same notation (x,y,z) for the elements in $M^{(3)}$ keeping in mind that $(x,y,z) = (u,v,w)$ in $M^{(3)}$ if and only if (x,y,z) is a permutation of (u,v,w).

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We state three conditions for such map:

$(M0)$ $d(x,x,x) = 0$, for any $x \in M$;
$(M1)$ $d(x,y,z) \leq d(x,a,b) + d(y,a,b) + d(z,a,b)$, for any $x,y,z,a,b \in M$;
(Ms) $d(x,x,y) = d(x,y,y)$, for any $x,y \in M$.

Let ρ be a subset of $M^{(3)}$. We consider the following two conditions for such a set:

$(E0)$ $(x,x,x) \in \rho$, for all $x \in M$;
$(E1)$ $(x,a,b),(a,y,b),(a,b,z) \in \rho$ implies $(x,y,z) \in \rho$, for any $x,y,z,a,b \in M$.

Definition 2.1. If ρ satisfies $(E0)$ and $(E1)$, we say that ρ is a $(3,2)$-equivalence.

Example 2.1. The set $\Delta = \{(x,x,x) \mid x \in M\}$ is a $(3,2)$-equivalence on M.

Example 2.2. The set $\rho_d = \{(x,y,z) \mid (x,y,z) \in M^{(3)}, d(x,y,z) = 0\}$, where d satisfies $(M0)$ and $(M1)$ is a $(3,2)$-equivalence.

Definition 2.2. Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d$ are as above.

i) If d satisfies $(M0)$ and $(M1)$, then we say that d is a $(3,2,\rho)$-metric on M and the pair (M,d) is said to be a $(3,2,\rho)$-metric space.

ii) If d satisfies $(M0)$,$(M1)$ and (Ms), then we say that d is a $(3,2,\rho)$-symmetric on M, and the pair (M,d) is said to be a $(3,2,\rho)$-symmetric space.

If $\rho = \Delta = \{(x,x,x) \mid x \in M\}$, then we write $(3,2)$ instead of $(3,2,\Delta)$.

Example 2.3. Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:

$$d(x,y,z) = \begin{cases} 0, & x = y = z, \\ 1, & \text{otherwise}, \end{cases}$$

is a $(3,2)$-metric on M (the discrete 3-metric).

Proposition 2.1. *If d is a $(3,2,\rho)$-metric on M, then*

(i) $d(x,x,y) \leq 2d(x,a,b) + d(y,a,b)$;
(ii) $d(x,x,y) \leq 2d(x,y,y)$;
(iii) $d(x,x,y) \leq 2d(x,z,z) + d(y,z,z)$,
for any $x,y,z,a,b \in M$.

Proof. Follows directly from Definition 2.2. □

Definition 2.3. Let d be a $(3,2,\rho)$-metric on M, $x,y \in M$ and $\epsilon > 0$. We define the following ϵ-balls as subsets of M:
i) $B(x,y,\epsilon) = \{z \mid z \in M, d(x,y,z) < \epsilon\}$ - with center at (x,y) and radius ϵ;
ii) $B(x,\epsilon) = \{z \mid z \in M$, there is a $v \in M$ such that $d(x,z,v) < \epsilon\}$ - with center at x and radius ϵ.

Proposition 2.2. *For any* $(3,2,\rho)$*-metric* d *on* M *and for any* $x \in M$, $\epsilon > 0$, $B(x,x,\epsilon) \subseteq B(x,\epsilon)$.

Proof. Follows directly from the previous definition. □

Definition 2.4. For a $(3,2,\rho)$-metric d on M and $U \subseteq M$, we define the topology $\tau(W,d)$ on M by: $U \in \tau(W,d)$ if and only if for any $x \in U$, there is an $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.

Definition 2.5. We say that a topological space (M,τ) is $(3,2)$-W-metrizable if there is a $(3,2)$-metric d on M such that $\tau = \tau(W,d)$.

Definition 2.6. We say that a topological space (M,τ) is $(3,2)$-W-symmetrizable if there is a $(3,2)$-symmetric d on M such that $\tau = \tau(W,d)$.

Proposition 2.3. *For any* $(3,2,\rho)$*-metric* d *and any sequence* $(x_n)_{n=1}^{+\infty}$, *the following conditions are equivalent:*
$(C1)$ $d(x_n,x_m,x_p) \to 0$ *as* $n,m,p \to +\infty$ *and*
$(C2)$ $d(x_n,x_m,x_m) \to 0$ *as* $n,m \to +\infty$.

Proof. Let d satisfy $(C1)$. For any $n,m \in \mathbb{N}$ we choose $p,q > \max\ \{m,n\}$. By the previous proposition we obtain

$$d(x_n,x_m,x_m) \leq d(x_n,x_p,x_q) + 2d(x_m,x_p,x_q).$$

Thus, $d(x_n,x_m,x_m) \to 0$, as $n,m \to +\infty$.

Let d satisfy the condition $(C2)$. For any $n,m,p \in \mathbb{N}$ we choose $q > \max\{m,n,p\}$ and we obtain

$$d(x_n,x_m,x_p) \leq d(x_n,x_q,x_q) + d(x_m,x_q,x_q) + d(x_p,x_q,x_q).$$

Thus, $d(x_n,x_m,x_p) \to 0$ as $n,m,p \to +\infty$. □

Definition 2.7. A sequence $(x_n)_{n=1}^{+\infty}$ in $(3,2,\rho)$-metric space (M,d) is called $(3,2)$-Cauchy if it satisfies $(C1)$ or $(C2)$.

In the following we use notations and results from [10].

Definition 2.8 ([10])**.** We say that a sequence $(x_n)_{n=1}^{+\infty}$ in a $(3,2,\rho)$-metric space (M,d):

(i) 1-converges to $x \in M$ if $d(x,x,x_n) \to 0$ as $n \to +\infty$;
(ii) 2-converges to $x \in M$ if $d(x,x_n,x_n) \to 0$ as $n \to +\infty$;
(iii) 3-converges to $x \in M$ if $d(x,x_n,x_m) \to 0$ as $n,m \to +\infty$.

Theorem 2.1 ([10])**.** *For any sequence $(x_n)_{n=1}^{+\infty}$ in $(3,2,\rho)$-metric space (M,d) the following conditions are equivalent:*

(i) $(x_n)_{n=1}^{+\infty}$ *1-converges to $x \in M$;*
(ii) $(x_n)_{n=1}^{+\infty}$ *2-converges to $x \in M$;*
(iii) $(x_n)_{n=1}^{+\infty}$ *3-converges to $x \in M$.*

Definition 2.9. We say that a sequence $(x_n)_{n=1}^{+\infty}$ in a $(3,2,\rho)$-metric space (M,d) is $(3,2)$-convergent if it satisfies any of the conditions in the previous theorem.

Lemma 2.1. *Let $x,y \in M$ and $(x_n)_{n=1}^{+\infty}$, $(y_n)_{n=1}^{+\infty}$ be sequences in M. For any $(3,2,\rho)$-symmetric d on M, if $d(x_n,x,x) \to 0$ and $d(y_n,y,y) \to 0$ as $n \to +\infty$, then $d(x_n,y,y) \to d(x,y,y)$ and $d(x_n,y_n,y_n) \to d(x,y,y)$ as $n \to +\infty$.*

Proof. From

$$d(x_n,y,y) = d(x_n,x_n,y) \leq 2d(x_n,x,x) + d(y,x,x) = 2d(x_n,x,x) + d(x,y,y),$$

we obtain

$$d(x_n,y,y) - d(x,y,y) \leq 2d(x_n,x,x). \tag{2.1}$$

From $d(x,y,y) = d(x,x,y) \leq 2d(x,x_n,x_n) + d(y,x_n,x_n) = 2d(x_n,x,x) + d(x_n,y,y)$, we obtain

$$d(x_n,y,y) - d(x,y,y) \geq -2d(x_n,x,x). \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$|d(x_n,y,y) - d(x,y,y)| \leq 2d(x_n,x,x),$$

from where $d(x_n,y,y) \to d(x,y,y)$ as $n \to +\infty$.

From $d(x_n,y_n,y_n) = d(x_n,x_n,y_n) \leq 2d(x_n,x,x) + d(y_n,x,x) = 2d(x_n,x,x) + d(y_n,y_n,x) \leq 2d(x_n,x,x) + 2d(y_n,y,y) + d(x,y,y)$ we obtain

$$d(x_n,y_n,y_n) - d(x,y,y) \leq 2(d(x_n,x,x) + d(y_n,y,y)). \tag{2.3}$$

From $d(x,y,y) = d(x,x,y) \leq 2d(x,x_n,x_n) + d(y,x_n,x_n) = 2d(x_n,x,x) + d(x_n,y,y) \leq 2d(x_n,x,x) + 2d(y,y_n,y_n) + d(x_n,y_n,y_n)$ we obtain

$$d(x_n,y_n,y_n) - d(x,y,y) \geq -2(d(x_n,x,x) + d(y_n,y,y)). \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$|d(x_n,y_n,y_n) - d(x,y,y)| \leq 2(d(x_n,x,x) + d(y_n,y,y)),$$

from where $d(x_n,y_n,y_n) \to d(x,y,y)$ as $n \to +\infty$. □

Lemma 2.2. *Let (M,τ)-be a $(3,2)$-W-metrizable space via $(3,2)$-metric d. Let $A \subseteq M, x \in M$ and $(x_n)_{n=1}^{+\infty}$ be a sequence in A. If $d(x_n,x,x) \to 0$ as $n \to +\infty$, then $x \in \overline{A}$.*

Proof. Let $(x_n)_{n=1}^{+\infty}$ be a sequence in A such that $d(x_n,x,x) \to 0$ as $n \to +\infty$. Let $U \in \tau$ and $x \in U$. Then there is an $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. There is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0, x_n \in B(x,x,\epsilon)$. So, $x_n \in B(x,x,\epsilon) \subseteq B(x,\epsilon) \subseteq U$ for $n \geq n_0$. Thus, $U \cap A \neq \emptyset$, i.e., $x \in \overline{A}$. □

Definition 2.10. Let (M,τ)-be a $(3,2)$-W-metrizable space via $(3,2)$-metric d. We say that (M,τ) is $(3,2)$-complete if any $(3,2)$-Cauchy sequence in M is $(3,2)$-convergent (with respect to the $(3,2)$-metric d).

3. Main results

Definition 3.1 ([12])**.** Let $\{A_i\}_{i=1}^{p}$ be a family of nonempty sets and $\mathcal{A} = \cup_{i=1}^{p} A_i$. We say that a mapping $f : \mathcal{A} \to \mathcal{A}$ is p-cyclic if $f(A_i) \subseteq A_{i+1}$, $i = 1,2,3,\ldots,p$, where $A_{p+1} = A_1$.

Definition 3.2. Let $\mathcal{F}$ denote the set of all lower semi-continuous functions $F : (\mathbb{R}^+)^6 \to \mathbb{R}$ satisfying the following conditions:

(F1) for all $x,y \in \mathbb{R}^+$, $x < y$, $F(a,b,c,d,e,x) \geq F(a,b,c,d,e,y)$ (non-increasing on the 6^{th} coordinate);

(F2) there is an $h \in [0,1)$ such that for all $u,v \geq 0$, $F(u,v,v,u,0,2u+v) \leq 0$ implies $u \leq hv$;

(F3) $F(t,t,0,0,t,t) > 0$ for $t > 0$.

Example 3.1. The function $F(a,b,c,d,e,f) = a - xb - y\max\{c,d,e,f\}$, where $x,y \geq 0$ and $x + 3y < 1$ is an element of $\mathcal{F}$.

$(F1)$ Obviously true.

$(F2)$ Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - xv - y\max\{u,v,2u+v\} \leq 0$. Then if $u > v$ we obtain that $u[1-(x+3y)] \leq 0$, which is a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = x+3y < 1$.

$(F3)$ $F(t,t,0,0,t,t) = t[1-(x+3y)] > 0$, for all $t > 0$.

Theorem 3.1. *Let (M,τ) be a $(3,2)$-complete $(3,2)$-W-symmetrizable space via $(3,2)$-symmetric d and $\{A_i\}_{i=1}^{p}$ be a family of nonempty closed subsets of M. Let $\mathcal{A} = \cup_{i=1}^{p} A_i$ and let $f : \mathcal{A} \to \mathcal{A}$ be a p-cyclic mapping such that for all $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,p$ and $F \in \mathcal{F}$*

$$F\begin{pmatrix} d(fx,fy,fy), d(x,y,y), d(x,fx,fx), \\ d(y,fy,fy), d(y,fx,fx), d(x,fy,fy) \end{pmatrix} \leq 0. \tag{3.1}$$

Then f has a unique fixed point in $\cap_{i=1}^{p} A_i$.

Proof. Let x_0 be an arbitrary point of A_1. We define $x_n = fx_{n-1}$, $n = 1,2,\ldots$ From Definition 3.1 and (3.1), for $x_0 \in A_1$ and $x_1 \in A_2$, we have $x_{p-1} = fx_{p-2} \in A_p$,

$x_p = fx_{p-1} \in A_{p+1} = A_1$, $x_{p+1} = fx_p \in A_2$ and

$$F\begin{pmatrix} d(fx_0, fx_1, fx_1), d(x_0, x_1, x_1), d(x_0, fx_0, fx_0), \\ d(x_1, fx_1, fx_1), d(x_1, fx_0, fx_0), d(x_0, fx_1, fx_1) \end{pmatrix} \leq 0,$$

i.e., $F(d(x_1, x_2, x_2), d(x_0, x_1, x_1), d(x_0, x_1, x_1), d(x_1, x_2, x_2), 0, d(x_0, x_2, x_2)) \leq 0$. Since d is $(3,2)$-symmetric, we have

$$d(x_0, x_2, x_2) \leq d(x_0, x_1, x_1) + 2d(x_2, x_1, x_1) = d(x_0, x_1, x_1) + 2d(x_1, x_2, x_2). \tag{3.2}$$

From (3.2) and (F1) we obtain

$$F\begin{pmatrix} d(x_1, x_2, x_2), d(x_0, x_1, x_1), d(x_0, x_1, x_1), \\ d(x_1, x_2, x_2), 0, d(x_0, x_1, x_1) + 2d(x_1, x_2, x_2) \end{pmatrix} \leq 0.$$

By (F2) we obtain

$$d(x_1, x_2, x_2) \leq hd(x_0, x_1, x_1).$$

Hence, we have

$$d(x_n, x_{n+1}, x_{n+1}) \leq hd(x_{n-1}, x_n, x_n) \leq \cdots \leq h^n d(x_0, x_1, x_1).$$

Then, for all $m, n \in \mathbb{N}$, $m > n$,

$$\begin{aligned} d(x_n, x_m, x_m) &\leq d(x_n, x_{n+1}, x_{n+1}) + 2d(x_m, x_{n+1}, x_{n+1}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) + 2d(x_m, x_{n+2}, x_{n+2}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + 4d(x_{n+2}, x_{n+3}, x_{n+3}) + 2d(x_m, x_{n+3}, x_{n+3}) \\ &\leq d(x_n, x_{n+1}, x_{n+1}) + 4d(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + 4d(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + 4d(x_{m-1}, x_m, x_m) \\ &\leq 4(h^n + h^{n+1} + \cdots + h^{m-1}) d(x_0, x_1, x_1) \\ &\leq 4\frac{h^n}{1-h} d(x_0, x_1, x_1). \end{aligned}$$

Thus, $d(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$, i.e., $(x_n)_{n=1}^{+\infty}$ is $(3,2)$-Cauchy sequence. Since (M, τ) is $(3,2)$-complete, there is $z \in M$, such that $d(x_n, z, z) \to 0$ as $n \to +\infty$. Then the sequences $(x_{np})_{n=0}^{+\infty}$,$(x_{np+1})_{n=0}^{+\infty}, \ldots, (x_{np+p-1})_{n=0}^{+\infty}$ converge to z. Since $x_{np+i-1} \in A_i$, $i = 1, 2, \ldots, p$, and the family $\{A_i\}_{i=1}^p$ is a family of nonempty closed subsets of M, by Lemma 2.2 we get $z \in \cap_{i=1}^p A_i$.

Next we will prove that z is a fixed point of f. If we set $x = x_n$ and $y = z$ at the inequality (3.1), we obtain

$$F\begin{pmatrix} d(fx_n, fz, fz), d(x_n, z, z), d(x_n, fx_n, fx_n), \\ d(z, fz, fz), d(z, fx_n, fx_n), d(x_n, fz, fz) \end{pmatrix} \leq 0,$$

i.e.,

$$F\begin{pmatrix} d(x_{n+1}, fz, fz), d(x_n, z, z), d(x_n, x_{n+1}, x_{n+1}), \\ d(z, fz, fz), d(z, x_{n+1}, x_{n+1}), d(x_n, fz, fz) \end{pmatrix} \leq 0. \tag{3.3}$$

It is obvious that $d(x_n, x_{n+1}, x_{n+1}) \to 0$ and $d(z, x_{n+1}, x_{n+1}) \to 0$ as $n \to +\infty$. From lemma 2.1 it follows that $d(x_{n+1}, fz, fz) \to d(z, fz, fz)$ and $d(x_n, fz, fz) \to d(z, fz, fz)$ as $n \to +\infty$. If we use all these combined with the fact that F is lower semi-continuous function, and let $n \to +\infty$ in (3.3), we obtain that

$$F(d(z, fz, fz), 0, 0, d(z, fz, fz), 0, d(z, fz, fz)) \le 0. \tag{3.4}$$

From (3.4) and the condition (F1) we get

$$F(d(z, fz, fz), 0, 0, d(z, fz, fz), 0, 2d(z, fz, fz)) \le 0.$$

And by (F2) we obtain that $d(z, fz, fz) = 0$, i.e., $fz = z$.

Next we will prove the uniqueness of point z. Suppose that there is another fixed point $z' \in \cap_{i=1}^{p} A_i$. If we set $x = z$ and $y = z'$ at the inequality (3.1), we obtain

$$F\begin{pmatrix} d(fz, fz', fz'), d(z, z', z'), d(z, fz, fz), \\ d(z', fz', fz'), d(z', fz, fz), d(z, fz', fz') \end{pmatrix} \le 0,$$

i.e.

$$F(d(z, z', z'), d(z, z', z'), 0, 0, d(z', z, z), d(z, z', z')) \le 0.$$

Since d is a $(3,2)$-symmetric, $d(z', z, z) = d(z, z', z')$. Hence,

$$F(d(z, z', z'), d(z, z', z'), 0, 0, d(z, z', z'), d(z, z', z')) \le 0.$$

From (F3) it follows that $d(z, z', z') = 0$. Thus, $z = z'$, i.e., z is the unique fixed point of f such that $z' \in \cap_{i=1}^{p} A_i$. $\square$

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1029–1040.

THE GROWTH OF GRADIENTS OF QC-MAPPINGS IN n-DIMENSIONAL EUCLIDEAN SPACE WITH BOUNDED LAPLACIAN

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Abstract. Here we review M. Mateljević's article [9], with some novelities. We focus on mappings between smooth domains which have bounded Laplacian. As an application, if these mappings are quasiconformal, we obtain some results on the behavior of their partial derivatives on the boundary. In the last part of this article, we announce one new result of the author of [9], which has been recently presented on Belgrade Seminary of Complex Analysis.

1. Introduction

In this article, we study quasiconformal mappings in the plane and space, which have a bounded Laplacian. As an application, we get some results which we can consider as spatial versions of Kellogg's theorem. This article is presentation of the part of the article [9]. The author of [9] pointed out that the ideas in that manuscript have been indicated in [6] in planar case and communicated at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, December 10–19, 2009 [16], see also paper cited here (in particular [15]) and the literature cited there. In [9], the author developed and proved some results announced and outlined in this communication. The main idea of this article is to present method of, so called, *Flattening the boundary*.

Also, in this article will be stated one new result of the author of [9], which can be regarded as a generalisation of series of previous results in this area. Namely, this result gives positive answer to the question weather qusaiconformal mapping between two $C^{1,\alpha}$ domains, which satisfies so called Laplacian-gradient inequality, is Lipshitz continuous. This

Key words and phrases. PDE of the second order, Laplacian-Gradient Inequalities, Quasiconformal harmonic mappings, Boundary behavior of partial derivatives.

2020 *Mathematics Subject Classification.* Primary: 30C62, 31C05.

DOI 10.46793/KgJMat2307.1029M

Received: September 29, 2022.

Accepted: December 27, 2022.

was subject of interest on Belgrade Seminary of Complex analysis, where M. Mateljević proposed one proof of one more general statement, where answer to above question arises as a corollary.

We write $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and by $|x|$ we denote Euclidean norm of vector x.

For $R > 0$, by $B(a, R)$ and $S(a, R)$ we denote the ball and the sphere in $\mathbb{R}^n$ with center in a of radius R. By $B(R)$ and $S(R)$ we denote $B(0, R)$ and $S(0, R)$. We use $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ for $B(1)$ and $S(1)$.

Let $\Omega \subset \mathbb{R}^n$, $\mathbb{R}_+ = [0, +\infty)$ and $f, g : \Omega \to \mathbb{R}_+$. If there is a positive constant c such that $f(x) \leqslant cg(x)$, $x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that $\frac{1}{c} g(x) \leqslant f(x) \leqslant cg(x)$, $x \in \Omega$, we write $f \approx g$ (or $f \asymp g$) on Ω.

Let Ω be a domain in $\mathbb{R}^n$ and u a $C^2(\Omega)$ function. The Laplacian (linear) partial differential operator, denoted by Δ, is defined with

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{1.1}$$

We say that function u is (Euclidean) harmonic in Ω if it satisfies *Laplace's equation*

$$\Delta u = 0.$$

Inhomogeneous form of Laplace's equation is called *Poisson's equation.* In this paper we will investigate the following Dirichlet's boundary value problem:

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ \quad u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Laplace's equation has a radially symmetric solution r^{2-n} for $n > 2$ and $\log r$ for $n = 2$, r being the radial distance from some fixed point. Let us fix a point y in Ω and introduce the normalized *fundamental solution* for Laplace's equation:

$$\Gamma(x - y) = \Gamma(|x - y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} \cdot \frac{1}{|x-y|^{n-2}}, & \text{for } n > 2, \\ \frac{1}{2\pi} \log |x - y|, & \text{for } n = 2, \end{cases} \tag{1.3}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. By simple computation we have that, for every $1 \leqslant i \leqslant n$

$$\frac{\partial}{\partial x_i} \Gamma(x - y) = \frac{1}{n\omega_n} \cdot \frac{x_i - y_i}{|x - y|^n}, \tag{1.4}$$

$$\left| \frac{\partial}{\partial x_i} \Gamma(x - y) \right| \leqslant \frac{1}{n\omega_n} \cdot \frac{1}{|x - y|^{n-1}}.$$

It is convenient to introduce the inversion with respect to the sphere $S(R)$ of the point $y \neq 0$ as

$$J_R(y) = \frac{R^2}{|y|^2} y. \tag{1.5}$$

Sometimes we write y^* instead $J_R(y)$. It is important to notice that $J_R^{-1} = J_R$. Set

$$G_{1,R}(x,\xi) := \Gamma(|x-\xi|) \quad \text{and} \quad G_{2,R}(x,\xi) := -\left(\frac{|\xi|}{R}\right)^{2-n} \Gamma(|x - J_R(\xi)|).$$

We define the Green function for *Dirichlet's problem* on the ball $B(R)$ as

$$\overline{g}_R(x,\xi) := G_{1,R}(x,\xi) + G_{2,R}(x,\xi).$$

The Green function G_Ω for the Dirichlet's problem on the domain Ω in $\mathbb{R}^n$ is chosen to satisfy

$$G_\Omega(x,y) = 0, \quad \text{for } x \in \partial\Omega.$$

For more information about Green functions, see Section 5. The Poisson kernel for the ball B_R is defined by

$$P_R(x,\xi) = \frac{R^2 - |x|^2}{n\omega_n R|x-\xi|^n}.$$

When $R = 1$, we omit R from the notation. Let us introduce the *Poisson's integral*

$$P[\varphi](x) := \int_{\mathbb{S}^{n-1}} P_R(x,y)\varphi(y)\,\mathrm{d}\sigma(y),$$

and the *Green potential*

$$G[f](x) := \int_{\mathbb{B}^n} \overline{g}_R(x,y) f(y)\,\mathrm{d}\nu(y).$$

2. Gradient Estimate of the Green Potential

We will give a short proof of an important result from [3].

Theorem 2.1. *Assume $u : \overline{B}(R) \to \mathbb{R}$ is continuous, belongs to $C^2(B(R))$, $u = \varphi$ on $S(R)$ and $f = \Delta u$ is bounded and locally Hölder continuous on $B(R)$. Then*

$$u(x) = P_R[\varphi](x) + G_R[f](x). \tag{2.1}$$

Lemma 2.1 ([9]). *If f is a bounded function on $\mathbb{B}^n$, then the partial derivatives of $G[f]$ are continuous on $\overline{\mathbb{B}^n}$.*

Proof. Let us define $u(x) = \int_{\mathbb{B}^n} \overline{g}(x,y) f(y)\,\mathrm{d}\nu(y)$. Then, for every $1 \leqslant i \leqslant n$, we have that

$$\begin{aligned}\frac{\partial}{\partial x_i} u(x) &= \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i} \overline{g}(x,y) f(y)\,\mathrm{d}\nu(y) \\ &= \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i} G_1(x,y) f(y)\,\mathrm{d}\nu(y) + \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i} G_2(x,y) f(y)\,\mathrm{d}\nu(y).\end{aligned}$$

If we define

$$I_{i,1}(x) := \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i} G_1(x,y) f(y)\,\mathrm{d}\nu(y) \quad \text{and} \quad I_{i,2}(x) := \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i} G_2(x,y) f(y)\,\mathrm{d}\nu(y),$$

we have that, for $k = 1, 2$,

$$I_{i,k}(x) = \int\limits_{|y| \leqslant 1/2} \frac{\partial}{\partial x_i} G_k(x,y) f(y)\,\mathrm{d}\nu(y) + \int\limits_{1/2 < |y| \leqslant 1} \frac{\partial}{\partial x_i} G_k(x,y) f(y)\,\mathrm{d}\nu(y).$$

Finally, let us introduce the notation

$$I_{i,k,1}(x) = \int\limits_{|y|\leqslant 1/2} \frac{\partial}{\partial x_i} G_k(x,y) f(y)\,\mathrm{d}\nu(y) \quad \text{and} \quad I_{i,k,2}(x) = \int\limits_{1/2<|y|\leqslant 1} \frac{\partial}{\partial x_i} G_k(x,y) f(y)\,\mathrm{d}\nu(y).$$

Let us prove that $I_{i,2}$ is continuous on the $\overline{\mathbb{B}^n}$. The proof that $I_{i,2}$ is continuous on the closed unit ball is analogous. It will be suffice to prove that $I_{i,2,k}$ is continuous for $k = 1, 2$.

Let us consider the function $I_{i,2,1}$ and assume that $|y| < 1/2$. Then for all $x \in \overline{\mathbb{B}^n}$, we have that $|x - y^*| \geqslant 1$. Now, using (1.4) we can check that

$$\left|\frac{\partial}{\partial x_i} G_2(x,y) f(y)\right| \preceq \frac{1}{|y|^{n-2}} \cdot \frac{1}{|x-y^*|^{n-1}} \preceq \frac{1}{|y|^{n-2}}.$$

This means that, for every $x_0 \in \overline{\mathbb{B}^n}$,

$$\lim_{x\to x_0} \int\limits_{|y|\leqslant 1/2} \frac{\partial}{\partial x_i} G_2(x,y) f(y)\,\mathrm{d}\nu(y) = \int\limits_{|y|\leqslant 1/2} \frac{\partial}{\partial x_i} G_2(x_0,y) f(y)\,\mathrm{d}\nu(y),$$

by Lebesgue dominance convergence theorem. This precisely means that function $I_{i,2,1}$ is continuous at the point x_0.

Now, we need to investigate continuity of the function $I_{i,2,2}$ on the closed unit ball. After introduction change of variable $y = J(z)$, where $J_J(z)$ denotes Jacobian determinant of the mapping J defined as in (1.5) we get

$$I_{i,2,2}(x) = \int\limits_{1<|z|<2} \frac{\partial}{\partial x_i} G_2(x,z^*) f(z^*) J_J(z)\,\mathrm{d}\nu(z). \tag{2.2}$$

After introducing change of variables $x - z = u$ in the integral on the right side of (2.2) we get

$$I_{i,2,2}(x) = \int\limits_{1<|u-x|<2} \frac{\partial}{\partial x_i} G_2(x,(u-x)^*) f((u-x)^*) J_J(u-x)\,\mathrm{d}\nu(u).$$

Again, after using formula (1.4) we get

$$\left|\frac{\partial}{\partial x_i} G_2(x,(u-x)^*) f((u-x)^*) J_J(u-x)\right| \preceq K_2(u) := |u-x|^{n-2} f((u-x)^*) J_J(u-x) \frac{1}{|u|^{n-1}}.$$

Since the function C defined as $C(z) := |z|^{n-2} f(z^*) J_J(z)$ is bounded for $1 < |z| < 2$, we have that the function C_1, $C_1(u) := C(u-x)$ is bounded on $1 < |u-x| < 2$ and

$$|K_2(x,u)| \preceq \frac{1}{|u|^{n-1}}, \quad \text{for } 1 < |u-x| < 2. \tag{2.3}$$

If we define the function

$$H(x,u) = \begin{cases} \frac{\partial}{\partial x_i} G_2(x,(u-x)^*) f((u-x)^*) J_J(u-x), & 1 < |x-u| < 2, \\ 0, & |u| < 3, |x-u| < 1, |x-u| > 2, \end{cases}$$

we have that $I_{i,2,2}(x) = \int\limits_{|u|<3} H(x,u)\,\mathrm{d}\nu(u)$. Using (2.3) we get that

$$\lim_{x\to x_0} \int\limits_{|u|<3} H(x,u)\,\mathrm{d}\nu(u) = \int\limits_{|u|<3} H(x_0,u)\,\mathrm{d}\nu(u), \quad \text{for every } |x_0| \leqslant 1,$$

by Lebesgue dominance convergence theorem, q.e.d. □

3. Local C^2-coordinate Method Flattening the Boundary

Let Ω be open subset of $\mathbb{R}^n$ and $C^k(\Omega)$ the set of functions having all derivatives of order less then or equal to k continuous in Ω. Next, let $C^k(\overline{\Omega})$ be the set of functions in $C^k(\Omega)$ all of whose derivatives of order less than or equal to k have continuous extensions to $\overline{\Omega}$.

Let $x_0 \in D$, where D is bounded subset of $\mathbb{R}^n$ and f is function defined on D. For $0 < \alpha < 1$, we say that f is *Hölder continuous* with exponent α at x_0, if

$$\sup_{x\in D} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < +\infty.$$

When $\alpha = 1$, we say that f is Lipschitz-continuous at x_0.

Suppose that D is not necessarily bounded. We say that f is *uniformly Hölder continuous* with exponent α in D if

$$\sup_{\substack{x,y\in D,\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty, \quad 0 < \alpha < 1.$$

Let Ω be an open set in $\mathbb{R}^n$ and k a non-negative integer. The *Hölder spaces* $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}(\overline{\Omega})$ are defined as the subspaces of $C^k(\Omega)$, resp. $C^k(\overline{\Omega})$, consisting of functions whose k-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent α in Ω. By $\mathrm{Lip}(\Omega)$ we denote class of function which are Lipshitz continuous on the set Ω.

Definition 3.1. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ belongs to the class $C^{k,\alpha}$, where $0\leqslant\alpha\leqslant1$, $k\in\mathbb{N}$, if its boundary belongs to the class $C^{k,\alpha}$, i.e., if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0, r_0)$ and a mapping $\psi : B \to D$ such that (cf. [3, page 95])

(a) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;
(b) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;
(c) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

We refer to ψ as a local coordinate diffeomorphism *flattening the boundary* in a neighborhood of x_0.

Proposition 3.1. *ψ is bi-Lipshitz on $B_1 \subset B$ if $k \geqslant 1$. Also, $\left|\frac{\partial^2}{\partial x_i \partial x_j}\psi\right|$, $1\leqslant i, j\leqslant n$, are bounded for $k\geqslant2$.*

In [9] the following lemma is proved. This lemma is an improvement of a similar result from [5], where only boundedness of the first partial derivatives on $\mathbb{B}^n$ is concluded.

Lemma 3.1. ([9, Claim 6]) *Let $u : \overline{\mathbb{B}^n} \to \mathbb{R}$ be a solution to the following Dirichlet's problem*

$$\begin{cases} \Delta u = f, & \text{in } \mathbb{B}^n, \\ u = \varphi, & \text{on } \mathbb{S}^{n-1}, \end{cases} \tag{3.1}$$

where $f \in L^\infty(\mathbb{B}^n)$ and $\varphi \in C^{1,\alpha}(\mathbb{S}^{n-1})$, $0 < \alpha < 1$. Then $u \in C^1(\overline{\mathbb{B}^n})$.

Lemma 3.2. ([9, Local Gradient Lemma Version 1, Lemma 2.2]) *For $x_0 \in \mathbb{S}^{n-1}$ and $0 < r_0 < 2$ let $V_0 = \mathbb{B}^n \cap B(x_0, r_0)$, $V(r) = \mathbb{B}^n \cap B(x_0, r)$ for $0 < r < r_0$ and $T_0 = \mathbb{S}^{n-1} \cap B(x_0, r_0)$. If $u \in C^2(V_0) \cap C(V_0 \cup T_0)$ is such that $\Delta u \in L^\infty(V_0)$ and $u \in C^{1,\alpha}(T_0)$, then*

$$\nabla u \in L^\infty(V(r)), \quad \text{for all } 0 < r < r_0.$$

Lemma 3.2 can be regarded as a local version of Lemma 3.1.

4. Quasiconformal and Quasiregular Mappings

Let D, D' and Ω be a domains in $\mathbb{R}^n$. If $f : D \to D'$ and $y{=}f(x)$ we write $y_i{=}f_i(x), 1{\leqslant}i{\leqslant}n$.

Definition 4.1. (1) Suppose that $f : D \to D'$ is a differentiable mapping at point $x \in D$. By $f'(x) : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ we denote the differential of the mapping f at point x, which can be identified with the matrix $\left(\frac{\partial}{\partial x_j} f_i(x)\right)$, and by $T_x\mathbb{R}^n$ we denote the tangent space at point x. We define

$$|f'(x)| = \max_{|h|=1} |f'(x)h| \quad \text{and} \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|.$$

(N) A homeormophism $f : D \to D'$ satisfies the condition (N) if $m(A) = 0$ implies $m(f(A)) = 0$. Here, by $f(A)$ we denote direct image of the set A by function f.

(2) A homeormophism $f : D \to D'$ is a K-*quasiconformal* (in the analytic sense) if f is absolutely continuous on lines, f is differentable a.e. in D and $|f'(x)|^n \leqslant K|J(x,f)|$ a.e. on D.

(3) Let $f : \Omega \to \mathbb{R}^n$ be continuous. We say that f is *quasiregular* if

(a) f belongs to Sobolev space $W^n_{1,\mathrm{loc}}(\Omega)$;

(b) there exists K, $1 \leqslant K < +\infty$, such that

$$|f'(x)|^n \leqslant K J_f(x) \text{ a.e.} \tag{4.1}$$

The smallest K in (4.1) is called the *outer dilatation* $K_O(f)$.

If f is quasiregular, then

$$J_f(x) \leqslant K' l(f'(x))^n \text{ a.e.} \quad \text{for some } K',\ 1 \leqslant K' < +\infty. \tag{4.2}$$

The smallest K' in (4.2) is called the *inner dilatation* $K_I(f)$ and $K(f) = \max(K_O(f), K_I(f))$ is called the *maximal diletation* of f. If $K(f) \leqslant K$, then f is called K-*quasiregular*. Here, we will only state a few basic results.

(i_1) If mapping $f : D \to D'$ is a qc, then mapping f^{-1} is a qc and both satisfies the (N) condition.

(i_2) (Change of variables) If mapping $f : D \to D'$ is a qc, and A is a measurable subset of D, then the set $f(A)$ is a measurable, and

$$m(f(A)) = \int_A |J_f(x)| \, \mathrm{d}\nu(x).$$

Furthermore, $J_f(x) \neq 0$ almost everywhere in D.

(i_3) (Reshetnyak's main theorem) Every non-constant quasiregular map is discrete and open.

In [21], it can be seen that, when qc mapping f is differentiable at point x, only two possibilities can emerge. Either $J_f(x) \neq 0$ either $f'(x) = 0$. It can be checked in, for example [20], that, in case $J_f(x) \neq 0$, $|f'(x)|$ and $l(f'(x))$ can be regarded as the greatest and the least *singular values* of non-singular matrix $f'(x)$.

Proposition 4.1. *If $f : D \to D'$ is a quasiconformal mapping, we have that*

$$l(f'(x)) \leqslant |\nabla f_i(x)| \leqslant |f'(x)|.$$

Proof. Let $x \in D$ and $\nabla f_i(x) \neq 0$. Then we have that $\nabla f_i(x) = f'(x)^T e_i$, where $f'(x)^T$ is (Euclidean) tanspose of matrix $f'(x)$ and $e_i = (0 \ldots, 0, 1, 0, \ldots, 0)$ is i-th coordinate vector, $1 \leqslant i \leqslant n$. Since, both non-singular matrix and it's transpose have the same singular values, we conclude that $l(f'(x)) \leqslant |\nabla f(x)| \leqslant |f'(x)|$. ☐

Theorem 4.1. ([9, Theorem 2.1]) *Let $D \subset \mathbb{R}^n$ be a C^2 domain and $f : \mathbb{B}^n \to D$ a C^2 K-qc mapping. If $\Delta f \in L^\infty(\mathbb{B}^n)$, then $f \in \mathrm{Lip}(\mathbb{B}^n)$.*

Proof. Let D, D' be domains in $\mathbb{R}^n$ and $f : D \to D'$ and $h : D' \to \mathbb{R}$ be C^2 functions and set $\hat{h} = h \circ f$. If $y = f(x)$ and $f_k(x) := y_k, 1 \leqslant k \leqslant n$, the following formulas hold:

$$\frac{\partial}{\partial x_k}\hat{h} = \sum_{i=1}^{n} \frac{\partial h}{\partial y_i}\frac{\partial f_i}{\partial x_k}, \tag{4.3}$$

and

$$\Delta \hat{h} = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2}|\nabla f_i|^2 + 2 \sum_{1 \leqslant i < j \leqslant n} \frac{\partial^2 h}{\partial x_i \partial x_j}\langle \nabla f_i, \nabla f_j\rangle + \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\Delta f_i. \tag{4.4}$$

Using Lemma 3.2, we get that $\tilde{f}_n$ is Lipshitz continuous in some neighbourhood V_0 of the point x_0. Next, Proposition 4.1 gives us that the whole function $\tilde{f}$ is Lipshitz continuous on V_0. Now, using Proposition 3.1 we get that function ψ is locally bi-Lipshitz, so $f = \psi^{-1} \circ f$ is Lipshitz continuous on V_0. From this we easily conclude that the function f is Lipshitz continuous on entire ball $\mathbb{B}^n$. See Figure 1. ☐

5. Further Results

Let D be a domain in $\mathbb{R}^n$ and $s : D \to R$. If

$$|\Delta s| \leqslant a|\nabla s|^2 + b, \quad \text{on } D,$$

then we say that s satisfies a, b - Laplacian-gradient inequality on D.

In [9] the author also studied the growth of gradient of mappings which satisfy certain PDE equations (or inequalities) using the Green-Laplacian formula for functions and their

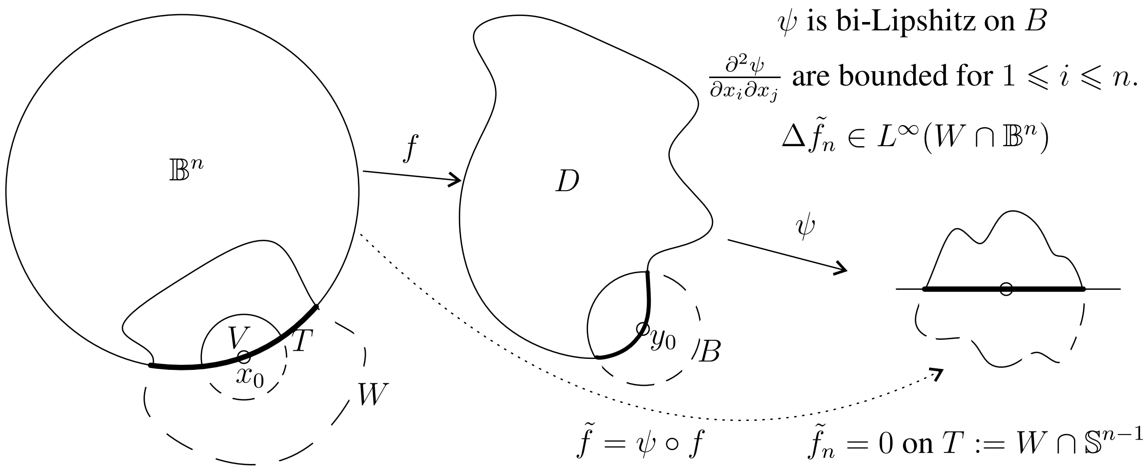

FIGURE 1. Flattening the boundary

derivatives. If in addition the considered mappings are quasiconformal (qc) between C^2 domains, M. Mateljević showed that they are Lipschitz. Some of the obtained results can be considered as versions of Kellogg-Warshawski type theorem for qc mappings. More precisely, developing further methods from Heinz paper [4]. See also Kalaj [5].

Theorem 5.1 ([9], Theorem D.). *Hypothesis:*

(1) *Let Ω be a domain in $\mathbb{R}^n$ with C^2 boundary and $f : \mathbb{B}^n \xrightarrow{onto} \Omega$ be a C^2 mapping, which has continuous extension on $\overline{\mathbb{B}^n}$.*
(2) *Suppose that f satisfy the Laplacian-gradient inequality on $B_0 = B(z_0, r_0) \cap \mathbb{B}^n$, where $x_0 \in \mathbb{S}^{n-1}$ and $r_0 > 0$ and f maps $B_0 \cap \mathbb{S}^{n-1}$ into $\partial\Omega$.*

Conclusion (VIII) (a) *There is $0 < r_1 < r_0$, $c > 0$ and a unit vector fields X on $V_1 = B(x_0, r_1) \cap \mathbb{B}^n$ (i.e., to each x we associate a unit vector $h = h(x)$ with initial point at x) such that $|\,\mathrm{d}f(x)h(x)| \leqslant c$ for every $x \in V_1$.*

(b) *If in addition f is qc in B_0, then f is Lipschitz continuous on V_1.*

It is also important to note following theorem, proved in [9]. The proof of this theorem is based on, so called "bootstrap" argument, which simplifies method used in [4]. For more about this, see [7–9].

Theorem 5.2 ([9] Theorem 3.2.). *Suppose the hypothesis* (1) *of the previous theorem holds, f is open and*

(3) *Δf is in $L^p(B_0)$ for some $p > n$ and*
(4) *f maps $B_0 \cap \mathbb{S}^{n-1}$ into $\partial\Omega$.*

Then Conclusion (VIII) (a) *holds.*

(c) *If in addition f is qr in B_0, then f is Lipschitz continuous on V_1.*

In [9] the author stated the following conjecture.

Proposition 5.1 ([9], Conjecture A.). *Let D and D_0 be $C^{1,\alpha}$, $0 < \alpha < 1$, domains (bounded) in $\mathbb{R}^n$ and $f : D \xrightarrow{onto} D_0$ a C^2 $K - qc$. If f satisfies the Laplacian-gradient inequality on D (in particular Δf is bounded), then f is Lip on D.*

It seems that local special C^2-coordinate method which works for C^2 domains needs to be modified. Namely if we work with $C^{1,\alpha}$, $0 < \alpha < 1$, domains the local coordinate ψ is $C^{1,\alpha}$, where $0 < \alpha < 1$, and therefore in general $\tilde{f}$ does not satisfy Laplacian-gradient inequality.

In the recent article [2] D. Kalaj and A. Gjokaj proved that
(i) if there is a $C^{1,\alpha}$ diffeomorphism $\phi : \overline{\mathbb{B}^n} \to \overline{D}$ and
(ii) f is a harmonic quasiconformal mapping between the unit ball in $\mathbb{R}^n$ and D,
then f is Lipschitz continuous in $\mathbb{B}^n$.

This generalizes some known results for $n = 2$ and improves some others in high dimensional case. Here we note that condition (i) is stronger than $C^{1,\alpha}$ condition on the domain D.

In the article in preparation [10] the author (of [10]) proposed proves of Theorem 5.3. This shows that conjecture proposed in Proposition 5.1 is true, under additional condition on the domain D.

Definition 5.1.

1. A function $g_D(x, \xi)$ defined on $\overline{D} \times D$ with the following properties:
   (1) g_D is harmonic in x in D except for $x = \xi$,
   (2) g_D is continuous in $\overline{D}$ except for $x = \xi$ and $g = 0$ on ∂D,
   (3) $g_D - |x - \xi|^{2-n}$ is harmonic for $x = \xi$,
   is called Green's function for D.
2. In mathematics, a function between topological spaces is called proper if inverse images of compact subsets are compact.
3. We say that D is good Green-ian domain if $\left|\frac{\partial}{\partial x_k} g_D(x, y)\right| \leqslant c\frac{1}{|x-y|^{n-1}}$, $x, y \in D$, $k = 1, \dots, n$, for some $c > 0$ and locally good Green-ian domain at $x_0 \in \partial D$ if for every $\delta > 0$ there is a C^{1+} domain $W = W_{x_0} \subset D \cap B(x_0, \delta)$ such that $x_0 \in \partial W$ and ∂W is an open set in ∂D.
4. Domain D has a C^{1+} boundary if there exists $\alpha \in (0, 1)$ such that D has $C^{1,\alpha}$ boundary.
5. D is a locally good Green-ian domain, if it is a locally good Green-ian domain at every $x_0 \in \partial D$.
6. Let $SC^1(G)$ be the class of functions $f \in C^1(G)$ such that $|f'(x)| \leqslant ar^{-1}\omega_f(x, r)$ for all $B(x, r) \subset G$, where $\omega_f(x, r) = \sup\{|f(y) - f(x)| : y \in B(x, r)\}$.

It seems that K. Widman proved that $C^{1,\alpha}$ domains are examples of good Green-ian domains. For more details see [22].

Let $d(x) = d_D(x)$, $x \in D$, be the distance of point x to the boundary of D.

Question 1. If D and G are domains with C^{1+} boundary, $f : D \xrightarrow{\text{onto}} G$ a C^2 and $K - qc$ and f_i, $i = 1, 2, \ldots, n$, satisfes Laplacian-gradient inequality (in particular f is harmonic) on D. Whether f is Lip on D?

Here, we state the following theorem, which proof will be omitted at this point.

Set $\gamma = 1 - \alpha$, $A = A_\gamma := d(z)^{-\gamma}$, $B = B_\gamma := |f'(z)|^{-\gamma}$ and $M = M_\gamma := AB$.

Theorem 5.3 ([10]). *Suppose that:*

(1) *D and G are domains with C^{1+} boundary, D is locally good Green-ian domain, $f : D \xrightarrow{onto} G$ proper and there is p such that $|\nabla f| \in L^p$, $p > n$,*
(2) *$f \in SC^1(D)$.*
(3) *Suppose in addition that G is $C^{1,\alpha}$ domain, f is a C^2 vector valued function, f_i, satisfy Laplacian-gradient inequality on D for $i = 1, 2, \ldots, n$.*
(4) *f is $K - qc$ on D.*

Conclusions:

(a) *If* (1) *holds, then $M_\gamma \in L^l$, for $l < l_0 = \frac{p}{2-\gamma+p\gamma}$.*
(b) *If* (1)–(4) *hold, then f is Lipschitz continuous on D.*

In the paper [17] is proved that harmonic quasi-regularity of function f implies condition (2) of the previous theorem. It is important to note that condition (2) is equivalent with Lipshitz continuty of function f wrt to quasi-hyperbolic metric.

6. Appendix

6.1. **Belgrade Seminary of Complex Analysis.** For detailed presentation of the eariest hystory of analysis school in Serbia, see article [13]. At this point, it is important to mention prof. Dajović, as the initiator of today Seminary. After the retirement of Professor Dajović, the group for complex analysis (M. Mateljević, M. Pavlović, M. Jevtić, M. Obradović) considered problems related to the spaces of analytic functions and slowly achieved international reputation. After returning from the USA in 1990, prof. Mateljević started working with N. Lakić in the field of quasiconformal mappings. Lakić soon left for the USA and obtained significant results in the field of Teichmüller spaces. The Seminary for Complex Analysis gains an international reputation, and there is talk of the Belgrade School (On conferences: Reich, Krushkal, Cazacu, Stanojević and others, especially Olli Martio during a visit to Belgrade in 2009.). It seemed that complex analysis had reached its highest point in Belgrade. But the surprises continue. V. Marković and V. Božin appear at the seminar. Together with Mateljević and Lakić, they solve Teichmüller's problem of extremal dilations. Today, V. Marković (In 2014, he was elected a member of the British Royal Society.) is a world leader in qc mapping theory and 3-dimensional topology and geometry and a proffessor at Univercity of Oxford. D. Kalaj and D. Šarić become internationaly recognized. D. Vukotić and N. Šešum also started at the seminar. Currently, M. Marković, M. Knežević, M. Svetlik, N. Mutavdžić, B. Karapetrović, P. Melentijević are actively participating in the seminar. For the seminary (or for the complex analysis group) are also connected M. Jevtić, M. Pavlović, M. Arsenović, S. Stević, I. Anić, M. Laudanović, O. Mihić, V. Manojlović, N. Babačev, A. Abaob, A. Shkheam, D. Đurčić, A. Bulatović, I. Petrović, S. Nikčević, V. Grujić as well as the students who presented on optional courses: J. Gajić, I. Savković, A. Savić, D. Fatić, M. Milović, N. Lelas, M. Lazarević, V. Stojisavljević, S. Gajović, F. Živanović, D. Kosanović, D. Špadijer, Z. Golubović and S. Radović. D. Kečkić, R. Živaljević, D. Milinković, D. Jocić, Đ. Milićević, D. Damjanović, D. Ranković, V. Baltić, N. Jozić (Baranović) and M. Albianić.

We briefly mention some facts related to the beginning of work on hqc mapping in Belgrade. During the visiting position at Wayne State University, Detroit, 1988/89, the author (of the revised article [9]) started considering hqc mappings. In particular, the author of [9] observed that the following results hold (see Proposition 6.1 and 6.2 below) and, when returned to Belgrade, used to talk on the seminary permanently and asked several open questions related to the subject. Many research papers are based on these communications.

Since not all of these researches have been published, it happens that some researchers discovered them later. Here we only discuss a few results from Revue Roum. Math. Pures Appl. **51**(5–6) (2006), 711–722.

Proposition 6.1 (Proposition 5 [11])**.** *If h is a harmonic univalent orientation preserving K-qc mapping of domain D onto D', then*

$$d(z)\Lambda_h(z) \leqslant 16\,K\,d_h(z) \quad \text{and} \quad d(z)\lambda_h(z) \geqslant \frac{1-k}{4} d_h(z)\,. \tag{6.1}$$

Proposition 6.2 (Corollary 1, Proposition 5 [11])**.** *Every e-harmonic quasi-conformal mapping of the unit disc* (*more generally of a strongly hyperbolic domain*) *is a quasi-isometry with respect to hyperbolic distances.*

The next theorem concerns harmonic maps onto a convex domain. For the planar version of Theorem 6.1 cf. [11,12], also [18, pp. 152-153]. The space version was communicated on International Conference on Complex Analysis and Related Topics (Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania), by Mateljević and stated in [11], also [14].

Theorem 6.1 (Theorem 1.3, [11])**.** *Suppose that h is an Euclidean harmonic mapping from the unit ball $\mathbb{B}^n$ onto a bounded convex domain $D = h(\mathbb{B}^n)$, which contains the ball $h(0) + R_0\mathbb{B}^n$. Then for any $x \in \mathbb{B}^n$*

$$d(h(x), \partial D) \geqslant (1 - \|x\|)R_0/2^{n-1}.$$

For further results of this type, see [14,17], and the literature cited there.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1041–1046.

THE INDEX FUNCTION OPERATOR FOR O-REGULARLY VARYING FUNCTIONS

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Abstract. The paper examines the functional transformation K of the class ORV_φ (see [3]) into the class of positive functions on interval $(0,+\infty)$ defined as follows:

$$K(f)=k_f, \tag{0.1}$$

where

$$k_f(\lambda)=\limsup_{x\to+\infty}\frac{f(\lambda x)}{f(x)},\quad \lambda\in(0,+\infty),$$

and $f\in ORV_\varphi$.

Let $f\in IRV_\varphi$ or SO_φ (see [4]), K be the transformation (0.1) and for any $n\in\mathbb{N}$, $K_n(f)=\underbrace{K(K\cdots(K}_{\text{n}}(f))\cdots)$, then the function $p(s)=\lim_{n\to+\infty}K_n(f)(s)$, $s>0$, is IRV_φ (and continuous) and SO_φ, respectively.

1. Introduction

The classic Karamata theory of regular variability has its beginnings in the 30s of the last century. Namely, studying the asymptotic properties of Riemann-Stieltjes (especially the Dirichlet and power series) Karamata observed the connection between the asymptotic properties of kernel of the Riemann-Stieltjes integral and the properties of that integral. Thus, asymptotic properties (serious and essential) for functions (and sequences) were perceived: regular variability and rapid variability; the study of the same in a qualitative sense and in applications began immediately (see [3]). These properties found a special place in the theory of summability, the theory of oscillations,

Key words and phrases. O-regularly varying function, index function.
2020 *Mathematics Subject Classification.* Primary: 26A12.
DOI 10.46793/KgJMat2307.1041DJ
Received: October 03, 2022.
Accepted: January 13, 2023.

the theory of Tauber properties, Fourier analysis, number theory, differential equations, etc (see [3]). In the 30s of the last century (and later) there were modifications of the classic Karamata theory of regular variability, depending on the needs of research. So, for example, the theory O-regular variability appears, which is an significant Tauber condition in very important Tauber-type theorems (see [1]). Recently, the classical Karamata theory of regular variability has a significant place in machine learning (especially in determining the direction of variation).

A function $f : [a, +\infty) \to (0, +\infty)$ is O-regularly varying function in the sense of Karamata (see [3] and [1]), if for some fixed $a > 0$ it is measurable and

$$\limsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = k_f(\lambda) < +\infty \tag{1.1}$$

holds, for every $\lambda > 0$. The function $k_f(\lambda)$, $\lambda > 0$, is called the index function of the function f and its characteristics give many of the asymptotic properties of the function f (see [2] and [4]). The function $k_f(\lambda)$, $\lambda > 0$, can be both measured and immeasurable. An example of the immeasurable index function is given by Rubel in [16] who has constructed the appropriate function f.

O-regularly varying functions in the sense of Karamata form a functional class ORV_φ, and elements of that class are very important objects in the qualitative analysis of divergent functional processes (see [3] and [1]).

1° Assume that $f \in ORV_\varphi$. Then $f \in IRV_\varphi$ (in some literature we can find that class IRV_φ is denoted by CRV_φ[4]) if $k_f(\lambda)$, $\lambda > 0$, is continuous. The importance of this class can be seen in the asymptotic analysis in points (e.g. [4, 5] and [7]).

2° Assume that $f \in ORV_\varphi$. Then $f \in ERV_\varphi$ (the class ERV_φ is so-called the Matuszewski class [14, 15] and [5], or the extended class of regularly varying functions in the sense of Karamata [3]), if $k_f(\lambda)$, $\lambda > 0$, for $\lambda = 1$ has finite one-sided derivatives. See [14] about the qualitative properties of the class ERV_φ.

3° Assume that $f \in ORV_\varphi$. Then $f \in RV_\varphi$ (RV_φ is a well-known class of regularly varying function in the sense of Karamata [9, 10]) if $k_f(\lambda)$, $\lambda > 0$, is differentiable function. An especially important subclass of RV_φ is the class SV_φ (slowly varying function in the sense of Karamata [1, 18]). For this function it holds that $k_f(\lambda) = 1$, for every $\lambda > 0$ (if $f \in SV_\varphi$).

It holds that (see [4])

$$SV_\varphi \subsetneq RV_\varphi \subsetneq ERV_\varphi \subsetneq IRV_\varphi \subsetneq ORV_\varphi. \tag{1.2}$$

Classes of functions in 1°, 2°, 3° are very important elements of Karamata's theory of regular variation (see [2] and [17]) and its applications (see [6, 8, 11–13] and [18]).

It is easy to prove the next lemma.

Lemma 1.1. *Assume that* $f \in ORV_\varphi$.

(a) *If* $k_f(\lambda)$, $\lambda > 0$, *is a measurable function, then* $k_f \in ORV_\varphi$.

(b) *If* $k_f(\lambda)$, $\lambda > 0$, *is a continuous function, then* $k_f \in IRV_\varphi$.

(*c*) *If* $k_f(\lambda)$, $\lambda > 0$, *has finite one-sided derivatives for* $\lambda = 1$, *then* $k_f \in ERV_\varphi$.
(*d*) *If* $k_f(\lambda)$, $\lambda > 0$, *is differentiable function, then* $k_f \in RV_\varphi$.
(*e*) *If* $k_f(\lambda)$, $\lambda > 0$, *is a constant function for* $\lambda > 0$, *then* $k_f \in SV_\varphi$.

2. The Main Result

Consider the functional transformation K on class ORV_φ into the class of positive functions defined on the interval $(0, +\infty)$, given as

$$K(f) = k_f. \tag{2.1}$$

The index function of function $f \in OR_\varphi$ (the operator $K(f) = k_f$) in the notation k_f carries with very important features for the function f. For example, upper and lower Karamata's index, also both Matuszewski's index for the observed function f. The characteristics of the index function k_f for the function $f \in ORV_\varphi$ describe the relation of the function f to the asymptotic equivalence relations and to the generalized inverse (see [3–8, 11] and [15]).

We can see that

$$K(K(f)) = K(f), \tag{2.2}$$

if $f \in RV_\varphi$.

The K transformation is called the index function operator. Its properties can be seen in Lemma 1.1. According to everything given above, it makes sense to consider the iterative process

$$K_n(f) = \underbrace{K(K \cdots (K}_{\text{n}}(f)) \cdots), \tag{2.3}$$

for $n \in \mathbb{N}$ on the class ORV_φ, IRV_φ, ERV_φ, RV_φ.

Let us consider the properties of the operator (2.1) in the sense of iterative process (2.3) on the class IRV_φ. On the class RV_φ for the operator (2.1) the iterative process (2.3) is described by (2.2).

In probability and statistics there is a great need for important characterizations using Seneta's functions (O-regular variable functions with a bounded index function). They are essential generalizations of slow varying functions: each of them is the product of a slow varying and bounded function that is positive.

A function $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, is called β-Seneta's function (see [17]), if there exist $\beta \geqslant 1$, $\beta \in \mathbb{R}$, such that

$$k_f(\lambda) \leqslant \beta, \tag{2.4}$$

for every $\lambda > 0$.

The class of β-Seneta's functions that satisfy (2.4) for given $\beta \geqslant 1$, $\beta \in \mathbb{R}$, we denote by SO_φ^β, and the class of all Seneta's functions by $SO_\varphi = \cup_{\beta \geqslant 1} SO_\varphi^\beta$.

This class is very important in approximation theory and probability theory (see [3] and [17]).

We have that $SV_\varphi \subseteq SO_\varphi \subseteq ORV_\varphi$. The class SO_φ can not be compared with classes IRV_φ and ERV_φ. We also have $SO_\varphi \cap (RV_\varphi \setminus SV_\varphi) = \emptyset$.

Lemma 2.1. *Let $f \in SO_\varphi$. If $k_f(\lambda)$, $\lambda > 0$, is a measurable function, then $K(f) \in SO_\varphi$. If $f \in SO_\varphi^\beta$ and $k_f(\lambda)$, $\lambda > 0$, is a measurable function, then $K(f) \in SO_\varphi^\beta$.*

Proof. If we give a proof for the second statement, then the first statement holds. Assume $f \in SO_\varphi^\beta$, for some real $\beta \geqslant 1$.

Then

$$0 < k_f(\lambda) = \limsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)} \leqslant \beta < +\infty, \tag{2.5}$$

for every $\lambda > 0$. Also, the function $k_f(\lambda)$, $\lambda > 0$, is measurable and for every $s > 0$ and every $t > 0$ and

$$\begin{aligned} 0 < k_f(st) &= \limsup_{x\to+\infty} \frac{f(stx)}{f(x)} = \limsup_{x\to+\infty} \left(\frac{f(stx)}{f(tx)} \cdot \frac{f(tx)}{f(x)} \right) \\ &\leqslant \limsup_{x\to+\infty} \frac{f(stx)}{f(tx)} \cdot \limsup_{x\to+\infty} \frac{f(tx)}{f(x)} \\ &= k_f(s) \cdot k_f(t) \end{aligned}$$

is satisfied. Actually, for every $t > 0$, we have that

$$0 < \limsup_{s\to+\infty} \frac{k_f(ts)}{k_f(s)} \leqslant k_f(t) \leqslant \beta < +\infty.$$

Hence, $k_f(\lambda)$, $\lambda > 0$, belongs to the class SO_φ^β. $\square$

From the above, we can conclude that for every $n \in \mathbb{N}$, $K_n(f) \in SO_\varphi$ is satisfied if the function $K_n(f)$ is measurable and $f \in SO_\varphi$.

Theorem 2.1. *Let $f \in ORV_\varphi$ and let operator K be given as in* (2.1). *Also, let functions $K_n(f)$, $n \in \mathbb{N}$, be given as in* (2.3) *are measurable. Then, for every $s > 0$, there is a function $p(s) = \lim_{n\to+\infty} K_n(f)(s)$ which belongs to class ORV_φ. Specially, if $f \in SO_\varphi^\beta \subsetneq ORV_\varphi$, then $p \in SO_\varphi^\beta$.*

Proof. Let $f \in ORV_\varphi$. Then according to Lemma 1.1 (a) function $K(f) \in ORV_\varphi$. Sequence of functions $K_n(f)(s)$, $s > 0$, is non-increasing sequence (supreme norm) of functions which are measurable and hold that $1 \leqslant K_n(f)(s) \cdot K(f)(\frac{1}{s}) < +\infty$ for every $n \in \mathbb{N}$ and every $s > 0$. That means, for every $s > 0$, sequence $(K_n(f)(s))$ converges to $0 < p(s) < +\infty$. The function $p(s)$, $s > 0$, is measurable as limit function of measurable functions.

As for every $s, t > 0$

$$p(s \cdot t) \leqslant p(s) \cdot p(t),$$

then for every $s > 0$

$$\limsup_{t\to+\infty} \frac{p(st)}{p(t)} = k_p(s) \leqslant p(s) < +\infty.$$

Hence, holds $p \in ORV_\varphi$. Specially, if $f \in SO_\varphi^\beta$, then according to Lemma 2.1 function $K(f) \in SO_\varphi^\beta$. Thus, for every $s > 0$, holds $k_p(s) \leqslant p(s) \leqslant K(f) \leqslant \beta$. Regarding, it is valid that $p \in SO_\varphi^\beta$. $\square$

Corollary 2.1. *If we observe class of Seneta's functions SO_φ instead of S_φ^β, the Theorem* 2.1 *still holds.*

Theorem 2.2. *Let $f \in IRV_\varphi$ and the operator K be given as* (2.1)*. Then the function $p(s) = \lim_{n\to+\infty} K_n(f)(s)$, $s > 0$, exists for $s > 0$, is continuous, and belongs to the class IRV_φ.*

Proof. Let $f \in IRV_\varphi$, then according to Lemma 1.1 (b), the function $K(f) \in IRV_\varphi$ is continuous on $(0, +\infty)$. Also, for every $n \in \mathbb{N}$, the function $K_n(f) \in IRV_\varphi$ is continuous on $(0, +\infty)$. If $s = 1$, then $p(s) = 1$. If $s > 0$, $s \neq 1$, then for every $n \in \mathbb{N}$ it holds

$$0 < \frac{1}{K_n(f)(\frac{1}{s})} \leqslant \frac{1}{K_{n+1}(f)(\frac{1}{s})} \leqslant K_{n+1}(f)(s) \leqslant K_n(f)(s) < +\infty.$$

Hence, the function $p(s)$ is finite and positive for $s > 0$. As $p(s) \leqslant K_1(f)(s)$ for every $s > 0$ and $\lim_{s\to1} K_1(f)(s) = 1$, then $\limsup_{s\to1} p(s) \leqslant 1$. Therefore, the function p is measurable on $(0, +\infty)$ as the limit value of a continuous function, and for every $s, t > 0$ we have

$$\begin{aligned} p(st) &= \lim_{n\to+\infty} K_n(f)(st) \\ &\leqslant \lim_{n\to+\infty} K_n(f)(s) \cdot \lim_{n\to+\infty} K_n(f)(t) \\ &= p(s) \cdot p(t). \end{aligned}$$

It means that the function p is continuous on $(0, +\infty)$. Since $K(p)(s) \leqslant p(s)$ for every $s > 0$ and $\limsup_{s\to1} K(p)(s) \leqslant 1$, then $K(p)$ is continuous on $(0, +\infty)$. It holds that $p \in IRV_\varphi$. $\square$

Remark 2.1. The continuinity of function p on $(0, +\infty)$ can be proved by using well-known Dini's theorem of uniform convergences.

Corollary 2.2. *Let $f \in ERV_\varphi$. Then $K(p) \in IRV_\varphi$, where the operator K is given by* (2.1) *and $p(s) = \lim_{n\to+\infty} K_n(f)(s)$, $s > 0$, ($K_n(f)$ is given by* (2.3)*, for every $n \in \mathbb{N}$).*

We finish with one open problem.

Remark 2.2. Does $p \in ERV_\varphi$ hold from Corollary 2.2?

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KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(7) (2023), PAGES 1047–1056.

ALMOST MULTI-DIAGONAL DETERMINANTS

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ABSTRACT. We found motivation for this paper in the conjectures about multi-diagonal determinants published in a few recent papers. Especially, we were interested in the case with a few non-zero elements in the lower left corner or/and in the upper right corner. Our research with changeable free elements lead us to the systems of partial differential equations. Also, we include some generalizations of the problems and conjectures.

1. INTRODUCTION

These determinants are of the theoretical and applicable interest. We can emphasize the computational problems related to the such matrices and their determinants as: the calculation of spectra, permanent, characteristic polynomial, inverse matrix, power, and decomposition of a matrix. They appear in the numerical methods for the differential equations. It is known that the three diagonal determinants are very important in the number theory and the theory of orthogonal polynomials and the five diagonal determinants in the statistics.

An almost (nearly) five constant diagonal determinant of ordinary order was considered in the paper [6], and the numerical methods for its numerical computing were developed. Similar problem was considered in the paper [7, 8].

Recently, [1] in 2020. Conjectures 6.1. and 6.2. about the almost four constant diagonal unit determinants were formulated. They caused a lot of attention and were proven a few months later in [9].

But, they initialized other considerations in that direction.

Key words and phrases. Determinants, multi-diagonal, closed form.
2020 *Mathematics Subject Classification.* Primary: 11C20. Secondary: 15A15, 47B36, 65F40.
DOI 10.46793/KgJMat2307.1047R
Received: September 29, 2022.
Accepted: January 17, 2023.

In the paper [4], the two sided almost constant multi-diagonal determinants were studied.

Papers about multi-diagonal matrices with equally spaced diagonals appeared soon. In the papers [10, 11], the multi-diagonal determinants with rare nonzero elements were considered.

This paper is organized as follows. In the Section 1, it is given the survey of the papers which deal with the multi-diagonal determinants and nearby multi-diagonal determinants. The preliminaries, i.e., definitions and known theorems were exposed in the Section 2. The last section is fulfilled with original contributions to the almost multi-diagonal determinants and their reduction to the systems of partial differential equations. We did not see any trial with such approach as we did in the Section 3. We believe that this point of view can be of interest for all which are investigating in this area.

2. Multi-Diagonal Determinants

In the paper [3], there is the following definition.

Definition 2.1. A square matrix $P_n(r,s) = [p_{i,j}]_{i,j=0}^{n-1}$ is (r,s)-*banded matrix* if

$$p_{i,j} = 0, \quad \text{for all } (i,j) : i-j > r \text{ or } j-i > s, \quad s, r \in \mathbb{N} : r+s < n. \tag{2.1}$$

The bandwidth of an (r,s)-banded matrix is $r+s+1$. In the expanded form, it can be written as

$$P_n(r,s) = \begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,s} & 0 & \cdots & 0 \\ p_{1,0} & p_{1,1} & & & p_{1,s+1} & & 0 \\ \vdots & & \ddots & & \ddots & & \\ p_{r,0} & & & p_{r,r} & & & \\ 0 & p_{r+1,1} & & & & & \\ \vdots & & \ddots & & \ddots & & 0 \\ & & & & & & p_{n-s-1,n-1} \\ 0 & & & & & \ddots & \vdots \\ 0 & & & & p_{n-1,n-r-1} & \cdots & p_{n-1,n-1} \end{bmatrix}.$$

Let us remind that a rational function $f(x_1,\dots,x_n)$ is *homogeneous* of degree k if

$$f(tx_1,\dots,tx_n) = t^k f(x_1,\dots,x_n), \quad \text{for all } (x_1,\dots,x_n) \in \mathbb{R}^n.$$

Lemma 2.1 ([16]). *Let $P_n(r,s) = [p_{i,j}]_{i,j=0}^{n-1}$ be an (r,s)-banded matrix with the principal minors π_k. Then, for every $n > \delta = \binom{r+s}{r}$, the sequence $\{\pi_k\}$ satisfies a nontrivial homogeneous linear recurrence relation of the form*

$$\pi_n = \sum_{k=1}^{\delta} R_k \pi_{n-k}, \tag{2.2}$$

where R_k is a homogeneous rational function of degree k with entries

$$\{a_{n-i,n-j}\}_{0\le i\le \delta-1;-s\le j\le r+\delta-1}.$$

In the continuation we will deal with the following matrices.

Definition 2.2. A square matrix $P_n(r,s;A)=[p_{i,j}]_{i,j=0}^{n-1}$ is (r,s)-*constant diagonal matrix* if it is (r,s)-*banded matrix* and

$$p_{i,i+j}=a_j,\quad j=-r,-r+1,\dots,s;i=0,1,\dots,n-1.$$

Consider the constant five-diagonal, i.e., $(2,2)$-banded determinants:

$$\pi_n=\pi_n(2,2;\mathbf{A_5})=\begin{vmatrix} a_0 & a_1 & a_2 & 0 & \cdots & 0 & 0 & 0 & \\ a_{-1} & a_0 & a_1 & a_2 & & 0 & 0 & 0 & \\ a_{-2} & a_{-1} & a_0 & a_1 & & 0 & 0 & 0 & \\ 0 & a_{-2} & a_{-1} & a_0 & & 0 & 0 & 0 & \\ \vdots & & & & \ddots & & & & \\ 0 & 0 & 0 & & & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & & & a_{-1} & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & & & a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & 0 & 0 & & & 0 & a_{-2} & a_{-1} & a_0 \end{vmatrix}_{n\times n},$$

where

$$\mathbf{A_5}=\{a_{-2},a_{-1},a_0;a_1,a_2\}.$$

Lemma 2.2 ([15])**.** *The sequence $\{\pi_n\}$, where $\pi_n=\pi_n(2,2;\mathbf{A_5})$, satisfies the seventh-term recurrence relation*

(2.3)
$$\begin{aligned}\pi_n=&a_0\pi_{n-1}+(a_2a_{-2}-a_1a_{-1})\pi_{n-2}+(a_2a_{-1}^2+a_1^2a_{-2}-2a_0a_2a_{-2})\pi_{n-3}\\ &+a_2a_{-2}(a_2a_{-2}-a_1a_{-1})\pi_{n-4}+a_0(a_2a_{-2})^2\pi_{n-5}-(a_2a_{-2})^3\pi_{n-6},\quad n=5,6,\dots\end{aligned}$$

Example 2.1. The three unit diagonal determinants $D_{3,n}=\pi_n(1,1;\{1,1,1\})$ satisfy the three-term recurrence relation

$$D_{3,n}=D_{3,n-1}-D_{3,n-2},\quad n\ge 5, \tag{2.4}$$

with initial values

$$D_{3,0}=1,\quad D_{3,1}=0. \tag{2.5}$$

The general solution of this difference equation and the initial values (2.5) give us the explicit form of the determinant $D_{3,n}$ with

$$D_{3,n}=\cos\left(\frac{n\pi}{3}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Even more, because of the presence of the cosine and sine function which have the periods, the determinants $D_{3,n}$ have the periodicity $T=6$, and the values:

$$D_{3,6n}=D_{3,6n+1}=1,\quad D_{3,6n+2}=0,\quad D_{3,6n+3}=D_{3,6n+4}=-1,\quad D_{3,6n+5}=0,$$

for $n=0,1,\dots$

Example 2.2. Let $\mathbf{I_4} = \{1, 1, 1, 1\}$. The four unit diagonal determinants

$$D_{4,n} = \pi_n(2, 1; \mathbf{I_4}) = |d_{i,j}|_{n\times n} : \quad d_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 1, \\ 1, & \text{if } i = j+2, \\ 0, & \text{others}, \end{cases} \tag{2.6}$$

satisfy the four-term recurrence relation

$$D_{4,n} = D_{4,n-1} - D_{4,n-2} + D_{4,n-3}.$$

Its general solution is

$$D_{4,n} = C_1 + C_2 \cos\frac{n\pi}{2} + C_3 \sin\frac{n\pi}{2}.$$

Using the initial values $D_{4,1} = 1$, $D_{4,2} = D_{4,3} = 0$, we find

$$D_{4,n} = \frac{1}{2}\left(1 + \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right).$$

Hence, its value is

$$D_{4,n} = \frac{1 + (-1)^{\lfloor n/2 \rfloor}}{2}, \quad n \in \mathbb{N}, \tag{2.7}$$

i.e.,

$$D_{4,n} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 4 \vee n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2 \pmod 4 \vee n \equiv 3 \pmod 4. \end{cases}$$

Remark 2.1. Notice that we will get the same value for the non-symmetric unit diagonal upper or lower with respect to the main diagonal. But, in some further considerations, it will be important for conclusions.

Example 2.3. The five unit diagonal determinants $D_{5,n} = \pi_n(2, 2; \{\mathbf{1}\})$ satisfy the seven-term recurrence relation

$$D_{5,n} = D_{5,n-1} + D_{5,n-5} - D_{5,n-6}, \quad n \ge 5, \tag{2.8}$$

with initial values

$$D_{5,0} = D_{5,1} = 1, \quad D_{5,k} = 0, \quad k = 2, 3, 4, \quad D_{5,5} = 1. \tag{2.9}$$

The general solution of this difference equation is

$$D_{5,n} = C_1 + C_2 n + C_3 \cos\frac{4n\pi}{5} - C_4 \sin\frac{4n\pi}{5} + C_5 \cos\frac{2n\pi}{5} + C_6 \sin\frac{2n\pi}{5}.$$

Including the initial values (2.9), we find the explicit form of the determinant $D_{5,n}$ with

$$C_1 = \frac{2}{5}, \quad C_2 = 0, \quad C_3 = \frac{3-\sqrt{5}}{10},$$
$$C_4 = -\frac{1}{5}\sqrt{\frac{5-\sqrt{5}}{2}}, \quad C_5 = \frac{3+\sqrt{5}}{10}, \quad C_6 = \frac{1}{5}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

Even more, the determinants $D_{5,n}$ have the periodicity $T = 5$ and the values:

$$D_{5,5n} = D_{5,5n+1} = 1, \quad D_{5,5n+k} = 0, \quad k = 2, 3, 4; n \in \mathbb{N}. \tag{2.10}$$

The computation of the exact values of the determinants $D_{k,n}$ for a lot of k's and n's, shows that we can establish the following conjecture.

Conjecture 2.1. *The determinants $\{D_{2k,n}\}_{n\in\mathbb{N}}$ have the periodicity $T = 2k$. The determinants $\{D_{2k+1,n}\}_{n\in\mathbb{N}}$ have the periodicity $T = 2k + 1$ or $T = 4k + 2$.*

Remark 2.2. A useful method for computing multi-diagonal determinants is, if it is possible, to decompose them into the product of lower and upper triangular matrix.

Remark 2.3. Many papers reals with the multi-diagonal determinants with the special numbers. For example, the role of the Fibonacci numbers in the nature and science induce that a lot attention is ascribed them. Numerous papers deal with their properties and representations (see [14]). They appear like values of special determinant sequences what was shown in the papers [2, 13] and [12].

Let $A(t)$ be a functional matrix

$$A(t) = [a_{i,j}(t)]_{n\times n}\,. \tag{2.11}$$

If we denote by $\hat{a}_k(t)$ the k^{th} row, we can write

$$\hat{a}_k(t) = \begin{bmatrix} a_{k,1}(t) & a_{k,2}(t) & \cdots & a_{k,n}(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \\ \vdots \\ \hat{a}_n(t) \end{bmatrix}. \tag{2.12}$$

The k^{th} derivative of the matrix $A(t)$ is

$$A^{(k)}(t) = \left[a_{i,j}^{(k)}(t)\right]_{n\times n}, \quad k \in \mathbb{N},$$

with assumption that all derivatives $a_{i,j}^{(k)}(t)$ exist.

Lemma 2.3 (Jacobi formula)**.** *The derivative of the determinant* (2.11) *can be expressed in the form*

$$D_t \det A(t) = \sum_{k=1}^{n} \mathcal{T}_k(A; t), \tag{2.13}$$

where

$$(2.14)\qquad \mathcal{T}_1(A;t)=\begin{vmatrix} D_t\hat{a}_1(t)\\ \hat{a}_2(t)\\ \vdots\\ \hat{a}_{n-1}(t)\\ \hat{a}_n(t)\end{vmatrix},\quad \mathcal{T}_k(A;t)=\begin{vmatrix} \hat{a}_1(t)\\ \vdots\\ \hat{a}_{k-1}(t)\\ \hat{D}_t\hat{a}_k(t)\\ \hat{a}_{k+1}(t)\\ \vdots\\ \hat{a}_{n-1}(t)\\ \hat{a}_n(t)\end{vmatrix},\quad k=2,\dots,n.$$

In more general form, we can find it in [5]:

$$D\det(A)(X)=\operatorname{tr}\left(\operatorname{adj}(A)X\right),$$

i.e.,

$$D\det(A)(X)=\sum_{i,j}\det M_{i,j}x_{i,j},$$

where $M_{i,j}$ is (i,j)-cofactor of A.

Denote by

$$(2.15)\qquad \nabla_{k,n}=D_{k,n}-D_{k.n-1}.$$

3. Some Almost Multi-Diagonal Determinants

There are determinants which have at least an element out of multi-diagonals. The Lagrange expansion was applied for some easier cases in a few papers (see, for example [4] and [9]). But, it requires a lot of computation and a lot of difficulties appear.

Here, we will use Jacobi formula for differentiation of determinants (2.11) for finding their closed form values.

Theorem 3.1. *The almost three unit diagonal determinant*

$$A_{3,n}=\begin{vmatrix} 1&1&0&0&\cdots&0&y&x\\ 1&1&1&0&&0&0&z\\ 0&1&1&0&&0&0&0\\ \vdots&&&\ddots&&&\vdots&\\ &&&&&&&\\ 0&0&0&&&1&1&0\\ 0&0&0&\cdots&&1&1&1\\ 0&0&0&&&0&1&1\end{vmatrix}_n\;:\quad a_{i,j}=\begin{cases}1, & \text{if } |i-j|\le 1,\\ x, & \text{if } i=1\wedge j=n,\\ y, & \text{if } i=1\wedge j=n-1,\\ z, & \text{if } i=2\wedge j=n,\\ 0, & \text{others,}\end{cases}$$

has the value

$$(3.1)\qquad A_{3,n}=(-1)^{n+1}(x-y-z)+D_{3,n}.$$

Proof. Applying the Jacobi formula for determinants (2.13), we get the system of partial differential equations

$$\frac{\partial A_{3,n}}{\partial x} = (-1)^{n+1}, \quad \frac{\partial A_{3,n}}{\partial y} = (-1)^n, \quad \frac{\partial A_{3,n}}{\partial z} = (-1)^n.$$

Integrating the first equation, we find

$$A_{3,n} = (-1)^{n+1}x + \varphi(y,z).$$

Hence, $\frac{\partial A_{3,n}}{\partial y} = \frac{\partial \varphi}{\partial y} = (-1)^n$ implies $\varphi = (-1)^n y + \psi(z)$. Now, we have

$$A_{3,n} = (-1)^{n+1}x + (-1)^n y + \psi(z).$$

By differentiation via z, we obtain $\frac{\partial A_{3,n}}{\partial z} = \frac{\partial \psi}{\partial z} = (-1)^n$ implies $\psi = (-1)^n z + C(n)$. Finally, we have

$$A_{3,n} = (-1)^{n+1}(x - y - z) + C_n.$$

Knowing that $A_{3,n}(0,0,0) = D_{3,n}$, we get the statement. $\square$

Theorem 3.2. *The upper almost four unit diagonal determinant*

$$A_{4,n} = |a_{i,j}|_{n\times n}: \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 1 \wedge \ i = j+2, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ 0, & \text{others}, \end{cases} \tag{3.2}$$

has the value

$$A_{4,n} = D_{4,n} + yz + (-1)^n \Big(-D_{4,n-1}x + (\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5})(y+z) \Big).$$

Proof. Derivative of a determinant is the sum of determinants provided by successive deriving the rows in the given determinant. Hence,

$$\begin{aligned} \frac{\partial A_{4,n}}{\partial x} &= (-1)^{n-1} D_{4,n-1}, \\ \frac{\partial A_{4,n}}{\partial y} &= z + (-1)^n (\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}), \\ \frac{\partial A_{4,n}}{\partial z} &= y + (-1)^n (\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}). \end{aligned}$$

Here, we have the system of three partial linear differential equations with unknown function $A_{4,n}(x,y,z)$. By integrating the first one, we get

$$A_{4,n} = (-1)^{n-1} D_{4,n-1}x + \varphi(y,z),$$

where $\varphi(y,z)$ is an arbitrary differentiable real function. Differentiating $A_{4,n}$ by y, we find

$$\frac{\partial A_{4,n}}{\partial y} = \frac{\partial \varphi}{\partial y} = z + (-1)^n (\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}),$$

wherefrom

$$\varphi = yz + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) y + \psi(z),$$

where $\psi(z)$ is an arbitrary differentiable real function. Hence,

$$A_{4,n} = (-1)^{n-1} D_{4,n-1} x + yz + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) y + \psi(z).$$

Finally, differentiating $A_{4,n}$ by z, we find

$$\frac{\partial A_{4,n}}{\partial z} = y + \psi'(z) = y + (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right),$$

wherefrom

$$\psi = (-1)^n \left(\nabla_{4,n-3} - \nabla_{4,n-4} + \nabla_{4,n-5}\right) z + C.$$

Knowing that $A_{4,n}(0,0,0) = D_{4,n}$, we get the statement. $\square$

Remark 3.1. The statement of the theorem can be written in the from

$$A_{4,n} = \begin{cases} (1-y)(1-z), & \text{if } n \equiv 0 \pmod 4, \\ 1 + x + yz, & \text{if } n \equiv 1 \pmod 4, \\ -x + y + (1+y)z, & \text{if } n \equiv 2 \pmod 4, \\ yz, & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Remark 3.2. When $x = b$ and $y = z = a$, we confirm the main result in the paper [9].

In the similar way, we can prove the following theorems.

Theorem 3.3. *The almost five unit diagonal determinant*

$$A_{5,n} = |a_{i,j}|_{n\times n} : \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 2, \\ x, & \text{if } i = 1 \land j = n, \\ y, & \text{if } i = 1 \land j = n-1, \\ z, & \text{if } i = 2 \land j = n, \\ 0, & \text{others}, \end{cases} \tag{3.4}$$

has the value

$$A_{5,n} = \begin{cases} (1-y)(1-z), & \text{if } n \equiv 0 \pmod 5, \\ 1 + x + yz, & \text{if } n \equiv 1 \pmod 5, \\ -x + y + (1+y)z, & \text{if } n \equiv 2 \pmod 5, \\ yz, & \text{if } n \equiv 3 \pmod 5, \\ yz, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Also, this method can be applied on the two sided almost multiple diagonal determinants considered in the paper [4].

Theorem 3.4. *The two sided almost five unit diagonal determinant*

$$A_n = \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & y & x \\ 1 & 1 & 1 & 1 & & 0 & 0 & z \\ 1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & & & 1 & 1 & 1 \\ v & 0 & 0 & \cdots & & 1 & 1 & 1 \\ u & w & 0 & & & 1 & 1 & 1 \end{vmatrix} : \quad a_{i,j} = \begin{cases} 1, & \text{if } |i-j| \le 2, \\ x, & \text{if } i = 1 \wedge j = n, \\ y, & \text{if } i = 1 \wedge j = n-1, \\ z, & \text{if } i = 2 \wedge j = n, \\ u, & \text{if } i = n \wedge j = 1, \\ v, & \text{if } i = n-1 \wedge j = 1, \\ w, & \text{if } i = n \wedge j = 2, \\ 0, & \text{others}, \end{cases}$$

has the value

$$\begin{aligned} A_{5n} &= (1-y)(1-z)(1-v)(1-w), \\ A_{5n+1} &= 1 + x + yz + u + vw, \\ A_{5n+2} &= -x + y + (1+y)z - u + v + (1+v)w - xu + zv + yw, \\ A_{5n+3} &= yz + vw + (-u + v + w + vw)x + (u-w)z + y(u - v + uz), \\ A_{5n+4} &= yz + vw - vwx + vwz + y(vw + (-u + v + w + vw)z). \end{aligned}$$

Proof. Applying again the Jacobi formula for determinants (2.13), we get the system of partial differential equations. For example, deriving by x, and after that by u, we find

$$\frac{\partial^2 A_n}{\partial x \partial u} = -D_{5,n-2}.$$

We will miss the whole proof because of its largeness. □

4. Conclusions

We researched the closed form for the multiple diagonal determinants with at most three elements in the opposite corners. Although it seems easy to be done by the Lagrange expansion, this method requires finding the recurrence relation with large depth. We pointed to the Jacobi formula for the derivation of the determinants as useful tool. It will be of interest to continue this research, for example, to examine the influence of a nonzero element at random position outside of the multiple diagonals on the determinant value.

Acknowledgements. This research was financially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Contract No. 451-03-68/2022-14/ 200109)

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KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(7) (2023), PAGES 1057–1063.

AN OPEN MAPPING THEOREM FOR ORDER-PRESERVING OPERATORS

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ABSTRACT. In the main result of this paper we prove a version of the well-known open mapping theorem for weakly additive, order-preserving operators between ordered real vector spaces with an order unit. We also provide a few examples to illustrate obtained results.

1. INTRODUCTION AND PRELIMINARIES

The open mapping theorem (known also as the Banach-Schauder theorem) is one of most important theorems in functional analysis [4], [14, Theorem 2.11] and has a number of applications in complex analysis [15, Theorem 4.4], topology [7,10,11] and in other mathematical disciplines (see, for instance, [1–3, 5, 6, 8, 9, 12, 13, 16, 17]). In this note we prove a version of this theorem for operators between ordered real vector spaces with an order unit.

We begin with definitions of notions that will be used in the sequel.

An element 1_E of an ordered real vector space E is said to be an *order unit* in E if for each $x \in E$ there is a real number $\varepsilon > 0$ such that $\varepsilon 1_E \geq x$.

In this article "spac" means "ordered real vector space".

Recall that a subset L of a space E with an order unit 1_E is said to be an A^{1_E}-*subspace* of E if $0_E \in L$, and $x \in L$ implies that $x + c1_E \in L$ for all $c \in \mathbb{R}$.

Key words and phrases. Ordered vector space, order unit, order-preserving mapping, weakly additive operator, open mapping theorem .

2020 *Mathematics Subject Classification.* Primary: 46A40. Secondary: 06F20, 46B40, 47B60, 54C10.

DOI 10.46793/KgJMat2307.1057K

Received: December 26, 2022.

Accepted: January 26, 2023.

The *order topology* on an ordered real vector space E with an order unit 1_E is the topology whose base is the collection of balls (with center x and radius ε)

$$B(x,\varepsilon) = \{y \in E\colon \|y - x\| < \varepsilon\}, \quad x \in E, \varepsilon > 0,$$

where for $x \in E$

$$\|x\| = \inf\{\lambda > 0\colon -\lambda 1_E \leq x \leq \lambda 1_E\}.$$

Recall that a mapping $f\colon X \to Y$ between topological spaces X and Y is said to be *open at* $x_0 \in X$ if for each open neighbourhood U of x_0 there exists an open neighbourhood V of $f(x_0)$, which lies in $f(U)$. A mapping $f\colon X \to Y$ is said to be *open* if it is open at every point $x \in X$, or, equivalently, if for any open set U in X its image $f(U)$ is an open set in Y.

Let E, F be spaces with order unit. An operator $f\colon E \to F$ is said to be:

(1) *order-preserving* if for any pair x, $y \in E$ the inequality $x \leq_E y$ implies $f(x) \leq_F f(y)$;

(2) *weakly additive* if the equality $f(x + \lambda 1_E) = f(x) + \lambda f(1_E)$ holds, for every $x \in E$ and every $\lambda \in \mathbb{R}$;

(3) *normed*, if $f(1_E) = 1_F$.

2. Results

We begin this section with some auxiliary results and examples.

Lemma 2.1. *Let E and F be spaces with order unit, $f : E \to F$ surjective, weakly additive order-preserving operator. If f is open at 0_E, then f is open over entire E.*

Proof. Let $x \in E$ be an arbitrary point and $B(x,\varepsilon) = x + B(0_E,\varepsilon)$ a neighbourhood of x. Since f is open in 0_E and $f(0_E) = 0_F$, there is $\mu > 0$ such that $B(0_F,\mu) \subset f(B(0_E,\varepsilon))$. We claim that

$$B(f(x),\mu) = f(x) + B(0_F,\mu) \subset f(B(x,\varepsilon)).$$

Let $y \in B(f(x),\mu)$. This means $y - f(x) \in B(0_F,\mu) \subset f(B(0_E,\varepsilon))$. It follows $y \in f(x) + f(B(0_E,\varepsilon))$, i.e., $y \in f(x + B(0_E,\varepsilon)) = f(B(x,\varepsilon))$. Therefore, f is open in $x \in E$. □

Recall that a metric on a vector space X is said to be *invariant* if

$$d(x + z, y + z) = d(x,y),$$

for all x, y, $z \in X$.

Lemma 2.2. *The metric generated by the order norm on a space with order unit is invariant.*

Proof. Let E be a space with order unit 1_E, $x, y \in E$. According to the definition of order norm we have

$$d(x,y) = \|y - x\| = \inf\{\lambda > 0\colon -\lambda 1_E \leq y - x \leq \lambda 1_E\}.$$

From here it follows $d(x + z, y + z) = d(x,y)$ for each vector $z \in E$. □

Recall that the graph of a mapping f of a set X into a set Y is the set of all pairs $(x, f(x))$ in the Cartesian product $X \times Y$. If X and Y are topological spaces, then in their product we will consider the usual product topology.

Let E and F be spaces with order unit. The product $E \times F$ becomes a space with order unit if one introduces on it coordinate-wise operations of addition and multiplication by a number:

$$\alpha(x_1,\, y_1) + \beta(x_2,\, y_2) = (\alpha x_1 + \beta x_2,\, \alpha y_1 + \beta y_2),$$

and coordinate-wise partial order:

$$(x_1,\, y_1) \leq_{E\times F} (x_2,\, y_2) \Leftrightarrow ((x_1 \leq_E x_2) \& (y_1 \leq_F y_2)).$$

Further in this article, we will use inequality signs without any indices and will imply from the context in which set they are defined.

The order norm on $E \times F$ is defined by the rule

$$\|(x_1,\, y_1)\| = \inf\{\lambda > 0\colon\, -\lambda(1_E,\, 1_F) \leq (x_1,\, y_1) \leq \lambda(1_E,\, 1_F)\}.$$

Here $(1_E,\, 1_F)$ is an order unit in $E \times F$. So, instead of the couple $(1_E,\, 1_F)$ one can use the symbol $1_{E\times F}$.

Lemma 2.3. *Let E and F be spaces with order unit, 1_E an order unit in E, $f\colon\, E \to F$ weakly additive, order-preserving operator. Then the graph G of operator f is an $A^{1_{E\times f(E)}}$-subspace in the space $E \times f(E)$ with the order unit $1_{E\times f(E)}$.*

Proof. We have $0_{E\times F} \equiv (0_E, 0_F) \in G$, since $f(0_E) = 0_F$. Let $(x, y) \in G$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}(x, y) + \lambda 1_{E\times f(E)} &= (x, f(x)) + \lambda 1_{E\times f(E)} \left(x + \lambda 1_E, f(x) + \lambda 1_{f(E)}\right)\\ &= (x + \lambda 1_E, f(x + \lambda 1_E))\,,\end{aligned}$$

and consequently, $\left((x, y) + \lambda 1_{E\times f(E)}\right) \in G$. $\square$

Corollary 2.1. *Let E and F be spaces with order unit, 1_E and 1_F, respectively, $f\colon\, E \to F$ a weakly additive, order-preserving, normed operator. Then the graph G of the operator f is $A^{1_{E\times F}}$-subspace of the space $E \times F$ with order unit $1_{E\times F}$.*

Remark 2.1. Note that in every topological vector space (in particular, in every space with an order unit) the only open subspace is the space itself. Unlike subspaces, A-subspaces of a space with an order unit can be open, closed, or everywhere dense.

Example 2.1. Consider the Euclidean plane $\mathbb{R}^2$ with the point-wise algebraic operations and the point-wise order. Then $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2\colon\, x_i \geq 0,\, i = 1, 2\}$ is a positive cone in $\mathbb{R}^2$. Arbitrary element of the set $\operatorname{Int}(\mathbb{R}^2_+) = \{(x_1,\, x_2) \in \mathbb{R}^2\colon\, x_i > 0,\, i = 1, 2\}$ can serve as an order unit. For precision, we fix $\mathbf{1} = (1, 1)$ as an order unit in $\mathbb{R}^2$. Then, as it is easy to check, the set

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2\colon\, x_1 - 1 < x_2 < x_1 + 1\right\}$$

is an open (with respect to the order topology) A-subspace in $\mathbb{R}^2$, and $C \neq \mathbb{R}^2$.

Example 2.2. Let $(\mathbb{R}^2, \mathbf{1})$ be the space with an order unit built in Example 2.1. Then the set

$$D = \Big\{(x_1, x_1 + r) \in \mathbb{R}^2 \colon r \in \mathbb{Q}\Big\},$$

where $\mathbb{Q}$ is the set of rational numbers, is a dense A-subspace in $\mathbb{R}^2$.

Example 2.3. Let $(\mathbb{R}^2, \mathbf{1})$ be the space in Example 2.1. It is clear that the set $\Lambda = \{\lambda\mathbf{1} : \lambda \in \mathbb{R}\}$ is a closed A-subspace in $\mathbb{R}^2$.

Remark 2.2. Note that every weakly additive, order-preserving operator $f\colon E \to F$ is automatically continuous.

Proposition 2.1. *Let E and F be spaces with order unit, and $f\colon E \to F$ be a weakly additive, order-preserving operator. Then the image $f(E)$ is an $A^{1_{f(E)}}$-subspace in F.*

Proof. Since $f(0_E) = 0_F$, then $0_F \in f(E)$. Let $y \in f(E)$, $\lambda \in \mathbb{R}$. Then there exists a vector $x \in E$, such that $y = f(x)$. We have

$$y + \lambda 1_{f(E)} = f(x) + \lambda 1_{f(E)} = f(x + \lambda 1_E),$$

i.e., $y + \lambda 1_{f(E)} \in f(E)$. Thus, $f(E)$ is $A^{1_{f(E)}}$-subspace in F. □

Finally, we formulate a version of the open mapping theorem for order-preserving operators.

Theorem 2.1. *Let E be a complete space with an order unit, F be a space with order unit and of the second category. If $f\colon E \to F$ is a surjective, weakly additive, order-preserving operator, then:*

(i) *the mapping f is open;*

(ii) *F is a complete space.*

Proof. (i) First we will show that $f(1_E)$ is an order unit in F.

Since E is complete, by the Baire category theorem we have $E = \bigcup_{m=1}^{\infty} mB(0_E, r)$ for every positive number r. Then one has $f(E) = F = \bigcup_{m=1}^{\infty} mf(B(0_E, r))$. Indeed, let $y \in F$. Since f is a surjective mapping, there exists $x \in E$ such that $y = f(x)$. There is such a positive integer m, that $-mr1_E < x < mr1_E$. Therefore, $-mrf(1_E) < f(x) < mrf(1_E)$, i.e., $y \in \bigcup_{m=1}^{\infty} mf(B(0_E, r))$.

So far we have $\operatorname{Int}(f(B(0_E, r))) \neq \emptyset$, i.e., the set $\operatorname{Int}(f(B(0_E, r)))$ is a neighbourhood of the zero of F. By definition of the order topology there exists σ such that $\sigma 1_F \in \operatorname{Int}(f(B(0_E, r))) \subset f(B(0_E, r))$. Hence, $-rf(1_E) < \sigma 1_F < rf(1_E)$, i.e., $f(1_E)$ is an order unit in F.

The arbitrariness of $r > 0$ guarantees that the operator f is open at O_E. But then, according to Lemma 2.1, the operator f is open at every point in E. So, the statement (i) is established.

(ii) Let $\{y_n\}$ be a Cauchy sequence in F, i.e., for every $\varepsilon > 0$ there exists n_ε such that for all $m \geq n_\varepsilon$ and $k \geq n_\varepsilon$ the double inequality

$$-\varepsilon 1_{f(E)} < y_m - y_k < \varepsilon 1_{f(E)}$$

holds. Without loss of generality, we can assume that for any positive integer n and any $m,\, k \geq n$ the following double inequality is fulfilled

$$-\tfrac{1}{n}1_{f(E)} < y_m - y_k < \tfrac{1}{n}1_{f(E)}. \tag{2.1}$$

Then $y_m - y_k \in B\left(0_F,\, \tfrac{1}{n}\right)$. According to the openness of the mapping f the set $f\left(B\left(0_E,\, \tfrac{1}{n}\right)\right)$ is an open neighbourhood of the zero in F. Moreover, we have

$$f\left(B\left(0_E,\, \tfrac{1}{n}\right)\right) = B\left(0_F,\, \tfrac{1}{n}\right). \tag{2.2}$$

Therefore, $y_m - y_k \in f\left(B\left(0_E,\, \tfrac{1}{n}\right)\right)$. It may turn out that for each pair m and k there exist a lot of pairs of vectors $x \in E$ and $x' \in E$, such that $f(x) = y_m$ and $f(x') = y_k$. As long as $y_m - y_k \in f\left(B\left(0_E,\, \tfrac{1}{n}\right)\right)$, then among such vector pairs must exist vectors $x \in E$ and $x' \in E$ with $f(x) = y_m$, $f(x') = y_k$ and $x - x' \in B\left(0_E,\, \tfrac{1}{n}\right)$.

For every positive integer n we denote by x_n any vector, which satisfies the following conditions:

1) $x_n \in f^{-1}(y_n)$;
2) for every $k \geq n$ there exists a vector $x \in f^{-1}(y_k)$ such that

$$x_n - x \in B\left(0_E,\, \tfrac{1}{n}\right).$$

Thus, we have built a sequence $\{x_n\}$ such that

$$f(x_n) = y_n, \quad n = 1, 2, \ldots, \tag{2.3}$$

on one side and, according to (2.1) and (2.2)

$$-\tfrac{1}{n}1_E < x_m - x_k < \tfrac{1}{n}1_E, \tag{2.4}$$

on the other side, for all n and for every pair of $m,\, k \geq n$.

By virtue of inequalities (2.4) we conclude, that $\{x_n\}$ is a Cauchy sequence in E. Since E is a complete space with an order unit, the sequence $\{x_n\}$ has to converge with respect to the order topology. Denote $x_0 = \lim\limits_{n\to\infty} x_n \in E$. Since f is a continuous mapping, then by (2.3) we have $f(x_0) = \lim\limits_{n\to\infty} y_n$. We put $y_0 = f(x_0)$. Then $y_0 = \lim\limits_{n\to\infty} y_n$. Thus, $\{y_n\}$ is a convergent sequence. Due to the arbitrariness of the chosen Cauchy sequence $\{y_n\}$, it follows that F is a complete space. □

Remark 2.3. Note that the openness principle for weakly additive, order-preserving case cannot be formulated similarly to the linear case. In contrast of the linear case, the conditions f is weakly additive and order-preserving in Theorem 2.1 do not guarantee the surjectivity of the mapping f. On the other hand, the image $f(E)$ is not obliged to be open in F. Finally, if we do not require surjectivity in Lemma 2.1, then the openness of a weakly additive, order-preserving operator at zero does not provide its openness on the whole space.

Example 2.4. Let $(\mathbb{R}^2, \mathbf{1})$ be the space with the order unit built in Example 2.1. We put $\overline{S} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 - 1 \leq x_2 \leq x_1 + 1\}$. Define the mapping $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$f(x_1, x_2) = \begin{cases} (x_1, x_1 - 1), & \text{if } x_2 \leq x_1 - 1, \\ (x_1, x_2), & \text{if } x_1 - 1 \leq x_2 \leq x_1 + 1, \\ (x_1, x_1 + 1), & \text{if } x_2 \geq x_1 + 1. \end{cases} \tag{2.5}$$

It is easy to check that f is a weakly additive mapping. We show that it is order-preserving. Since this property holds for the identity mappings, then f is order-preserving on $\overline{S}$. So, we have to check the first and the third cases in (2.5). But, the first case and the third case are checked similarly. That is why we will verify only the third case.

Let $x_2 \geq x_1 + 1$. Take any vector $(y_1,\, y_2) \in \mathbb{R}^2$ such that $(x_1,\, x_2) \leq (y_1,\, y_2)$. The last inequality is equivalent to $x_1 \leq y_1$ and $x_2 \leq y_2$.

The following three cases are possible.

1^0 $y_2 \geq y_1 + 1$. Then

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (\text{since } x_1 \leq y_1) \leq (y_1, y_1 + 1) = f(y_1, y_2).$$

2^0 $y_1 - 1 \leq y_2 \leq y_1 + 1$. We have $x_1 + 1 \leq y_2$. Therefore,

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (y_1, y_2) = f(y_1, y_2).$$

3^0 $y_2 \leq y_1 - 1$. But $x_1 + 1 \leq y_1 - 1$. Consequently,

$$f(x_1, x_2) = (x_1, x_1 + 1) \leq (y_1, y_1 - 1) = f(y_1, y_2).$$

So, f is order-preserving on $\mathbb{R}^2$.

For the operator f we have $f(\mathbb{R}^2) = \overline{S} \neq \mathbb{R}^2$, although the operator f is weakly additive, order-preserving, and the image $f(\mathbb{R}^2)$ is a set of the second category in $\mathbb{R}^2$. Clearly, $f(\mathbb{R}^2)$ is not open in $\mathbb{R}^2$. Moreover, it is easy to see that the mapping f is open at zero, but it is not open on $\mathbb{R}^2$. Indeed, for the open neighbourhood $B((2,4);1) = \{(x_1, x_2) \in \mathbb{R}^2 \colon 1 < x_1 < 3,\, 3 < x_2 < 5\}$ of $(2,4) \in \mathbb{R}^2$ its image $f(B((2,4);1)) = \{(x_1, x_1 + 1) \colon 1 < x_1 < 3\}$ is not open in $f(\mathbb{R}^2)$.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1065–1074.

VERTICAL TIME: SOUND AND VISION

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Abstract. The concept of poetic time, experienced through poetry, music and plastic arts, has firstly been individuated by Gaston Bachelard in the 1930s. The spectator's sensation of time rupture and ecstasy has been defined as time rearrangement into a vertical structure, linking to eternity. Although this particular perception of time has been described as triggered by spectator's experience of sublime art forms, this paper finds earlier evidence of the phenomenon in Giacomo Leopardi's poem *L'Infinito* (1819), as result of nature contemplation. Analysis of the sound that induces the vision of vertical time in Leopardi's poetry is performed by signal processing techniques. Relations between time, frequency and lag representations of the signal are established, finding correspondence of the signal power spectral density and time verticality in the lag domain.

1. Introduction

The concept of vertical time has been introduced by the French philosopher of science Gaston Bachelard (1884–1962). Bachelard's work, pioneering historical epistemology, could not but be permeated by reflection on time [15]. Impressing time thought with a primordial intuition, Bachelard individuates different temporalities that structure life.

Whether Bachelard identifies horizontal time, cadenzed by clocks, social, phenomenal and vital frameworks of duration, as prosodic time, the time of poetry is vertical [2]. Vertical time, accessed through poetry, is the surging up of the instant, possessing metaphysical scope. In [2], he writes, "*Every true poem can reveal the elements of suspended time, meterless time - a time we shall call vertical in order to distinguish*

Key words and phrases. Vertical time, spectral analysis, white noise, delta function.
2020 *Mathematics Subject Classification.* Primary: 00A64. Secondary: 42A38, 44A05, 60G10.
DOI 10.46793/KgJMat2307.1065S
Received: October 03, 2022.
Accepted: January 31, 2023.

it from everyday time, which sweeps along horizontally with the streaming waters and the blowing winds".

This rupture of the horizontal axis that spreads into the dormant depth of temporality is described by Bachelard as an ecstatic moment, into which past and future converge. In [1], Bachelard refers the capability of vertical time to possesses eternity when he writes "*This line running perpendicular to the temporal axis of life alone in fact gives consciousness of the present the means to flee and escape, to expand and deepen which have very often led to the present instant being linked to an eternity*".

Although the concept of vertical time has been identified by Bachelard, the experience of time reverberation from the horizontal axis to a vertical one linking to eternity has been vividly described by the Italian poet Giacomo Leopardi (1798–1837) in his poem *L'Infinito* [8]. Whether Bachelard, and later other authors [11, 19], identify in poetry and art in general the trigger of vertical time experience, Leopardi finds it in the sound of nature.

In his work Zibaldone [9], Leopardi states "*Pure, simple reason and mathematics have never been able, and will never be able, to discover anything poetic. Because everything that is poetic is felt rather than being known or understood, or perhaps we should say, is known or understood in being felt; nor indeed may it be known, discovered, or understood save by being felt. But pure reason and mathematics have no sensorium whatsoever*". Therefore, Leopardi claims that poetry cannot be inspired by pure reason and mathematics, but does not rule out the contrary.

This paper explores the relation between sound and vertical time by an analysis based on signal processing.

2. The Infinite: Form and Content

L'Infinito was written in Recanati in 1819, and published in the volume *Versi* in 1826 [7]. The poem is a *canzone libera*, an Italian metric form with uneven stanzas and free rhyme. Subtitled by the author as *Idyll I*, it recalls the Hellenistic tradition of short pastoral poems [5]. The term *idyll*, deriving from the diminutive of the ancient Greek "seeing, image", stands for small image or scene. The author himself, by labeling the poem, deceives the reader, by imposing the expectation of a pastoral's search for the sensory world. Despite, the fifteen verses are rather a description of a sublime sensory void, an idyll without figures [3, 10].

It is here reported with literal translation alongside.

L'Infinito	The Infinite
Sempre caro mi fu quest'ermo colle,	Always dear was to me this solitary hill,
E questa siepe, che da tanta parte	And this hedge, which from so much part
Dell'ultimo orizzonte il guardo esclude.	Of the last horizon the look excludes.
Ma sedendo e mirando, interminati	But sitting and staring, endless
Spazi di là da quella, e sovrumani	Spaces beyond that, and superhuman
Silenzi, e profondissima quiete	Silences, and deepest quietness
Io nel pensier mi fingo; ove per poco	I pretend in thinking; where almost
Il cor non si spaura. E come il vento	My heart scares. And like the wind
Odo stormir tra queste piante, io quello	I hear rustling through these trees, that
Infinito silenzio a questa voce	Infinite silence to this voice
Vo comparando: e mi sovvien l'eterno,	I go comparing: and I recall eternity,
E le morte stagioni, e la presente	And the dead seasons, and the present
E viva, e il suon di lei. Così tra questa	And alive, and its sound. So between this
Immensità s'annega il pensier mio:	Immensity drowns my thought:
E il naufragar m'è dolce in questo mare.	And shipwreck is sweet to me in this sea.

In the three opening verses, the author carries thrift to extreme in the material scenery description; an anonymous hill and a hedge that hinders the view of the horizon are the representation of the finite, recalling the need for an interior escape from the limits of the matter. By abandoning the sensory world of the first three verses, the syntactic structure frees from the meter, producing ten verses of incomparable discursive and imagery continuity. While the introductory verses were characterized by a uniform rhythm, suggesting a regular meter, the following ten verses (4–13) are lacking of pauses, forming a single movement with irregular dilatations and accelerations.

The obstacle (hedge) imposed to the horizon is rejected in a subjective monologue, generating a series of images unbounded by the limit of human perception (endless/Spaces, superhuman/Silences, and deepest quiet), converging into an overwhelming sensory oblivion. The loss of physical references flashes on a perceptive limb of missing content (where almost/My heart scares). It is worth noticing that in Buzzati's *Il deserto dei Tartari* [4] the horizon is the most insidious limit for the spectator's eye, a whirl absorbing all the observer references into a voluntary self-annihilation.

The tension by the vicinity of a void abyss releases by an abrupt incursion of the tangible. The sound of wind rustling in the trees, transiently rending the bare canvas of perception, evokes remembrance of the eternity (And like the wind/ I hear rustling among these plants, that/Infinite silence to this voice/I go comparing: and I recall eternity). All the sensory void is fulfilled by an atopic, yet absolute experience of temporal awareness.

The moment of free flow and apparent disorganization settles in the rhythmicity of the closing verses. The sense of abandonment is framed by a consolatory acceptance

Figure 1. View from the Colle Dell'Infinito, Recanati, Italy (left) [17], view from Mountain Avala, Belgrade, Serbia

of the new condition (So between this/Immensity drowns my thought:/And shipwreck is sweet to me in this sea). Recurrence of the initial composed rhythm produces a sensation of closure and circularity (Always dear was to me this solitary hill/And shipwreck is sweet to me in this sea), introducing a parallelism between space and time.

3. A Signal Processing Insight

The poem *L'Infinito* initially introduces the reader into a poetry of space, however the rising intensity of the verses reveals the intention of the author to address an ephemeral experience of gazing into the abyss of time. Remembrance of eternity unambiguously recalls the concept of vertical time, individuated by Bachelard as the time of poetry and permeating arts in general [11, 19]. However, in Leopardi's poem it is intimately related to the contemplation of nature. In fact, the sensation of time verticality is straightforwardly linked to the sound of wind rustling in the trees, psithurism (from the Ancient Greek $\psi\iota\theta\acute{\upsilon}\rho\iota\sigma\mu\alpha$ – whisper). This explicit relation opens to the possibility of a signal processing approach in the study of vertical time.

3.1. **Materials and methods.** It is known that the *idyll* was composed at the top of Monte Tabor, a hill in Recanati, Italy, known today as Colle Dell'Infinito (the Hill of The Infinite), Figure 1 (left). The name strikingly coincides to the Mount Tabor whereon the Christ transfiguration has taken place, which is an infinitive experience as well. The traditional icon of such an event is temporally organized in terms of vertical structuring [12]. Taking into account vegetation features of the original site (mixture of evergreens and caducous trees), psithurism samples have been collected at Mountain Avala near Belgrade, Serbia, Figure 1 (right). Samples were recorded with sampling frequency $f_s = 44100$ Hz, and a bit depth of 16 bits per sample.

A signal is a function of time (a real-valued function of time, throughout this paper), that conveys information about a phenomenon. An electrical analog signal is the result of conversion of the physical waveform (in this case, fluctuation of air pressure) by means of a transducer [16]. The normalized instantaneous power of such an electrical signal $x(t)$ (based on the Ohm's law and assuming a unitary resistor) is

$$p(t) = x^2(t), \tag{3.1}$$

with $x(t)$ being, equivalently, the transducer's output current or voltage.

The energy of a signal is defined as

$$E_x = \lim_{T\to+\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\, dt, \tag{3.2}$$

while the signal average power is

$$P_x = \lim_{T\to+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\, dt. \tag{3.3}$$

A signal is classified as *energy signal*, if and only if, it has nonzero but finite energy $0 < E_x < +\infty$ (and thus zero average power). On the other hand, a signal is referred to as *power signal* if it has nonzero but finite power $0 < P_x < +\infty$ (and thus infinite energy). The two categories are mutually exclusive. While signals that are both deterministic and nonperiodic are classified as *energy signals*, periodic, and random signals are treated as *power signals* [16].

3.1.1. *Energy spectral density.* The total energy (from $-\infty$ to $+\infty$) of an *energy signal*, is maintained in the frequency domain, according to Parseval's theorem [16], as

$$E_x = \int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} |\hat{x}(f)|^2\, df, \tag{3.4}$$

with $\hat{x}(f)$ being the Fourier transform of the signal $x(t)$:

$$\hat{x}(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-i2\pi ft}dt. \tag{3.5}$$

The squared magnitude spectrum, $|\hat{x}(f)|^2$, thus, describes the signal's energy per unit bandwidth, and will be referred to as *energy spectral density* (ESD).

The autocorrelation function of the energy signal,

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)\, dt, \tag{3.6}$$

is related to the ESD by means of the Fourier transform, as

$$\mathcal{F}\{R_x(\tau)\} = |\hat{x}(f)|^2. \tag{3.7}$$

3.1.2. *Power spectral density.* For power signals, such as sample signals $x(t)$ of a wide-sense-stationary (WSS) random process $X(t)$, which extend over infinite time, the Fourier transform might not exist. To deal with such signals in the frequency domain, the Fourier transform of the random process truncated by some window T, $\mathcal{F}\{X_T(t)\}$, should be considered. According to the Wiener-Khinchin theorem [6], the autocorrelation function of the WSS random process has its Fourier pair in the *power spectral density* (PSD) $G(f)$, as

$$G(f) = \mathcal{F}\{R_x(\tau)\} = \lim_{T\to+\infty} \frac{1}{T}\mathrm{E}\{|\hat{X}_T(f)|^2\}, \tag{3.8}$$

with $\mathrm{E}\{\cdot\}$ denoting the expected value.

If the WSS random process is also ergodic in the autocorrelation, in terms of one sample signal $x(t)$ for $\tau = 0$ (taking into account Exp. (3.8)), the autocorrelation takes the form

$$R_x(0) = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,dt = \int_{-\infty}^{+\infty} G(f)\,df. \tag{3.9}$$

Since the left-hand-term and the middle term in Eq. (3.9) represent the average power of the signal, the PSD in the right-hand-term describes the distribution of the signal's power in the frequency domain.

3.2. **Signals in different domains - an interpretation of vertical time.** The frequency representation of one sample of psithurism, being an *energy signal*, is reported in Figure 2 (a) by means of its ESD. Even though the energy distribution presents slightly dominant components in the lower frequency range, the ESD appears essentially flat within the interval of audible frequencies [14] (the ESD peak close to 5 kHz is the result of a cricket chirp, clearly audible in the recording and recognizable from the spectrogram). However, evenly distributed energy in the frequency domain cannot be considered a feature of the sound of wind in general; for comparison, the sound of wind in an open space produces an ESD with more pronounced characteristics of flicker noise [18], Figure 2 (b).

3.2.1. *The impulse function model.* Since we would acquiesce that, in the audible frequency range, psithurism is perceived as white noise, its characteristics will be analysed in different domains based on the theoretical white noise model [16]. White noise is modeled as a WSS, zero-mean, ergodic random process, with its PSD being constant for all frequencies:

$$G_N(f) = \frac{N_0}{2}, \quad \mathrm{W/Hz}, \tag{3.10}$$

where the factor of 2 indicates the $G(f)$ is a two-sided PSD.

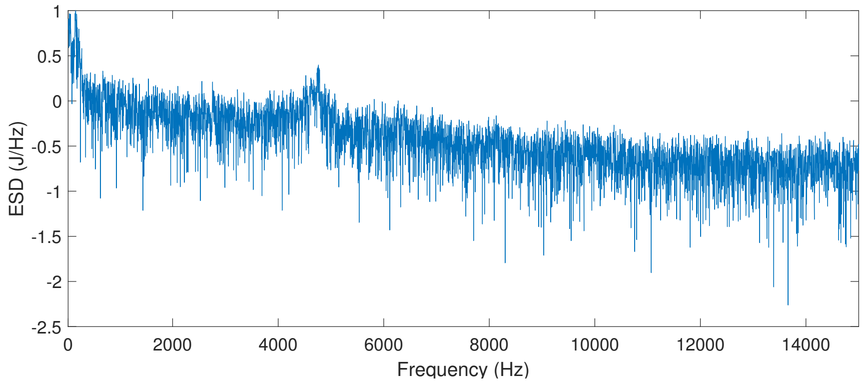
The autocorrelation function of white noise, given as the inverse Fourier transform of the PSD $G(f)$ (3.8), is denoted as:

$$R_N(\tau) = \mathcal{F}^{-1}\{G_N(f)\} = \frac{N_0}{2}\delta(\tau). \tag{3.11}$$

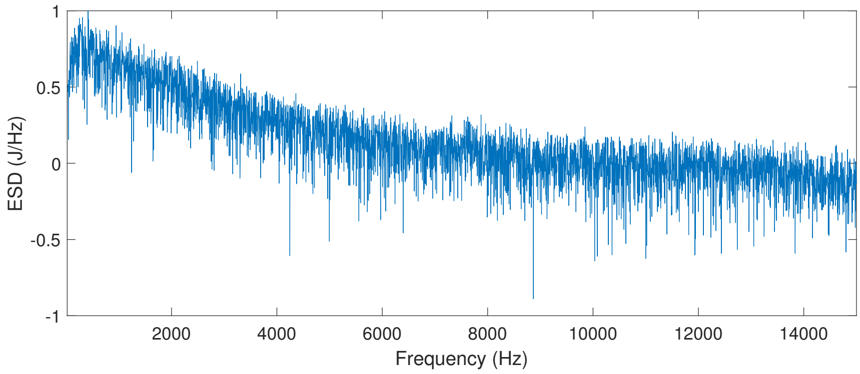
This relation is represented in Figure 3.

The Dirac delta function, defined by

$$\delta(\tau) = \begin{cases} +\infty, & \tau = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.12}$$



(a)



(b)

FIGURE 2. ESD of one sample of sound of wind in trees from Mountain Avala location (a), ESD of one sample of sound of wind in open space (b)

and

$$\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1 \tag{3.13}$$

is an infinite impulse in the lag domain, supported only by $\tau = 0$ [13].

Expression (3.11), thus, realizes the implicit link between the perceived sound of psithurism and the experience of an infinite impulse, materialized in the domain of the time lag. This relation between sound and the vertical impulse, however, is conceded only for sound with spectral characteristics of white noise.

3.2.2. *The sifting property of the impulse function.* The delta function, thus, meets the static description of vertical time as a meterless impulse [2]. However, it is through its dynamics that vertical time materializes eternity. As described by Bachelard, vertical time runs perpendicularly to the temporal axis, linking the present instant to eternity [1].

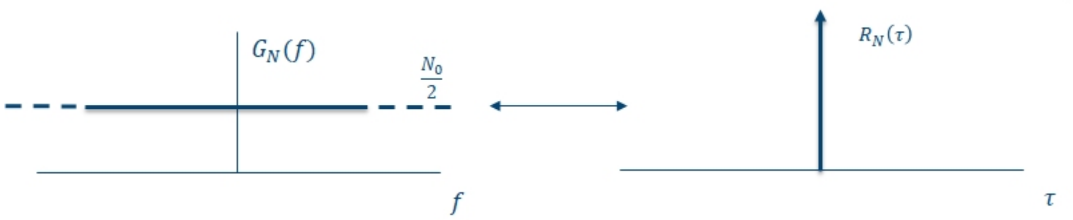


FIGURE 3. White noise in the frequency and lag domains

In order to enlighten the described behavior of vertical time across the horizontal axis, which according to Bachelard gives access to all time at once, a deeper insight into the delta function model and its properties will be required. As obvious from (3.12), $\delta(\tau)$ is not a function in the ordinary mathematical sense. It would be more appropriate to refer to $\delta(\tau)$ as to a functional quantity with a certain well-defined symbolic meaning [13]. For instance, one could define a sequence of functions $\delta(\tau,\epsilon)$ [13]

$$\delta(\tau,\epsilon) = \epsilon\,\mathrm{rect}(\epsilon\tau) \equiv \begin{cases} \dfrac{\epsilon}{2}, & |\tau| < \dfrac{1}{\epsilon}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

with

$$\int_{-\infty}^{+\infty} \delta(\tau,\epsilon)\,d\tau = 1. \tag{3.15}$$

With increasing values of the parameter ϵ, the sequence of functions $\delta(\tau,\epsilon)$ differ from zero only on a decreasing interval about the origin, with

$$\lim_{\epsilon\to+\infty} \int_{-\infty}^{+\infty} \delta(\tau,\epsilon)\,d\tau = 1 \tag{3.16}$$

still holding true. Thus any operation involving the delta function $\delta(\tau)$ should be performed with a function $\delta(\tau,\epsilon)$ and the limit $\epsilon \to +\infty$ introduced at the conclusion of the calculation.

Let us now consider a continuous and well-behaved function $f(\tau)$, and the value of the integral

$$\int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau-a)\,d\tau = \lim_{\epsilon\to+\infty} \int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau-a,\epsilon)\,d\tau. \tag{3.17}$$

The value of the integral on the right-hand side depends on the behavior of $f(\tau)$ in the vicinity of the $\tau = a$. Thus, taking an arbitrarily large ϵ allows the error originated from replacing $f(\tau)$ by $f(a)$ to become negligibly small. In accordance

$$\lim_{\epsilon\to+\infty} \int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau-a,\epsilon)\,d\tau = f(a) \lim_{\epsilon\to+\infty} \int_{-\infty}^{+\infty} \delta(\tau-a,\epsilon)\,d\tau, \tag{3.18}$$

and hence

$$\int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau - a)\,d\tau = f(a). \tag{3.19}$$

Equation (3.19), referred to as the delta function *sifting property*, completes the explication of vertical time by means of the delta function. In fact, if the function $f(\tau) = \tau$ is considered in (3.19), as

$$\int_{-\infty}^{+\infty} \tau\,\delta(\tau - a)\,d\tau = a, \tag{3.20}$$

the delta function gives access to any time instant. By its *sifting property*, the delta function draws from the infinity of the horizontal axis. Furthermore, elimination of the privileged position of the present instant from the perspective of vertical time ("*This line running perpendicular to the temporal axis of life alone gives consciousness of the present the means to flee and escape...*" [1]), finds correspondence in the delta function model.

Disanchoring from the referent *now* by the vertical time perspective, reveals the unbearable depth of temporality, as described by Leopardi in the last verses of *L'Infinito* (So between this/ Immensity drowns my thought:/ And shipwreck is sweet to me in this sea).

4. Conclusion

Vertical time has been described by Bachelard as an impulse which runs perpendicular on the temporal axis, being in possession of eternity within an instant. This moment of metaphysical perspective, in the poem *L'Infinito* by Leopardi, is induced by the annihilation of the material world, except for the sound of wind in trees.

The direct link between sound and vision of vertical time in Leopardi's poetry allows to establish a correspondence between signals in different domains by means of spectral analysis. The moment of rupture of the temporal axis, generating an infinite impulse, has been identified as a delta function in the domain of time lag. The ability of vertical time to reach for any time form the past and future is reflected in the delta function *sifting property.* In Leopardi's poetry spatial and temporal dimensions are fluid and interchangeable, since the all-embracing view from the top of the solitary hill becomes a panoramic of the eternity from an infinite impulse standing above time.

Acknowledgements. This work is dedicated to Professor Marisa Slanina.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1075–1086.

TOTALLY WEAKLY CHAIN SEPARATED SETS IN A TOPOLOGICAL SPACE

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Abstract. In this article, by using the notion of chain, we give some characterizations of totally separated spaces. Then, we give some examples, study the properties of those spaces and give new proofs.

Furthermore, by using the notion of chain, we introduce the notions of totally weakly chain separated and totally chain separated sets in a topological space, we state some useful aspects of these sets as well as the various relationships between them and by using these notions we give some characterizations of discrete and totally separated spaces.

1. Introduction

Unlike the standard definition of connectedness, which is given by a negative sentence, the characterization of connectedness by using the coverings is given by an affirmative sentence (see [4]), and it is a useful tool for proving some particular properties of connected spaces. In [3, 5] connectedness is generalized to the notion of chain connected set in a topological space and some properties are obtained. In [3] a pair of chain separated sets, and in [7] a pair of weakly chain separated sets in a topological space are introduced and, by using these notions, two characterizations of connected space are obtained. In [1] the notion of isolated point in a T_1 space is characterized by using coverings. In [7] a totally separated space and the discrete space are characterized by coverings.

So, by using the notion of chains in coverings we can successfully characterize some topological notions and study their properties.

Key words and phrases. Chain connected sets, totally chain separated sets, totally weakly chain separated set.

2020 *Mathematics Subject Classification.* Primary: 54D05.

DOI 10.46793/KgJMat2307.1075D

Received: February 10, 2023.

Accepted: March 20, 2023.

In this paper we continue from the articles [1, 3, 5–7] to investigate the notions related to connectedness and its generalizations by using the notion of chain. The statements in this section, except the last two paragraphs, are from these articles.

The basic notions related to chain connectedness together with some important results are introduced in the first chapter. In the second chapter we give a characterization of totally separated spaces by using the notion of chain in a covering, which later we use to study the properties of those spaces. The discrete space and totally separated spaces are also characterized, in the fourth chapter, with the help of the newly introduced notions of totally chain separated and totally weakly chain separated sets in a topological space, respectively. The fifth chapter introduces the space of chain components of a topological space and the space of chain component of a set in a topological space.

In this paper by a covering we understand an open covering. By a covering $\mathcal{U}$ of X, if it is not otherwise stated, we mean a covering $\mathcal{U}$ of X in X.

The following definition is given in [4].

Definition 1.1. Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. *A chain in* $\mathcal{U}$ *that connects* x *and* y (from x to y or from y to x) is a finite sequence of sets $U_1, U_2, \ldots, U_n$ of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, \ldots, n-1$.

Let X be a topological space and $C \subseteq X$.

Definition 1.2. The set C is *chain connected* in X, if for every covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

The following theorems are proved in [3, 5].

Theorem 1.1. *Let* $C \subseteq Y \subseteq X$. *If* C *is chain connected in* Y, *then* C *is chain connected in* X.

Theorem 1.2. *If* C *is chain connected in* X *and* $f : X \to Y$ *is a continuous function, then* $f(C)$ *is chain connected in* $f(X)$.

We denote by $V_{CX}(x, \mathcal{U})$ the set that consists of all elements $y \in C$ such that there exists a chain in $\mathcal{U}$, that connects x and y. If $C = X$, we use the notation $V(x, \mathcal{U})$ instead of $V_{CX}(x, \mathcal{U})$.

Theorem 1.3. *The set* $V_{CX}(x, \mathcal{U})$ *is nonempty, open, and closed in* C.

Definition 1.3. Let $x, y \in X$. The element x is *chain related* to y in X, and we denote it by $x \underset{X}{\sim} y$ or $x \sim y$, if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.

If x is not chain related to y in X we use the notation $x \underset{X}{\nsim} y$ or $x \nsim y$.

The chain relation in a topological space X is an equivalence relation, and it depends on the set X and the topology τ of X. The chain component $V_{CX}(x)$ of the element x of C in X is the largest chain connected set in C that contains x.

When $C = X$ we use the notation $V_X(x)$ or $V(x)$ for $V_{CX}(x)$.

Let X be a topological space. The *quasicomponent* of $x \in X$, is the intersection of all clopen (closed and open) neighborhoods of x. We denote that with $Q_X(x)$ or $Q(x)$. Quasicomponents are closed sets.

Theorem 1.4. *Let X be a topological space and $C \subseteq X$. Then, for every $x \in C$,*

$$V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} Q_C(y).$$

So, chain components of C in X are a union of quasicomponents of the set and if the set agrees with the space, the chain components match with the quasicomponents, i.e., for every $x \in X$,

$$V_X(x) = Q_X(x).$$

Theorem 1.5. *The topological space X is connected if and only if X is chain connected in X.*

Theorem 1.6. *Let X be a topological space and $C \subseteq X$. If the set C is chain connected in X, then every subset of C is chain connected in X.*

Definition 1.4. Let X be a topological space and $A, B \subseteq X$. The nonempty sets A and B are *weakly chain separated in* X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

Definition 1.5. Let X be a topological space and $A, B \subseteq X$. The nonempty sets A and B are *chain separated in* X, if there exists a covering $\mathcal{U}$ of X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

The following definitions are given in many textbooks about connectedness, as in [2].

A subset of a topological space is disconnected if it is not connected. The topological space X is *totally disconnected* if all subsets with more than one element are disconnected. So, the only connected subsets of X are the singletons and the empty set. Equivalently, the topological space X is totally disconnected if and only if the connected components of X are the singletons.

The topological space X is *totally separated* if its quasicomponents are singletons. Equivalently, the topological space X is totally separated if and only if for every pair of distinct points $x, y \in X$ there exists a separation $X = U \cup V$ (i.e., X is represented as the union of a pair of disjoint open and closed sets U and V) such that $x \in U$ and $y \in V$.

2. Criterion for Totally Separated Spaces by Using the Notion of Chain

The next theorem gives a criterion for totally separated spaces by using the notion of chain.

It enables the study of these spaces by using the coverings of the space and chains on them (Chapter 3). The relation of the theorem with other notions enables the characterization of totally separated spaces through chain components, i.e., quasi-components, chain relation, chain separated and weakly chain separated sets. Some examples of topological spaces explained through the characterization given by this theorem will be considered.

Theorem 2.1. *The topological space X is totally separated if and only if for every two distinct points $x, y \in X$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.*

Proof. Let X be totally separated and $x, y \in X$. It follows that there exists a separation $X = U \cup V$ such that $x \in U$ and $y \in V$. Then for the covering $\mathcal{U} = \{U, V\}$ there is no chain in $\mathcal{U}$ that connects x and y.

Conversely, for every two distinct points $x, y \in X$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Let $U = V(x, \mathcal{U})$ and $V = X \backslash U$. It follows firstly that $x \in U$, $y \in V$ and U is an open and closed set in X and secondly that V is open and closed set in X, i.e., $X = U \cup V$ is a separation. Hence, X is a totally separated space. $\square$

The following proposition is given in [7].

Proposition 2.1. *The topological space X is totally separated if and only if every two distinct singletons of X are weakly chain separated in X.*

The next proposition follows directly from the definition of totally separated spaces and Theorem 1.4.

Proposition 2.2. *The topological space X is totally separated if and only if the only chain components of X are singletons, i.e., for every $x \in X$, $V(x) = \{x\}$.*

From Proposition 2.2 it follows that the topological space X is totally separated if and only if the only chain connected sets are the singletons.

By $CovX$ we mean the set of all coverings of the space X. Note that by a covering in this paper we understand an open covering.

Since from the definition of chain components it follows that $V(x) = \bigcap_{\mathcal{U} \in \mathrm{Cov}\, X} V(x, \mathcal{U})$, the next statement holds.

Proposition 2.3. *X is a totally separated space if and only if for every $x \in X$,*

$$\{x\} = \bigcap_{\mathcal{U} \in \mathrm{Cov}\, X} V(x, \mathcal{U}).$$

Proposition 2.4. *The topological space X is totally separated if and only if every two distinct singletons of X are not in a chain relation, i.e., for every distinct $x, y \in X$, $x \not\sim y$.*

Proof. Let $x \in X$. If $x \not\sim y$ for every $y \in X$, $y \neq x$, it follows that for every $y \in X$ there exists a covering $\mathcal{U}_y$ of X such that there is no chain in $\mathcal{U}_y$ that connects x and y, i.e., $y \notin V(x, \mathcal{U}_y)$. Then $V(x) \subseteq \bigcap_{y \in X \setminus \{x\}} V(x, \mathcal{U}_y) = \{x\}$, i.e., $V(x) = \{x\}$. From arbitrariness of $x \in X$, it follows that X is totally separated.

If the topological space X is totally separated then from Proposition 2.2 it follows that for every $x \in X$, $V(x) = \{x\}$, i.e., for every $x \in X$ and every $y \in X$, $y \neq x$, it follows that $x \not\sim y$. □

Proposition 2.5. *The topological space X is totally separated if and only if every two distinct singleton sets of X are chain separated in X.*

Proof. X is totally separated and $x, y \in X$, $x \neq y$ if and only if there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y, i.e., by Definition 1.5, $\{x\}$ and $\{y\}$ are chain separated in X. From the arbitrariness of $x, y \in X$, $x \neq y$, it follows the accuracy of the statement of the theorem. □

Some examples of totally separated spaces explained using Theorem 2.1 follow.

Example 2.1. a) The discrete space X is totally separated space. Indeed, if $x, y \in X$, $x \neq y$, then for the covering $\mathcal{U} = \{\{x\} \mid x \in X\}$ there is no chain in $\mathcal{U}$ that connects x and y.

b) The space of rational numbers $\mathbb{Q}$ with the standard topology is totally separated space. Namely, if $x, y \in \mathbb{Q}$ then there exists an irrational number z such that $x < z < y$ and for the covering $\mathcal{U} = \{(-\infty, z) \cap \mathbb{Q}, (z, \infty) \cap \mathbb{Q}\}$ of $\mathbb{Q}$ there is no chain in $\mathcal{U}$ that connects x and y. From the arbitrariness of x and y it follows that $\mathbb{Q}$ is totally separated space.

c) The Cantor set C is totally separated space. Indeed, let $x, y \in C$. Then there exists $z \notin C$ such that $x < z < y$ and $\mathcal{U} = \{(-\infty, z) \cap C, (z, \infty) \cap C\}$ is a covering of C such that there is no chain in $\mathcal{U}$ that connects x and y. From the arbitrariness of x and y it follows that C is totally separated space.

d) Sorgenfrey line $\mathbb{R}_l$ is totally separated. Namely, let $a, b \in \mathbb{R}$, $a < b$, and let $c \in (a, b)$. Then $\mathcal{U} = \{(-\infty, c), [c, \infty)\}$ is a covering of $\mathbb{R}_l$ such that there is no chain in $\mathcal{U}$ that connects a and b. It follows that $\mathbb{R}_l$ is totally separated space.

3. Properties of Totally Separated Spaces

In this section we obtain some new proofs for some properties of totally separated spaces by using the criteria from Theorem 2.1 and Propositions 2.1–5.

Theorem 3.1. *If X is a totally separated space, then X is totally disconnected.*

Proof. Let X be a totally separated space, i.e., for every $x \in X$, $V(x) = \{x\}$, where $V(x)$ is a chain component of X that contains x. Since $C(x) \subseteq V(x)$, where $C(x)$ is a connected component of X that contains x, then $C(x) = \{x\}$ holds for every $x \in X$, i.e., X is a totally disconnected space. □

All spaces in Example 2.1 are totally disconnected. The next example (see [8]) shows the existence of a totally disconnected space which is not totally separated.

Example 3.1. Let C be the Cantor set in the unit interval at x-axis and $M\left(\frac{1}{2}, \frac{1}{2}\right)$ be a point in the plane. Let $L(N)$ be the segments with endpoints in M and $N \in C$, $E \subseteq C$ be the set of endpoints of the removed intervals obtained by the construction of the Cantor set and $F = C \backslash E$. Define:

$$X_E = \cup\{L(N) \mid N \in E\}, \quad X_F = \cup\{L(N) \mid N \in F\},$$

$$Y_E = \{(x, y) \in X_E \mid y \in \mathbb{Q}\} \quad \text{and} \quad Y_F = \{(x, y) \in X_F \mid y \neq \mathbb{Q}\}.$$

The Knaster-Kuratowski fan (Figure 1) is the set $Y = Y_E \cup Y_F$.

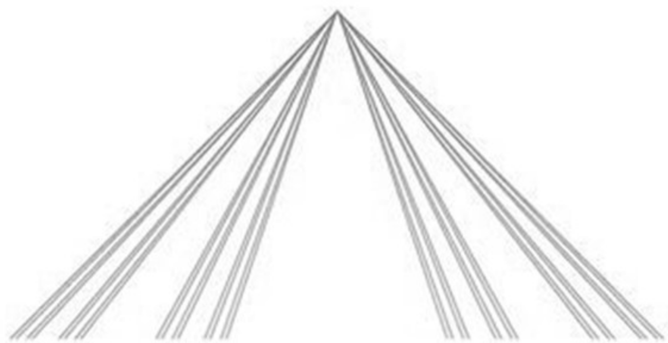

FIGURE 1. The Knaster-Kuratowski fan

The Knaster-Kuratowski fan with the removed point, $Y^* = Y \backslash \{M\}$, is a totally disconnected space (see [8]). However, Y^* is not totally separated, since for every $N \in C$, $L(N) \cap Y^*$ is contained in one quasicomponent, i.e., the chain component $V(N)$ (see Example 129, page 145–147 in [8]).

Theorem 3.2. *Let $\{X_i\}_{i \in I}$ be a family of disjoint totally separated spaces. Then, the disjoint union (sum) $X = \coprod_{i \in I} X_i$ is a totally separated space if and only if X_i are totally separated spaces for every $i \in I$.*

Proof. Let X_i be sets such that for all $i, j \in I$, $i \neq j$, $X_i \cap X_j = \emptyset$. We assume that X is a totally separated space. Then, by Theorem 2.1, X_i are totally separated spaces for all $i \in I$.

Conversely, let X_i, $i \in I$, be totally separated spaces. Let A be an arbitrary chain connected subset in X. We assume that there exist $i, j \in I$, $i \neq j$, such that $A \cap X_i \neq \emptyset$ and $A \cap X_j \neq \emptyset$. In this case there is no chain from $x \in A \cap X_i$ to $y \in A \cap X_j$ in the covering $\mathcal{U} = \{X_i \mid i \in I\}$, which is opposite of the assumption that A is a chain connected set in X. Therefore, there exist only one index $i \in I$ such that $A \subseteq X_i$, and since X_i is a totally separated space, A is a singleton. From the arbitrariness of A, it follows that X is totally separated. $\square$

We notice that the sufficient condition in the previous theorem is valid also if $\{X_i\}_{i\in I}$ is not a family of disjoint spaces. Moreover, Theorem 3.2 is true if only we consider the sum of topological spaces. Specifically, this theorem is not valid for $X = \bigcup_{x\in[0,1]}\{x\}$, where $[0,1]$ is considered with the standard topology.

Theorem 3.3. *Let $f : X \to Y$ be an injective continuous function. If Y is a totally separated space, then X is totally separated.*

Proof. Let $f : X \to Y$ be an injective continuous function and Y be a totally separated space. Let C be a chain connected set in X. Then $f(C)$ is chain connected in Y, and since Y is totally separated, $f(C)$ is singleton. Since f is an injection, the set C is a singleton. Hence, all chain connected sets in X are singletons, i.e., X is a totally separated space. □

The following example shows why injectivity of the function is a necessary condition on the previous theorem.

Example 3.2. Let $X = [0,1] \cup \{2\}$, $Y = \{1,2\}$ and $f : X \to Y$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ 2, & \text{if } x = 2. \end{cases}$$

Then f is a continuous non-injective function, Y is a totally separated space, but X is not.

Theorem 3.4. *Let $f : X \to Y$ be a homeomorphism. Then X is totally separated space if and only if Y is totally separated.*

Proof. Let X be a totally separated space. Then the chain components of X are singletons.

Let C be a chain connected set in Y. Then $f^{-1}(C)$ is a chain connected set in X, and therefore, $f^{-1}(C)$ is a singleton. Since f is bijection, it follows that the set C is a singleton. From the arbitrariness of C it follows that all chain connected sets in Y are singletons, i.e., Y is a totally separated space.

The converse statement can be proved analogously, if we work with f instead of f^{-1}. □

However, if X and Y are homotopic equivalent, it doesn't imply that both spaces are totally separated. This statement is proved by the following example.

Example 3.3. Let $X = \{1,2\}$ and $Y = [0,1] \cup [2,3]$. Then, X and Y are homotopic equivalent and X is totally separated but Y is not.

Theorem 3.5. *Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \subset \tau_2$. Then, if (X,τ_1) is a totally separated space, so is (X,τ_2).*

Proof. Assume that (X,τ_2) is not a totally separated space, i.e., there exist $x, y \in X$ such that for all coverings of (X,τ_2) there exists a chain from x to y. Since $\tau_1 \subset \tau_2$, all coverings of (X, τ_1) are also coverings of (X, τ_2). Therefore, for any covering of (X, τ_1) there exists a chain from x to y, i.e., (X, τ_1) is not totally separated space. $\square$

In order to point out that the converse statement of the above theorem is not valid we consider the real line $\mathbb{R}$ with the standard topology and the Sorgenfrey line $\mathbb{R}_l$. Namely, $\mathbb{R} \subset \mathbb{R}_l$, $\mathbb{R}_l$ is a totally separated space but $\mathbb{R}$ is not totally separated via Theorem 3.1, since $\mathbb{R}$ is connected.

4. Totally Chain Separated and Totally Weakly Chain Separated Sets in a Topological Space

Now, we will define the notion of a totally weakly chain separated set in a topological space.

Let X be a topological space and $C \subseteq X$.

Definition 4.1. The set C is totally weakly chain separated in X if for every two distinct points $x, y \in C$ there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

The next statement follows from Definition 4.1 and Theorem 2.1.

Corollary 4.1. *The topological space X is totally separated if and only if X is totally weakly chain separated in X.*

Proposition 4.1. *The set C is totally weakly chain separated in X if and only if every two distinct singletons in C are weakly chain separated in X, i.e., if and only if every two distinct singletons in C are chain separated in X.*

Proof. The set C is totally weakly chain separated in X, i.e., for every two distinct points $x, y \in C$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y if and only if from Definition 1.5 every two distinct singletons in C are chain separated in X. Clearly, two singletons are weakly chain separated in X if and only if they are chain separated in X. $\square$

Proposition 4.2. *The set C is totally weakly chain separated in X if and only if the only chain components of C in X are the singletons, i.e., for every $x \in C$, $V_{CX}(x) = \{x\}$.*

Proof. Let $x \in C$. The element $y \in V_{CX}(x)$, $y \neq x$; if and only if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y, i.e., C is not a totally weakly chain separated in X. $\square$

Theorem 4.1. *Every subset of a totally weakly chain separated set in X is a totally weakly chain separated set in X.*

Proof. Let C be a totally weakly chain separated set in X and $D \subseteq C$. It follows that for every $x, y \in C$ and, as a consequence, for every $x, y \in D$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y, i.e., D is a totally weakly chain separated set in X. $\square$

Theorem 4.2. *Let $C \subseteq Y \subseteq X$. If C is a totally weakly chain separated set in X, then C is a totally weakly chain separated in Y.*

Proof. Let C be a totally weakly chain separated set in X and let $x, y \in C$. It follows that there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Then for the covering $\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y \mid U \in \mathcal{U}\}$ there is no chain in Y that connects x and y, i.e., C is totally weakly chain separated in Y. $\square$

Corollary 4.2. *If C is totally weakly chain separated set in X, then C is totally separated.*

Proof. If C is totally weakly chain separated set in X, then C is totally weakly chain separated set in C by Theorem 4.2 and so, by Corollary 4.1, C is totally separated. $\square$

The next example shows that the converse statement of Corollary 4.2 is not true in general.

Example 4.1. Let $X = [0,1]$ and $C = \{0,1\}$. Then C is totally separated since C is the discrete, i.e., $V_C(0) = \{0\}$ and $V_C(1) = \{1\}$, but it is not totally weakly chain separated in X since X is connected, i.e., from Theorem 1.5, X is chain connected in X, and from Theorem 1.6, C is chain connected in X, i.e., for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects 0 and 1. The conclusion can be done directly, for arbitrary covering $\mathcal{U}$ of X, since X is compact, there exists a finite subcovering from which we can chose a chain that connects 0 and 1.

Corollary 4.3. *The set C is totally weakly chain separated in X if for every distinct $x, y \in C$, $x \not\sim y$.*

Proof. Obvious from Corollary 4.1 and Proposition 2.2. $\square$

We want to consider the set that is defined similarly as the totally weakly chain separated set where the separation is reinforced by the rotation of the quantifiers.

Definition 4.2. The set C is totally chain separated in X if there exists a covering $\mathcal{U}$ of X such that for every two distinct points $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y.

The difference between Definition 4.1 and Definition 4.2 is that quantifiers are rotated. A totally chain separated set is separated by one covering, i.e., the separation is strong. If C is a totally chain separated set, then there exists a covering $\mathcal{U}$ such that the set $\mathcal{U} \cap C$ consists of singletons. A totally weakly chain separated set in general case does not have to be separated by one covering, i.e., the separation is weak.

Clearly, if the set C is totally chain separated in X, then C is totally weakly chain separated in X. The next example shows that the converse case does not hold in general.

Example 4.2. The sets $C_{n_0} = \{0\} \cup \left\{\frac{1}{n} \mid n \geq n_0\right\}$, $n_0 \in \mathbb{N}$, are totally weakly chain separated in $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Namely for arbitrary elements $x = 0$ or $x = \frac{1}{n_2}$, and $y = \frac{1}{n_1}$, $n_2 > n_1 \geq n_0$, for the covering

$$\mathcal{U} = \left\{\{0\} \cup \left\{\frac{1}{n} \mid n > n_2\right\}, \left\{\frac{1}{n_2}\right\}, \left\{\frac{1}{n_2 - 1}\right\}, \ldots, \left\{\frac{1}{n_0}\right\}, \left\{\frac{1}{n_0 - 1}\right\}, \ldots, \{1\}\right\}$$

there is no chain in $\mathcal{U}$ that connects x and y. But the sets C_{n_0}, $n_0 \in \mathbb{N}$, are not totally chain separated in X. Namely, if $\mathcal{U}$ is a covering of X, then the element $U \in \mathcal{U}$ that contains 0, contains also an element $z = \frac{1}{n_3}$, $n_3 \in \mathbb{N}$, and U is a chain in $\mathcal{U}$ that connects 0 and z.

Theorem 4.3. *Let $C \subseteq Y \subseteq X$. If C is a totally chain separated set in X, then C is totally chain separated in Y.*

Proof. Let C be a totally chain separated set in X, i.e., there exists a covering $\mathcal{U}$ of X such that for every distinct $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y. It follows that $\mathcal{U}_Y = \mathcal{U} \cap Y$ is a covering of Y such that there is no chain in $\mathcal{U}_Y$ that connects x and y for every distinct $x, y \in C$, i.e., C is totally chain separated in Y. □

Theorem 4.4. *The set C is totally chain separated in C if and only if C is the discrete space.*

Proof. Let C be totally chain separated in C, i.e., there exists a covering $\mathcal{U}$ of C such that for every distinct $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y. It follows that for every $x \in C$ the chain component $V(x) = V(x, \mathcal{U}) = \{x\}$ is an open singleton, i.e., C is the discrete space.

Conversely, let C be the discrete space, i.e., every singleton in C is open. Then for the covering $\mathcal{U} = \{\{x\} \mid x \in C\}$ there is no chain in $\mathcal{U}$ that connects x and y, for every distinct $x, y \in C$, i.e., C is totally chain separated in C. □

If the set C is totally chain separated in X, then C is the discrete space. Example 4.1 shows that even if C is a discrete space, it may not be a totally chain separated in X.

The discrete space is characterized by chain in [1, 7]. Here we give a new characterization. According to genesis of the notion, by using this criterion, the discrete space also can be called totally chain separated space.

5. The Space of Chain Components of a Set in a Topological Space

Let X be a topological space and $C \subseteq X$.

The space of quasicomponents QX of a topological space X consists of the all quasicomponents of X equipped with the topology generated by the base composed from the sets $QF = \{A \mid A \in QX, A \subseteq F\}$ where F is clopen in X.

The next statement is given below Theorem 2.2 in [3].

Proposition 5.1. *The nonempty set A is clopen in X if and only if there exists a point $x \in X$ and a covering $\mathcal{U}$ of X such that $A = V(x, \mathcal{U})$.*

Proof. Let A be a clopen set and $x \in A$. Then, $X \backslash A$ is also a clopen set and for the covering $\mathcal{U} = \{A, X \backslash A\}$ it follows that $A = V(x, \mathcal{U})$.

If for the set A holds $A = V(x, \mathcal{U})$ for some covering $\mathcal{U}$ of X and $x \in X$, since, by Theorem 1.3, $V(x, \mathcal{U})$ is nonempty and clopen in X, it follows that A is clopen in X. □

Let VX be the set of all chain components of the space X. Clearly $QX = VX$.

Definition 5.1. *A space of chain components of* X is the set VX with the topology generated by the base composed from the sets:

$$\{A \mid A \in VX, A \subseteq V(x, \mathcal{U})\}, \quad x \in X, \mathcal{U} \in \operatorname{Cov} X.$$

Since, from Proposition 5.1 it follows that for every nonempty clopen set A in X there exists a covering $\mathcal{U}$ of X and a point $x \in X$, such that $A = V(x, \mathcal{U})$, the space of chain components of a topological space X is well defined and matches with the space of quasicomponents. So, Definition 5.1 is one more interpretation of the space of quasicomponents.

If the space X is a totally separated space, then the elements of the corresponding space of chain components, VX, are singletons $\{x\}$, $x \in X$.

In the next definition we generalise the notion of a space of chain components to a space of chain components of a set in a space.

Let VCX be the set of all chain components of the set C in X.

Definition 5.2. *A space of chain components of a set* C in a topological space X is the set VCX with the topology generated by the base composed from the sets:

$$\{A \mid A \in VCX, A \subseteq V_{CX}(x, \mathcal{U})\}, \quad x \in X, \mathcal{U} \in \operatorname{Cov} X.$$

Since a chain component of a set in a topological space in general is a union of quasicomponents [3], the space of chain components of a set in a topological space in general differs from a space of chain components.

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Kragujevac Journal of Mathematics
Volume 47(7) (2023), Pages 1087–1092.

ON THE LIMITS OF PROXIMATE SEQUENCES

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Abstract. We investigate the continuity of the pointwise limits of proximate sequences. Both general proximate sequences and a subclass are considered. We obtain some results related to the fixed points of the limit functions and fixed point like properties of the proximate sequences.

1. Introduction

Theory of shape is shown to be a good alternative to homotopy theory for locally complicated spaces.

The first definitions of shape theory are mainly using external spaces, first in Borsuk's approach the spaces are embedded in the Hilbert cube and after in the categorical approach pioneered by Mardesic and Segal we see the use of inverse systems of polyhedra-again external spaces.

Recently, in the last decades some intrinsic descriptions of shape emerged. In the latter approach, functions between original spaces are investigated. From [2], in compact metric spaces there exists a cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of finite coverings.

The morphisms in the intrinsic approach, as described in [3], are characterized by sequences of functions (f_n) that map objects in the category to one another. In the case of compact metric spaces, these functions are continuous over the members of a cofinal sequence of coverings for the space. As the index of the function in the sequence increases, its level of continuity improves, moving closer and closer to a state of being completely continuous. This idea is intuitive and straightforward to understand.

Key words and phrases. intrinsic shape, proximate sequence, limit function, fixed point property.
2020 *Mathematics Subject Classification.* Primary: 54C56. Secondary: 54C10.
DOI 10.46793/KgJMat2307.1087B
Received: September 29, 2022.
Accepted: April 05, 2023.

Usually while investigating natural processes we can observe them only approximately. If the resolution of the observation becomes higher we get higher approximation - by the same analogy in the proximate approach increasing the index of covering means increasing the precision. The advantage here is that we can compute those processes because there are only finite many members of each covering.

Considering the intrinsic approach allows us to look at the limit of these functions. In this paper we investigate a special class of proximate sequences and we obtain some relations between proximate sequences and their limit functions.

2. Definition of Proximate Sequences

Along this paper by a covering we mean an open covering of the space.

For arbitrary space W by $Cov(W)$ we denote the set of all open coverings of W.

In this chapter we will introduce the intrinsic definition for shape. For more detailed explanations about intrinsic approach of shape we suggest [3].

If $\mathcal{U}, \mathcal{V}$ are two coverings of the space X, then $\mathcal{V}$ is refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$.

If $U \in \mathcal{U}$, then the star of U is the set $St(U, \mathcal{U}) = \cup\{W \in \mathcal{U} \mid W \cap U \neq \emptyset\}$ and by $St\mathcal{U}$ will be denoted the collection of all $St(U, \mathcal{U}), U \in \mathcal{U}$.

Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}$ - near to f if for every $x \in X$, $f(x)$ and $g(x)$ lie in the same member of $\mathcal{V}$. It is denoted by $f =_{\mathcal{V}} g$.

Let $f : X \to X$ be a function and let $\mathcal{U}$ be a covering of X. We say that the point $x \in X$ is $\mathcal{U}$ - invariant for f if there exists $U \in \mathcal{U}$ such that $x, f(x) \in U$.

Definition 2.1. Let $\mathcal{V}$ be a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$ - continuous at the point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$. A function $f : X \to Y$ is $\mathcal{V}$-continuous on X if it is $\mathcal{V}$ - continuous at every point $x \in X$.

(The family of all such U_x forms a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}$ - continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.)

Definition 2.2. For arbitrary covering $\mathcal{V}$ of the space Y, we say that two functions $f, g : X \to Y$ are $\mathcal{V}$ - homotopic, if there exists a function $F : X \times I \to Y$ such that:

1) $F : X \times I \to Y$ is $st\mathcal{V}$ - continuous;
2) $F : X \times I \to Y$ is $\mathcal{V}$ - continuous at all points of $X \times \partial I$;
3) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

We denote this by $f \overset{\mathcal{V}}{\sim} g$.

Proposition 2.1. *The relation "* $\overset{\mathcal{V}}{\sim}$ *" is an equivalence relation.*

Proof. See [2, 4]. □

Further on we will work only with compact metric spaces. In this case it is enough to work with finite coverings.

Definition 2.3 ([3])**.** The sequence (f_n) of functions $f_n : X \to Y$ is a proximate sequence from X to Y, if there exists a cofinal sequence of finite coverings of Y, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ and for all indices $m \geq n$, f_n and f_m are $\mathcal{V}_n$ - homotopic. In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.

Definition 2.4 ([3])**.** If (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences (f_n) and $(f'_n) : X \to Y$ are homotopic if for some cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$, and for all integers f_n and f'_n are $\mathcal{V}_n$ - homotopic.

Now we will define the composition of proximate sequences.

Definition 2.5. If $(f_n) : X \to Y$ is a proximate sequence over $(\mathcal{V}_n)$ and $(g_k) : Y \to Z$ is a proximate sequence over $(\mathcal{W}_k)$, for a covering $\mathcal{W}_k$ of Z, there exists a covering $\mathcal{V}_{n_k}$ of Y such that $g_k(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$. Then, the composition is the proximate sequence $(h_k) = (g_k \circ f_{n_k}) : X \to Z$.

Compact metric spaces and homotopy classes of proximate sequences form the shape category, i.e., isomorphic spaces in this category have the same shape. See [4].

Now we will define regular coverings from [4].

Definition 2.6. Let X be a set and $V = \{V_i \mid i = 1, 2, \ldots, n\}$ be a finite set of subsets of X. If $V \in \mathcal{V}$, we define depth of V in $\mathcal{V}$, to be the biggest number $k \in \mathbb{N}$ such that there exists sequence of elements of $\mathcal{V}$ such that $V \subset V_2 \subset V_3 \subset \cdots \subset V_k$. (If V is not a proper subset of any element in $\mathcal{V}$, then depth of V is 1). It will be denoted by depth(V).

Definition 2.7. A covering $\mathcal{V}$ of the topological space Y is regular if it satisfies the following conditions.

1) If $V \in \mathcal{V}$, then $V \cap Y \neq \emptyset$.
2) If $U, V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

Since for every finite covering of the space there are finitely many nonempty intersections of the elements we have the following property. If Y is compact metric space, then there exists a cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of regular coverings.

Definition 2.8. Let $(f_n) : X \to Y$ be a proximate sequence over the coverings $(\mathcal{V}_n)$. We will call (f_n) super proximate sequence if for $m \geq n$, it follows $f_m =_{\mathcal{V}_n} f_n$.

It is clear that if a sequence of functions $(f_n) : X \to Y$ fulfills the property for $m \geq n$, $f_m =_{\mathcal{V}_n} f_n$ than it will be proximate sequence. This follows from the fact that every $\mathcal{V}$ - near functions are $\mathcal{V}$ - homotopic [4].

3. Limits of Proximate Sequences

Since, in the intrinsic shape - proximate sequences from X to Y consist of functions with codomain Y, we can investigate the limit

$$\lim_{n\to+\infty} f_n(x).$$

For the general situation the limit function is not even Darboux even if the component functions are continuous surjections.

Example 3.1. The pointwise limit of proximate sequences is not continuous in general. Take $X = Y = I$ with Euclidean topology and let $f_n(x) = x^n$. It is clear that the sequence (f_n) is proximate sequence, but the limit function is:

$$f(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1, & \text{if } x = 1, \end{cases}$$

which, clearly is not Darboux hence not almost continuous. (Every almost continuous function on the closed unit interval is Darboux, see [5]).

Remark. Let $(\mathcal{V}_n)$ be a cofinal sequence of (regular) coverings for the space Y. Lets define e sequence $(\mathcal{W}_n)$ of coverings of the space Y by $\mathcal{W}_n = st(\mathcal{V}_n)$. From $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, it follows that $st(\mathcal{V}_1) \succ st(\mathcal{V}_2) \succ \cdots$. At the other side, if $\mathcal{W}$ is an open covering of Y, from the fact that in compact (paracompact) space for every cover has an open star refinement there exists an open covering $\mathcal{K}$ of Y such that $st(\mathcal{K}) \prec \mathcal{W}$. From cofinality of $(\mathcal{V}_n)$ there exists a member $\mathcal{V}_{n_0}$ such that $\mathcal{V}_{n_0} \prec \mathcal{K}$. Hence, we have $st(\mathcal{V}_{n_0}) \prec st(\mathcal{K}) \prec \mathcal{W}$, i.e., $(st(\mathcal{V}_n))$ is also cofinal sequence.

Theorem 3.1. *The pointwise limit of every super proximate sequence is a continuous function.*

Proof. Let $(f_n) : X \to Y$ be a super proximate sequence and let $\lim_{n\to+\infty} f_n(x) = f(x)$. Let $\mathcal{V}$ be arbitrary finite covering of Y. From the fact that Y is compact and from the Remark we can choose a covering $\mathcal{V}_{n'}$ such that $st(\mathcal{V}_{n'}) \prec \mathcal{V}$. Now, let's take arbitrary $x \in X$ and let V be an element from $\mathcal{V}_{n'}$ such that $f(x) \in V$. Now, there exists $n'' > n'$ such that $f_{n''}(x) \in V$. On the other hand, from $f_{n'} =_{\mathcal{V}_{n'}} f_{n''}$, there exists $V' \in \mathcal{V}$ with property $f_{n'}(x), f_{n''}(x) \in V'$. Finally we have that $f(x), f_{n'}(x) \in st(V, \mathcal{V}_{n'})$ and we can write $f =_{st(\mathcal{V}_{n'})} f_{n'}$. Using the fact $st(\mathcal{V}_{n'}) \prec \mathcal{V}$ we can say the function f is $\mathcal{V}$-near to the $\mathcal{V}$-continuous function $f_{n'}$. From [6, Lemma 4.3.] it follows that f is continuous. $\square$

The following example ensures us that in general super proximate sequences need not to have continuous component functions.

Example 3.2. Take the space I and the proximate sequence $(f_n(x)) : I \to I$ defined by:

$$f_n(x) = \begin{cases} x, & \text{if } x > 0, \\ 1/n, & \text{if } x = 0. \end{cases}$$

We define a sequence $(\mathcal{V}_n)$ of coverings of Y in the following way.

Let $\mathcal{V}_1 = \{[0, 1/2)\} \cup \{B_i^1 \mid i \in 1, 2, \dots, m_1\}$ where B_i^1 are balls with radius smaller than $1/2$ such that $0 \notin B_i^1$.

Now, by Lebesgue lemma choose $\mathcal{V}_2 = \{[0, 1/3)\} \cup \{B_i^2 \mid i \in 1, 2, \dots, m_2\}$ such that $\mathcal{V}_2 \prec \mathcal{V}_1$, B_i^2 have radius smaller than $1/3$ and the only element of $\mathcal{V}_2$ that contains zero is $[0, 1/3)$.

Inductively, define $\mathcal{V}_n = \{[0, 1/(n+1))\} \cup \{B_i^n | i \in 1, 2, \dots, m_n\}$ where $\mathcal{V}_n \prec \mathcal{V}_{n-1} \prec \dots \prec \mathcal{V}_2 \prec \mathcal{V}_1$ and B_i^n are balls with radius smaller than $1/(n+1)$ such that $0 \notin B_i^n$.

We can see that $(f_n(x))$ is a super proximate sequence over $(\mathcal{V}_n)$ with noncontinuous components.

We will show now that for every continuous function can be expressed as limit of a nontrivial super proximate sequence.

Theorem 3.2. *Let $f : X \to Y$ be a continuous function where X, Y are compact, Haussdorf spaces. There exists a cofinal sequence $(\mathcal{W}_n)$ of Y and a super proximate sequence over $(\mathcal{W}_n)$ such that $f_n \to f$, $n \to +\infty$.*

Proof. Let's define the super proximate sequence $(f_n) : X \to Y$ in the following way.

Lets fix $n \in \mathbb{N}$ and take $x \in X$. Choose W_x a member of $\mathcal{V}_n$ with the maximal depth that is contained in $st(f(x), \mathcal{V}_n)$. We define $f_n(x)$ to be one fixed selected element from W_x.

1) f_n is $st(\mathcal{V}_n)$ - continuous.

Let $x \in X$, take $V \in \mathcal{V}_n$ be the element of $\mathcal{V}_n$ such that $f(x) \in V$, from the continuity of f we can choose an open set $U \subset X$ such that $f(U) \subset V$. We have $f_n(U) \subset st(V, \mathcal{V}_n)$, so f_n is $st(\mathcal{V}_n)$ - continuous.

2) If $m > n$, then $f_m =_{st(\mathcal{V}_n)} f_n$. This follows from the fact that $st(f(x), \mathcal{V}_m) \subset st(f(x), \mathcal{V}_n)$.

3) $\lim_{n\to+\infty} f_n(x) = f(x)$. For this part let O be open neighborhood of $f(x)$ in Y. Take the covering $\mathcal{O} = \{O, Y \setminus f(x)\}$ of Y. There exists n_0 such that $st^2(\mathcal{V}_{n_0}) \prec \mathcal{O}$ and there exists $V \in \mathcal{V}_{n_0}$ such that $f(x) \in V$. Now, let $m > n_0$. From $f_m =_{st(\mathcal{V}_{n_0})} f_{n_0}$ and from the fact that $st(V, \mathcal{V}_{n_0})$ is an element of $st(\mathcal{V}_{n_0})$ that contains $f(x)$ we have $f_m(x) \in st^2(V, \mathcal{V}_{n_0}) \subset O$. □

In the proof we can ommit the requirement the covering to be regular, but in this way if a neighborhood U_x of some point x has the property $st(u, \mathcal{V}_n) = st(x, \mathcal{V}_n)$ for all $u \in U_x$ then function f_n will be constant at that neighborhood.

4. Fixed Point Property of Limits

In this section we will establish a connection between fixed points of the limit function and some properties of the corresponding super proximate sequence.

Theorem 4.1. *Let $(f_n) : X \to X$ be a super proximate sequence over the cofinal sequence $(\mathcal{V}_n)$, where X is compact metric space and let $\lim_{n\to+\infty} f_n(x) = f(x)$. The following statements are equivalent.*

1) *f has fixed point.*

2) *For every $n \in \mathbb{N}$ there exists $V_n \in \mathcal{V}_n$ and $x_n \in X$ such that $f_n(x_n)$, x_n lie in the same set of $st(\mathcal{V}_n)$.*

Proof. 1) $\Rightarrow$ 2) Let $f : X \to X$ be the limit of super proximate sequence $(f_n) : X \to X$ and $f(x') = x'$ for a point $x' \in X$. Lets assume the opposite, that there exists $n_0 \in N$ such that $x', f_{n_0}(x') \notin st(V, \mathcal{V}_{n_0})$ for all $V \in \mathcal{V}_{n_0}$. There exists $n_1 > n_0$ such that $f(x') = x', f_{n_1}(x') \in V_{n_0}$ for some $V_{n_0} \in \mathcal{V}_{n_0}$. But, $f_{n_1} =_{\mathcal{V}_{n_0}} f_{n_0}$, so there exists element V'_{n_0} in $\mathcal{V}_{n_0}$ such that $f_{n_1}(x'), f_{n_0}(x') \in V'_{n_0}$ so $f_{n_0}(x'), x' \in st(V_{n_0}, \mathcal{V}_{n_0})$, which is contradiction.

2) $\Rightarrow$ 1) From compactness of X the sequence (x_n) has a convergent sub-sequence (x_{n_k}) in X. Let's assume that $\lim_{k\to+\infty} x_{n_k} = x'$. We claim that $f(x') = x'$. If we suppose the contrary, i.e., $f(x') \neq x'$, then from the fact that X is Hausdorff there exist open sets U' and U'' such that $f(x') \in U'$, $x' \in U''$, $f(U'') \subset U'$ and $U' \cap U'' = \emptyset$. Take the covering $\mathcal{O} = \{X \setminus \{f(x')\}, U'\}$ of X. There exists $n_0 \in \mathbb{N}$ such that:

$$st(\mathcal{V}_{n_0}) \prec \mathcal{O}, x_{n_0} \in U'' \quad \text{and} \quad f(x_{n_0}) \in U'.$$

From the fact that f_{n_0} is $\mathcal{V}_{n_0}$ - continuous it follows that there exist a neighborhood $U_{x'}$ of x' and an element V_{n_0} from $\mathcal{V}_{n_0}$ that contains $f_{n_0}(x')$ such that $f_{n_0}(U_{x'}) \subset V_{n_0}$. Now, choose $n_1 > n_0$ to be large enough such that $f_{n_1}(x'), f(x')$ lie in same element of $\mathcal{V}_{n_0}$ and $x_{n_1} \in U_{x'}$ it follows that $f_{n_0}(x_{n_1}) \in V_{n_0}$. From $f_{n_1} =_{\mathcal{V}_{n_0}} f_{n_0}$ we have that $f_{n_1}(x_{n_1}), f_{n_0}(x_{n_1})$ lie in the same element of $\mathcal{V}_{n_0}$, i.e., $f_{n_1}(x_{n_1}) \in st(V_{n_0}, \mathcal{V}_{n_0})$.

Considering the fact that $f(x') \in st(V_{n_0}, \mathcal{V}_{n_0})$, we have $f_{n_1}(x_{n_1}), x_{n_1}$ must lie in different elements of $st(\mathcal{V}_{n_0})$, which is contradiction. □

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