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TOTALLY WEAKLY CHAIN SEPARATED SETS IN A TOPOLOGICAL SPACE

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ABSTRACT. In this article, by using the notion of chain, we give some characterizations of totally separated spaces. Then, we give some examples, study the properties of those spaces and give new proofs.

Furthermore, by using the notion of chain, we introduce the notions of totally weakly chain separated and totally chain separated sets in a topological space, we state some useful aspects of these sets as well as the various relationships between them and by using these notions we give some characterizations of discrete and totally separated spaces.

1. INTRODUCTION

Unlike the standard definition of connectedness, which is given by a negative sentence, the characterization of connectedness by using the coverings is given by an affirmative sentence (see [4]), and it is a useful tool for proving some particular properties of connected spaces. In [3, 5] connectedness is generalized to the notion of chain connected set in a topological space and some properties are obtained. In [3] a pair of chain separated sets, and in [7] a pair of weakly chain separated sets in a topological space are introduced and, by using these notions, two characterizations of connected space are obtained. In [1] the notion of isolated point in a T_1 space is characterized by using coverings. In [7] a totally separated space and the discrete space are characterized by coverings.

So, by using the notion of chains in coverings we can successfully characterize some topological notions and study their properties.

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In this paper we continue from the articles [1, 3, 5–7] to investigate the notions related to connectedness and its generalizations by using the notion of chain. The statements in this section, except the last two paragraphs, are from these articles.

The basic notions related to chain connectedness together with some important results are introduced in the first chapter. In the second chapter we give a characterization of totally separated spaces by using the notion of chain in a covering, which later we use to study the properties of those spaces. The discrete space and totally separated spaces are also characterized, in the fourth chapter, with the help of the newly introduced notions of totally chain separated and totally weakly chain separated sets in a topological space, respectively. The fifth chapter introduces the space of chain components of a topological space and the space of chain component of a set in a topological space.

In this paper by a covering we understand an open covering. By a covering $\mathcal{U}$ of X, if it is not otherwise stated, we mean a covering $\mathcal{U}$ of X in X.

The following definition is given in [4].

Definition 1.1. Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. *A chain in* $\mathcal{U}$ *that connects* x *and* y (from x to y or from y to x) is a finite sequence of sets $U_1, U_2, \dots, U_n$ of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, \dots, n-1$.

Let X be a topological space and $C \subseteq X$.

Definition 1.2. The set C is *chain connected* in X, if for every covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

The following theorems are proved in [3, 5].

Theorem 1.1. *Let $C \subseteq Y \subseteq X$. If C is chain connected in Y, then C is chain connected in X.*

Theorem 1.2. *If C is chain connected in X and $f : X \to Y$ is a continuous function, then $f(C)$ is chain connected in $f(X)$.*

We denote by $V_{CX}(x, \mathcal{U})$ the set that consists of all elements $y \in C$ such that there exists a chain in $\mathcal{U}$, that connects x and y. If $C = X$, we use the notation $V(x, \mathcal{U})$ instead of $V_{CX}(x, \mathcal{U})$.

Theorem 1.3. *The set $V_{CX}(x, \mathcal{U})$ is nonempty, open, and closed in C.*

Definition 1.3. Let $x, y \in X$. The element x is *chain related* to y in X, and we denote it by $x \underset{X}{\sim} y$ or $x \sim y$, if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.

If x is not chain related to y in X we use the notation $x \underset{X}{\nsim} y$ or $x \nsim y$.

The chain relation in a topological space X is an equivalence relation, and it depends on the set X and the topology τ of X. The chain component $V_{CX}(x)$ of the element x of C in X is the largest chain connected set in C that contains x.

When $C = X$ we use the notation $V_X(x)$ or $V(x)$ for $V_{CX}(x)$.

Let X be a topological space. The *quasicomponent* of $x \in X$, is the intersection of all clopen (closed and open) neighborhoods of x. We denote that with $Q_X(x)$ or $Q(x)$. Quasicomponents are closed sets.

Theorem 1.4. *Let X be a topological space and $C \subseteq X$. Then, for every $x \in C$,*

$$V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} Q_C(y).$$

So, chain components of C in X are a union of quasicomponents of the set and if the set agrees with the space, the chain components match with the quasicomponents, i.e., for every $x \in X$,

$$V_X(x) = Q_X(x).$$

Theorem 1.5. *The topological space X is connected if and only if X is chain connected in X.*

Theorem 1.6. *Let X be a topological space and $C \subseteq X$. If the set C is chain connected in X, then every subset of C is chain connected in X.*

Definition 1.4. Let X be a topological space and $A, B \subseteq X$. The nonempty sets A and B are *weakly chain separated in* X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

Definition 1.5. Let X be a topological space and $A, B \subseteq X$. The nonempty sets A and B are *chain separated in* X, if there exists a covering $\mathcal{U}$ of X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

The following definitions are given in many textbooks about connectedness, as in [2].

A subset of a topological space is disconnected if it is not connected. The topological space X is *totally disconnected* if all subsets with more than one element are disconnected. So, the only connected subsets of X are the singletons and the empty set. Equivalently, the topological space X is totally disconnected if and only if the connected components of X are the singletons.

The topological space X is *totally separated* if its quasicomponents are singletons. Equivalently, the topological space X is totally separated if and only if for every pair of distinct points $x, y \in X$ there exists a separation $X = U \cup V$ (i.e., X is represented as the union of a pair of disjoint open and closed sets U and V) such that $x \in U$ and $y \in V$.

2. Criterion for Totally Separated Spaces by Using the Notion of Chain

The next theorem gives a criterion for totally separated spaces by using the notion of chain.

It enables the study of these spaces by using the coverings of the space and chains on them (Chapter 3). The relation of the theorem with other notions enables the characterization of totally separated spaces through chain components, i.e., quasi-components, chain relation, chain separated and weakly chain separated sets. Some examples of topological spaces explained through the characterization given by this theorem will be considered.

Theorem 2.1. *The topological space X is totally separated if and only if for every two distinct points $x, y \in X$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.*

Proof. Let X be totally separated and $x, y \in X$. It follows that there exists a separation $X = U \cup V$ such that $x \in U$ and $y \in V$. Then for the covering $\mathcal{U} = \{U, V\}$ there is no chain in $\mathcal{U}$ that connects x and y.

Conversely, for every two distinct points $x, y \in X$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Let $U = V(x, \mathcal{U})$ and $V = X \backslash U$. It follows firstly that $x \in U$, $y \in V$ and U is an open and closed set in X and secondly that V is open and closed set in X, i.e., $X = U \cup V$ is a separation. Hence, X is a totally separated space. □

The following proposition is given in [7].

Proposition 2.1. *The topological space X is totally separated if and only if every two distinct singletons of X are weakly chain separated in X.*

The next proposition follows directly from the definition of totally separated spaces and Theorem 1.4.

Proposition 2.2. *The topological space X is totally separated if and only if the only chain components of X are singletons, i.e., for every $x \in X$, $V(x) = \{x\}$.*

From Proposition 2.2 it follows that the topological space X is totally separated if and only if the only chain connected sets are the singletons.

By $CovX$ we mean the set of all coverings of the space X. Note that by a covering in this paper we understand an open covering.

Since from the definition of chain components it follows that $V(x) = \bigcap_{\mathcal{U} \in \mathrm{Cov}\, X} V(x, \mathcal{U})$, the next statement holds.

Proposition 2.3. *X is a totally separated space if and only if for every $x \in X$,*

$$\{x\} = \bigcap_{\mathcal{U} \in \mathrm{Cov}\, X} V(x, \mathcal{U}).$$

Proposition 2.4. *The topological space X is totally separated if and only if every two distinct singletons of X are not in a chain relation, i.e., for every distinct $x, y \in X$, $x \not\sim y$.*

Proof. Let $x \in X$. If $x \not\sim y$ for every $y \in X$, $y \neq x$, it follows that for every $y \in X$ there exists a covering $\mathcal{U}_y$ of X such that there is no chain in $\mathcal{U}_y$ that connects x and y, i.e., $y \notin V(x, \mathcal{U}_y)$. Then $V(x) \subseteq \bigcap_{y \in X \setminus \{x\}} V(x, \mathcal{U}_y) = \{x\}$, i.e., $V(x) = \{x\}$. From arbitrariness of $x \in X$, it follows that X is totally separated.

If the topological space X is totally separated then from Proposition 2.2 it follows that for every $x \in X$, $V(x) = \{x\}$, i.e., for every $x \in X$ and every $y \in X$, $y \neq x$, it follows that $x \not\sim y$. □

Proposition 2.5. *The topological space X is totally separated if and only if every two distinct singleton sets of X are chain separated in X.*

Proof. X is totally separated and $x, y \in X$, $x \neq y$ if and only if there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y, i.e., by Definition 1.5, $\{x\}$ and $\{y\}$ are chain separated in X. From the arbitrariness of $x, y \in X$, $x \neq y$, it follows the accuracy of the statement of the theorem. □

Some examples of totally separated spaces explained using Theorem 2.1 follow.

Example 2.1. a) The discrete space X is totally separated space. Indeed, if $x, y \in X$, $x \neq y$, then for the covering $\mathcal{U} = \{\{x\} \mid x \in X\}$ there is no chain in $\mathcal{U}$ that connects x and y.

b) The space of rational numbers $\mathbb{Q}$ with the standard topology is totally separated space. Namely, if $x, y \in \mathbb{Q}$ then there exists an irrational number z such that $x < z < y$ and for the covering $\mathcal{U} = \{(-\infty, z) \cap \mathbb{Q}, (z, \infty) \cap \mathbb{Q}\}$ of $\mathbb{Q}$ there is no chain in $\mathcal{U}$ that connects x and y. From the arbitrariness of x and y it follows that $\mathbb{Q}$ is totally separated space.

c) The Cantor set C is totally separated space. Indeed, let $x, y \in C$. Then there exists $z \notin C$ such that $x < z < y$ and $\mathcal{U} = \{(-\infty, z) \cap C, (z, \infty) \cap C\}$ is a covering of C such that there is no chain in $\mathcal{U}$ that connects x and y. From the arbitrariness of x and y it follows that C is totally separated space.

d) Sorgenfrey line $\mathbb{R}_l$ is totally separated. Namely, let $a, b \in \mathbb{R}$, $a < b$, and let $c \in (a, b)$. Then $\mathcal{U} = \{(-\infty, c), [c, \infty)\}$ is a covering of $\mathbb{R}_l$ such that there is no chain in $\mathcal{U}$ that connects a and b. It follows that $\mathbb{R}_l$ is totally separated space.

3. Properties of Totally Separated Spaces

In this section we obtain some new proofs for some properties of totally separated spaces by using the criteria from Theorem 2.1 and Propositions 2.1–5.

Theorem 3.1. *If X is a totally separated space, then X is totally disconnected.*

Proof. Let X be a totally separated space, i.e., for every $x \in X$, $V(x) = \{x\}$, where $V(x)$ is a chain component of X that contains x. Since $C(x) \subseteq V(x)$, where $C(x)$ is a connected component of X that contains x, then $C(x) = \{x\}$ holds for every $x \in X$, i.e., X is a totally disconnected space. □

All spaces in Example 2.1 are totally disconnected. The next example (see [8]) shows the existence of a totally disconnected space which is not totally separated.

Example 3.1. Let C be the Cantor set in the unit interval at x-axis and $M\left(\frac{1}{2},\frac{1}{2}\right)$ be a point in the plane. Let $L(N)$ be the segments with endpoints in M and $N \in C$, $E \subseteq C$ be the set of endpoints of the removed intervals obtained by the construction of the Cantor set and $F = C\backslash E$. Define:

$$X_E = \cup\{L(N) \mid N \in E\}, \quad X_F = \cup\{L(N) \mid N \in F\},$$

$$Y_E = \{(x,y) \in X_E \mid y \in \mathbb{Q}\} \quad \text{and} \quad Y_F = \{(x,y) \in X_F \mid y \neq \mathbb{Q}\}.$$

The Knaster-Kuratowski fan (Figure 1) is the set $Y = Y_E \cup Y_F$.

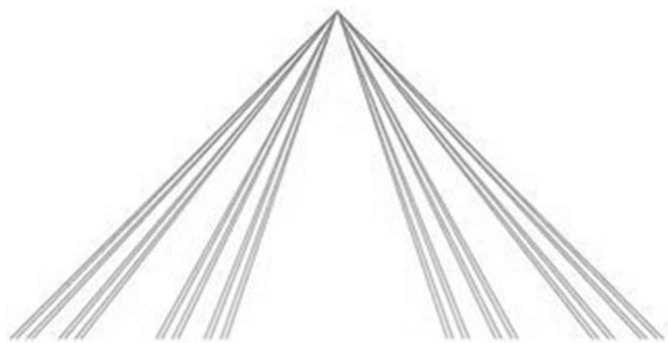

FIGURE 1. The Knaster-Kuratowski fan

The Knaster-Kuratowski fan with the removed point, $Y^* = Y\backslash\{M\}$, is a totally disconnected space (see [8]). However, Y^* is not totally separated, since for every $N \in C$, $L(N) \cap Y^*$ is contained in one quasicomponent, i.e., the chain component $V(N)$ (see Example 129, page 145–147 in [8]).

Theorem 3.2. *Let $\{X_i\}_{i\in I}$ be a family of disjoint totally separated spaces. Then, the disjoint union (sum) $X = \coprod_{i\in I} X_i$ is a totally separated space if and only if X_i are totally separated spaces for every $i \in I$.*

Proof. Let X_i be sets such that for all $i,j \in I$, $i \neq j$, $X_i \cap X_j = \emptyset$. We assume that X is a totally separated space. Then, by Theorem 2.1, X_i are totally separated spaces for all $i \in I$.

Conversely, let X_i, $i \in I$, be totally separated spaces. Let A be an arbitrary chain connected subset in X. We assume that there exist $i,j \in I$, $i \neq j$, such that $A \cap X_i \neq \emptyset$ and $A \cap X_j \neq \emptyset$. In this case there is no chain from $x \in A \cap X_i$ to $y \in A \cap X_j$ in the covering $\mathcal{U} = \{X_i \mid i \in I\}$, which is opposite of the assumption that A is a chain connected set in X. Therefore, there exist only one index $i \in I$ such that $A \subseteq X_i$, and since X_i is a totally separated space, A is a singleton. From the arbitrariness of A, it follows that X is totally separated. □

We notice that the sufficient condition in the previous theorem is valid also if $\{X_i\}_{i\in I}$ is not a family of disjoint spaces. Moreover, Theorem 3.2 is true if only we consider the sum of topological spaces. Specifically, this theorem is not valid for $X = \bigcup_{x\in[0,1]}\{x\}$, where $[0,1]$ is considered with the standard topology.

Theorem 3.3. *Let $f : X \to Y$ be an injective continuous function. If Y is a totally separated space, then X is totally separated.*

Proof. Let $f : X \to Y$ be an injective continuous function and Y be a totally separated space. Let C be a chain connected set in X. Then $f(C)$ is chain connected in Y, and since Y is totally separated, $f(C)$ is singleton. Since f is an injection, the set C is a singleton. Hence, all chain connected sets in X are singletons, i.e., X is a totally separated space. □

The following example shows why injectivity of the function is a necessary condition on the previous theorem.

Example 3.2. Let $X = [0,1] \cup \{2\}$, $Y = \{1,2\}$ and $f : X \to Y$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ 2, & \text{if } x = 2. \end{cases}$$

Then f is a continuous non-injective function, Y is a totally separated space, but X is not.

Theorem 3.4. *Let $f : X \to Y$ be a homeomorphism. Then X is totally separated space if and only if Y is totally separated.*

Proof. Let X be a totally separated space. Then the chain components of X are singletons.

Let C be a chain connected set in Y. Then $f^{-1}(C)$ is a chain connected set in X, and therefore, $f^{-1}(C)$ is a singleton. Since f is bijection, it follows that the set C is a singleton. From the arbitrariness of C it follows that all chain connected sets in Y are singletons, i.e., Y is a totally separated space.

The converse statement can be proved analogously, if we work with f instead of f^{-1}. □

However, if X and Y are homotopic equivalent, it doesn't imply that both spaces are totally separated. This statement is proved by the following example.

Example 3.3. Let $X = \{1,2\}$ and $Y = [0,1] \cup [2,3]$. Then, X and Y are homotopic equivalent and X is totally separated but Y is not.

Theorem 3.5. *Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \subset \tau_2$. Then, if (X,τ_1) is a totally separated space, so is (X,τ_2).*

Proof. Assume that (X,τ_2) is not a totally separated space, i.e., there exist $x, y \in X$ such that for all coverings of (X,τ_2) there exists a chain from x to y. Since $\tau_1 \subset \tau_2$, all coverings of (X, τ_1) are also coverings of (X, τ_2). Therefore, for any covering of (X, τ_1) there exists a chain from x to y, i.e., (X, τ_1) is not totally separated space. $\square$

In order to point out that the converse statement of the above theorem is not valid we consider the real line $\mathbb{R}$ with the standard topology and the Sorgenfrey line $\mathbb{R}_l$. Namely, $\mathbb{R} \subset \mathbb{R}_l$, $\mathbb{R}_l$ is a totally separated space but $\mathbb{R}$ is not totally separated via Theorem 3.1, since $\mathbb{R}$ is connected.

4. Totally Chain Separated and Totally Weakly Chain Separated Sets in a Topological Space

Now, we will define the notion of a totally weakly chain separated set in a topological space.

Let X be a topological space and $C \subseteq X$.

Definition 4.1. The set C is totally weakly chain separated in X if for every two distinct points $x, y \in C$ there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

The next statement follows from Definition 4.1 and Theorem 2.1.

Corollary 4.1. *The topological space X is totally separated if and only if X is totally weakly chain separated in X.*

Proposition 4.1. *The set C is totally weakly chain separated in X if and only if every two distinct singletons in C are weakly chain separated in X, i.e., if and only if every two distinct singletons in C are chain separated in X.*

Proof. The set C is totally weakly chain separated in X, i.e., for every two distinct points $x, y \in C$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y if and only if from Definition 1.5 every two distinct singletons in C are chain separated in X. Clearly, two singletons are weakly chain separated in X if and only if they are chain separated in X. $\square$

Proposition 4.2. *The set C is totally weakly chain separated in X if and only if the only chain components of C in X are the singletons, i.e., for every $x \in C$, $V_{CX}(x) = \{x\}$.*

Proof. Let $x \in C$. The element $y \in V_{CX}(x)$, $y \neq x$; if and only if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y, i.e., C is not a totally weakly chain separated in X. $\square$

Theorem 4.1. *Every subset of a totally weakly chain separated set in X is a totally weakly chain separated set in X.*

Proof. Let C be a totally weakly chain separated set in X and $D \subseteq C$. It follows that for every $x, y \in C$ and, as a consequence, for every $x, y \in D$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y, i.e., D is a totally weakly chain separated set in X. $\square$

Theorem 4.2. *Let $C \subseteq Y \subseteq X$. If C is a totally weakly chain separated set in X, then C is a totally weakly chain separated in Y.*

Proof. Let C be a totally weakly chain separated set in X and let $x, y \in C$. It follows that there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Then for the covering $\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y \mid U \in \mathcal{U}\}$ there is no chain in Y that connects x and y, i.e., C is totally weakly chain separated in Y. □

Corollary 4.2. *If C is totally weakly chain separated set in X, then C is totally separated.*

Proof. If C is totally weakly chain separated set in X, then C is totally weakly chain separated set in C by Theorem 4.2 and so, by Corollary 4.1, C is totally separated. □

The next example shows that the converse statement of Corollary 4.2 is not true in general.

Example 4.1. Let $X = [0, 1]$ and $C = \{0, 1\}$. Then C is totally separated since C is the discrete, i.e., $V_C(0) = \{0\}$ and $V_C(1) = \{1\}$, but it is not totally weakly chain separated in X since X is connected, i.e., from Theorem 1.5, X is chain connected in X, and from Theorem 1.6, C is chain connected in X, i.e., for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects 0 and 1. The conclusion can be done directly, for arbitrary covering $\mathcal{U}$ of X, since X is compact, there exists a finite subcovering from which we can chose a chain that connects 0 and 1.

Corollary 4.3. *The set C is totally weakly chain separated in X if for every distinct $x, y \in C$, $x \not\sim y$.*

Proof. Obvious from Corollary 4.1 and Proposition 2.2. □

We want to consider the set that is defined similarly as the totally weakly chain separated set where the separation is reinforced by the rotation of the quantifiers.

Definition 4.2. The set C is totally chain separated in X if there exists a covering $\mathcal{U}$ of X such that for every two distinct points $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y.

The difference between Definition 4.1 and Definition 4.2 is that quantifiers are rotated. A totally chain separated set is separated by one covering, i.e., the separation is strong. If C is a totally chain separated set, then there exists a covering $\mathcal{U}$ such that the set $\mathcal{U} \cap C$ consists of singletons. A totally weakly chain separated set in general case does not have to be separated by one covering, i.e., the separation is weak.

Clearly, if the set C is totally chain separated in X, then C is totally weakly chain separated in X. The next example shows that the converse case does not hold in general.

Example 4.2. The sets $C_{n_0} = \{0\} \cup \left\{\frac{1}{n} \mid n \geq n_0\right\}$, $n_0 \in \mathbb{N}$, are totally weakly chain separated in $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Namely for arbitrary elements $x = 0$ or $x = \frac{1}{n_2}$, and $y = \frac{1}{n_1}$, $n_2 > n_1 \geq n_0$, for the covering

$$\mathcal{U} = \left\{\{0\} \cup \left\{\frac{1}{n} \mid n > n_2\right\}, \left\{\frac{1}{n_2}\right\}, \left\{\frac{1}{n_2 - 1}\right\}, \ldots, \left\{\frac{1}{n_0}\right\}, \left\{\frac{1}{n_0 - 1}\right\}, \ldots, \{1\}\right\}$$

there is no chain in $\mathcal{U}$ that connects x and y. But the sets C_{n_0}, $n_0 \in \mathbb{N}$, are not totally chain separated in X. Namely, if $\mathcal{U}$ is a covering of X, then the element $U \in \mathcal{U}$ that contains 0, contains also an element $z = \frac{1}{n_3}$, $n_3 \in \mathbb{N}$, and U is a chain in $\mathcal{U}$ that connects 0 and z.

Theorem 4.3. *Let $C \subseteq Y \subseteq X$. If C is a totally chain separated set in X, then C is totally chain separated in Y.*

Proof. Let C be a totally chain separated set in X, i.e., there exists a covering $\mathcal{U}$ of X such that for every distinct $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y. It follows that $\mathcal{U}_Y = \mathcal{U} \cap Y$ is a covering of Y such that there is no chain in $\mathcal{U}_Y$ that connects x and y for every distinct $x, y \in C$, i.e., C is totally chain separated in Y. □

Theorem 4.4. *The set C is totally chain separated in C if and only if C is the discrete space.*

Proof. Let C be totally chain separated in C, i.e., there exists a covering $\mathcal{U}$ of C such that for every distinct $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y. It follows that for every $x \in C$ the chain component $V(x) = V(x, \mathcal{U}) = \{x\}$ is an open singleton, i.e., C is the discrete space.

Conversely, let C be the discrete space, i.e., every singleton in C is open. Then for the covering $\mathcal{U} = \{\{x\} \mid x \in C\}$ there is no chain in $\mathcal{U}$ that connects x and y, for every distinct $x, y \in C$, i.e., C is totally chain separated in C. □

If the set C is totally chain separated in X, then C is the discrete space. Example 4.1 shows that even if C is a discrete space, it may not be a totally chain separated in X.

The discrete space is characterized by chain in [1, 7]. Here we give a new characterization. According to genesis of the notion, by using this criterion, the discrete space also can be called totally chain separated space.

5. The Space of Chain Components of a Set in a Topological Space

Let X be a topological space and $C \subseteq X$.

The space of quasicomponents QX of a topological space X consists of the all quasicomponents of X equipped with the topology generated by the base composed from the sets $QF = \{A \mid A \in QX, A \subseteq F\}$ where F is clopen in X.

The next statement is given below Theorem 2.2 in [3].

Proposition 5.1. *The nonempty set A is clopen in X if and only if there exists a point $x \in X$ and a covering $\mathcal{U}$ of X such that $A = V(x, \mathcal{U})$.*

Proof. Let A be a clopen set and $x \in A$. Then, $X \backslash A$ is also a clopen set and for the covering $\mathcal{U} = \{A, X \backslash A\}$ it follows that $A = V(x, \mathcal{U})$.

If for the set A holds $A = V(x, \mathcal{U})$ for some covering $\mathcal{U}$ of X and $x \in X$, since, by Theorem 1.3, $V(x, \mathcal{U})$ is nonempty and clopen in X, it follows that A is clopen in X. □

Let VX be the set of all chain components of the space X. Clearly $QX = VX$.

Definition 5.1. *A space of chain components of* X is the set VX with the topology generated by the base composed from the sets:

$$\{A \mid A \in VX, A \subseteq V(x, \mathcal{U})\}, \quad x \in X, \mathcal{U} \in \operatorname{Cov} X.$$

Since, from Proposition 5.1 it follows that for every nonempty clopen set A in X there exists a covering $\mathcal{U}$ of X and a point $x \in X$, such that $A = V(x, \mathcal{U})$, the space of chain components of a topological space X is well defined and matches with the space of quasicomponents. So, Definition 5.1 is one more interpretation of the space of quasicomponents.

If the space X is a totally separated space, then the elements of the corresponding space of chain components, VX, are singletons $\{x\}$, $x \in X$.

In the next definition we generalise the notion of a space of chain components to a space of chain components of a set in a space.

Let VCX be the set of all chain components of the set C in X.

Definition 5.2. *A space of chain components of a set* C in a topological space X is the set VCX with the topology generated by the base composed from the sets:

$$\{A \mid A \in VCX, A \subseteq V_{CX}(x, \mathcal{U})\}, \quad x \in X, \mathcal{U} \in \operatorname{Cov} X.$$

Since a chain component of a set in a topological space in general is a union of quasicomponents [3], the space of chain components of a set in a topological space in general differs from a space of chain components.

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